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"Inequality order for Zenga distribution"

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Inequality order for Zenga distribution

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Abstract

The aim of this paper is to investigate the roles of the parameters in the inequality-analysis framework for the recently introduced Zenga distribution. The main results are summarized in two different theorems, which state that the two shape parameters of such distribution are inequality indicators. The inequality-evaluating tools considered are Lorenz curve with the related Gini concentration ratio and Zenga inequality curve with the related inequality index I.

Keywords: Zenga distribution, Lorenz order, order based on $I(p)$ curve, convex order, inequality $I(p)$ curve.

1 Introduction

In literature many distributions have been introduced in order to model particular financial, economic or actuarial variables. All of them have some important features, suitable to describe the variables of interest.

In Zenga (2010) a new distribution has been proposed: so far it seems to be very useful to model economics variables like wealth and income distribution. The advantages of such distribution have been described in several works, like Zenga (2010), Zenga et al. (2010a) and Zenga et al. (2010b). Just to report some of them:

1. the capability to assume very different shapes;
2. the guaranteed (for any admissable choice of the parameters) finiteness of the expectation;
3. the paretian right-tail;
4. the fitting to real data.

For further details about the previous subjects, especially for the last one, refer to Arcagni (2011).

In this paper the meaning of the parameters is investigated. Particular attention is devoted to their interpretation as inequality indicators. This subject is fundamental in inequality analysis, since it allows to understand how the inequality of the distribution changes as a parameter varies.

The paper is organized as follows. In the next section some preliminary definitions are recalled. In Section 3 three partial orders are introduced and the links among them investigated. In Section 4 two ordering theorems for Zenga distribution are stated and proved in detail. In the final section, a conclusive assessment about the consequences for inequality analysis is provided.

2 Preliminaries

The new distribution model introduced in Zenga (2010) has the probability density function depending on the three positive parameters μ, α, θ and it is given by:

$$f(x;\mu,\alpha,\theta) = \begin{cases} \dfrac{1}{2\mu B(\alpha,\theta)} \left(\dfrac{x}{\mu}\right)^{-3/2} \displaystyle\int_0^{\frac{x}{\mu}} k^{\alpha-1/2}(1-k)^{\theta-2}dk & 0 < x \leq \mu \\ \dfrac{1}{2\mu B(\alpha,\theta)} \left(\dfrac{\mu}{x}\right)^{3/2} \displaystyle\int_0^{\frac{\mu}{x}} k^{\alpha-1/2}(1-k)^{\theta-2}dk & x > \mu \end{cases}$$

where $B(a, b)$ denotes the mathematical Beta function

$$\begin{aligned} B(a,b) &= \int_0^1 t^{a-1}(1-t)^{b-1}\,dt \\ &= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \end{aligned} \qquad (1)$$

where $a > 0$ and $b > 0$ and Γ denotes the Gamma function

$$\Gamma(y) = \int_0^{+\infty} e^{-t}t^{y-1}\,dt \qquad y > 0,$$

A complete analysis on Beta and Gamma functions can be found in Lebedev (1972).

In the following $X \sim Zenga(\mu, \alpha, \theta)$ will be used to denote that the random variable X follows Zenga distribution with density $f(x; \mu, \alpha, \theta)$.

As shown in detail in Zenga et al. (2010a) and in Zenga et al. (2010b) the density of Zenga distribution is very flexible: it is easy to see how many behaviours can have, changing the values of the parameters. In Figures 1 and 2 some probability density functions and the corresponding distribution functions are showed: in the first one, the value of the parameters μ and α is fixed and equal to 2, while in the second one, α changes and the other two parameters are set to 2.

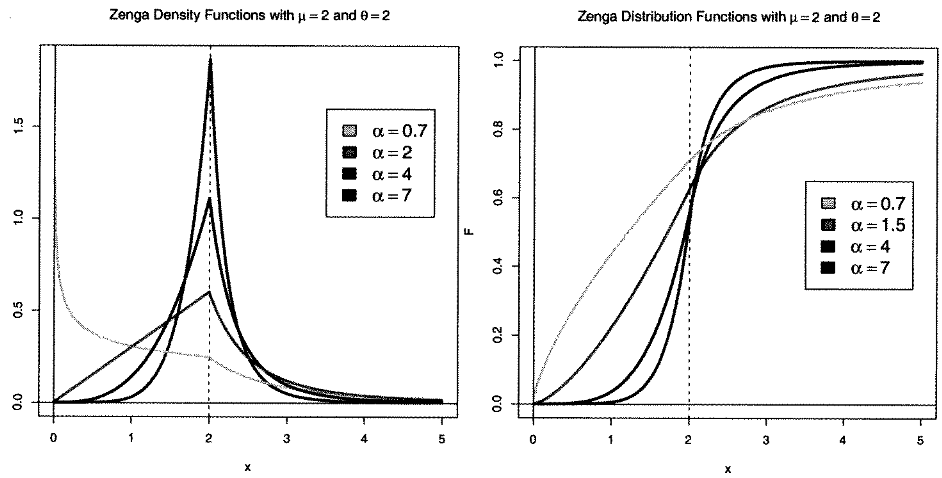

Figure 1: Density (on the left) and distribution (on the right) functions of Zenga model with $\mu = 2$ and $\theta = 2$

