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Perturbative and non-perturbative infrared behavior of supersymmetric gauge theories

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Riassunto della tesi

L'accensione del Large Hadron Collider pone nuove sfide sulla presenza di nuova fisica oltre il Modello Standard. Tra la pletora di modelli costruiti per spiegare i suoi risultati sperimentali, la supersimmetria è considerata la miglior candidata tra tutta la nuova fisica che si potrebbe scoprire. La supersimmetria risolve la soluzione al problema della gerarchia, porta all'unificazione delle costanti di accoppiamento e contiene candidati per la materia oscura.

La miglior candidata per una teoria consistente di gravità quantistica è la teoria di stringa. Le teorie di campo supersimmetriche e la teoria delle stringhe sono intimamente connesse. I gradi di libertà principali nella teoria delle stringhe sono stringhe vibranti i cui estremi sono vincolati ad oggetti estesi detti brane. La teoria di bassa energia con supporto su un insieme di brane coincidenti è una teoria supersimmetrica di Yang-Mills. Quindi è molto importante studiare il comportamento delle teorie di gauge supersimmetriche per capire di più sulla teoria delle stringhe.

Una completa comprensione del limite di bassa energia delle teorie di gauge non è semplice, perché le teorie quantistiche di campo sono spesso fortemente accoppiate nel loro limite infrarosso. Tuttavia, la supersimmetria ci permette di conoscere molto sulle teorie efficaci di bassa energia.

La prima parte di questa tesi studia un modello in cui la supersimmetria è parzialmente esplicitamente rotta mediante tecniche perturbative in superspazio $N = 1$. Il modello in considerazione sorge come limite di bassa energia di una configurazione della teoria di stringa in dieci dimensioni che rompe metà delle supercariche rompendo le regole di anticommutazione nell'algebra di supersimmetria. In questo modo solo una parte dei teoremi di non-rinormalizzazione vale ancora e l'invarianza di gauge e la rinormalizzabilita sono profondamente legate tra loro. Nel formalismo ordinario la supersimmetria residua non è manifesta e ciò ha portato a conclusioni errate sulla proprietà di rinormalizzabilitá della teoria. Qui si discute di come il problema viene risolto in un formalismo manifestamente supersimmetrico. Si calcolano inoltre le funzioni beta della teoria a un loop e si trova che essa possiede alcuni punti fissi non banali. I flussi del gruppo di rinormalizzazione verso questi punti fissi possono essere studiati analizzando attentamente la matrice di stabilitá in un loro intorno.

La congettura AdS/CFT è una corrispondenza tra una teoria quantistica della gravità, come la stringa (o la M-) teoria, e una teoria di campo senza gravità in dimensione più bassa. Poiché la dualitá lega una teoria fortemente accoppiata con una debolmente accoppiata, i calcoli perturbativi nelle teorie di campo portano ad una maggiore comprensione di configurazioni fortemente accoppiate di teoria in stringa.

Nella seconda parte della mia tesi ho applicato le tecniche perturbative del superspazio $N = 1$ a teorie di campo in tre dimensioni. Queste descrivono il limite di bassa energia di un insieme di M2 brane coincidenti; le M2 brane sono i gradi di libertà fondamentali della M-teoria undicidimensionale. Secondo la corrispondenza AdS/CFT, teorie tridimensionali di Chern-Simons supersimmetriche accoppiate con campi di materia sono la descrizione corretta di una particolare configurazione della M-teoria. Qui discutiamo una classe molto generale di teorie di Chern-Simons e le loro proprietá di rinormalizzazione e ricaviamo tutte le funzioni beta del modello. Quindi discutiamo i flussi del gruppo di rinormalizzazione e dimostriamo che tutti i punti fissi sono stabili nell'infrarosso. Questo è in pieno accordo con la AdS/CFT. Discutiamo inoltre l'insieme di operatori esattamente marginali ricavati dalle funzioni beta a due-loop.

La spettro del Modello Standard non è supersimmetrico, e quindi la supersimmetria, se esiste, deve essere rotta. Le sue principali caratteristiche sono mantenute solo se la rottura é spontanea, piuttosto che esplicita. Inoltre, è preferibile che il meccanismo di rottura di simmetria sia dinamico, perché in questo modo la scala elettrodebole viene esponenzialmente soppressa rispetto alla scala di cut-off (che puó essere la scala di Planck o di grande unificazione), risolvendo così il problema della gerarchia. Sebbene la letteratura al riguardo sia vasta, non abbiamo ancora raggiunto una buona comprensione di questo problema.

Un potente strumento offerto dalla supersimmetria è la comprensione del regime di accoppiamento forte di una teoria di campo per mezzo di una dualità. In teorie di gauge supersimmetriche esiste una naturale dualitá elettrica/magnetica. Nella terza parte della tesi ci concentriamo sulle dualità in teorie $N = 1$ supersimmetriche, dette dualitá di Seiberg. Queste legano una teoria debolmente accoppiata con il limite di bassa energia di una teoria asintoticamente libera: é dunque un esempio di dualitá tra una teoria fortemente accoppiata e una debolmente accoppiata. Le proprietà non-perturbative della teoria microscopica possono essere calcolate perturbativamente nella teoria duale, o magnetica. Tramite la dualità di Seiberg, presentiamo un modello di rottura dinamica di supersimmetria che si basa su un'estensione della QCD supersimmetrica. Il limite di bassa energia di questa teoria è una teoria supersimmetrica di gauge fortemente accoppiata e il suo duale magnetico è debolmente accoppiato. Quindi i calcoli perturbativi sono affidabili, e discutiamo l'emergere di vuoti metastabili in cui la supersimmetria viene spontaneamente rotta, e di come la loro vita può essere resa più lunga della vita del nostro Universo. Inoltre, analizziamo un modello tridimensionale di rottura della supersimmetria, che é utile ad accrescere la nostra conoscenza della AdS/CFT e ad acquisire nuove conoscenze sulla gravità quantistica in quattro dimensioni.

Nell'ultima parte della tesi discutiamo una recente applicazione della corrispondenza AdS/CFT. Alcuni fenomeni critici quantistici in sistemi della fisica dello stato solido sono ben descritti da teorie conformi fortemente accoppiate. Secondo la AdS/CFT, ad una determinata teoria conforme corrisponde una teoria classica di gravitá. Qui mostriamo come sia possibile costruire una teoria di gravità che descriva un modello di superconduttivitá e portare a termine un'analisi dettagliata delle sue caratteristiche.

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Introduction and outline

The advent of the Large Hadron Collider poses new theoretical challenges on the physics beyond the Standard Model. Among the plethora of models built to explain its experimental signatures, supersymmetry is considered the most compelling new physics that could be discovered. It addresses the solution to the hierarchy problem, leads to coupling constant unification and contains dark matter candidates.

The Standard Model of particle physics successfully describes all the observed non-gravitational phenomena in the framework of quantum gauge theories, while gravity is described by Einstein's General Relativity. Providing a framework in which quantum gravity and quantum gauge theories are described in a unified way is the dream of theoretical physics. The best candidate for this would-be theory of everything is string theory. Consistency of string theory naturally demands that supersymmetry should be a symmetry of the spacetime we live in. While a deep understanding of string theory is far to be achieved, it is widely understood that supersymmetric field theories and string theory are intimately related. The main degrees of freedom in string theory are vibrating strings which can end on extended objects also called branes. The low energy theory living on a stack of branes is a supersymmetric Yang-Mills theory. Thus the physics of string theory at long distances is well described by a supersymmetric local quantum field theory. Then it turns out that it is very important to understand the behavior of supersymmetric gauge theories to learn more about string theory.

A complete understanding of the low energy limit of gauge theories is not a simple task, because quantum field theories are often strongly coupled as they approach their infrared limit. Nevertheless, supersymmetry ensures better renormalization properties and a much more powerful non-perturbative analysis than ordinary quantum field theories. In other words, we can gain many insights on the behavior of the low energy effective theories if we allow supersymmetry to be a fundamental symmetry of Nature. This is the first step towards a deeper understanding of the same topics in theories with broken supersymmetries, with the ultimate goal of the application of similar techniques to ordinary field theories.

The first part of my Thesis studies an explicit supersymmetry breaking model by means of perturbative techniques in $N = 1$ superspace. The particular model arises as the low energy limit of a ten dimensional string theory background which breaks half of the supersymmetries by deforming the supersymmetry algebra down to a non-anticommuting one. In this setting only a subset of the non-renormalization theorems holds and the interplay of gauge invariance and renormalizability is highly nontrivial. In a component field formalism the residual supersymmetry is not manifest and this led to a wrong conclusion about the renormalizability property of the theory. I discuss how the problem is solved in a manifest supersymmetric setting. I also compute the β-functions of the theory to one-loop order and find they possess some nontrivial fixed points. The RG flows towards these fixed points can be studied by carefully analyzing the stability matrix around them.

The AdS/CFT conjecture is a remarkable correspondence between a quantum theory of gravity, such as string (or M-)theory, and a lower dimensional field theory without gravity. Because it is a weak/strong duality, perturbative field theories computations open the way to the understanding of strongly coupled string theoretical backgrounds and quantum gravity.

In the second part of my Thesis I apply the perturbative techniques of $N = 1$ superspace to three-dimensional field theories. The latter have been shown to describe the low energy limit of a stack of M2 branes, the fundamental degrees of freedom of eleven-dimensional M-theory. By means of the AdS/CFT correspondence, supersymmetric three-dimensional Chern-Simons matter theories are the right description of M-theory in a particular background. I discuss in a very general setting the renormalization of Chern-Simons matter theories and present a complete treatment of their β-functions. Then I consider the RG flows and show that the fixed points are all infrared stable. This is in full agreement with the AdS/CFT conjecture. I also discuss the set of exactly marginal operators which arise at the two-loop level.

The Standard Model particle spectrum is not supersymmetric, and therefore supersymmetry, if it exists, must be broken. Its main features are preserved only if it is spontaneously broken, rather than explicitly broken. Furthermore, a dynamical symmetry breaking mechanism is preferred, because it triggers the electroweak scale to be exponentially smaller than the cutoff (Planck or grand unified) scale, thus solving the hierarchy problem. Although many efforts have been made in this direction, a complete understanding of all these issues has not been achieved yet.

A powerful tool offered by supersymmetry is the understanding of the strong coupling regime of a field theory by means of dualities. In supersymmetric gauge theories there exists a natural electric/magnetic duality. In the third part of of my Thesis I concentrate on the $N = 1$ case, namely Seiberg duality. It relates a weakly coupled dual description of the low energy limit of an asymptotically free theory, thus it is an example of a weak/strong duality. Non-perturbative properties of the microscopic theory at low energies can be perturbatively calculated in the dual magnetic theory. By the use of the Seiberg duality, I present a model of dynamical supersymmetry breaking based on an extension of the supersymmetric QCD. The low energy limit of this theory is a strongly coupled supersymmetric gauge theory and its magnetic dual is weakly coupled. Then the perturbative low energy computations are reliable, and I discuss how metastable vacua in which supersymmetry is broken are found, and how their lifetime can be made larger than the life of our Universe. I also analyze supersymmetry breaking in three dimensions, that should be useful to further explore the AdS/CFT correspondence and to gain new insights into four-dimensional quantum gravity.

This Thesis is organized as follows. Part I is devoted to the study of nonanticommutative field theories. In Chapter 1 we begin by discussing the starting motivations for the birth of the nonanticommutative deformation as arising from string theory in the presence of a Ramond-Ramond flux. In Chapter 2 we construct the natural arena for the deformation, namely the nonanticommutative superspace. We show how a suitable $*$-product can be obtained in the general superspace setup starting from the request for the most general structure compatible with the assumption of associativity. We also discuss the subtleties related to the right choice of complex conjugation rules, allowed only in Euclidean superspace, and we discuss the non-renormalization theorems which hold due to the unbroken supersymmetry. A general discussion concerning the Wess-Zumino model is contained in Chapter 3. We discuss the quantum properties of the model up to two-loop, and give an all loop argument for its renormalizability. In Chapter 4 we discuss in all details the case of gauge theories, both in the case of pure Super Yang-Mills theories and in the case of interacting matter. We develop a manifestly background gauge invariance method for computing loop corrections, by extending the background field method of ordinary supersymmetric theories. A complete one-loop analysis is carried out in all cases for non-Abelian unitary gauge groups. We conclude this Part by explicitly calculating all the β-functions for the Abelian gauge theory coupled to adjoint chiral superfields. We consider both the case with one chiral superfield and with three different flavors, thus our treatment includes the deformation of the $N = 4$ supersymmetric Yang-Mills theory. Although the Abelian $N = 4$ theory is a free theory, we find that the nonanticommutative deformation leads to a nontrivial highly interacting action.

In Part II we explain our results in three-dimensional field theories. Chapter 5 contains a basic description of three-dimensional field theories with a Chern-Simons term. Motivated by the AdS/CFT correspondence, we specialize to the case where the gauge group is the product of two unitary groups. In this class of theories, the maximally supersymmetric theory is conjectured to describe the dynamics of a stack of membranes probing a Calabi-Yau four-fold. The quantization of Chern-Simons theories coupled to matter fields is carried out in Chapter 6. We compute the two-loop β-functions and identify the fixed points of a large class of models. We study their infrared stability and their RG trajectories.

Part III is devoted to supersymmetry breaking models. The basic material to relate them to the signatures of the visible (Supersymmetric) Standard Model sector is explained in Chapter 7. In Chapter 8 we review the Supersymmetric Quantum Chromodynamics (SQCD), and show how supersymmetry allows us to compute exact, non-perturbative, quantum effects. The concept of non-Abelian electric/magnetic duality is introduced, and we show that it is useful to describe the low energy dynamics of a supersymmetric gauge theory. While the low energy effective theory is often strongly coupled, its magnetic dual is weakly coupled. This allows us to use reliable perturbation theory and obtain calculable predictions on the non-holomorphic part of the theory. In particular, we show that SQCD possesses non-supersymmetric metastable vacua in Chapter 9. We compute the lifetime of the metastable vacua in the magnetic free range, and find they are long lived. We argue that this does not hold for the conformal window, because the RG flow combines the dimensionful parameters of the theory in such a way that the lifetime does not depend upon them. We overcome this problem by adding appropriate relevant deformations and show that a viable model of supersymmetry breaking can be realized in the conformal window. We end this Chapter by showing that supersymmetry breaking models also exist in three-dimensional field theories. The latter are a useful laboratory for the study of four-dimensional quantum gravity through the AdS/CFT correspondence.

Some Appendices follow. In Appendix A we list our group theory and superspace conventions, both for four- and three-dimensional field theories. A list of useful integrals for evaluating the

Feynman diagrams is also given. In Appendix B we collect the Feynman rules for perturbative computations of nonanticommutative theories. Details on supersymmetry breaking computations can be found in Appendix C.

Part I

Nonanticommutative theories

Chapter 1

Basics and motivations

1.1 Introduction

String theories naturally produce supersymmetric field theories as their low energy limits. Different configurations in string theory lead to different well-known field theories in the limit in which the string length shrinks to zero. For instance, the effective field theory on the worldvolume of N $D3$-branes in type IIB string theory compactified on a Calabi-Yau 3-fold is a $N = 1$ $U(N)$ supersymmetric Yang-Mills theory. Different amounts of supersymmetry can be obtained by changing the details of the configuration.

Supersymmetric field theories can be defined on a superspace, where spacetime is enlarged by adding fermionic, i.e. anticommuting, coordinates. A remarkable discovery was the fact that string theory admits as its low energy limit a particular configuration in which the superspace coordinates algebra is modified. Considering a configuration of strings and branes in which a constant Ramond-Ramond self-dual graviphoton background field strength is turned on, in a suitable low energy limit the fermionic coordinates obey to the Clifford algebra

$$\left\{\theta^{\alpha}, \theta^{\beta}\right\} = 2\mathcal{F}^{\alpha\beta} \tag{1.1.1}$$

with the remaining superspace structure unchanged [1]. Successively Seiberg began the study of this kind of nonanticommutative field theories [2] and since then a lot of work has been devoted to the study of the new peculiar effects of these models.

First of all, we have strong constraints on the signature of the manifold: this deformation is only possible on Euclidean superspace. This is evident from constraints on the conjugation relations, but it is confirmed by its stringy origin. The low energy limit giving nonanticommutative deformation in fact, is allowed because the self-dual graviphoton field is an exact solution to the string equation of motion (without backreaction): only in Euclidean space we can turn on the self-dual part while setting to zero the anti-self-dual one. Given the deformed structure of superspace, for nonanticommutative field theories one has to define a Moyal $*$-product. It has to be defined by a fermionic differential operator and, starting from $N = 1$ supersymmetry, we have two different possibilities. By defining the $*$-product with covariant (with respect to the supersymmetry transformations) derivatives, we maintain all the supersymmetries of the theory, but a consistent definition of chirality can no more be given. Since the chirality is a fundamental ingredient in building sensible supersymmetric field theories, this choice is not suitable for the $N = 1$ case. The second possibility is given by choosing the supercharges as generators of the $*$-product. In this case, although chirality can be consistently defined, deformed field theories show a partially broken supersymmetry. In particular, if one starts from a $N = 1$ superspace, the resulting theories will have $N = \frac{1}{2}$ supersymmetry. Actually, one can try to give similar deformations also in superspace with extended supersymmetry: what happens is that more possibilities are involved. In the $N = 2$ case, for example, we can preserve the entire supersymmetry and Euclidean group.

The breaking of half of the supersymmetry has important consequences on the dynamics of these field theories. In particular, since supersymmetry plays a fundamental role in determining the quantum properties of a field theory, the well known no-renormalization theorems are no longer valid, and a systematic analysis is compelling. The simple deformation of the original theories, obtained by trading the usual product with the Moyal one, is no longer renormalizable. However, the remaining unbroken supersymmetry preserves these models from the catastrophe, and a suitable completion of the action, with a finite number of new terms, is always possible in order to cure the divergences. Moreover, the new divergences arising from supersymmetry breaking for these theories maintain their logarithmic nature: this strongly suggests the fact that the breaking of the supersymmetry obtained in the string setup by turning on the flux of $F^{\alpha\beta}$ is a soft breaking.

1.2 Nonanticommutativity from superstrings

In this section we show how the concept of nonanticommutativity arises in string theory when a constant graviphoton background is considered. The four-dimensional theory is obtained by compactifying the ten-dimensional Type II superstring on a three-fold Calabi-Yau manifold [3, 2, 4]. A similar derivation can be carried out for the full ten-dimensional superstring theory [5].

In what follows, we make use of Berkovits' formalism [6] which furnishes a very compact derivation. In this setup, the target-space supersymmetry of the superstring theory is manifest. Moreover, it allows for a covariant quantization of the superstring theory.

The four-dimensional superspace coordinates the usual bosonic ones x^μ and the fermionic θ^α, $\theta^{\dot{\alpha}}$, $\bar{\theta}^\alpha$ and $\bar{\theta}^{\dot{\alpha}}$. In our notations, a dot over an index indicates target space Weyl spinor chirality, while a bar indicates worldsheet holomorphicity. The relevant part of the Lagrangian density is (we set $\alpha' = 1$)

$$\mathcal{L} = \frac{1}{2}\partial x^\mu \bar{\partial} x_\mu + p_\alpha \bar{\partial}\theta^\alpha + p_{\dot{\alpha}} \bar{\partial}\theta^{\dot{\alpha}} + \bar{p}_\alpha \partial\bar{\theta}^\alpha + \bar{p}_{\dot{\alpha}} \partial\bar{\theta}^{\dot{\alpha}} \tag{1.2.2}$$

where p_α, $\bar{p}_\alpha$, $p_{\dot{\alpha}}$ and $\bar{p}_{\dot{\alpha}}$ are the conjugate momenta to the fermionic variables they couple to. Since we are working in the euclidean signature, they are independent variables. The worldsheet is parametrized by z and $\bar{z}$ which are complex conjugate of each other.

The Lagrangian (1.2.2) describes a free conformal field theory: all the fields satisfy free equations of motion, which are second order for the bosonic coordinates and first order for the fermionic ones. Moreover, the theory exhibits $N = 2$ target space supersymmetry. It is useful to define the worldsheet versions of the covariant derivatives d_α and $d_{\dot{\alpha}}$ and of the supercharges q_α

and $q_{\dot{\alpha}}$

$$y^\mu = x^\mu + i\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \theta^{\dot{\alpha}} + i\bar{\theta}^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}}$$

$$d_{\dot{\alpha}} = p_{\dot{\alpha}} - i\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \partial x_\mu - \frac{1}{2}\theta^\alpha \theta_\alpha \partial\theta_{\dot{\alpha}} + \frac{1}{4}\theta_{\dot{\alpha}} \partial(\theta^\alpha\theta_\alpha) \tag{1.2.3}$$

$$q_\alpha = -p_\alpha - i\sigma^\mu_{\alpha\dot{\alpha}} \theta^{\dot{\alpha}} \partial x_\mu + \frac{1}{4}\theta^{\dot{\alpha}}\theta_{\dot{\alpha}} \partial\theta_\alpha - \frac{3}{4}\partial(\theta_\alpha \theta^{\dot{\alpha}}\theta_{\dot{\alpha}}) \tag{1.2.4}$$

and similarly for the others, in terms of which the Lagrangian density becomes

$$\mathcal{L} = \frac{1}{2}\partial y^\mu \bar{\partial} y_\mu - q_\alpha \bar{\partial}\theta^\alpha + d_{\dot{\alpha}} \bar{\partial}\theta^{\dot{\alpha}} - \bar{q}_\alpha \partial\bar{\theta}^\alpha + \bar{d}_{\dot{\alpha}} \partial\bar{\theta}^{\dot{\alpha}} + \text{total} \quad \text{derivative} \tag{1.2.5}$$

Therefore, q and d represent the conjugate momenta to the θ's at fixed y exactly as the p's represent the conjugate momenta at fixed x.

If the worldsheet ends on a D-brane, the boundary conditions are easily found by imposing that there is no surface term in the equations of motion. For a boundary at $z = \bar{z}$, the boundary conditions are

$$\theta(z) = \bar{\theta}(\bar{z}) \qquad q(z) = \bar{q}(\bar{z}) \qquad d(z) = \bar{d}(\bar{z}) \tag{1.2.6}$$

which break half of the supersymmetries preserving only $Q_\alpha + \bar{Q}_\alpha$ and $Q_{\dot{\alpha}} + \bar{Q}_{\dot{\alpha}}$.

Now consider the system in the background of a constant graviphoton field strength $F_{\alpha\beta}$. Differently from the Lorentzian signature case, in euclidean signature we are allowed to take a self-dual field strength

$$F_{\alpha\beta} \neq 0 \qquad\qquad F_{\dot{\alpha}\dot{\beta}} = 0 \tag{1.2.7}$$

We take a self-dual F because such a background is a solution of the spacetime equations of motion without backreaction of the metric. Indeed, a purely self-dual field strength does not contribute to the energy momentum tensor and therefore does not lead to a source in the gravity equations of motion. Moreover, since the kinetic term of the graviphoton does not contain the dilaton, the background $F_{\alpha\beta}$ does not lead to a source in the dilaton equation.

The graviphoton background is represented in the worldsheet Lagrangian by adding the interaction with the supersymmetry currents

$$F^{\alpha\beta} q_\alpha \bar{q}_\beta \tag{1.2.8}$$

which makes manifest that the worldsheet theory remains free in this background. While the original coordinates x couple to F, the y's remain free and independent of the background.

The nontrivial part of the Lagrangian now reads (we reinstate the right powers of α')

$$\mathcal{L} = -q_\alpha \bar{\partial}\theta^\alpha - \bar{q}_\alpha \partial\bar{\theta}^\alpha + F^{\alpha\beta} q_\alpha \bar{q}_\beta \tag{1.2.9}$$

The fields q and $\bar{q}$ can be integrated out using their equations of motion

$$\begin{aligned} \bar{\partial}\theta^\alpha &= \alpha' F^{\alpha\beta} \bar{q}_\beta \\ \partial\bar{\theta}^\alpha &= -\alpha' F^{\alpha\beta} q_\beta \end{aligned} \tag{1.2.10}$$

to obtain the effective Lagrangian

$$\mathcal{L}_{eff} = \left(\frac{1}{\alpha' F}\right)_{\alpha\beta} \partial\bar{\theta}^{\alpha}\bar{\partial}\theta^{\beta} \tag{1.2.11}$$

When we consider a system with a boundary (along $z = \bar{z}$ for an euclidean worldsheet) we need to find the appropriate boundary conditions. These are determined from the surface term in the equations of motion

$$\left(\frac{1}{F}\right)_{\alpha\beta} \left(\bar{\partial}\theta^{\alpha}\delta\bar{\theta}^{\beta} + \partial\bar{\theta}^{\alpha}\delta\theta^{\beta}\right)\Big|_{z=\bar{z}} = 0 \tag{1.2.12}$$

Then, we impose

$$\begin{aligned} \theta^{\alpha}(z=\bar{z}) &= \bar{\theta}^{\alpha}(z=\bar{z}) \\ \partial\bar{\theta}^{\alpha}(z=\bar{z}) &= -\bar{\partial}\theta^{\alpha}(z=\bar{z}) \end{aligned} \tag{1.2.13}$$

The first condition states that the superspace has half the number of θ's. This means that half of the supersymmetries has been broken. The second condition, through equation (1.2.10), that the supersymmetry charges q_{α} and $\bar{q}_{\alpha}$ are equal on the boundary.

By imposing the proper singularity at coincident points, Fermi statistics and the aforementioned boundary conditions, we find the fermionic propagators

$$\begin{aligned} \langle\theta^{\alpha}(z)\theta^{\beta}(\omega)\rangle &= \frac{\alpha'^2 F^{\alpha\beta}}{2\pi i}\log\frac{\bar{z}-\omega}{z-\bar{\omega}} \\ \langle\bar{\theta}^{\alpha}(z)\bar{\theta}^{\beta}(\omega)\rangle &= \frac{\alpha'^2 F^{\alpha\beta}}{2\pi i}\log\frac{\bar{z}-\omega}{z-\bar{\omega}} \end{aligned} \tag{1.2.14}$$

$$\langle\theta^{\alpha}(z)\theta^{\beta}(\omega)\rangle = \frac{\alpha'^2 F^{\alpha\beta}}{2\pi i}\log\frac{(z-\omega)(\bar{z}-\bar{\omega})}{(z-\bar{\omega})^2} \tag{1.2.15}$$

Since the branch cuts of the logarithms are outside the worldsheet, for two points on the boundary $z = \bar{z} \equiv \tau$ and $\omega = \bar{\omega} \equiv \tau'$ we get

$$\langle\theta^{\alpha}(\tau)\theta^{\beta}(\tau')\rangle = \frac{\alpha'^2 F^{\alpha\beta}}{2}\text{sign}\,(\tau-\tau') \tag{1.2.16}$$

corresponding to the algebra

$$\{\theta^{\alpha}, \theta^{\beta}\} = \alpha'^2 F^{\alpha\beta} \equiv 2\mathcal{F}^{\alpha\beta} \neq 0 \tag{1.2.17}$$

which represents a Clifford algebra for the θ's. It is important that since the coordinates $\bar{\theta}$ and y were not affected by the background coupling (1.2.8) they remain commuting. In particular $[y,^{\mu}, y^{\nu}] = 0$ and therefore $[x^{\mu}, x^{\nu}] \neq 0$ and $[x, \theta] \neq 0$. This is also consistent with the purely field theoretical point of view (see [1] and the next chapter).

It is easy to see that the Clifford algebra deformation breaks half of the supersymmetries left by the D-brane boundary conditions. Since the equation of motion of θ states that q is holomorphic, it extends to a holomorphic field $\bar{q}(\bar{z}) = q(z)$, and therefore the supersymmetry charge $Q_\alpha + \bar{Q}_\alpha$ is conserved. On the other hand, θ and $\bar{\theta}$ are not holomorphic objects and do not extend to holomorphic fields through the boundary (because of (1.2.15)). Therefore, the supercharges $Q_{\dot{\alpha}} + \bar{Q}_{\dot{\alpha}}$ are broken by the deformation.

As a final remark, note that the deformation (1.2.17) does not survive the field theory limit $\alpha' \to 0$ unless we simultaneously take $F^{\alpha\beta} \to \infty$ with $\alpha'^2 F^{\alpha\beta}$ kept fixed. It is possible to make sense of this limit since the constant self-dual graviphoton background is an exact solution to the string equations of motion for any value of $F^{\alpha\beta}$.

Chapter 2

Nonanticommutative superspace

As far as it is recognized that the symmetries play an important role in the study of the properties of a physical system, a formalism to make them linearly realized has becomed the key to simplify computations and gain more insights in the system at hand. As we are dealing with (partially) broken supersymmetries, we need a formalism which only makes manifest the survived supersymmetries. This is the nonanticommutative superspace.

In this section we analyze the basics of non(anti)commutative geometry, and look for the most general consistent four-dimensional superspace which realizes it. We consider the deformation of the Minkowski superspace, then we move to the euclidean case, which will be of much more interest for us. We conclude this section by giving the new (non)-renormalization theorems which arise in this context.

2.1 Minkowski

We start by considering the four-dimensional $N = 1$ Minkowski superspace. We denote by $z^A = (x^{\alpha ad}, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$ the superspace coordinates, where $x^{\alpha\dot{\alpha}}$ are the four bosonic coordinates and $\theta^\alpha, \bar{\theta}^{\dot{\alpha}}$ are the Weyl fermions which satisfy the conjugation rule $(\theta^\alpha)^\dagger = \bar{\theta}^{\dot{\alpha}}$. We deform the $N = 1$ algebra given by

$$
\begin{aligned}
\{\theta^\alpha, \theta^\beta\} &= \{\bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\} = \{\theta^\alpha, \bar{\theta}^{\dot{\alpha}}\} = 0 \\
[x^{\alpha\dot{\alpha}}, \theta^\beta] &= [x^{\alpha\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}] = 0 \\
[x^{\alpha\dot{\alpha}}, x^{\beta\dot{\beta}}] &= 0
\end{aligned}
\tag{2.1.1}
$$

to the most general set of (anti)commutation relations

$$
\begin{aligned}
\left\{\theta^\alpha, \theta^\beta\right\} &= \mathcal{A}^{\alpha\beta}(x,\theta,\bar{\theta}) \quad , \quad \left\{\bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\right\} = \bar{\mathcal{A}}^{\dot{\alpha}\dot{\beta}}(x,\theta,\bar{\theta}) \\
\left\{\theta^\alpha, \bar{\theta}^{\dot{\alpha}}\right\} &= \mathcal{B}^{\alpha\dot{\alpha}}(x,\theta,\bar{\theta}) \\
\left[x^{\underline{a}}, \theta^\beta\right] &= i\mathcal{C}^{\underline{a}\beta}(x,\theta,\bar{\theta}) \quad , \quad \left[x^{\underline{a}}, \bar{\theta}^{\dot{\beta}}\right] = i\bar{\mathcal{C}}^{\underline{a}\dot{\beta}}(x,\theta,\bar{\theta}) \\
\left[x^{\underline{a}}, x^{\underline{b}}\right] &= i\mathcal{D}^{\underline{ab}}(x,\theta,\bar{\theta})
\end{aligned}
\tag{2.1.2}
$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are functions of the superspace variables, $\bar{\mathcal{A}}, \bar{\mathcal{C}}$ are the hermitian conjugates of $\mathcal{A}$ and $\mathcal{C}$ respectively and $\bar{\mathcal{B}}, \bar{\mathcal{D}}$ are hermitian operators.

While the covariance under Lorentz transformation is manifest, we require (2.1.2) to be covariant under (super)translations

$$\begin{aligned} \theta'^{\alpha} &= \theta^{\alpha} + \epsilon^{\alpha} \\ \bar{\theta}'^{\dot{\alpha}} &= \bar{\theta}^{\dot{\alpha}} + \bar{\epsilon}^{\dot{\alpha}} \\ x'^{\alpha\dot{\alpha}} &= x^{\alpha\dot{\alpha}} + a^{\alpha\dot{\alpha}} - \frac{i}{2}\left(\epsilon^{\alpha}\bar{\theta}^{\dot{\alpha}} + \bar{\epsilon}^{\dot{\alpha}}\theta^{\alpha}\right) \end{aligned} \tag{2.1.3}$$

which constraints the functional dependence of the functions $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ upon the coordinates. By performing a supertranslation on the coordinates (2.1.2) and computing the new algebra, one finds that $\mathcal{A}$ and $\mathcal{B}$ are independent of the coordinates and $\mathcal{C}$ and $\mathcal{D}$ are independent of the bosonic coordinates. More precisely, we get the algebra

$$\begin{aligned} \left\{\theta^{\alpha}, \theta^{\beta}\right\} &= A^{\alpha\beta} \quad , \quad \left\{\bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\right\} = \bar{A}^{\dot{\alpha}\dot{\beta}} \quad , \quad \left\{\theta^{\alpha}, \bar{\theta}^{\dot{\alpha}}\right\} = B^{\alpha\dot{\alpha}} \\ \left[x^{\alpha\dot{\alpha}}, \theta^{\beta}\right] &= i\mathcal{C}^{\alpha\dot{\alpha}\beta}(\theta, \bar{\theta}) \\ \left[x^{\alpha\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\right] &= i\bar{\mathcal{C}}^{\alpha\dot{\alpha}\dot{\beta}}(\theta, \bar{\theta}) \\ \left[x^{\alpha\dot{\alpha}}, x^{\beta\dot{\beta}}\right] &= i\mathcal{D}^{\alpha\dot{\alpha}\beta\dot{\beta}}(\theta, \bar{\theta}) \end{aligned} \tag{2.1.4}$$

where

$$\begin{aligned} \mathcal{C}^{\alpha\dot{\alpha}\beta}(\theta, \bar{\theta}) &= C^{\alpha\dot{\alpha}\beta} - \frac{1}{2}\theta^{\alpha}B^{\beta\dot{\alpha}} - \frac{1}{2}\bar{\theta}^{\dot{\alpha}}A^{\alpha\beta} \\ \mathcal{D}^{\alpha\dot{\alpha}\beta\dot{\beta}}(\theta, \bar{\theta}) &= D^{\alpha\dot{\alpha}\beta\dot{\beta}} - \frac{i}{2}\left(\theta^{\beta}\bar{C}^{\alpha\dot{\alpha}\dot{\beta}} - \bar{\theta}^{\dot{\alpha}}C^{\beta\dot{\beta}\alpha} - \theta^{\alpha}\bar{C}^{\beta\dot{\beta}\dot{\alpha}} + \bar{\theta}^{\dot{\beta}}C^{\alpha\dot{\alpha}\beta}\right) \\ &- \frac{i}{4}\left(\theta^{\alpha}\bar{A}^{\dot{\alpha}\dot{\beta}}\theta^{\beta} + \theta^{\alpha}B^{\beta\dot{\alpha}}\bar{\theta}^{\dot{\beta}} + \bar{\theta}^{\dot{\alpha}}B^{\alpha\dot{\beta}}\theta^{\beta} + \bar{\theta}^{\dot{\alpha}}A^{\alpha\beta}\bar{\theta}^{\dot{\beta}}\right) \end{aligned} \tag{2.1.5}$$

and A, B, C and D are constant functions.

2.2 Euclidean

In the Minkowski case it is impossible to break only half of the supersymmetries due to the conjugation rules of the spinorial variables. It is known that relaxing these constraints amounts to deal with extended supersymmetry or consider a space with euclidean signature. The main difference with respect to the Minkowski one relies on the reality conditions satisfied by the fermionic variables. In euclidean signature a reality condition on spinors is applicable only in the case of extended supersymmetry. The simplest case is then the $N = 2$ superspace.

The spinors of the $N = 2$ euclidean superspace satisfy the pseudo-Majorana condition

$$(\theta^{\alpha}_{i})^{*} = \theta^{i}_{\alpha} \equiv C^{ij}\,\theta^{\beta}_{j}\,C_{\beta\alpha} \quad , \quad (\bar{\theta}^{\dot{\alpha},i})^{*} = \bar{\theta}_{\dot{\alpha},i} \equiv \bar{\theta}^{\dot{\beta},j}\,C_{\dot{\beta}\dot{\alpha}}\,C_{ji} \tag{2.2.6}$$

where we defined the charge conjugation matrix

$$C^{ij} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \tag{2.2.7}$$

We then follow the same procedure as for the Minkowski space and find that the the covariance under superspace translations give the following constraints on the algebra

$$\begin{aligned}
\left\{\theta_i^\alpha, \theta_j^\beta\right\} &= {A_1}^{\alpha\beta,}{}_{ij} \quad , \quad \left\{\bar\theta^{\dot\alpha,i}, \bar\theta^{\dot\beta,j}\right\} = A_2^{\dot\alpha\dot\beta,ij} \quad , \quad \left\{\theta_i^\alpha, \bar\theta^{\dot\alpha,j}\right\} = B^{\alpha\dot\alpha,}{}_i{}^j \\
\left[x^{\alpha\dot\alpha}, \theta_i^\beta\right] &= i{\mathcal{C}_1}^{\underline{a}\beta,}{}_i(\theta,\bar\theta) \\
\left[x^{\alpha\dot\alpha}, \bar\theta^{\dot\beta,i}\right] &= i\mathcal{C}_2^{\alpha\dot\alpha\dot\beta,i}(\theta,\bar\theta) \\
\left[x^{\alpha\dot\alpha}, x^{\beta\dot\beta}\right] &= i\mathcal{D}^{\alpha\dot\alpha\beta\dot\beta}(\theta,\bar\theta)
\end{aligned} \tag{2.2.8}$$

where

$$\begin{aligned}
{\mathcal{C}_1}^{\alpha\dot\alpha\beta,}{}_i(\theta,\bar\theta) &\equiv {C_1}^{\alpha\dot\alpha\beta,}{}_i + \frac{i}{2}\theta_j^\alpha B^{\beta\dot\alpha,}{}_i{}^j + \frac{i}{2}\bar\theta^{\dot\alpha,j}{A_1}^{\alpha\beta,}{}_{ji} \\
\mathcal{C}_2^{\alpha\dot\alpha\dot\beta,i}(\theta,\bar\theta) &\equiv C_2^{\alpha\dot\alpha\dot\beta,i} + \frac{i}{2}\theta_j^\alpha A_2^{\dot\alpha\dot\beta,ji} + \frac{i}{2}\bar\theta^{\dot\alpha,j}B^{\alpha\dot\beta,}{}_j{}^i \\
\mathcal{D}^{\alpha\dot\alpha\beta\dot\beta}(\theta,\bar\theta) &\equiv D^{\alpha\dot\alpha\beta\dot\beta} \\
&\quad + \frac{1}{2}\left(\theta_i^\alpha C_2^{\beta\dot\beta\dot\alpha,i} - \theta_i^\beta C_2^{\alpha\dot\alpha\dot\beta,i} + \bar\theta^{\dot\alpha,i}{C_1}^{\beta\dot\beta\alpha,}{}_i - \bar\theta^{\dot\beta,i}{C_1}^{\alpha\dot\alpha\beta,}{}_i\right) \\
&\quad + \frac{i}{4}\left(\theta_i^\alpha A_2^{\dot\alpha\dot\beta,ij}\theta_j^\beta + \theta_i^\alpha B^{\beta\dot\alpha,}{}_j{}^i\bar\theta^{\dot\beta,j} + \bar\theta^{\dot\alpha,i}B^{\alpha\dot\beta,}{}_i{}^j\theta_j^\beta + \bar\theta^{\dot\alpha,i}{A_1}^{\alpha\beta,}{}_{ij}\bar\theta^{\dot\beta,j}\right)
\end{aligned} \tag{2.2.9}$$

with A_1, A_2, B, C_1, C_2 and D constant.

Finally, we discuss how to obtain an *associative* algebra. By imposing the generalized Jacobi identities hold we get

$$\begin{aligned}
\left\{\theta_i^\alpha, \theta_j^\beta\right\} &= {A_1}^{\alpha\beta,}{}_{ij} \quad , \quad \left\{\bar\theta^{\dot\alpha,i}, \bar\theta^{\dot\beta,j}\right\} = 0 \quad , \quad \left\{\theta_i^\alpha, \bar\theta^{\dot\beta,j}\right\} = 0 \\
\left[x^{\alpha\dot\alpha}, \theta_i^\beta\right] &= i{C_1}^{\alpha\dot\alpha\beta,}{}_i - \frac{1}{2}\bar\theta^{\dot\alpha,j}{A_1}^{\alpha\beta,}{}_{ji} \\
\left[x^{\alpha\dot\alpha}, \bar\theta^{\dot\beta,i}\right] &= 0 \\
\left[x^{\alpha\dot\alpha}, x^{\beta\dot\beta}\right] &= iD^{\alpha\dot\alpha\beta\dot\beta} + \frac{i}{2}\left(\bar\theta^{\dot\alpha,i}{C_1}^{\beta\dot\beta\alpha,}{}_i - \bar\theta^{\dot\beta,i}{C_1}^{\alpha\dot\alpha\beta,}{}_i\right) - \frac{1}{4}\bar\theta^{\dot\alpha,i}{A_1}^{\alpha\beta,}{}_{ij}\bar\theta^{\dot\beta,j}
\end{aligned} \tag{2.2.10}$$

or equivalently

$$\begin{aligned}
\left\{\theta_i^\alpha, \theta_j^\beta\right\} &= 0 \quad , \quad \left\{\bar\theta^{\dot\alpha,i}, \bar\theta^{\dot\beta,j}\right\} = A_2^{\dot\alpha\dot\beta,ij} \quad , \quad \left\{\theta_i^\alpha, \bar\theta^{\dot\beta,j}\right\} = 0 \\
\left[x^{\alpha\dot\alpha}, \theta_i^\beta\right] &= 0 \\
\left[x^{\alpha\dot\alpha}, \bar\theta^{\dot\beta,i}\right] &= iC_2^{\alpha\dot\alpha\dot\beta,i} - \frac{1}{2}\theta_j^\alpha A_2^{\dot\alpha\dot\beta,ji} \\
\left[x^{\alpha\dot\alpha}, x^{\beta\dot\beta}\right] &= iD^{\alpha\dot\alpha\beta\dot\beta} + \frac{i}{2}\left(\theta_i^\alpha C_2^{\beta\dot\beta\dot\alpha,i} - \theta_i^\beta C_2^{\alpha\dot\alpha\dot\beta,i}\right) - \frac{1}{4}\theta_i^\alpha A_2^{\dot\alpha\dot\beta,ij}\theta_j^\beta
\end{aligned} \tag{2.2.11}$$

which are the final forms we were looking for.

In the case of euclidean signature a non-trivial anticommutator is allowed in the algebra of fermionic coordinates. This reflects the fact that they satisfy different hermiticity conditions from the Minkowski case. Moreover, we note that associativity imposes less restrictive conditions because of the absence of conjugation relations between A_1 and A_2.

We move to the definition of $N = 1$ euclidean superspace. The basic idea is to temporarly double the fermionic degrees of freedom and to use the conjugation operator to halve them. We remark the conjugation rules $(\theta^\alpha_i)^* = i\theta_{\alpha i}$, $(\bar\theta^{\dot\alpha i})^* = i\bar\theta^i_{\dot\alpha}$ which do not mix θ_1 and θ_2. Then, the description of $N = 1$ euclidean superspace is formally equivalent to euclidean $N = 2$.

We define the independent fermionic degrees of freedom and their relations as

$$\begin{aligned} (\theta^\alpha)^* &= i\theta_\alpha \\ (\theta_\alpha)^* &= -i\theta_\alpha \end{aligned} \tag{2.2.12}$$

where the second identity follows from the first. In particular, the double conjugation operator

$$()^{**} = -1 \tag{2.2.13}$$

when acting on fermionic variables. Similar identities hold for the covariant derivatives.

Now that we defined the space in which we are working, we look for a formalism that automatically reproduces the above relations. This is achieved by defining the Moyal product acting on superfields

$$\begin{aligned} \phi * \psi &= \phi e^{-\overleftarrow{\partial}_\alpha \mathcal{F}^{\alpha\beta} \overrightarrow{\partial}_\beta} \psi \\ &= \phi\psi - \phi \overleftarrow{\partial}_\alpha \mathcal{F}^{\alpha\beta} \overrightarrow{\partial}_\beta \psi + \frac{1}{2} \phi \overleftarrow{\partial}_\alpha \overleftarrow{\partial}_\gamma \mathcal{F}^{\alpha\beta} \mathcal{F}^{\gamma\delta} \overrightarrow{\partial}_\delta \overrightarrow{\partial}_\beta \psi \\ &= \phi\psi - \phi \overleftarrow{\partial}_\alpha \mathcal{F}^{\alpha\beta} \overrightarrow{\partial}_\beta \psi - \frac{1}{2} \mathcal{F}^2 \partial^2 \phi \partial^2 \psi \end{aligned} \tag{2.2.14}$$

where we have defined $\mathcal{F}^2 = \mathcal{F}^{\alpha\beta}\mathcal{F}_{\alpha\beta}$. It is easy to see that the ccovariant derivatives are still derivatives under this product, and the set of chiral fields forms a ring.

If we work in the chiral representation, the covariant derivative algebra is the same as the ordinary case, while the superspace algebra is modified to

$$\begin{aligned} \left\{\theta^\alpha, \theta^\beta\right\}_* &= 2\mathcal{F}^{\alpha\beta} \quad , \quad \left\{\bar\theta^{\dot\alpha}, \bar\theta^{\dot\beta}\right\}_* = \left\{\theta^\alpha, \bar\theta^{\dot\beta}\right\}_* = 0 \\ \left[x^{\alpha\dot\alpha}, \theta^\beta\right]_* &= -2i\mathcal{F}^{\alpha\beta}\bar\theta^{\dot\alpha} \\ \left[x^{\alpha\dot\alpha}, \bar\theta^{\dot\beta}\right]_* &= 0 \\ \left[x^{\alpha\dot\alpha}, x^{\beta\dot\beta}\right]_* &= 2\bar\theta^{\dot\alpha}\mathcal{F}^{\alpha\beta}\bar\theta^{\dot\beta} \end{aligned} \tag{2.2.15}$$

We further introduce the suitable change of variable $y^{\alpha\dot\alpha} = x^{\alpha\dot\alpha} - i\theta^\alpha\bar\theta^{\dot\alpha}$ to set to zero all the commutators. From now on, the bosonic coordinates will be taken to be commuting, unless otherwise specified. For future reference, we quote here the final form of the algebra we will use:

$$\begin{aligned} \left\{\theta^\alpha, \theta^\beta\right\}_* &= 2\mathcal{F}^{\alpha\beta} \quad , \quad \left\{\bar\theta^{\dot\alpha}, \bar\theta^{\dot\beta}\right\}_* = \left\{\theta^\alpha, \bar\theta^{\dot\beta}\right\}_* = 0 \\ \left[y^{\alpha\dot\alpha}, \theta^\beta\right]_* &= \left[y^{\alpha\dot\alpha}, \bar\theta^{\dot\beta}\right]_* = \left[y^{\alpha\dot\alpha}, y^{\beta\dot\beta}\right]_* = 0 \end{aligned} \tag{2.2.16}$$

The explicit supersymmetry breaking can be seen by computing the anticommutation relations among the supercharges. We quote here the result

$$\begin{array}{rcl} \{Q_\alpha, Q_\beta\}_* = & 0 \quad , & \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\}_* = i\partial_{\alpha\dot{\alpha}} \\ \left\{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\right\}_* = & & 2\mathcal{F}^{\alpha\beta}\partial_{\alpha\dot{\alpha}}\partial_{\beta\dot{\beta}} \end{array} \tag{2.2.17}$$

2.3 Non-renormalization theorems

We are interested in how the deformation of the algebra modifies the quantum dynamics of supersymmetric field theories. It is well known that supersymmetry has a number of interesting features, the most of them originating from the so-called non-renormalization theorems. Let us review them and their direct consequences before moving to the deformed theory:

- `Theorem 1:` *Each term in the effective action is expressible as a superspace integral over a single $d^2\theta d^2\bar{\theta}$.*

- `Theorem 2:` *The general structure of the effective action is given as*

$$\Gamma[\Phi, \bar{\Phi}] = \sum_n \int \prod_{j=1}^n d^4x_j \int d^2\theta d^2\bar{\theta}\, G_n(x_1, ..., x_n) F_1(x_1, \theta) ... F_n(x_n, \theta).$$

  where $G_n(x_1, ..., x_n)$ are translation-invariant functions on Grassmann-even coordinates and $F(x, \theta, \bar{\theta})$ are local operators of $\Phi, \bar{\Phi}$ and their covariant derivatives:

$$F(x, \theta, \bar{\theta}) = F(\Phi, \bar{\Phi}, D\Phi, \bar{D}\bar{\Phi}, ...)$$

The above theorems, especially Theorem 2, lead immediately to the following results: (1) energy density of supersymmetric vacuum is zero because, in this case, there are no $F(x, \theta, \bar{\theta})$ field insertions in the effective action, so the $\int d^2\theta d^2\bar{\theta}$ integral gives zero; (2) the holomorphic and antiholomorphic parts are not renormalized. The reason is that to get holomorphic part one needs to integrate out $\int d^2\bar{\theta}$. However, as there is no $\Box^{-1}$ in the effective action, one cannot do that by combining it with the D^2 operator. A similar argument holds for the antiholomorphic part.

Now in the deformed theory at hand, Theorem 1 is not modified. The proof goes exactly the same as the ordinary theory. However, Theorem 2 is modified crucially by the Moyal product which explicitly inserts the supercharges in the action. Thus we derive the following new theorem.

- `Theorem 2` [after deformation]: *The general structure of the effective action is given as*

$$\Gamma[\Phi, \bar{\Phi}] = \sum_n \int \prod_{j=1}^n d^4x_j \int d^2\theta d^2\bar{\theta}\, G_n(x_1, ..., x_n; \bar{\theta}\bar{\theta}) F_1(x_1, \theta, \bar{\theta}) ... F_n(x_n, \theta, \bar{\theta}), \tag{2.3.18}$$

  where $G_n(x_1, ..., x_n; \bar{\theta}\bar{\theta})$ are translation-invariant functions on Grassmann-even coordinates **and** *possible insertion of $\bar{\theta}\bar{\theta}$ resulting of superspace loop integrals, while $F(x, \theta, \bar{\theta})$ are local* *operators of the background fields* $\Phi, \bar{\Phi}$*, their covariant derivatives,* **and** *the action of the chiral supercharge* Q *but* **not** $\bar{Q}$*:*

$$F(x, \theta, \bar{\theta}) = F(\Phi, \bar{\Phi}, D\Phi, \bar{D}\bar{\Phi}, Q\Phi, Q\bar{\Phi}...) \,.$$

Using the modified Theorem 2, we are now able to derive the following results:

- The vacua $|\text{vac}\rangle, \langle\text{vac}|$ that preserve the $\mathcal{N} = \frac{1}{2}$ supersymmetry are characterized by a set of critical points of the antiholomorphic superpotential, $\bar{W}'(\bar{A}) = 0$;
- the energy density of supersymmetric vacuum is still zero. Although the $\bar{\theta}^2$-dependence $G(x_1, \cdots, x_n; \bar{\theta}^2)$ would be able to render the $\int d^2\bar{\theta}$ integral nonzero, in the absence of any $F(x, \theta, \bar{\theta})$ insertions, the $\int d^2\theta$ integral still vanishes;
- the antiholomorphic part is still not renormalized, because the $\int d^2\theta$ are not absorbable, for the same reason as in the ordinary supersymmetric theories;
- the holomorphic part *is renormalized*. The reason is that now we have the θ^2 insertion from $G(x_1, \cdots, x_n; \bar{\theta}\bar{\theta})$, which can absorb the $\int d^2\bar{\theta}$ integral. Because of this, the D-terms with pure chiral fields and holomorphic F-terms are not distinguishable, and in fact both D-terms and F-terms are unified in $\mathcal{N} = \frac{1}{2}$ supersymmetry. We emphasize that this is the feature that was not evident from the classical consideration [2], but was revealed only after full quantum effects are taken into account.

Now, the main point is to understand when the deformed theories are renormalizable or not. If the holomorphic part is not prevented from being renormalized, quantum corrections can *a* priori generate an infinite number of divergent counterterms. Indeed, we introduced the new dimensionful coupling constant $\mathcal{F}$ in the theory, whit dimension -1. In the following chapters we will analyze the Wess-Zumino to illustrate how superspace techniques works in the $N = 1/2$ case, then we will move to the gauge theory. We will see that, despite the simplicity of the Wess-Zumino model, the Yang-Mills theory is a much more complicated subject.

Chapter 3

The Wess-Zumino model

The Wess-Zumino model is the first, prototypical example of a supersymmetric field theory. Its field content only consists of chiral superfields which interact through a superpotential. Studying this model is the first step before moving to the more complicated gauge theory, which will be discussed subsequently.

The nonanticommutative Wess-Zumino model is easily defined on the nonanticommutative superspace previously defined. As explained in the previous section, the consistent way to deform a supersymmetric theory is to promote all the products between fields to Moyal products (2.2.14), and expand them to get new interaction vertices. The effect of the nonanticommutativity is very easy to describe in this case: the action, when projected on the component fields, is the usual Wess-Zumino action augmented by a cubic term in the auxiliary field of the chiral multiplet. Due to the non appareance of any antiholomorphic part, only half of the supersymmetries are broken, and some of the original features of the original model are still valid, in accordance with the results in the previous section. In this section, we show this explicitly in the superspace approach. It has been also demonstrated in the component field approach [7].

3.1 Superspace approach

3.1.1 Generalities

In this section we concentrate on the superspace formulation of the nonanticommutative Wess-Zumino model. A similar computation is possible even in the component formalism, but the evaluation of the Feyman diagrams revealed much more complicated [7].

On the nonanticommutative superspace (2.2.16) we define the Wess-Zumino (WZ) model as given by the ordinary cubic action where products of superfields are generalized to the star product (2.2.14).

We start from the classical action

$$S = \int d^8 z \bar{\Phi}\Phi - \frac{m}{2}\int d^6 z \Phi^2 - \frac{\bar{m}}{2}\int d^6 \bar{z} \bar{\Phi}^2$$
$$-\frac{g}{3}\int d^6 z \Phi * \Phi * \Phi - \frac{\bar{g}}{3}\int d^6 \bar{z} \bar{\Phi} * \bar{\Phi} * \bar{\Phi} \qquad (3.1.1)$$

We stress that we are working with the euclidean space where no hermitian conjugation relations are assumed for fields, masses and couplings. The general strategy to analyze a deformed theory is to expand the Moyal product between the fields; note that in doing that, one can always neglect total spacetime derivatives. Moreover, the $*$-product has no effect when acting on a quadratic term in the action, as it would contribute only for a total fermionic derivative which vanishes by definition when integrated in the fermionic coordinates.

To deal with a well-defined superspace expression for all the terms in the lagrangian, we introduce an external constant spurion superfield $U = \theta^2\bar{\theta}^2\mathcal{F}^2$. The action takes the form

$$S = \int d^8z\bar{\Phi}\Phi - \frac{m}{2}\int d^6z\Phi^2 - \frac{\bar{m}}{2}\int d^6\bar{z}\bar{\Phi}^2 - \frac{g}{3}\int d^6z\Phi^3 - \frac{\bar{g}}{3}\int d^6\bar{z}\bar{\Phi}^3 + \frac{g}{6}\int d^8zU(D^2\Phi)^3 \tag{3.1.2}$$

Note that the term in the last line can be equivalently written as $\int d^6z\Phi(D^2\Phi)^2$, but the integrand is not chiral and it is not clear why it should be expressed as an F-term. Th use of the constant spurion superfield allows us to employ all the superspace perturbation theory machinery.

By using $\int d^2\theta\ldots = D^2(\ldots)$ and the projection method, we write the component action as

$$\begin{aligned} S \;=\; \int d^4x\Big[&\phi\Box\bar{\phi} + F\bar{F} - GF - \bar{G}\bar{F} + \frac{g}{6}C^2F^3 \\ &+\psi^\alpha i\partial_\alpha^{\dot{\alpha}}\bar{\psi}_{\dot{\alpha}} - \frac{m}{2}\psi^\alpha\psi_\alpha - \frac{\bar{m}}{2}\bar{\psi}^{\dot{\alpha}}\bar{\psi}_{\dot{\alpha}} - g\phi\psi^\alpha\psi_\alpha - \bar{g}\bar{\phi}\bar{\psi}^{\dot{\alpha}}\bar{\psi}_{\dot{\alpha}}\Big] \end{aligned} \tag{3.1.3}$$

where

$$\begin{aligned} G &= m\phi + g\phi^2 \\ \bar{G} &= \bar{m}\bar{\phi} + \bar{g}\bar{\phi}^2 \end{aligned} \tag{3.1.4}$$

The auxilary fields F and $\bar{F}$ satisfy the algebraic equations of motion (EOM)

$$F = \bar{G} \qquad , \qquad \bar{F} = G - \frac{g}{2}C^2F^2 = G - \frac{g}{2}C^2\bar{G}^2 \tag{3.1.5}$$

To compute quantum perturbative corrections it is convenient to perform the quantum-background splitting $\Phi \to \Phi + \Phi_q$ and integrating over the quantum fluctuations Φ_q. The expansion produces the ordinary cubic vertices and two vertices involving the spurion field U and depicted in Fig. 3.1.

The scalar propagators are

$$\begin{aligned} \langle\Phi\bar{\Phi}\rangle &= \frac{1}{p^2 + m\bar{m}}\delta^{(4)}(\theta - \theta') \\ \langle\Phi\Phi\rangle &= -\frac{\bar{m}D^2}{p^2(p^2 + m\bar{m})}\delta^{(4)}(\theta - \theta') \\ \langle\bar{\Phi}\bar{\Phi}\rangle &= -\frac{m\bar{D}^2}{p^2(p^2 + m\bar{m})}\delta^{(4)}(\theta - \theta') \end{aligned} \tag{3.1.6}$$

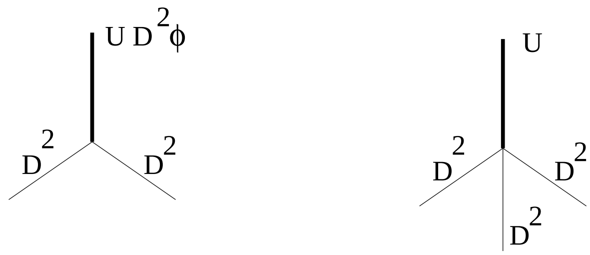

Figure 3.1: New vertices proportional to the external U superfield

To compute quantum corrections at a given loop, we write down all the Feynman diagrams up to the given order and insert an extra $\bar{D}^2(D^2)$ derivative on each chiral (antichiral) line leaving a vertex except for one of the lines at a completely (anti)chiral vertex.