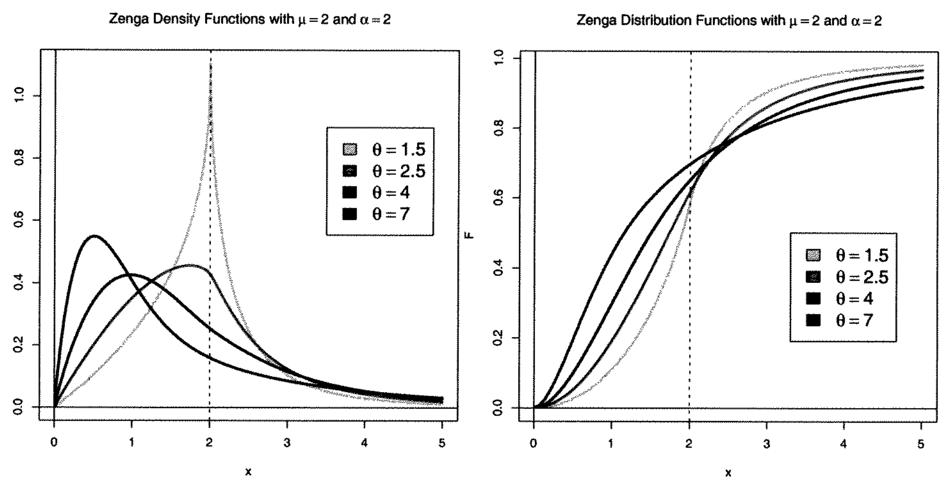

Figure 2: Density (on the left) and distribution (on the right) functions of Zenga model with $\mu = 2$ and $\alpha = 2$

Zenga distribution is obtained as a mixture of the following particular truncated Pareto distributions, introduced by Polisicchio (2008)

$$f(x;\mu|k) = \begin{cases} \frac{\sqrt{\mu}}{2} k^{1/2}(1-k)^{-1}x^{-3/2} & \mu k \leq x \leq \frac{\mu}{x} \\ 0 & \text{otherwise} \end{cases}$$

with mixing function on the parameter k given by a Beta density with parameters $\alpha > 0$ and $\theta > 0$:

$$g(k;\alpha,\theta) = \begin{cases} \frac{k^{\alpha-1}(1-k)^{\theta-1}}{B(\alpha,\theta)} & 0 < k < 1 \\ 0 & \text{otherwise.} \end{cases}$$

A first important feature of the conditional densities is that all of them have the same expectation $\mu > 0$: this implies that μ is also the expectation of the mixture.

A second characteristic is that each of them has uniform $I(p)$ curve with inequality level equal to $1 - k$ with $k \in (0,1)$. (see Polisicchio (2008) for further details and proofs).

Altrought, the natural range of the parameter θ is $]0, +\infty[$, in the following only the case $\theta > 1$ will be considered. It is opinion of the author that, with some changes in the proofs, the results can be extended to the case $\theta \in]0,1]$, but further investigations are necessary.

3 Inequality orders

In literature several stochastic orderings for distributions have been introduced and extensively studied. The most suitable orders for inequality analysis are the one based on Lorenz curve and the one based on inequality $I(p)$ curve. Another useful order is the convex one. In the following the definitions of the aforementioned orders are recalled, and the links among them is presented. A complete overview on this subject can be found in Shaked and Shanthikumar (2007). It is important to underline that these three are partial orders and not total ones.

Definition 1 *Let X_1 and X_2 be two continuous non-negative random variables with finite expectations. X_1 is said to be larger (or more unequal) than X_2 in the Lorenz ordering (and it is note $X_1 \geq_L X_2$), iff*

$$L_{X_1}(p) \leq L_{X_2}(p) \qquad \forall p \in (0,1)$$

where $L_{X_i}(p)$, $i = 1, 2$, is the value assumed by the Lorenz curve of X_i in p (with $i = 1, 2$).

In analogy to the order based on Lorenz curve, Porro (2008) introduced the order based on $I(p)$ curve by the following definition.

Definition 2 *Let X_1 and X_2 be two continuous non-negative random variables with finite expectations. X_1 is said to be larger (or more unequal) than X_2 in the ordering based on $I(p)$ curve (and it is note $X_1 \geq_I X_2$), iff*

$$I_{X_1}(p) \geq I_{X_2}(p) \qquad \forall p \in (0,1)$$

where $I_{X_i}(p)$, $i = 1, 2$, is the value assumed by the inequality $I(p)$ curve of X_i in p (with $i = 1, 2$).

The partial order based on $I(p)$ curve and the partial order based on $L(p)$ curve are equivalent, since as shown in Porro (2008), the next equivalence lemma holds.

Lemma 1 (Equivalence Lemma) *Let X_1 and X_2 be two continuous non-negative random variables, both with finite and positive expectation. Then:*

$$X_1 \geq_L X_2 \Leftrightarrow X_1 \geq_I X_2.$$

The last considered ordering is the well-known convex order, which is approached in the next section. Unlike the previous ones, the convex order needs the equality of the expectattions of the two random variables involved.

Definition 3 *Let X_1 and X_2 be two continuous non-negative random variables with finite expectations. X_1 is said to be larger than X_2 in the convex order (and it is note $X_1 \geq_{CX} X_2$), iff*

$$\mathbb{E}[\phi(X_1)] \geq \mathbb{E}[\phi(X_2)]$$

for all the convex functions $\phi : \mathbb{R} \to \mathbb{R}$, such that the expectations exist.

An important result involving the above partial orders holds: it will be applied in the next section to prove the two main ordering theorems (see for further details Shaked and Shanthikumar (2007) and Porro (2008)). It is a generalization of the Equivalence Lemma and it can be stated as follows:

Proposition 1 *Let X_1 and X_2 be two continuous non-negative random variables with the same finite expectation. Then*

$$X_2 \leq_{CX} X_1 \quad \Leftrightarrow \quad X_2 \leq_L X_1 \quad \Leftrightarrow \quad X_2 \leq_I X_1.$$

4 Ordering theorems

In this section the main results of the papar are presented and proved. In order to semplify the presentation, some useful lemmas and related corollaries are introduced.

Lemma 2 *Let $y > 0$ and $\mathbb{R}_0^+ = \{x \in \mathbb{R} : x > 0\}$. Then the function h_y:*

$$\begin{array}{rccc} h_y : & \mathbb{R}_0^+ & \longrightarrow & \mathbb{R}_0^+ \\ & x & \longmapsto & \dfrac{\Gamma(x)}{\Gamma(x+y)} \end{array}$$

is monotonically strictly decreasing.

Proof of Lemma 2
The derivative of the function $h_y = h_y(x)$ with respect to x is:

$$\frac{dh_y(x)}{dx} = \frac{\Gamma'(x)\Gamma(x+y) - \Gamma(x)\Gamma'(x+y)}{[\Gamma(x+y)]^2}.$$

Recalling the definition of the digamma function ψ:

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

it follows that:

$$\begin{aligned} \frac{dh_y(x)}{dx} &= \frac{\Gamma(x)\psi(x)\Gamma(x+y) - \Gamma(x)\psi(x+y)\Gamma(x+y)}{[\Gamma(x+y)]^2} \\ &= \frac{\Gamma(x)\Gamma(x+y)}{[\Gamma(x+y)]^2}\left[\psi(x) - \psi(x+y)\right]. \end{aligned}$$

It is well-known (see for example Abramowitz and Stegun (1964)) that for $x > 0$, the digamma function $\psi(x)$ is monotonically increasing: it implies that $\psi(x) - \psi(x+y) < 0$ and therefore $\dfrac{dh_y(x)}{dx} < 0\ \forall x \in \mathbb{R}_0^+$. So it is proved that $h_y(x)$ is a monotonically decreasing function in $\mathbb{R}_0^+$.

Corollary 1 *If $0 < \alpha_1 < \alpha_2$ and $y > 0$, then $\dfrac{\Gamma(\alpha_2)\Gamma(\alpha_1+y)}{\Gamma(\alpha_2+y)\Gamma(\alpha_1)} < 1$.*

Proof of Corollary 1
By the lemma 2, if $0 < \alpha_1 < \alpha_2$ it follows that:

$$h_y(\alpha_2) < h_y(\alpha_1),$$

that is, by definition of function h_y:

$$\frac{\Gamma(\alpha_2)}{\Gamma(\alpha_2+y)} < \frac{\Gamma(\alpha_1)}{\Gamma(\alpha_1+y)}$$

or, equivalently:

$$\frac{\Gamma(\alpha_2)\Gamma(\alpha_1+y)}{\Gamma(\alpha_2+y)\Gamma(\alpha_1)} < 1.$$

Lemma 3 *Let $f:\mathbb{R}\rightarrow\mathbb{R}_0^+$ and $g:\mathbb{R}\rightarrow\mathbb{R}_0^+$ two derivable, positive and strictly decreasing functions. Then the function*

$$\begin{array}{rccl} p: & \mathbb{R} & \longrightarrow & \mathbb{R}_0^+ \\ & x & \longmapsto & p(x)=f(x)g(x) \end{array}$$

is monotonically strictly decreasing.