The loop integrals are evaluated in the dimensional regularization ($n = 4 - 2\epsilon$) and minimal subtraction scheme. Divergent integrals are regularized using the so-called G-scheme

$$\int d^4k f(k) \to G(\epsilon) \int d^nk f(k) \tag{3.1.7}$$

where $G(\epsilon) = (4\pi)^{-\epsilon}\Gamma(1-\epsilon)$. Factors of 4π are always neglected along the calculations and a $(4\pi)^2$ factor is inserted for each momentum loop in the final result.

3.1.2 One-loop divergencies

At one loop divergencies appear that involve the two-, three-, and four point functions. The last two ones are due to the deformation of the theory and also contain the spurion superfield.

The divergent two-point function is the ordinary self-energy diagram which gives the wave function renormalization. Its contribution is

$$A_0 \;\to\; \frac{2}{\epsilon} g\bar{g} \int d^8z \Phi\bar{\Phi} \tag{3.1.8}$$

No divergencies with more than one insertion of the U vertices can appear; then, the only divergent topologies are the ones given in Fig. 3.2.

The result for the self-energy momentum integral is

$$\int d^4k \frac{1}{(k^2 + m\bar{m})[(p-k)^2 + m\bar{m}]} \to \frac{1}{\epsilon} \tag{3.1.9}$$

with which one obtain the results

$$\begin{aligned} A_1 &\;\to\; -\frac{1}{\epsilon} g^2\bar{m}^2 \int d^8z U(D^2\Phi)^2 \;=\; -\frac{1}{\epsilon} g^2\bar{m}^2 C^2 \int d^4x\; F^2 \\ A_2 &\;\to\; -\frac{4}{\epsilon} g^2\bar{g}\bar{m} \int d^8z U(D^2\Phi)^2\bar{\Phi} \;=\; -\frac{4}{\epsilon} g^2\bar{g}\bar{m} C^2 \int d^4x F^2\bar{\phi} \\ A_3 &\;\to\; -\frac{4}{\epsilon} g^2\bar{g}^2 \int d^8z U(D^2\Phi)^2\bar{\Phi}^2 \;=\; -\frac{4}{\epsilon} g^2\bar{g}^2 C^2 \int d^4x F^2\bar{\phi}^2 \end{aligned} \tag{3.1.10}$$

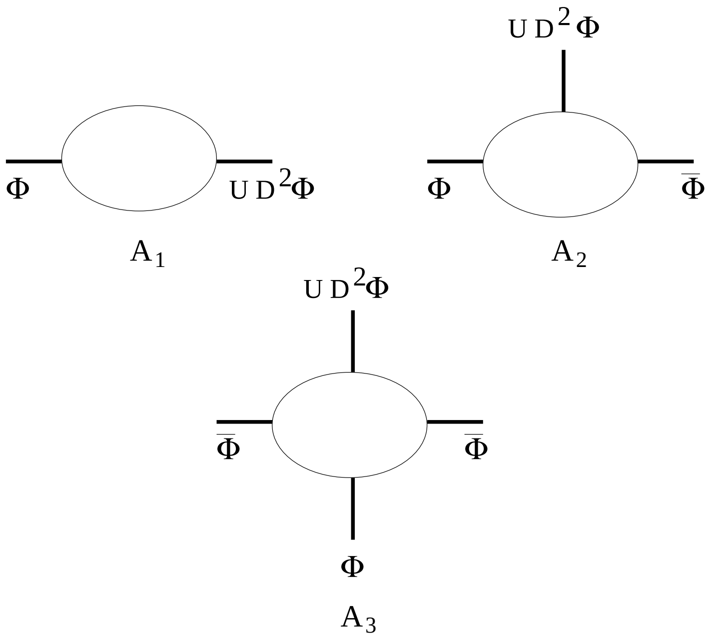

Figure 3.2: One-loop divergent diagrams with one insertion of the $U(D^2\Phi)^3$-vertex

Summing them up, and using the equations of motion (3.1.5)[1], the result can be recasted as

$$-\frac{1}{\epsilon}g^2\int d^8z\Big[\bar{m}^2U(D^2\Phi)^2+4\bar{g}U(D^2\Phi)^3\Big] \qquad (3.1.11)$$

Because (3.1.11) contains a divergent term which is not present in the classical action, the theory considered so far is not renormalizable. One has to modify the classical action by adding appropriate counterterms at tree level. The modified, renormalizable action takes the form

$$\begin{aligned} S=&\int d^8z\bar{\Phi}\Phi-\frac{m}{2}\int d^6z\Phi^2-\frac{\bar{m}}{2}\int d^6\bar{z}\bar{\Phi}^2-\frac{g}{3}\int d^6z\Phi^3-\frac{\bar{g}}{3}\int d^6\bar{z}\bar{\Phi}^3\\ &+\frac{g}{6}\int d^8zU(D^2\Phi)^3+k_1\bar{m}^4\int d^8zUD^2\Phi+k_2\bar{m}^2\int d^8zU(D^2\Phi)^2 \end{aligned} \qquad (3.1.12)$$

where two new coupling constants appear. The choice of $\bar{m}$ is a simplifying assumption. The new divergent diagrams of Fig. 3.3 give the following contributions

$$\begin{aligned} \widetilde{A}_1 &\rightarrow -\frac{2}{\epsilon}\,k_2\,g\bar{m}^4\int d^8zU(D^2\Phi) = -\frac{2}{\epsilon}\,k_2\,g\bar{m}^4C^2\int d^4x\;F\\ \widetilde{A}_2 &\rightarrow -\frac{8}{\epsilon}\,k_2\,g\bar{g}\bar{m}^3\int d^8zU(D^2\Phi)\bar{\Phi} = -\frac{8}{\epsilon}\,k_2\,g\bar{g}\bar{m}^3C^2\int d^4xF\bar{\phi}\\ \widetilde{A}_3 &\rightarrow -\frac{8}{\epsilon}\,k_2\,g\bar{g}^2\bar{m}^2\int d^8zU(D^2\Phi)\bar{\Phi}^2 = -\frac{8}{\epsilon}\,k_2\,g\bar{g}^2\bar{m}^2C^2\int d^4xF\bar{\phi}^2 \end{aligned} \qquad (3.1.13)$$

[1] the use of the classical EOM has been justified in [8] and it is valid only in this case. In general, one has to use the full quantum equations.

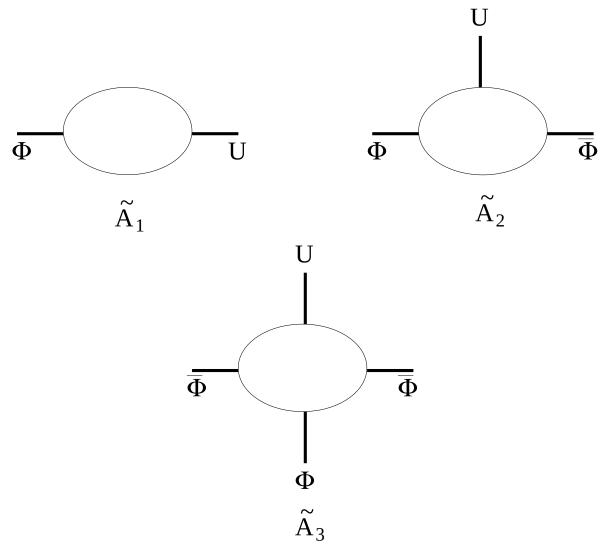

Figure 3.3: One-loop divergent diagrams with one insertion of the $U(D^2\Phi)^2$-vertex

and they sum up to the following divergent expression

$$\widetilde{A}_1 + \widetilde{A}_2 + \widetilde{A}_3 \ = \ -\frac{2}{\epsilon} k_2 g \bar{m}^2 C^2 \int d^4x \Big[\bar{m}^2 F \ + \ 4\bar{g} F \bar{G} \Big] \tag{3.1.14}$$

Using the classical EOM (3.1.5) for the F-field the second term is an F^2 contribution.

Therefore, summing everything and reinserting (4π) factors, the total one–loop divergence in superspace language is

$$-\frac{1}{\epsilon}\frac{1}{(4\pi)^2} \int d^8z \Big[2k_2 g \bar{m}^4 U(D^2\Phi) + \bar{m}^2(g^2 + 8k_2 g\bar{g}) U(D^2\Phi)^2 + 4g^2\bar{g} U(D^2\Phi)^3 \Big] \tag{3.1.15}$$

The two loop corrections and the β-functions of the model was also determined, but they are not illuminating for our purposes. We only comment on the main results:

- by direct inspection and using general (D-algebra) arguments, it was proven that divergent diagrams contain only one insertion of the spurion field. This means that divergent contributions to the effective action are only proportional up to $\mathcal{F}^2$;
- the classical model in which the ordinary product is replaced with the Moyal product is not renormalizable. Moreover, all the divergent counterterms arise at one loop yet. As we will see, the gauging preserves this features as well;
- divergencies are only logarithmic. Then, the predictive power of supersymmetry is preserved even if half of it is explicitly broken;
- the results shown here confirm the non-renormalization theorem stated in a previous section. In particular, only one between the holomorphic and antiholomorphic part undergoes renormalization;

	dim	$U(1)_R$	$U(1)_\Phi$		dim	$U(1)_R$	$U(1)_\Phi$
Φ	1	1	1	$\bar{\Phi}$	1	-1	-1
U	-4	4	0	$d^4\theta$	2	0	0
D_α	1/2	-1	0	$\bar{D}_{\dot{\alpha}}$	1/2	1	0
D^2	1	-2	0	$\bar{D}^2$	1	2	0
g	0	-1	-3	$\bar{g}$	0	1	3
m	1	0	-2	$\bar{m}$	1	0	2
κ_1	0	0	0	κ_2	0	0	0

Table 3.1: Global $U(1)$ charge assignment in superspace

- unless not explicitly shown, the $*$-product does not get deformed at the quantum level; indeed, the effective action can be written by keeping the product implicit.

3.2 An all loop argument

The renormalizability of the WZ model was further discussed in subsequent papers and an all loop argument was given [9, 10, 11]. We review here the superspace formulation of their argument [11].

The action (3.1.12) possesses two global $U(1)$ pseudo-symmetries, namely a $U(1)_\Phi$ flavor symmetry and the $U(1)_R$ R-symmetry [12]. The charge assignment are given in Table 3.1.

In particular, the couplings κ_1 and κ_2 are neutral under both the symmetries and dimensionless, so we will not indicate them explicitly from now on.

The most general divergent term in the effective action can be written as

$$\int d^4x \Gamma_{\mathcal{O}} = \lambda \int d^4x d^4\theta \, (D^2)^\gamma \, (\bar{D}^2)^\delta \, (D_\alpha \partial^{\alpha\dot{\alpha}} \bar{D}_{\dot{\alpha}})^\eta \, \Box^\zeta \, U^\rho \, \Phi^\alpha \, \bar{\Phi}^\beta \tag{3.2.16}$$

with γ, δ, η, ζ, ρ, α, β non-negative integers. Every derivative is intended as acting on a (spurion) superfield, and we can also use the commutation relations to lower the number of spinorial derivatives when necessary. The coefficient λ has dimension d and charges R and S under the $U(1)_R$ and $U(1)_\Phi$ respectively, and takes the form

$$\lambda \sim \Lambda^d g^{x-R} \bar{g}^x \left(\frac{m}{\Lambda}\right)^y \left(\frac{\bar{m}}{\Lambda}\right)^{y+\frac{S-3R}{2}} \lambda_2^{\omega_2} \tag{3.2.17}$$

where Λ is an ultraviolet momentum cutoff. Note that the independence on λ_1 is fixed by the fact that we cannot form a 1PI connected diagram from $\int U(D^2\Phi$. MOreover, $\omega_2 \leq \rho$ because λ_2 only appears coupled to U.

The constraints that (3.2.16) has dimension 4 and zero charge translate into

$$\begin{aligned} d &= 2 + 4\rho - \alpha - \beta - \gamma - \delta - 2\eta - 2\zeta \\ R &= \beta - \alpha + 2\gamma - 2\delta - 4\rho \\ S &= \beta - \alpha \, . \end{aligned} \tag{3.2.18}$$

and we define the overall power of Λ as

$$P = d - 2y - \frac{1}{2}(S - 3R) \tag{3.2.19}$$

which reduces to

$$P = 2 + 2\gamma - 2\rho - 2\alpha - 4\delta - 2y - 2\eta - 2\zeta \geq 0 \tag{3.2.20}$$

where the last inequality is the condition to have a divergent diagram. We proceed by analyzing case by case:

- $\rho = 0$. In this case we have the ordinary Wess-Zumino terms.
- $\rho = 1$. Then

$$\gamma - \alpha - 2\delta - y - \eta - \zeta \geq 0 \tag{3.2.21}$$

  From the fact that U has only the highest component, and that the chiral covariant derivatives only act on chiral superfields up to the commutation relations, we conclude that $\gamma \leq \alpha$. Then, (3.2.21) is only satisfied when

$$\gamma = \alpha \qquad \delta = y = \eta = \zeta = 0 \tag{3.2.22}$$

  and $P = 0$, i.e., only logarithmic divergencies appear. The general divergent term

$$\int d^8 z U (D^2\Phi)^\alpha \bar{\Phi}^\beta \,. \tag{3.2.23}$$

- $\rho = 1 + n$, $n > 0$

  Since the U superfield has only the $\theta^2\bar{\theta}^2$ component, we need at least n D^2 and n $\bar{D}^2$. Therefore

$$\gamma = n + \gamma_1 \,, \qquad \delta = n + \delta_1 \tag{3.2.24}$$

  and then

$$\gamma_1 - \alpha - 2n - 2\delta_1 - y - \eta - \zeta \geq 0 \,. \tag{3.2.25}$$

  Since $\gamma_1 \leq \alpha$ (as in the previous case), and $n > 0$, we see that eq. (3.2.25) cannot be satisfied.

The previous discussion shows that there are only loarithmic divergencies. To show that there are only finitely many divergent terms, we take the solution (3.2.22) with $\lambda \sim g^{x-R}\, \bar{g}^x\, \bar{m}^{(S-3R)/2}\, \lambda_2^{\omega_2}$ and $(S - 3R)/2 = -\beta - 2\alpha + 6$. In dimensional regularization, the evaluation of the integral cannot depend on the mass parameter; this means that powers of the mass in the coupling λ can appear from the λ_2 vertex and from the massive propagators. Because the number of the latter is always nonnegative and using $\omega_2 \leq \rho$, we obtain

- $\omega_2 = 0 \quad \rightarrow \quad -\beta - 2\alpha + 6 \geq 0 \quad \rightarrow \quad \beta + 2\alpha \leq 6$

- $\omega_2 = 1 \quad \rightarrow \quad -\beta - 2\alpha + 4 \geq 0 \quad \rightarrow \quad \beta + 2\alpha \leq 4$

along with $\alpha \geq 1$ which comes from the D-algebra of every supergraph.

In conclusion, we found that at any loop order, there is only a finite number of logarithmic divergent counterterms of the form

$$\int d^8 z U (D^2\Phi)^\alpha \bar{\Phi}^\beta \,, \quad \alpha \geq 1 \,, \qquad \beta + 2\alpha \leq 6 - 2\omega_2 \,, \quad \omega_2 = 0, 1 \,. \tag{3.2.26}$$

By using the EOM, they can be recasted to obtain their explicit form [8, 11]

$$\int U(D^2\Phi), \int U(D^2\Phi)^2, \int U(D^2\Phi)^3 \tag{3.2.27}$$

which are suuficient to renormalize the theory at any order of perturbation theory, in agreement with the results of the previous section.

Chapter 4

Gauge theories

The most interesting quantum field theories which are realized in particle physics are gauge theories. In particular, supersymmetric gauge theories arise as field theory limits of superstring theories when D-branes are embedded in some background.

Gauge theories, even in their supersymmetric form, are much more difficult to analyze then the models of scalar (super)fields we discussed in the previous chapter. Nevertheless, the use of a formalism which makes manifest the (super)symmetries of the theory drastically reduces the computational efforts to analyze the physical system.

Four-dimensional supersymmetric Yang-Mills theories are renormalizable. As a direct consequence of supersymmetry, the divergences only appear at logarithmic order. In this chapter we discuss the generalization to gauge theories with partially broken supersymmetries, where the role of the order parameter of broken supersymmetry is played by the lack of anticommutativity between the fermionic coordinates of superspace. It is known that, in the ordinary case, one can perform renormalization in any gauge, since (super)gauge invariance is not spoiled by quantum corrections. In component field formalism, the best gauge choice is the so-called Wess-Zumino gauge, because it has the advantage that the expansion of the gauge superfield strength in components gives rise to a finite number of terms; this is true also in the nonanticommutative case. However, in the latter case there exist no arguments to guarantee that (super)gauge invariance will survive at the quantum level since the star product is defined in terms of non-covariant derivatives. Therefore, the quantum properties proved in the Wess-Zumino gauge cannot be safely extended to any gauge without a deeper understanding of the relation among supersymmetry breaking and (super)gauge invariance at the quantum level.

The issue of performing a perturbative analysis directly in superspace requires the development of a different approach to the deformed Yang-Mills theories; the $N = 1/2$ superspace is the appropriate setting for this study. The nontrivial structure of the star product modifies the Feynman rules and now every vertex comes with extra spinorial derivatives, thus also modifying the D-algebra and the resulting loop momentum integrals. Moreover, since the gauge transformation rules are also affected by the deformation, to keep the efficient properties of the superspace techniques one has to generalize the background field method to the nonanticommutative case. The generalization is not straightforward for two main reasons: the change of conjugation rules among the fermionic variables and the lack of simple identities among the covariant derivatives. However, a modified version of the method exists which allows for a manifestly covariant quantization of gauge theories.

Our task is to compute one-loop divergent contributions to the effective action. Our starting theory is $N = 1/2$ gauge theory with chiral matter in the adjoint representation and no superpotential. In the ordinary case, only the two point function is divergent, stemming for the fact that the gauge coupling runs and the gauge invariance is not modified at the quantum level. In the deformed case, the modified Feynman rules introduce new kind of divergences in the three- and four-point functions. This is a direct consequence of the modified gauge transformation rules: it turns out that in the presence of the star product $SU(N)$ is no more a group, and one is forced to take into account the $U(1)$ part of the (super)fields. Consequently, the quantum corrected two-point function is not gauge invariant, but its gauge variation is proportional to the gauge variation of the three- and four-point functions named before. The one-loop effective action contains all this terms in such a way the cancellation among these non-vanishing gauge variations occurs, leading to a gauge invariant result. Thus, despite the gauge invariance of the quantum theory is not manifest, it is a property of the theory even at the quantum level.

The theory is complicated when a superpotential is added. In this case, the chiral (super)fields also acquire a nontrivial interaction among themselves, and their two-point function becomes non-vanishing. Analogously to the discussion above, this does not constitute a gauge invariant contribution to the effective action, but the full result comes out in a particular linear combination of two- and three-point functions that makes the effective action a gauge invariant quantity. The way to achieve this in some way different from the gauge sector. In this case we cannot quantize the fields as in the ordinary case, but we only have to modify the antichiral propagator, thus breaking the chiral-antichiral symmetry of the theory. However, there is no *a priori* argument that forbids this asymmetry: indeed, in the euclidean signature we are working with there are no conjugation rules between the chiral and antichiral quantities. Thus, to deal with a consistent quantum theory the Kähler potential is modified in a such a way the antichiral (super)field plays a privileged role. This is analogous to what happens with the fermionic variables of the nonanticommutative superspace, where the chiral anticommutation rules are modified.

This chapter is organized as follows. First, we review the supersymmetric gauge theories defined on the nonanticommutative superspace, and generalize the background field method to this case. Then, we compute on the one-loop computations of the two-, three- and four-point functions for the gauge theory, proving the one-loop (super)gauge invariance of the divergent part of the effective action. Then, we move to the analysis of the theory when a nontrivial superpotential is added. We conclude with the explicit example of the $U(1)$ case, where most of the difficulties are missing.

4.1 Supersymmetric gauge theories on $N = 1/2$ superspace

We define supersymmetric gauge theories on the nonanticommutative superspace (2.2.16)

$$\left\{\theta^{\alpha}, \theta^{\beta}\right\} = 2\mathcal{F}^{\alpha\beta} \tag{4.1.1}$$

where $\mathcal{F}^{\alpha\beta}$ is a 2×2 symmetric constant matrix. As already explained, we work in euclidean signature since this algebra is only consistent in the case where the chiral and the antichiral sectors are not related by hermitian conjugation.

The deformation (4.1.1) can be realized by introducing the so-called star product (2.2.14)

$$\begin{aligned}\phi * \psi &= \phi e^{-\overleftarrow{\partial}_\alpha \mathcal{F}^{\alpha\beta}\overrightarrow{\partial}_\beta}\psi \\ &= \phi\psi - \phi\overleftarrow{\partial}_\alpha \mathcal{F}^{\alpha\beta}\overrightarrow{\partial}_\beta\psi + \frac{1}{2}\phi\overleftarrow{\partial}_\alpha\overleftarrow{\partial}_\gamma \mathcal{F}^{\alpha\beta}\mathcal{F}^{\gamma\delta}\overrightarrow{\partial}_\delta\overrightarrow{\partial}_\beta\psi \\ &= \phi\psi - \phi\overleftarrow{\partial}_\alpha \mathcal{F}^{\alpha\beta}\overrightarrow{\partial}_\beta\psi - \frac{1}{2}\mathcal{F}^2\partial^2\phi\partial^2\psi \end{aligned} \tag{4.1.2}$$

under which the class of (anti)chiral superfields is closed.

A key property of the star product is the following: the trace of more than two fields is not cyclic in general unless it is integrated over $d^2\theta$. In particular, we have

$$\mathrm{Tr}(A * B * C) \neq \mathrm{Tr}(C * A * B) \tag{4.1.3}$$

$$\int d^4x d^2\bar{\theta}\ \mathrm{Tr}(A * B * C) \neq \int d^4x d^2\bar{\theta}\ \mathrm{Tr}(C * A * B) \tag{4.1.4}$$

but

$$\int d^4x d^2\theta\ \mathrm{Tr}(A * B * C) = \int d^4x d^2\theta\ \mathrm{Tr}(C * A * B) \tag{4.1.5}$$

However, even under $d^2\theta$ integration the cyclicity property gets spoiled when the trace appears multiplied by an extra function. In particular,

$$\int d^2\theta\ \mathrm{Tr}(A * B)\mathrm{Tr}(C) \neq \int d^2\theta\ \mathrm{Tr}(B * A)\mathrm{Tr}(C) \tag{4.1.6}$$

These properties can be easily proved by expanding the star product itself.

We now turn to the description of supersymmetric gauge theories in the nonanticommutative superspace. The gauge fields, the fields strengths and their superpartners can be organized into superfields, all expressed in terms of a prepotential V which is a scalar superfield in the adjoint representation of the gauge group ($V \equiv V^a T^a$, T^a being the group generators). In euclidean signature, it is convenient for technical reasons to choose the prepotential to be pure immaginary. The supergauge transformations are given in terms of two independent chiral and antichiral parameter superfields $\Lambda, \overline{\Lambda}$

$$e_*^V \to e_*^{V'} = e_*^{i\overline{\Lambda}} * e_*^V * e_*^{-i\Lambda} \tag{4.1.7}$$

The corresponding covariant derivatives (in gauge chiral representation) are given by

$$\nabla_A \equiv (\nabla_\alpha, \nabla_{\dot{\alpha}}, \nabla_{\alpha\dot{\alpha}}) \ = \ (e_*^{-V} * D_\alpha\, e_*^V\ ,\ \overline{D}_{\dot{\alpha}}\ ,\ -i\{\nabla_\alpha, \nabla_{\dot{\alpha}}\}_*) \tag{4.1.8}$$

whereas in gauge antichiral representation they are defined as

$$\overline{\nabla}_A \equiv (\overline{\nabla}_\alpha, \overline{\nabla}_{\dot{\alpha}}, \overline{\nabla}_{\alpha\dot{\alpha}}) \ = \ (D_\alpha\ ,\ e_*^V * \overline{D}_{\dot{\alpha}}\, e_*^{-V}\ ,\ -i\{\overline{\nabla}_\alpha, \overline{\nabla}_{\dot{\alpha}}\}_*) \tag{4.1.9}$$

satisfying $\overline{\nabla}_A = e_*^V * \nabla_A * e_*^{-V}$.

They can be expressed in terms of ordinary supercovariant derivatives $D_A, \overline{D}_A$ and a set of connections, as $\nabla_A \equiv D_A - i\Gamma_A$ or $\overline{\nabla}_A \equiv \overline{D}_A - i\overline{\Gamma}_A$. Nontrivial connections are then

$$\Gamma_\alpha = ie_*^{-V} * D_\alpha \, e_*^V \qquad , \qquad \Gamma_{\alpha\dot\alpha} = -i\overline{D}_{\dot\alpha}\Gamma_\alpha \tag{4.1.10}$$

or

$$\overline{\Gamma}_{\dot\alpha} = ie_*^{V} * \overline{D}_{\dot\alpha} \, e_*^{-V} \qquad , \qquad \overline{\Gamma}_{\alpha\dot\alpha} = -iD_\alpha\overline{\Gamma}_{\dot\alpha} \tag{4.1.11}$$

The field strengths are defined as $*$–commutators of supergauge covariant derivatives

$$W_\alpha = -\frac{1}{2}[\nabla^{\dot\alpha}, \nabla_{\alpha\dot\alpha}]_* \qquad , \qquad \widetilde{W}_{\dot\alpha} = -\frac{1}{2}[\nabla^{\alpha}, \nabla_{\alpha\dot\alpha}]_* \tag{4.1.12}$$

or

$$\widetilde{W}_\alpha = -\frac{1}{2}[\overline{\nabla}^{\dot\alpha}, \overline{\nabla}_{\alpha\dot\alpha}]_* \qquad , \qquad \overline{W}_{\dot\alpha} = -\frac{1}{2}[\overline{\nabla}^{\alpha}, \overline{\nabla}_{\alpha\dot\alpha}]_* \tag{4.1.13}$$

and satisfy the Bianchi's identities $\nabla^\alpha * W_\alpha + \nabla^{\dot\alpha} * W_{\dot\alpha} = 0$ or $\overline{\nabla}^\alpha * \overline{W}_\alpha + \overline{\nabla}^{\dot\alpha} * \overline{W}_{\dot\alpha} = 0$. The superfield strenghts in antichiral representation are related to the ones in chiral representation as

$$\widetilde{W}_\alpha = e_*^V * W_\alpha * e_*^{-V} \qquad\qquad \bar{W}_{\dot\alpha} = e_*^V * \widetilde{W}_{\dot\alpha} * e_*^{-V} \tag{4.1.14}$$

While W_α and $\bar{W}_{\dot\alpha}$ are ordinary chiral and antichiral superfields, the tilde quantities are covariantly (anti)chiral.

Under supergauge transformations (4.1.7) all the superfield strengths transform covariantly. For infinitesimal transformations we have

$$\begin{aligned} \delta W_\alpha = i[\Lambda, W_\alpha]_* \qquad &, \qquad \delta\widetilde{W}_{\dot\alpha} = i[\Lambda, \widetilde{W}_{\dot\alpha}]_* \\ \delta\widetilde{W}_\alpha = i[\overline{\Lambda}, \widetilde{W}_\alpha]_* \qquad &, \qquad \delta\overline{W}_{\dot\alpha} = i[\overline{\Lambda}, \overline{W}_{\dot\alpha}]_* \end{aligned} \tag{4.1.15}$$

If we expand $W_\alpha = W_\alpha^A T^A$ where T^A are the group generators, use the definitions (A.1.3, A.1.4) and the identity (A.1.7) we can rewrite

$$\delta W_\alpha^A = \frac{i}{2} d_{ABC}[\Lambda^B, W_\alpha^C]_* - \frac{1}{2} f_{ABC}\{\Lambda^B, W_\alpha^C\}_* \tag{4.1.16}$$

and similarly for the others. In the particular case of $U(N)$, given the explicit expressions (A.1.4) for d_{ABC}, the first term in δW_α^A mixes $U(1)$ and $SU(N)$ fields. In particular, the abelian, $U(1)$ field strength W_α^0 transforms nontrivially under $SU(N)$ and its transform is given in terms of both $U(1)$ and $SU(N)$ fields. In the commutative limit this term goes to zero and we are back to the ordinary theory where $SU(N)$ fields only transform under $SU(N)$ transformations while the abelian field is a singlet. As we shall see this is the source of significant complications.

In the ordinary anticommutative superspace, in the absence of instantonic effects, any of the following actions

$$\begin{aligned} S = \int d^4x d^2\theta \; \mathrm{Tr}(W^\alpha W_\alpha) \quad &; \quad \widetilde{S} = \int d^4x d^2\bar\theta \; \mathrm{Tr}(\widetilde{W}^{\dot\alpha}\widetilde{W}_{\dot\alpha}) \\ \overline{S} = \int d^4x d^2\bar\theta \; \mathrm{Tr}(\overline{W}^{\dot\alpha}\overline{W}_{\dot\alpha}) \quad &; \quad \widetilde{\overline{S}} = \int d^4x d^2\theta \; \mathrm{Tr}(\widetilde{W}^{\alpha}\widetilde{W}_{\alpha}) \end{aligned} \tag{4.1.17}$$

can be used to describe pure gauge theory. In fact, any of these actions is gauge invariant and, when reduced to components in the WZ gauge, describes the correct dynamics of the physical degrees of freedom (gluons and gluinos) [13]. In particular, the actions S and $\widetilde{\widetilde{S}}$, as well as $\overline{S}$ and $\widetilde{S}$, are trivially identical as can be easily understood by using the relations (4.1.14) and the cyclicity of the trace. Instead, S and $\overline{S}$ differ by surface terms which are zero if we do not include instantons. The equivalence of the actions (4.1.17) holds for any gauge group, $U(1)$ included.

A peculiarity of the NAC case is that in the presence of star products it is no longer true that the four actions (4.1.17) are all equivalent. For example, let us consider $\widetilde{S}$ versus $\overline{S}$. By using the relations (4.1.14) we have the following chain of relations

$$\begin{aligned}\widetilde{S} = \int d^4x d^2\bar{\theta}\ \mathrm{Tr}(\widetilde{W}^{\dot{\alpha}}\widetilde{W}_{\dot{\alpha}}) &= \int d^4x d^2\bar{\theta}\ \mathrm{Tr}(e_*^{-V} * \overline{W}^{\dot{\alpha}}\overline{W}_{\dot{\alpha}} * e_*^{V}) \\ &\neq \int d^4x d^2\bar{\theta}\ \mathrm{Tr}(\overline{W}^{\dot{\alpha}}\overline{W}_{\dot{\alpha}}) = \overline{S} \qquad (4.1.18)\end{aligned}$$

since in this case the trace is not cyclic, as follows from (4.1.4). What is interesting from the physical point of view is that the non–equivalence of the two actions has important consequences for their gauge invariance. In fact, it is easy to show that under transformations (4.1.15) the action $\overline{S}$ *is* gauge invariant whereas $\widetilde{S}$ *is not*. For what concerns S and $\widetilde{\widetilde{S}}$ instead, they are still equivalent and both gauge invariant since they are defined as chiral integrals and the cyclicity of the trace can be used in this case. Finally, as in the ordinary case, the two gauge invariant actions S and $\overline{S}$ are equivalent up to instantonic effects when reduced to components in the WZ gauge [2].

The situation is even worse if we consider only the $U(1)$ part of the actions (4.1.17). We note that this part can be separated out in the form of a product of single traces. Looking at the $U(N)$ transformations of the abelian superfield strengths as given in (4.1.16) one can prove that among the abelian actions

$$\begin{aligned} S_0 &= \int d^4x d^2\theta\ \mathrm{Tr}(W^\alpha)\mathrm{Tr}(W_\alpha) \quad ; \quad \widetilde{S}_0 = \int d^4x d^2\bar{\theta}\ \mathrm{Tr}(\widetilde{W}^{\dot{\alpha}})\mathrm{Tr}(\widetilde{W}_{\dot{\alpha}}) \\ \overline{S}_0 &= \int d^4x d^2\bar{\theta}\ \mathrm{Tr}(\overline{W}^{\dot{\alpha}})\mathrm{Tr}(\overline{W}_{\dot{\alpha}}) \quad ; \quad \widetilde{\widetilde{S}}_0 = \int d^4x d^2\theta\ \mathrm{Tr}(\widetilde{W}^{\alpha})\mathrm{Tr}(\widetilde{W}_{\alpha}) \qquad (4.1.19)\end{aligned}$$

only $\widetilde{\widetilde{S}}_0$ is gauge invariant, whereas the others are *not* and need to be completed by extra terms in order to restore gauge invariance. In particular, we will be interested in the gauge invariant completion of $\overline{S}_0$ which reads [14]

$$\int d^4x\ d^2\overline{\theta}\ \mathrm{Tr}(\overline{W}^{\dot{\alpha}})\mathrm{Tr}(\overline{W}_{\dot{\alpha}}) + 4i\mathcal{F}^{\rho\gamma} \int d^4x\ d^4\theta\ \ \overline{\theta}^2\ \mathrm{Tr}\left(\partial_{\rho\dot{\rho}}\overline{\Gamma}^{\dot{\alpha}}\right) \mathrm{Tr}\left(\overline{W}_{\dot{\alpha}}\overline{\Gamma}_{\gamma}^{\ \dot{\rho}}\right) \qquad (4.1.20)$$

where $\mathcal{F}^{\rho\gamma}$ is the NAC parameter. We note that the lack of invariance of the abelian actions in (4.1.19) is due to the fact that the abelian gauge field transforms nontrivially under $SU(N)$ and its variation is proportional to the $SU(N)$ gauge fields (see eq. (4.1.16)). We also note that despite the nontrivial variation of the $U(1)$ part as described in $\overline{S}_0$, the total action $\overline{S}$ which describes the propagation of $U(1)$ and $SU(N)$ fields *is* gauge invariant since the gauge variation of the $U(1)$ fields gets compensated by the gauge variation of the $SU(N)$ fields. This is peculiar to the NAC case and does not have direct correspondence in the ordinary anticommutative case.

Given the asymmetry between chiral and antichiral representations introduced by the nonanticommutativity, it turns out that the choice of one representation with respect to the other may be preferable from the point of view of technical convenience. We find it preferable to work in antichiral representation for the following reason: in the ordinary, anticommuting case we often switch between full superspace integrals and chiral (or antichiral) integrals by using the equivalence $\int d^4xd^4\theta\ \mathrm{Tr}[F(z)] \equiv \int d^4xd^2\theta\ \mathrm{Tr}[\overline{\nabla}^2 F(z)] \equiv \int d^4xd^2\overline{\theta}\ \mathrm{Tr}[\nabla^2 F(z)]$. However, in the NAC case the second equality fails if one is working in chiral representation, as can be seen by examining its derivation in the following sequence of equalities (star products understood in the NAC case):

$$\int d^4xd^4\theta\ \mathrm{Tr}[F(z)] = \int d^4xd^4\theta\ \mathrm{Tr}\{e^{-V}[F(z)]e^V\} = \int d^4xd^2\overline{\theta}\ \mathrm{Tr}\{D^2e^{-V}[F(z)]e^V\}$$
$$= \int d^4xd^2\overline{\theta}\ \mathrm{Tr}\{e^{-V}e^VD^2e^{-V}[F(z)]e^V\} = \int d^4xd^2\overline{\theta}\ \mathrm{Tr}\{e^{-V}\nabla^2[F(z)]e^V\} \qquad (4.1.21)$$

(Note that $\nabla^2e^V = 0$.) In the ordinary case one can use the cyclicity of the trace to remove the exponentials after the last equality and thus establish the required equivalence. However, in the NAC case we know that the cyclicity of the trace does not hold since a $d^2\theta$ integration is lacking. Thus the first step above, which introduces the exponentials, is valid; however, after the last step the exponentials cannot be removed and the usual equivalence fails. By working in antichiral representation we generally manage to avoid this problem since $\nabla_\alpha = D_\alpha$. We note that the same problem does not occur for $\overline{\nabla}_{\dot\alpha}$ since the surviving $d^2\theta$ integration makes the trace cyclic as in the ordinary case.

Therefore, from now on we choose to describe the gauge sector of the theory in antichiral representation with the classical action

$$S_{inv} = \frac{1}{2g^2}\int d^4xd^2\bar{\theta}\ \mathrm{Tr}(\overline{W}^{\dot\alpha}\overline{W}_{\dot\alpha}) \qquad (4.1.22)$$

or more generally with

$$S_{inv} = \frac{1}{2g^2}\int d^4xd^2\bar{\theta}\ \mathrm{Tr}(\overline{W}^{\dot\alpha}\overline{W}_{\dot\alpha}) \qquad (4.1.23)$$
$$+\frac{1}{2g_0^2}\int d^4x\ d^4\theta\ \Big[\ \mathrm{Tr}(\overline{\Gamma}^{\dot\alpha})\mathrm{Tr}(\overline{W}_{\dot\alpha}) + 4i\mathcal{F}^{\rho\gamma}\overline{\theta}^2\ \mathrm{Tr}\left(\partial_{\rho\dot\rho}\overline{\Gamma}^{\dot\alpha}\right)\mathrm{Tr}\left(\overline{W}_{\dot\alpha}\overline{\Gamma}_\gamma{}^{\dot\rho}\right)\Big]$$

if we are interested in assigning different coupling constants to the $SU(N)$ and $U(1)$ gauge fields. An equivalently convenient choice would make use of $\widetilde{\overline{S}}$ in (4.1.17) and $\widetilde{\overline{S}}_0$ in (4.1.19).

In the presence of chiral matter in the adjoint representation of the gauge group $\mathcal{G}$ the SYM action is

$$S = \int d^4xd^4\theta\ \mathrm{Tr}(e_*^{-V} * \overline{\Phi} * e_*^V * \Phi) + \frac{1}{2g^2}\int d^4xd^2\theta\ \mathrm{Tr}(W^\alpha W_\alpha)$$
$$-\frac{1}{2}m\int d^4xd^2\theta\ \Phi^2 - \frac{1}{2}\overline{m}\int d^4xd^2\overline{\theta}\ \overline{\Phi}^2 \qquad (4.1.24)$$

where all the superfield can be consistently taken to be real (we are in Euclidean superspace). In what follows we work with $\mathcal{G} = U(N)$.

We now generalize the background field method [15, 16, 17, 13] to the case of NAC super Yang–Mills theories with chiral matter in a real representation of the gauge group. We perform the nonlinear splitting $e_*^V \rightarrow e_*^\Omega * e_*^V$ where Ω is the background prepotential, and write the covariant derivatives (in gauge-chiral representation) as

$$\nabla_\alpha = e_*^{-V} * \boldsymbol{\nabla}_\alpha * e_*^V \qquad , \qquad \nabla_{\dot\alpha} \equiv \boldsymbol{\nabla}_{\dot\alpha} = \overline{D}_{\dot\alpha} \tag{4.1.25}$$

with similar expressions for $(\overline{\nabla}_\alpha, \overline{\nabla}_{\dot\alpha})$. These derivatives transform covariantly with respect to two types of gauge transformations: quantum transformations

$$\begin{gathered} e_*^V \rightarrow e_*^{i\overline{\Lambda}} * e_*^V * e_*^{-i\Lambda} \\ \nabla_A \rightarrow e_*^{i\Lambda} * \nabla_A * e_*^{-i\Lambda} \qquad , \qquad \boldsymbol{\nabla}_A \rightarrow \boldsymbol{\nabla}_A \\ \overline{\nabla}_A \rightarrow e_*^{i\overline{\Lambda}} * \overline{\nabla}_A * e_*^{-i\overline{\Lambda}} \qquad , \qquad \overline{\boldsymbol{\nabla}}_A \rightarrow \overline{\boldsymbol{\nabla}}_A \end{gathered} \tag{4.1.26}$$

with background covariantly (anti)chiral parameters, $\boldsymbol{\nabla}_\alpha * \overline{\Lambda} = \boldsymbol{\nabla}_{\dot\alpha}\Lambda = 0$, and background transformations

$$\begin{gathered} e_*^V \rightarrow e_*^{iK} * e_*^V * e_*^{-iK} \\ \nabla_A \rightarrow e_*^{iK} * \nabla_A * e_*^{-iK} \qquad , \qquad \boldsymbol{\nabla}_A \rightarrow e_*^{iK} * \boldsymbol{\nabla}_A * e_*^{-iK} \\ \overline{\nabla}_A \rightarrow e_*^{iK} * \overline{\nabla}_A * e_*^{-iK} \qquad , \qquad \overline{\boldsymbol{\nabla}}_A \rightarrow e_*^{iK} * \overline{\boldsymbol{\nabla}}_A * e_*^{-iK} \end{gathered} \tag{4.1.27}$$

with real parameter K.

Full covariantly (anti)chiral superfields $\nabla_{\dot\alpha}\Phi = \nabla_\alpha * \overline{\Phi} = 0$ are expressed in terms of background (anti)chiral superfields as $\Phi = \Phi_0$, $\overline{\Phi} = \overline{\Phi}_0 * e_*^V$, $\overline{\boldsymbol{\nabla}}_{\dot\alpha} * \Phi_0 = 0$, $\boldsymbol{\nabla}_\alpha * \overline{\Phi}_0 = 0$ and then linearly split into a background and a quantum part. Under quantum transformations the fields transform as $\Phi' = e_*^{i\Lambda} * \Phi$, $\overline{\Phi}' = \overline{\Phi} * e_*^{-i\overline{\Lambda}}$, whereas under background transformations they transform as $\Phi' = e_*^{iK} * \Phi$, $\overline{\Phi}' = \overline{\Phi} * e_*^{-iK}$.

The classical action (4.1.24) is invariant under the transformations (4.1.26, 4.1.27). Background field quantization consists in performing gauge–fixing which explicitly breaks the (4.1.26) gauge invariance while preserving manifest invariance under (4.1.27). The procedure follows closely the ordinary one [13] by simply replacing products with star products. It leads to a gauge–fixed action $S_{tot} = S_{inv} + S_{GF} + S_{gh}$ where S_{gh} is given in terms of background covariantly (anti)chiral FP and NK ghost superfields and the quadratic part reads

$$S_{gh} = \int d^4x d^4\theta \, \left[\overline{c}' c - c' \overline{c} + \overline{b} b \right] \tag{4.1.28}$$

From the rest of the action we read the V propagator which in the Feynman gauge is

$$\langle V_a(z) V_b(z') \rangle = g^2 \frac{\delta_{ab}}{\Box_0} \, \delta^{(4)}(\theta - \theta') \tag{4.1.29}$$

and the pure gauge interaction terms useful for one–loop calculations

$$-\frac{1}{2g^2}\int d^4xd^4\theta\ \mathrm{Tr}V\Big[-i[\Gamma^{\alpha\dot\alpha},\partial_{\alpha\dot\alpha}V]_* - i\{W^\alpha, D_\alpha V\}_* - i\{W^{\dot\alpha},\overline{D}_{\dot\alpha}V\}_* \\ -\frac{1}{2}[\Gamma^{\alpha\dot\alpha},[\Gamma_{\alpha\dot\alpha},V]]_* - \{W^\alpha,[\Gamma_\alpha,V]\}_* - \{W^{\dot\alpha},[\Gamma_{\dot\alpha},V]\}_*\Big] \tag{4.1.30}$$

We now turn to the action for matter in a real representation of the gauge group. In particular, ghosts fall in this category so the following procedure can be applied to the action (4.1.28).

In the ordinary case, in terms of covariantly chiral and antichiral superfields Φ and $\overline{\Phi}$ (related by complex conjugation)

$$S = \int d^4xd^4\theta\ \overline{\Phi}\Phi \tag{4.1.31}$$

The corresponding equations of motion

$$\mathcal{O}\begin{pmatrix}\Phi\\ \bar\Phi\end{pmatrix} = 0 \qquad\qquad \mathcal{O} = \begin{pmatrix}0 & \overline{D}^2\\ \nabla^2 & 0\end{pmatrix} \tag{4.1.32}$$

can be formally derived from the functional determinant

$$\Delta \ = \int \mathcal{D}\Psi e^{\overline{\Psi}\mathcal{O}\Psi} \sim (\det\mathcal{O})^{-\frac{1}{2}} \tag{4.1.33}$$

where Ψ is the column vector $\begin{pmatrix}\Phi\\ \bar\Phi\end{pmatrix}$. If we perform the change of variables $\Psi = \sqrt{\mathcal{O}}\Psi'$, whose jacobian is $\det\sqrt{\mathcal{O}} = \Delta^{-\frac{1}{2}}$, we can write

$$\Delta = \int \mathcal{D}\Psi'\ \Delta^{-1}\ e^{\overline{\Psi}'\mathcal{O}^2\Psi'} \tag{4.1.34}$$

or equivalently

$$\Delta^2 = \int \mathcal{D}\Psi e^{\overline{\Psi}\mathcal{O}^2\Psi} \tag{4.1.35}$$

where

$$\mathcal{O}^2 = \begin{pmatrix}\overline{D}^2\nabla^2 & 0\\ 0 & \nabla^2\overline{D}^2\end{pmatrix} \tag{4.1.36}$$

is a diagonal matrix. Therefore, defining the actions

$$S' = \frac{1}{2}\int d^4xd^4\theta\ \Phi\nabla^2\Phi \\ \overline{S}' = \frac{1}{2}\int d^4xd^4\theta\ \overline{\Phi}\overline{D}^2\overline{\Phi} \tag{4.1.37}$$

it is easy to see that the following chain of identities holds [13]

$$\Delta^2 \ = \int \mathcal{D}\Phi\mathcal{D}\overline{\Phi}\ e^{S'+\overline{S}'} \ = \Big|\int \mathcal{D}\Phi e^{S'}\Big|^2 \ = \ \Big(\int \mathcal{D}\Phi e^{S'}\Big)^2 \tag{4.1.38}$$

where we have used $\overline{S}' = (S')^\dagger$ and the fact that they both contribute in the same way to Δ [13]. Therefore, when Δ is real, we can identify the original Δ with $\int \mathcal{D}\Phi e^{S'}$ and derive from here the Feynman rules [13].

We now extend the previous derivation to the case of NAC euclidean superspace where all the h.c. relations are relaxed and Φ, $\overline{\Phi}$ are two independent but *real* superfields. The matter action is still given by (4.1.31) and we can still define Δ as in (4.1.34). Therefore, we write

$$\Delta_* = \int \mathcal{D}\Psi \; e^{\Psi^T * \mathcal{O} * \Psi} \sim (\det(\mathbf{O}))^{-\frac{1}{2}} \tag{4.1.39}$$

We can then proceed as before and square the functional integral to obtain

$$\Delta_*^2 = \int \mathcal{D}\Psi^T e^{\Psi * \mathbf{O}^2 * \Psi} \tag{4.1.40}$$

where $\mathbf{O}^2$ is given in (4.1.36) with the products promoted to star products. Now, if we introduce

$$\Delta_1 = \int \mathcal{D}\Phi e^{S'} \qquad , \qquad \Delta_2 = \int \mathcal{D}\bar{\Phi} e^{\overline{S}'} \tag{4.1.41}$$

with S', $\overline{S}'$ still given in (4.1.37) we can finally write

$$\Delta_*^2 = \Delta_1 \Delta_2 \tag{4.1.42}$$

In contradistinction to the ordinary case, now $\overline{S}' \neq (S')^\dagger$. Moreover, the star product, when expanded, could in principle generate different terms in the two actions. Therefore, the chain of identities (4.1.38) is not immediately generalizable to the NAC case and we cannot identify $\Delta_* = \Delta_1$.

However, given the equality (4.1.42) the Feynman rules for Δ_* can be still inferred from $\Delta_{1,2}$ order by order. In fact, we consider $\Delta_*, \Delta_1, \Delta_2$ as functions of the coupling constant g and perform a perturbative expansion

$$\begin{aligned} \Delta_*[g] &= \Delta_*[0] + g^2 \Delta_*'[0] + \cdots \\ \Delta_1[g] &= \Delta_1[0] + g^2 \Delta_1'[0] + \cdots \\ \Delta_2[g] &= \Delta_2[0] + g^2 \Delta_2'[0] + \cdots \end{aligned} \tag{4.1.43}$$

Normalizing the functionals as $\Delta_*[0] = \Delta_1[0] = \Delta_2[0] = 1$ and expanding the identity (4.1.42) in powers of g we obtain

$$\Delta_*^2 = \left(1 + g^2 \Delta_*'[0] + \cdots\right)^2 = \left(1 + g^2 \Delta_1'[0] + \cdots\right)\left(1 + g^2 \Delta_2'[0] + \cdots\right) \tag{4.1.44}$$

In particular, since we are interested in computing one–loop contributions to the effective action at order g^2 we find

$$2\Delta_*'[0] = \Delta_1'[0] + \Delta_2'[0] \tag{4.1.45}$$

Therefore at one loop Δ_* is given by the sum of the contributions from S' and $\overline{S}'$.

Following closely the ordinary case [13] we derive the Feynman rules from S' and $\overline{S}'$ by first extracting the quadratic part of the actions and then reading the vertices from the rest.

Since for the chiral action the identities involving covariant derivatives are formally the same except for the products which are now star products, the procedure to obtain the analytic expressions associated to the vertices is formally the same. We then refer the reader to Ref. [13] for details while reporting here only the final rules:

- Propagator

$$\langle \Phi(z)\Phi(z')\rangle = -\frac{1}{\Box_0}\delta^{(4)}(\theta-\theta') \tag{4.1.46}$$

- Chiral vertices: at one loop the prescription requires associating with one vertex

$$\frac{1}{2}\overline{D}^2(\nabla^2 - D^2) \tag{4.1.47}$$

and with the other vertices

$$\frac{1}{2}(\Box_+ - \Box_0) \tag{4.1.48}$$

where $\Box_+ \;=\; \Box_{cov} - iW^\alpha * \nabla_\alpha - \frac{i}{2}(\nabla^\alpha * W_\alpha), \qquad \Box_{cov} \;=\; \frac{1}{2}\nabla^{\alpha\dot\alpha} * \nabla_{\alpha\dot\alpha}.$

The procedure can be easily extended to the case of massive chirals by simply promoting the propagators (4.1.46) to massive propagators $-1/(\Box_0 - m\overline{m})$. We also note that these rules are strictly one-loop rules. At higher orders there are no difficulties and ordinary rules apply, as described in [13] with obvious modifications required by noncommutativity.

We can write down a formal effective interaction lagrangian that corresponds to the one-loop rules above. In the case of massive matter (chirals with mass m and antichirals with $\overline{m}$) in the adjoint representation of the gauge group, it is given by (from now on we avoid indicating star products when no confusion arises)

$$S_0 + S_1 + S_2 \equiv \int d^4x d^4\theta \ \mathrm{Tr}\left\{\overline{\psi}(\Box_0 - m\overline{m})\psi + \frac{1}{2}\left[\overline{\psi}\ \overline{D}^2(\nabla^2 - D^2)\psi \ + \ \overline{\psi}(\Box_+ - \Box_0)\psi\right]\right\} \tag{4.1.49}$$

where $\psi, \overline{\psi}$ are *quantum unconstrained* superfields and the first vertex has to appear once and only once in any one–loop diagram.
By writing everything explicitly in terms of connections and field strengths and performing some integrations by parts it can be rewritten as (we neglect terms with lower powers of $\overline{D}$ which would not contribute in one–loop calculations)

$$\begin{aligned} S_1 &= \int d^4x d^4\theta \ \mathrm{Tr}\left\{\left(\frac{i}{4}\Gamma^\alpha[\overline{D}^2\psi, D_\alpha\overline{\psi}] - \frac{i}{4}\Gamma^\alpha[\overline{D}^2 D_\alpha\psi, \overline{\psi}]\right) + \left(-\frac{1}{4}\overline{\psi}\{\Gamma^\alpha[\Gamma_\alpha, \overline{D}^2\psi]\}\right)\right\} \\ &\equiv S_1 + S_1' \end{aligned} \tag{4.1.50}$$

$$\begin{aligned} S_2 &= \int d^4x d^4\theta \ \mathrm{Tr}\left\{\left(\frac{i}{4}\Gamma^{\alpha\dot\alpha}[\psi, \partial_{\alpha\dot\alpha}\overline{\psi}] - \frac{i}{4}\Gamma^{\alpha\dot\alpha}[\partial_{\alpha\dot\alpha}\psi, \overline{\psi}]\right) + \left(\frac{i}{4}W^\alpha[\psi, D_\alpha\overline{\psi}] - \frac{i}{4}W^\alpha[D_\alpha\psi, \overline{\psi}]\right)\right. \\ &\qquad \left. + \ \left(\frac{1}{4}[\Gamma^\alpha, \psi][W_\alpha, \overline{\psi}] + \frac{1}{4}[W^\alpha, \psi][\Gamma_\alpha, \overline{\psi}]\right) + \left(\frac{1}{4}[\Gamma^{\alpha\dot\alpha}, \psi][\Gamma_{\alpha\dot\alpha}, \overline{\psi}]\right)\right\} \\ &\equiv S_2 + S_2' + S_2'' + S_2''' \end{aligned} \tag{4.1.51}$$

To extract the Feynman rules for the antichiral sector, we have to go carefully through the whole procedure since some of the identities used in the ordinary case do not hold anymore because of the noncommutative product. We find convenient to express the action $\overline{S}'$ in terms of ordinary (not covariantly) antichiral superfields. Using cyclicity under trace and $d^4\theta$ integration we find

$$\overline{S}' = \frac{1}{2}\int d^4x d^4\theta \ \mathrm{Tr}(\overline{\Phi} e^V \overline{D}^2 e^{-V}\overline{\Phi}) = \frac{1}{2}\int d^4x d^2\overline{\theta} \ \mathrm{Tr}(\overline{\Phi} D^2 \overline{\nabla}^2 \overline{\Phi}) \tag{4.1.52}$$

where $\overline{\nabla}^2 = e^V_* * \overline{D}^2 e^{-V}_*$. Using covariant derivatives in the antichiral representation we can formally follow the same procedure of the chiral sector by changing bar quantities into unbar ones, and viceversa. Therefore, the Feynman rules are:

- Propagator

$$\langle \overline{\Phi}(z)\overline{\Phi}(z')\rangle = -\frac{1}{\Box_0}\delta^{(4)}(\theta-\theta') \tag{4.1.53}$$

- one vertex: $\frac{1}{2}D^2(\overline{\nabla}^2 - \overline{D}^2)$
- other vertices: $\frac{1}{2}(\overline{\Box}_+ - \Box_0)$

  with $\overline{\Box}_+ = \overline{\Box}_{cov} - i\overline{W}^{\dot\alpha} * \overline{\nabla}_{\dot\alpha} - \frac{i}{2}(\overline{\nabla}^{\dot\alpha} * \overline{W}_{\dot\alpha})$ (here $\overline{W}_{\dot\alpha}$ is the antichiral field strength and $\overline{\Box}_{cov} = \frac{1}{2}\overline{\nabla}^{\alpha\dot\alpha} * \overline{\nabla}_{\alpha\dot\alpha}$).