Proof of Lemma 3
The derivative of p is

$$\frac{d}{dx}p(x) = f'(x)g(x) + f(x)g'(x).$$

By hypothesis, $f' < 0$, $g' < 0$, $f > 0$, and $g > 0$, then it follows that dp/dx is negative and therefore the thesis holds.

Another important result is the following (see Shaked and Shanthikumar (2007) for proof and further details): it is a sufficient condition for the convex order.

Lemma 4 *Let X_1 and X_2 two continuous random variables with the same expectation and probability density function f_1 and f_2 respectively. If the function $f_2 - f_1$ changes sign twice and the sign sequence is $+,-,+$, then $X_1 \leq_{CX} X_2$.*

Now, it is possible to start the investigation about the roles of the distribution parameters from the inequality point of view. First, it is interesting to point out that for Zenga distribution μ is a scale parameter (see Zenga (2010)), then it is not relevant from the inequality point of view. This means that every reasonable inequality-measuring tool, such as inequality curves, inequality indexes, etc., must not be depending on it. It is clear therefore, why the investigation about the role of the distribution parameters has to be restricted to α and θ, considering μ a fixed value.

The statement of the first ordering theorem is the following.

Theorem 1 (First ordering theorem) *Let X_1 and X_2 two continuous random variables such that:*

- $X_1 \sim$ *Zenga* (μ, α_1, θ)
- $X_2 \sim$ *Zenga* (μ, α_2, θ)

where $\theta > 1$, $0 < \alpha_1 < \alpha_2$ and $\mu > 0$. Then $X_2 \leq_{CX} X_1$.

Proof of Theorem 1

Obviously, the random variables X_1 and X_2 have the same expected value μ.

Now, let f_i denote the probability density function of X_i $i = 1, 2$. In order to apply Lemma 4, the number of sign changes of $f_2 - f_1$ has to be determinated. To do that, let g be the ratio between the two density functions, that is:

$$g(x) = \frac{f_1}{f_2}(x).$$

The function g for $x \in]0, \mu]$ is given by:

$$\begin{aligned} g(x) &= \frac{\dfrac{1}{2\mu B(\alpha_1, \theta)} \left(\dfrac{x}{\mu}\right)^{-3/2} \displaystyle\int_0^{x/\mu} k^{\alpha_1 - 1/2}(1-k)^{\theta-2}\, dk}{\dfrac{1}{2\mu B(\alpha_2, \theta)} \left(\dfrac{x}{\mu}\right)^{-3/2} \displaystyle\int_0^{x/\mu} k^{\alpha_2 - 1/2}(1-k)^{\theta-2}\, dk} \\ &= \frac{B(\alpha_2, \theta)}{B(\alpha_1, \theta)} \cdot \frac{\displaystyle\int_0^{x/\mu} k^{\alpha_1 - 1/2}(1-k)^{\theta-2}\, dk}{\displaystyle\int_0^{x/\mu} k^{\alpha_2 - 1/2}(1-k)^{\theta-2}\, dk}, \end{aligned}$$

while for $x > \mu$ coincides with

$$\begin{aligned} g(x) &= \frac{\dfrac{1}{2\mu B(\alpha_1, \theta)} \left(\dfrac{\mu}{x}\right)^{3/2} \displaystyle\int_0^{\mu/x} k^{\alpha_1 - 1/2}(1-k)^{\theta-2}\, dk}{\dfrac{1}{2\mu B(\alpha_2, \theta)} \left(\dfrac{\mu}{x}\right)^{3/2} \displaystyle\int_0^{\mu/x} k^{\alpha_2 - 1/2}(1-k)^{\theta-2}\, dk} \\ &= \frac{B(\alpha_2, \theta)}{B(\alpha_1, \theta)} \cdot \frac{\displaystyle\int_0^{\mu/x} k^{\alpha_1 - 1/2}(1-k)^{\theta-2}\, dk}{\displaystyle\int_0^{\mu/x} k^{\alpha_2 - 1/2}(1-k)^{\theta-2}\, dk}. \end{aligned}$$

It is important to note that g is continuous, since functions f_1 and f_2 are continuous. Consider first the case $x \in]0, \mu]$.

The value of g as x approaches the extreme 0 is:

$$\lim_{x \to 0+} g(x) = \lim_{x \to 0+} \frac{B(\alpha_2, \theta)}{B(\alpha_1, \theta)} \cdot \frac{\displaystyle\int_0^{x/\mu} k^{\alpha_1 - 1/2}(1-k)^{\theta-2}\, dk}{\displaystyle\int_0^{x/\mu} k^{\alpha_2 - 1/2}(1-k)^{\theta-2}\, dk},$$

and then, by De l'Hopital rule, it becomes:

$$\lim_{x \to 0+} g(x) = \lim_{x \to 0+} \frac{B(\alpha_2, \theta)}{B(\alpha_1, \theta)} \cdot \frac{\dfrac{1}{\mu} \left(\dfrac{x}{\mu}\right)^{\alpha_1 - 1/2} \left(1 - \dfrac{x}{\mu}\right)^{\theta-2}}{\dfrac{1}{\mu} \left(\dfrac{x}{\mu}\right)^{\alpha_2 - 1/2} \left(1 - \dfrac{x}{\mu}\right)^{\theta-2}}$$

$$= \lim_{x\to 0^+} \frac{B(\alpha_2,\theta)}{B(\alpha_1,\theta)} \cdot \left(\frac{1}{\mu}\right)^{\alpha_1-\alpha_2} x^{\alpha_1-\alpha_2} = +\infty.$$

As x approaches to μ:

$$\begin{aligned}
\lim_{x\to\mu^-} g(x) &= \lim_{x\to\mu^-} \frac{B(\alpha_2,\theta)}{B(\alpha_1,\theta)} \cdot \frac{\int_0^{x/\mu} k^{\alpha_1-1/2}(1-k)^{\theta-2}\,dk}{\int_0^{x/\mu} k^{\alpha_2-1/2}(1-k)^{\theta-2}\,dk} \\
&= \frac{B(\alpha_2,\theta)}{B(\alpha_1,\theta)} \frac{B(\alpha_1+1/2,\theta-1)}{B(\alpha_2+1/2,\theta+1)} \\
&= \frac{\Gamma(\alpha_2)\Gamma(\alpha_2+\theta-1/2)\Gamma(\alpha_1+1/2)\Gamma(\alpha_1+\theta)}{\Gamma(\alpha_2+1/2)\Gamma(\alpha_2+\theta)\Gamma(\alpha_1)\Gamma(\alpha_1+\theta-1/2)}.
\end{aligned}$$

Considering now the functions

$$\begin{aligned}
h_{1/2} : \mathbb{R}_0^+ &\longrightarrow \mathbb{R}_0^+ \\
x &\longmapsto \frac{\Gamma(x)}{\Gamma(x+1/2)}
\end{aligned}$$

and

$$\begin{aligned}
h_{\theta+1/2} : \mathbb{R}_0^+ &\longrightarrow \mathbb{R}_0^+ \\
x &\longmapsto \frac{\Gamma(x)}{\Gamma(x+\theta+1/2)}.
\end{aligned}$$

By Lemma 2 they are monotonically decreasing and consequently, the function $h_{1/2} \cdot h_{\theta+1/2}$ is decreasing (by Lemma 3). So, for $0 < \alpha_1 < \alpha_2$, it holds that:

$$\frac{\Gamma(\alpha_2)\Gamma(\alpha_2+\theta-1/2)}{\Gamma(\alpha_2+1/2)\Gamma(\alpha_2+\theta)} < \frac{\Gamma(\alpha_1)\Gamma(\alpha_1+\theta-1/2)}{\Gamma(\alpha_1+1/2)\Gamma(\alpha_1+\theta)}$$

and therefore

$$\lim_{x\to\mu-} g(x) < 1.$$

Now, the derivative of function g is:

$$\begin{aligned}
\frac{d}{dx}g(x) &= \frac{d}{dx}\left[\frac{B(\alpha_2,\theta)}{B(\alpha_1,\theta)} \cdot \frac{\int_0^{x/\mu} k^{\alpha_1-1/2}(1-k)^{\theta-2}\,dk}{\int_0^{x/\mu} k^{\alpha_2-1/2}(1-k)^{\theta-2}\,dk}\right] \\
&= c \cdot \left[\left(\frac{x}{\mu}\right)^{\alpha_1-1/2} \int_0^{x/\mu} k^{\alpha_2-1/2}(1-k)^{\theta-2}\,dk - \left(\frac{x}{\mu}\right)^{\alpha_2-1/2} \int_0^{x/\mu} k^{\alpha_1-1/2}(1-k)^{\theta-2}\,dk\right] \\
&= c \cdot \int_0^{x/\mu} (1-k)^{\theta-2} \left[\left(\frac{x}{\mu}\right)^{\alpha_1-1/2} k^{\alpha_2-1/2} - \left(\frac{x}{\mu}\right)^{\alpha_2-1/2} k^{\alpha_1-1/2}\right] dk \\
&= c \cdot \int_0^{x/\mu} (1-k)^{\theta-2} k^{\alpha_1-1/2} \left(\frac{x}{\mu}\right)^{\alpha_1-1/2} \left[k^{\alpha_2-\alpha_1} - \left(\frac{x}{\mu}\right)^{\alpha_2-\alpha_1}\right] dk \qquad (2)
\end{aligned}$$

where

$$c = \frac{B(\alpha_2,\theta)}{\mu B(\alpha_1,\theta)} \left(1-\frac{x}{\mu}\right)^{\theta-2} \left[\int_0^{x/\mu} k^{\alpha_2-1/2}(1-k)^{\theta-2}\,dk\right]^{-2} > 0.$$