Again, it is convenient to introduce an effective action in terms of quantum unconstrained superfields ξ and $\overline{\xi}$

$$\overline{S}_0 + \overline{S}_1 + \overline{S}_2 \equiv \int d^4x d^4\theta \ \mathrm{Tr}\left\{\overline{\xi}(\Box_0 - m\overline{m})\xi + \frac{1}{2}\left[\overline{\xi} D^2(\overline{\nabla}^2 - \overline{D}^2)\xi \ + \ \overline{\xi}(\overline{\Box}_+ - \Box_0)\xi\right]\right\} \tag{4.1.54}$$

In terms of connections and field strengths it can be rewritten as

$$\begin{aligned}\overline{S}_1 &= \int d^4x d^4\theta \ \mathrm{Tr}\left\{\left(\frac{i}{4}\overline{\Gamma}^{\dot\alpha}[\xi, \overline{D}_{\dot\alpha} D^2\overline{\xi}] - \frac{i}{4}\overline{\Gamma}^{\dot\alpha}[\overline{D}_{\dot\alpha}\xi, D^2\overline{\xi}]\right) + \left(-\frac{1}{4}\overline{\xi}\{\overline{\Gamma}^{\dot\alpha}[\overline{\Gamma}_{\dot\alpha}, D^2\xi]\}\right)\right\} \\ &\equiv \overline{S}_1 + \overline{S}_1' \end{aligned}\tag{4.1.55}$$

$$\begin{aligned}\overline{S}_2 &= \int d^4x d^4\theta \ \mathrm{Tr}\left\{\left(\frac{i}{4}\overline{\Gamma}^{\alpha\dot\alpha}[\xi, \partial_{\alpha\dot\alpha}\overline{\xi}] - \frac{i}{4}\overline{\Gamma}^{\alpha\dot\alpha}[\partial_{\alpha\dot\alpha}\xi, \overline{\xi}]\right) + \left(\frac{i}{4}\overline{W}^{\dot\alpha}[\xi, \overline{D}_{\dot\alpha}\overline{\xi}] - \frac{i}{4}\overline{W}^{\dot\alpha}[\overline{D}_{\dot\alpha}\xi, \overline{\xi}]\right)\right. \\ &\quad \left. + \left(\frac{1}{4}[\overline{\Gamma}^{\dot\alpha}, \xi][\overline{W}_{\dot\alpha}, \overline{\xi}] + \frac{1}{4}[\overline{W}^{\dot\alpha}, \xi][\overline{\Gamma}_{\dot\alpha}, \overline{\xi}]\right) + \left(\frac{1}{4}[\overline{\Gamma}^{\alpha\dot\alpha}, \xi][\overline{\Gamma}_{\alpha\dot\alpha}, \overline{\xi}]\right)\right\} \\ &\equiv \overline{S}_2 + \overline{S}_2' + \overline{S}_2'' + \overline{S}_2''' \end{aligned}\tag{4.1.56}$$

4.2 Pure gauge theory

In this section, using the techniques described above, we compute the one–loop divergent contributions to the gauge effective action. In the ordinary case, in background field method only the

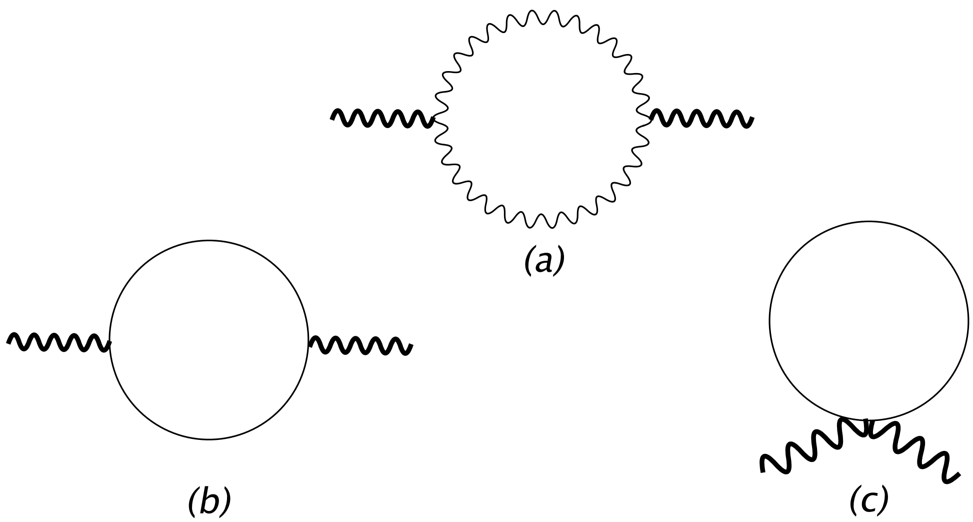

Figure 4.1: Gauge one-loop two-point functions.

two–point function with chiral loop is divergent [15, 16, 17, 13]. In the NAC case, instead, we find divergent contributions up to the 4–point function for the gauge field due to the nontrivial modifications to the D–algebra induced by the star product.

We list only the relevant results. The reader is reffered to [18] for a more complete treatment and technical details. Calculations are performed in dimensional regularization ($n = 4 - 2\epsilon$) and the integrals useful for our goal are listed in Appendix A.2. In particular, all the divergences are expressed in terms of a tadpole integral $\mathcal{T}$ and a self-energy $\mathcal{S}$ defined in (A.2.8) and (A.2.9).

4.2.1 Two–point function

Divergent contributions to the two–point function are listed in Fig. 4.1.

Diagram (4.1a) with a vector loop can be computed by using Feynman rules (4.1.29) and (4.1.30). The divergent contribution turns out to be proportional to $\mathcal{F}^2$

$$-2\mathcal{F}^2 \int d^4x d^4\theta \ \mathrm{Tr}(\partial^2 W^{\dot\alpha})\mathrm{Tr}(W_{\dot\alpha}) = -2\mathcal{F}^2 \int d^4x d^4\theta \ \partial^2 \mathrm{Tr}(e_*^{-V} * \overline{W}^{\dot\alpha} * e_*^V)\mathrm{Tr}(e_*^{-V} * \overline{W}_{\dot\alpha} * e_*^V) \tag{4.2.57}$$

where we have expressed the covariantly antichiral field strength $W_{\dot\alpha}$ in terms of the ordinary antichiral one.
Given the particular group structure one can prove that this contribution is equal to

$$-2\mathcal{F}^2 \ \mathcal{S} \int d^4x d^4\theta \ (\partial^2 \mathrm{Tr}\overline{W}^{\dot\alpha})\Big[\mathrm{Tr}\overline{W}_{\dot\alpha} + 2\mathcal{F}^{\alpha\beta}\mathrm{Tr}(\partial_\alpha e_*^V \partial_\beta(e_*^{-V} * \overline{W}_{\dot\alpha}))\Big] \tag{4.2.58}$$

and actually vanishes once integrated in $d^2\theta$.

We now consider matter loops (4.1b,4.1c) following the Feynman rules (4.1.29), (4.1.49) and

(4.1.54) for the chiral superfields and the analogous ones for the antichirals. This also covers contributions from ghosts up to an overall sign.
We focus on the chiral and the antichiral sectors separately.

• Chiral sector:
Order $\mathcal{F}^0$: This is the contribution which is already present in the the ordinary case. It is obtained by taking the D^2 factor inside the loop entirely from the covariant derivatives. Performing the explicit calculation we find

$$\mathcal{D}^{(2)} = \frac{1}{2}\mathcal{S} \int d^8z \Big[\, N \; \mathrm{Tr}\,(\Gamma^\alpha \; W_\alpha) - \mathrm{Tr}\,(\Gamma^\alpha)\,\mathrm{Tr}\,(W_\alpha) \Big] \tag{4.2.59}$$

Order $\mathcal{F}$: Contributions proportional to a single power of $\mathcal{F}$ come from diagrams which have already a single D from the vertices, whereas a second factor D is produced by linearly expanding the star product. However, it is easy to realize that this expansion is always proportional to a trivial vanishing colour factor.
Order $\mathcal{F}^2$: These contributions are associated to nonplanar diagrams where no derivatives come from the vertices and the star products are expanded up to second order. After a bit of calculations we obtain

$$\begin{aligned} \frac{\mathcal{F}^2}{4} \int d^8z \, \Big\{ & - \mathrm{Tr}\left((4\mathcal{T} - 4m\overline{m}\mathcal{S} + \Box\, \mathcal{S})\partial^2 \frac{D^2}{\Box}\Gamma^\alpha \right) \mathrm{Tr}(\overline{D}^2\Gamma_\alpha) \\ & + \frac{2}{3} i \mathrm{Tr}\left((2\mathcal{T} - 4m\overline{m}\mathcal{S} + \Box\, \mathcal{S})\partial^2 D_\beta \frac{\partial^{\beta\dot{\alpha}}}{\Box}\Gamma^\alpha \right) \mathrm{Tr}(\overline{D}_{\dot{\alpha}}\Gamma_\alpha) \\ & + \frac{i}{3} \mathrm{Tr}\left((4\mathcal{T} + 4m\overline{m}\mathcal{S} - \Box\, \mathcal{S})\frac{D^\alpha}{\Box} \; \partial^2\Gamma_\alpha \right) \mathrm{Tr}(\partial^{\beta\dot{\beta}}\overline{D}_{\dot{\beta}}\Gamma_\beta) \\ & - 4 \; \mathcal{T} \; \mathrm{Tr}(\partial^2\Gamma^\alpha) \; \mathrm{Tr}(\Gamma_\alpha) \Big\} \end{aligned} \tag{4.2.60}$$

It is possible to prove that this expression actually vanishes due to the particular structure of the star product hidden in Γ_α and the fact that in (4.2.60) only the $U(1)$ part of the connection appears.

• Antichiral sector:
Order $\mathcal{F}^0$: Also in this case this is the contribution which is already present in the ordinary case.

$$\overline{\mathcal{D}}^{(2)} = \frac{1}{2}\mathcal{S} \int d^8z \Big[\, N \; \mathrm{Tr}\left(\overline{\Gamma}^{\dot{\alpha}} \; \overline{W}_{\dot{\alpha}}\right) - \mathrm{Tr}\left(\overline{\Gamma}^{\dot{\alpha}}\right) \mathrm{Tr}\left(\overline{W}_{\dot{\alpha}}\right) \Big] \tag{4.2.61}$$

Order $\mathcal{F}$: As in the chiral sector, this contribution is proportional to a trivial vanishing colour factor, then there is no contribution at this order.
Order $\mathcal{F}^2$: Divergent contributions proportional to $\mathcal{F}^2$ have the form

$$\mathcal{F}^2 \int d^8z \overline{\theta}^2 \, \Box \, \partial^2 \overline{\Gamma}^{\dot{\alpha}} \; \overline{W}_{\dot{\alpha}} = \mathcal{F}^2 \int d^8z \overline{\theta}^2 \, \Box \, D^2 \overline{\Gamma}^{\dot{\alpha}} \; \overline{W}_{\dot{\alpha}} \tag{4.2.62}$$

and trivially vanish when integrated in $d^2\theta$.

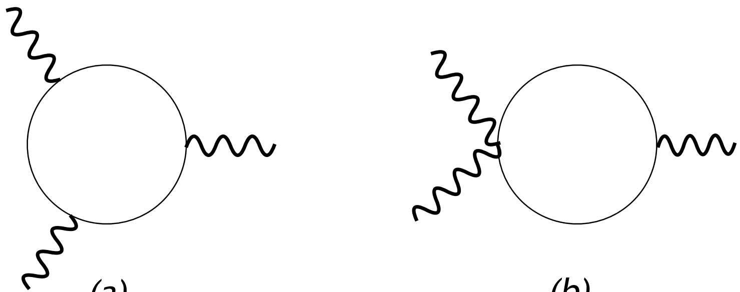

Figure 4.2: Gauge one-loop three-point functions.

4.2.2 Three–point function

Divergent contributions to the three–point function are listed in Fig. 4.2.

Three-point diagrams with vector loop are finite, then we focus on matter loops. Both in the chiral and in the antichiral sectors, divergent terms are of order $\mathcal{F}$. Indeed, even if in principle we can have divergences also at order $\mathcal{F}^2$, the explicit calculation of the phase structures shows that these contributions cancel.

• Chiral sector:
We obtain

$$\mathcal{D}^{(3)} = i\mathcal{S}\int d^4x d^2\theta \; \mathcal{F}^{\beta\gamma}[\mathrm{Tr}(\partial_\beta W^\alpha) * \mathrm{Tr}(\Gamma_\gamma * W_\alpha) - \mathrm{Tr}(\partial_\beta W^\alpha) * \mathrm{Tr}(W_\alpha * \Gamma_\gamma) \\ -\mathrm{Tr}(\partial_\beta \Gamma_\gamma) * \mathrm{Tr}(W^\alpha * W_\alpha)]_{\overline{\theta}=0} \qquad (4.2.63)$$

We have already performed the $\overline{\theta}$ integration since it allows for some cancellation among various terms. One can prove that the star products in (4.2.63) are actually ordinary products and the result can be rewritten as

$$\mathcal{D}^{(3)} = i\mathcal{S}\int d^4x d^2\theta \; \mathcal{F}^{\beta\gamma}[2\mathrm{Tr}(D_\beta W^\alpha)\mathrm{Tr}(\Gamma_\gamma W_\alpha) - \mathrm{Tr}(D_\beta \Gamma_\gamma)\mathrm{Tr}(W^\alpha W_\alpha)]_{\overline{\theta}=0} \qquad (4.2.64)$$

• Antichiral sector:
We obtain

$$\overline{\mathcal{D}}^{(3)} = \frac{2}{3}\mathcal{S}\mathcal{F}^{\rho\gamma}\int d^8z \; \overline{\theta}^{\dot\beta}\Big[\mathrm{Tr}\left(\partial_\rho \overline{\Gamma}^{\dot\alpha}\right)\mathrm{Tr}\left(\overline{W}_{\dot\alpha}\overline{\Gamma}_{\gamma\dot\beta}\right) + \mathrm{Tr}\left(\partial_\rho \overline{W}^{\dot\alpha}\right)\mathrm{Tr}\left(\overline{\Gamma}_{\dot\alpha}\overline{\Gamma}_{\gamma\dot\beta}\right) \\ +\mathrm{Tr}\left(\partial_\rho \overline{\Gamma}_{\gamma\dot\beta}\right)\mathrm{Tr}\left(\overline{W}^{\dot\alpha}\overline{\Gamma}_{\dot\alpha}\right)\Big] \qquad (4.2.65)$$

In this case we perform both the θ and $\overline{\theta}$ integrations, in order to allow for some cancellations

$$\overline{\mathcal{D}}^{(3)} = i\mathcal{S}\mathcal{F}^{\rho\gamma}\int d^4x\Big[\; 2\;\partial_{\rho\dot\rho}\mathrm{Tr}\left(\overline{W}^{\dot\alpha}\right)\mathrm{Tr}\left(\overline{W}_{\dot\alpha}\overline{\Gamma}_\gamma^{\;\dot\rho}\right) + \partial_{\rho\dot\rho}\mathrm{Tr}\left(\overline{\Gamma}_\gamma^{\;\dot\rho}\right)\mathrm{Tr}\left(\overline{W}^{\dot\alpha}\overline{W}_{\dot\alpha}\right)\Big]_{\theta=\overline{\theta}=0} \qquad (4.2.66)$$

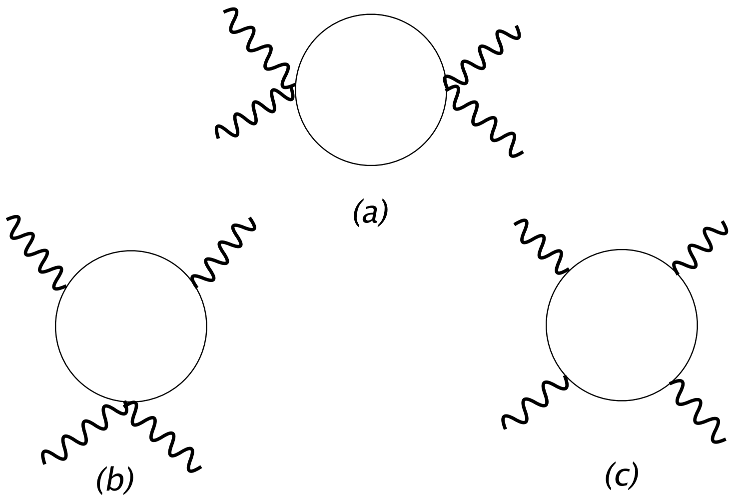

Figure 4.3: Gauge one-loop four-point functions.

4.2.3 Four–point function

Divergent contributions to the four–point function are listed in Fig. 4.3.

Again, diagrams with vector loops give finite contributions. Considering matter loops we find divergences only at order $\mathcal{F}^2$.

• Chiral sector:

After $\overline{\theta}$ integration, we obtain

$$\mathcal{D}^{(4)} = \mathcal{S}\mathcal{F}^2 \int d^4x d^2\theta \, \Big[\frac{1}{2}\partial^2 \mathrm{Tr}\left(\Gamma^\alpha * \Gamma_\alpha\right) \mathrm{Tr}\left(W^\beta * W_\beta\right) - \partial^2 \mathrm{Tr}\left(\Gamma^\alpha * W^\beta\right) \mathrm{Tr}\left(\Gamma_\alpha * W_\beta\right) \\ - \mathrm{Tr}\left(\partial^2 \Gamma^\alpha\right) \mathrm{Tr}\left(\Gamma_\alpha * W^\beta * W_\beta\right) - \mathrm{Tr}\left(\partial^2 W^\alpha\right) \mathrm{Tr}\left(W_\alpha * \Gamma^\beta * \Gamma_\beta\right) \Big]_{\overline{\theta}=0} \tag{4.2.67}$$

In this case it would be possible to replace all the star products with ordinary products in the first two terms, but not in the last two.

• Antichiral sector:

We obtain

$$\begin{aligned} \overline{\mathcal{D}}^{(4)} &= -\frac{1}{12}\mathcal{S}\mathcal{F}^2 \int d^8z \, \overline{\theta}^2 \Big[\partial^2 \mathrm{Tr}\left(\overline{W}^{\dot\alpha} * \overline{\Gamma}_{\dot\alpha}\right) \mathrm{Tr}\left(\overline{\Gamma}^{\gamma\dot\gamma} * \overline{\Gamma}_{\gamma\dot\gamma}\right) \\ &\qquad + 8 \, \partial^2 \mathrm{Tr}\left(\overline{\Gamma}^{\dot\alpha} * \overline{W}^{\dot\beta}\right) \mathrm{Tr}\left(\overline{W}_{\dot\beta} * \overline{\Gamma}_{\dot\alpha}\right) \Big] \\ &= -\frac{1}{2}\mathcal{S}\mathcal{F}^2 \int d^4x \Big[\mathrm{Tr}\left(\overline{W}^{\dot\alpha}\overline{W}_{\dot\alpha}\right) \mathrm{Tr}\left(\overline{W}^{\dot\beta}\overline{W}_{\dot\beta}\right) \Big]_{\theta=\overline{\theta}=0} \end{aligned} \tag{4.2.68}$$

4.2.4 (Super)gauge invariance

Collecting all the results of the previous section and performing the θ and $\overline{\theta}$ integrations for simplicity, the divergent part of the one–loop gauge effective action reads

$$
\begin{aligned}
\Gamma^{(1)}_{gauge} = & \frac{1}{2}(-3+N_f)\mathcal{S}\int d^4x \Bigg\{ \frac{1}{2}D^2 \Big[N \,\mathrm{Tr}\,(W^\alpha \, W_\alpha) - \mathrm{Tr}\,(W^\alpha)\,\mathrm{Tr}\,(W_\alpha) \Big] \\
& + \frac{1}{2}\overline{D}^2 \Big[N \,\mathrm{Tr}\left(\overline{W}^{\dot\alpha} \, \overline{W}_{\dot\alpha}\right) - \mathrm{Tr}\left(\overline{W}^{\dot\alpha}\right)\mathrm{Tr}\left(\overline{W}_{\dot\alpha}\right) \Big] \\
& + i\mathcal{F}^{\rho\gamma} D^2 \left[2\mathrm{Tr}(D_\rho W^\alpha)\mathrm{Tr}(\Gamma_\gamma W_\alpha) - \mathrm{Tr}(D_\rho \Gamma_\gamma)\mathrm{Tr}(W^\alpha W_\alpha)\right] \\
& + i\mathcal{F}^{\rho\gamma}\Big[2 \,\partial_{\rho\dot\rho}\mathrm{Tr}\left(\overline{W}^{\dot\alpha}\right)\mathrm{Tr}\left(\overline{W}_{\dot\alpha}\overline{\Gamma}_\gamma^{\ \dot\rho}\right) + \partial_{\rho\dot\rho}\mathrm{Tr}\left(\overline{\Gamma}_\gamma^{\ \dot\rho}\right)\mathrm{Tr}\left(\overline{W}^{\dot\alpha}\overline{W}_{\dot\alpha}\right)\Big] \\
& + \mathcal{F}^2 \Big[\frac{1}{2}D^2\mathrm{Tr}\,(\Gamma^\alpha\Gamma_\alpha)\, D^2\mathrm{Tr}\left(W^\beta W_\beta\right) - D^2\mathrm{Tr}\left(\Gamma^\alpha W^\beta\right) D^2\mathrm{Tr}\,(\Gamma_\alpha W_\beta) \\
& \quad - \mathrm{Tr}\left(D^2\Gamma^\alpha\right) D^2\mathrm{Tr}\left(\Gamma_\alpha * W^\beta * W_\beta\right) - \mathrm{Tr}\left(D^2 W^\alpha\right) D^2\mathrm{Tr}\left(W_\alpha * \Gamma^\beta * \Gamma_\beta\right)\Big] \\
& - \frac{1}{2}\mathcal{F}^2\Big[\mathrm{Tr}\left(\overline{W}^{\dot\alpha}\overline{W}_{\dot\alpha}\right)\mathrm{Tr}\left(\overline{W}^{\dot\beta}\overline{W}_{\dot\beta}\right)\Big]\Bigg\}_{\theta=\overline{\theta}=0} \\
\equiv & \frac{1}{2}(-3+N_f)\big[\mathcal{D}^{(2)} + \overline{\mathcal{D}}^{(2)} + \mathcal{D}^{(3)} + \overline{\mathcal{D}}^{(3)} + \mathcal{D}^{(4)} + \overline{\mathcal{D}}^{(4)}\big]
\end{aligned}
\tag{4.2.69}
$$

where factor $\frac{1}{2}$ comes from (4.1.45), (-3) is the contribution from the ghosts whereas N_f comes from matter. We note that the contributions to the two-point functions are independent of $\mathcal{F}$, three-point functions are linear in $\mathcal{F}$ and four-point functions are quadratic in $\mathcal{F}$.
We consider the variation of $\Gamma^{(1)}_{gauge}$ under supergauge transformation $e_*^{V'} = e_*^{i\overline{\Lambda}} * e_*^V * e_*^{-i\Lambda}$. Superfield strengths and superconnections transform as

$$
\begin{aligned}
&\delta\Gamma_\alpha = D_\alpha\Lambda + i[\Lambda,\Gamma_\alpha]_* \qquad \qquad \delta W_\alpha = i[\Lambda, W_\alpha]_* \\
&\delta\overline{\Gamma}_{\dot\beta} = \overline{D}_{\dot\beta}\overline{\Lambda} + i[\overline{\Lambda},\overline{\Gamma}_{\dot\beta}]_* \qquad \delta\overline{\Gamma}_{\beta\dot\beta} = \partial_{\beta\dot\beta}\overline{\Lambda} + i[\overline{\Lambda},\overline{\Gamma}_{\beta\dot\beta}]_* \qquad \delta\overline{W}_{\dot\beta} = i[\overline{\Lambda},\overline{W}_{\dot\beta}]_*
\end{aligned}
\tag{4.2.70}
$$

from which we can easily infer the transformation rules of the components appearing in (4.2.69). By expanding the $*$-product, after a long but straightforward calculation, it is possible to show that

$$
\delta\mathcal{D}^{(2)} = A\ \mathcal{S} \qquad \delta\mathcal{D}^{(3)} = -\left(A+B\right)\ \mathcal{S} \qquad \delta\mathcal{D}^{(4)} = B\ \mathcal{S} \tag{4.2.71}
$$

with

$$A = 2i\mathcal{F}^{\rho\gamma} \int d^4x \Big[\mathrm{Tr}\left(D_\gamma \Lambda D_\rho W^\alpha\right) \mathrm{Tr}\left(D^2 W_\alpha\right) - \mathrm{Tr}\left(D^2 \Lambda D_\rho W^\alpha\right) \mathrm{Tr}\left(D_\gamma W_a\right) + \mathrm{Tr}\left(D_\gamma \Lambda D^2 W^\alpha\right) \mathrm{Tr}\left(D_\rho W_\alpha\right) \Big]_{\theta=\bar{\theta}=0} \tag{4.2.72}$$

$$\begin{aligned} B = &-\mathcal{F}^2 \int d^4x \Big[2\mathrm{Tr}\left(D^2 \Lambda D_\beta W^\alpha\right) D^2 \mathrm{Tr}\left(\Gamma^\beta W_\alpha\right) - 2\mathrm{Tr}\left(D_\beta \Lambda D^2 W^\alpha\right) D^2 \mathrm{Tr}\left(\Gamma^\beta W_\alpha\right) \\ &- \mathrm{Tr}\left(D^2 \Lambda D_\beta \Gamma^\beta\right) D^2 \mathrm{Tr}\left(W^\alpha W_\alpha\right) + \mathrm{Tr}\left(D_\beta \Lambda D^2 \Gamma^\beta\right) D^2 \mathrm{Tr}\left(W^\alpha W_\alpha\right) \\ &+ \mathrm{Tr}\left(D^2 W^\alpha\right) \mathrm{Tr}\left(\left\{D^\beta \Gamma^\gamma, D_\gamma \Lambda\right\} D_\beta W_\alpha\right) - \mathrm{Tr}\left(D^2 W^\alpha\right) \mathrm{Tr}\left(\Gamma^\gamma \left\{D^2 \Lambda, D_\gamma W_\alpha\right\}\right) \\ &+ \mathrm{Tr}\left(D^2 W^\alpha\right) \mathrm{Tr}\left(\Gamma^\gamma \left[D_\gamma \Lambda, D^2 W_\alpha\right]\right) + \mathrm{Tr}\left(D^2 W^\alpha\right) \mathrm{Tr}\left(\left\{D^2 \Lambda, D_\gamma \Gamma^\gamma\right\} W_\alpha\right) \\ &+ \mathrm{Tr}\left(D^2 W^\alpha\right) \mathrm{Tr}\left(\left[D^2 \Gamma^\gamma, D_\gamma \Lambda\right] W_\alpha\right) + \mathrm{Tr}\left(D^2 \Gamma^\gamma\right) \mathrm{Tr}\left(D_\gamma \Lambda D^2 W^\alpha W_\alpha\right) \\ &+ \mathrm{Tr}\left(D^2 \Gamma^\gamma\right) \mathrm{Tr}\left(D_\gamma \Lambda W^\alpha D^2 W_\alpha\right) + \mathrm{Tr}\left(D^2 \Gamma^\gamma\right) \mathrm{Tr}\left(D_\gamma \Lambda D_\beta W^\alpha D^\beta W_\alpha\right) \\ &- \mathrm{Tr}\left(D^2 \Gamma^\gamma\right) \mathrm{Tr}\left(D^2 \Lambda D_\gamma W^\alpha W_\alpha\right) + \mathrm{Tr}\left(D^2 \Gamma^\gamma\right) \mathrm{Tr}\left(D^2 \Lambda W^\alpha D_\gamma W_\alpha\right) \Big]_{\theta=\bar{\theta}=0} \\ &- \mathcal{F}^2 \mathcal{F}^{\rho\gamma} \int d^4x \Big[\mathrm{Tr}\left(D^2 W^\alpha\right) \mathrm{Tr}\left(D_\rho \Gamma_\gamma \left[D^2 \Lambda, D^2 W_\alpha\right]\right) \\ &+ \mathrm{Tr}\left(D^2 W^\alpha\right) \mathrm{Tr}\left(\left[D^2 \Gamma_\gamma, D^2 \Lambda\right] D_\rho W_\alpha\right) + \mathrm{Tr}\left(D^2 \Gamma_\rho\right) \mathrm{Tr}\left(D^2 \Lambda D^2 W^\alpha D_\gamma W_\alpha\right) \\ &+ \mathrm{Tr}\left(D^2 \Gamma_\rho\right) \mathrm{Tr}\left(D^2 \Lambda D_\gamma W^\alpha D^2 W_\alpha\right) \Big]_{\theta=\bar{\theta}=0} \end{aligned} \tag{4.2.73}$$

whereas

$$\begin{aligned} \delta \overline{\mathcal{D}}^{(2)} &= -\delta \overline{\mathcal{D}}^{(3)} \\ &= -2i\mathcal{S}\mathcal{F}^{\rho\gamma} \int d^4x \ \mathrm{Tr}\left(\partial_{\rho\dot{\rho}} \overline{W}^{\dot{\alpha}}\right) \ \mathrm{Tr}\left(\partial_\gamma{}^{\dot{\rho}} \overline{\Lambda} \ \overline{W}_{\dot{\alpha}}\right) \Big|_{\theta=\bar{\theta}=0} \end{aligned} \tag{4.2.74}$$

$$\delta \overline{\mathcal{D}}^{(4)} = 0 \tag{4.2.75}$$

We note that in the chiral sector the gauge variation, when evaluated in components, is proportional to $D_\gamma \Lambda|$ and $D^2 \Lambda|$ but not to $\Lambda|$. Therefore, in this sector ordinary gauge invariance is preserved term by term, whereas the supergauge one seems to be broken. In the antichiral sector instead, the gauge variation of each term is proportional to $\overline{\Lambda}|$ so breaking ordinary gauge invariance. However it is easy to see that all the variations sum up to zero and we find

$$\delta \Gamma^{(1)}_{gauge} = 0 \tag{4.2.76}$$

We have then proved the supergauge invariance of the one–loop gauge effective action.

4.2.5 The renormalizable gauge action

In the previous section we computed the divergent contributions to the pure gauge sector of the NAC $SU(N) \otimes U(1)$ SYM theory. It turns out that the classical action (4.1.24) is not renormalizable since further divergent configurations arise at one-loop which are $N = 1/2$ supersymmetric and supergauge invariant. However, it is possible to deform the classical action in such a way as to produce a one-loop renormalizable theory [18]. The right manner in which one has to proceed is to start *ab initio* with a deformed action containing all possible terms allowed by gauge invariance, R-symmetry, and dimensional analysis. Then, one has to compute all the one-loop divergences produced by the new action. This procedure determines a one-loop renormalizable action depending on a number of arbitrary coupling constants. Computing the β-functions one finds that they allow for specific restrictions on these constants. In particular, two different choices for minimal deformed actions are allowed which are one-loop renormalizable

$$\begin{aligned}
S^{(1)}_{gauge} = & \frac{1}{2\,g^2}\int d^4x\; d^4\theta\; \mathrm{Tr}\left(\overline{\Gamma}^{\dot\alpha} * \overline{W}_{\dot\alpha}\right) \\
& +\frac{1}{2\,g_0^2\,N}\int d^4x\; d^4\theta \Bigg[\; \mathrm{Tr}\left(\overline{\Gamma}^{\dot\alpha}\right) * \mathrm{Tr}\left(\overline{W}_{\dot\alpha}\right) \\
& \qquad +4i\mathcal{F}^{\rho\gamma}\overline{\theta}^2\; \mathrm{Tr}\left(\partial_{\rho\dot\rho}\overline{\Gamma}^{\dot\alpha}\right) * \mathrm{Tr}\left(\overline{W}_{\dot\alpha} * \overline{\Gamma}_{\gamma}^{\;\dot\rho}\right) \\
& \qquad -\mathcal{F}^2\overline{\theta}^2\; \mathrm{Tr}\left(\overline{\Gamma}^{\dot\alpha} * \overline{W}_{\dot\alpha}\right) \mathrm{Tr}\left(\overline{W}^{\dot\beta} * \overline{W}_{\dot\beta}\right) \Bigg] \\
& +\frac{1}{l^2}\,\mathcal{F}^2\int d^4x\; d^4\theta\; \overline{\theta}^2\; \mathrm{Tr}\left(\overline{\Gamma}^{\dot\alpha} * \overline{W}_{\dot\alpha} * \overline{W}^{\dot\beta} * \overline{W}_{\dot\beta}\right)
\end{aligned} \tag{4.2.77}$$

or

$$\begin{aligned}
S^{(2)}_{gauge} = & \frac{1}{2\,g^2}\int d^4x\; d^4\theta \Bigg[\; \mathrm{Tr}\left(\overline{\Gamma}^{\dot\alpha} * \overline{W}_{\dot\alpha}\right) \\
& \qquad +\mathcal{F}^2\,\overline{\theta}^2\; \mathrm{Tr}\left(\overline{\Gamma}^{\dot\alpha} * \overline{W}_{\dot\alpha}\right) * \mathrm{Tr}\left(\overline{W}^{\dot\beta} * \overline{W}_{\dot\beta}\right) \Bigg] \\
& +\frac{1}{2\,g_0^2\,N}\int d^4x\; d^4\theta \Bigg[\; \mathrm{Tr}\left(\overline{\Gamma}^{\dot\alpha}\right) * \mathrm{Tr}\left(\overline{W}_{\dot\alpha}\right) \\
& \qquad +4i\mathcal{F}^{\rho\gamma}\,\overline{\theta}^2\; \mathrm{Tr}\left(\partial_{\rho\dot\rho}\overline{\Gamma}^{\dot\alpha}\right) * \mathrm{Tr}\left(\overline{W}_{\dot\alpha} * \overline{\Gamma}_{\gamma}^{\;\dot\rho}\right) \Bigg] \\
& +\frac{1}{l^2}\,\mathcal{F}^2\int d^4x\; d^4\theta\; \overline{\theta}^2\; \mathrm{Tr}\left(\overline{\Gamma}^{\dot\alpha} * \overline{W}_{\dot\alpha} * \overline{W}^{\dot\beta} * \overline{W}_{\dot\beta}\right)
\end{aligned} \tag{4.2.78}$$

In both cases the theory contains three independent coupling constants. While g, g_0 are the $SU(N) \otimes U(1)$ couplings already present in the ordinary theory, the appearance of the third coupling l is strictly related to the NAC deformation we have performed. We note that g, g_0 must be different reflecting the fact that, as in the ordinary case, $SU(N)$ and $U(1)$ fields renormalize differently. The results (4.2.77) and (4.2.78) agree with those found in components [19, 20, 21].

4.3 Renormalization with interacting matter

At the classical level, a NAC SYM theory with interacting chiral matter in the adjoint representation of $SU(N) \otimes U(1)$ can be described by the following action [18, 22]

$$\begin{aligned} S = & \frac{1}{2g^2}\int d^4x d^2\bar{\theta}\ \mathrm{Tr}(\overline{W}^{\dot{\alpha}} * \overline{W}_{\dot{\alpha}}) \\ & + \frac{1}{2g_0^2}\int d^4x\ d^4\theta\ \Big[\ \mathrm{Tr}(\overline{\Gamma}^{\dot{\alpha}}) * \mathrm{Tr}(\overline{W}_{\dot{\alpha}}) + 4i\mathcal{F}^{\rho\gamma}\bar{\theta}^2\ \mathrm{Tr}\left(\partial_{\rho\dot{\rho}}\overline{\Gamma}^{\dot{\alpha}}\right) * \mathrm{Tr}\left(\overline{W}_{\dot{\alpha}} * \overline{\Gamma}_{\gamma}^{\ \dot{\rho}}\right)\Big] \\ & + \int d^4x\ d^4\theta\ \mathrm{Tr}(\overline{\Phi} * \Phi) \\ & + h\int d^4x\ d^2\theta\ \mathrm{Tr}(\Phi * \Phi * \Phi) + \bar{h}\int d^4x\ d^2\bar{\theta}\ \mathrm{Tr}(\overline{\Phi} * \overline{\Phi} * \overline{\Phi}) \end{aligned} \tag{4.3.79}$$

where $\Phi \equiv e_*^V * \phi * e_*^{-V}$, $\bar{\Phi} = \overline{\phi}$ are covariantly (anti)chiral superfields expressed in terms of ordinary (anti)chirals. Therefore, the quadratic matter action contains nontrivial couplings between gauge and chiral superfields.

The action is invariant under the infinitesimal supergauge transformations

$$\begin{aligned} \delta\Phi = i[\overline{\Lambda}, \Phi]_* \qquad &, \qquad \delta\overline{\Phi} = i[\overline{\Lambda}, \overline{\Phi}]_* \\ \delta\overline{\Gamma}_{\alpha\dot{\alpha}} = [\overline{\nabla}_{\alpha\dot{\alpha}}, \overline{\Lambda}]_* \qquad &, \qquad \delta\overline{W}_{\dot{\alpha}} = i[\overline{\Lambda}, \overline{W}_{\dot{\alpha}}]_* \end{aligned} \tag{4.3.80}$$

As discussed in [18] the term proportional to $\bar{\theta}^2$ in (4.3.79) is necessary in order to restore gauge invariance of $\int \mathrm{Tr}(\overline{\Gamma}^{\dot{\alpha}})\mathrm{Tr}(\overline{W}_{\dot{\alpha}})$.

Under the quantum-background splitting of the Euclidean prepotential $e_*^V \to e_*^V * e_*^U$ where U is the background prepotential and V its quantum counterpart, the covariantly (anti)chiral superfields in the adjoint representation are expressed in terms of background covariantly (anti)chiral objects as

$$\overline{\Phi} = \overline{\mathbf{\Phi}} \qquad , \qquad \Phi = e_*^V * \mathbf{\Phi} * e_*^{-V} = e_*^V * (e_*^U * \phi * e_*^{-U}) * e_*^{-V} \tag{4.3.81}$$

and then splitted as $\mathbf{\Phi} \to \mathbf{\Phi} + \mathbf{\Phi}_q$ and $\bar{\mathbf{\Phi}} \to \bar{\mathbf{\Phi}} + \bar{\mathbf{\Phi}}_q$, where $\mathbf{\Phi}, \bar{\mathbf{\Phi}}$ are background fields and $\mathbf{\Phi}_q, \bar{\mathbf{\Phi}}_q$ their quantum fluctuations.

We perform quantum-background splitting in the action (4.3.79) and extract the Feynman rules necessary for one-loop calculations. We concentrate on the chiral sector, since the gauge sector has been already discussed.

We first express the full covariantly (anti)chiral superfields in terms of background covariantly (anti)chiral superfields according to (4.3.81). Expanding in powers of V we have (we use the notation $\Phi_*^3 \equiv \Phi * \Phi * \Phi$)

$$\begin{aligned} S_0 + S_{int} = & \int d^4x d^4\theta\ \bar{\mathbf{\Phi}} * \mathbf{\Phi} + \int d^4x d^4\theta\ \left(\bar{\mathbf{\Phi}}[V, \mathbf{\Phi}]_* + \frac{1}{2}\bar{\mathbf{\Phi}}[V, [V, \mathbf{\Phi}]_*]_* + \dots\right) \\ & + h\int d^4x d^2\theta\ \mathrm{Tr}(\mathbf{\Phi}_*^3) + \overline{h}\int d^4x d^2\bar{\theta}\ \mathrm{Tr}(\bar{\mathbf{\Phi}}_*^3) \end{aligned} \tag{4.3.82}$$

where the trace over group indices has been omitted since the quantization procedure works independently of the color structure. After the shift $\mathbf{\Phi} \to \mathbf{\Phi} + \mathbf{\Phi}_q$, $\bar{\mathbf{\Phi}} \to \bar{\mathbf{\Phi}} + \bar{\mathbf{\Phi}}_q$ only terms with two quantum superfields need to be considered for one–loop calculations.

Quantization is accomplished by adding source terms

$$\begin{aligned} S_j &= \int d^4x d^2\theta \; j * \mathbf{\Phi}_q + \int d^4x d^2\bar{\theta} \; \bar{\mathbf{\Phi}}_q * \bar{j} \\ &= \int d^4x d^4\theta \; \left(j * \frac{1}{\Box_+} * \mathbf{\nabla}^2 \mathbf{\Phi}_q + \bar{\mathbf{\Phi}}_q * \frac{1}{\Box_-} * \overline{\mathbf{\nabla}}^2 * \bar{j} \right) \end{aligned} \tag{4.3.83}$$

where, for any (anti)chiral superfield, we have defined

$$\begin{aligned} \bar{\nabla}^2 * \nabla^2 * \Phi = \Box_+ * \Phi \qquad & \Box_+ = \Box_{cov} - i\widetilde{W}^\alpha * \nabla_\alpha - \frac{i}{2}(\nabla^\alpha * \widetilde{W}_\alpha) \\ \nabla^2 * \bar{\nabla}^2 * \bar{\Phi} = \Box_- * \bar{\Phi} \qquad & \Box_- = \Box_{cov} - i\overline{W}^{\dot{\alpha}} * \bar{\nabla}_{\dot{\alpha}} - \frac{i}{2}(\bar{\nabla}^{\dot{\alpha}} * \overline{W}_{\dot{\alpha}}) \end{aligned} \tag{4.3.84}$$

and performing the gaussian integral in

$$Z = \int \mathcal{D}\mathbf{\Phi}_q \mathcal{D}\bar{\mathbf{\Phi}}_q e^{S_{int}(\frac{\delta}{\delta j}, \frac{\delta}{\delta \bar{j}})} e^{\int d^4x d^4\theta \; (\bar{\mathbf{\Phi}}_q * \mathbf{\Phi}_q + j * \frac{1}{\Box_+} * \mathbf{\nabla}^2 \mathbf{\Phi}_q + \bar{\mathbf{\Phi}}_q * \frac{1}{\Box_-} * \overline{\mathbf{\nabla}}^2 * \bar{j})} \tag{4.3.85}$$

The Feynman rules can then be read from

$$Z = \Delta \; e^{S_{int}(\frac{\delta}{\delta j}, \frac{\delta}{\delta \bar{j}})} e^{-\int d^4x d^4\theta \; j * \frac{1}{\Box_-} * \bar{j}} \tag{4.3.86}$$

where $\Delta \equiv \int \mathcal{D}\mathbf{\Phi}_q \mathcal{D}\bar{\mathbf{\Phi}}_q e^{S_0}$. In particular, we obtain the covariant scalar propagator

$$\langle \Phi^A \bar{\Phi}^B \rangle = - \left(\frac{1}{\Box_-} \right)^{AB} \tag{4.3.87}$$

At one–loop, from the matter sector we have two different contributions to the effective action. A first contribution to the gauge effective action comes from the perturbative evaluation of Δ. This can be worked out by using the doubling trick procedure introduced in [13] for ordinary SYM theories and generalized to NAC theories in [18]. The corresponding Feynman rules are collected in Ref. [18]. A second contribution comes from the perturbative expansion of $e^{S_{int}}$ from which we can read gauge–chiral vertices. Further interaction vertices arise from the expansion of $1/\Box_-$ in powers of the background fields (see Appendix B.1).

4.3.1 One–loop divergences: The matter sector

We now study the structure of one-loop divergent contributions to the matter sector.

The matter part of the classical action (4.3.79) is

$$S_{matter} = \int d^4x \; d^4\theta \; \mathrm{Tr}(\overline{\Phi} * \Phi) + h \int d^4x \; d^2\theta \; \mathrm{Tr}(\Phi^3_*) + \bar{h} \int d^4x \; d^2\bar{\theta} \; \mathrm{Tr}(\overline{\Phi}^3_*) \tag{4.3.88}$$

for covariantly (anti)chiral superfields. Applying background field method we evaluate one-loop diagrams with external matter.

The quadratic action

Divergent diagrams contributing to the two–point function are given in Fig. 4.4 where the internal lines correspond to ordinary $1/\Box$ propagators (straight lines correspond to chiral propagators, whereas waved lines correspond to vectors). It turns out that divergent contributions to the quadratic term come only from vertices not including the deformation parameter. Therefore, they coincide with the ones of the underformed theory and are given by

$$\mathcal{S}\int d^4xd^4\theta\,\left[\left(9h\overline{h}-2g^2\right)N\ \mathrm{Tr}\left(\bar{\mathbf{\Phi}}*\mathbf{\Phi}\right)+\left(9h\overline{h}+2g^2\right)\ \mathrm{Tr}\bar{\mathbf{\Phi}}*\mathrm{Tr}\mathbf{\Phi}\right] \tag{4.3.89}$$

where $\mathcal{S}$ is the self-energy divergent integral (A.2.9).

We note that a new trace structure appears reflecting the fact that $SU(N)$ and $U(1)$ superfields acquire different contributions. In fact, considering only the kinetic term, the previous result reads (using eq. (4.3.81))

$$\mathcal{S}\int d^4xd^4\theta\ \left[\left(9h\overline{h}-2g^2\right)N\bar{\phi}^a\phi^a+18h\overline{h}N\bar{\phi}^0\phi^0\right] \tag{4.3.90}$$

In particular, corrections to the abelian kinetic term coming from the gauge–chiral loop cancel in agreement with the calculation done in components [23].

In the ordinary case, the appearance of the double–trace term is harmless since it is supergauge invariant. In the NAC case this is no longer true since its variation is

$$\delta\,\mathrm{Tr}\bar{\mathbf{\Phi}}*\mathrm{Tr}\mathbf{\Phi}=2i\bar{\theta}^2\mathcal{F}^{\alpha\beta}\left[\mathrm{Tr}\left(\partial_\alpha^{\ \dot\alpha}\bar{\Lambda}*\partial_{\beta\dot\alpha}\bar{\mathbf{\Phi}}\right)*\mathrm{Tr}\mathbf{\Phi}+\mathrm{Tr}\bar{\mathbf{\Phi}}*\mathrm{Tr}\left(\partial_\alpha^{\ \dot\alpha}\bar{\Lambda}*\partial_{\beta\dot\alpha}\mathbf{\Phi}\right)\right] \tag{4.3.91}$$

and does not vanish when integrated on superspace coordinates.

On general grounds, it is easy to see that there are two possible gauge completions for $\int\mathrm{Tr}\bar{\mathbf{\Phi}}*\mathrm{Tr}\mathbf{\Phi}$. In fact, the following expressions (both for background covariantly and full covariantly (anti)chiral superfields)

$$\mathrm{Tr}\bar{\Phi}*\mathrm{Tr}\Phi+2i\bar{\theta}^2\mathcal{F}^{\alpha\beta}\ \mathrm{Tr}(\overline{\Gamma}_\alpha^{\ \dot\alpha}*\bar{\Phi})*\mathrm{Tr}(\partial_{\beta\dot\alpha}\Phi)+2i\bar{\theta}^2\mathcal{F}^{\alpha\beta}\ \mathrm{Tr}(\overline{\Gamma}_\alpha^{\ \dot\alpha}*\Phi)*\mathrm{Tr}(\partial_{\beta\dot\alpha}\bar{\Phi}) \tag{4.3.92}$$

and

$$\begin{aligned}\mathrm{Tr}\bar{\Phi}*\mathrm{Tr}\Phi\ -\ 2i\mathcal{F}^{\alpha\beta}\bar{\theta}^2\mathrm{Tr}\left[\overline{\Gamma}_\alpha^{\ \dot\alpha}*\left(\partial_{\beta\dot\alpha}\bar{\Phi}-\frac{i}{2}[\overline{\Gamma}_{\beta\dot\alpha},\bar{\Phi}]_*\right)\right]*\mathrm{Tr}\left(\Phi\right)\\ -\ 2i\mathcal{F}^{\alpha\beta}\bar{\theta}^2\mathrm{Tr}\left[\overline{\Gamma}_\alpha^{\ \dot\alpha}*\left(\partial_{\beta\dot\alpha}\Phi-\frac{i}{2}[\overline{\Gamma}_{\beta\dot\alpha},\Phi]_*\right)\right]*\mathrm{Tr}\left(\bar{\Phi}\right)\end{aligned} \tag{4.3.93}$$

are both gauge invariant when integrated. While the first expression involves only gauge–chiral cubic terms in addition to the quadratic term, the second one involves also quartic couplings. Therefore, we have to investigate whether at one–loop the theory develops further divergent terms cubic and/or quartic in the background fields which provide the gauge completion of $\int\mathrm{Tr}\mathbf{\Phi}*\mathrm{Tr}\bar{\mathbf{\Phi}}$.

Divergences proportional to gauge–chiral cubic terms are still obtained from diagrams in Fig. 4.4 where the internal lines correspond to covariant $1/\Box_-$ and $1/\hat{\Box}$ propagators expanded up to

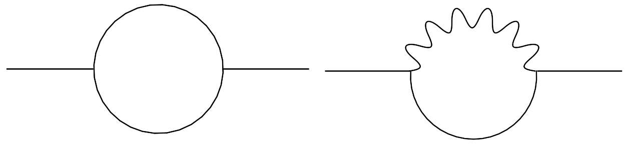

Figure 4.4: One–loop two–point functions with chiral external fields.

quadratic order in the background gauge superfields (see eqs. (B.1.17, B.1.34)). Summing the contributions coming from both diagrams in Fig. 4.4 we obtain

$$(9h\overline{h}+2g^2)\ \mathcal{S}\int d^4xd^4\theta\ 2i\ \mathcal{F}^{\alpha\beta}\bar{\theta}^2\left[\mathrm{Tr}(\overline{\Gamma}_\alpha^{\ \dot{\alpha}}*\bar{\mathbf{\Phi}})*\mathrm{Tr}(\partial_{\beta\dot{\alpha}}\mathbf{\Phi})+\mathrm{Tr}(\overline{\Gamma}_\alpha^{\ \dot{\alpha}}*\mathbf{\Phi})*\mathrm{Tr}(\partial_{\beta\dot{\alpha}}\bar{\mathbf{\Phi}})\right]$$
$$-2i(9h\overline{h}-2g^2)\ \mathcal{S}\int d^4xd^4\theta\ \ \mathcal{F}^{\alpha\beta}\bar{\theta}^2\mathrm{Tr}(\partial_{\beta\dot{\alpha}}\overline{\Gamma}_\alpha^{\ \dot{\alpha}})*\mathrm{Tr}(\mathbf{\Phi}*\bar{\mathbf{\Phi}}) \tag{4.3.94}$$

The first line is exactly the gauge completion of (4.3.89) according to (4.3.92). In addition, a second divergent term appears in the second line. Since it is gauge invariant it is allowed by super Ward identities.

We should not expect divergent four–point functions proportional to $\overline{\Gamma}_{\alpha\dot{\alpha}}$ connections since there is no need to saturate gauge–variation of two–point divergences. In fact, from a direct inspection one can realize that only structures of the form

$$\mathcal{F}^2\ \int d^4xd^4\theta\ \bar{\theta}^2\ \mathbf{\Phi}*\bar{\mathbf{\Phi}}*\bar{W}^{\dot{\alpha}}*\bar{W}_{\dot{\alpha}} \tag{4.3.95}$$

can be divergent. For any kind of trace structure all these terms are gauge–invariant and do not interfere with the previous structures.

To summarize, the evaluation of one–loop divergences reveals that the action (4.3.88) we started with is not renormalizable because of the appearance of new one–loop structures not originally present.

At this stage it is easy to generalize the classical action to a renormalizable one in a gauge invariant way: It is sufficient to start with a classical quadratic action of the form

$$\int d^4xd^4\theta\ \ \Big\{\mathrm{Tr}\left(\bar{\Phi}*\Phi\right) \tag{4.3.96}$$
$$+\Big[\mathrm{Tr}\bar{\Phi}*\mathrm{Tr}\Phi+2i\bar{\theta}^2\mathcal{F}^{\alpha\beta}\mathrm{Tr}(\overline{\Gamma}_\alpha^{\ \dot{\alpha}}*\bar{\Phi})*\mathrm{Tr}(\partial_{\beta\dot{\alpha}}\Phi)+2i\bar{\theta}^2\mathcal{F}^{\alpha\beta}\mathrm{Tr}(\overline{\Gamma}_\alpha^{\ \dot{\alpha}}*\Phi)*\mathrm{Tr}(\partial_{\beta\dot{\alpha}}\bar{\Phi})\Big]\Big\}$$

supplemented by the gauge invariant terms appearing in the second line of (4.3.94) and in (4.3.95).

We stress once again that the divergent contributions (4.3.89) to the quadratic action would be present also in the ordinary, not deformed theory. Therefore, also in that case we would be forced to generalize the classical quadratic action to contain a double–trace part, in order to make the theory renormalizable. The crucial difference is that the double–trace term would be gauge invariant and no gauge completion would be required.

As already mentioned, in the NAC case the double trace quadratic action has in principle two possible gauge completions. From direct inspection, the theory seems to prefer the gauge invariant structure (4.3.92) rather than (4.3.93).

4.3.2 The superpotential problem

As we now describe, when a chiral superpotential is turned on the generalization (4.3.96) for the quadratic matter action is not sufficient to make the theory renormalizable.

Since the nonrenormalization theorem for chiral integrals works also in the NAC case [12, 8, 24], the cubic superpotential in (4.3.88) does not get corrected by new diagrams proportional to Φ^3 and/or $\bar{\Phi}^3$. As in the ordinary case, the renormalization of the chiral coupling constant is induced by the wave–function renormalization under the requirement that $Z_h Z_\Phi^{-3/2} = 1$ (a similar relation holds for the antichiral coupling). On the other hand, $SU(N)$ and $U(1)$ chiral superfields renormalize differently, so should do the corresponding chiral couplings. Therefore, a cubic superpotential as the one in (4.3.88) which assigns the same coupling to the $SU(N)$, $U(1)$ and mixed interaction vertices is inconsistent with the request of renormalizability. We note that this problem is not peculiar of the NAC deformation being present already in the ordinary case.

The way out is once again the generalization of the classical action to include different couplings for different cubic vertices. Exploiting the fact that in Euclidean space $Z_{\bar{\Phi}}$ is not necessarily equal to Z_Φ, we can trigger the renormalization in such a way that for instance all the renormalization asymmetry between non–abelian and abelian fields is confined to the antichiral sector. As a consequence, we can consistently choose the ordinary $h \int d^4x\ d^2\theta \mathrm{Tr}(\Phi_*^3)$ superpotential in the chiral sector, but generalize the one for the antichiral sector to

$$\int d^4x\ d^2\bar{\theta}\ \left[\bar{h}_1 \mathrm{Tr}(\bar{\Phi}_*^3) + \bar{h}_2 \mathrm{Tr}\bar{\Phi} * \mathrm{Tr}(\bar{\Phi}_*^2) + \bar{h}_3 (\mathrm{Tr}\bar{\Phi})_*^3\right] \tag{4.3.97}$$

However, while in the ordinary case the different structures are separately gauge invariant, in the NAC case the addition of the $\bar{h}_2, \bar{h}_3$ terms breaks gauge invariance. In fact, due to the lack of θ–integration, the traces are no longer cyclic and $\delta \int (\mathrm{Tr}\bar{\Phi} * \mathrm{Tr}(\bar{\Phi}_*^2))$ and $\delta \int (\mathrm{Tr}\bar{\Phi})_*^3$ are non–vanishing.

The gauge completion of these terms reads

$$\begin{aligned} \bar{h}_2 \int d^4x d^2\bar{\theta}\ \Big\{ & \mathrm{Tr}\Big(\bar{\Phi} - 2i\bar{\theta}^2 \mathcal{F}^{\alpha\beta}\ \overline{\Gamma}_\alpha^{\ \dot{\alpha}} * \left\{\partial_{\beta\dot{\alpha}}\bar{\Phi} - \tfrac{i}{2}\left[\overline{\Gamma}_{\beta\dot{\alpha}}, \bar{\Phi}\right]_*\right\}\Big) * \mathrm{Tr}\left(\bar{\Phi}_*^2\right) \\ & + \mathrm{Tr}\bar{\Phi} * \mathrm{Tr}\Big(\bar{\Phi}_*^2 - 2i\bar{\theta}^2 \mathcal{F}^{\alpha\beta}\ \overline{\Gamma}_\alpha^{\ \dot{\alpha}} * \left\{\partial_{\beta\dot{\alpha}}\bar{\Phi}_*^2 - \tfrac{i}{2}\left[\overline{\Gamma}_{\beta\dot{\alpha}}, \bar{\Phi}_*^2\right]_*\right\}\Big)\Big\} \end{aligned} \tag{4.3.98}$$

and

$$\bar{h}_3 \int d^4x d^2\bar{\theta}\, \mathrm{Tr}\Big(\bar{\Phi} - 6i\bar{\theta}^2 \mathcal{F}^{\alpha\beta} \overline{\Gamma}_\alpha^{\ \dot{\alpha}} * \Big\{\partial_{\beta\dot{\alpha}}\bar{\Phi} - \frac{i}{2}\left[\overline{\Gamma}_{\beta\dot{\alpha}}, \bar{\Phi}\right]_*\Big\}\Big) * \mathrm{Tr}\big(\bar{\Phi}\big) * \mathrm{Tr}\big(\bar{\Phi}\big) \tag{4.3.99}$$

respectively.

The terms proportional to $\overline{\Gamma}_{\alpha\dot{\alpha}}$ in the previous expressions break supersymmetry completely since they are given by non–antichiral expressions integrated over an antichiral measure. Therefore, one–loop renormalizability, gauge invariance and $N = 1/2$ supersymmetry seem to be incompatible. This is the translation in superspace language of the negative result already found in components [23].

4.3.3 The solution to the superpotential problem

Fortunately, generalizing the superpotential to contain more than one coupling constant does not seem to be the only possibility for constructing a renormalizable action. In fact, an alternative procedure exists for treating the diverse renormalization of the abelian fields in a consistent way. The idea is to start with a classical quadratic action of the form (4.3.96) but with a new coupling in front of the double–trace term

$$\int d^4x d^4\theta \, \Big\{ \mathrm{Tr}\left(\bar\Phi * \Phi\right) + \frac{\kappa - 1}{N}\Big[\mathrm{Tr}\bar\Phi * \mathrm{Tr}\Phi + 2i\bar\theta^2 \mathcal{F}^{\alpha\beta}\mathrm{Tr}(\overline{\Gamma}_\alpha^{\ \dot\alpha} * \bar\Phi) * \mathrm{Tr}(\partial_{\beta\dot\alpha}\Phi) + 2i\bar\theta^2 \mathcal{F}^{\alpha\beta}\mathrm{Tr}(\overline{\Gamma}_\alpha^{\ \dot\alpha} * \Phi) * \mathrm{Tr}(\partial_{\beta\dot\alpha}\bar\Phi)\Big]\Big\} \tag{4.3.100}$$

and tune the renormalization of κ with the wave–function renormalization in order to make $SU(N)$ and $U(1)$ superfields to renormalize in the same way. Consequently, a cubic superpotential of the form $h\int \mathrm{Tr}\Phi_*^3 + \bar h \int \mathrm{Tr}\bar\Phi_*^3$ can be safely added, with no need of further terms like the ones in (4.3.97).

As discussed in details in Appendix B.1, the background field method can be easily generalized to the action (4.3.100) by performing a change of variables $\mathbf{\Phi}_q \to \mathbf{\Phi}'_q = (\mathbf{\Phi}_q^a, \kappa_1\mathbf{\Phi}_q^0)$ and $\bar{\mathbf{\Phi}}_q \to \bar{\mathbf{\Phi}}'_q = (\bar{\mathbf{\Phi}}_q^a, \kappa_2\bar{\mathbf{\Phi}}_q^0)$, $\kappa_1\kappa_2 = \kappa$, in the functional integral. The net result is a rescaling of the covariant propagators according to eqs. (B.1.29-B.1.32). Expanding the propagators in powers of the background gauge fields (see Appendix B.1) this is equivalent to a rescaling of the abelian propagator

$$\langle \bar\phi^0 \phi^0 \rangle = \frac{1}{\kappa}\,\frac{1}{\Box_0} \tag{4.3.101}$$

and a rescaling of all gauge–chiral interaction vertices involving abelian superfields. Precisely, vertices containing $\mathbf{\Phi}^0$, $\bar{\mathbf{\Phi}}^0$ acquire an extra coupling constant $1/\kappa_1$, $1/\kappa_2$, respectively.