The integrand function in equation (2) is strictly smaller than 0 for $k \in]0, x/\mu[$ and it is zero for $k = x/\mu$: that implies therefore that the value of the integral is strictly smaller than 0: this means that $dg(x)/dx$ is negative and so g is a monotonically strictly decreasing function if $x \in]0, \mu]$. Here a recap for $x \in]0, \mu]$:

- $\lim_{x\to 0^+} g(x) = +\infty$;
- $\lim_{x\to \mu^-} g(x) < 1$;
- g is a continuous strictly monotonic decreasing function,

therefore it follows that

$$\exists\,!\; x_0 \in]0,\mu] \;:\; g(x_0) = \frac{f_1}{f_2}(x_0) = 1$$

that is

$$\exists\,!\; x_0 \in]0,\mu] \;:\; (f_2 - f_1)(x_0) = 0. \tag{3}$$

Equation (3) states that there exists only one sign change of function $f_2 - f_1$ in the interval $]0, \mu]$ and the sign sequence is $-, +$.

Let now x be greater than μ (case $x > \mu$).
As before, the limits of function g if x approaches to the extreme values of the interval are

$$\begin{aligned}\lim_{x\to\mu+} g(x) &= \lim_{x\to\mu+} \frac{B(\alpha_2,\theta)}{B(\alpha_1,\theta)} \cdot \frac{\int_0^{\mu/x} k^{\alpha_1-1/2}(1-k)^{\theta-2}\,dk}{\int_0^{\mu/x} k^{\alpha_2-1/2}(1-k)^{\theta-2}\,dk} \\ &= \frac{\Gamma(\alpha_2)\Gamma(\alpha_2+\theta-1/2)\Gamma(\alpha_1+1/2)\Gamma(\alpha_1+\theta)}{\Gamma(\alpha_2+1/2)\Gamma(\alpha_2+\theta)\Gamma(\alpha_1)\Gamma(\alpha_1+\theta-1/2)} < 1\end{aligned}$$

and

$$\lim_{x\to+\infty} g(x) = \lim_{x\to+\infty} \frac{B(\alpha_2,\theta)}{B(\alpha_1,\theta)} \cdot \frac{\int_0^{\mu/x} k^{\alpha_1-1/2}(1-k)^{\theta-2}\,dk}{\int_0^{\mu/x} k^{\alpha_2-1/2}(1-k)^{\theta-2}\,dk}$$

that is, by De l'Hopital rule, it results that

$$\begin{aligned}\lim_{x\to+\infty} g(x) &= \lim_{x\to+\infty} \frac{B(\alpha_2,\theta)}{B(\alpha_1,\theta)} \frac{\left(-\frac{\mu}{x^2}\right)\left(\frac{\mu}{x}\right)^{\alpha_1-1/2}\left(1-\frac{\mu}{x}\right)^{\theta-2}}{\left(-\frac{\mu}{x^2}\right)\left(\frac{\mu}{x}\right)^{\alpha_2-1/2}\left(1-\frac{\mu}{x}\right)^{\theta-2}} \\ &= \lim_{x\to+\infty} \frac{B(\alpha_2,\theta)}{B(\alpha_1,\theta)} \cdot \left(\frac{x}{\mu}\right)^{\alpha_2-\alpha_1} = +\infty.\end{aligned}$$

Now, the derivative of function g is:

$$\frac{d}{dx}g(x) = \frac{d}{dx}\left[\frac{B(\alpha_2,\theta)}{B(\alpha_1,\theta)} \cdot \frac{\int_0^{\mu/x} k^{\alpha_1-1/2}(1-k)^{\theta-2}\,dk}{\int_0^{\mu/x} k^{\alpha_2-1/2}(1-k)^{\theta-2}\,dk}\right]$$

$$\begin{aligned}&= a\cdot\left[\left(\frac{\mu}{x}\right)^{\alpha_1-1/2}\int_0^{\mu/x} k^{\alpha_2-1/2}(1-k)^{\theta-2}\,dk - \left(\frac{\mu}{x}\right)^{\alpha_2-1/2}\int_0^{\mu/x} k^{\alpha_1-1/2}(1-k)^{\theta-2}\,dk\right] \\ &= a\cdot\int_0^{\mu/x}(1-k)^{\theta-2}\left[\left(\frac{\mu}{x}\right)^{\alpha_1-1/2}k^{\alpha_2-1/2} - \left(\frac{\mu}{x}\right)^{\alpha_2-1/2}k^{\alpha_1-1/2}\right]dk \\ &= a\cdot\int_0^{\mu/x}(1-k)^{\theta-2}k^{\alpha_1-1/2}\left(\frac{\mu}{x}\right)^{\alpha_1-1/2}\left[k^{\alpha_2-\alpha_1} - \left(\frac{\mu}{x}\right)^{\alpha_2-\alpha_1}\right]dk \qquad (4)\end{aligned}$$

where

$$a = \frac{B(\alpha_2,\theta)}{B(\alpha_1,\theta)}\left(1-\frac{\mu}{x}\right)^{\theta-2}\left(-\frac{\mu}{x^2}\right)\left[\int_0^{\mu/x} k^{\alpha_2-1/2}(1-k)^{\theta-2}\,dk\right]^{-2} < 0.$$

As the previous case, the integrand function in equation (4) is strictly smaller than 0 for $k \in]0, \mu/x[$ and equal to 0 for $k = \mu/x$, therefore the value of the integral is strictly smaller than 0: that implies that $dg(x)/dx$ is positive and so g is a monotonically increasing function if $x > \mu$.

So, for $x > \mu$:

- $\lim_{x\to\mu^+} g(x) < 1$;
- $\lim_{x\to+\infty} g(x) + \infty$;
- g is a continuous strictly monotonic increasing function,

therefore it follows that

$$\exists\,!\, x_0 > \mu \;:\; g(x_0) = \frac{f_1}{f_2}(x_0) = 1$$

that is

$$\exists\,!\, x_0 > \mu \;:\; (f_2 - f_1)(x_0) = 0. \tag{5}$$

Equation (5) states that there exists only one sign change of function $f_2 - f_1$ for $x > \mu$ and the sign sequence is $+, -$. The matching of the restrictions (3) and (5), gives that if $x \in \mathbb{R}_0^+$ the function $f_2 - f_1$ has only two sign changes, with sign sequence $-, +, -$, therefore, by Lemma 4, it follows that $X_2 \leq_{CX} X_1$, and then the thesis of Theorem 1 is achieved.

The following result is an immediate consequence of Proposition 1.

Corollary 2 *Under the same hypothesis of Theorem 1, it follows that:*

$$X_2 \leq_L X_1 \qquad \textit{and} \qquad X_2 \leq_I X_1.$$

The previous result states that for Zenga distribution, Lorenz curves and the $I(p)$ curves are nested if one parameter changes. In detail, it proves that as the parameter θ is fixed, then α is an *inverse* inequality indicator for Lorenz curve and for the related Gini concentration ratio. That holds also for the inequality $I(p)$ curve and the related index I: the bigger α, the less unequal the distribution.

The role of the parameter θ is summarized in the next result.

Theorem 2 (Second ordering theorem) *Let X_1 and X_2 two continuous random variables, such that:*

- $X_1 \sim$ *Zenga* (μ, α, θ_1)
- $X_2 \sim$ *Zenga* (μ, α, θ_2)

where $1 < \theta_1 < \theta_2$, $\alpha > 0$ and $\mu > 0$. Then $X_1 \leq_{CX} X_2$.

Proof of Theorem 2

The proof is very similar to proof of Theorem 1. Also in this case the ratio g of the two density functions f_i of X_i $(i = 1, 2)$ is considered:

$$g(x) = \frac{f_1}{f_2}(x) = \begin{cases} \dfrac{B(\alpha,\theta_2)}{B(\alpha,\theta_1)} \dfrac{\displaystyle\int_0^{x/\mu} k^{\alpha-1/2}(1-k)^{\theta_1-2}\,dk}{\displaystyle\int_0^{x/\mu} k^{\alpha-1/2}(1-k)^{\theta_2-2}\,dk} & 0 < x \leq \mu \\[2ex] \dfrac{B(\alpha,\theta_2)}{B(\alpha,\theta_1)} \dfrac{\displaystyle\int_0^{\mu/x} k^{\alpha-1/2}(1-k)^{\theta_1-2}\,dk}{\displaystyle\int_0^{\mu/x} k^{\alpha-1/2}(1-k)^{\theta_2-2}\,dk} & x > \mu. \end{cases}$$

The behavior of function g at the extreme values of the domain is described by the following two limits:

$$\lim_{x\to 0^+} g(x) = \frac{\Gamma(\theta_2)\Gamma(\alpha+\theta_1)}{\Gamma(\alpha+\theta_2)\Gamma(\theta_1)}$$

$$\lim_{x\to +\infty} g(x) = \frac{\Gamma(\theta_2)\Gamma(\alpha+\theta_1)}{\Gamma(\alpha+\theta_2)\Gamma(\theta_1)}$$

and, by Corollary 1 it holds that:

$$\frac{\Gamma(\theta_2)\Gamma(\alpha+\theta_1)}{\Gamma(\alpha+\theta_2)\Gamma(\theta_1)} < 1.$$

Instead, if x approaches to μ:

$$\lim_{x\to\mu^-} g(x) = \lim_{x\to\mu^+} g(x) = \frac{\Gamma(\theta_2)\Gamma(\alpha+\theta_1)\Gamma(\theta_1-1)\Gamma(\alpha+\theta_2-1/2)}{\Gamma(\alpha+\theta_2)\Gamma(\theta_1)\Gamma(\alpha+\theta_1-1/2)\Gamma(\theta_2-1)}. \quad (6)$$