It is important to note that in the covariant propagators the κ_1, κ_2 couplings appear only in terms proportional to the deformation parameter. Therefore, the dependence on these two couplings would disappear in the ordinary $N = 1$ supersymmetric case. In that case, as it is well known, the rescaling (4.3.101) of the abelian propagator would be the only effect of choosing a modified quadratic lagrangian for the abelian superfields.

To summarize, we begin with a NAC classical gauge theory whose gauge sector is still described by (4.2.77) or (4.2.78) , whereas the matter action is given by (4.3.100) supplemented by the single–trace cubic superpotential. However, as appears from one–loop calculations, extra couplings need be considered which are consistent with $N = 1/2$ supersymmetry and supergauge invariance. In the next Section we will select all possible couplings which can be added at classical level.

4.3.4 The most general gauge invariant action

Before entering the study of renormalization properties, we will select all possible divergent structures which could come out at quantum level on the basis of dimensional analysis and global symmetries of the theory.

Dimensional analysis and global symmetries

The most general divergent term which may arise at quantum level has the form

$$\int d^4x d^4\theta \ \bar{\theta}^{\bar{\tau}} \ \mathcal{F}^{\alpha} \ \Lambda^{\beta} \ D^{\gamma} \ \bar{D}^{\bar{\gamma}} \ \partial^{\delta} \ \overline{\Gamma}^{\bar{\sigma}} \ \Phi^{n} \ \bar{\Phi}^{m} \ h^{r} \ \bar{h}^{s} \tag{4.3.102}$$

where all the exponents are non negative integers. Of course, powers of the gauge coupling g can appear. However, its presence is irrelevant for our discussion, being g adimensional and with zero R–symmetry charge. Therefore, in what follows we will neglect it.

We make the following simplifications:

- We can choose the connections to be the bosonic $\overline{\Gamma}^{\alpha\dot{\alpha}}$. In fact, thanks to the relation $\overline{\Gamma}^{\alpha\dot{\alpha}} = -iD_{\alpha}\overline{\Gamma}_{\dot{\alpha}}$, switching from bosonic to fermionic connections would amount to shifting $\gamma \to \gamma + \bar{\sigma}$.

- The parameter $\bar{\tau}$ takes the values $0, 1, 2$. However, we can fix it to be 2 by writing $\bar{\theta}^{\dot{\alpha}} = \bar{D}^{\dot{\alpha}}\bar{\theta}^2 \to \bar{\theta}^2\bar{D}^{\dot{\alpha}}$ and $-1 = \bar{D}^2\bar{\theta}^2 \to \bar{\theta}^2\bar{D}^2$ where we think of integrating by parts the antichiral derivatives.

- Assuming that the NAC deformation is a soft supersymmetry breaking mechanism we set $\beta = 0$.

- At one–loop, the Φ^3 vertex provides a single power of the h coupling and one external Φ-field. Taking into account that further external chirals can come from gauge–chiral vertices, we have the constraint $r \leq n$. Similarly, for the antichiral vertex it must be $s \leq m$.

Therefore, the general structure for divergences can be reduced to the following form

$$\int d^4x d^4\theta \ \bar{\theta}^2 \ \mathcal{F}^{\alpha} \ \nabla^{\gamma} \ \bar{D}^{\bar{\gamma}}\partial^{\delta} \ \overline{\Gamma}^{\bar{\sigma}} \ \Phi^{n} \ \bar{\Phi}^{m} \ h^{r} \ \bar{h}^{s} \qquad r \leq n \quad , \quad s \leq m \tag{4.3.103}$$

where the number of ∇–derivatives should not exceed $(\bar{\sigma}+2(n-1))$ in order to avoid the integrand to be a total ∇–derivative. Further constraints on the exponents come from imposing the global symmetries as listed in Table 4.1, in addition to the request for the integrand to have mass dimension 2. Moreover, we need impose the number of dotted and undotted indices to be even from the requirement that they contract among themselves to generate a supersymmetry singlet. Finally, we impose $\alpha \geq 1$ to allow for a non–trivial dependence on the nonanticommutative parameter.

	dim	R-charge	Φ-charge
$\overline{\Gamma}^{\alpha\dot{\alpha}}$	1	0	0
$D_\alpha \equiv \nabla_\alpha$	1/2	1	0
$\bar{D}_{\dot{\alpha}}$	1/2	−1	0
$\bar{\theta}$	−1/2	1	0
$\partial_{\alpha\dot{\alpha}}$	1	0	0
$\mathcal{F}^{\rho\gamma}$	−1	−2	0
Φ	1	−1	1
h	0	1	−3
$\bar{\Phi}$	1	1	−1
$\bar{h}$	0	−1	3

Table 4.1: Dimensions, R and Φ-charge assignments of $N = 1/2$ operators.

With the charge assignements given in Table 4.1 the set of constraints read

$$\begin{array}{ll}
\text{Dimensions:} & -3-\alpha+\frac{\gamma}{2}+\frac{\bar{\gamma}}{2}+\delta+\bar{\sigma}+n+m=0 \\
\text{R-charge:} & 2-2\alpha+\gamma-\bar{\gamma}-n+m+r-s=0 \\
\text{Index contraction:} & 2\alpha+\gamma+\delta+\bar{\sigma}=2l+4 \\
 & \bar{\gamma}+\delta+\bar{\sigma}=2l' \\
\text{Derivatives:} & \gamma \leq \bar{\sigma}+2n-2 \\
\Phi\text{–symmetry:} & n-m+3(s-r)=0 \\
\text{One–loop rules:} & r \leq n \\
 & s \leq m
\end{array} \qquad (4.3.104)$$

where $l, l' \geq 0$ are integer numbers.

Combining the first two equations we get

$$8-4l' = 3n+m-r+s \geq 3n+m-r \geq 2n+m \geq 0 \qquad (4.3.105)$$

from which we derive the conditions

$$l' \leq 2 \qquad 2n+m \leq 8-4l' \qquad (4.3.106)$$

A simple constraint on l can be obtained from merging the third, the forth and the sixth equations in (4.3.104)

$$\begin{aligned}
2(l-l')+4 &= 2\alpha+\gamma-\bar{\gamma} \leq 2\alpha+\bar{\sigma}+2n-2-\bar{\gamma} \\
&= 2\alpha+2l'-\delta+2n-2\bar{\gamma}-2 \\
\Rightarrow l &\leq \alpha+2l'-3-\frac{1}{2}\delta+n-\bar{\gamma}
\end{aligned} \qquad (4.3.107)$$

Then, using the first constraint and the previous bound we find

$$2n+m = 3+\alpha+n-\frac{1}{2}\gamma-\frac{1}{2}\bar{\gamma}-2l'+\bar{\gamma} \leq 8-4l' \qquad (4.3.108)$$

which, after a bit of trivial algebra, provides a constraint on α

$$1 \leq \alpha \leq 4 - l' - \bar{\gamma} \tag{4.3.109}$$

Finally, using this condition we can constrain l even more and obtain

$$0 \leq l \leq 5 - \delta - l' - 2\bar{\gamma} \tag{4.3.110}$$

Now we are ready to list divergent contributions. We assign values $0, 1, 2$ to l' according to (4.3.106), and we fix δ, $\bar{\sigma}$ and $\bar{\gamma}$, which are bounded by l' itself. Then we can vary l into the range given by (4.3.110) and α in the range (4.3.109), while the value of γ follows immediately from the third equation in (4.3.104). Finally, the remaining parameters (n, m, r, s) are varied according with the set of equations (4.3.104).

A detailed investigation reveals that, independently of their particular trace structure, the only allowed terms are (for the moment we forget about $*$–products)

1. Matter sector. These structures are obtained by setting $\bar{\sigma} = 0$ when $l' = 0, 1$ and correspond to

   - $$\bar{h}(h\bar{h})^r \mathcal{F}^2 \int d^4x d^4\theta \; \bar{\theta}^2 \Phi \bar{\Phi}^4 \qquad r = 0, 1 \tag{4.3.111}$$
   - $$(h\bar{h})^r \mathcal{F}^2 \int d^4x d^4\theta \; \bar{\theta}^2 \Phi (\nabla^2 \Phi) \bar{\Phi}^2 \qquad r \leq 2 \tag{4.3.112}$$
   - $$h \; \mathcal{F}^2 \int d^4x d^4\theta \; \bar{\theta}^2 \Phi (\nabla^2 \Phi)^2 \tag{4.3.113}$$
   - $$h \; \mathcal{F}^{\alpha\beta} \int d^4x d^4\theta \; \bar{\theta}^2 (\nabla_\alpha \Phi)(\nabla_\beta \Phi) \Phi \tag{4.3.114}$$

   Powers of the gauge coupling g are also allowed. The first three terms are non–vanishing whatever the color structure is. In the abelian case they correspond to the actual structures which arise at one and two loops in the ungauged NAC WZ model [8, 24, 10, 11, 7]. The last term, instead, is nontrivial only when $\nabla_\alpha \Phi$ and $\nabla_\beta \Phi$ have different color index. Therefore, it is present only when gauging the WZ model with a non–abelian group.

2. Mixed sector. All structures selected correspond to the case $l' = 1$ and are given by

   - $$(h\bar{h})^r \mathcal{F}^{\alpha\beta} \int d^4x d^4\theta \; \bar{\theta}^2 \partial_\beta{}^{\dot{\alpha}} \overline{\Gamma}_{\alpha\dot{\alpha}} \Phi \bar{\Phi} \tag{4.3.115}$$
   - $$(h\bar{h})^r \mathcal{F}^{\alpha\beta} \int d^4x d^4\theta \; \bar{\theta}^2 \overline{\Gamma}_\beta{}^{\dot{\alpha}} \overline{\Gamma}_{\alpha\dot{\alpha}} \Phi \bar{\Phi} \tag{4.3.116}$$
   - $$\bar{h} \; \mathcal{F}^2 \int d^4x d^4\theta \; \bar{\theta}^2 \overline{\Gamma}^{\alpha\dot{\alpha}} \overline{\Gamma}_{\alpha\dot{\alpha}} \bar{\Phi} \bar{\Phi} \bar{\Phi} \tag{4.3.117}$$
   - $$(h\bar{h})^r \mathcal{F}^2 \int d^4x d^4\theta \; \bar{\theta}^2 \overline{W}^{\dot{\alpha}} \overline{W}_{\dot{\alpha}} \Phi \bar{\Phi} \tag{4.3.118}$$

   where in (4.3.115) the space–time derivative can act on any of the three fields.

   At one–loop, we can only have $r = 0, 1$. When $r = 0$ a g^2 factor is present and corresponds to contributions generated by mixed gauge–chiral vertices. When $r = 1$ we have divergent terms generated by pure (anti)chiral vertices.

3. Gauge sector. This case corresponds to $l' = 2$ because of the bound $2n + m \leq 8 - 4l' = 0$ which implies $n = m = 0$, i.e. no external (anti)chiral fields. The structures we find are exactly the ones found in [18].

The previous analysis can be generalized to the case $\beta \neq 0$ in (4.3.102) allowing for positive powers of the UV cut–off. It is not difficult to see that for any positive value of β non–trivial structures which satisfy all the constraints cannot be constructed. This proves that even in the presence of interacting matter supersymmetry is softly broken.

Gauge invariance

The previous structures have been selected without requiring supergauge invariance. We expect that imposing it as a further constraint, only particular linear combinations of the previous terms with specific color structures will survive.

In the matter sector, thanks to the presence of the $\bar{\theta}^2$ factor, the (anti)chiral interaction terms (4.3.111–4.3.113) are gauge–invariant, independently of their color structure. The term (4.3.114) is non–vanishing only when it is single–trace and it is gauge invariant.

Focusing on the mixed sector, it is easy to see that the general terms (4.3.117, 4.3.118) are always gauge invariant, independently of their trace structure.

Terms (4.3.115, 4.3.116), instead, give rise to different gauge invariant combinations depending on their trace structure. The only invariant single–trace operator which can arise at one–loop is

$$\mathcal{F}^{\alpha\beta}\bar{\theta}^2\mathrm{Tr}\left(\partial_{\beta\dot{\alpha}}\overline{\Gamma}_{\alpha}^{\ \dot{\alpha}}\{\Phi,\bar{\Phi}\} - \frac{i}{2}[\overline{\Gamma}_{\beta\dot{\alpha}},\overline{\Gamma}_{\alpha}^{\ \dot{\alpha}}]_*\{\Phi,\bar{\Phi}\}\right) \tag{4.3.119}$$

where the explicitly indicated $*$-product is the only non–trivial $*$–product which appears. Looking at double–trace operators, we already know that structures of the form (4.3.115, 4.3.116) combine with the double–trace 2pt function in order to make it gauge invariant (see eqs. (4.3.92, 4.3.93)). Further gauge invariant combinations from (4.3.115, 4.3.116) are

$$\mathcal{F}^{\alpha\beta}\bar{\theta}^2\ \mathrm{Tr}\left(\partial_{\beta\dot{\alpha}}\overline{\Gamma}_{\alpha}^{\ \dot{\alpha}}\Phi - \frac{i}{2}[\overline{\Gamma}_{\beta\dot{\alpha}},\overline{\Gamma}_{\alpha}^{\ \dot{\alpha}}]_*\Phi\right)\mathrm{Tr}(\bar{\Phi}) \tag{4.3.120}$$

$$\mathcal{F}^{\alpha\beta}\bar{\theta}^2\ \mathrm{Tr}\left(\partial_{\beta\dot{\alpha}}\overline{\Gamma}_{\alpha}^{\ \dot{\alpha}}\bar{\Phi} - \frac{i}{2}[\overline{\Gamma}_{\beta\dot{\alpha}},\overline{\Gamma}_{\alpha}^{\ \dot{\alpha}}]_*\bar{\Phi}\right)\mathrm{Tr}(\Phi) \tag{4.3.121}$$

$$\mathcal{F}^{\alpha\beta}\bar{\theta}^2\ \mathrm{Tr}\left(\partial_{\beta\dot{\alpha}}\overline{\Gamma}_{\alpha}^{\ \dot{\alpha}}\right)\mathrm{Tr}\left(\Phi\bar{\Phi}\right) \tag{4.3.122}$$

while there is no way to saturate the gauge variation of $\mathcal{F}^{\alpha\beta}\bar{\theta}^2\mathrm{Tr}\left(\overline{\Gamma}_{\alpha}^{\ \dot{\alpha}}\right)\mathrm{Tr}\left((\partial_{\beta\dot{\alpha}}\Phi)\bar{\Phi}\right)$ or, similarly, of the term obtained by exchanging $\Phi \leftrightarrow \bar{\Phi}$. Indeed, only the combination

$$\left[\mathcal{F}^{\alpha\beta}\bar{\theta}^2\mathrm{Tr}\left(\overline{\Gamma}_{\alpha}^{\ \dot{\alpha}}\right)\mathrm{Tr}\left((\partial_{\beta\dot{\alpha}}\Phi)\bar{\Phi}\right) + \mathcal{F}^{\alpha\beta}\bar{\theta}^2\mathrm{Tr}\left(\overline{\Gamma}_{\alpha}^{\ \dot{\alpha}}\right)\mathrm{Tr}\left(\Phi(\partial_{\beta\dot{\alpha}}\bar{\Phi})\right)\right]$$

is gauge invariant. However, integrating by parts, this reduces to (4.3.122).

Using similar arguments, we find that the only triple–trace gauge–invariant operator is

$$\mathcal{F}^{\alpha\beta}\bar{\theta}^2\ \mathrm{Tr}\left(\partial_{\beta\dot{\alpha}}\overline{\Gamma}_{\alpha}^{\ \dot{\alpha}}\right)\mathrm{Tr}(\Phi)\mathrm{Tr}(\bar{\Phi}) \tag{4.3.123}$$

Finally, looking at the gauge sector, once we impose gauge invariance only terms corresponding to all NAC structures present in (4.2.77, 4.2.78) are selected.

The general action

We are now ready to propose the most general classical action for a NAC gauge theory with massless matter in the adjoint of $SU(N) \otimes U(1)$. Introducing the greatest number of coupling constants compatible with gauge invariance, we write

$$S = S_{gauge} + S_{matter} + S_{\overline{\Gamma}} + S_{\overline{W}} \tag{4.3.124}$$

where S_{gauge} is given in (4.2.77) (or equivalently (4.2.78)),

$$\begin{aligned}
S_{matter} \; = \; & \int d^4x d^4\theta \, \Big\{ \mathrm{Tr} \left(\bar{\Phi} * \Phi \right) + \frac{\kappa - 1}{N} \Big[\mathrm{Tr}\bar{\Phi} * \mathrm{Tr}\Phi \\
& \quad + 2i\bar{\theta}^2 \mathcal{F}^{\alpha\beta} \mathrm{Tr}(\overline{\Gamma}_{\alpha}^{\;\dot{\alpha}} * \bar{\Phi}) * \mathrm{Tr}(\partial_{\beta\dot{\alpha}}\Phi) + 2i\bar{\theta}^2 \mathcal{F}^{\alpha\beta} \mathrm{Tr}(\overline{\Gamma}_{\alpha}^{\;\dot{\alpha}} * \Phi) * \mathrm{Tr}(\partial_{\beta\dot{\alpha}}\bar{\Phi}) \Big] \Big\} \\
& + h \int d^4x d^2\theta \; \mathrm{Tr}\Phi_*^3 + \bar{h} \int d^4x d^2\bar{\theta} \; \mathrm{Tr}\bar{\Phi}_*^3 + \tilde{h}_3 \; \mathcal{F}^{\alpha\beta} \int d^4x d^4\theta \; \bar{\theta}^2 \mathrm{Tr}((\nabla_\alpha \Phi)(\nabla_\beta \Phi)\Phi) \\
& + \sum_{j=1}^{3} h_3^{(j)} \, \mathcal{C}_j^{ABC} \, \mathcal{F}^2 \int d^4x d^4\theta \; \bar{\theta}^2 \Phi^A (\nabla^2 \Phi^B)(\nabla^2 \Phi^C) \\
& + \sum_{j=1}^{10} h_4^{(j)} \, \mathcal{D}_j^{ABCD} \, \mathcal{F}^2 \int d^4x d^4\theta \; \bar{\theta}^2 \Phi^A (\nabla^2 \Phi^B) \bar{\Phi}^C \bar{\Phi}^D \\
& + \sum_{j=1}^{12} h_5^{(j)} \, \mathcal{E}_j^{ABCDE} \, \mathcal{F}^2 \int d^4x d^4\theta \; \bar{\theta}^2 \Phi^A \bar{\Phi}^B \bar{\Phi}^C \bar{\Phi}^D \bar{\Phi}^E
\end{aligned} \tag{4.3.125}$$

and $S_{\overline{\Gamma}}, S_{\bar{W}}$ contain all possible gauge invariant mixed terms proportional to the bosonic connection

$$\begin{aligned}
S_{\overline{\Gamma}} = \; & t_1 \; \mathcal{F}^{\alpha\beta} \int d^4x d^4\theta \; \bar{\theta}^2 \mathrm{Tr} \left(\partial_{\beta\dot{\alpha}} \overline{\Gamma}_{\alpha}^{\;\dot{\alpha}} \right) \; \mathrm{Tr} \left(\bar{\Phi}\Phi \right) \\
& + t_2 \; \mathcal{F}^{\alpha\beta} \int d^4x d^4\theta \; \bar{\theta}^2 \mathrm{Tr} \left(\partial_{\beta\dot{\alpha}} \overline{\Gamma}_{\alpha}^{\;\dot{\alpha}} \right) \; \mathrm{Tr}\bar{\Phi} \; \mathrm{Tr}\Phi \\
& + t_3 \; \mathcal{F}^{\alpha\beta} \int d^4x d^4\theta \; \bar{\theta}^2 \mathrm{Tr} \left((\partial_{\beta\dot{\alpha}} \overline{\Gamma}_{\alpha}^{\;\dot{\alpha}} - \frac{i}{2} [\overline{\Gamma}_{\beta\dot{\alpha}}, \overline{\Gamma}_{\alpha}^{\;\dot{\alpha}}]_*) \; \{\bar{\Phi}, \Phi\} \right) \\
& + t_4 \; \mathcal{F}^{\alpha\beta} \int d^4x d^4\theta \; \bar{\theta}^2 \mathrm{Tr} \left((\partial_{\beta\dot{\alpha}} \overline{\Gamma}_{\alpha}^{\;\dot{\alpha}} - \frac{i}{2} [\overline{\Gamma}_{\beta\dot{\alpha}}, \overline{\Gamma}_{\alpha}^{\;\dot{\alpha}}]_*) \; \Phi \right) \; \mathrm{Tr}\bar{\Phi} \\
& + t_5 \; \mathcal{F}^{\alpha\beta} \int d^4x d^4\theta \; \bar{\theta}^2 \mathrm{Tr} \left((\partial_{\beta\dot{\alpha}} \overline{\Gamma}_{\alpha}^{\;\dot{\alpha}} - \frac{i}{2} [\overline{\Gamma}_{\beta\dot{\alpha}}, \overline{\Gamma}_{\alpha}^{\;\dot{\alpha}}]_*) \; \bar{\Phi} \right) \; \mathrm{Tr}\Phi \\
& + \sum_{j=1}^{18} \tilde{t}_6^{(j)} \, \mathcal{G}_j^{ABCDE} \, \mathcal{F}^2 \int d^4x d^4\theta \; \bar{\theta}^2 \, \overline{\Gamma}^{A\,\alpha\dot{\alpha}} \overline{\Gamma}^{B}_{\alpha\dot{\alpha}} \bar{\Phi}^C \bar{\Phi}^D \bar{\Phi}^E
\end{aligned} \tag{4.3.126}$$

and to the field–strength

$$S_{\overline{W}} = \sum_{j=1}^{12} l_j \, \mathcal{H}_j^{ABCD} \, \mathcal{F}^2 \int d^4x d^4\theta \; \bar{\theta}^2 \; \overline{W}^{A\,\dot{\alpha}} \overline{W}^{B}_{\dot{\alpha}} \Phi^C \bar{\Phi}^D \tag{4.3.127}$$

We have introduced the following group tensors to take into account all possible color structures (we use the shorten notation $\mathrm{Tr}(T^A) = (A)$ for any group matrix)

$$\begin{aligned}
&\mathcal{C}_1^{ABC} = (ABC) \quad \mathcal{C}_2^{ABC} = (AB)(C) \quad \mathcal{C}_3^{ABC} = (A)(B)(C) \\
&\\
&\mathcal{D}_1^{ABCD} = (ABCD) \quad \mathcal{D}_2^{ABCD} = (ACBD) \\
&\mathcal{D}_3^{ABCD} = (A)(BCD) \quad \mathcal{D}_4^{ABCD} = (C)(ABD) \\
&\mathcal{D}_5^{ABCD} = (AB)(CD) \quad \mathcal{D}_6^{ABCD} = (AC)(BD) \\
&\mathcal{D}_7^{ABCD} = (AB)(C)(D) \quad \mathcal{D}_8^{ABCD} = (AC)(B)(D) \quad \mathcal{D}_9^{ABCD} = (A)(B)(CD) \\
&\mathcal{D}_{10}^{ABCD} = (A)(B)(C)(D) \\
&\\
&\mathcal{E}_1^{ABCDE} = (ABCDE) \quad \mathcal{E}_2^{ABCDE} = (ABCD)(E) \quad \mathcal{E}_3^{ABCDE} = (BCDE)(A) \\
&\mathcal{E}_4^{ABCDE} = (ABC)(DE) \quad \mathcal{E}_5^{ABCDE} = (BCD)(AE) \quad \mathcal{E}_6^{ABCDE} = (ABC)(D)(E) \\
&\mathcal{E}_7^{ABCDE} = (BCD)(A)(E) \quad \mathcal{E}_8^{ABCDE} = (AB)(CD)(E) \quad \mathcal{E}_9^{ABCDE} = (BC)(DE)(A) \\
&\mathcal{E}_{10}^{ABCDE} = (A)(BC)(D)(E) \quad \mathcal{E}_{11}^{ABCDE} = (AB)(C)(D)(E) \\
&\mathcal{E}_{12}^{ABCDE} = (A)(B)(C)(D)(E) \\
&\\
&\mathcal{G}_1^{ABCDE} = (ABCDE) \quad \mathcal{G}_2^{ABCDE} = (ACBDE) \\
&\mathcal{G}_3^{ABCDE} = (ABCD)(E) \quad \mathcal{G}_4^{ABCDE} = (ACBD)(E) \quad \mathcal{G}_5^{ABCDE} = (BCDE)(A) \\
&\mathcal{G}_6^{ABCDE} = (ABC)(DE) \quad \mathcal{G}_7^{ABCDE} = (BCD)(AE) \quad \mathcal{G}_8^{ABCDE} = (AB)(CDE) \\
&\mathcal{G}_9^{ABCDE} = (ABC)(D)(E) \quad \mathcal{G}_{10}^{ABCDE} = (BCD)(A)(E) \quad \mathcal{G}_{11}^{ABCDE} = (A)(B)(CDE) \\
&\mathcal{G}_{12}^{ABCDE} = (AB)(CD)(E) \quad \mathcal{G}_{13}^{ABCDE} = (BC)(DE)(A) \quad \mathcal{G}_{14}^{ABCDE} = (BC)(AD)(E) \\
&\mathcal{G}_{15}^{ABCDE} = (A)(BC)(D)(E) \quad \mathcal{G}_{16}^{ABCDE} = (AB)(C)(D)(E) \\
&\mathcal{G}_{17}^{ABCDE} = (A)(B)(CD)(E) \quad \mathcal{G}_{18}^{ABCDE} = (A)(B)(C)(D)(E) \\
&\\
&\mathcal{H}_1^{ABCD} = (ABCD) \quad \mathcal{H}_2^{ABCD} = (ACBD) \\
&\mathcal{H}_3^{ABCD} = (A)(BCD) \quad \mathcal{H}_4^{ABCD} = (C)(ABD) \quad \mathcal{H}_5^{ABCD} = (D)(ABC) \\
&\mathcal{H}_6^{ABCD} = (AB)(CD) \quad \mathcal{H}_7^{ABCD} = (AC)(BD) \\
&\mathcal{H}_8^{ABCD} = (AB)(C)(D) \quad \mathcal{H}_9^{ABCD} = (AC)(B)(D) \quad \mathcal{H}_{10}^{ABCD} = (AD)(B)(C) \\
&\mathcal{H}_{11}^{ABCD} = (A)(B)(CD) \quad \mathcal{H}_{12}^{ABCD} = (A)(B)(C)(D)
\end{aligned} \tag{4.3.128}$$

Whenever in the action the $*$–product is not explicitly indicated the products are indeed ordinary products. This happens in most terms above because of the presence of the $\bar{\theta}^2$ factor.

4.3.5 One–loop renormalizability and gauge invariance

In this Section we will provide general arguments in support of the one–loop renormalizability of the action (4.3.124).

The action (4.3.124) has been obtained by including all possible divergent structures which can appear at one–loop. Therefore, one might be tempted to conclude that it is *a fortiori* renormalizable. However, some of these terms need enter particular linear combinations in order to insure gauge–invariance. Such terms are identified by couplings $(\kappa-1)$ and t_3, t_4, t_5. Therefore, proving one–loop renormalizability amounts to prove that quantum corrections maintain the correct gauge–invariant combinations. In what follows we will be mainly focused on these terms and find the conditions under which gauge invariance is maintained at quantum level.

In order to perform one–loop calculations we use the background–field method revised in Section 4.1 and Appendix B.1 and applied to the general action (4.3.124). In Appendix B.1 the necessary Feynman rules are collected.

When drawing possible divergent diagrams we make use of the following observations: First of all, from the dimensional analysis performed in Section 4.3.4, one-loop divergences may be proportional to the non–anticommutation parameter $\mathcal{F}$ at most quadratically. Therefore, we do not take into account diagrams which give higher powers of $\mathcal{F}$. Moreover, the structures we are mainly interested in (the ones associated to the couplings $(\kappa-1)$ and t_3, t_4, t_5) are proportional to $\mathcal{F}^{\alpha\beta}$, so they cannot receive corrections from diagrams which contain vertices proportional to $\mathcal{F}^2$.

For each supergraph we perform ∇–algebra [13, 15, 16, 17] in order to reduce it to an ordinary momentum graph and read the background structures associated to the divergent integrals. We discuss renormalizability of the different sectors, separately.

Pure gauge sector

In the absence of a superpotential term, the one–loop effective action for the gauge sector has been already computed in [18].

With the addition of the cubic superpotential and the related modifications of the classical action, the gauge effective action could, *a priori*, get corrected because of two different reasons: The modification of the chiral propagators to include different couplings for the abelian superfields which might affect the evaluation of Δ in (4.3.86), and the presence of new mixed gauge–chiral interaction vertices from S_{int} in (4.3.85) as coming from $S_{\overline{\Gamma}}$ and $S_{\bar{W}}$ and the second line of (4.3.125).

The former modification is harmless because of the reparametrization invariance of Δ under the change of variables $\mathbf{\Phi}^A \to \mathbf{\Phi}'^A \equiv (\mathbf{\Phi}^a, \kappa_1 \mathbf{\Phi}^0)$, $\bar{\mathbf{\Phi}}^A \to \bar{\mathbf{\Phi}}'^A \equiv (\bar{\mathbf{\Phi}}^a, \kappa_2 \bar{\mathbf{\Phi}}^0)$, $\kappa = \kappa_1\kappa_2$

$$\Delta = \int D\mathbf{\Phi} D\bar{\mathbf{\Phi}} \; \exp \int d^4x d^4\theta \, \left(\mathrm{Tr}\bar{\mathbf{\Phi}}\mathbf{\Phi} + \frac{\kappa-1}{N} \mathrm{Tr}\bar{\mathbf{\Phi}} \mathrm{Tr}\mathbf{\Phi} \right)$$
$$\sim \int D\mathbf{\Phi}' D\bar{\mathbf{\Phi}}' \; \exp \int d^4x d^4\theta \; \mathrm{Tr}\bar{\mathbf{\Phi}}'\mathbf{\Phi}' \qquad (4.3.129)$$

The κ–independence of Δ can be also checked by explicit calculations, noting that in its one–loop expansion abelian superfields never enter.

The other source of possible modifications for the gauge effective action is the appearance of new gauge–chiral vertices in $S_{\overline{\Gamma}}$ and $S_{\bar{W}}$, eqs. (4.3.126) and (4.3.127), and second line of (4.3.125). In any case the new vertices produce tadpole–like diagrams when contracting the matter superfields leaving gauge fields as background fields. After ∇–algebra, the tadpole provides the covariant propagator $1/\Box_{cov}$ which can be expanded as in (B.1.34) up to second order in $\overline{\Gamma}$ producing divergent contributions. It is easy to prove that these divergences cancel exactly as in the ordinary case.

We conclude that the addition of a cubic superpotential and related modifications does not change the results in [18] for the divergent part of the one–loop gauge effective action. Therefore, if we start with a classical action as the one in (4.2.77) or (4.2.78) we can multiplicatively renormalize all the divergences of the gauge sector (see Ref. [18] for the detailed calculation).

Gauge–matter sector

We now study one–loop divergent contributions to the rest of the action, i.e. $S_{matter} + S_{\overline{\Gamma}} + S_{\bar{W}}$ (see eqs. (4.3.125–4.3.127)). The contributions identified by the couplings $(\kappa - 1)$ and t_3, t_4, t_5, whose gauge invariance is under discussion, belong to this sector. Therefore, we concentrate primarily on this kind of terms.

Divergent contributions come from diagrams in Fig. 4.5 where internal lines are covariant gauge and chiral propagators (see eqs. (B.1.10, B.1.29–B.1.32)). Expanding the propagators in powers of the background superfields we find two, three and four–point divergences, whereas higher powers give rise to finite contributions.

We analyze the diagrams separately.

Diagram (4.5a)

Diagram (4.5a) is obtained by joining two vertices in Fig. (B.1a) by one chiral propagator $1/\Box_-$ and one vector propagator $1/\hat{\Box}$. Expanding the propagators at the lowest order, $1/\Box_-, 1/\hat{\Box} \sim 1/\Box$, we obtain the ordinary divergent quadratic term when the $*$–product at the vertices is neglected. Quadratic terms with a nontrivial dependence on $\mathcal{F}$ are finite. Instead, divergent three–point functions exhibiting a linear dependence on $\mathcal{F}$ come from the first order expansion of the propagators (see eqs.(B.1.17, B.1.34)). Their dependence on the NAC parameter comes either when expanding the $*$–product at the vertices or from $\mathcal{F}^{\alpha\beta}$ terms in eqs. (B.1.17, B.1.34). Combining all contributions, diagram (4.5a) gives rise to

$$\Gamma_2^{(1)}(g) + \Gamma_3^{(1)}(g) + \Gamma_3'^{(1)}(g) + \Gamma_4^{(1)}(g) \tag{4.3.130}$$

where

$$\Gamma_2^{(1)}(g) + \Gamma_3^{(1)}(g) = 2g^2\mathcal{S}\int d^4x d^4\theta \Big[- N\mathrm{Tr}\left(\bar{\mathbf{\Phi}}\mathbf{\Phi}\right) \tag{4.3.131}$$
$$+\mathrm{Tr}\bar{\mathbf{\Phi}} * \mathrm{Tr}\mathbf{\Phi} + 2i\bar{\theta}^2\mathcal{F}^{\alpha\beta}\ \mathrm{Tr}(\overline{\Gamma}_\alpha{}^{\dot{\alpha}} * \bar{\mathbf{\Phi}}) * \mathrm{Tr}(\partial_{\beta\dot{\alpha}}\mathbf{\Phi}) + 2i\bar{\theta}^2\mathcal{F}^{\alpha\beta}\ \mathrm{Tr}(\overline{\Gamma}_\alpha{}^{\dot{\alpha}} * \mathbf{\Phi}) * \mathrm{Tr}(\partial_{\beta\dot{\alpha}}\bar{\mathbf{\Phi}})\Big]$$

and

$$\Gamma_3'^{(1)}(g) = 4ig^2\mathcal{S}\mathcal{F}^{\alpha\beta}\int d^4x d^4\theta\ \bar{\theta}^2\mathrm{Tr}(\partial_{\beta\dot{\alpha}}\overline{\Gamma}_\alpha{}^{\dot{\alpha}})\mathrm{Tr}(\mathbf{\Phi}\bar{\mathbf{\Phi}}) \tag{4.3.132}$$

Four–point functions $\Gamma_4^{(1)}(g)$ come from the second order expansion of the product of the two propagators. They are divergent but always proportional to $\mathcal{F}^2$, therefore automatically gauge–invariant.

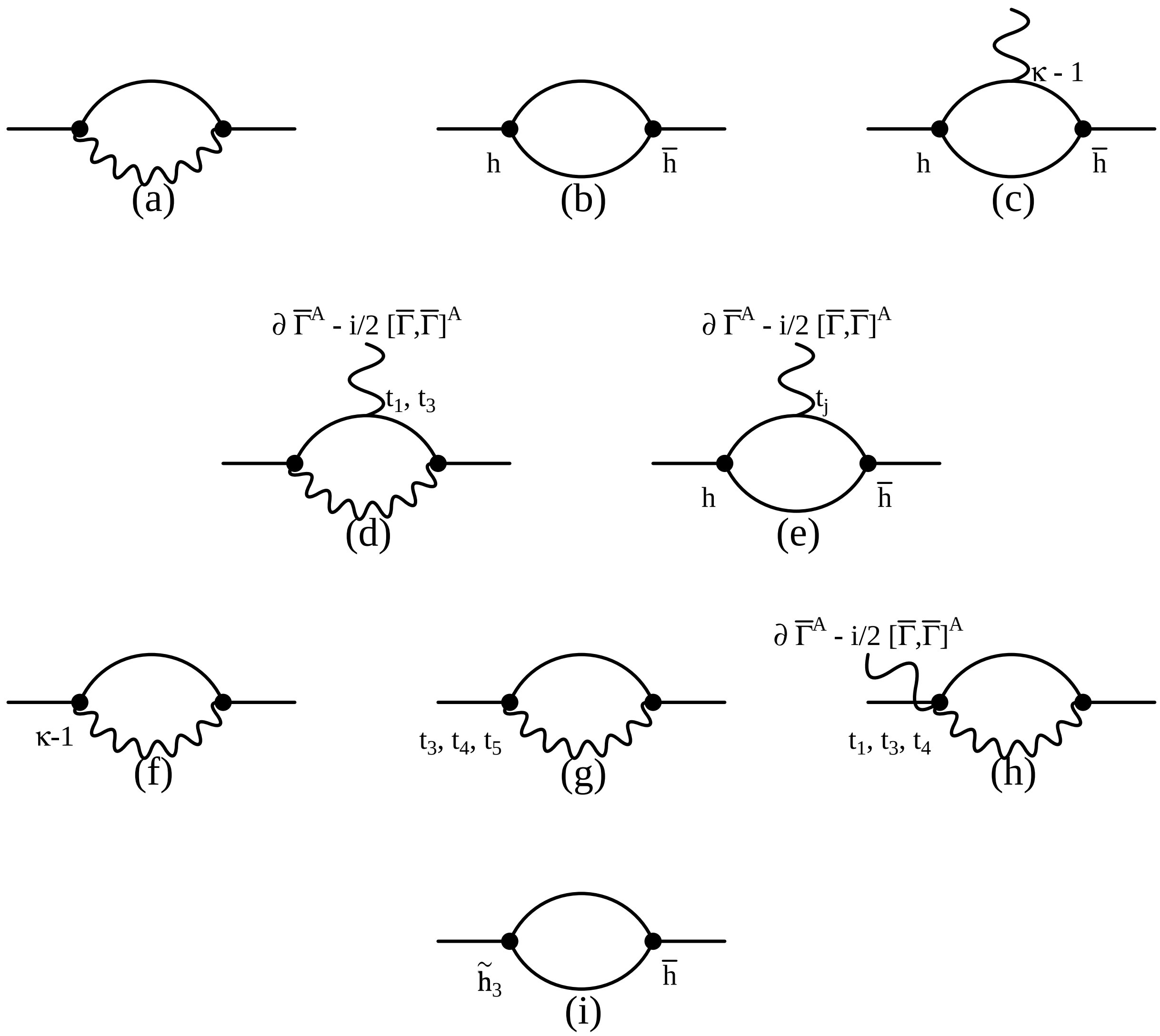

Figure 4.5: Master diagrams which, after the expansion of the covariant propagators, give rise to two, three and four–point divergent contributions.

We note that the divergences (4.3.131, 4.3.132) come in the right linear combinations for preserving gauge–invariance.

Diagrams (4.5b, 4.5c)

With the aim of discussing gauge invariance, it is convenient to consider the sum of diagrams (4.5b) and (4.5c). Diagram (4.5b) is obtained by joining one h and one $\bar{h}$ vertices in Fig. (B.1h, B.1j) by two $1/\Box_-$ propagators, whereas diagram (4.5c) is generated from diagram (4.5b) by the insertion of an extra $(\kappa - 1)$–vertex in Figs. (B.1c, B.1d). Expanding the chiral propagators at the lowest order, $1/\Box_- \sim 1/\Box$ and neglecting the $*$–product at the vertices, from diagram (4.5b) we obtain the ordinary divergent quadratic term and from diagram (4.5c) a three–point divergent contribution linear in $\mathcal{F}$. Further three and four–point contributions come from the higher order expansion of the propagators in both diagrams. In diagram (4.5b) the linear dependence in the NAC parameter comes either from terms in the propagator expansion or from the $*$–product at the vertices.

Combining all contributions, the sum of the two diagrams gives rise to

$$\Gamma_2^{(1)}(h,\bar{h}) + \Gamma_3^{(1)}(h,\bar{h}) + \Gamma_3'^{(1)}(h,\bar{h}) + \Gamma_4^{(1)}(h,\bar{h}) \tag{4.3.133}$$

where

$$\begin{aligned}\Gamma_2^{(1)}(h,\bar{h}) + \Gamma_3^{(1)}(h,\bar{h}) = \mathcal{S}\int d^4x d^4\theta \, \Big\{ 9h\overline{h}\left(N + 4\frac{1-\kappa}{N\kappa}\right) \mathrm{Tr}\left(\bar{\mathbf{\Phi}} * \mathbf{\Phi}\right) \\ + \, 9h\overline{h}\left(1 + 2\left(\frac{1-\kappa}{N\kappa}\right)^2\right)\Big[\mathrm{Tr}\bar{\mathbf{\Phi}} * \mathrm{Tr}\mathbf{\Phi} + 2i\bar{\theta}^2\mathcal{F}^{\alpha\beta}\ \mathrm{Tr}(\overline{\Gamma}_\alpha^{\ \dot{\alpha}} * \bar{\mathbf{\Phi}}) * \mathrm{Tr}(\partial_{\beta\dot{\alpha}}\mathbf{\Phi}) \\ + 2i\bar{\theta}^2\mathcal{F}^{\alpha\beta}\ \mathrm{Tr}(\overline{\Gamma}_\alpha^{\ \dot{\alpha}} * \mathbf{\Phi}) * \mathrm{Tr}\left(\partial_{\beta\dot{\alpha}}\bar{\mathbf{\Phi}}\right)\Big]\Big\}\end{aligned} \tag{4.3.134}$$

$$\begin{aligned}
\Gamma_3'^{(1)}(h,\overline{h}) =& \left[\frac{54}{N} - \frac{18}{\kappa N} - \frac{18}{\kappa_1 N} - \frac{18}{\kappa_2 N} - 36\frac{1-\kappa}{\kappa N}\right] i \\
& \times h\overline{h}\mathcal{S}\mathcal{F}^{\alpha\beta}\int d^4x d^4\theta\ \bar{\theta}^2 \mathrm{Tr}(\partial_{\beta\dot{\alpha}}\overline{\Gamma}_\alpha^{\ \dot{\alpha}}\{\mathbf{\Phi},\bar{\mathbf{\Phi}}\}) \\
& + \left[-\frac{36}{N^2} + \frac{36}{\kappa_1 N^2} + \frac{36}{\kappa N^2} - \frac{36}{\kappa_1\kappa N^2}\right] i \\
& \times h\overline{h}\mathcal{S}\mathcal{F}^{\alpha\beta}\int d^4x d^4\theta\ \bar{\theta}^2 \mathrm{Tr}(\partial_{\beta\dot{\alpha}}\overline{\Gamma}_\alpha^{\ \dot{\alpha}}\bar{\mathbf{\Phi}})\ \mathrm{Tr}\mathbf{\Phi} \\
& + \left[-\frac{36}{N^2} + \frac{36}{\kappa_2 N^2} + \frac{36}{\kappa N^2} - \frac{36}{\kappa_2\kappa N^2} - 36\left(\frac{1-\kappa}{\kappa N}\right)^2\right] i \\
& \times h\overline{h}\mathcal{S}\mathcal{F}^{\alpha\beta}\int d^4x d^4\theta\ \bar{\theta}^2 \mathrm{Tr}(\partial_{\beta\dot{\alpha}}\overline{\Gamma}_\alpha^{\ \dot{\alpha}}\mathbf{\Phi})\ \mathrm{Tr}\bar{\mathbf{\Phi}} \\
& + \left[-\frac{36}{N^2} + \frac{36}{\kappa_1 N^2} + \frac{36}{\kappa_2 N^2} - \frac{36}{\kappa N^2} - 72\left(\frac{1-\kappa}{\kappa N}\right)^2\right] i \\
& \times h\overline{h}\mathcal{S}\mathcal{F}^{\alpha\beta}\int d^4x d^4\theta\ \bar{\theta}^2 \mathrm{Tr}(\partial_{\beta\dot{\alpha}}\overline{\Gamma}_\alpha^{\ \dot{\alpha}})\ \mathrm{Tr}(\mathbf{\Phi}\bar{\mathbf{\Phi}}) \\
& + \left[\frac{36}{N^3}\frac{1-\kappa}{\kappa}\left(-1 + \frac{1}{\kappa_1} + \frac{1}{\kappa_2} - \frac{1}{\kappa}\right) - 72\left(\frac{1-\kappa}{\kappa N}\right)^3\right] i \\
& \times h\overline{h}\mathcal{S}\mathcal{F}^{\alpha\beta}\int d^4x d^4\theta\ \bar{\theta}^2 \mathrm{Tr}(\partial_{\beta\dot{\alpha}}\overline{\Gamma}_\alpha^{\ \dot{\alpha}})\ \mathrm{Tr}\mathbf{\Phi}\ \mathrm{Tr}\bar{\mathbf{\Phi}} \qquad (4.3.135)
\end{aligned}$$

and

$$\begin{aligned}
\Gamma_4^{(1)}(h,\overline{h}) =& -36\left(\frac{1-\kappa}{\kappa N}\right) h\overline{h}\mathcal{S}\mathcal{F}^{\alpha\beta}\int d^4x d^4\theta\ \bar{\theta}^2 \mathrm{Tr}([\overline{\Gamma}_{\beta\dot{\alpha}}, \overline{\Gamma}_\alpha^{\ \dot{\alpha}}]\{\mathbf{\Phi},\bar{\mathbf{\Phi}}\}) \\
& -36\left(\frac{1-\kappa}{\kappa N}\right)^2 h\overline{h}\mathcal{S}\mathcal{F}^{\alpha\beta}\int d^4x d^4\theta\ \bar{\theta}^2 \mathrm{Tr}([\overline{\Gamma}_{\beta\dot{\alpha}}, \overline{\Gamma}_\alpha^{\ \dot{\alpha}}]\mathbf{\Phi})\ \mathrm{Tr}\bar{\mathbf{\Phi}} \qquad (4.3.136)
\end{aligned}$$

We note that $\Gamma_2^{(1)}(h,\bar{h}) + \Gamma_3^{(1)}(h,\bar{h})$ gives a gauge–invariant correction to the quadratic action. On the other hand, in $\Gamma_3'^{(1)}(h,\bar{h})$ the first three lines are not gauge invariant. Possible gauge completions for these terms are contained in $\Gamma_4^{(1)}(h,\bar{h})$ if the corresponding factors satisfy the following constraints

$$-\frac{i}{2}\left[\frac{54}{N} - \frac{18}{\kappa N} - \frac{18}{\kappa_1 N} - \frac{18}{\kappa_2 N} - 36\frac{1-\kappa}{\kappa N}\right] i = -36\left(\frac{1-\kappa}{\kappa N}\right) \qquad (4.3.137)$$

$$-\frac{i}{2}\left[-\frac{36}{N^2} + \frac{36}{\kappa_1 N^2} + \frac{36}{\kappa N^2} - \frac{36}{\kappa_1\kappa N^2}\right] i = 0 \qquad (4.3.138)$$

$$-\frac{i}{2}\left[-\frac{36}{N^2} + \frac{36}{\kappa_2 N^2} + \frac{36}{\kappa N^2} - \frac{36}{\kappa_2\kappa N^2} - 36\left(\frac{1-\kappa}{\kappa N}\right)^2\right] i = -36\left(\frac{1-\kappa}{\kappa N}\right)^2 \qquad (4.3.139)$$

Having introduced two independent couplings κ_1, κ_2 we have the freedom to fix them in order to satisfy this set of equations. It is easy to see that a non–trivial solution is given by

$$\kappa_1 = 1 \qquad , \qquad \kappa_2 = \kappa \tag{4.3.140}$$

with no further requests on κ. Therefore, these conditions provide the right prescription for computing (4.5b,4.5c)–type contributions to the effective action while preserving background gauge invariance.

Given the solution (4.3.140) and recalling eq. (B.1.22) we conclude that the extra coupling in front of the abelian quadratic action origins entirely from a rescaling of the antichiral superfields.

Diagrams (4.5d)

Diagrams of type (4.5d) are obtained by inserting in diagram (4.5a) one t_1 or one t_3 vertex (the insertion of t_2, t_4, t_5 vertices would give diagrams with vanishing color factors). Expanding the propagators and considering only divergent terms linear in the deformation parameter, it is easy to see that the diagram with the insertion of one t_1 vertex gives divergent contributions of the form t_1, t_2 in $S_{\overline{\Gamma}}$, whereas the diagram with one t_3 vertex contributes to the t_1, t_3, t_4, t_5 structures. They all come out automatically in the right gauge–invariant combinations.

Diagrams (4.5e)

Diagrams of type (4.5e) are obtained by inserting in diagram (4.5b) one of the t_j vertices. Expanding the propagators and considering only divergent terms linear in the deformation parameter, from diagrams with t_1, t_2 vertices gauge–invariant structures associated to t_1 and t_2 in $S_{\overline{\Gamma}}$ arise. From diagrams with the insertion of vertices t_3, t_4, t_5 the background structure proportional to $\partial_{\beta}^{\dot{\alpha}} \overline{\Gamma}_{\alpha\dot{\alpha}}$ combines with the structure $[\overline{\Gamma}_{\beta}^{\dot{\alpha}}, \overline{\Gamma}_{\alpha\dot{\alpha}}]$ to give gauge–invariant divergent contributions of the form $t_1, \cdots, t_5$.

Diagrams (4.5f)

This kind of diagrams are obtained by contracting the $(\kappa - 1)$ vertices with a quantum gauge V–field (see Figs. (B.1e, B.1f, B.1g)) with the ordinary vertex in Fig. (B.1a). Expanding the covariant propagators it is easy to see that they are either vanishing or finite.

Diagrams (4.5g)

This class of diagrams is constructed by contracting a t_3, t_4, t_5–vertex in Fig. (B.1p) with the ordinary vertex (B.1a) (diagrams with t_1 and t_2 vertices vanish for color reasons). Explicit calculations reveal that nontrivial cancellations occur, so that no divergent contributions arise proportional to t_4 and t_5, whereas a non–vanishing term is generated by t_3 which is automatically in the right linear combination to respect gauge invariance. Precisely, it corrects t_1, t_3, t_4, t_5 couplings.

Diagrams (4.5h)

These diagrams are obtained by contracting one vertex (B.1o) with the ordinary vertex (B.1a). In all cases divergences arise when expanding the propagators at lowest order (self–energy diagrams). They are automatically gauge invariant and correct the t_1, t_3, t_4, t_5 couplings.

Diagram (4.5i)

Finally, possible divergent contributions come from contracting the $\tilde{h}_3$ vertex with the ordinary $\bar{h}$–vertex in Fig. (B.1j). They come from expanding the propagators up to the first order in $\overline{\Gamma}$. Even in this case non–trivial cancellations occur and the final result is the sum of non–vanishing, but gauge invariant contributions to the t_1, t_3, t_4, t_5 couplings.

The list of diagrams we have analyzed includes all possible divergent diagrams linear in the deformation parameter. Any other divergence is necessarily proportional to $\mathcal{F}^2$ and comes either from the expansion of the $*$–products in the previous diagrams or from new diagrams constructed from $\mathcal{F}^2$–vertices in (4.3.124). Since we know that any single $\mathcal{F}^2$ term is automatically gauge–invariant and appears in the action with its own coupling, we can immediately conclude that the $\mathcal{F}^2$ sector of the action is one–loop renormalizable.

In conclusion, we have provided evidence that the general action (4.3.124) is multiplicatively renormalizable. Its renormalization can be then performed by setting

$$
\begin{aligned}
&\Phi^a_B = Z^{\frac{1}{2}}\Phi^a \quad , \quad \bar{\Phi}^a_B = \bar{Z}^{\frac{1}{2}}\bar{\Phi}^a \\
&\Phi^0_B = Z^{\frac{1}{2}}\Phi^0 \quad , \quad \bar{\Phi}^0_B = \bar{Z}^{\frac{1}{2}}\bar{\Phi}^0 \\
&(\kappa-1)_B = Z_\kappa(\kappa-1) \\
&h_B = Z_h h \quad , \quad \bar{h}_B = Z_{\bar{h}}\bar{h} \\
&\tilde{h}_{3\,B} = Z_{\tilde{h}_3}\tilde{h}_3 \\
&h^{(j)}_{3\,B} = Z_{h^{(j)}_3}h^{(j)}_3 \quad , \quad h^{(j)}_{4\,B} = Z_{h^{(j)}_4}h^{(j)}_4 \quad , \quad h^{(j)}_{5\,B} = Z_{h^{(j)}_5}h^{(j)}_5 \\
&t_{n\,B} = Z_{t_n}t_n \qquad n=1,\dots,5 \\
&\tilde{h}^{(j)}_{6\,B} = Z_{\tilde{h}^{(j)}_6}\tilde{h}^{(j)}_6 \\
&l_{n\,B} = Z_{l_n}l_n \qquad n=1,\dots,12
\end{aligned}
\tag{4.3.141}
$$

where we have assigned the same renormalization function to the abelian and non–abelian scalar superfields.

We consider for instance the nontrivial renormalization of the quadratic matter action, first two lines of eq. (4.3.125). At one–loop, in terms of renormalized superfields, we can write

$$
\begin{aligned}
\Gamma_{1loop} \to &\int d^4x d^4\theta \,\Big\{ \left((Z\bar{Z})^{\frac{1}{2}} - 1 + \frac{a}{\epsilon}\right)\mathrm{Tr}\left(\bar{\Phi}*\Phi\right) + \\
&\left((Z\bar{Z})^{\frac{1}{2}}Z_\kappa - 1 + \frac{b}{\epsilon}\right)\frac{\kappa-1}{N}\Big[\mathrm{Tr}\bar{\Phi}*\mathrm{Tr}\Phi + 2i\bar{\theta}^2\mathcal{F}^{\alpha\beta}\mathrm{Tr}(\overline{\Gamma}_\alpha^{\ \dot{\alpha}}*\bar{\Phi})*\mathrm{Tr}(\partial_{\beta\dot{\alpha}}\Phi) \\
&\qquad + 2i\bar{\theta}^2\mathcal{F}^{\alpha\beta}\mathrm{Tr}(\overline{\Gamma}_\alpha^{\ \dot{\alpha}}*\Phi)*\mathrm{Tr}(\partial_{\beta\dot{\alpha}}\bar{\Phi})\Big]\Big\}
\end{aligned}
\tag{4.3.142}
$$

where, from eqs. (4.3.131, 4.3.134) we read

$$
\begin{aligned}
a &= \frac{1}{(4\pi)^2}\left[-2g^2N + 9h\bar{h}\left(N + 4\frac{1-\kappa}{\kappa N}\right)\right] \\
b &= \frac{1}{(4\pi)^2}\frac{1}{\kappa-1}\left[2g^2N + 9h\bar{h}\left(1 + 2\left(\frac{1-\kappa}{\kappa N}\right)^2\right)\right]
\end{aligned}
\tag{4.3.143}
$$

In order to cancel divergences we can set

$$Z = \bar{Z} = 1 - \frac{1}{(4\pi)^2}\frac{1}{\epsilon}\left[-2g^2N + 9h\bar{h}\left(N + 4\frac{1-\kappa}{\kappa N}\right)\right] \tag{4.3.144}$$
$$Z_\kappa = 1 + \frac{1}{(4\pi)^2}\frac{1}{\epsilon}\left[-2g^2N\frac{\kappa}{\kappa-1} + 9h\bar{h}N\left(\frac{\kappa-2}{\kappa-1}\right) - 18\frac{h\bar{h}}{\kappa^2 N}(2\kappa^2-\kappa-1)\right]$$

Different choices with $Z \neq \bar{Z}$ are also allowed.

Renormalization of the rest of the couplings then follows, accordingly.

4.4 An explicit case: the $U(1)$ theory

In the previous section we gave sufficient evidence for one–loop renormalizability, but the complete renormalization has not been carried out yet. In fact, due to the non-trivial group structure, the form of the action is quite complicated and the calculation of all one–loop divergent contributions would imply the evaluation of a large number of diagrams.

In order to avoid technical complications related to the group structure, in this section we focus on the $U_*(1)$ case [25]. The noncommutative $U_*(1)$ gauge theory is obtained from the non(anti)commutative $U(N)$ theory in the limit $N \to 1$. Despite the abelian nature of the generator algebra the resulting gauge theory is highly interacting as a consequence of the non(anti)commutative nature of the $*$–product.

In this case complications related to the different renormalization undergone by non–abelian and abelian superfields [22] are absent and the general structure of SYM theories with matter in the adjoint representation of the gauge group is rather simpler.

We first consider the case of a single matter superfield interacting with a cubic superpotential. We complete the one–loop renormalization of the theory and compute the corresponding beta–functions.

We then generalize the calculation to the case of three adjoint chiral superfields in interaction through the superpotential

$$h_1\int d^4x d^2\theta\ \Phi_1 * \Phi_2 * \Phi_3 - h_2\int d^4x d^2\theta\ \Phi_1 * \Phi_3 * \Phi_2 \tag{4.4.145}$$
$$+\bar{h}_1\int d^4x d^2\bar{\theta}\ \bar{\Phi}^1 * \bar{\Phi}^2 * \bar{\Phi}^3 - \bar{h}_2\int d^4x d^2\bar{\theta}\ \bar{\Phi}^1 * \bar{\Phi}^3 * \bar{\Phi}^2$$

For $h_1 = h_2, \bar{h}_1 = \bar{h}_2$ it exhibits a global $SU(3)$ invariance and can be interpreted as a nontrivial NAC deformation of the ordinary abelian $N = 4$ SYM theory. Turning on nonanticommutativity breaks $N = 4$ to $N = 1/2$. More generally, for $h_1 \neq h_2$ and/or $\bar{h}_1 \neq \bar{h}_2$ the $SU(3)$ symmetry is lost and the superpotential (4.4.145) describes the NAC generalization of a marginally deformed [26, 27] $N = 4$ SYM theory.

We find that at one–loop the theory with equal couplings is *finite* exactly like the ordinary $N = 4$ counterpart. Using perturbative arguments based on dimensional considerations and symmetries of the theory we provide evidence that the theory should be finite at all loop orders. On the other hand, in the presence of marginal deformations UV divergences arise which in general prevent the theory from being at a fixed point.

Both for the one and three–flavor cases we study the spectrum of fixed points and the RG flows in the parameter space. We find that nonanticommutativity always renders the fixed points IR and UV unstable. Compared to the ordinary case, we loose the IR stability of the fixed point corresponding to the free theory ($h = \bar{h} = 0$ and $h_1 = h_2, \bar{h}_1 = \bar{h}_2$). This is due to the fact that in the NAC case the parameter space gets enlarged and new directions appear which drive the theories away from the fixed point.