The limit in (6) is greater than 1, since by Lemma 2 it holds that

$$\frac{\Gamma(\alpha+\theta_2-1/2)}{\Gamma(\alpha+\theta_1-1/2)} > \frac{\Gamma(\alpha+\theta_2)}{\Gamma(\alpha+\theta_1)},$$

and that

$$\frac{\Gamma(\theta_1-1)}{\Gamma(\theta_2-1)} > \frac{\Gamma(\theta_1)}{\Gamma(\theta_2)},$$

therefore

$$\frac{\Gamma(\alpha+\theta_2-1/2)}{\Gamma(\alpha+\theta_1-1/2)}\cdot\frac{\Gamma(\theta_1-1)}{\Gamma(\theta_2-1)} > \frac{\Gamma(\alpha+\theta_2)}{\Gamma(\alpha+\theta_1)}\cdot\frac{\Gamma(\theta_1)}{\Gamma(\theta_2)},$$

that is

$$\frac{\Gamma(\theta_2)\Gamma(\alpha+\theta_1)\Gamma(\theta_1-1)\Gamma(\alpha+\theta_2-1/2)}{\Gamma(\alpha+\theta_2)\Gamma(\theta_1)\Gamma(\alpha+\theta_1-1/2)\Gamma(\theta_2-1)} > 1.$$

Here the importance of the hypothesis $\theta > 1$ is evident: the previous equations make sense only under this assumption.

It can be proved that g is strictly monotonic increasing in the interval $]0,\mu]$, and strictly monotonic decreasing if $x > \mu$, since for $x \in]0,\mu]$:

$$\frac{d}{dx}g(x) = z\cdot\int_0^{x/\mu} k^{\alpha-1/2}(1-k)^{\theta_1-2}\left(1-\frac{x}{\mu}\right)^{\theta_1-2}\left[(1-k)^{\theta_2-\theta_1}-\left(1-\frac{x}{\mu}\right)^{\theta_2-\theta_1}\right] dk$$

with

$$z = \frac{B(\alpha,\theta_2)}{\mu B(\alpha,\theta_1)}\left(\frac{x}{\mu}\right)^{\alpha-1/2}\left[\int_0^{x/\mu} k^{\alpha-1/2}(k)^{\theta_2-2}\, dk\right]^{-2} > 0,$$

then $dg(x)/dx > 0$ if $x \in [0; \mu]$; while for $x > \mu$:

$$\frac{d}{dx}g(x) = w \cdot \int_0^{\mu/x} k^{\alpha-1/2}(1-k)^{\theta_1-2}\left(1-\frac{\mu}{x}\right)^{\theta_1-2}\left[(1-k)^{\theta_2-\theta_1} - \left(1-\frac{\mu}{x}\right)^{\theta_2-\theta_1}\right] dk$$

with

$$w = \frac{B(\alpha,\theta_2)}{B(\alpha,\theta_1)}\left(\frac{\mu}{x}\right)^{\alpha-1/2}\left(-\frac{\mu}{x}\right)\left[\int_0^{\mu/x} k^{\alpha-1/2}(k)^{\theta_2-2}\, dk\right]^{-2} < 0,$$

and therefore $dg(x)/dx < 0$ for $x > \mu$.

So, as in Theorem 1, it follows that:

- in $]0, \mu]$ there exists only one point where the function $f_2 - f_1$ changes sign, and the function has the sign sequence $+, -$;
- if $x > \mu$ there exists only one point where the function $f_2 - f_1$ changes sign, and the function has the sign sequence $-, +$,

therefore $f_2 - f_1$ changes sign only twice in $\mathbb{R}_0^+$ with sign sequence $+, -, +$: by Lemma 4 it holds that $X_1 \leq_{CX} X_2$.

As the previous case, the following result can easily achieved, using Proposition 1.

Corollary 3 *Under the same hypothesis of Theorem 2, it follows that:*

$$X_1 \leq_L X_2 \qquad \textit{and} \qquad X_1 \leq_I X_2.$$

The previous result states that for Zenga distribution, Lorenz curves and the $I(p)$ curves are nested if parameter α is fixed and θ changes: this means that θ is a *direct* inequality indicator for Gini concentration ratio (index related to Lorenz curve) and for the inequality index I (related to the $I(p)$ curve): the bigger θ, the more unequal the distribution.

5 Conclusions

In this paper, the roles of the parameters of Zenga distribution in terms of inequality have been investigated. The distribution parameter μ has been excluded, since it is a scale parameter, therefore it does not affect the inequality level at all. The other two parameters has been analyzed and it has been proved that as θ is fixed, α is an inverse inequality indicator, while θ is a direct inequality indicator, as the α is fixed. These results are very important, since Zenga distribution seems to be largely useful to model wealth, financial, actuarial and especially, income distributions: in all these cases it is fundamental to understand how a change of the parameters affects the inequality level.

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