4.4.1 $U_*(1)$ NAC SYM theories

Specializing the results of [22] to the $U_*(1)$ case the most general renormalizable action for a NAC SYM theory with one self–interacting chiral superfield in the adjoint representation of the gauge group is given by (for simplicity we consider massless matter)

$$\begin{aligned}
S = & \frac{1}{2g^2}\int d^4x d^2\bar\theta \; \bar W^{\dot\alpha} * \bar W_{\dot\alpha} \\
& + \int d^4x d^4\theta \; \Phi * \bar\Phi + h\int d^4x d^2\theta \; \Phi_*^3 + \bar h \int d^4x d^2\bar\theta \; \bar\Phi_*^3 \\
& + it_1 \mathcal{F}^{\alpha\beta}\int d^4x d^4\theta \; \bar\theta^2 \; \partial_\alpha{}^{\dot\alpha}\overline{\Gamma}_{\beta\dot\alpha} * \Phi * \bar\Phi + t_2\mathcal{F}^2\int d^4x d^4\theta \; \bar\theta^2 \; \overline{\Gamma}^{\alpha\dot\alpha} * \overline{\Gamma}_{\alpha\dot\alpha} * \bar\Phi_*^3 \\
& + t_3\mathcal{F}^2\int d^4x d^4\theta \; \bar\theta^2 \; \bar W^{\dot\alpha} * \bar W_{\dot\alpha} * \Phi * \bar\Phi \\
& + h_3\mathcal{F}^2\int d^4x d^4\theta \; \bar\theta^2 \; \Phi * \nabla^2\Phi * \nabla^2\Phi \\
& + h_4\mathcal{F}^2\int d^4x d^4\theta \; \bar\theta^2 \; \nabla^2\Phi * \Phi * \bar\Phi_*^2 + h_5\mathcal{F}^2\int d^4x d^4\theta \; \bar\theta^2 \; \Phi * \bar\Phi_*^4
\end{aligned} \tag{4.4.146}$$

where $\Phi \equiv e_*^V * \phi * e_*^{-V}$, $\bar\Phi = \overline{\phi}$ are covariantly (anti)chiral superfields expressed in terms of ordinary (anti)chirals. We choose to indicate explicitly the $*$–product everywhere without distinguishing the cases where it actually coincides with the ordinary product. For example, it is easy to see that $\int d^4x d^2\bar\theta \; \bar\Phi_*^3 = \int d^4x d^2\bar\theta \; \bar\Phi^3$ up to superspace total derivatives.

We note that in contrast with the $SU(N)\otimes U(1)$ case [18] the pure gauge action contains only the NAC generalization of the standard quadratic term. In fact, it is easy to see that all the extra terms which need be taken into account in the $SU(N)\otimes U(1)$ case for insuring renormalizability and gauge invariance are identically zero in the $U_*(1)$ limit.

More generally, we consider a theory with three different flavors in the (anti)fundamental representation of $SU(3)$, still interacting through a cubic superpotential. Again, using the results of [22] the most general renormalizable action which respects two global $U(1)$ symmetries is

$$
\begin{aligned}
S = \frac{1}{2g^2} \int & d^4x d^2\bar\theta \ \bar W^{\dot\alpha} * \bar W_{\dot\alpha} + \int d^4x d^4\theta \ \Phi_i * \bar\Phi^i \\
& + \int d^4x d^2\theta \ \left(h_1 \Phi_1 * \Phi_2 * \Phi_3 - h_2 \Phi_1 * \Phi_3 * \Phi_2 \right) \\
& + \int d^4x d^2\bar\theta \ \left(\bar h_1 \bar\Phi^1 * \bar\Phi^2 * \bar\Phi^3 - \bar h_2 \bar\Phi^1 * \bar\Phi^3 * \bar\Phi^2 \right) \\
& + i t_1 \mathcal{F}^{\alpha\beta} \int d^4x d^4\theta \ \bar\theta^2 \ \partial_\alpha{}^{\dot\alpha} \overline\Gamma_{\beta\dot\alpha} * \Phi_i * \bar\Phi^i + t_2 \mathcal{F}^2 \int d^4x d^4\theta \ \bar\theta^2 \ \overline\Gamma^{\alpha\dot\alpha} * \overline\Gamma_{\alpha\dot\alpha} * \bar\Phi^1 * \bar\Phi^2 * \bar\Phi^3 \\
& + t_3 \mathcal{F}^2 \int d^4x d^4\theta \ \bar\theta^2 \ \bar W^{\dot\alpha} * \bar W_{\dot\alpha} * \Phi_i * \bar\Phi^i \\
& + \tilde h_3 \mathcal{F}^{\alpha\beta} \int d^4x d^4\theta \ \bar\theta^2 \ \nabla_\alpha \Phi_1 * \nabla_\beta \Phi_2 * \Phi_3 + h_3 \mathcal{F}^2 \int d^4x d^4\theta \ \bar\theta^2 \ \Phi_1 * \nabla^2 \Phi_2 * \nabla^2 \Phi_3 \\
& + h_4^{(=)} \mathcal{F}^2 \int d^4x d^4\theta \ \bar\theta^2 \ \sum_{i=1}^{3} \nabla^2 \Phi_i * \Phi_i * \bar\Phi^i * \bar\Phi^i \\
& + h_4^{(\neq)} \mathcal{F}^2 \int d^4x d^4\theta \ \bar\theta^2 \ \sum_{i<j} \nabla^2 \Phi_i * \Phi_j * \bar\Phi^i * \bar\Phi^j \\
& + h_5 \mathcal{F}^2 \int d^4x d^4\theta \ \bar\theta^2 \ \Phi_i * \bar\Phi^i * \bar\Phi^1 * \bar\Phi^2 * \bar\Phi^3
\end{aligned}
\qquad (4.4.147)
$$

in terms of covariantly (anti)chiral superfields Φ_i, $\bar\Phi^i$. We note that one extra coupling $\tilde h_3$ is allowed in this case which would be trivially zero in the action (4.4.146), for symmetry reasons. The two global $U(1)$ charges for the matter superfields are $(1,-1,0)$ and $(0,1,-1)$ respectively, whereas antichiral superfields carry opposite charges.

The two actions are invariant under the following gauge transformations

$$
\begin{aligned}
\delta\Phi_i = i[\overline\Lambda, \Phi_i]_* \qquad &, \qquad \delta\bar\Phi^i = i[\overline\Lambda, \bar\Phi^i]_* \\
\delta\overline\Gamma_{\alpha\dot\alpha} = [\overline\nabla_{\alpha\dot\alpha}, \overline\Lambda]_* \qquad &, \qquad \delta\overline W_{\dot\alpha} = i[\overline\Lambda, \overline W_{\dot\alpha}]_*
\end{aligned}
\qquad (4.4.148)
$$

We note that except for the transformation of $\overline\Gamma$ the right hand sides vanish when $\mathcal{F}^{\alpha\beta} = 0$, as it should in the ordinary $U(1)$ case (when taking the commutative limit matter in the adjoint representation of $U_*(1)$ is mapped into $U(1)$ singlets).

In general, the cubic superpotential of (4.4.147) is a function of four independent couplings $h_1, h_2, \bar h_1, \bar h_2$. If we set $h_1 = h_2$ and $\bar h_1 = \bar h_2$ the action (4.4.147) has a global $SU(3)$ invariance which can be thought of as related to the R–symmetry of an ordinary $N = 4$ SYM theory. Therefore, we study the theory (4.4.147) as a non–trivial NAC deformation of the abelian $N = 4$ SYM [1]. We note that, while the ordinary $U(1)$ $N = 4$ theory is a free theory of one vector superfield plus three chiral gauge singlets in the fundamental of $SU(3)$, the NAC deformation we propose is highly interacting.

[1] At classical level, the NAC generalization of $N = 4$ SYM theories has been studied in [28] starting from an action which is the ordinary $N = 4$ action with products promoted to $*$–products.

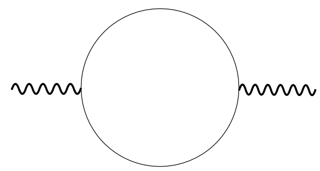

Figure 4.6: Gauge self–energy diagram.

More generally, if we set $h_1 = he^{i\pi\beta}, h_2 = he^{-i\pi\beta}$ and $\bar{h}_1 = \bar{h}e^{-i\pi\bar{\beta}}, \bar{h}_2 = \bar{h}e^{i\pi\bar{\beta}}$ only the two global $U(1)$'s survive and we have the NAC generalization of beta–deformed theories [27]. We note that, being the theory in euclidean space with strictly real matter superfields, we need take the deformation parameters $\beta, \bar{\beta}$ to be pure imaginary in order to guarantee the reality of the action. In the ordinary anticommutative case supersymmetric theories with pure imaginary β have been studied in [29, 30, 31].

Both in the $N = 4$ case and in its less supersymmetric marginal deformations, supersymmetry is broken to $N = 1/2$ by the NAC superspace structure.

4.4.2 One flavor case: Renormalization and β–functions

We first concentrate on the theory described by the action (4.4.146) and perform one–loop renormalization.

Using Feynman rules listed in the Appendix we draw all possible one–loop divergent diagrams. A useful selection rule arises by looking at the overall power of the NAC parameter for a given diagram. In fact, as it is clear from the dimensional analysis of Refs. [18, 22] divergent contributions can be at most quadratic in $\mathcal{F}^{\alpha\beta}$. Since powers of $\mathcal{F}$ come from vertices and from the expansion of covariant propagators (see eqs. (B.2.50), (B.2.65)) it is easy to count the overall power of the NAC parameter and withdraw diagrams with too many $\mathcal{F}$'s.

According to standard D–algebra arguments, in the NAC case as in the ordinary one divergent contributions to the gauge effective action come only from diagrams with a chiral matter/ghost quantum loop [18]. For the $U_*(1)$ theory the only potentially divergent contribution comes from the two–point diagram in Fig. 4.6 with interaction vertices arising from the expansion (B.2.50) of the covariant chiral propagator. Being the vertices of order $\mathcal{F}$ the result would be of order $\mathcal{F}^2$. Since dimensional analysis does not allow for self–energy divergent contributions proportional to the NAC parameter we expect the divergent part of this diagram to vanish. In fact, by direct inspection it is easy to see that after D–algebra it reduces to a tadpole thus giving a vanishing contribution in dimensional regularization. Therefore, the gauge action does not receive any one–loop contributions. This is consistent with the result of [18] specialized to the $N = 1$ case.

We then concentrate on the renormalization of the gauge/matter part of the action (4.4.146). Using Feynman rules in the Appendix we select diagrams in Figs. 4.7, 4.8 as the only one–loop divergent diagrams. Diagrams $(4.7a, 4.7c, 4.7d, 4.7e)$ are obtained from diagram $(4.8a)$ by expanding $1/\Box_{cov}$ as in (B.2.50) and writing $\overline{W} \sim D\overline{\Gamma}$. All internal lines are associated to ordinary $1/\Box$ propagators.

By direct calculation it turns out that diagrams (4.7d) and (4.7e) cancel one against the other whereas the rest, after performing D–algebra, leads to the following one–loop effective action

$$
\begin{aligned}
\Gamma^{(1)}_{div} = & \frac{1}{2g^2}\int d^4x d^2\bar\theta\ \bar{\mathbf{W}}^{\dot\alpha} * \bar{\mathbf{W}}_{\dot\alpha} + \int d^4x d^4\theta\ \mathbf{\Phi} * \bar{\mathbf{\Phi}}\ \left[1 + 18h\bar h\,\mathcal{S}\right] \\
& + h\int d^4x d^2\theta\ \mathbf{\Phi}^3_* + \bar h\int d^4x d^2\bar\theta\ \bar{\mathbf{\Phi}}^3_* \\
& + i\,\mathcal{F}^{\alpha\beta}\int d^4x d^4\theta\ \bar\theta^2\ \partial_\alpha^{\ \dot\alpha}\bar{\mathbf{\Gamma}}_{\beta\dot\alpha} * \mathbf{\Phi} * \bar{\mathbf{\Phi}}\ \left[t_1 + 36\,(h\bar h - h\bar h t_1)\,\mathcal{S}\right] \\
& + t_2\,\mathcal{F}^2\int d^4x d^4\theta\ \bar\theta^2\ \bar{\mathbf{\Gamma}}^{\alpha\dot\alpha} * \bar{\mathbf{\Gamma}}_{\alpha\dot\alpha} * \bar{\mathbf{\Phi}}^3_* \\
& + \mathcal{F}^2\int d^4x d^4\theta\ \bar\theta^2\ \bar{\mathbf{W}}^{\dot\alpha} * \bar{\mathbf{W}}_{\dot\alpha} * \mathbf{\Phi} * \bar{\mathbf{\Phi}}\ \left[t_3 + 36\,(h\bar h - h\bar h t_3 - h t_2)\,\mathcal{S}\right] \\
& + \mathcal{F}^2\int d^4x d^4\theta\ \bar\theta^2\ \mathbf{\Phi} * (\nabla^2\mathbf{\Phi})^2_*\ \left[h_3 + (12\,g^2 h - 12\,g^2 t_1 h + 3\,g^2 t_1^2 h + 6\,h h_4)\,\mathcal{S}\right] \\
& + \mathcal{F}^2\int d^4x d^4\theta\ \bar\theta^2\ \nabla^2\mathbf{\Phi} * \mathbf{\Phi} * \bar{\mathbf{\Phi}}^2_*\ \left[h_4 + (72\,h\bar h g^2 t_1 - 36\,h\bar h g^2 t_1^2 + 648\,h_3 h\bar h^2\right. \\
& \qquad\qquad \left. + 324\,h^2\bar h^2 - 144\,h\bar h h_4 + 36\,h h_5)\,\mathcal{S}\right] \\
& + \mathcal{F}^2\int d^4x d^4\theta\ \bar\theta^2\ \mathbf{\Phi} * \bar{\mathbf{\Phi}}^4_*\ \left[h_5 + (108\,h\bar h^2 g^2 t_1^2 + 216\,h\bar h^2 h_4 - 144\,h\bar h h_5)\,\mathcal{S}\right]
\end{aligned} \qquad (4.4.149)
$$

where $\mathcal{S}$ is given in (A.2.9).

Few comments are in order. First of all we note that the matter quadratic term does not receive gauge contributions. This is consistent with the results of Ref. [22] where it was already shown that the abelian gauge quadratic term does not correct by terms proportional to g^2. The superpotential does not renormalize thanks to the non–renormalization theorem which holds also in the NAC case. A similar behavior is exhibited by the t_2–term which, at least at one–loop, seems to be protected from renormalization. However, in this case we do not have any argument for expecting such a protection beyond one–loop.

We now proceed to the renormalization of the theory by defining renormalized coupling constants as

$$
\begin{array}{lll}
\Phi = Z^{-\frac12}\Phi_B & \bar\Phi = \bar Z^{-\frac12}\bar\Phi_B & \\
g = \mu^{-\epsilon}Z_g^{-1}g_B & h = \mu^{-\epsilon}Z_h^{-1}h_B & \bar h = \mu^{-\epsilon}Z_{\bar h}^{-1}\bar h_B \\
t_1 = Z_{t_1}^{-1}t_{1\,B} & t_2 = \mu^{-\epsilon}Z_{t_2}^{-1}t_{2\,B} & t_3 = Z_{t_3}^{-1}t_{3\,B} \\
h_3 = \mu^{-\epsilon}Z_{h_3}^{-1}h_{3\,B} & h_4 = \mu^{-2\epsilon}Z_{h_4}^{-1}h_{4\,B} & h_5 = \mu^{-3\epsilon}Z_{h_5}^{-1}h_{5\,B}
\end{array} \qquad (4.4.150)
$$

where powers of the renormalization mass μ have been introduced in order to deal with dimensionless renormalized couplings. In order to cancel the divergences in (4.4.149) we set

$$
\begin{aligned}
&Z = \bar{Z} = 1 - 18\,\frac{h\bar{h}}{(4\pi)^2}\,\frac{1}{\epsilon} \\
&Z_h h = h + 27\,\frac{h^2\bar{h}}{(4\pi)^2}\,\frac{1}{\epsilon} \;\equiv\; h + \frac{h^{(1)}}{\epsilon} \\
&Z_{\bar{h}} \bar{h} = \bar{h} + 27\,\frac{h\bar{h}^2}{(4\pi)^2}\,\frac{1}{\epsilon} \;\equiv\; \bar{h} + \frac{\bar{h}^{(1)}}{\epsilon} \\
&Z_{h_3}\, h_3 = h_3 + \frac{27\, h\bar{h}h_3 - 12\, g^2 h + 12\, g^2 h t_1 - 3\, g^2 h t_1^2 - 6\, h h_4}{(4\pi)^2}\,\frac{1}{\epsilon} \;\equiv\; h_3 + \frac{h_3^{(1)}}{\epsilon} \\
&Z_{t_1}\, t_1 = t_1 + 18\,(3\,t_1 - 2)\,\frac{h\bar{h}}{(4\pi)^2}\,\frac{1}{\epsilon} \;\equiv\; t_1 + \frac{t_1^{(1)}}{\epsilon} \\
&Z_{t_2}\, t_2 = t_2 + \frac{27\, t_2 h\bar{h}}{(4\pi)^2}\,\frac{1}{\epsilon} \;\equiv\; t_2 + \frac{t_2^{(1)}}{\epsilon} \\
&Z_{t_3}\, t_3 = t_3 + \frac{54\, t_3 h\bar{h} - 36\, h\bar{h} + 36\, h t_2}{(4\pi)^2}\,\frac{1}{\epsilon} \;\equiv\; t_3 + \frac{t_3^{(1)}}{\epsilon} \\
&Z_{h_4}\, h_4 = h_4 + \frac{180\, h\,\bar{h}\, h_4 - 36\, h h_5 + 36\, h\bar{h} g^2 t_1^2 - 72\, h\bar{h} g^2 t_1 - 648\, h_3 h\bar{h}^2 - 324\,(h\bar{h})^2}{(4\pi)^2}\,\frac{1}{\epsilon} \\
&\qquad \equiv\; h_4 + \frac{h_4^{(1)}}{\epsilon} \\
&Z_{h_5}\, h_5 = h_5 - \frac{108\, h\bar{h}^2 g^2 t_1^2 + 216\, h\bar{h}^2 h_4 - 189\, h\bar{h} h_5}{(4\pi)^2}\,\frac{1}{\epsilon} \;\equiv\; h_5 + \frac{h_5^{(1)}}{\epsilon}
\end{aligned}
\tag{4.4.151}
$$

We have chosen to renormalize the chiral and the antichiral superfields in the same way, although this is not forced by any symmetry of the theory. We note that divergences can be cancelled without renormalizing the NAC parameter $\mathcal{F}^{\alpha\beta}$. Therefore, the star product does not get deformed by quantum corrections.

We compute the beta–functions according to the general prescription

$$
\beta_{\lambda_j} = -\epsilon\, \alpha_j\, \lambda_j - \alpha_j\, \lambda_j^{(1)} + \sum_i \left(\alpha_i\, \lambda_i \frac{\partial \lambda_j^{(1)}}{\partial \lambda_i} \right)
\tag{4.4.152}
$$

where λ_j stands for any coupling of the theory and α_j is its naive dimension. Reading the single pole coefficients $\lambda_j^{(1)}$ in eq. (4.4.151) we finally obtain

$$
\begin{aligned}
\beta_g &= 0 \\
\beta_h &= \frac{54\, h^2\, \bar{h}}{(4\pi)^2} \\
\beta_{\bar{h}} &= \frac{54\, h\, \bar{h}^2}{(4\pi)^2} \\
\beta_{h_3} &= \frac{1}{(4\pi)^2}\left(54\, h\bar{h}h_3 - 24\, g^2 h + 24\, g^2 h t_1 - 6\, g^2 h t_1^2 - 12\, h h_4\right) \\
\beta_{t_1} &= \frac{36}{(4\pi)^2}\,(3\, t_1 - 2)\, h\bar{h} \\
\beta_{t_2} &= \frac{54\, t_2 h\bar{h}}{(4\pi)^2} \\
\beta_{t_3} &= \frac{1}{(4\pi)^2}\left(108\, t_3 h\bar{h} - 72\, h\bar{h} + 72\, h t_2\right) \\
\beta_{h_4} &= \frac{1}{(4\pi)^2}\left(72\, h\bar{h}g^2 t_1^2 - 144\, h\bar{h}g^2 t_1 - 1296\, h_3 h\bar{h}^2 - 648\,(h\bar{h})^2 + 360\, h\bar{h}h_4 - 72\, h h_5\right) \\
\beta_{h_5} &= \frac{1}{(4\pi)^2}\left(-216\, h\bar{h}^2 g^2 t_1^2 - 432\, h\bar{h}^2 h_4 + 378\, h\bar{h}h_5\right)
\end{aligned}
\tag{4.4.153}
$$

4.4.3 Three–flavor case: Renormalization and β-functions

In this Section we consider the case of the NAC $U_*(1)$ gauge theory in interaction with matter in the adjoint representation of the gauge group and in the fundamental representation of a flavor $SU(3)$ group. Its action is given in (4.4.147). We note that in the case $h_1 = h_2, \bar{h}_1 = \bar{h}_2$, setting $\mathcal{F}^{\alpha\beta} = 0$ turns off all the interactions and we are back to the ordinary free U(1) $N = 4$ SYM theory. On the other hand, the noncommutative nature of the star product allows us to construct even in the "abelian" case non–trivial interacting theories which can be studied as NAC deformations of $N = 4$ SYM. More generally, we will consider $h_1 \neq h_2, \bar{h}_1 \neq \bar{h}_2$ in order to take into account also marginal deformations.

We perform one–loop renormalization of the theory. Comparing to the case of a single chiral field, we note that the couplings are exactly of the same kind but dressed by flavor except for the extra coupling $\tilde{h}_3$ which in the previous case was trivially zero. Therefore, in order to evaluate divergent diagrams, it is sufficient to generalize the previous calculations to take into account non–trivial flavor factors and add possible new contributions arising from the contraction of a $\tilde{h}_3$ vertex with the rest. Since the $\tilde{h}_3$ vertex has the same structure of the vertex obtained when first order expanding the $*$–product in the superpotential (see vertices $(B.2f)$ and $(B.2h)$ in (B.2.65)), the topologies of divergent diagrams are still the ones in Fig. 4.7, 4.8.

From a direct evaluation of all the contributions, for the one–loop divergent part of the effective action we find (in order to shorten the notation we define $h_{12} \equiv h_1 - h_2$ and $\bar{h}_{12} \equiv \bar{h}_1 - \bar{h}_2$)

$$\begin{aligned}
\Gamma^{(1)}_{div} &= \frac{1}{2g^2}\int d^4x d^2\bar\theta\ \overline{W}^{\dot\alpha}\overline{W}_{\dot\alpha} + \int d^4x d^2\theta\ \Phi_i\bar\Phi^i\ \left[1 + h_{12}\bar h_{12}\,\mathcal{S}\right] \qquad (4.4.154)\\
&+h_1\int d^4x d^2\theta\ \Phi_1\Phi_2\Phi_3 - h_2\int d^4x d^2\theta\ \Phi_1\Phi_3\Phi_2\\
&+\bar h_1\int d^4x d^2\bar\theta\ \bar\Phi^1\bar\Phi^2\bar\Phi^3 - \bar h_2\int d^4x d^2\theta\ \bar\Phi_1\bar\Phi_3\bar\Phi_2\\
&+\tilde h_3\,\mathcal{F}^{\alpha\beta}\int d^4x d^4\theta\ \bar\theta^2\ \nabla_\alpha\Phi_1 * \nabla_\beta\Phi_2 * \Phi_3\\
&+\mathcal{F}^2\int d^4x d^4\theta\ \bar\theta^2\ \Phi_1\nabla^2\Phi_2\nabla^2\Phi_3\left[h_3 + \left(12\,g^2h_{12} - 6\,g^2t_1h_{12} + 3\,g^2t_1^2h_{12} + 3h_{12}h_4^{(\neq)}\right)\,\mathcal{S}\right]\\
&+i\,\mathcal{F}^{\alpha\beta}\int d^4x d^4\theta\ \bar\theta^2\ \partial_\alpha{}^{\dot\alpha}\overline{\Gamma}_{\beta\dot\alpha}\Phi_i\bar\Phi^i\left[t_1 + 2\,h_{12}\bar h_{12}(1-t_1)\mathcal{S}\right]\\
&+t_2\,\mathcal{F}^2\int d^4x d^4\theta\ \bar\theta^2\ \overline{\Gamma}^{\alpha\dot\alpha}\overline{\Gamma}_{\alpha\dot\alpha}\bar\Phi^1\bar\Phi^2\bar\Phi^3\\
&+\mathcal{F}^2\int d^4x d^4\theta\ \bar\theta^2\ \bar W^{\dot\alpha}\bar W_{\dot\alpha}\Phi_i\bar\Phi^i\left[t_3 + 2\left(h_{12}\bar h_{12} - h_{12}\bar h_{12}t_3 - h_{12}t_2\right)\mathcal{S}\right]\\
&+\mathcal{F}^2\int d^4x d^4\theta\ \bar\theta^2\ \sum_i \nabla^2\Phi_i\Phi_i\bar\Phi^i\bar\Phi^i\Big\{h_4^{(=)} + \Big[(h_3+\tilde h_3)h_{12}\bar h_{12}^2\\
&\qquad -2\,h_1h_2\bar h_{12}^2 - 2\,h_{12}\bar h_{12}\,h_4^{(\neq)} + h_{12}h_5 - \frac{1}{2}\left(\tilde h_3^2 + 2\,(h_1+h_2)\tilde h_3\right)\bar h_{12}^2\Big]\,\mathcal{S}\Big\}\\
&+\mathcal{F}^2\int d^4x d^4\theta\ \bar\theta^2\ \sum_{i<j}\nabla^2\Phi_i\Phi_j\bar\Phi^i\bar\Phi^j\Big\{h_4^{(\neq)} + \Big[8h_{12}\bar h_{12}g^2t_1 - 4h_{12}\bar h_{12}g^2t_1^2 - 8h_{12}\bar h_{12}h_4^{(=)}\\
&\qquad +2(h_1^2+h_2^2)\bar h_{12}^2 + 2(h_3+\tilde h_3)h_{12}\bar h_{12}^2 + \left(\tilde h_3^2 + 2(h_1+h_2)\tilde h_3\right)\bar h_{12}^2\\
&\qquad -4h_{12}\bar h_{12}h_4^{(\neq)} + 4h_{12}h_5\Big]\,\mathcal{S}\Big\}\\
&+\mathcal{F}^2\int d^4x d^4\theta\ \bar\theta^2\ \Phi_i\bar\Phi^i\bar\Phi^1\bar\Phi^2\bar\Phi^3\Big[h_5 + \Big(4\,h_{12}\bar h_{12}^2g^2t_1^2 + 2\,h_{12}\bar h_{12}^2h_4^{(=)}\\
&\qquad +3\,h_{12}\bar h_{12}^2h_4^{(\neq)} - 6\,h_{12}\bar h_{12}h_5\Big)\,\mathcal{S}\Big]
\end{aligned}$$

As in the previous case the gauge sector of the theory does not receive divergent contributions. Moreover, the quadratic matter action does not receive contributions from quantum gauge fields.

Renormalization is still performed by using renormalized field functions and coupling constants as defined in (4.4.150). Choosing the same renormalization constants for the three (anti)chiral superfields, in minimal subtraction scheme we set

$$
\begin{aligned}
&Z_i = \bar{Z}_i = 1 - \frac{h_{12}\bar{h}_{12}}{(4\pi)^2}\frac{1}{\epsilon} \\
&Z_{h_1} = Z_{\bar{h}_1} = Z_{h_2} = Z_{\bar{h}_2} = Z_{\tilde{h}_3} = 1 + \frac{3\,h_{12}\bar{h}_{12}}{2(4\pi)^2}\frac{1}{\epsilon} \\
&Z_{h_3}\,h_3 = h_3 + \frac{3\,h_3 h_{12}\bar{h}_{12} - 24\,g^2 h_{12} + 12\,g^2 t_1 h_{12} - 6\,g^2 t_1^2 h_{12} - 6\,h_{12}h_4^{(\neq)}}{2(4\pi)^2}\frac{1}{\epsilon} \\
&Z_{t_1}\,t_1 = t_1 + (3\,t_1 - 2)\,\frac{h_{12}\bar{h}_{12}}{(4\pi)^2}\frac{1}{\epsilon} \\
&Z_{t_2}\,t_2 = t_2 + \frac{3\,h_{12}\bar{h}_{12}t_2}{2(4\pi)^2}\frac{1}{\epsilon} \\
&Z_{t_3}\,t_3 = t_3 + \frac{3\,h_{12}\bar{h}_{12}t_3 - 2\,h_{12}\bar{h}_{12} + 2\,h_{12}t_2}{(4\,\pi)^2}\frac{1}{\epsilon} \\
&Z_{h_4^{(=)}}\,h_4^{(=)} = h_4^{(=)} + \frac{1}{(4\pi)^2}\Big[2\,h_{12}\bar{h}_{12}h_4^{(\neq)} - h_{12}h_5 + 2\,h_{12}\bar{h}_{12}h_4^{(=)} \\
&\qquad\qquad +2\,h_1\,h_2\,\bar{h}_{12}^2 - (h_3 + \tilde{h}_3)h_{12}\bar{h}_{12}^2 + \frac{1}{2}\left(\tilde{h}_3^2 + 2\,(h_1{+}h_2)\tilde{h}_3\right)\bar{h}_{12}^2\Big]\frac{1}{\epsilon} \\
&Z_{h_4^{(\neq)}}\,h_4^{(\neq)} = h_4^{(\neq)} + \frac{1}{(4\pi)^2}\Big[4\,h_{12}\bar{h}_{12}g^2 t_1^2 - 8\,h_{12}\bar{h}_{12}g^2 t_1 - 2(h_1^2 + h_2^2)\bar{h}_{12}^2 + 8\,h_{12}\bar{h}_{12}h_4^{(=)} \\
&\qquad +6\,h_{12}\bar{h}_{12}h_4^{(\neq)} - 4\,h_{12}h_5 - 2(h_3 + \tilde{h}_3)h_{12}\bar{h}_{12}^2 - \left(\tilde{h}_3^2 + 2\,(h_1{+}h_2)\tilde{h}_3\right)\bar{h}_{12}^2\Big]\frac{1}{\epsilon} \\
&Z_{h_5}\,h_5 = h_5 - \frac{1}{(4\pi)^2}\left(4\,h_{12}\bar{h}_{12}^2 g^2 t_1^2 + 2\,h_{12}\bar{h}_{12}^2 h_4^{(=)} + 3\,h_{12}\bar{h}_{12}^2 h_4^{(\neq)} - \frac{17}{2}\,h_{12}\bar{h}_{12}\bar{h}_5\right)\frac{1}{\epsilon}
\end{aligned}
\tag{4.4.155}
$$

Finally, applying the prescription (4.4.152) we find the beta–functions of the theory

$$\beta_g = 0$$

$$\beta_{h_1} = \frac{3}{(4\pi)^2} h_1 h_{12}\bar{h}_{12} \qquad \beta_{\bar{h}_1} = \frac{3}{(4\pi)^2}\bar{h}_1 h_{12}\bar{h}_{12}$$

$$\beta_{h_2} = -\frac{3}{(4\pi)^2} h_2 h_{12}\bar{h}_{12} \qquad \beta_{\bar{h}_2} = -\frac{3}{(4\pi)^2}\bar{h}_2 h_{12}\bar{h}_{12}$$

$$\beta_{\tilde{h}_3} = \frac{3}{(4\pi)^2}\, h_{12}\bar{h}_{12}\tilde{h}_3$$

$$\beta_{h_3} = \frac{1}{(4\,\pi)^2}\left(3\, h_{12}\bar{h}_{12}h_3 - 24 g^2 h_{12} + 12\, g^2 t_1 h_{12} - 6\, g^2 t_1^2 h_{12} - 6\, h_{12} h_4^{(\neq)}\right)$$

$$\beta_{t_1} = \frac{2}{(4\pi)^2}(3\, t_1 - 2)\, h_{12}\bar{h}_{12}$$

$$\beta_{t_2} = \frac{3}{(4\pi)^2}\, h_{12}\bar{h}_{12} t_2$$

$$\beta_{t_3} = \frac{1}{(4\pi)^2}\left(6\, h_{12}\bar{h}_{12} t_3 - 4\, h_{12}\bar{h}_{12} + 4\, h_{12} t_2\right)$$

$$\beta_{h_4^{(=)}} = \frac{1}{(4\,\pi)^2}\Big[4\, h_{12}\bar{h}_{12} h_4^{(\neq)} + 4\, h_{12}\bar{h}_{12} h_4^{(=)} - 2\, h_{12} h_5 + 4\, h_1 h_2 \bar{h}_{12}^2$$
$$-2\,(h_3 + \tilde{h}_3) h_{12}\bar{h}_{12}^2 + \left(\tilde{h}_3^2 + 2\,(h_1 + h_2)\tilde{h}_3\right)\bar{h}_{12}^2\Big]$$

$$\beta_{h_4^{(\neq)}} = \frac{1}{(4\,\pi)^2}\Big[8\, h_{12}\bar{h}_{12} g^2 t_1^2 - 16\, h_{12}\bar{h}_{12} g^2 t_1 + 16\, h_{12}\bar{h}_{12} h_4^{(=)}$$
$$+12\, h_{12}\bar{h}_{12} h_4^{(\neq)} - 8\, h_{12} h_5 - 4\,(h_1^2 + h_2^2)\bar{h}_{12}^2$$
$$-4\,(h_3 + \tilde{h}_3) h_{12}\bar{h}_{12}^2 - 2\left(\tilde{h}_3^2 + 2\,(h_1 + h_2)\tilde{h}_3\right)\bar{h}_{12}^2\Big]$$

$$\beta_{h_5} = -\frac{1}{(4\,\pi)^2}\left(8\, h_{12}\bar{h}_{12}^2 g^2 t_1^2 + 4\, h_{12}\bar{h}_{12}^2 h_4^{(=)} + 6\, h_{12}\bar{h}_{12}^2 h_4^{(\neq)} - 17\, h_{12}\bar{h}_{12} h_5\right)$$

4.4.4 Finiteness, fixed points and IR stability

We now discuss the previous results for different choices of the chiral couplings. We recall that we are working with euclidean theories which are not subject to hermitian conjugation constraints. In particular, Φ and $\bar{\Phi}$ are independent real superfields as well as the corresponding couplings h and $\bar{h}$.

We first consider the case of the theory with a single chiral superfield. Referring to the results (4.4.149) we note that all the divergences are proportional to (powers of) the superpotential coupling h. Therefore, setting $h = 0$ the theory turns out to be one–loop finite and we have no need to add all possible couplings in order to get a renormalizable theory. Precisely, the following action

$$S = \frac{1}{2g^2}\int d^4x d^2\bar{\theta}\ \bar{W}^{\dot{\alpha}} * \bar{W}_{\dot{\alpha}} + \int d^4x d^4\theta\ \Phi * \bar{\Phi} + \bar{h}\int d^4x d^2\bar{\theta}\ \bar{\Phi}_*^3 \qquad (4.4.156)$$

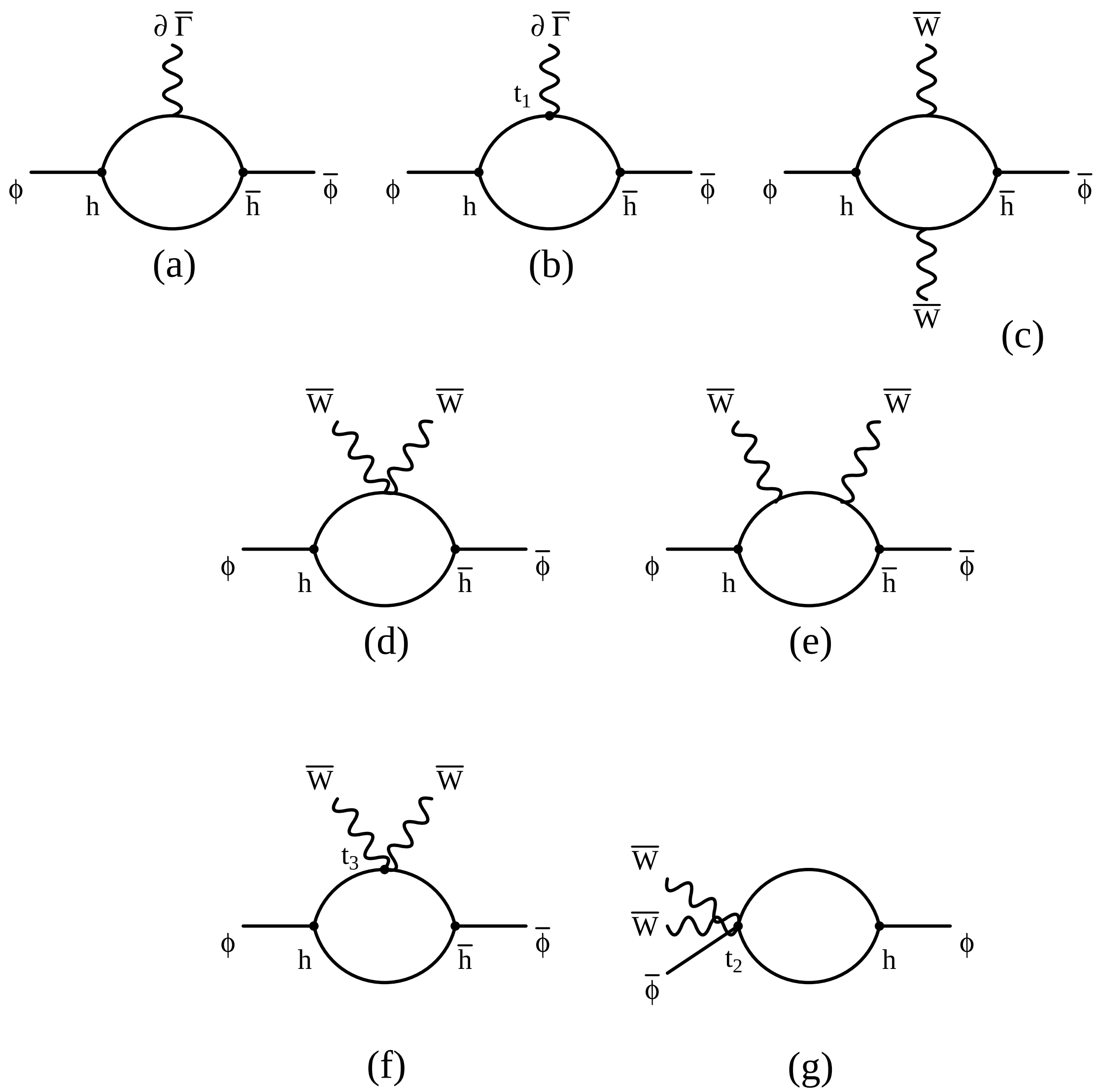

Figure 4.7: One–loop diagrams contributing to the mixed sector.

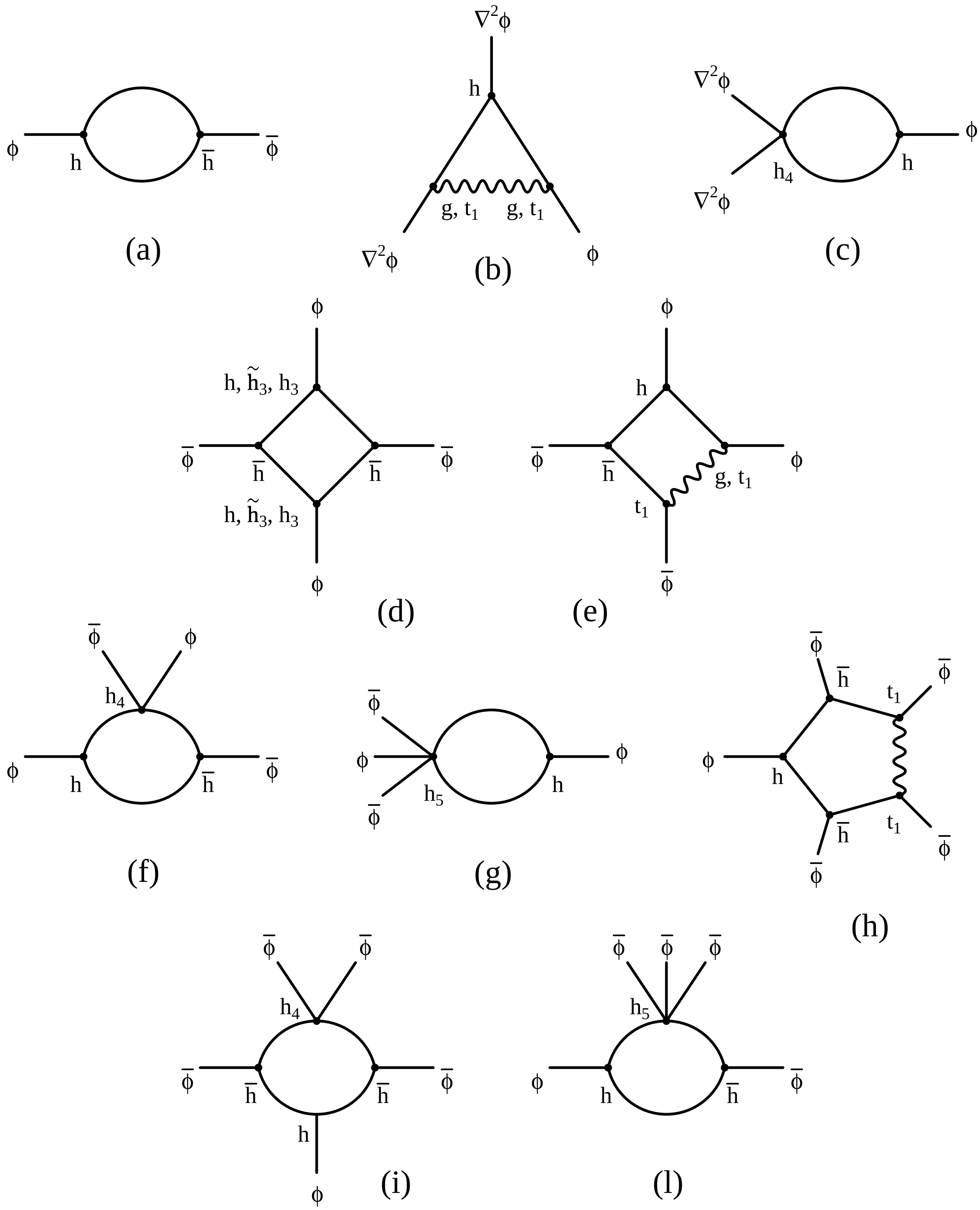

Figure 4.8: One–loop diagrams contributing to the matter sector.

is perfectly consistent at quantum level and one–loop finite.

Conversely, if we set $\bar{h} = 0$ while keeping the chiral superpotential on we have few divergent contributions surviving in (4.4.149). Making the minimal choice of setting to zero all the extra couplings which do not get renormalized we find that the following action is one–loop renormalizable

$$S = \frac{1}{2g^2}\int d^4x d^2\bar{\theta}\ \bar{W}^{\dot{\alpha}} * \bar{W}_{\dot{\alpha}} + \int d^4x d^4\theta\ \Phi * \bar{\Phi} + h\int d^4x d^2\theta\ \Phi^3_* + h_3\mathcal{F}^2\int d^4x d^4\theta\ \bar{\theta}^2\ \Phi * \nabla^2\Phi * \nabla^2\Phi \qquad (4.4.157)$$

but not finite. This result is consistent with what has been found [12, 8, 24, 10, 11] for the NAC ungauged Wess-Zumino model.

The fact that the theory is finite when we turn off the superpotential in the chiral sector while tolerating a superpotential for antichirals but not viceversa is a manifestation of the asymmetry between the chiral and the antichiral sectors induced by the star product.

We now discuss the spectrum of fixed points for the most general case where all the couplings are turned on. As already seen, the theory is one–loop finite when we set $h = 0$, independently of the value of the other couplings. Therefore, $h = 0$ defines an eight dimensional hypersurface of fixed points.

However, $h = 0$ does not exhaust the spectrum of fixed points. In fact, by a quick look at the beta functions in (4.4.153) we can easily see that taking $h \neq 0$ there is another hypersurface of fixed points given by

$$\bar{h} = h_5 = t_2 = 0$$
$$2h_4 + g^2(t_1 - 2)^2 = 0 \qquad (4.4.158)$$

In any case, from the requirement for $\beta_h, \beta_{\bar{h}}$ to vanish we are forced to set either h or $\bar{h}$ equal to zero. This is due to the fact that, despite the non–trivial gauge/matter interaction, the matter quadratic term does not get corrections from gauge quantum fields. As a consequence, we do not have non–trivial $h(g), \bar{h}(g)$ functions which describe marginal flows as it happens in ordinary non–abelian SYM theories.

We study the stability of fixed points and compare the present situation with the corresponding anticommutative case, that is an ordinary abelian SYM theory perturbed by a cubic superpotential

$$h\int d^4x d^2\theta\ \Phi^3 + \bar{h}\int d^4x d^2\bar{\theta}\ \bar{\Phi}^3 \qquad (4.4.159)$$

where hermiticity requires $\bar{h}$ to be the complex conjugate of h.

In the ordinary case the theory is simply a free gauge theory plus a massless Wess–Zumino model. The corresponding one–loop β–functions go like $\beta_h \sim |h|^2 h$ and $\beta_{\bar{h}} \sim |h|^2\bar{h}$. Therefore, the only fixed point of the theory is $h = \bar{h} = 0$. The RG trajectories are drawn in Fig. 4.9 where only the first and third quadrants have to be considered ($h\bar{h} = |h|^2 \geq 0$). Therefore, the origin corresponds to an IR stable fixed point.

We now consider the NAC case described by the general action (4.4.146). The great number of coupling constants forbids plotting global RG trajectories; however, we can study the IR

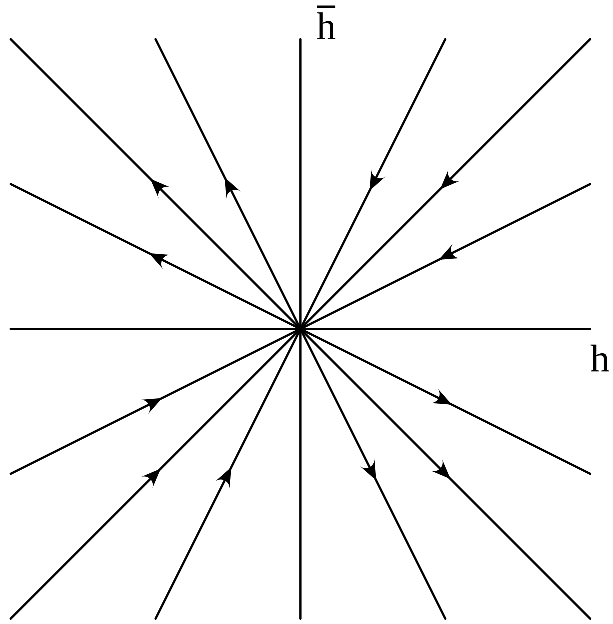

Figure 4.9: Renormalization group trajectories near the $h = \bar{h} = 0$ fixed point. Arrows indicate the IR flows.

behavior of the theory on lower dimensional hypersurfaces by temporarily keeping a certain number of couplings fixed. First of all, since $\beta_g = 0$ we can sit on hypersurfaces $g = \bar{g}$ where $\bar{g}$ is a small constant. Moreover, we can identify the flows associated to β_h and $\beta_{\bar{h}}$ as a closed subset of equations.

The main difference compared to the ordinary case is that now h and $\bar{h}$ are two *real* independent couplings. This has two consequences: 1) The spectrum of fixed points is now given by the two lines $h = 0$ and $\bar{h} = 0$; 2) Since the product $h\bar{h}$ can be either positive or negative we need extend the study of RG trajectories to the whole $(h, \bar{h})$ plane.

The configuration of RG trajectories is given in Fig. 4.9 where arrows indicate the IR flow. It is easy to see that the two axes $h = 0$ and $\bar{h} = 0$ are lines of unstable fixed points.

In particular, we see that in this case the origin is neither an infrared nor an ultraviolet attractor. This is in contrast with the ordinary case where, as discussed above, the origin is an IR stable fixed point. The different behavior of the two theories can be traced back to the different hermiticity conditions which constrain the (anti)chiral coupling constants.

Although the failure of the origin to be an IR attractor is conclusive, we can restrict the couplings to have the same sign (then studying the flows in the first and third quadrants) and investigate whether we can identify a region in the parameter space for which the origin is an infrared attractor.

The $(h, \bar{h}) = (0, 0)$ fixed point spans a seven dimensional hypersurface of fixed points corresponding to all possible values of the other couplings. We study RG trajectories on this hypersurface by linearizing in the rest of the couplings.

The system of linearized equations we consider is $\mu dh_i/d\mu = \beta_{h_i}, i = 3, 4, 5$, while the remaining equations decouple and have stability matrix with positive eigenvalues. Keeping $(h, \bar{h})$ slightly away from the fixed point, the eigenvalues of the stability matrix for the subset (h_3, h_4, h_5) are approximatively

$$\rho_1 = -1.608\, h\, \bar{h} \qquad \rho_2 = 232.788\, h\, \bar{h} \qquad \rho_3 = 560.82\, h\, \bar{h} \tag{4.4.160}$$

We see that the matrix vanishes at the fixed point but, as soon as we move away from the fixed point, there is at least one negative eigenvalue in any quadrant of the $(h, \bar{h})$–plane. The corresponding eigenvector represents an instability direction and leads to the conclusion that the origin is never an IR attractor whatever the range for $(h, \bar{h})$ is.

We now consider the more interesting case of three flavors. As already stressed, the theory (4.4.147) describes a NAC generalization of the abelian $N = 4$ SYM theory and theories obtained from it by adding marginal deformations.

We remind that the ordinary abelian $N = 4$ SYM theory is a free theory, then necessarily finite. Marginal deformations can be added of the form (we write them in a form which can be easily generalized to the NAC case)

$$\int d^4xd^2\theta \;\left(h_1\Phi_1\Phi_2\Phi_3 - h_2\Phi_1\Phi_3\Phi_2\right) + \int d^4xd^2\bar{\theta}\;\left(\bar{h}_1\bar{\Phi}^1\bar{\Phi}^2\bar{\Phi}^3 - \bar{h}_2\bar{\Phi}^1\bar{\Phi}^3\bar{\Phi}^2\right) \tag{4.4.161}$$

which break supersymmetry down to $N = 1$. In our notation $N = 4$ supersymmetry is recovered for $h_1 = h_2$ ($\bar{h}_1 = \bar{h}_2$ are the hermitian conjugates). The deformed theory is no longer finite since a divergent self–energy contribution to the (anti)chirals appears at one–loop, proportional to $h_{12}\bar{h}_{12}$. It is easy to see that the free $N = 4$ theory is a stable IR fixed point.

We now study what happens in the NAC case. Looking at the results (4.4.154) the first important observation is that the gauge beta–function is identically zero and all the other divergences are proportional to powers of h_{12} and $\bar{h}_{12}$. Therefore, setting $h_1 = h_2$ and $\bar{h}_1 = \bar{h}_2$ kills all the divergences and the theory is one–loop finite. It follows that, at least at one–loop, the NAC deformation does not affect the finiteness properties of the $N = 4$ SYM theory.

It is not difficult to provide general arguments for extending this analysis to all loops. First of all, the gauge sector cannot receive loop corrections at any perturbative order. In fact, for dimensional and symmetry reasons [22] in the $U_*(1)$ case the only local background structure which might be produced is the quadratic term $\int \bar{W}^{\dot{\alpha}}\bar{W}_{\dot{\alpha}}$ with no powers of the NAC parameter in front. As already discussed, any loop diagram that we can draw contributing to the gauge sector is proportional to powers of $\mathcal{F}^{\alpha\beta}$ and then necessarily finite.

In the mixed and matter sectors, the constraints on the maximal power of $\mathcal{F}^{\alpha\beta}$ that we can have in divergent diagrams imply that at least one chiral or one antichiral vertex from the superpotential needs be present at order zero in the NAC parameter, therefore carrying a coupling h_{12} or $\bar{h}_{12}$. Then, it is a matter of fact that in the case of equal couplings all contributions vanish.

Therefore, on general grounds we conclude that the $U_*(1)$ deformation of the abelian $N = 4$ SYM theory is all loop finite.

Exactly marginal deformations are obtained by adding marginal operators to the action which do not affect the vanishing of the beta–functions. In our case, taking $h_1 \neq h_2$ and/or $\bar{h}_1 \neq \bar{h}_2$ means adding marginal operators. However, not all of them turn out to be exactly marginal, at least at one–loop. In fact, in order to have vanishing beta–functions away from the symmetric point $h_{12} = \bar{h}_{12} = 0$ we need require either

$$h_1 = h_2 \qquad (\tilde{h}_3 + 2h_1) = 0 \tag{4.4.162}$$

or

$$\bar{h}_1 = \bar{h}_2 \qquad t_2 = h_5 = 0 \qquad g^2(t_1^2 - 2t_1 + 4) + h_4^{(\neq)} = 0 \tag{4.4.163}$$

In order to study the stability of the fixed points we can perform an analysis similar to the previous one with the obvious substitutions $h \to h_{12}$ and $\bar{h} \to \bar{h}_{12}$ plus the additional couplings which were not present in the one–flavor case.

The flow equations for h_{12} and $\bar{h}_{12}$ still decouple from the rest of the system and we can first study the IR behavior of the theory restricted to the $(h_{12}, \bar{h}_{12})$ plane. With the suitable substitutions Fig. 4.9 is still valid and provides two lines $h_{12} = 0$ and $\bar{h}_{12} = 0$ of unstable fixed points.

Restricting the range of $(h_{12}, \bar{h}_{12})$ within the first and third quadrants and neglecting t_2 which has a trivial β–function, we are left with a system of seven equations whose stability matrix can be studied in a neighborhood of the origin. The corresponding eigenvalues are approximatively

$$\begin{array}{ccc} \rho_1 = 3\, h_{12}\, \bar{h}_{12} & \rho_{2,3} = 6\, h_{12}\, \bar{h}_{12} & \rho_4 = -0.626\, h_{12}\, \bar{h}_{12} \\ \rho_5 = 3.936\, h_{12}\, \bar{h}_{12} & \rho_6 = 11.674\, h_{12}\, \bar{h}_{12} & \rho_7 = 25.017\, h_{12}\, \bar{h}_{12} \end{array} \tag{4.4.164}$$

Again, the appearance of at least one negative eigenvalue for any choice of the couplings leads to the conclusion that the $N = 4$ theory is not an IR attractor. This result is similar to what happens in the ordinary non–abelian SYM theories with gauge group $SU(N \geq 3)$ [32, 33], even if the two theories are not directly mappable one onto the other.

4.5 Summary and conclusions

In this Chapter we have studied the problem of the renormalizability for nonanticommutative $N = 1/2$ SYM theories in the presence of interacting matter. The introduction of a superpotential for (anti)chiral superfields complicates the investigation of the quantum properties of the gauge theory, not only from a technical point of view. In fact, at a first sight the non–trivial interplay between partial breaking of supersymmetry, gauge invariance of the action and renormalization procedure leads to drastic consequences for the theory: In NAC geometry only $SU(\mathcal{N}) \otimes U(1)$ gauge theories are well defined and, as in the ordinary case, the renormalization of the kinetic term requires a different renormalization function for the $SU(\mathcal{N})$ and $U(1)$ wave–functions. Consequently, superpotential terms proportional to the abelian fields need appear with different coupling constants. In superspace formalism this can be realized by generalizing the single–trace (anti)chiral interaction to contain different trace structures, each one with its own coupling. However, the addition of multi–trace terms, while completely harmless in the ordinary SYM theories, in the NAC case affects the theory in a non–trivial way. In fact, these terms are no longer gauge singlets and require suitable completions which break explicitly the residual $N = 1/2$ supersymmetry.

The way–out we have proposed amounts to re–establish perfect equivalence between $SU(\mathcal{N})$ and $U(1)$ wave–function renormalizations by multiplying the abelian quadratic term by an extra coupling constant. As a consequence, a single–trace superpotential is allowed which respects $N = 1/2$ supersymmetry and supergauge invariance. Basically, we have shifted the problem of deforming the action from the superpotential to the Kähler potential or, in other words, from an integral on chiral variables to an integral on the whole superspace. This has the nice effect to leave the residual $N = 1/2$ supersymmetry unbroken. It is important to stress that in contradistinction with the ordinary case where rescaling the abelian kinetic term or suitably rescaling the superpotential couplings lead to equivalent theories, in the NAC case this is no longer true. In one case we obtain a consistent $N = 1/2$ theory whereas in the other case we loose completely supersymmetry. The ultimate cause is the non–trivial NAC gauge transformations undergone by the abelian superfields.

Having solved the main problem of adding a matter cubic superpotential we have studied the most general divergent structures which could arise at loop level selecting them on the basis of dimensional considerations and global symmetries. We have then proposed the action (4.3.124) as the most general renormalizable gauge–invariant $N = 1/2$ deformation of the ordinary SYM field theory with interacting matter. The next steps should be the complete study of one–loop renormalization, the computation of the β–functions and the implementation of the massive case. Moreover, strictly speaking our results hold only at one–loop. Higher loop calculations would be necessary to further confirm the good renormalization properties of our action.

Generalizing in an obvious way our construction to include more than one (anti)chiral superfields would lead to a consistent non-Abelian NAC generalization of the $N = 4$ SYM. This would be an important step towards clarifying the stringy origin of NAC deformations and deformations of the AdS/CFT correspondence. In particular, it would be nice to investigate how robust properties of $N = 4$ SYM like finiteness and integrability might be affected by NAC deformations.

We have further continued the investigation of NAC SYM theories by performing the one–loop renormalization of $U_*(1)$ SYM theories with matter in the adjoint representation of the gauge group, motivated by the idea of finding NAC generalizations of ordinary SYM with extended supersymmetry. In general, the actions are not simply obtained from the ordinary ones by deforming the products, but contain suitable completions given in terms of all classical marginal operators which respect a given set of global symmetries.

We have first considered a SYM theory with a single chiral field self–interacting through a cubic superpotential. Then, we have extended our analysis to the case of three matter fields interacting through a cubic superpotential which depends on four coupling constants, $h_1, h_2, \bar{h}_1, \bar{h}_2$. For $h_1 = h_2$ and $\bar{h}_1 = \bar{h}_2$ the classical action exhibits a global $SU(3)$ symmetry and can be interpreted as a NAC generalization of the ordinary $N = 4$ SYM theory. More generally, for $h_1 \neq h_2$ and/or $\bar{h}_1 \neq \bar{h}_2$ it looks like the NAC generalization of marginal deformations of $N = 4$ SYM.

Since in the ordinary case $N = 4$ SYM is finite, one of the questions we have addressed is whether finiteness survives in the NAC case. We note that, while in the ordinary $U(1)$ case finiteness is a trivial statement, being the theory free, its NAC generalization is highly interacting and the question becomes interesting. We have found that at one–loop the theory with $h_1 = h_2, \bar{h}_1 = \bar{h}_2$ is indeed *finite*. Moreover, based on general arguments we have provided a proof for the all–loop finiteness of the theory.

More generally, we have considered theories in the presence of marginal deformations. In this case UV divergences arise which in general set the theory away from a fixed point. In the parameter space we have studied the spectrum of fixed points and the renormalization group flows. We have found that, while in the ordinary $N = 4$ case $h_1 = h_2, \bar{h}_1 = \bar{h}_2$ is an IR stable fixed point (free theory), in our case nonanticommutativity makes all the fixed points unstable. This is due to the fact that in the presence of extra marginal operators proportional to $\mathcal{F}^{\alpha\beta}$, the parameter space gets enlarged and new lines of instability are allowed. Even if our analysis is based on one–loop calculations, we have already enough information for drawing qualitative conclusions on the effects that this kind of geometrical deformations have on the RG flows: NAC theories resemble the non–abelian $SU(N \geq 3)$ ordinary theories for which $N = 4$ SYM is neither an IR nor an UV attractor. We focused only on massless theories but it is easy to convince that the addition of a mass term should not change the main features of the theories.

In order to simplify the analysis, we considered the $U_*(1)$ case. From the point of view of studying how renormalization works these theories are not too trivial. In fact, as already stressed, they are highly interacting. Therefore, the results obtained on the finiteness in a subspace of the parameter space and, more generally, on the role of nonanticommutativity on their UV and IR behavior are actually not *a priori* expected.

However, considering this example we have lost the non–trivial coupling between non–abelian and abelian superfields which is a peculiar feature of the NAC gauge theories. It would be then very interesting to consider the non–trivial $SU(\mathcal{N}) \otimes U(1)$ case and investigate whether the obtained results survive. In particular, it would be interesting to address the question of finiteness. In fact, we expect that at one–loop the gauge sector would not receive divergent corrections since matter loops would cancel ghost loops, still giving $\beta_g^{(1)} = 0$. In the matter sector new contributions proportional to g^2 would arise for the two and higher point functions. Therefore, as in the ordinary non abelian cases, we expect non–trivial surfaces of fixed points of the form $h_{12} = h_{12}(g), \bar{h}_{12} = \bar{h}_{12}(g)$. The non–trivial question is whether this is only a one–loop effect or it would arise as an actual feature of the whole quantum lagrangian.

From a stringy point of view, our results are a further step towards a better understanding of the dynamics of D3-branes in the presence of non-vanishing RR forms and provide few hints for constructing gravity duals.

Part II

Three-dimensional field theories

Chapter 5

$N = 2$ Chern-Simons matter theories

Gauge theories in three spacetime dimensions have several features not seen in their four-dimensional counterparts. In particular, the possible appearance of the Chern-Simons term in the action is connected with the parity anomaly [34]. The original motivation for their study was their resemblance to the high temperature limit of four-dimensional gauge theories. Another motivation is that they could describe condensed matter phenomena, as condensed matter systems often relies on three-dimensional electrodynamics and involves a Chern-Simons term.

The Chern-Simons term has been widely studied from a field theoretical point of view. The bosonic theory provides a powerful tool to define and study knot invariants in arbitrary three-manifolds [35] and explain many of the properties of two-dimensional field theory. In this setting, general arguments about large gauge invariance as well as a direct inspection using perturbation theory [36, 37] revealed that quantum corrections only arise at one-loop and they can only shift the Chern-Simons level, which is constrained to assume only integer values.

Recently, a renew interest in three dimensional Chern–Simons (CS) theories has been triggered by the formulation of the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence between CS matter theories and M/string theory. While pure CS is a topological theory [38, 35], the addition of matter degrees of freedom makes it dynamical and can be used to describe nontrivial 3D systems. The addition of matter can be also exploited to formulate theories with extended supersymmetry [39, 40, 41]. Chern–Simons matter theories corresponding to a single gauge group can be at most $\mathcal{N} = 3$ supersymmetric [42], while the use of direct products of groups and matter in the bifundamental representation allows to increase supersymmetry up to $\mathcal{N} = 8$ [43, 44].

This has led to the precise formulation of the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence which in its original form [45] states that M-theory on $\mathrm{AdS}_4 \times S^7/\mathcal{Z}_k$ describes the strongly coupled dynamics of a two–level $\mathcal{N} = 6$ supersymmetric Chern–Simons theory with $U(N)_k \times U(N)_{-k}$ gauge group and $SU(2) \times SU(2)$ invariant matter in the bifundamental. This is the field theory generated at low energies by a stack of N M2–branes probing a $\mathcal{C}^4/\mathcal{Z}_k$ singularity. In the decoupling limit $N \to \infty$ with $\lambda \equiv N/k$ large and fixed, choosing $N \ll k^5$, the radius of the eleventh dimension in M-theory shrinks to zero and the dual description is given in terms of a type IIA string theory on $\mathrm{AdS}_4 \times \mathcal{CP}^3$ background [45]. In the particular case of $N = 2$, supersymmetry gets enhanced to $\mathcal{N} = 8$ and the strings provide a dual description of the Bagger–Lambert–Gustavsson (BLG) model [43, 46, 46, 47]. Enhancement of supersymmetry occurs also for $k = 1, 2$ where the

ABJM theory describes the low energy dynamics of N membranes in flat space and in $\mathcal{R}^8/\mathcal{Z}_2$, respectively [45, 48, 49].

Since the original formulation of the correspondence, a lot of work has been done for studying the dynamical properties of this particular class of CS matter theories, such as integrability [50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60], the structure of the chiral ring and the operatorial content [45], [61, 62, 63, 64, 65, 66, 67] and dynamical supersymmetry breaking [68, 69]. Many efforts have been also devoted to the generalization of the correspondence to different gauge groups [70, 71], to less (super)symmetric backgrounds [72, 70, 71, 73, 74, 75, 76, 77, 78, 79, 80, 81] and to include flavor degrees of freedom [82, 83, 84, 85, 86]. Theories with two different CS levels (k_1, k_2) have been also introduced [39, 79] which correspond to turning on a Romans mass [87] in the dual background.

5.1 The ABJM model

In this section we review the construction and the basic properties of the ABJM model.

Pure Chern-Simons theories in $2 + 1$ dimensions are topological theories. When they are coupled to matter fields they are no longer topological, but they can be still conformally invariant. An example is the $N = 2$ case with no superpotential. Its field content includes a vector multiplet V in the adjoint representation of the gauge group G and chiral multiplets Φ_i in some representations R_i. The multiplets and their kinetic terms are the dimensional reduction of the four dimensional case, with the kinetic term for the vector multiplet replaced by a Chern-Simons term. In the Wess-Zumino gauge it is written as

$$S_{CS}^{N=2} = \frac{k}{4\pi} \int \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} A^3 - \chi\bar{\chi} + 2D\sigma \right) \tag{5.1.1}$$

where χ is the gaugino, D is the auxiliary field of the vector multiplet and σ is the real scalar field in the vector multiplet which arises from the A_3 component of the dimensionally reduced gauge field in 3+1 dimensions. For non-abelian gauge groups the Chern-Simons level k is quantized, and it takes integer values for unitary gauge groups if the trace is in the fundamental representation.

None of the fields in the vector multiplet has kinetic terms. Therefore, they are all auxiliary fields and can be integrated out by using their equations of motion. The component action takes the form

$$\begin{aligned} S = \int \Big\{ & \frac{k}{4\pi} \left(A \wedge dA + \frac{2}{3} A^3 \right) + D_\mu \bar{\phi}_i D^\mu \phi_i + i \bar{\psi}_i \gamma^\mu D_\mu \psi_i \\ & - \frac{16\pi^2}{k^2} (\bar{\phi}_i T^a_{R_i} \phi_i)(\bar{\phi}_j T^b_{R_j} \phi_j)(\bar{\phi}_k T^a_{R_k} T^b_{R_k} \phi_k) - \frac{4\pi}{k} (\bar{\phi}_i T^a_{R_i} \phi_i)(\bar{\psi}_j T^a_{R_j} \psi_j) \\ & - \frac{8\pi}{k} (\bar{\psi}_i T^a_{R_i} \phi_i)(\bar{\phi}_j T^a_{R_j} \psi_j) \Big\} \end{aligned} \tag{5.1.2}$$

Since k is an integral number it cannot receive quantum corrections; hence, this action preserves conformal invariance also at the quantum level.

The $N = 3$ supersymmetric theories are obtained by considering the field content of an $N = 4$ theory. We add to the vector superfield an additional auxiliary chiral multiplet ϕ in the adjoint representation of the gauge group and assume that the chiral multiplets come in pairs Φ_i, $\tilde{\Phi}_i$ in conjugate representations of the gauge group. Note that this procedure closely resembles the construction of the $N = 2$ theory in $3 + 1$ dimensions, where the combination of Φ and $\tilde{\Phi}$ forms a hypermultiplet. We add the superpotential terms $\tilde{\Phi}_i \phi \Phi^i$ and $-\frac{k}{8\pi}\mathrm{Tr}\phi^2$. The latter breaks the supersymmetry to $N = 3$. Since ϕ is an auxiliary field and has not any kinetic term we can integrate it out. The superpotential now reads

$$W = \frac{4\pi}{k}(\tilde{\Phi}_i T^a_{R_i} \Phi_i)(\tilde{\Phi}_j T^a_{R_j} \Phi_j) \tag{5.1.3}$$

The full theory is the sum of (5.1.2) (with the addition of the same terms for the conjugate chiral multiplets) and (5.1.2). The relative coefficients are fixed by supersymmetry.

Let us now specialize to the case of $U(N)_k \times U(N)_{-k}$ gauge group and two hypermultiplets (A_1, B_1) and (A_2, B_2) in the bifundamental representation of the gauge group. The pedices of the two $U(N)$ factors of the gauge group indicate that we are taking the two Chern-Simons levels equal but opposite in sign. The above construction leads us to the following superpotential

$$W = \frac{2\pi}{k}\epsilon^{ab}\epsilon^{\dot{a}\dot{b}}\mathrm{Tr}\left(A_a B_{\dot{a}} A_b B_{\dot{b}}\right) \tag{5.1.4}$$

which exhibits an $SU(2) \times SU(2)$ symmetry acting separately on the A's and on the B's. Moreover, theories with N superconformal symmetries in 2+1 dimensions have an $SO(N)$ R-symmetry which in this case is $SO(3) \simeq SU(2)_R$. It is realized on the component fields in the following way. The vector multiplet fermions form a triplet and a singlet of $SU(2)_R$, and the three scalar fields a triplet. The lowest component of the chiral superfields A_1 and B_1^* transform as a doublet, as A_2 and B_2^* do. The R-symmetry does not commute with the global $SU(2) \times SU(2)$ symmetry, and they together form an $SU(4)_R \simeq SO(6)_R$ R-symmetry. It is a R-symmetry because the supercharges cannot be singlets under it. Thus, the theory (5.1.2) with the superpotential (5.1.4) enjoys $N = 6$ supersymmetry.

The coupling constant in (5.1.4) is $1/k$ and for large k it is weakly coupled. In the large N limit with N/k fixed one can expand in $1/N^2$ and the effective coupling constant in the leading order, planar diagrams is the 't Hooft coupling $\lambda \equiv N/k$. Thus, the theory is weakly coupled for $k \gg N$ and strongly coupled for $k \ll N$.

5.2 Generalizations

In this section we review some generalizations of the ABJM model to different gauge groups and to less supersymmetric field theories. We focus on unitary gauge groups, but many other possibilities have been analyzed [88].

The first generalization we consider is to $U(N)_k \times U(M)_{-k}$ Chern-Simons matter theories, with $M \neq N$ but the same matter content and interactions as in the ABJM model [88]. In complete analogy with the previous section, the theory is constructed from the $N = 3$ Chern-Simons matter theory and the special form of the superpotential (5.1.4) leads to the conclusion that it has an enhanced $N = 6$ superconformal symmetry. The field theory has been named the ABJ model, after the authors who explained its gravity dual [70].

The second generalization we consider is to change the Chern-Simons levels so that they are no longer equal and opposite in sign. In other words, we are interested in $U(N)_{k_1} \times U(M)_{k_2}$ Chern-Simons matter theories studied in [79]. Along the lines of the construction of the ABJM model, one is lead to the following action

$$S_{N=3} = \frac{k_1}{4\pi} S_{CS}(V_1) + \frac{k_2}{4\pi} S_{CS}(V_2) + \int d^4x d^4\theta \ \mathrm{Tr}\left(e^{-V_1} A_i^\dagger e^{V_2} A_i + e^{-V_1} B_i e^{V_2} B_i^\dagger\right) + \int d^4x d^2\theta \ \left(\frac{2\pi}{k_1} \mathrm{Tr}\,(B_i A_i)^2 + \frac{2\pi}{k_2} \mathrm{Tr}\,(A_i B_i)^2\right) \tag{5.2.5}$$

Due to the unequal Chern-Simons levels, the $SU(2) \times SU(2)$ global symmetry of the ABJM model is broken to the diagonal $SU(2)$ that rotates the A_i and B_i simultaneously. The action (5.2.5) still has an $SO(3)_R \simeq SU(2)_R$ R-symmetry. Therefore, it enjoys $N = 3$ superconformal symmetry. We conclude that unequal CS levels break some supersymmetries.

One can further break supersymmetry by allowing a more general superpotential:

$$W = c \int d^4x d^2\theta \ \epsilon^{ij} \epsilon^{kl} A_i B_k A_j B_l \tag{5.2.6}$$

This is the most general $SO(4)$ global symmetry preserving superpotential we can write. Note that the coupling c is now free to assume any complex value; therefore, the theory is no longer superconformal, and supersymmetry is broken to $N = 2$.

If we want to preserve $N = 2$ supersymmetry without adding any new superfields, (5.2.6) can only be modified by breaking the $SO(4)$ global symmetry, thus broadening the parameter space one can study. We discuss this in details in the next chapter.

The addition of fundamental flavor fields [82, 83, 84] to the ABJM action (5.1.4) broadens the class of field theories one can study. In addition to the $N = 4$ vector multiplets (V_i, Φ_i) and the bifundamental hypermultiplets $(A_i, B_i^\dagger)$ we introduce $2N_f$ fundamental hypermultiplets $(Q_{1r}, \tilde{Q}_{1r}^\dagger)$ and $(Q_{2r}, \tilde{Q}_{2r}^\dagger)$ with $r = 1, \ldots, N_f$ with superpotential

$$W_{flavor} = \alpha_1 \tilde{Q}_{1r} \Phi_1 Q_{1r} + \alpha_2 \tilde{Q}_{2r} \Phi_2 Q_{2r} \tag{5.2.7}$$

which preserves $N = 2$ supersymmetry for arbitrary values of the coupling constants α_1 and α_2. The component fields of the $N = 4$ vector multiplets are all auxiliary fields and can be integrate out giving rise to (we suppress flavor indices)

$$W_{flavor} = \frac{4\pi\alpha_1}{k} \tilde{Q}_1 (A_i B^i) Q_1 - \frac{4\pi\alpha_2}{k} \tilde{Q}_2 (B^i A_i) Q_2 + \frac{2\pi\alpha_1^2}{k} (Q_1 \tilde{Q}_1)^2 - \frac{2\pi\alpha_2^2}{k} (\tilde{Q}_2 Q_2)^2 \tag{5.2.8}$$

For the particular values $\alpha_1 = -\alpha_2 = 1$ supersymmetry is enhanced to $N = 3$. As in the ABJM model, this is accompanied by an enhancement of the $U(1)_R$ R-symmetry to $SU(2)_R$. In addition, the theory also inherits the $SU(2)_D$ diagonal subgroup of the $SU(2) \times SU(2)$ of (5.1.4). In this case the two $SU(2)$'s commute with each other. Note that this symmetries survive when we choose the two CS levels to be unequal.

The superpotential (5.2.8) clearly exhibits an $U(N_f) \times U(N_f)$ flavor symmetry. Given its structure, one can easily extend it to $U(N_f) \times U(N_f')$ with $N_f \neq N_f'$. There is one more operator one can add while preserving all the global symmetries

$$\Delta W = Q_1 \tilde{Q}_1 Q_2 \tilde{Q}_2 \tag{5.2.9}$$

In the next chapter we study the quantum effects of the theory obtained by adding (5.2.6), (5.2.8) and (5.2.9) and modifying them so that only the $U(N_f) \times U(N_f')$ global symmetry survives.

Chapter 6

Quantization, fixed points and RG flows

CS matter theories involved in the AdS_4/CFT_3 correspondence are of course at their superconformal fixed point [1]. Compactification of type IIA supergravity on $AdS_4 \times \mathcal{CP}^3$ does not contain scalar tachyons [91]. Since these states are dual to relevant operators in the corresponding field theory, AdS_4/CFT_3 correspondence leads to the prediction that in the far IR fixed points should be stable.

As a nontrivial check of the correspondence, it is then interesting to investigate the properties of these fixed points in the quantum field theory in order to establish whether they are isolated fixed points or they belong to a continuum surface of fixed points, whether they are IR stable and which are the RG trajectories which intersect them. Since for $k \gg N$ the CS theory is weakly coupled, a perturbative approach is available.

With these motivations in mind, we consider a $\mathcal{N} = 2$ supersymmetric two–level CS theory for gauge group $U(N) \times U(M)$ with matter in the bifundamental representation and flavor degrees of freedom in the fundamental, perturbed by the most general matter superpotential compatible with $\mathcal{N} = 2$ supersymmetry. For particular values of the couplings the model reduces to the $\mathcal{N} = 6$ ABJ/ABJM superconformal theories [45, 70] ($\mathcal{N} = 8$ BLG theory [43, 44, 46] for $N = M = 2$) or to the superconformal $\mathcal{N} = 2, 3$ theories with different CS levels studied in [79], in all cases with and without flavors. More generally, it describes marginal (but not exactly marginal) perturbations which can drive the theory away from the superconformal points.

At two loops, we compute the beta–functions and determine the spectrum of fixed points. In the absence of flavors the condition of vanishing beta–functions necessarily implies the vanishing of anomalous dimensions for all the elementary fields of the theory. Therefore, the set of superconformal fixed points coincides with the set of superconformal finite theories. When flavors are present this is no longer true and in the space of the couplings we determine a surface of fixed points where the theory is superconformal but not two–loop finite.

When flavors are turned off we determine a continuum surface of fixed points which contains as non–isolated fixed points the BLG, the ABJ and ABJM theories. The case of theories with equal

[1] A classification of a huge landscape of superconformal Chern–Simons matter theories in terms of matter representations of global symmetries has been given in [89, 90].

CS levels and $U(1)_A \times U(1)_B$ symmetry preserving perturbations has been already investigated in [92]. In this chapter we provide details for that class of theories and generalize the results to the case of no–symmetry preserving perturbations [93]. When the CS levels are different the surface contains a $\mathcal{N} = 2$, $SU(2)_A \times SU(2)_B$ invariant and a $\mathcal{N} = 3$ superconformal theories. This result confirms the existence of the superconformal points conjectured in [79]. Moreover, we prove that the two theories are connected by a line of $\mathcal{N} = 2$ fixed points, as conjectured there.

We extend our analysis to the case of complex couplings, so including fixed points corresponding to beta–deformed theories [77].

In the presence of flavor matter the spectrum of fixed points spans a seven dimensional hypersurface in the space of the couplings which contains the fixed point corresponding to the ABJM/ABJ models with flavors studied in [82, 83, 84]. More generally, we find a fixed point which describes a $\mathcal{N} = 3$ theory with different CS levels with the addition of flavor degrees of freedom [83]. As a generalization of the pattern arising in the unflavored case, we find that it is connected by a four dimensional hypersurface of $\mathcal{N} = 2$ fixed points to a line of $\mathcal{N} = 2$ fixed points with $SU(2)_A \times SU(2)_B$ invariance in the bifundamental sector.

We then study RG trajectories around these fixed points in order to investigate their IR stability. The pattern which arises is common to all these theories, flavors included or not, and can be summarized as follows.

- Infrared stable fixed points always exist and we determine the RG trajectories which connect them to the UV stable fixed point (free theory).
- In general these fixed points belong to a continuum surface. The surface is globally stable since RG flows always point towards it.
- Locally, each single fixed point has only one direction of stability which corresponds to perturbations along the RG trajectory which intersects the surface at that point. In the ABJ/ABJM case this direction coincides with the maximal flavor symmetry preserving perturbation [92]. Along any other direction, perturbations drive the system away from the original point towards a different point on the surface. This is what we call local instability.
- When flavors are added, stability is guaranteed by the presence of nontrivial interactions between flavors and bifundamental matter. The fixed point corresponding to setting these couplings to zero is in fact unstable.

6.1 Quantization of $\mathcal{N} = 2$ Chern–Simons matter theories

In three dimensions, we consider a $\mathcal{N} = 2$ supersymmetric $U(N) \times U(M)$ Chern–Simons theory for vector multiplets $(V, \hat{V})$ coupled to chiral multiplets A^i and B_i, $i = 1, 2$, in the $(N, \bar{M})$ and $(\bar{N}, M)$ representations of the gauge group respectively, and flavor matter described by two couples of chiral superfields $Q_i, \tilde{Q}_i$, $i = 1, 2$ charged under the gauge groups and under a global $U(N_f)_1 \times U(N_f')_2$.

The vector multiplets $V, \hat{V}$ are in the adjoint representation of the gauge groups $U(N)$ and $U(M)$ respectively, and we write $V^b{}_a \equiv V^A (T_A)^b{}_a$ and $\hat{V}^{\hat{b}}{}_{\hat{a}} \equiv \hat{V}^A (T_A)^{\hat{b}}{}_{\hat{a}}$. Bifundamental matter carries global $SU(2)_A \times SU(2)_B$ indices $A^i, \bar{A}_i, B_i, \bar{B}^i$ and local $U(N)\times U(M)$ indices $A^a{}_{\hat{a}}, \bar{A}^{\hat{a}}{}_a, B^{\hat{a}}{}_a, \bar{B}^a{}_{\hat{a}}$. Flavor matter carries (anti)fundamental gauge and global indices, $(Q_1^r)^a, (\tilde{Q}_{1,r})_a, (Q_2^{r'})^{\hat{a}}, (\tilde{Q}_2$ with $r=1,\cdots N_f$, $r'=1,\cdots N_f'$.

In $\mathcal{N}=2$ superspace the action reads (for superspace conventions see Appendix)

$$\mathcal{S} = \mathcal{S}_{\rm CS} + \mathcal{S}_{\rm mat} + \mathcal{S}_{\rm pot} \tag{6.1.1}$$

with

$$\begin{aligned} \mathcal{S}_{\rm CS} &= K_1 \int d^3x\, d^4\theta \int_0^1 dt\ {\rm Tr}\Big[V \bar{D}^\alpha \left(e^{-tV} D_\alpha e^{tV}\right)\Big] \\ &+ K_2 \int d^3x\, d^4\theta \int_0^1 dt\ {\rm Tr}\Big[\hat{V} \bar{D}^\alpha \left(e^{-t\hat{V}} D_\alpha e^{t\hat{V}}\right)\Big] \end{aligned} \tag{6.1.2}$$

$$\begin{aligned} \mathcal{S}_{\rm mat} &= \int d^3x\, d^4\theta\ {\rm Tr}\left(\bar{A}_i e^V A^i e^{-\hat{V}} + \bar{B}^i e^{\hat{V}} B_i e^{-V}\right) \\ &+ \int d^4x d^4\theta\ {\rm Tr}\left(\bar{Q}_r^1 e^V Q_1^r + \bar{\tilde{Q}}^{1,r} \tilde{Q}_{1,r} e^{-V} + \bar{Q}^2_{r'} e^{\hat{V}} Q_2^{r'} + \bar{\tilde{Q}}^{2,r'} \tilde{Q}_{2,r'} e^{-\hat{V}}\right) \end{aligned} \tag{6.1.3}$$

$$\begin{aligned} \mathcal{S}_{\rm pot} &= \int d^3x\, d^2\theta\ {\rm Tr}\big[h_1(A^1B_1)^2 + h_2(A^2B_2)^2 + h_3(A^1B_1A^2B_2) + h_4(A^2B_1A^1B_2) \\ &+\lambda_1(Q_1\tilde{Q}_1)^2 + \lambda_2(Q_2\tilde{Q}_2)^2 + \lambda_3 Q_1\tilde{Q}_1Q_2\tilde{Q}_2 \\ &+\alpha_1\tilde{Q}_1A^1B_1Q_1 + \alpha_2\tilde{Q}_1A^2B_2Q_1 + \alpha_3\tilde{Q}_2B_1A^1Q_2 + \alpha_4\tilde{Q}_2B_2A^2Q_2\Big] + h.c. \end{aligned} \tag{6.1.4}$$

Here $2\pi K_1$, $2\pi K_2$ are two independent integers, as required by gauge invariance of the effective action. In the perturbative regime we take $K_1, K_2 \gg N, M$. The superpotential (6.1.4) is the most general classically marginal perturbation which respects $\mathcal{N}=2$ supersymmetry but allows only for a $U(N_f)\times U(N_f')$ global symmetry in addition to a global $U(1)$ under which the bifundamentals have for example charges $(1,0,-1,0)$.

For generic values of the couplings, the action (6.1.1) is invariant under the following gauge transformations

$$e^V \to e^{i\bar{\Lambda}_1} e^V e^{-i\Lambda_1} \qquad e^{\hat{V}} \to e^{i\bar{\Lambda}_2} e^{\hat{V}} e^{-i\Lambda_2} \tag{6.1.5}$$

$$\begin{aligned} A^i &\to e^{i\Lambda_1} A^i e^{-i\Lambda_2} \qquad & B_i &\to e^{i\Lambda_2} B_i e^{-i\Lambda_1} \\ Q_1 &\to e^{i\Lambda_1} Q_1 & \tilde{Q}_1 &\to \tilde{Q}_1 e^{-i\Lambda_1} \\ Q_2 &\to e^{i\Lambda_2} Q_2 & \tilde{Q}_2 &\to \tilde{Q}_2 e^{-i\Lambda_2} \end{aligned} \tag{6.1.6}$$

where Λ_1, Λ_2 are two chiral superfields parametrizing $U(N)$ and $U(M)$ gauge transformations, respectively. Antichiral superfields transform according to the conjugate of (6.1.6).

For special values of the couplings we can have enhancement of global symmetries and/or R–symmetry with consequent enhancement of supersymmetry. We list the most important cases we will be interested in.

<u>Theories without flavors</u>

Turning off flavor matter ($N_f = N'_f = 0$, $\alpha_j = \lambda_j = 0$) and setting

$$K_1 = -K_2 \equiv K \qquad , \qquad h_1 = h_2 = 0 \tag{6.1.7}$$

we have $\mathcal{N} = 2$ ABJM/ABJ–like theories already studied in [92]. In this case the theory is invariant under two global $U(1)$'s in addition to $U(1)_R$. The transformations are

$$\begin{aligned} U(1)_A : \quad & A^1 \to e^{i\alpha} A^1 \qquad , \qquad U(1)_B : \quad B_1 \to e^{i\beta} B_1 \\ & A^2 \to e^{-i\alpha} A^2 \qquad , \qquad \qquad \qquad \; B_2 \to e^{-i\beta} B_2 \end{aligned} \tag{6.1.8}$$

When $h_3 = -h_4 \equiv h$, the global symmetry becomes $U(1)_R \times SU(2)_A \times SU(2)_B$ and gets enhanced to $SU(4)_R$ for $h = 1/K$ [72]. For this particular values of the couplings we recover the $\mathcal{N} = 6$ superconformal ABJ theory [70] and for $N = M$ the ABJM theory [45].

More generally, we can select theories corresponding to complex couplings

$$h_3 = h e^{i\pi\beta} \quad , \quad h_4 = -h e^{-i\pi\beta} \tag{6.1.9}$$

These are $\mathcal{N} = 2$ β–deformations of the ABJ–like theories. For particular values of h and β we find a superconformal invariant theory.

Going back to real couplings, we now consider the more general case $K_1 \neq -K_2$. Setting

$$h_1 = h_2 = \frac{1}{2}(h_3 + h_4) \tag{6.1.10}$$

the corresponding superpotential reads

$$\mathcal{S}_{\rm pot} = \frac{1}{2} \int d^3x\, d^2\theta \, {\rm Tr}\left[h_3 (A^i B_i)^2 + h_4 (B_i A^i)^2 \right] \, + \, h.c. \tag{6.1.11}$$

This is the class of $\mathcal{N} = 2$ theories studied in [79] with $SU(2)$ invariant superpotential, where $SU(2)$ rotates simultaneously A^i and B_i.

When $h_3 = -h_4$, that is $h_1 = h_2 = 0$, we have the particular set of $\mathcal{N} = 2$ theories with global $SU(2)_A \times SU(2)_B$ symmetry [79]. This is the generalization of ABJ/ABJM–like theories to $K_1 \neq -K_2$. According to AdS/CFT, for particular values of $h_3 = -h_4$ we should find a superconformal invariant theory.

Another interesting fixed point should correspond to $h_3 = \frac{1}{K_1}$ and $h_4 = \frac{1}{K_2}$. The $U(1)_R$ R–symmetry is enhanced to $SU(2)_R$ and the theory is $\mathcal{N} = 3$ superconformal [79].

Theories with flavors

Setting

$$\begin{aligned} & K_1 = -K_2 \equiv K \qquad , \qquad h_1 = h_2 = 0 \qquad , \qquad h_3 = -h_4 = \frac{1}{K} \\ & \lambda_1 = \frac{a_1^2}{2K} \qquad , \qquad \lambda_2 = -\frac{a_2^2}{2K} \qquad , \qquad \lambda_3 = 0 \\ & \alpha_1 = \alpha_2 = \frac{a_1}{K} \qquad , \qquad \alpha_3 = \alpha_4 = \frac{a_2}{K} \end{aligned} \tag{6.1.12}$$

with a_1, a_2 arbitrary, our model reduces to the class of $\mathcal{N}=2$ theories studied in [82]. Choosing in particular $a_1 = -a_2 = 1$ there is an enhancement of R–symmetry and the theory exhibits $\mathcal{N}=3$ supersymmetry. This set of couplings should correspond to a superconformal fixed point [82, 83, 84].

In the more general case of $K_1 \neq -K_2$, in analogy with the unflavored case we consider the class of theories with

$$h_1 = h_2 = \frac{1}{2}(h_3 + h_4) \qquad ; \qquad \alpha_1 = \alpha_2 \quad , \quad \alpha_3 = \alpha_4 \tag{6.1.13}$$

For generic couplings these are $\mathcal{N}=2$ theories with a $SU(2)$ symmetry in the bifundamental sector which rotates simultaneously A^i and B_i. When $h_3 = -h_4$ this symmetry is enhanced to $SU(2)_A \times SU(2)_B$. The flavor sector has only $U(N_f) \times U(N_f')$ flavor symmetry.

Within this class of theories we can select the one corresponding to

$$\begin{aligned} \lambda_1 = \frac{h_3}{2} \qquad , \qquad \lambda_2 = \frac{h_4}{2} \qquad , \qquad \lambda_3 = 0 \\ \alpha_1 = \alpha_2 = h_3 \qquad , \qquad \alpha_3 = \alpha_4 = h_4 \end{aligned} \tag{6.1.14}$$

The values $h_3 = \frac{1}{K_1}, h_4 = \frac{1}{K_2}$ give the $\mathcal{N}=3$ superconformal theory with flavors mentioned in [83]. It corresponds to flavoring the $\mathcal{N}=3$ theory of [79].

We now proceed to the quantization of the theory in a manifest $\mathcal{N}=2$ setup.
In each gauge sector we choose gauge-fixing functions $\bar{F} = D^2 V$, $F = \bar{D}^2 V$ and insert into the functional integral the factor

$$\int \mathcal{D}f\mathcal{D}\bar{f}\, \Delta(V)\Delta^{-1}(V) \exp\Big\{ -\frac{K}{2\alpha}\int d^3x d^2\theta \mathrm{Tr}(ff) - \frac{K}{2\alpha}\int d^3x d^2\bar{\theta}\mathrm{Tr}(\bar{f}\bar{f})\Big\} \tag{6.1.15}$$

where $\Delta(V) = \int d\Lambda d\bar{\Lambda}\delta(F(V,\Lambda,\bar{\Lambda}) - f)\delta(\bar{F}(V,\Lambda,\bar{\Lambda}) - \bar{f})$ and the weighting function has been chosen in order to have a dimensionless gauge parameter α. We note that the choice of the weighting function is slightly different from the four dimensional case [13] where we usually use $\int \mathcal{D}f\mathcal{D}\bar{f} \exp\Big\{-\frac{1}{g^2\alpha}\int d^4x d^4\theta \mathrm{Tr}(f\bar{f})\Big\}$.

The quadratic part of the gauge–fixed action reads

$$\begin{aligned} S_{CS} + S_{gf} \to \frac{1}{2}K_1 \int & d^4x d^4\theta \; \mathrm{Tr}\, V \left(\bar{D}^\alpha D_\alpha + \frac{1}{\alpha}D^2 + \frac{1}{\alpha}\bar{D}^2\right) V \\ +\frac{1}{2}K_2 \int & d^4x d^4\theta \; \mathrm{Tr}\, \hat{V} \left(\bar{D}^\alpha D_\alpha + \frac{1}{\alpha}D^2 + \frac{1}{\alpha}\bar{D}^2\right) \hat{V} \end{aligned} \tag{6.1.16}$$

and leads to the gauge propagators

$$\langle V^A(1)\, V^B(2)\rangle = -\frac{1}{K_1}\frac{1}{\Box}\left(\bar{D}^\alpha D_\alpha + \alpha D^2 + \alpha\bar{D}^2\right)\, \delta^4(\theta_1 - \theta_2)\, \delta^{AB} \tag{6.1.17}$$

$$\langle \hat{V}^A(1)\, \hat{V}^B(2)\rangle = -\frac{1}{K_2}\frac{1}{\Box}\left(\bar{D}^\alpha D_\alpha + \alpha D^2 + \alpha\bar{D}^2\right)\, \delta^4(\theta_1 - \theta_2)\, \delta^{AB} \tag{6.1.18}$$

In our calculations we will use the analog of the Landau gauge, $\alpha = 0$.

Expanding $S_{CS}+S_{gf}$ at higher orders in $V, \hat{V}$ we obtain the interaction vertices. For two–loop calculations we need

$$\begin{aligned} S_{CS}+S_{gf} \rightarrow & \frac{i}{6}K_1 f^{ABC}\int d^4x d^4\theta \, \left(\bar{D}^\alpha V^A \, V^B \, D_\alpha V^C\right) \\ & -\frac{1}{24}K_1 f^{ABE} f^{ECD}\int d^4x d^4\theta \, \left(\bar{D}^\alpha V^A \, V^B \, D_\alpha V^C \, V^D\right) \\ & +\frac{i}{6}K_2 f^{ABC}\int d^4x d^4\theta \, \left(\bar{D}^\alpha \hat{V}^A \, \hat{V}^B \, D_\alpha \hat{V}^C\right) \\ & -\frac{1}{24}K_2 f^{ABE} f^{ECD}\int d^4x d^4\theta \, \left(\bar{D}^\alpha \hat{V}^A \, \hat{V}^B \, D_\alpha \hat{V}^C \, \hat{V}^D\right) \end{aligned} \qquad (6.1.19)$$

The ghost action is the same as the one of the four dimensional $\mathcal{N}=1$ case [13]

$$S_{gh} = \mathrm{Tr}\int d^4x d^4\theta \left[\overline{c}'c - c'\overline{c} + \frac{1}{2}(c'+\overline{c}')[V,(c+\overline{c})]\right] + \mathcal{O}(V^2) \qquad (6.1.20)$$

and gives ghost propagators

$$\langle \overline{c}'(1)\, c(2)\rangle = \langle c'(1)\, \overline{c}(2)\rangle = -\frac{1}{\Box}\delta^4(\theta_1-\theta_2) \qquad (6.1.21)$$

and cubic interaction vertices

$$\frac{i}{2}f^{ABC}\int d^4x d^4\theta \left(c'^A V^B c^C + \overline{c}'^A V^B c^C + c'^A V^B \overline{c}^C + \overline{c}'^A V^B \overline{c}^C\right) \qquad (6.1.22)$$

We now quantize the matter sector. From the quadratic part of the action (6.1.3) we read the propagators

$$\begin{aligned} & \langle \bar{A}^{\hat{a}}{}_a(1)\, A^b{}_{\hat{b}}(2)\rangle = -\frac{1}{\Box}\delta^4(\theta_1-\theta_2)\, \delta^{\hat{a}}{}_{\hat{b}}\, \delta_a{}^b \qquad (6.1.23) \\ & \langle \bar{B}^a{}_{\hat{a}}(1)\, B^{\hat{b}}{}_b(2)\rangle = -\frac{1}{\Box}\delta^4(\theta_1-\theta_2)\, \delta^a{}_b\, \delta_{\hat{a}}{}^{\hat{b}} \\ & \langle (\bar{Q}^1_r)_a(1)\, (Q^q_1)^b(2)\rangle = -\frac{1}{\Box}\delta^4(\theta_1-\theta_2)\, \delta_a{}^b \delta_r{}^q \\ & \langle (\tilde{Q}_{1,r})_a(1)\, (\bar{\tilde{Q}}^{1,q})^b(2)\rangle = -\frac{1}{\Box}\delta^4(\theta_1-\theta_2)\, \delta_a{}^b \delta_r{}^q \qquad r,q = 1,\cdots,N_f \\ & \langle (\bar{Q}^2_{r'})_{\hat{a}}(1)\, (Q^{q'}_2)^{\hat{b}}(2)\rangle = -\frac{1}{\Box}\delta^4(\theta_1-\theta_2)\, \delta_{\hat{a}}{}^{\hat{b}} \delta_{r'}{}^{q'} \\ & \langle (\tilde{Q}_{2,r'})_{\hat{a}}(1)\, (\bar{\tilde{Q}}^{2,q'})^{\hat{b}}(2)\rangle = -\frac{1}{\Box}\delta^4(\theta_1-\theta_2)\, \delta_{\hat{a}}{}^{\hat{b}} \delta_{r'}{}^{q'} \qquad r',q' = 1,\cdots,N'_f \end{aligned}$$

Figure 6.1: One–loop diagrams for scalar propagators.

From the expansion of (6.1.3) mixed gauge/matter vertices entering two–loop calculations are

$$S_{\rm mat} \to \int d^4x d^4\theta \, {\rm Tr} \left(\bar{A} V A - \bar{A} A \hat{V} + \bar{B} \hat{V} B - \bar{B} B V \right) \tag{6.1.24}$$
$$+ \int d^4x d^4\theta \, {\rm Tr} \left(\frac{1}{2} \bar{A} V V A + \frac{1}{2} \bar{A} A \hat{V} \hat{V} - \bar{A} V A \hat{V} + \frac{1}{2} \bar{B} \hat{V} \hat{V} B + \frac{1}{2} \bar{B} B V V - \bar{B} \hat{V} B V \right)$$
$$+ \int d^4x d^4\theta \, {\rm Tr} \left(\bar{Q}^1_r V Q^r_1 - \bar{\tilde{Q}}^{1,r} \tilde{Q}_{1,r} V + \bar{Q}^2_{r'} \hat{V} Q^{r'}_2 - \bar{\tilde{Q}}^{2,r'} \tilde{Q}_{2,r'} \hat{V} \right)$$
$$+ \int d^4x d^4\theta \, {\rm Tr} \left(\frac{1}{2} \bar{Q}^1_r V V Q^r_1 + \frac{1}{2} \bar{\tilde{Q}}^{1,r} \tilde{Q}_{1,r} V V + \frac{1}{2} \bar{Q}^2_{r'} \hat{V} \hat{V} Q^{r'}_2 + \frac{1}{2} \bar{\tilde{Q}}^{2,r'} \tilde{Q}_{2,r'} \hat{V} \hat{V} \right)$$

Pure matter vertices can be read from the superpotential (6.1.4).

6.2 Two–loop renormalization and β–functions

It is well known that even in the presence of matter chiral superfields the CS actions cannot receive loop divergent corrections [37, 94]. In fact, gauge invariance requires $2\pi K_1, 2\pi K_2$ to be integers, so preventing any renormalization except for a finite shift. In particular, for the $\mathcal{N} = 2$ case it has been proved [94] that even finite renormalization is absent.

Divergent contributions are then expected only in the matter sector. Since a non–renormalization theorem still holds for the superpotential (in $\mathcal{N} = 2$ superspace perturbative calculations one can never produce local, chiral divergent contributions) divergences arise only in the Kähler sector and lead to field functions renormalization.

In odd spacetime dimensions there are no UV divergences at odd loops. Therefore, the first non trivial tests for the perturbative quantum properties of the theory arise at two loops.

6.2.1 One loop results

We first compute the finite quantum corrections to the scalar and gauge propagators which then enter two-loop computations.

The only diagrams contributing to the matter field propagators are the ones given in Fig. 6.1. It is easy to verify that they vanish for symmetry reasons.

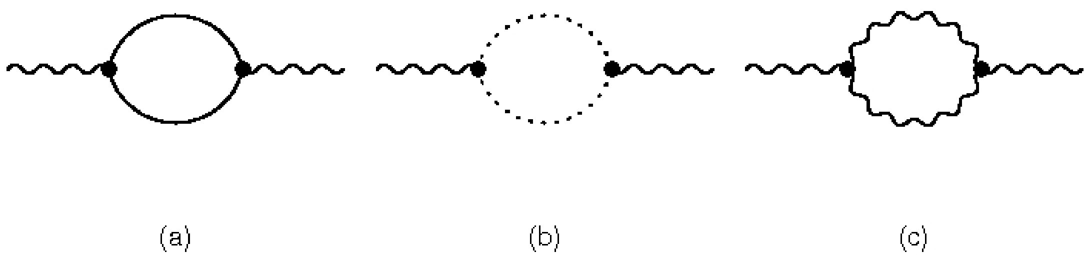

Figure 6.2: One–loop diagrams for gauge propagators.

We then move to the gauge propagator. Gauge one-loop self–energy contributions come from diagrams in Fig. 6.2 where chiral, gauge and ghost loops are present.

Performing the calculation in momentum space and using the superspace projectors [13]

$$\Pi_0 \equiv -\frac{1}{k^2}\{D^2, \bar{D}^2\}(k) \quad , \quad \Pi_{1/2} \equiv \frac{1}{k^2}\bar{D}^\alpha D^2 \bar{D}_\alpha(k) \qquad\qquad \Pi_0 + \Pi_{1/2} = 1 \tag{6.2.25}$$

we find the following finite contributions to the quadratic action for the gauge fields

$$\begin{aligned}
\Pi^{(1)}_{gauge}(2a) &= \frac{1}{8} f^{ABC} f^{A'BC} \int \frac{d^3k}{(2\pi)^3} d^4\theta \; B_0(k)\, k^2\, V^A(k)\, \Pi_0\, V^{A'}(-k) \\
\Pi^{(1)}_{gauge}(2b) &= -\frac{1}{8} f^{ABC} f^{A'BC} \int \frac{d^3k}{(2\pi)^3} d^4\theta \; B_0(k)\, k^2\, V^A(k) \left(\Pi_0 + \Pi_{1/2}\right) V^{A'}(-k) \\
\Pi^{(1)}_{gauge}(2c) &= \left(M + \frac{N_f}{2}\right) \delta^{AA'} \int \frac{d^3k}{(2\pi)^3} d^4\theta \; B_0(k)\, k^2\, V^A(k)\, \Pi_{1/2}\, V^{A'}(-k)
\end{aligned} \tag{6.2.26}$$

$$\begin{aligned}
\hat{\Pi}^{(1)}_{gauge}(2a) &= \frac{1}{8} \hat{f}^{ABC} \hat{f}^{A'BC} \int \frac{d^3p}{(2\pi)^3} d^4\theta \; B_0(p)\, p^2\, \hat{V}^A(p)\, \Pi_0\, \hat{V}^{A'}(-p) \\
\hat{\Pi}^{(1)}_{gauge}(2b) &= -\frac{1}{8} \hat{f}^{ABC} \hat{f}^{A'BC} \int \frac{d^3p}{(2\pi)^3} d^4\theta \; B_0(p)\, p^2\, \hat{V}^A(p) \left(\Pi_0 + \Pi_{1/2}\right) \hat{V}^{A'}(-p) \\
\hat{\Pi}^{(1)}_{gauge}(2c) &= \left(N + \frac{N'_f}{2}\right) \delta^{AA'} \int \frac{d^3p}{(2\pi)^3} d^4\theta \; B_0(p)\, p^2\, \hat{V}^A(p)\, \Pi_{1/2}\, \hat{V}^{A'}(-p)
\end{aligned} \tag{6.2.27}$$

$$\tilde{\Pi}^{(1)}_{gauge}(2c) \;=\; -2\sqrt{NM}\delta^{A0}\delta^{A'0} \int \frac{d^3p}{(2\pi)^3} d^4\theta \; B_0(p)\, p^2\, V^A(p)\, \Pi_{1/2}\, \hat{V}^{A'}(-p) \tag{6.2.28}$$

where $B_0(p) = 1/(8|p|)$ is the three dimensional bubble scalar integral (see (A.3.26)).

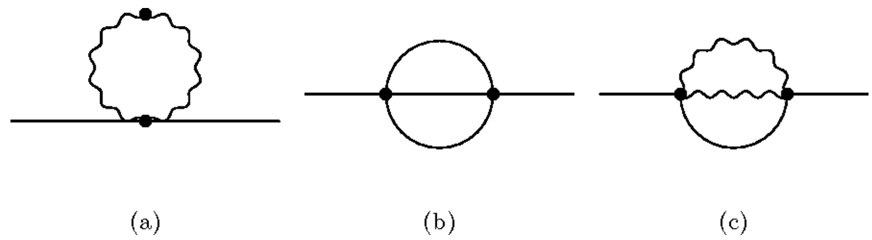

Figure 6.3: Two–loop divergent diagrams contributing to the matter propagators.

Summing all the contributions we see that the gauge loop cancels against part of the ghost loop as in the 4D $\mathcal{N}=1$ case [15] and we find the known results [15, 95]

$$\Pi^{(1)}_{gauge} = \left[-\frac{1}{8} f^{ABC} f^{A'BC} + \left(M + \frac{N_f}{2}\right)\delta^{AA'}\right] \int \frac{d^3p}{(2\pi)^3} d^4\theta \; B_0(p)\, p^2\, V^A(p)\, \Pi_{1/2}\, V^{A'}(-p)$$
$$\hat{\Pi}^{(1)}_{gauge} = \left[-\frac{1}{8} f^{ABC} f^{A'BC} + \left(N + \frac{N'_f}{2}\right)\delta^{AA'}\right] \int \frac{d^3p}{(2\pi)^3} d^4\theta \; B_0(p)\, p^2\, \hat{V}^A(p)\, \Pi_{1/2}\, \hat{V}^{A'}(-p) \quad (6.2.29)$$

together with $\tilde{\Pi}^{(1)}_{gauge}$ in (6.2.28) which mixes the two $U(1)$ gauge sectors.

6.2.2 Two-loop results

We are now ready to evaluate the matter self–energy contributions at two loops. Both for the bifundamental and the flavor matter the divergent diagrams are given in Fig. 6.3.

Evaluation of each diagram proceeds in the standard way by first performing D–algebra in order to reduce supergraphs to ordinary Feynman graphs and evaluate them in momentum space and dimensional regularization ($d = 3 - 2\epsilon$). Separating the contributions of each diagram, the results for the bifundamental matter are

$$\Pi^{(2)}_{bif}(3a) = -\left[\frac{1}{K_1^2}\left(2NM + NN_f - \frac{1}{2}\left(N^2-1\right)\right) + \right.$$
$$\left. + \frac{1}{K_2^2}\left(2NM + MN'_f - \frac{1}{2}\left(M^2-1\right)\right) + \frac{4}{K_1 K_2}\right] F(0)\ \mathrm{Tr}\left(\bar{A}_i A^i + \bar{B}^i B_i\right)$$

$$\Pi^{(2)}_{bif}(3b) = \Big[4|h_1|^2(MN+1) + (|h_3|^2 + |h_4|^2)MN + (h_3\bar{h}_4 + h_4\bar{h}_3)$$
$$+ (|\alpha_1|^2 N N_f + |\alpha_3|^2 M N'_f)\Big] F(p)\ \mathrm{Tr}\left(\bar{A}_1 A^1 + \bar{B}^1 B_1\right)$$
$$+ \Big[4|h_2|^2(MN+1) + (|h_3|^2 + |h_4|^2)MN + (h_3\bar{h}_4 + h_4\bar{h}_3)$$
$$+ (|\alpha_2|^2 N N_f + |\alpha_4|^2 M N'_f)\Big] F(p)\ \mathrm{Tr}\left(\bar{A}_2 A^2 + \bar{B}^2 B_2\right)$$

$$\Pi^{(2)}_{bif}(3c) = -\frac{1}{2}\left[\frac{N^2+1}{K_1^2} + \frac{M^2+1}{K_2^2} + \frac{4NM}{K_1 K_2}\right] F(p)\ \mathrm{Tr}\left(\bar{A}_i A^i + \bar{B}^i B_i\right) \quad (6.2.30)$$

where $F(p)$ is the two–loop self–energy integral given in (A.3.28).

Analogously, for fundamental matter we find

$$\begin{aligned}
\Pi^{(2)}_{fund1}(3a) &= -\frac{1}{K_1^2}\left(2NM + NN_f - \frac{1}{2}\left(N^2-1\right)\right) F(0)\,\mathrm{Tr}\left(\bar{Q}^1 Q_1 + \bar{\tilde{Q}}^1 \tilde{Q}_1\right) \\
\Pi^{(2)}_{fund2}(3a) &= -\frac{1}{K_2^2}\left(2NM + MN'_f - \frac{1}{2}\left(M^2-1\right)\right) F(0)\,\mathrm{Tr}\left(\bar{Q}^2 Q_2 + \bar{\tilde{Q}}^2 \tilde{Q}_2\right) \\
\Pi^{(2)}_{fund1}(3b) &= \Big[4|\lambda_1|^2(NN_f+1) + |\lambda_3|^2 MN'_f \\
&\qquad + \left(|\alpha_1|^2 + |\alpha_2|^2\right) MN\Big]\, F(p)\,\mathrm{Tr}\left(\bar{Q}^1 Q_1 + \bar{\tilde{Q}}^1 \tilde{Q}_1\right) \\
\Pi^{(2)}_{fund2}(3b) &= \Big[4|\lambda_2|^2(MN'_f+1) + |\lambda_3|^2 NN_f \\
&\qquad + \left(|\alpha_3|^2 + |\alpha_4|^2\right) NM\Big]\, F(p)\,\mathrm{Tr}\left(\bar{Q}^2 Q_2 + \bar{\tilde{Q}}^2 \tilde{Q}_2\right) \\
\Pi^{(2)}_{fund1}(3c) &= -\frac{N^2+1}{2K_1^2}\, F(p)\,\mathrm{Tr}\left(\bar{Q}^1 Q_1 + \bar{\tilde{Q}}^1 \tilde{Q}_1\right) \\
\Pi^{(2)}_{fund2}(3c) &= -\frac{M^2+1}{2K_2^2}\, F(p)\,\mathrm{Tr}\left(\bar{Q}^2 Q_2 + \bar{\tilde{Q}}^2 \tilde{Q}_2\right)
\end{aligned} \qquad (6.2.31)$$

where $F(p)$ is still given in (A.3.28).

We now proceed to the renormalization of the theory. We define renormalized fields as

$$\Phi = Z_\Phi^{-\frac{1}{2}}\Phi_B \qquad , \qquad \bar{\Phi} = \bar{Z}_{\bar{\Phi}}^{-\frac{1}{2}}\bar{\Phi}_B \qquad (6.2.32)$$

where Φ stands for any chiral field of the theory, and coupling constants as

$$\begin{aligned}
h_j &= \mu^{-2\epsilon} Z_{h_j}^{-1} h_{j\,B} & \bar{h}_j &= \mu^{-2\epsilon} Z_{\bar{h}_j}^{-1} \bar{h}_{j\,B} \\
\lambda_j &= \mu^{-2\epsilon} Z_{\lambda_j}^{-1} \lambda_{j\,B} & \bar{\lambda}_j &= \mu^{-2\epsilon} Z_{\bar{\lambda}_j}^{-1} \bar{\lambda}_{j\,B} \\
\alpha_j &= \mu^{-2\epsilon} Z_{\alpha_j}^{-1} \alpha_{j\,B} & \bar{\alpha}_j &= \mu^{-2\epsilon} Z_{\bar{\alpha}_j}^{-1} \bar{\alpha}_{j\,B}
\end{aligned}$$

together with $K_1 = \mu^{2\epsilon} K_{1\,B}, K_2 = \mu^{2\epsilon} K_{2\,B}$. Powers of the renormalization mass μ have been introduced in order to deal with dimensionless renormalized couplings.

In order to cancel the divergences in (6.2.30) and (6.2.31) we choose

$$Z_{A^1} = Z_{\bar{A}_1} = Z_{B_1} = Z_{\bar{B}^1} = 1 - \tag{6.2.33}$$
$$\frac{1}{64\pi^2}\Big[-\frac{2NM+NN_f+1}{K_1^2} - \frac{2NM+MN_f'+1}{K_2^2} - \frac{2NM+4}{K_1K_2}$$
$$+4|h_1|^2(MN+1) + (|h_3|^2+|h_4|^2)MN + (h_3\bar{h}_4 + h_4\bar{h}_3) + (|\alpha_1|^2 NN_f + |\alpha_3|^2 MN_f')\Big]\frac{1}{\epsilon}$$

$$Z_{A^2} = Z_{\bar{A}_2} = Z_{B_2} = Z_{\bar{B}^2} = 1 -$$
$$\frac{1}{64\pi^2}\left[-\frac{2NM+NN_f+1}{K_1^2} - \frac{2NM+MN_f'+1}{K_2^2} - \frac{2NM+4}{K_1K_2}\right.$$
$$\left.+4|h_2|^2(MN+1) + (|h_3|^2+|h_4|^2)MN + (h_3\bar{h}_4 + h_4\bar{h}_3) + (|\alpha_2|^2 NN_f + |\alpha_4|^2 MN_f')\right]\frac{1}{\epsilon}$$

$$Z_{Q_1} = Z_{\bar{Q}^1} = Z_{\tilde{Q}_1} = Z_{\bar{\tilde{Q}}^1} = 1 -$$
$$\frac{1}{64\pi^2}\left[-\frac{2NM+NN_f+1}{K_1^2} + 4|\lambda_1|^2\,(NN_f+1) + |\lambda_3|^2 MN_f' + (|\alpha_1|^2+|\alpha_2|^2)MN\right]\frac{1}{\epsilon}$$

$$Z_{Q_2} = Z_{\bar{Q}^2} = Z_{\tilde{Q}_2} = Z_{\bar{\tilde{Q}}^2} = 1 -$$
$$\frac{1}{64\pi^2}\left[-\frac{2NM+MN_f'+1}{K_2^2} + 4|\lambda_2|^2\,(MN_f'+1) + |\lambda_3|^2 NN_f + (|\alpha_3|^2+|\alpha_4|^2)MN\right]\frac{1}{\epsilon}$$

Thanks to the non-renormalization theorem for the superpotential, the renormalization of the couplings is a consequence of the field renormalization. In particular, we set

$$Z_{\nu_j} = \prod_{\Phi_i} Z_{\Phi_i}^{-\frac{1}{2}} \tag{6.2.34}$$

where ν_j stands for any coupling of the theory and the sum is extended to all the Φ_i fields coupled by ν_j.

The anomalous dimensions and the beta-functions are given by the general prescription

$$\gamma_{\Phi_j} \equiv \frac{1}{2}\frac{\partial \log Z_{\Phi_j}}{\partial \log \mu} = -\frac{1}{2}\sum_i d_i\,\nu_i\,\frac{\partial Z^{(1)}_{\Phi_j}}{\partial \nu_i} \tag{6.2.35}$$

$$\beta_{\nu_j} = -d_j\,\nu_j^{(1)} + \sum_i \left(d_i\,\nu_i\,\frac{\partial \nu_j^{(1)}}{\partial \nu_i}\right) = \nu_j(\mu)\sum_i \gamma_i \tag{6.2.36}$$

where d_j is the bare dimension of the ν_j–coupling and $Z^{(1)}_{\Phi_j}$ is the coefficient of the $1/\epsilon$ pole in Z_{Φ_j}. The last equality in (6.2.36) follows from (6.2.35) and (6.2.34).

Reading the single pole coefficient $Z^{(1)}_{\Phi_j}$ in eqs. (6.2.33) we finally obtain

$$
\begin{aligned}
\gamma_{A^1} = \gamma_{B_1} = \frac{1}{32\pi^2}\Big[& -\frac{2NM+NN_f+1}{K_1^2} - \frac{2NM+MN'_f+1}{K_2^2} - \frac{2NM+4}{K_1K_2} \\
& +4|h_1|^2(MN+1) + (|h_3|^2+|h_4|^2)MN + (h_3\bar{h}_4 + h_4\bar{h}_3) \\
& +(|\alpha_1|^2 NN_f + |\alpha_3|^2 MN'_f)\Big] \\
\gamma_{A^2} = \gamma_{B_2} = \frac{1}{32\pi^2}\Big[& -\frac{2NM+NN_f+1}{K_1^2} - \frac{2NM+MN'_f+1}{K_2^2} - \frac{2NM+4}{K_1K_2} \\
& +4|h_2|^2(MN+1) + (|h_3|^2+|h_4|^2)MN + (h_3\bar{h}_4 + h_4\bar{h}_3) \\
& +(|\alpha_2|^2 NN_f + |\alpha_4|^2 MN'_f)\Big] \\
\gamma_{Q_1} = \gamma_{\tilde{Q}_1} = \frac{1}{32\pi^2}\Big[& -\frac{2NM+NN_f+1}{K_1^2} \\
& +4|\lambda_1|^2\,(NN_f+1) + |\lambda_3|^2 MN'_f + (|\alpha_1|^2+|\alpha_2|^2)MN\Big] \\
\gamma_{Q_2} = \gamma_{\tilde{Q}_2} = \frac{1}{32\pi^2}\Big[& -\frac{2NM+MN'_f+1}{K_2^2} \\
& +4|\lambda_2|^2\,(MN'_f+1) + |\lambda_3|^2 NN_f + (|\alpha_3|^2+|\alpha_4|^2)MN\Big]
\end{aligned}
\tag{6.2.37}
$$

whereas the corresponding beta–functions are given by

$$
\begin{aligned}
&\beta_{h_1} = 4h_1\gamma_{A^1} && \beta_{h_2} = 4h_2\gamma_{A^2} \\
&\beta_{h_3} = 2h_3(\gamma_{A^1}+\gamma_{A^2}) && \beta_{h_4} = 2h_4(\gamma_{A^1}+\gamma_{A^2}) \\
&\beta_{\lambda_1} = 4\lambda_1\gamma_{Q_1} && \beta_{\lambda_2} = 4\lambda_2\gamma_{Q_2} \\
&\qquad\beta_{\lambda_3} = 2\lambda_3(\gamma_{Q_1}+\gamma_{Q_2}) && \\
&\beta_{\alpha_1} = 2\alpha_1(\gamma_{A_1}+\gamma_{Q_1}) && \beta_{\alpha_2} = 2\alpha_2(\gamma_{A_2}+\gamma_{Q_1}) \\
&\beta_{\alpha_3} = 2\alpha_3(\gamma_{A_1}+\gamma_{Q_2}) && \beta_{\alpha_4} = 2\alpha_4(\gamma_{A_2}+\gamma_{Q_2})
\end{aligned}
\tag{6.2.38}
$$

6.3 The spectrum of fixed points

In this Section we study solutions to the equations $\beta_{\nu_j} = 0$ where the beta–functions are given in (6.2.38). We consider separately the cases with and without flavor matter.

6.3.1 Theories without flavors

We begin by considering the class of theories without flavors. In eqs. (6.2.37) we set $N_f = N'_f = 0$, $\lambda_j = \alpha_j = 0$ and solve the equations

$$
\begin{aligned}
&\beta_{h_1} = 4h_1\gamma_{A^1} = 0 && \beta_{h_2} = 4h_2\gamma_{A^2} = 0 \\
&\beta_{h_3} = 2h_3(\gamma_{A^1}+\gamma_{A^2}) = 0 && \beta_{h_4} = 2h_4(\gamma_{A^1}+\gamma_{A^2}) = 0
\end{aligned}
\tag{6.3.39}
$$

When $h_j \neq 0$ for any j the conditions (6.3.39) are equivalent to $\gamma_{A^1} = \gamma_{A^2} = 0$, that is no UV divergences appear at two–loops. On the other hand, if we restrict to the surface $h_1 = h_2 = 0$, the beta–functions are zero when $\gamma_{A^1} + \gamma_{A^2} = 0$, which in principle would not require the anomalous dimensions to vanish. However, it is easy to see from (6.2.37) that for $h_1 = h_2 = 0$ we have $\gamma_{A^1} = \gamma_{A^2}$ and again $\beta_{h_3} = \beta_{h_4} = 0$ imply the vanishing of all the anomalous dimensions. Therefore, at two loops the request for vanishing beta–functions is equivalent to the request of finiteness.

We first study the class of theories with $h_1 = h_2 = 0$. In this case we find convenient to redefine the couplings as [92]

$$y_1 = h_3 + h_4 \qquad , \qquad y_2 = h_3 - h_4 \tag{6.3.40}$$

In fact, writing the superpotential in terms of the new couplings

$$\int d^4x d^2\theta \left[\frac{y_1}{2} \mathrm{Tr}(A^1 B_1 A^2 B_2 + A^2 B_1 A^1 B_2) \; + \; \frac{y_2}{4} \epsilon_{ij} \epsilon^{kl} \mathrm{Tr}(A^i B_k A^j B_l) \right] \tag{6.3.41}$$

it is easy to see that y_1 is associated to a $SU(2)_A \times SU(2)_B$ breaking perturbation, whereas y_2 is symmetry preserving.

When $K_1 = -K_2 \equiv K$, the spectrum of nontrivial fixed points is given by the condition

$$y_1^2(MN+1) + y_2^2(MN-1) = \frac{4}{K^2}(MN-1) \tag{6.3.42}$$

which describes an ellipse in the space of the couplings, as already found in [95]. We note that in the large M, N limit and for $K \gg 1$ this becomes a circle with center in the origin and radius infinitesimally small. Therefore, the solutions fall inside the region of validity of the perturbative description. The particular point $(0, 2/K)$ corresponds to the ABJ/ABJM models.

The result (6.3.42) read in terms of the original couplings (h_1, h_2) states that in the class of scalar, dimension–two composite operators of the form

$$\mathcal{O} = h_1 \, \mathrm{Tr}(A^1 B_1 A^2 B_2) + h_2 \, \mathrm{Tr}(A^2 B_1 A^1 B_2) \tag{6.3.43}$$

there is only one exactly marginal operator. This is the operator which allows the system to move along the fixed line.

More generally, for real couplings the anomalous dimensions vanish when

$$y_1^2(MN+1) + y_2^2(MN-1) = 2(2MN+1)\left(\frac{1}{K_1^2} + \frac{1}{K_2^2}\right) + 2\frac{2MN+4}{K_1 K_2} \tag{6.3.44}$$

This describes an ellipse in the parameter space. For $K_{1,2}$ sufficiently large it is very closed to the origin and solutions fall in the perturbative regime. The ellipse degenerates to a circle in the large M, N limit. Fixed points corresponding to $y_1 \neq 0$ ($h_4 \neq -h_3$) describe $\mathcal{N} = 2$ superconformal theories with $U(1)_A \times U(1)_B$ global symmetry (6.1.8).

A more symmetric conformal point is obtained by solving (6.3.44) under the condition $y_1 = 0$. The solution

$$h_3 = -h_4 = \sqrt{\frac{2MN+1}{2(MN-1)}\left(\frac{1}{K_1^2} + \frac{1}{K_2^2}\right) + \frac{MN+2}{MN-1}\frac{1}{K_1 K_2}} \tag{6.3.45}$$

corresponds to a superconformal theory with $SU(2)_A \times SU(2)_B$ global symmetry. This is the theory conjectured in [79]. When $K_1 = -K_2 \equiv K$ it reduces to $h_3 = -h_4 = 1/K$ and we recover the $\mathcal{N} = 6$ ABJ model [70] and, for $N = M$, the ABJM one [45].

More generally, we study fixed points with $h_j \neq 0$ for any j. In this case we have two equations, $\gamma_{A^1} = \gamma_{A^2} = 0$, for four unknowns. The spectrum of fixed points then spans a two dimensional surface which for real couplings is given by

$$h_1^2 = h_2^2 = \frac{1}{4(MN+1)}\Big[(2MN+1)\left(\frac{1}{K_1^2} + \frac{1}{K_2^2}\right) + 2\frac{MN+2}{K_1 K_2} - MN(h_3^2 + h_4^2) - 2h_3 h_4\Big] \tag{6.3.46}$$

This equation describes an ellipsoid in the four dimensional h–space as given in Fig. 6.4, localized in the subspace $h_1 = h_2$ (or equivalently $h_1 = -h_2$). A particular point on this surface corresponds to $h_3 = 1/K_1$ and $h_4 = 1/K_2$ with, consequently, $h_1 = h_2 = \frac{1}{2}(\frac{1}{K_1} + \frac{1}{K_2})$. This is the $\mathcal{N} = 3$ superconformal theory discussed in [79] [2].

The locus $h_1 = h_2 = 0$, $h_3 = -h_4$ of this surface is the $\mathcal{N} = 2$, $SU(2)_A \times SU(2)_B$ invariant superconformal theory (6.3.45). Therefore, the $\mathcal{N} = 3$ and the $\mathcal{N} = 2$, $SU(2)_A \times SU(2)_B$ superconformal points are continuously connected by the surface (6.3.46).

We can select a particular line of fixed points interpolating between the two theories, by setting

$$h_1 = h_2 = \frac{1}{2}(h_3 + h_4) \tag{6.3.47}$$

and, consequently

$$h_3^2 + h_4^2 + 2\frac{MN+2}{2MN+1}\, h_3 h_4 = \frac{1}{K_1^2} + \frac{1}{K_2^2} + 2\frac{MN+2}{K_1 K_2\,(2MN+1)} \tag{6.3.48}$$

These are $SU(2)$ invariant, $\mathcal{N} = 2$ superconformal theories with superpotential (6.1.11). The existence of a line of $SU(2)$ invariant fixed points interpolating between the two theories was already conjectured in [79].

So far we have considered real solutions to the equations $\beta_{\nu_j} = 0$. We now discuss the case of complex couplings focusing in particular on the so–called β–deformations.

In the class of theories with $h_1 = h_2 = 0$ we look for solutions of the form

$$h_3 = h e^{i\pi\beta} \qquad , \qquad h_4 = -h e^{-i\pi\beta} \tag{6.3.49}$$

which implies $y_1 = 2h \sin \pi\beta$, $y_2 = 2h \cos \pi\beta$ in (6.3.40). The condition for vanishing beta–functions then reads

$$h^2 MN - h^2 \cos 2\pi\beta = \frac{1}{2}(2MN+1)\left(\frac{1}{K_1^2} + \frac{1}{K_2^2}\right) + \frac{MN+2}{K_1 K_2} \tag{6.3.50}$$

This describes a line of fixed points which correspond to superconformal beta–deformations of the $SU(2)_A \times SU(2)_B$ invariant theory (6.3.45). For $\beta \neq 0$ the global symmetry is broken to

[2] Finiteness properties of $\mathcal{N} = 3$ CS–matter theories have also been discussed in [81] within the $\mathcal{N} = 3$ harmonic superspace setup.

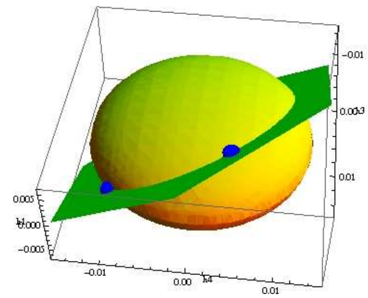

Figure 6.4: The exactly marginal surface of fixed points in the space of h_i couplings, restricted to the subspace $h_1 = h_2$. The parameters have been chosen as $K_1 = 150, K_2 = 237, N = 43, M = 30$. The dots denote the $\mathcal{N} = 3$ and the $\mathcal{N} = 2$, $SU(2)_A \times SU(2)_B$ fixed points belonging to the ellipsoid. The plane represents the class of theories (6.1.11) with $SU(2)$ global symmetry and its intersection with the ellipsoid is the line described by (6.3.48).

$U(1)_A \times U(1)_B$ in (6.1.8) and the deformed theory is only $\mathcal{N} = 2$ supersymmetric. In particular, setting $K_1 = -K_2$ we obtain the β–deformed ABJM/ABJ theories studied in [77].

In the large M, N limit the β–dependence of equation (6.3.50) disappears, consistently with the fact that in planar Feynman diagrams the effects of the deformation are invisible [96]. In this limit the condition for superconformal invariance reads

$$h^2 = \frac{1}{K_1^2} + \frac{1}{K_2^2} + \frac{1}{K_1 K_2} \tag{6.3.51}$$

which reduces to $h = 1/K$ for opposite CS levels.

The analysis of β–deformations can be extended to theories with $h_1, h_2 \neq 0$. Since they enter the anomalous dimensions only through $|h_1|^2$ and $|h_2|^2$ we can take them to be arbitrarily complex and still make the ansatz (6.3.49) for h_3, h_4. The surface of fixed points is then given by

$$\begin{aligned} |h_1|^2 = |h_2|^2 = \frac{1}{4(MN+1)} \Big[& (2MN+1)\left(\frac{1}{K_1^2} + \frac{1}{K_2^2}\right) + 2\frac{MN+2}{K_1 K_2} \\ & -2h^2 MN + 2h^2 \cos 2\pi\beta \Big] \end{aligned} \tag{6.3.52}$$

and describes superconformal β–deformations of $\mathcal{N} = 2$ invariant theories.

The results of this Section agree with the ones in [95] obtained by using the three–algebra formalism.

6.3.2 Theories with flavors

As in the previous case, when all the couplings are non-vanishing, the request for zero beta–functions implies the finiteness conditions $\gamma_{\Phi_i} = 0$. These provide four constraints on a set of eleven unknowns (see eqs. (6.2.37)). Therefore, in the space of the coupling constants the spectrum of fixed points spans a seven dimensional hypersurface given by the equations

$$|\alpha_2|^2 = \frac{1}{NN_fK_1^2K_2^2}\Big\{K_2^2\,(2NM+NN_f+1)+K_1^2\left(2NM+MN_f'+1\right)$$
$$+2K_1K_2\,(NM+2)-4|h_2|^2\,K_1^2K_2^2\,(MN+1)$$
$$-\,K_1^2K_2^2\left[(|h_3|^2+|h_4|^2)MN+(h_3\bar{h}_4+h_4\bar{h}_3)+|\alpha_4|^2MN_f'\right]\Big\}$$

$$|\alpha_3|^2 = \frac{1}{MN_f'K_1^2K_2^2}\Big\{K_2^2\,(2NM+NN_f+1)+K_1^2\left(2NM+MN_f'+1\right)$$
$$+2K_1K_2\,(NM+2)-4|h_1|^2\,K_1^2K_2^2\,(MN+1)$$
$$-\,K_1^2K_2^2\left[(|h_3|^2+|h_4|^2)MN+(h_3\bar{h}_4+h_4\bar{h}_3)+|\alpha_1|^2NN_f\right]\Big\}$$

$$|\lambda_1|^2 = \frac{1}{4(NN_f+1)K_1^2}\Big\{2NM+NN_f+1-K_1^2\left[|\lambda_3|^2MN_f'+(|\alpha_1|^2+|\alpha_2|^2)MN\right]\Big\}$$
$$|\lambda_2|^2 = \frac{1}{4(MN_f'+1)K_2^2}\Big\{2NM+MN_f'+1-K_2^2\left[|\lambda_3|^2NN_f+(|\alpha_3|^2+|\alpha_4|^2)MN\right]\Big\} \tag{6.3.53}$$

When $K_1=-K_2\equiv K$ a particular point on this surface corresponds to

$$\begin{aligned} &h_1=h_2=0 \quad , \quad h_3=-h_4=\frac{1}{K}\\ &\lambda_1=-\lambda_2=\frac{1}{2K} \quad , \quad \lambda_3=0\\ &\alpha_1=\alpha_2=\frac{1}{K} \quad , \quad \alpha_3=\alpha_4=-\frac{1}{K} \end{aligned} \tag{6.3.54}$$

and describes the $\mathcal{N}=3$ ABJ/ABJM models with flavor matter [82, 83, 84].

More generally, allowing $K_2\neq -K_1$ we find the fixed point

$$\begin{aligned} &h_1=h_2=\frac{1}{2}\left(\frac{1}{K_1}+\frac{1}{K_2}\right) \quad , \quad h_3=\frac{1}{K_1} \quad , \quad h_4=\frac{1}{K_2}\\ &\lambda_1=\frac{1}{2K_1} \quad , \quad \lambda_2=\frac{1}{2K_2} \quad , \quad \lambda_3=0\\ &\alpha_1=\alpha_2=\frac{1}{K_1} \quad , \quad \alpha_3=\alpha_4=\frac{1}{K_2} \end{aligned} \tag{6.3.55}$$

which corresponds to a superconformal theory obtained from the $\mathcal{N}=3$ theory of [79] by the addition of flavor matter [83]. The superpotential

$$\mathcal{S}_{\rm pot} = \int d^3x\, d^2\theta\, {\rm Tr} \Big\{ \frac{1}{2}\left(\frac{1}{K_1}+\frac{1}{K_2}\right)\left[(A^1B_1)^2+(A^2B_2)^2\right] + \frac{1}{K_1}(A^1B_1A^2B_2) + \frac{1}{K_2}(A^2B_1A^1B_2) + \frac{1}{2K_1}(Q_1\tilde{Q}_1)^2 + \frac{1}{2K_2}(Q_2\tilde{Q}_2)^2 + \frac{1}{K_1}\left[\tilde{Q}_1A^iB_iQ_1\right] + \frac{1}{K_2}\left[\tilde{Q}_2B_iA^iQ_2\right]\Big\} + h.c. \qquad (6.3.56)$$

can be thought of as arising from the action

$$\mathcal{S} = \mathcal{S}_{\rm CS} + \mathcal{S}_{\rm mat} + \int d^3x\, d^2\theta \left[-\frac{K_1}{2}{\rm Tr}(\Phi_1^2) + {\rm Tr}(B_i\Phi_1A^i) + {\rm Tr}(\tilde{Q}_1\Phi_1Q_1)\right] + \int d^3x\, d^2\theta \left[-\frac{K_2}{2}{\rm Tr}(\Phi_2^2) + {\rm Tr}(A^i\Phi_2B_i) + {\rm Tr}(\tilde{Q}_2\Phi_2Q_2)\right] + h.c. \qquad (6.3.57)$$

after integration on the Φ_1, Φ_2 chiral superfields belonging to the adjoint representations of the two gauge groups and giving the $\mathcal{N}=4$ completion of the vector multiplet. Therefore, as in the unflavored case, the theory exhibits $\mathcal{N}=3$ supersymmetry with the couples $(A,B^\dagger)_i$, $(Q,\tilde{Q}^\dagger)_1^r$ and $(Q,\tilde{Q}^\dagger)_2^{r'}$ realizing $(2+N_f+N_f')$ $\mathcal{N}=4$ hypermultiplets (The CS terms break $\mathcal{N}=4$ to $\mathcal{N}=3$).

As already discussed, in the absence of flavors the $\mathcal{N}=3$ superconformal theory is connected by the line of fixed points (6.3.48) to a $\mathcal{N}=2$, $SU(2)_A\times SU(2)_B$ invariant theory. We now investigate whether a similar pattern arises even when flavors are present.

To this end, we first choose

$$h_1 = h_2 = \frac{1}{2}(h_3+h_4) \quad , \quad \alpha_1=\alpha_2 \quad , \quad \alpha_3=\alpha_4 \qquad (6.3.58)$$

with λ_j arbitrary. This describes a set of $\mathcal{N}=2$ theories with global $SU(2)$ invariance in the bifundamental sector.

Solving the equations $\beta_{\nu_j}=0$ for real couplings we find a whole line of $SU(2)_A\times SU(2)_B$ invariant fixed points parametrized by the unconstrained coupling λ_3

$$\begin{aligned} \alpha_1 = \alpha_3 &= 0 \\ h_3 = -h_4 &= \sqrt{\frac{(2NM+MN_f'+1)K_1^2 + 2(MN+2)K_1K_2 + (2MN+NN_f+1)K_2^2}{2(MN-1)K_1^2K_2^2}} \\ \lambda_1^2 &= \frac{2MN+NN_f+1-K_1^2MN_f'\lambda_3^2}{4K_1^2(NN_f+1)} \\ \lambda_2^2 &= \frac{2MN+MN_f'+1-K_2^2NN_f\lambda_3^2}{4K_2^2(MN_f'+1)} \end{aligned} \qquad (6.3.59)$$

A four dimensional hypersurface of $\mathcal{N}=2$ fixed points given by

$$\begin{aligned}
\alpha_1^2 &= \frac{1}{2MNK_1^2}\Big[-4K_1^2(NN_f+1)\lambda_1^2+NN_f+2MN+1-MN_f'\lambda_3^2K_1^2\Big] \\
\alpha_3^2 &= \frac{1}{2MNK_2^2}\Big[-4K_2^2(MN_f'+1)\lambda_2^2+MN_f'+2MN+1-NN_f\lambda_3^2K_2^2\Big] \\
h_3 &= -\frac{1}{2MN+1}\Bigg\{(MN+2)h_4\pm\Bigg[(2MN+1)\Bigg(MN_f'\left(-\alpha_3^2+4\lambda_2^2+\lambda_3^2\right) \\
&\qquad +2NM(\alpha_1^2+\alpha_3^2+1)+NN_f\left(-\alpha_1^2+4\lambda_1^2+\lambda_3^2\right) \\
&\qquad +4\left(\lambda_1^2+\lambda_2^2\right)+\frac{4}{K_1K_2}\Bigg)-3h_4^2\left(M^2N^2-1\right)\Bigg]^{1/2}\Bigg\}
\end{aligned} \tag{6.3.60}$$

connects the line of $\mathcal{N}=2$, $SU(2)_A\times SU(2)_B$ invariant theories (6.3.59) to the $\mathcal{N}=3$ theory (6.3.55). This is the analogous of the fixed line (6.3.48) found in the unflavored theories.

Before closing this Section we address the question of superconformal invariance versus finiteness for theories with flavor matter. In the bifundamental sector, the only possibility to have vanishing beta–functions without vanishing anomalous dimensions is by setting $h_1=h_2=0$. When flavor matter is present, this does not necessarily imply $\gamma_{A^1}=\gamma_{A^2}$, so we can solve for $\beta_{h_3,h_4}=\gamma_{A^1}+\gamma_{A^2}=0$ without requiring the two γ's to vanish separately. Once these equations have been solved in the bifundamental sector, in the flavor sector we choose $\lambda_1=\lambda_2=0$ and $\alpha_1=\alpha_4=0$ (or equivalently, $\alpha_2=\alpha_3=0$) in order to avoid $\gamma_{Q_1}=\gamma_{Q_2}=0$. We are then left with five couplings subject to the three equations $\gamma_{A^1}+\gamma_{A^2}=0$, $\gamma_{A^1}+\gamma_{Q_2}=0$ and $\gamma_{A^2}+\gamma_{Q_1}=0$. Solutions correspond to superconformal but *not finite* theories. We note that this is true as long as we work with M,N finite. In the large M,N limit with $N_f,N_f'\ll M,N$ we are back to $\gamma_{A^1}=\gamma_{A^2}$, as flavor contributions are subleading. In this case superconformal invariance requires finiteness.

6.4 Infrared stability

We now study the RG flows around the fixed points of main interest in order to establish whether they are IR attractors or repulsors. In particular, we concentrate on the ABJ/ABJM theories, $\mathcal{N}=3$ and $SU(2)_A\times SU(2)_B$ $\mathcal{N}=2$ superconformal points, in all cases with and without flavors.

The behavior of the system around a given fixed point ν_0 is determined by studying the stability matrix

$$\mathcal{M}_{ij}\equiv\frac{d\beta_i}{d\nu_j}(\nu_0) \tag{6.4.61}$$

Diagonalizing $\mathcal{M}$, positive eigenvalues correspond to directions of increasing β–functions, whereas negative eigenvalues give decreasing betas. It follows that the fixed point is IR stable if $\mathcal{M}$ has all positive eigenvalues, whereas negative eigenvalues represent directions where a classically marginal operator becomes relevant.

If null eigenvalues are present we need compute derivatives of the stability matrix along the directions individuated by the corresponding eigenvectors. If along a null direction the second

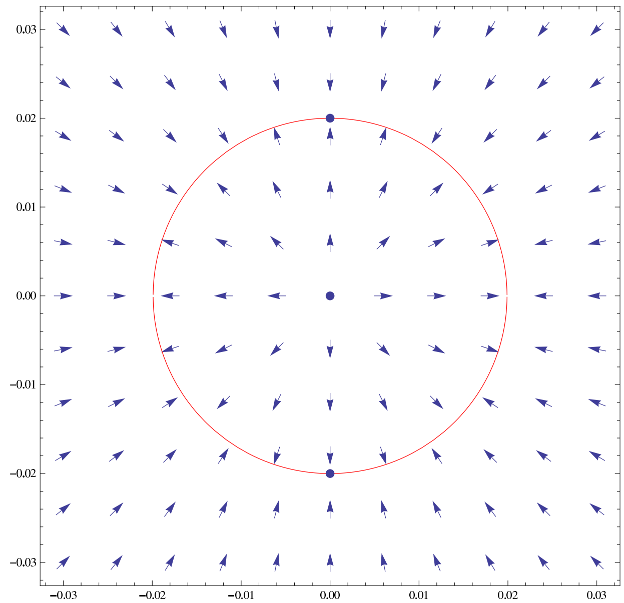

Figure 6.5: Line of fixed points and RG trajectories. The arrows indicate flows towards the IR. We have chosen $K = 100$, $N = 10$, $M = 20$.

derivative of the beta–function is different from zero, then the function has a parabolic behavior and the system is unstable.

We apply these criteria to the two–loop beta–functions (6.2.38).

6.4.1 Theories without flavors

We begin with the $\mathcal{N} = 2$ theories without flavor discussed in Section 6.3.1. As shown, the nontrivial fixed points lie on a two dimensional ellipsoid and particular points on it are the $\mathcal{N} = 3$ and the $\mathcal{N} = 2$ $SU(2)_A \times SU(2)_B$ invariant theories. Since the ellipsoid is localized in the subspaces $h_1 = \pm h_2$ we restrict our discussion to the $h_1 = h_2$ case.

When $K_1 = -K_2 \equiv K$ and $h_1 = h_2 = 0$ we can study the RG flows by solving the RG equations exactly [92]. Using eq. (6.3.42) we can write

$$\frac{dy_2}{dy_1} = \frac{y_2}{y_1} \tag{6.4.62}$$

and the most general solution is $y_2 = Cy_1$ with C arbitrary. Therefore, in this case the ellipsoid reduces to an ellipse in the (y_1, y_2) plane with y_1 and y_2 defined in eq. (6.3.40). In the (y_1, y_2) plane and at the order we are working the RG trajectories are all the straight lines passing through the origin and intersecting the ellipse.

Infrared flows can be easily determined by plotting the vector $(-\beta_{y_1}, -\beta_{y_2})$ in each point of the (y_1, y_2) plane. The result is given in Fig. 6.5 where a number of interesting features arise.

First of all the origin, corresponding to the free theory, is always an unstable point in the IR. Second, the line of fixed points is stable in the sense that the system always flows towards it. However, every single fixed point has only one direction of stability which corresponds to the RG trajectory passing through it. For any other direction of perturbation it is unstable since in general a small perturbation will drive the system to a different point on the line.

In particular, the ABJ/ABJM fixed point is IR stable against small perturbations which respect the $SU(2)_A \times SU(2)_B$ symmetry (that is along the vertical line $y_1 = 0$), whereas if the perturbation breaks $SU(2)_A \times SU(2)_B$ the system will flow to a less symmetric fixed point.

This can be understood by computing the stability matrix at $h_1 = h_2 = 0$, $h_3 = -h_4 = 1/K$ and diagonalizing it. We find that mutual orthogonal directions are $(h_1 = h_2, y_1, y_2)$ and the corresponding eigenvalues are

$$\mathcal{M} = diag\Big\{0, 0, \frac{MN-1}{2\pi^2 K^2}\Big\} \tag{6.4.63}$$

For $M, N > 1$ the third eigenvalue is positive, so the ABJ/ABJM theory is an attractor along the y_2–direction.

Solving the degeneracy of null eigenvalues requires computing the matrix of second derivatives. In particular, looking at the y_1–direction we find

$$\frac{\partial^2 \beta_{y_1}}{\partial y_1^2} = \frac{1-MN}{2\pi^2 K} \tag{6.4.64}$$

Since it is non–vanishing, the y_1 coordinate is a line of instability. Therefore, when perturbed by a $SU(2)_A \times SU(2)_B$ violating operator the system leaves the ABJM fixed point and flows to a less symmetric fixed point along a RG trajectory.

We now generalize the analysis to the case of different CS levels. In this case we refer to the surface of fixed points in Fig. 6.6 where for clearness only half of the ellipsoid has been drawn. The black line corresponds to $\mathcal{N} = 2$ superconformal theories with $h_1 = h_2 = 0$, where the green point is the $SU(2)_A \times SU(2)_B$ invariant model. The red point is instead the $\mathcal{N} = 3$ superconformal theory.

From eq. (6.2.37) we see that in the $h_1 = h_2$ subsector we have $\gamma_{A^1} = \gamma_{A^2}$. As a consequence, all the beta–functions are equal and the RG flow equations simplify to

$$\frac{dh_i}{dh_j} = \frac{h_i}{h_j} \tag{6.4.65}$$

In the three dimensional parameter space $(h_1 = h_2, h_3, h_4)$, solutions are all the straight lines passing through the origin and intersecting the ellipsoid.

Infrared flows can be easily studied by plotting the vector $(-\beta_{h_1}, -\beta_{h_3}, -\beta_{h_4})$ in each point. The result is given in Fig. 6.6 where it is clear that the entire surface is globally IR stable.

In order to study the local behavior of the system in proximity of a given fixed point, we compute the stability matrix at the point (6.3.46) and diagonalize it. Surprisingly, the eigenvalues turn out to be independent of the particular point on the surface

$$\mathcal{M} = diag\left\{0, 0, \frac{K_1^2 + 4K_1K_2 + K_2^2 + 2\left(K_1^2 + K_1K_2 + K_2^2\right)MN}{4K_1^2K_2^2\pi^2}\right\} \tag{6.4.66}$$

The two null eigenvalues characterize directions of instability. In fact, we can solve the degeneracy by computing the matrix of second derivatives respect to the corresponding eigenvectors. It turns

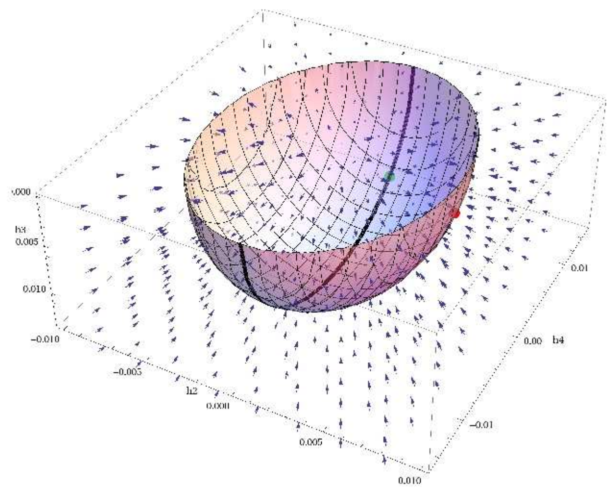

Figure 6.6: The ellipsoid of fixed points and the RG flows for $\mathcal{N}=2$ theories in the space of couplings $(h_1 = h_2, h_3, h_4)$. Arrows point towards IR directions. The parameters are $K_1 = 150$, $K_2 = 237$, $M = 30$ and $N = 43$.

out that in all cases the beta functions have a parabolic behavior along those directions and the system is unstable.

For example, at the $\mathcal{N}=2$, $SU(2)_A \times SU(2)_B$ invariant fixed point (green dot in the Figure), these eigenvectors are $\{0,1,1\}$ and $\{1,0,0\}$, which are precisely the directions $h_3 = h_4$ and h_1, tangent to the surface at that point. It is clear from Fig. 6.6 that if we perturb the system along these directions it will intercept a RG trajectory which leads it to another fixed point.

The stability properties of the β–deformed theories are easily inferred from the previous discussion. In fact, performing the following rotation of the couplings

$$h\cos(\pi\beta) = \frac{x}{2}, \qquad h\sin(\pi\beta) = \frac{y}{2} \tag{6.4.67}$$

the condition (6.3.50) for vanishing beta–functions becomes

$$\frac{1}{4}(MN-1)x^2 + \frac{1}{4}(MN+1)y^2 = \frac{1}{2}(2MN+1)\left(\frac{1}{K_1^2}+\frac{1}{K_2^2}\right) + \frac{MN+2}{K_1K_2} \tag{6.4.68}$$

This is exactly the ellipse (6.3.44) of the undeformed case. Therefore, the infrared stability properties of this curve are precisely the ones discussed before.

6.4.2 Theories with flavors

We now turn to flavored theories introduced in Section 4.2. The form of the stability matrix is quite cumbersome, but we can analyze the effects of the interactions with flavor multiplets by

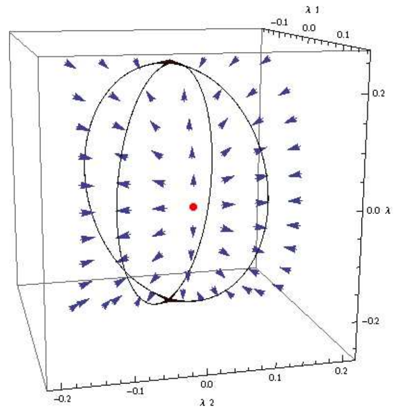

Figure 6.7: A sketch of the RG flow for the λ_i couplings only. A curve of fixed points is shown, which is IR stable. The red dot represents an isolated IR unstable fixed point. Here, the parameters are $K_1 = K_2 = 20$, $M = N = 10$, $N_f = N'_f = 1$.

studying particular examples.

As the simplest case we consider the class of theories described by the superpotential (6.1.4) where only λ_i couplings have been turned on. The β-functions of the theory split into two completely decoupled sectors: The former is the four dimensional space of couplings h_1, h_2, h_3, h_4, whose stability was addressed in the previous subsection; the latter is the three dimensional space of λ_i couplings.

Looking at the λ_i sector, nontrivial solutions to $\beta_i = 0$ describe a curve of fixed points given by expressing λ_1 and λ_2 as functions of λ_3 (see eqs. (6.3.53)). It is the two–branch curve of Fig. 6.7. The most general solution includes also isolated points where either λ_1 or λ_2 vanish.

Drawing the vector $(-\beta_{\lambda_1}, -\beta_{\lambda_2}, -\beta_{\lambda_3})$ in each point of the parameter space we obtain the RG flow configurations as given in Fig. 6.7. It is then easy to see that the isolated fixed points are always unstable since the RG flows drive the theory to one of the two branches in the IR.

This behavior can be also inferred from the structure of the stability matrix. In fact, one can check that when evaluated on the curve the matrix has two positive eigenvalues, whereas when evaluated at the isolated solutions it has negative eigenvalues. As before, theories living on the curve have directions of local instability signaled by the presence of a null eigenvalue which can be solved at second order in the derivatives. The direction of instability is tangent to the curve.

Finally, we consider the more complicated case of theories with superpotential (6.1.4) where only the α_i couplings are non–vanishing. This time the β–functions for the h_i sector do not decouple from the β–functions of the α_i sector and the analysis of fixed points becomes quite complicated.

In order to effort the calculation we restrict to the class of $U(N) \times U(N)$ theories (therefore $N_f = N'_f$) with $|K_1| = |K_2|$. This allows to choose α_i all equal to α. Moreover, we set $h_1 = h_2$ and $y_1 = 0$ in (6.3.40). The spectrum of fixed points and the RG trajectories are then studied in the three-dimensional space of parameters (α, h_1, y_2).

The β–functions vanish for vanishing couplings (free theory) and for $\gamma_{A^1} = \gamma_{Q_1} = 0$. Non-

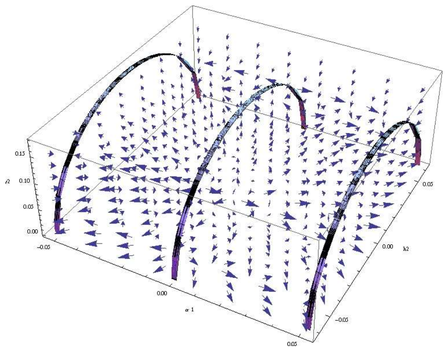

Figure 6.8: The three ellipses of fixed points and the RG flows for $\mathcal{N}=2$ theories with α couplings turned on. Arrows point towards IR directions. The parameters are $K_1 = K_2 = 20$, $M = N = 10$, $N_f = N'_f = 1$.

trivial solutions for α are obtained from $\gamma_{Q_1} = 0$. Using eqs. (6.2.37), for real couplings we find

$$\alpha = \pm\sqrt{\frac{2N^2 + NN_f + 1}{2N^2 K_1^2}} \tag{6.4.69}$$

Fixing α to be one of the three critical values (zero or one of these two values) we can solve $\gamma_{A^1} = 0$. As in the previous cases this describes an ellipse on the (h_1, y_2) plane localized at $\alpha = \text{const}$. For theories with $K_1 = K_2$ the configuration of fixed points is given in Fig. 6.8 where we have chosen to draw only half ellipses.

Renormalization group flows are obtained by plotting the vector $(-\beta_\alpha, -\beta_{h_1}, -\beta_{y_2})$. The stability of fixed points is better understood by projecting RG trajectories on orthogonal planes. Looking for instance at the $h_1 = 0$ plane we obtain the configurations in Fig. 6.9 and 6.10 where the red dots indicate the origin and the intersections of the three ellipses with the plane.

From this pictures we immediately infer that the free theory is an IR unstable fixed point since the system is always driven towards nontrivial fixed points. Among them, the ones corresponding to $\alpha \neq 0$ are attractors, whereas $\alpha = 0$ does not seem to be a preferable point for the theory. In fact, it is reached flowing along the $\alpha = 0$ trajectory, but as soon as we perturb the system with a marginal operator corresponding to $\alpha \neq 0$ it will flow to one of the two nontrivial points. We conclude that if we add flavor degrees of freedom the system requires a nontrivial interaction with bifundamental matter in order to reach a stable superconformal configuration in the infrared region.

6.5 A relevant perturbation

As a concluding remark, we consider $\mathcal{N} = 2$ CS–matter theories with two extra propagating chiral superfields in the adjoint and a superpotential given by

$$\int d^3x\, d^2\theta\, \left[s\,\mathrm{Tr}(\Phi_1^3) + s\,\mathrm{Tr}(\Phi_2^3) + t\,\mathrm{Tr}(B_i \Phi_1 A^i) + t\,\mathrm{Tr}(A^i \Phi_2 B_i) \right] + \text{h.c.} \tag{6.5.70}$$

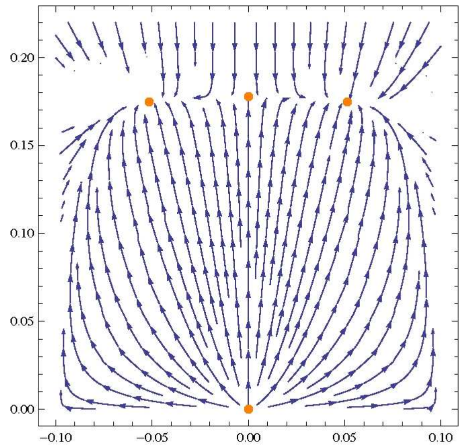

Figure 6.9: RG trajectories on the $h_1 = 0$ plane for $\mathcal{N} = 2$ theories with α couplings turned on. Arrows point towards IR directions. The parameters are $K_1 = K_2 = 20$, $M = N = 10$, $N_f = N_f' = 1$.

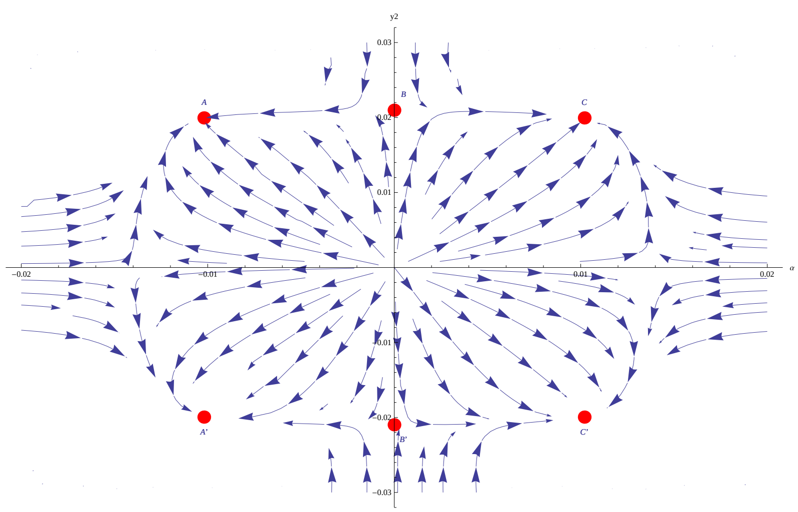

Figure 6.10: RG trajectories for the flavored model with $\lambda_i = 0$ and equal non-vanishing α couplings. The arrows indicate flows towards the IR. We have chosen $K_1 = -K_2 = 100$, $M = N = 50$, $N_f = N_{f'} = 5$.

This kind of theories should flow to a strongly coupled fixed point in the IR since, as conjectured in [97], they have a dual description in terms of a AdS4 $\times V_{5,2}/\mathcal{Z}_k$ supergravity solution.

In the UV region (6.5.70) is a relevant perturbation with dimension–$\frac{1}{2}$ couplings. A perturbative evaluation of the beta–functions requires computing the two–loop diagrams of Figs. 6.3a), 6.3c) with (Φ_i, A^i, B_i) as external fields. Setting $N = M$ for simplicity, in the large N limit the result is

$$\begin{aligned} \beta_s &= 3s\,\gamma_\Phi & \gamma_\Phi &= -\frac{1}{8\pi^2}\frac{N^2}{K^2} \\ \beta_t &= t\,(\gamma_\Phi + \gamma_A + \gamma_B) & \gamma_A = \gamma_B &= -\frac{1}{16\pi^2}\frac{N^2}{K^2} \end{aligned} \qquad (6.5.71)$$

The only perturbatively accessible fixed point is $s = t = 0$ which, according to the sign of the beta–functions, is reached at high energies. Therefore, the theory is free in the UV but naturally flows to a strongly coupled system in the IR. Such a behavior is similar to what we have found for the ABJ–like theories. However, in contrast with the previous case where under suitable requirements on the gauge coupling the IR fixed points are visible perturbatively, for the present theory they are not and other methods need be used to establish the existence of superconformal points [97].

We note that our conclusions are not an artifact of the two–loop approximation. In fact, by dimensional analysis it is easy to realize that there are no contributions to the beta–functions proportional to the chiral couplings (s, t) at higher orders, being the gauge corrections the only possible sources of divergences. Therefore, no extra fixed points other than the free theory can be found perturbatively. This is an obvious consequence of supersymmetry and of the dimensionful nature of the chiral couplings. Even the addition of a SYM term in the original action which would not be excluded by the IR results of [97] cannot change the analysis since in Feynman diagrams the replacement of CS vertices with SYM vertices improves the convergence of the integrals.

Part III

Supersymmetry breaking

Chapter 7

Basics and motivations

The Standard Model of particle physics contains all the information we know about the elementary particles and their strong and electroweak interactions. From the theoretical point of view, the Standard Model is an incomplete theory. It does not describe the gravitational interactions and it leads to quadratically divergent quantum corrections which are at the origin of the hierarchy problem. Supersymmetry is the most compelling new physics which can be discovered at the TeV scale, since both it solves the hierarchy problem and it naturally arises in ultraviolet complete theories. The non-renormalization theorems guarantee that the quadratic divergences cancel out at all orders in perturbation theory.

Since supersymmetry relates fermionic and bosonic degrees of freedom, any Standard Model fermion (boson) will have a bosonic (fermionic) partner with the same quantum numbers. Since the mass operator commutes with the supersymmetry generators, all particles in a given supersymmetry multiplet necessarily have the same mass. This is trivially excluded by experiments, and we conclude that supersymmetry must be broken, either explicitly or spontaneously. We are mainly interested in a spontaneous breaking mechanism, since it preserves the predictive power of supersymmetry and it voids the quadratic divergences.

It turns out that the particle masses are constrained by a sum rule even if supersymmetry is spontaneously broken. This is the content of the so-called supertrace theorem. In particular, it implies that at least one of the scalar partners of the quarks is lighter than the up or down quarks. This is obviously excluded by the current experiments. Thus, we have to violate the sum rule, by violating one of its hypothesis. The main goal of this Chapter is to show how it is possible to overcome the phenomenological problem imposed by the sum rule, and how supersymmetry breaking is communicated to (some supersymmetric extension of) the Standard Model.

7.1 The supertrace theorem

The supersymmetry algebra contains the translation operator P_μ

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \tag{7.1.1}$$

where the q's are the supersymmetry generators. Therefore, for a supersymmetric field theory the Hamiltonian can be written as

$$H \equiv P^0 = \frac{1}{4}\left(\{\bar{Q}_1, Q_1\} + \{\bar{Q}_2, Q_2\}\right) \tag{7.1.2}$$

If supersymmetry is unbroken in the vacuum state it is annihilated by the supersymmetry generators, hence its vacuum energy vanishes:

$$E = \langle 0|H|0\rangle = 0 \tag{7.1.3}$$

Conversely, if supersymmetry is spontaneously broken in the vacuum state, then by definition it is not invariant under supersymmetry transformations and $Q_\alpha|0\rangle \neq 0$ and $\bar{Q}_{\dot{\alpha}}|0\rangle \neq 0$. It follows that

$$E = \langle 0|H|0\rangle = \frac{1}{4}\left(||Q_1|0\rangle||^2 + ||\bar{Q}_1|0\rangle||^2 + ||Q_2|0\rangle||^2 + ||\bar{Q}_2|0\rangle||^2\right) > 0 \tag{7.1.4}$$

since the Hilbert space is to have positive norm. Thus, the order parameter for global supersymmetry breaking is the vacuum energy.

If we ask the vacuum to be a Lorentz invariant configuration, then all the spacetime derivatives and all the non-scalar fields must vanish. This means that only scalar fields can have a vacuum expectation value:

$$\langle A^a_\mu\rangle = \langle\lambda^a\rangle = \langle\psi^i\rangle = \partial_\mu\langle\phi_i\rangle = 0 \tag{7.1.5}$$

where A^a_μ are the gauge fields, λ^a their fermionic superpartners (the gauginos), and ψ^i are the fermionic partners of the scalar fields ϕ^i. The scalar vacuum expectation value is given by the solution to the equation

$$\frac{\partial V}{\partial\phi^i} = \frac{\partial V}{\partial\phi^\dagger_i} = 0 \tag{7.1.6}$$

In a general renormalizable supersymmetric field theory the scalar potential is given by the sum of two contributes

$$V = F^\dagger_i F^i + \frac{1}{2}D^a D^a \tag{7.1.7}$$

where

$$F^\dagger_i = \frac{\partial W}{\partial\phi^i} \qquad\qquad D^a = -g\phi^\dagger_i (T^a)^i_j\,\phi^j \tag{7.1.8}$$

Here, T^a are the gauge group generators and W is a holomorphic function called the superpotential.

The spontaneous breaking of a global symmetry always implies a massless Goldstone mode with the same quantum numbers as the broken symmetry generator. In the case of global supersymmetry, the broken generator is the fermionic charge Q_α. Therefore, the Goldstone particle is a massless Weyl fermion, called the goldstino. In the basis (λ^a, ψ^i), the fermion mass matrix has the form

$$m_f = \begin{pmatrix} 0 & -g\langle\phi_l^\dagger\rangle\,(T^a)_i^l \\ -g\langle\phi_l^\dagger\rangle\,(T^b)_j^l & \langle W_{ji}\rangle \end{pmatrix} \tag{7.1.9}$$

where we defined $W_i \equiv \partial W/\partial\phi^i$ and so on. Now, the condition for the minimum of the scalar potential reads

$$0 = \frac{\partial V}{\partial\phi^i} = F^j\,\frac{\partial^2 W}{\partial\phi^i\partial\phi^j} - gD^a\,\phi_j^\dagger\,(T^a)_i^j \tag{7.1.10}$$

while the condition that the superpotential is gauge invariant

$$0 = \delta^a_{gauge}W = \frac{\partial W}{\partial\phi^i}\delta^a_{gauge}\phi^i = F_i^\dagger\,(T^a)_j^i\,\phi^j \tag{7.1.11}$$

Using these two equations it is easy to show that the vector

$$\tilde{G} = \begin{pmatrix} \langle D^a\rangle \\ \langle F_i\rangle \end{pmatrix} \tag{7.1.12}$$

is annihilated by the fermion mass matrix (7.1.9). Hence (7.1.9) has a vanishing eigenvalue. The corresponding eigenvector is proportional to the goldstino wavefunction

$$\psi_G \propto \sum_i \langle F_i\rangle\psi_i + \sum_a \langle D^a\rangle\lambda^a \tag{7.1.13}$$

and it is nontrivial if and only if at least one of the auxiliary fields F and D has a vacuum expectation value, thus breaking supersymmetry. This proves that if global supersymmetry is spontaneously broken, then there must be a massless goldstino, and that its components among the various fermions in the theory are just proportional to the corresponding auxiliary field vacuum expectation values.

We now move to the bosonic sector. If supersymmetry is unbroken all particles within a supermultiplet have the same mass. If supersymmetry is broken this is no longer true, but the mass splitting in the multiplet can be computed as a function of the supersymmetry breaking parameters, i.e. the VEVs of the auxiliary fields. Let us introduce the notation

$$D_i^a = \frac{\partial D^a}{\partial\phi^i} = -g\phi_j^\dagger\,(T^a)_i^j \qquad D^{ia} = \frac{\partial D^a}{\partial\phi_i^\dagger} = -g(T^a)_j^i\phi_i \qquad D_j^{ai} = -g(T^a)_j^i \tag{7.1.14}$$

$$F^{ij} = \frac{\partial^2 W}{\partial\phi_i^\dagger\partial\phi_j^\dagger} \qquad\qquad F_{ij} = \frac{\partial^2 W}{\partial\phi^i\partial\phi^j} \tag{7.1.15}$$

The squared masses of the real scalar degrees of freedom are the eigenvalues of the matrix

$$m_s^2 = \begin{pmatrix} \langle F_{il}\rangle\langle F^{lk}\rangle + \langle D^{ak}\rangle\langle D_i^a\rangle + \langle D^a\rangle D_i^{ak} & \langle F^l\rangle\langle F_{ijl}\rangle + \langle D_i^a\rangle\langle D_j^a\rangle \\ \langle F_l^\dagger\rangle\langle F^{pkl}\rangle + \langle D^{ap}\rangle\langle D^{ak}\rangle & \langle F_{jl}\rangle\langle F^{pl}\rangle + \langle D^{ap}\rangle\langle D_j^a\rangle + \langle D^a\rangle D_i^{ap} \end{pmatrix} \tag{7.1.16}$$

It follows that the sum of the real scalar squared mass eigenvalues is

$$\mathrm{Tr}(m_s^2) = 2\langle F_{il}\rangle\langle F^{il}\rangle + 2g^2\left[C(R_i)\phi_i^\dagger\phi^i + \mathrm{Tr}(T^a)D^a\right] \tag{7.1.17}$$

where $C(R_i)$ is the quadratic Casimir for the representation R_i. The vector squared masses are

$$m_V^2 = 2\langle D_i^a\rangle\langle D^{bi}\rangle \tag{7.1.18}$$

and from (7.1.9) the fermion squared mass matrix is

$$m_f m_f^\dagger = \begin{pmatrix} 2\langle D_j^a\rangle\langle D_i^b\rangle & -\sqrt{2}i\langle D_i^b\rangle\langle F_{ij}\rangle \\ \sqrt{2}i\langle D_j^a\rangle\langle F^{ij}\rangle & \langle F_{ij}\rangle\langle F^{ij}\rangle \end{pmatrix} \tag{7.1.19}$$

which gives

$$\mathrm{Tr}(m_f m_f^\dagger) = F_{ij}F^{ij} + 4g^2 C(R_i)\phi_i^\dagger\phi^i \tag{7.1.20}$$

It follows that the supertrace of the tree-level squared mass eigenvalues, defined in general as the weighted sum over all particles with spin j

$$\mathrm{STr}(m^2) \equiv \sum_j (-1)^j\,(2j+1)\,\mathrm{Tr}(m_j^2) \tag{7.1.21}$$

satisfies the sum rule

$$\mathrm{STr}(m^2) = \mathrm{Tr}(m_s^2) - 2\mathrm{Tr}(m_f m_f^\dagger) + 3\mathrm{Tr}(m_V^2) = 2g^2\,\mathrm{Tr}(T^a)\langle D^a\rangle \tag{7.1.22}$$

which vanishes either if $\langle D^a\rangle = 0$ or if all the gauge group generators are traceless or if the traces of the $U(1)$ charges over the chiral superfields vanish. The latter condition holds for the Standard Model hypercharge and in general for any non-anomalous gauge symmetry.

The supertrace theorem then states that the sum of the squared masses of all bosonic degrees of freedom equals the sum over the fermion ones. In a supersymmetry breaking theory this is a very strong condition, which constraints the spectrum too much and leads to unobserved light particles. Therefore, the sum rule has to be violated.

7.2 Non-renormalizable interactions

The sum rule can be avoided by violating one of its hypothesis. The first one is the renormalizability of the theory. The second one is that the supersymmetry breaking is communicated via tree-level interactions to the Standard Model. The effective procedure to parametrize non-renormalizable interactions consists in adding terms in the Lagrangian that are not invariant under the supersymmetry transformations. Typical terms of this kind are masses for some components in a supermultiplet, breaking the mass degeneracy. One of the main advantages of supersymmetric field theories is their renormalizability properties, which follow from cancellations in the perturbative computation between fermionic and bosonic degrees of freedom. We would like to preserve this property when adding explicit supersymmetry breaking terms. It turns out that they can be parametrized by a few terms in the Lagrangian [98]. The interactions that can be added to a supersymmetric Lagrangian without introducing quadratic divergences are called soft terms. All the soft interactions are given by the following recipe [98]: given any renormalizable superspace action, add to it terms which are product of ordinary superfields (and their derivatives) and of an spacetime-independent but θ-dependent superfield (i.e. a spurion), restricted by the condition that if the spurion is set to 1, the resulting term leads to a renormalizable action or is a total derivative. This is a very powerful statement, because it allows us to completely classify the soft terms. Define the spurion field S such that $\langle S\rangle = \theta^2\langle F\rangle$. Then the soft non-renormalizable terms are

$$\begin{aligned}
&\int d^4x d^2\theta\ S\Phi^2 \longrightarrow F\left(\phi^2 + (\phi^\dagger)^2\right)\\
&\frac{1}{M}\int d^4x d^2\theta\ S\Phi^3 \longrightarrow \frac{F}{M}\left(\phi^3 + (\phi^\dagger)^3\right)\\
&\frac{1}{M}\int d^4x d^2\theta\ SW^\alpha W_\alpha \longrightarrow \frac{F}{M}\lambda_\alpha\lambda^\alpha\\
&\frac{1}{M^2}\int d^4x d^4\theta\ \overline{S}S\bar{\Phi}\Phi \longrightarrow \left(\frac{F}{M}\right)^2\phi^\dagger\phi\\
&\frac{1}{M^2}\int d^4x d^4\theta\ \overline{S}S\, D^\alpha(\Phi W_\alpha) + \text{h.c.} \longrightarrow \left(\frac{F}{M}\right)^2(\psi^\alpha\lambda_\alpha + \text{h.c.})
\end{aligned} \tag{7.2.23}$$

where we introduced a cutoff scale M. The first term is a holomorphic mass for the scalar component field of a chiral multiplet, and it is called b-term. The second line is called A-term. The third term provides a mass for the gaugino field. The fourth term is a non-holomorphic mass for the scalar component field. If the chiral field is the Higgs field, this term is the so-called Bμ-term.

7.3 Mediating the supersymmetry breaking effects: an overview

The non-renormalizable interactions discussed in the previous Section naturally arise in the context of gauge-mediated supersymmetry breaking. The supersymmetry breaking mechanism is realized in a separate, *hidden* sector, and its effects are then communicated to the MSSM through interactions with heavy fields. The low energy effective theory one obtains by integrating them out effectively contains non-renormalizable interactions, therefore it evades the supertrace theorem. While the the so-called *messenger* sector obeys the sum rule, this does not constitute a problem, if some non-renormalizable interactions are generated at low energies. The messenger fields couple to the supersymmetry breaking sector through renormalizable interactions, and they are taken to be charged under the MSSM gauge group, so they couple to the *visible* sector through the MSSM gauge interactions.

From the discussion of the previous paragraph, the standard scenario of gauge mediation can be summarized as follows:

- the *visible* sector: this is a supersymmetric extension of the Standard Model. The MSSM is commonly considered;

- the *hidden* sector: this is the sector where supersymmetry breaking occurs. The fields should be singlets under the visible sector gauge transformations. Details of this sector are model dependent. The effects of this sector can be summarized by considering a set of visible sector gauge singlets chiral superfields S_i which acquire a non-vanishing vacuum expectation value for both their scalar and auxiliary components

$$\langle S_i \rangle = s_i + \theta^2 F_i \tag{7.3.24}$$

  where the F_i set the supersymmetry breaking scale. The Goldstino is a linear combination of the S_i;

- the *messenger* sector: the messenger (chiral) superfields Φ, $\tilde{\Phi}$ which belong to this sector are charged under the visible sector gauge group and couple through tree-level interactions to the hidden sector superfields

$$W \sim \sum_i \lambda_i S_i \Phi_i \tilde{\Phi}_i \tag{7.3.25}$$

  The superpotential (7.3.25) together with the s_i vacuum expectation values for the hidden sector superfields S_i lead to a supersymmetric mass term for the messenger fields. The supersymmetry breaking F_i terms lead to a non-supersymmetric mass splitting between the fermion and the scalar component of the messengers.

In gauge mediated theories the effects of gravity are usually discarded. This allows to study models with field theoretical tools only, without have to deal with quantum gravity. This is particularly interesting because lot of progress have been made in understanding non-perturbative aspects of supersymmetric gauge theories, where supersymmetry breaking mechanisms, and their communication to the visible sector, can be investigated.

Gravity can be also considered as the force which mediates the effects of supersymmetry breaking. Since gravity is a non-renormalizable theory, the supertrace theorem does not hold. Although this is a widely studied scenario, there are some stringent bounds coming from the flavor changing neutral currents. In this case, there is no obvious reason why gravity arranges its interactions to be diagonal in the same basis in which the Higgs couples to the visible sector fermions. The supersymmetry breaking masses for the squarks and the sleptons can be non-flavor invariant even at tree-level. This non-universality of the masses can lead to flavor changing neutral currents excluded by experiments. This problem is automatically avoided by the gauge interactions, which are known to be flavor blind.

7.4 R-parity and R-symmetry

The Standard Model renormalizable gauge-invariant interactions automatically lead to the conserved baryon B and lepton L numbers. This is in good agreement with the experiments, because the proton decay has not been observed. In the MSSM the baryon and lepton numbers are not automatically conserved quantum numbers. The stringent experimental bounds have to be taken into account both in the hidden and in the visible sectors. While the hidden sector is highly model dependent, we can add a new discrete symmetry to the visible sector which eliminates the possible baryon and lepton violating terms in the renormalizable superpotential. This symmetry is called R-parity or matter parity. The matter parity quantum numbers are given by

$$P_M = (-1)^{3(B-L)} \tag{7.4.26}$$

under which the quark and lepton supermultiplets have $P_M = -1$ and the Higgs and vector supermultiplets have $P_M = 1$. It is easy to check that it is equivalent to the R-parity

$$P_R = (-1)^{3(B-L)+2s} \tag{7.4.27}$$

where s is the spin of the particle. The advantage of this definition is that the Standard Model particles have $P_R = 1$ while their supersymmetric superpartners have $P_R = -1$.

Supersymmetric field theories also possess the continuous R-symmetry, under which the gaugino is usually assigned to have charge 1. This continuous symmetry has nothing to do with the discrete R-parity. It is of phenomenological interest that R-symmetry is broken in the hidden sector. By looking at the gaugino mass term in (7.2.23) it is easy to see that it does not preserve R-symmetry. Hence if a supersymmetric theory have a vacuum which breaks supersymmetry but preserves R-symmetry, the radiative corrections which generate the non-renormalizable soft interactions (7.2.23) cannot generate a gaugino mass term. Then, the low energy spectrum would have a massless fermion in the adjoint representation, which is not observed.

Even if the R-symmetry is anomalous and broken by quantum effects to a discrete subgroup, this problem is not solved unless this discrete subgroup is reduced to Z_2, and in R-symmetry preserving vacua a gaugino mass is forbidden.

Chapter 8

Supersymmetric QCD and Seiberg duality

Since four-dimensional supersymmetric Yang-Mills theories are expected to play a central role in the discovery of new physics in future experiments, it is essential to understand their main features. Their (eventual) application to the Nature strongly relies on our knowledge of how they may overcome the stringent bounds on new physics.

As happens for many physical systems, included QCD, supersymmetric gauge theories present many phases, most of which are strongly coupled field theories. In this range of parameters where the perturbative expansion breaks down, we have to make use of other computational methods. In this section we analyze the physics of supersymmetric Yang-Mills theories and their low-energy dynamics through the analysis of their effective Lagrangians. We concentrate on $N = 1$ supersymmetric field theories, as they do not suffer from the rigidity of extended supersymmetry while they still present the powerful of supersymmetric field theories, and can be embedded quite easy in string theory.

We do not plan to be exhaustive. There are many textbooks [13, 99, 100] and reviews [101, 102, 103, 104, 105] on this subject. In Section 8.1 we give an overview on the general features of $N = 1$ supersymmetric field theories. In particular, we describe their phases, we explain how to write down a supersymmetric Lagrangian and find its low energy limit, and we show how to find its classical vacuum states. Section 8.2 is devoted to the case of Supersymmetric QCD (SQCD). We start by describing the low energy dynamics of pure Super Yang-Mills theory, then we apply the general arguments of Section 8.1 to the case of interacting matter fields, and argue that an electric/magnetic duality, namely Seiberg duality, exists. We discuss various aspects of this duality. We present a marginal deformation of SQCD in Section 8.3 and discuss how Seiberg duality is deformed in this case.

8.1 Generalities

A gauge theory and its behavior are characterized by their symmetries and the way they are (or not) realized in the vacuum state. The original local symmetry of a gauge theory can be, for instance, completely spontaneously broken; otherwise, the vector bosons mediate long-range interactions. This leads us to the concept of the phases of the gauge theories.

The phases of gauge theories are characterized by the potential $V(r)$ between test charges at a large distance r. According to how the gauge symmetry is realized, we have

- Coulomb phase: the vector bosons remain massless and mediate interactions with $V(r) \sim 1/r$;
- Higgs phase: the gauge group is spontaneously broken and all the vector bosons get masses. In this case the potential is constant;
- confining phase: the sources of the gauge group are bound into singlets; $V(r) \sim r$.

The Coulomb phase is also present in non-Abelian Yang-Mills theories: when the theory is not asymptotically free the long-range potential between the charges is of the form $e(r)/r$, where $e(r)$ is the renormalized charge which decreases as the logarithm of the distance. In this case, the theory flows to a fixed point of the renormalization group: the infrared theory is a nontrivial conformal field theory.

The above discussion applies to electric charges. In addition, one can also have magnetic charges in the theory. At large distances, their potential behaves as

- Coulomb phase: $V(r) \sim 1/r$;
- Higgs phase: $V(r) \sim r$;
- confining phase: $V(r) \sim constant$.

The relation between these phases is particularly well understood in the Abelian case. The Dirac quantization condition relates the electric charge $e(r)$ to the magnetic charge $g(r)$ by $e(r)g(r) \sim 1$. When an electrically charged field acquires a vacuum expectation value (Higgs phase), the magnetically charges are confined in flux tubes. Analogously, when a magnetically charged field acquires a vacuum expectation value, the electric charges are confined in flux tubes, i.e., we are in the confining phase.

In the non-Abelian cases the relation we described cannot be made manifest. When there are matter fields in the fundamental representation of the gauge group, virtual pairs can completely screen the charges. In fact, there is no invariant distinction between the Higgs and confinement phases: the flux tube in which the charges are confined can break. Usually, one interprets the theory for large expectation values as being in the Higgs phase, while for small expectation values as being in the confining phase. However, one can still smoothly interpolate between the two.

What we just described is the electric-magnetic duality, which exchanges the electric charges with the magnetic ones. When the theory is in its Coulomb phase, the duality does not bring the theory to another phase. In particular, if the microscopic, electric theory is strongly coupled, its description in terms of dual magnetic charges is weakly coupled. Because our main goal is to found low-energy models of supersymmetry breaking, and because supersymmetric gauge theories are often strongly coupled at low energies, it is very convenient to have a consistent description of their behavior in terms of the dual magnetic degrees of freedom. Then, in the next sections we describe supersymmetric Yang-Mills theories with fundamental matter and how the duality transformations allow us to perform reliable computations of dynamical quantities at large distances.

8.1.1 Supersymmetric Lagrangians

Before moving to the issue of supersymmetry breaking, let us review how a supersymmetric theory is constructed, how the global supersymmetry restricts the form of the low-energy actions and which powerful tools it introduces for their analysis. This also represents a strong motivation for the study of supersymmetric field theories.

When a given symmetry is linearly realized in a field theory, the best way to write down its Lagrangian is making the symmetry manifest. In the case of supersymmetry, this is achieved by working in terms of superfields: they can be classified as chiral superfields Φ, antichiral superfields $\bar{\Phi}$ and vector superfields V. The former contains as its components a scalar field ϕ and a Weyl fermion ψ; the antichiral superfield is its hermitian conjugate. They are in some representation of the gauge group. The physical states of the vector superfield are the gauge field and its fermionic superpartner, and it belongs to the adjoint representation. The most general Lagrangian with at most two derivatives is written as

$$\mathcal{L} = \int d^4x d^4\theta \ K(\bar{\Phi}, \exp V\Phi) + \left(\frac{-i}{16\pi^2}\right) \int d^4x d^2\theta \ \tau(\Phi) W^\alpha W_\alpha + \text{h.c.} + \int d^4x d^2\theta \ W(\Phi) \quad (8.1.1)$$

where K is the Kähler potential, W^α is the gauge superfield strength and W is the superpotential. In general, the Kähler potential can be an arbitrary real function of its arguments. However, for our purposes, and in order to void the issue of non-renormalizability of the theory, we mainly work with a canonical Kähler potential

$$K(\bar{\Phi}, \Phi) = \bar{\Phi}\Phi \quad (8.1.2)$$

which gives the standard kinetic terms for the component fields. In the low-energy theory, we should take care that higher order corrections are not so strong to destroy this form.

The second term in the Lagrangian is the kinetic term for the gauge and the gaugino fields. More precisely, $W^\alpha W_\alpha$ is the supersymmetric completion of $F^2 + iF\tilde{F}$, so the coefficient of this term is the combination of the gauge coupling and the θ parameter

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2} \quad (8.1.3)$$

Since the renormalization effects depend on the nature of the vacuum state, τ generally depends on the values of the chiral superfields that indicate which vacuum has been chosen. The main point is that it is a holomorphic function of its arguments. This means that it can often be exactly determined.

The last term of the supersymmetric Lagrangian contains the superpotential. This term leads to the non-derivative interactions of the chiral fields. In particular, it yields to a potential for the component scalar fields and a Yukawa type interaction between the scalars and the fermions. An important property is that the superpotential is a holomorphic function of the chiral superfields. Quantum mechanically, the low-energy effective superpotential can also depend upon the effective coupling constants and the dynamical scale of the theory Λ

$$W_{eff} = W_{eff}(\Phi, g_i, \Lambda) \quad (8.1.4)$$

Supersymmetry constraint the quantum superpotential to be a holomorphic function of its arguments. When doing this, we think of all the coupling constants and the dynamical scale as background fields. Then, in general, W_{eff} is constrained by

- symmetries: we assign transformation laws both to the fields and to the coupling constants. In this case, the symmetry group of the Lagrangian is called the pseudo-symmetry. The effective superpotential is invariant under it;
- holomorphy: the effective superpotential is holomorphic in all the fields, even in the background fields. This is a direct consequence of supersymmetry. As an immediate corollary, we obtain the non-renormalization theorems which hold perturbatively. Then, the only corrections to the superpotential come from non-perturbative effects;
- limits: when the superpotential is analyzed at weak coupling, some singularities can appear. They have physical meaning and can be controlled.

8.1.2 The vacuum state

Another direct consequence of supersymmetry is that any state has positive semidefinite energy. Thus, the supersymmetric vacuum state of the theory, if it exists, has vanishing energy.

To find the vacuum state from the given Lagrangian, we have to know the potential energy and minimize it. Assuming a canonical Kähler potential for the fields Φ^i, $i = 1, \ldots, N_f$, the equations of motion for the auxiliary field F of the chiral multiplet and the auxiliary field D of the vector multiplet are

$$\bar{F}_i = \frac{\partial W}{\partial \Phi^i} \tag{8.1.5}$$

$$D^a = \sum_i \bar{\Phi}^i T^a \Phi^i \tag{8.1.6}$$

where T^a, $a = 1, \ldots, \dim(G)$ are the generators of the gauge group G. The potential energy is written as

$$V = \bar{F}_i F^i + \frac{1}{2} g^2 (D^a)^2 \tag{8.1.7}$$

with g the gauge coupling constant. Being a sum of two positive quantities, it is immediately realized that the supersymmetric vacua satisfy

$$\langle F_i \rangle = \langle D^a \rangle = 0 \tag{8.1.8}$$

for every i, a. If one of these conditions is not satisfied, supersymmetry is broken. The conditions $F = 0$ and $D = 0$ are called F-flatness and D-flatness conditions, respectively.

Classical gauge theories often have directions in the field space where the D-flatness conditions are satisfied for a certain range of non-vanishing vacuum expectation value $\langle \phi_i \rangle$ for some i. These theories are said to have classical moduli space of degenerate vacua. Let us show this by a simple example, which will be useful in the following. Consider a $U(1)$ gauge theory coupled with two chiral superfields Q and $\tilde{Q}$ of charges 1 and -1 respectively. The scalar potential is $V = (\bar{Q}Q - \bar{\tilde{Q}}\tilde{Q})^2$ and thus if we choose $\langle Q \rangle = \langle \tilde{Q} \rangle = a$, modulo gauge transformations, we obtain a continuum set of degenerate vacua labeled by the arbitrary complex parameter a. For $a \neq 0$ the gauge group is broken by the supersymmetric analogue of the Higgs mechanism: the gauge field eats one degree of freedom from the matter fields and gets a mass of order $|a|$. The other matter degree of freedom remains massless. It can be described by the gauge invariant combination $X = Q\tilde{Q}$, with $\langle X \rangle = a^2$ in the vacuum. Because a is arbitrary, there is no classical potential for X, and it is called a modulus field whose expectation value labels the classical moduli space of degenerate vacua. Because we took the classical Kähler potential to be canonical in the original fields, $K = \bar{Q}e^V Q + \bar{\tilde{Q}}e^{-V}\tilde{Q}$, we obtain $K = 2\sqrt{\bar{X}X}$ which is singular at $X = 0$. This singularity has physical meaning. In general, a singularity in a low-energy effective action signals the presence of extra massless states in the spectrum of the theory. Indeed, the singularity at $X = 0$ corresponds to the fact that the gauge group is unbroken for $a = 0$ and all the original fields are massless there.

The above example shows the essential features of supersymmetric theories. The D-flatness equations are not holomorphic, but their solutions can be parametrized by gauge-invariant combinations of holomorphic fields. Furthermore, they can be always be given a gauge-invariant description in terms of the expectation values of gauge invariant polynomials in the fields, as it is the case for $\langle X \rangle$ above. This is because the parameter of the gauge transformations of a supersymmetric gauge theory is a chiral superfield, and thus its bosonic part is a complex parameter. Fixing this complex extension of the gauge group is equivalent to fixing the gauge symmetry and imposing the D-flatness conditions [106]. The F-flatness conditions are invariant under the complexified gauge group. The gauge invariant chiral polynomials correspond to the matter fields which remain massless after the Higgs mechanism and are classical moduli with vanishing (super)potential.

The vacuum degeneracy of the classical moduli space of vacua is not protected by any symmetry. Vacua with different expectation values of the fields are physically inequivalent: for stance, the masses of the gauge fields depend on them. Therefore, this degeneracy is accidental and can be lifted in the quantum theory by a dynamically generated quantum superpotential W_{eff}. For this reason, the classical moduli are called pseudo-moduli. In many cases the effective superpotential can be determined by the constraints supersymmetry imposes on it.

8.2 Supersymmetric QCD

8.2.1 Lagrangian

Supersymmetric QCD is a supersymmetric $SU(N_c)$ gauge theory with N_f flavors of quark superfields in the fundamental representation of the gauge group. The matter superfields are called $Q_i, \tilde{Q}_i$, $i = 1, \ldots, N_f$ and belongs to the $(N_c + \bar{N}_c)$ representation of the gauge group.

The Lagrangian is

$$\mathcal{L} = \int d^4x d^4\theta \, \left(\bar{Q}_i e^V Q_i + \tilde{Q}_i e^V \bar{\tilde{Q}}_i \right) - \frac{i}{16\pi} \int d^4x d^2\theta \, \tau W^\alpha W_\alpha + \text{h.c.} \tag{8.2.9}$$

and no superpotential. At the classical level, this theory has a large global symmetry group:

$$G = SU(N_f)_L \times SU(N_f)_R \times U(1)_A \times U(1)_B \times U(1)_{R'} \tag{8.2.10}$$

In the quantum theory both the axial symmetry $U(1)_A$ and the R-symmetry $U(1)_{R'}$ are anomalous; however, one can take a non-anomalous linear combination of them R-symmetry $U(1)_R$. The full quantum global symmetry is then $G = SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_R$ where the quarks transform as

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_R$
Q	$\square$	1	1	$\frac{N_f-N_c}{N_f}$
$\tilde{Q}$	1	$\bar{\square}$	-1	$\frac{N_f-N_c}{N_f}$

(8.2.11)

In a supersymmetric Yang-Mills theory with gauge group G_c and matter superfields in the representations r_i the β-function for the gauge coupling g is given by the NSVZ function [107, 108, 109, 110]

$$\beta_g = -\frac{g^3}{16\pi}\frac{3C_2(G_c) - \sum_i C_2(r_i)(1-2\gamma_i)}{1 - g^2\frac{N}{8\pi}} \tag{8.2.12}$$

where $C_2(r_i)$ is the quadratic Casimir of the representation r_i. For $SU(N_c)$, $C_2(G_c) = N_c$ and $C_2(N_c) = 1/2$. In the case of supersymmetric QCD

$$b_0 \equiv 3C_2(G_c) - \sum_i C_2(r_i) = 3N_c - N_f \tag{8.2.13}$$

and we obtain the effective running coupling constant of the theory in terms of the fundamental coupling constant g defined at the large mass scale M as

$$\frac{4\pi}{g^2(\mu)} = \frac{4\pi i}{g^2} - \frac{b_0}{2\pi}\log\frac{M}{\mu} \tag{8.2.14}$$

The dynamical scale Λ is defined as the mass scale at which this expression formally diverges

$$\Lambda = Me^{-8\pi^2/b_0 g^2} \tag{8.2.15}$$

Note that the sign of b_0 determines whenever a theory is asymptotically free or not. For $N_f < 3N_c$ it is, while for greater values of N_f it is not ultraviolet complete.

8.2.2 $N_f = 0$

Let us begin by considering the pure supersymmetric Yang-Mills theory with no flavor fields. The matter content is composed by the gauge fields and their fermionic superpartners, the gauginos. In terms of component fields the kinetic terms are

$$\mathcal{L}^{(2)} = \mathrm{Tr}\left(-\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} + \frac{i}{g^2}\bar{\lambda}\not{D}\lambda + \frac{i\theta}{32\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu}\right) \tag{8.2.16}$$

Because there are no flavors, it is not possible to build a continuous anomaly free R-symmetry; however, a discrete subgroup is left unbroken. The basic reason is that the chiral rotation of the gaugino field is a symmetry if we combine it with a shift of the θ parameter $\theta \to \theta + 2N_c\alpha$, or in terms of $\tau = \theta/2\pi + 4\pi i/g^2$, $'\tau \to \tau + N_c\alpha/\pi$. Since the physics is periodic in θ with period 2π this is a quantum symmetry if α is a multiple of π/N_c. Thus, a Z_{2N_c} subgroup of the canonical R-symmetry is a quantum symmetry.

The above discussion can be stated in another way. If the quantum theory is not invariant under the full R-symmetry, there is a term in the effective quantum superpotential which breaks it. The property of holomorphy and the symmetries constrain the quantum superpotential to be of the form

$$W_{eff} = cM^3 e^{2\pi i\tau/N_c} \tag{8.2.17}$$

where c is a dimensionless number to be determined and the powers of the large scale M give the superpotential the right mass dimension. Using the definition (8.2.15), we write

$$W_{eff} = c\Lambda^3 \tag{8.2.18}$$

where we used holomorphy to trade g^2 for the full gauge coupling τ. The gaugino condensate is computed by

$$\langle\lambda\lambda\rangle = 16\pi i \frac{\partial}{\partial\tau} W_{eff} = -\frac{32\pi^2}{N_c} c\Lambda^3 \tag{8.2.19}$$

and it is set by the nonperturbative scale Λ.

By performing the chiral transformation $\lambda \to e^{i\alpha}\lambda$ with $\alpha = \pi m/N_c$ we note that only $\alpha = \pi$ leaves the gaugino condensate invariant. This means that the gaugino condensate ultimately breaks the R-symmetry down to the Z_2 subgroup. The symmetry $\tau \to \tau + 1$ has been broken down to $\tau \to \tau + N_c$: as a result, there are N_c inequivalent vacuum states (8.2.19) in which the gaugino condensate assumes N_c distinct values. This is in complete agreement with the Witten index [111], which suggests that the picture above is indeed correct. However, we are not able to compute the quantum superpotential in this theory, or even to say if it really arises from quantum computations. This is the main reason that leads us to consider theories with flavor fundamental quarks.

8.2.3 $N_f < N_c$

When $N_f < N_c$ SQCD has the global symmetry group $G = SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_R$ with quantum numbers given in (8.2.11). Recall that the gaugino is defined to have R-charge 1.

The unique gauge invariant chiral superfield that one can build from Q and $\tilde{Q}$ is the meson field

$$M_{ij} = Q_i \tilde{Q}_j \tag{8.2.20}$$

which is a $N_f \times N_f$ matrix and transforms as $(\square, \bar{\square})$ under the $SU(N_f)_L \times SU(N_f)_R$ global symmetries. Its determinant is also invariant under these nonabelian symmetries and its non-anomalous R-charge is $2N_f - 2N_c$.

The other symmetry invariant holomorphic object at our disposal is the dynamical scale Λ. From (8.2.15) it carries no non-anomalous R-charge.

The low energy effective theory is described by a low energy effective superpotential which should respect all the symmetries of the theory. The unique dimension-three, symmetry invariant superpotential which carries two units of R-charge we can build is of the form

$$W_{eff} = c\left(\frac{\Lambda^{b_0}}{\det M}\right)^{1/(N_c-N_f)} \tag{8.2.21}$$

where c is a dimensionless constant which eventually depends on N_c and N_f. The superpotential (8.2.21) is called the Affleck-Dine-Seiberg (ADS) superpotential [112]. It is non-perturbative and thus it is not in conflict with the perturbative non-renormalization theorem. Note that (8.2.21) does not exist when $N_f \geq N_c$ because either the exponent diverges or the determinant vanishes.

Differentiating the effective superpotential with respect to Q to obtain the F-terms we find

$$F_{Q_i} \sim \frac{1}{Q}\left(\frac{1}{\det Q}\right)^{1/(N_c-N_f)} \tag{8.2.22}$$

which is a decreasing function of Q. This means that the scalar potential monotonically decreases as Q becomes large, asymptotically approaching zero at infinity. This in turn means that the vacuum state of the theory is at infinity, or that the theory has no vacua at all.

We show some consistency checks on the superpotential (8.2.21). First of all, note that the fermionic components of the quarks cannot condense in pairs as the gaugino, because this is not consistent with supersymmetry. Indeed, a quark bilinear is the F-term of the meson superfield, and its non-vanishing vacuum expectation value would break supersymmetry with a positive definite potential. Since the potential at infinity is zero, this state would not be the ground state of the theory.

Next consider giving the quarks a vacuum expectation value, in a way which satisfies the D-flatness conditions

$$D^a = \bar{Q}T^aQ - \tilde{Q}T^a\bar{\tilde{Q}} \tag{8.2.23}$$

Using the gauge and global symmetries, the result can always be casted in the form

$$Q = \tilde{Q} = \begin{pmatrix} a_1 & & 0 & & 0 & \ddots \\ & a_2 & & & 0 & \\ & & \ddots & & & \\ & & & a_{N_f-1} & 0 & \\ & 0 & & & a_{N_f} \; 0 & \ddots \end{pmatrix} \tag{8.2.24}$$

with arbitrary a_i. For large vacuum expectation values, the gauge symmetry is spontaneously broken to $SU(N_c{-}N_f)$, and all the bosons and fermions which transform under the residual gauge group obtain mass. at energy scales well below the a_is the theory looks like pure supersymmetric Yang-Mills.

Consider the case $a_1 = \ldots = a_{N_f} \equiv v$ for a while. For energy scales $E > v$ the theory is SQCD with N_f flavors, while for $E < v$ the theory is pure gauge theory with gauge group $SU(N_c - N_f)$. We can compute the running gauge coupling constant in the two theories and then match them at $E = v$. The high energy coupling obeys the one-loop relation

$$\frac{4\pi}{g^2(E)} = \frac{3N_c - N_f}{2\pi} \log \frac{E}{\Lambda} \tag{8.2.25}$$

At energies below v the one-loop coefficient of the gauge theory is given by $b_0 = 3(N_c - N_f)$, and we indicate the holomorphic scale by Λ_{eff}. Thus,

$$\frac{4\pi}{g^2(E)} = \frac{3(N_c - N_f)}{2\pi} \log \frac{E}{\Lambda_{eff}} \tag{8.2.26}$$

Matching the above equations at the scale $E = v$ we obtain Λ_{eff} as a function of the high energy parameters

$$\Lambda_{eff} = v \left(\frac{\Lambda}{v}\right)^{\frac{3N_c - N_f}{3(N_c - N_f)}} \tag{8.2.27}$$

This way, the gauge coupling constant changes continuously from the low energy to the high energy theory. Recalling that in the low energy theory gaugino condensation implies the non-perturbative superpotential

$$W_{eff} = c\Lambda_{eff}^3 = c\left(\frac{\Lambda^{3N_c - N_f}}{v^{2N_f}}\right)^{\frac{1}{N_c - N_f}} \tag{8.2.28}$$

we recover the ADS superpotential (8.2.21) in the high energy theory (note that $v^{2N_f} = \det M$). Said in another way, the ADS superpotential correctly describes gaugino condensation in the low energy theory where the massive quarks have been integrated out and the gauge group is spontaneously broken.

The constant c in the low energy $SU(N_c' = N_c - N_f)$ theory only depends upon $N_c' = N_c - N_f$. The fact that the high energy ADS superpotential depends on the same constant c means that c is a function of the difference $N_c - N_f$. This will allow us to compute it in a particular case.

The holomorphic decoupling offers us a second way to check the validity of the ADS superpotential. Consider giving a mass term for the N_fth flavor

$$W = mQ_{N_f}\tilde{Q}_{N_f} = mM_{N_f N_f} \tag{8.2.29}$$

The full superpotential now is

$$W = c\left(\frac{\Lambda^{3N_c - N_f}}{\det M}\right)^{\frac{1}{N_c - N_f}} + mM_{N_f N_f} \tag{8.2.30}$$

Once the D-flatness conditions have been satisfied, one has to consider the F-terms. The vanishing of the F-terms leads to consider the meson field to assume the form

$$M = \begin{pmatrix} \tilde{M} & 0 \\ 0 & t \end{pmatrix} \tag{8.2.31}$$

The F-flatness condition for t is

$$t = \left(\frac{N_c - N_f}{c} m \left(\frac{\Lambda^{b_0}}{\det \tilde{M}} \right)^{1/(N_c - N_f)} \right)^{(N_c - N_f)/(N_c - N_f + 1)} \tag{8.2.32}$$

Putting this back into the superpotential we obtain

$$W = c' \left(\frac{m \Lambda^{3N_c - N_f}}{\det \tilde{M}} \right)^{1/(N_c - N_f + 1)} \tag{8.2.33}$$

where

$$\left(\frac{c'}{N_c - N_f + 1} \right)^{(N_c - N_f + 1)} = \left(\frac{c}{N_c - N_f} \right)^{(N_c - N_f)} \tag{8.2.34}$$

which implies that, in general, $c = (N_c - N_f) C^{1/(N_c - N_f)}$ with C a universal constant. The superpotential (8.2.33) is exactly the ADS superpotential for the low energy $SU(N_c)$ theory with $N_f - 1$ flavors, in which the N_fth flavor has been integrated out. We can match the low energy and the high energy gauge coupling constant at the scale m where we integrate out the massive flavor field obtaining

$$\Lambda_{eff}^{3N_c - N_f + 1} = m \Lambda^{3N_c - N_f} \tag{8.2.35}$$

Once again this is consistent with (8.2.33).

We showed that the family of superpotentials (8.2.21) are consistent with one another for every value of $N_f < N_c$. First of all, we have to show that it arises from a non-perturbative effect. Secondly, we have to compute the universal constant C for at least one couple of values (N_c, N_f).

A direct derivation of the ADS superpotential can be given for $N_f = N_c - 1$. In this case, the expectation values for the quark fields break the gauge symmetry completely via the Higgs mechanism. In this setting the leading order non-perturbative contribution is the weakly coupled, and then perfectly reliable, one-instanton contribution. The universal constant C can be computed for $N_c = 2$, $N_f = 1$ with the result $C = 1$ [113]. Thus, the dynamically generated superpotential in the case $N_f < N_c$ is

$$W_{eff} = (N_c - N_f) \left(\frac{\Lambda^{3N_c - N_f}}{\det Q\tilde{Q}} \right)^{1/(N_c - N_f)} \tag{8.2.36}$$

Since a non-perturbative superpotential is generated in the quantum theory, the classical moduli space of vacua (8.2.24) is completely lifted. In particular, we saw that the quantum theory does not have a ground state. A ground state can be found by giving a small mass to the quark superfields through the superpotential

$$W = \mathrm{Tr} m Q\tilde{Q} = \mathrm{Tr}\, m M \tag{8.2.37}$$

Solving the F-term equations we find N_c vacua at

$$\langle M_{ij} \rangle = \left(\det m \Lambda^{3N_c - N_f} \right)^{1/N_c} \left(\frac{1}{m} \right)_{ij} \tag{8.2.38}$$

8.2.4 $N_f \geq 3N_c$

In this range the theory is not asymptotically free: the coupling constant becomes smaller at low energies. The infrared spectrum of the theory consists of elementary quarks and gluons and their superpartners and can be read off from the Lagrangian.

since it is not asymptotically free, it is not an ultraviolet complete theory and its description breaks down when the Landau pole Λ is reached. However, it can be a well defined low energy description of another theory.

8.2.5 $\frac{3}{2}N_c < N_f < 3N_c$

In the range $\frac{3}{2}N_c < N_f < 3N_c$ $SU(N_c)$ SQCD is asymptotically free. It was argued [114, 115, 116] that along the RG flow, its gauge coupling gets larger, until it reaches a finite value. Thus, the infrared theory sits at a fixed point of the renormalization group. To see this, consider the NSVZ β-function

$$\begin{aligned} \beta(g) &= -\frac{g^3}{16\pi^2}\frac{3N_c - N_f + N_f\gamma(g)}{1 - N_c\frac{g^2}{8\pi^2}} \\ \gamma(g) &= -\frac{g^2}{8\pi^2}\frac{N_c^2-1}{N_c} + \mathcal{O}(g^4) \end{aligned} \tag{8.2.39}$$

where $\gamma(g)$ is the anomalous dimension of the quarks. For some values of N_f and N_c the one-loop β-function is negative, while the two-loop contribution is positive. For example, consider the large N_c and N_f limit with $N_c g^2$ and $N_f/N_c = 3 - \epsilon$ fixed. In this limit one finds that the β-function vanishes at $N_c g_*^2 \simeq \frac{8\pi^2}{3}\epsilon$ and a non-trivial fixed point exists. It was argued [116] that this fixed point exists in the full window $\frac{3}{2}N_c < N_f < 3N_c$.

Given that such a fixed point exists, we can use the superconformal algebra to derive some exact results about the theory. This algebra includes an R symmetry. It follows from the algebra that the dimensions of the operators satisfy

$$D \geq \frac{3}{2}|R| \tag{8.2.40}$$

The inequality is saturated for chiral operators, for which $D = \frac{3}{2}R$, and for anti-chiral operators, for which $D = -\frac{3}{2}R$. Its main consequence is that chiral operators form a ring. Too see this, consider the operator product of two chiral operators, $\mathcal{O}_1(x)\mathcal{O}_2(0)$. All the operators in the resulting expansion have $R = R(\mathcal{O}_1) + R(\mathcal{O}_2)$ and hence $D \geq D(\mathcal{O}_1) + D(\mathcal{O}_2)$. Therefore, there is no singularity in the expansion at $x = 0$ and we can define the product of the two operators by simply taking the $x \to 0$ limit. If this limit does not vanish, it leads to a new chiral operator $\mathcal{O}_3$ whose dimension is $D(\mathcal{O}_3) = D(\mathcal{O}_1) + D(\mathcal{O}_2)$.

The R symmetry of the superconformal fixed point is not anomalous and commutes with the flavor $SU(N_f) \times SU(N_f) \times U(1)_B$ symmetry. Hence the gauge invariant mesonic operators have

$$D(M) = \frac{3}{2}R(\widetilde{Q}Q) = 3\frac{N_f - N_c}{N_f} \tag{8.2.41}$$

and the baryonic ones have

$$D(B) = D(\widetilde{B}) = \frac{3N_c(N_f - N_c)}{2N_f} \tag{8.2.42}$$

The gauge invariant operators at the fixed point should be in unitary representations of the superconformal algebra. The main constraint on the representations is that spinless operators have $D \geq 1$, except the identity operator with $D = 0$, and the bound is saturated for free fields. For $D < 1$ ($D \neq 0$) a highest weight representation includes a negative norm state which is not consistent in a unitary theory.

8.2.6 Duality

The physics of the interacting fixed point has an equivalent description in terms of dual degrees of freedom. The so-called magnetic theory is based on the gauge group $SU(N = N_f - N_c)$, with N_f flavors of elementary magnetic quarks q_i and $\tilde{q}^i$ and elementary gauge singlets M^i_j. There is no contradictions in having different gauge symmetries in the two descriptions of the theory; indeed, the gauge symmetry is a redundancy of the description of the physics. On the other hand, global symmetries are physical and are indeed the same in the electric and magnetic descriptions.

The magnetic degrees of freedom transforms according to table (8.2.43).

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_R$
q	$\bar{\square}$	1	$\frac{N_f-N}{N}$	$\frac{N_f-N}{N_f}$
$\tilde{q}$	1	$\square$	$-\frac{N_f-N}{N}$	$\frac{N_f-N}{N_f}$
M	$\square$	$\bar{\square}$	0	$2\frac{N}{N_f}$

(8.2.43)

The most general superpotential compatible with the symmetries is

$$W = \frac{1}{\hat{\Lambda}} M^i_j q_i \tilde{q}^j \tag{8.2.44}$$

This superpotential is required by the duality. Indeed, without it, the magnetic theory would also flow to a non-Abelian Coulomb phase fixed point because $\frac{3}{2}N < N_f < 3N$ for the above range of N_f. At this fixed point M is a free field of dimension one and $D(q\tilde{q}) = 3/2R(q\tilde{q}) = 3(N_f - N)/N_f$. Because the dimensions of chiral operators add, the superpotential (8.2.44) has dimension $D = 1 + 3(N_f - N)/N_f < 3$ at the fixed point of the magnetic gauge theory and is thus a relevant perturbation, driving the theory to a new fixed point. This new fixed point is identical to that of the original, "electric," $SU(N_c)$ theory.

The intermediate scale $\hat{\Lambda}$ in (8.2.44) is necessary for the following reason. In the electric description the meson field M has dimension two at the UV fixed point and acquires an anomalous dimension at the IR fixed point. In the magnetic description, the elementary mesonic field M_m has dimension one at the UV fixed point but it flows to the same operator of the electric theory at the IR fixed point. The relation between the magnetic and the electric description in the UV is given by $M = \hat{\Lambda} M_m$.

The magnetic theory has a dynamical scale $\tilde{\Lambda}$ which is related to the scale Λ of the electric theory by

$$\Lambda^{3N_c-N_f}\tilde{\Lambda}^{3(N_f-N_c)-N_f} = (-1)^{N_f-N_c}\hat{\Lambda}^{N_f} \tag{8.2.45}$$

This scale matching shows that as the electric theory becomes stronger the magnetic theory becomes weaker and vice versa. The phase $(-1)^{N_f-N_c}$ is necessary, because it implies that if we dualize the magnetic theory we are back with the electric theory again. Moreover, the field strengths of the electric and the magnetic theories are related as $W_\alpha^2 = -\widetilde{W}_\alpha^2$. Note the minus sign, which is common in electric/magnetic type dualities, which map $E^2 - B^2 = -(\widetilde{E}^2 - \widetilde{B}^2)$.

8.2.7 $N_c + 2 \leq N_f \leq \frac{3}{2}N_c$

Supersymmetric QCD can also be described by its magnetic dual in the window $N_c + 1 < N_f \leq \frac{3}{2}N_c$. However in this range the theory does not flow to an IR fixed point. The one-loop coefficient of the magnetic gauge β-function is negative

$$\tilde{b}_0 = 3N - N_f = 2N_f - 3N - c < 0 \tag{8.2.46}$$

and the theory is not asymptotically free. Note that the superpotential (8.2.44) is irrelevant. Therefore, the low energy spectrum of the theory consists of the $SU(N)$ gauge fields and the magnetic fields M, q, and $\tilde{q}$. These magnetic massless states are composites of the elementary electric degrees of freedom. Because there are massless magnetically charged fields, the theory is in a non-Abelian free magnetic phase.

8.2.8 Deformations

We now consider how the duality relates the electric and the magnetic theory when some deformations are added in the microscopic description.

Consider adding to the SQCD Lagrangian a mass term for a quark (the generalization to more massive quarks is straightforward)

$$W_e = mQ_{N_f}\tilde{Q}^{N_f} = mM_{N_f}^{N_f} \tag{8.2.47}$$

where M is the electric meson superfield. The low energy theory has $N_f - 1$ light flavors and a dynamical scale Λ_L related to the original scale Λ of the high energy theory by the relation

$$\Lambda_L^{3N_c-(N_f-1)} = m\Lambda^{3N_c-N_f} \tag{8.2.48}$$

which shows that the low energy electric theory flows to a more strongly coupled fixed point.

In the magnetic theory the mass deformation maps to a linear term for the elementary meson M

$$W_m = M_j^i q_i \tilde{q}^j + \hat{\Lambda} m M_{N_f}^{N_f} \tag{8.2.49}$$

The equations of motion of $M_{N_f}^{N_f}$, $M_{N_f}^i$ and $M_i^{N_f}$

$$q_{N_f}\tilde{q}^{N_f} = -\hat{\Lambda}m \qquad q_i\tilde{q}^{N_f} = q_{N_f}\tilde{q}^i = 0 \tag{8.2.50}$$

show that the magnetic gauge group is Higgsed to $SU(N_f - N_c - 1)$ with $N_f - 1$ massless quarks. The equations of motion of the massive quarks read

$$M^{N_f}_{N_f} = M^i_{N_f} = M^{N_f}_i = 0 \tag{8.2.51}$$

and substituting them in (8.2.49) we obtain the low energy superpotential as

$$W = M^i_j q_i \tilde{q}^j \tag{8.2.52}$$

where now $i, j = 1, \ldots, N_f - 1$ and $q_i, \tilde{q}^j$ are the massless quarks which remain in the low energy theory.

The scale of the magnetic theory is modified from $\tilde{\Lambda}$ to $\tilde{\Lambda}_L$ by

$$\tilde{\Lambda}_L^{3(N-1)-(N_f-1)} = \frac{\tilde{\Lambda}^{3N-N_f}}{\hat{\Lambda}m} \tag{8.2.53}$$

The low energy magnetic theory is at weaker coupling at low energies and is the dual of the low energy limit of the electric theory. Thus, the duality is preserved when a mass deformation is introduced and exchanges a more strongly coupled electric description with a more weakly coupled magnetic one.

For completeness, we include the case $N_f = N_c + 2$. The mass term for the flavor field completely breaks the magnetic gauge group. The low energy spectrum contains the mesons M_i^j and the massless singlet quarks q_i and $\tilde{q}^i$, with $i, j = 1, \ldots, N_c + 1$. The superpotential is

$$W_m = \frac{1}{\Lambda_L^{2N_c-1}} M q \tilde{q} \tag{8.2.54}$$

Because the $SU(2)$ magnetic gauge group is completely Higgsed, we have to include the instanton contribution

$$W_{inst} = -\frac{\det M}{\Lambda_L^{2N_c-1}} \tag{8.2.55}$$

which is the superpotential of the electric theory in the case $N_f = N_c + 1$. In the electric description (8.2.55) is a strong coupling effect. In the magnetic description it is associated with an instanton contribution at weak coupling.

8.3 SQCD with singlets: SSQCD

We now extend the discussion of the above section to a new set of theories, first defined in [117]. Consider $SU(N_c)$ SQCD with N_f fundamental flavors Q_i and $\tilde{Q}_i$, $i = 1, \ldots, N_f$ and N_f' additional flavors $Q'_{i'}$ and $\tilde{Q}'_{i'}$, $i' = 1, \ldots, N_f'$ coupled to $N_f'^2$ gauge singlets $S^{i'j'}$ by the superpotential

$$W = hSQ'\tilde{Q}' \tag{8.3.56}$$

For $h = 0$, the resulting theory is SQCD with $N_f + N_f'$ flavors which flows to a nontrivial fixed point when $\frac{3}{2}N_c < N_f + N_f' < 3N_c$. The superpotential (8.3.56) is a relevant deformation which drives the RG flow away from the original fixed point to a new family of conformal field theories in the infrared.

The global symmetry of the theory is $SU(N_f) \times SU(N_f) \times SU(N'_f) \times SU(N'_f) \times U(1)_B \times U(1)_{B'} \times U(1)_F \times U(1)_R$ enhanced to $SU(N_f + N'_f) \times SU(N_f + N'_f) \times U(1)_B \times U(1)_R$ when $h = 0$. The magnetic dual of the theory can be obtained by deforming the Seiberg dual of SQCD with a mass term which pairs up the gauge singlets S with the $N_f'^2$ mesons $M' \sim Q'\tilde{Q}'$ and integrating them out. We are left with an $SU(N = N_f + N'_f - N_c)$ gauge theory with the magnetic quarks q'_i, $\tilde{q}'_i$, $q_{i'}$ and $\tilde{q}_{i'}$ and N_f^2 gauge singlets M_{ij} and $2N_f N'_f$ singlets $K_{ij'}$ and $L_{ij'}$ with superpotential

$$W = Mq'\tilde{q}' + Kq'\tilde{q} + L\tilde{q}'q \tag{8.3.57}$$

The first term is similar to the superpotential of the electric theory. The additional terms distinguish the magnetic duals from the original electric theory.

The duality maps mesons and singlet fields as

$$Q\tilde{Q} \to M \qquad Q\tilde{Q}' \to K \qquad Q'\tilde{Q} \to L \tag{8.3.58}$$

As a consequence, at the infrared fixed point the superconformal R-charges of the electric and magnetic descriptions should satisfy

$$2R(Q) = R(M) \qquad R(S) = 2R(q) \qquad R(Q) + R(Q') = R(P) \tag{8.3.59}$$

together with the constraints from anomaly freedom

$$N_c + N_f(R(Q) - 1) + N'_f(R(Q') - 1) = 0 \qquad R(S) + 2R(Q') = 2 \tag{8.3.60}$$

which follow from the vanishing of the NSVZ β-function and the condition that the superpotential (8.3.56) is exactly marginal. Thus, we also have

$$R(q') = 1 - R(Q) \qquad R(q) = 1 - R(Q') \tag{8.3.61}$$

Note that in the above formulas we made use of $R(Q) = R(\tilde{Q})$ and $R(Q') = R(\tilde{Q}')$ and similarly for the magnetic quarks, because of the global symmetries. The $U(1)_F$ factor can mix with the canonical $U(1)_R$ R-symmetry at the superconformal fixed point and thus its presence allows $R(Q)$ and $R(Q')$ to differ.

Chapter 9

Non-supersymmetric vacua

In this chapter we look for nonsupersymmetric vacua of the supersymmetric QCD models described previously. If supersymmetry exists, it has to be broken. To preserve its appealing features and its predictive power, this breaking must be spontaneous rather than explicit: that is, the Lagrangian should be invariant under the action of supersymmetry, but the vacuum state should not. Furthermore, we would like the mechanism to be dynamical [118].

There are many challenges in trying to implement realistic realizations of dynamical supersymmetry breaking. A first one directly follows from the Witten index [111], which counts the number of supersymmetric vacua of a field theory. If we request the nonsupersymmetric vacuum to be the true, static, global vacuum of the theory, the Witten index has to vanish. Models with this property are quite non-generic and look rather complicate. In particular, any $N = 1$ supersymmetric gauge theory with massive, vector-like matter has supersymmetric vacua, and one has to look for theories with either chiral or massless matter.

Another challenge is the relation between R-symmetry and supersymmetry breaking, the so-called Nelson-Seiberg theorem [119]. Generically, supersymmetry breaking is possible if and only if there is a R-symmetry. With generically we mean that the superpotential of the theory should be the most general superpotential admitted by the symmetries of the theory. Analogously, there is broken supersymmetry in a metastable state if and only if there is an approximate R-symmetry. An unbroken R-symmetry is problematic from different point of views; it cannot be an exact symmetry of the low-energy theory, because it forbids Majorana gaugino masses, thus leading to an unobserved light particle in the spectrum if we want to preserve the gauge mediation mechanism of supersymmetry breaking. Having an exact but spontaneously broken R-symmetry leads to a light R-axion. Thus the R-symmetry should be explicitly broken. Another possibility would be that of keeping the unbroken R-symmetry in the theory and choose a (semi-)direct gauge mediation mechanism.

Based on these observations, Intriligator, Seiberg and Shih (ISS) first discussed a new way to obtain consistent nonsupersymmetric vacua [120]. The basic idea is to abandon the request the theory must have no supersymmetric vacua, and to accept metastable vacua in the supersymmetry breaking sector. It is a phenomenologically viable possibility that we live in a metastable, false, vacuum whose lifetime is longer than the age of the Universe. The true supersymmetric vacuum is located elsewhere in the field space, but it is too far away to observe decays into it.

If we accept metastable states, the hypothesis mentioned above cease to be necessary. In particular, the Witten index is the main obstruction to the construction of dynamical supersymmetry breaking models. Allowing for a non-vanishing Witten index leads us to also consider very simple models such as the supersymmetric QCD. This is the main example in [120].

Phenomenologically, we would like the lifetime of the metastable state to be longer than the age of the Universe. Moreover, the notion of metastable state is meaningful only when it is parametrically long lived. Thus, almost any models of dynamical supersymmetry breaking in metastable states have a dimensionless parameter whose parametric smallness guarantees the longevity of the metastable state.

The rest of the chapter is organized as follows. We first recall some generalities about supersymmetry breaking and review some of the basic models. Then we explain the ISS model of supersymmetry breaking to set the basis for the more complicated case of the SQCD with singlets, which is explained in all details [121]. Finally, we discuss the issue of supersymmetry breaking in three-dimensional field theories [69], which can be useful to understand quantum gravity in four dimensions.

9.1 Generalities and basic examples

Consider a theory of chiral superfields which interact through a superpotential W. For simplicity, we are ignoring the possibility of adding gauge fields, and taking the Kähler potential to be canonical. Since a direct consequence of supersymmetry is that the vacuum state of the theory has vanishing energy, and recalling (8.1.7), supersymmetry is spontaneously broken when the F-component of a chiral superfield X acquires a vacuum expectation value

$$\langle F_X \rangle = \frac{\partial W}{\partial X} \neq 0 \tag{9.1.1}$$

It is easy to construct a linear sigma model using a single chiral superfield. The superpotential is simply

$$W = fX \tag{9.1.2}$$

where f has units of mass square. Supersymmetry is broken by the expectation value of the F-component of the superfield X, and the potential is evaluated to be $V = |f|^2$. Because it is independent of X there are classical vacua for any $\langle X \rangle$.

Supersymmetric theories often have a continuous manifold of supersymmetric vacua which are usually referred to as moduli space of vacua. When supersymmetry is broken, this degeneracy of vacua rarely survives once radiative corrections are taken into account, because it is no longer protected by any (super)symmetry. Therefore, this manifold is called pseudo-moduli space of vacua. Quantum corrections to the potential often lift the degeneracy. This is not the case of the example above, because it is free and the space of vacua remains present even in the quantum theory.

The analogous of the Goldstone theorem for spontaneously broken global fermionic charges states that every time global supersymmetry is spontaneously broken the spectrum of the theory contains a fermionic massless state, the Goldstino. In this case, this fermionic zero mode is the fermionic component of the superfield X, namely ψ_X, while its scalar partner X is the classically massless pseudo-modulus.

Note that there is a R-symmetry, under which X has charge 2. According to the Nelson-Seiberg theorem [119], the presence of a R-symmetry and the fact that (9.1.2) is the most general superpotential compatible with the symmetries lead us to a supersymmetry breaking vacuum. The Nelson-Seiberg theorem also states that if R-symmetry is explicitly broken then the theory has a supersymmetric vacuum. The simple example just discussed gives us the opportunity to see this. Consider deforming the superpotential (9.1.2) by adding $\Delta W = \frac{1}{2}\epsilon X^2$. The classical equation for X now becomes

$$\langle \bar{F}_X \rangle = f + \epsilon X = 0 \tag{9.1.3}$$

which has the supersymmetric solution $\langle X \rangle = -f/\epsilon$. Now the potential reads $V = |f + \epsilon X|^2$ and depends upon X: the pseudo-moduli space has been lifted by the R-symmetry breaking deformation ΔW and we are only left with a supersymmetric vacuum. According to the Goldstone theorem there is no massless Goldstino: indeed the fermionic field ψ_X now has mass ϵ.

Consider again the theory (9.1.2) with a more general Kähler potential $K(X, \bar{X})$. Although it is not renormalizable, it can be viewed as the low energy effective theory of another, microscopic, theory which is valid at energies larger than the cutoff scale Λ.

Let us suppose that the Kähler is smooth. Then the potential

$$V = \left(\frac{\partial^2 K}{\partial X \partial \bar{X}} \right)^{-1} |f|^2 \tag{9.1.4}$$

never vanishes at finite $\langle X \rangle$ and lifts the degeneracy along the pseudo-moduli space. Nevertheless, before concluding that supersymmetry is broken, we should consider the behavior at large vacuum expectation value. If there is any direction along which $\lim_{|} X| \to \infty \partial_X \partial_{\bar{X}} K$ diverges then V approaches zero at infinity and it has no global minimum: the theory does not have a ground state. If $\lim_{|} X| \to \infty \partial_X \partial_{\bar{X}} K$ vanishes in all directions the potential rises at infinity and it has a supersymmetry breaking global minimum at some finite $\langle X \rangle$. Finally, if the large VEV limit is finite, the potential approaches a constant and a more detailed analysis is needed.

Let us assume that the Kähler potential can be expanded around $X = 0$ as

$$K = \bar{X} X - \frac{(\bar{X} X)^2}{|\Lambda|^2} + \dots \tag{9.1.5}$$

where Λ is the UV cutoff. Equations (9.1.2) and (9.1.5) define the so-called Polonyi model [122]. There is a nonsupersymmetric vacuum at $X = 0$ in which the scalar component of X gets mass

$$m_X^2 = \frac{4|f|^2}{|\Lambda|^2} \tag{9.1.6}$$

and its fermionic partner is the exactly massless Goldstino. Note that the Polonyi model possesses an unbroken R-symmetry.

The Polonyi model serves as the effective low energy theory for a wide class of supersymmetry breaking models. Amongst them, the O'Raifeartaigh-type models play an important role. The basic idea is to find a set of chiral fields and a superpotential such that the F-conditions cannot be solved simultaneously.

The basic O'Raifeartaigh model [123] consists of three chiral superfields with canonical Kähler and superpotential

$$W = \frac{1}{2}hX\phi_1^2 + m\phi_1\phi_2 + fX \tag{9.1.7}$$

When $m^2 \gg f$ the ϕ_i fields can be integrated out and the theory reduces to (9.1.2). For non-vanishing m the equations

$$\begin{aligned} \bar{F}_X &= f + \frac{1}{2}h\phi_1^2 = 0 \\ \bar{F}_{\phi_1} &= hX\phi_1 + m\phi_2 = 0 \\ \bar{F}_{\phi_2} &= m\phi_1 = 0 \end{aligned} \tag{9.1.8}$$

cannot be simultaneously satisfied and supersymmetry is broken. The minima of the potential depend on the parameter

$$y \equiv \left|\frac{hf}{m^2}\right| \tag{9.1.9}$$

We only consider the case $y < 1$ for the moment. Then the potential has a degenerate space of minima at $\phi_1 = \phi_2 = 0$ and arbitrary X:

$$V_{min} = |f|^2 \tag{9.1.10}$$

The scalar component of X is the classical pseudo-modulus, and its fermionic partner is the Goldstino. The classical mass spectrum of the other fields can be easily computed. The two Weyl fermions have mass

$$m_{1/2}^2 = \frac{1}{4}(|hX| \pm \sqrt{|hX|^2 + 4|m|^2})^2 \tag{9.1.11}$$

and the four real scalars have mass

$$m_0^2 = \left(|m|^2 + \frac{1}{2}\eta|hf| + \frac{1}{2}|hX|^2 \pm \frac{1}{2}\sqrt{|hf|^2 + 2\eta|hf||hX|^2 + 4|m|^2|hX|^2 + |hX|^4}\right) \tag{9.1.12}$$

where $\eta = \pm 1$. Since the spectrum changes along the pseudo-moduli space parameterized by X, these vacua are physically distinct.

The parameter y sets the relative size of the mass splittings. For $y \ll 1$, corresponding to $m^2 \gg f$, the fields ϕ_1 and ϕ_2 have been integrated out at a scale higher than the supersymmetry breaking scale and their spectrum is approximately supersymmetric. For $y = 1$ the mass splitting is maximized.

The above example is straightforwardly generalized. Consider a theory of three chiral superfields X_1, X_2 and ϕ with canonical Kähler potential and superpotential

$$W = X_1 g_1(\phi) + X_2 g_2(\phi) \tag{9.1.13}$$

To ensure renormalizability we choose g_1 and g_2 to be at most quadratic. The theory has a R-symmetry with $R(X_1) = R(X_2) = 2$ and ϕ is neutral. The supersymmetry conditions are

$$\begin{aligned} \bar{F}_{X_1} &= g_1(\phi) = 0 \\ \bar{F}_{X_2} &= g_2(\phi) = 0 \\ \bar{F}_{\phi} &= X_1 g_1'(\phi) + X_2 g_2'(\phi) = 0 \end{aligned} \tag{9.1.14}$$

We can always set $F_\phi = 0$ by choosing appropriate X_1 and X_2. But, for generic functions $g_1(\phi)$ and $g_2(\phi)$, we cannot simultaneously solve the first two conditions. Hence supersymmetry is generically broken. Since only one linear combination of X_1 and X_2 is constrained by $F_\phi = 0$, the other linear combination is a classical pseudo-modulus. On this pseudo-moduli space we have to find the minimum of the potential

$$V = |g_1(\phi)|^2 + |g_2(\phi)|^2 \tag{9.1.15}$$

which requires to solve

$$\left(\overline{g_1(\phi)} g_1'(\phi) + \overline{g_2(\phi)} g_2'(\phi)\right)\Big|_{\phi=\langle\phi\rangle} = 0 \tag{9.1.16}$$

Writing $X_i = \langle X_i \rangle + \delta X_i$, $\phi = \langle \phi \rangle + \delta\phi$ and expanding to quadratic order in δX_1, δX_2 and $\delta\phi$ yields the mass matrix of the scalar fields; the eigenvalues of this matrix have to be all non-negative in order to void tachyonic modes. The fermion mass terms are given by

$$\left(X_1 g_1''(\phi) + X_2 g_2''(\phi)\right) \psi_\phi \psi_\phi + \left(g_1'(\phi)\psi_{X_1} + g_2'(\phi)\psi_{X_2}\right) \psi_\phi \tag{9.1.17}$$

Note that one linear combination of ψ_{X_1} and ψ_{X_2} remains massless: this is the Goldstino field.

The generalized O'Raifeartaigh models are the effective theories of many models of dynamical supersymmetry breaking. Once it is established that a model reduces to a O'Raifeartaigh-type model at low energies, one can immediately conclude that there is a supersymmetry breaking vacuum. In the next sections, we use this technique to infer the presence of a supersymmetry breaking vacuum at low energies.

9.2 The ISS model

In this section we discuss the prototypical example of the dynamical supersymmetry breaking models [120]. It is an extremely simple model: $N = 1$ supersymmetric $SU(N_c)$ QCD with N_f massive fundamental flavors. In order to have control over the infrared theory, N_f is chosen in the free magnetic range $N_c + 1 < N_f < \frac{3}{2}N_c$. The $SU(N_c)$ SQCD with $N_f < 3N_c$ flavors is asymptotically free, then it becomes strongly coupled in the infrared. In order to perform reliable calculations at low energies, we switch to its magnetic weakly coupled description. The powerful of the duality allows us to determine quantum corrections to non-holomorphic quantities in a well-posed perturbative framework.

As expected from the Witten index argument, the low energy theory has supersymmetric vacua, nevertheless, it also possesses metastable long-lived nonsupersymmetric vacua. The lifetime of the latter depends upon a particular ratio between the two parameters of the theory, namely

$$\epsilon \sim \sqrt{\frac{m}{\Lambda}} \tag{9.2.18}$$

which can be made parametrically small. Here, m is the fundamental flavors mass and Λ is the dynamical scale of the theory (8.2.15).

Let us define the model more precisely. The $N = 1$ supersymmetric $SU(N_c)$ QCD Lagrangian reads

$$S = \int d^4xd^4\theta \,\left(\bar{Q}_i e^V Q_i + \tilde{Q}_i e^V \bar{\tilde{Q}}_i\right) - \frac{i}{16\pi}\int d^4xd^2\theta \;\tau W^\alpha W_\alpha + \text{h.c.} \tag{9.2.19}$$

where $i = 1, \ldots, N_f$ is a flavor index. We limit ourselves to the case $N_c + 1 < N_f < \frac{3}{2}N_c$. We take for the tree-level superpotential

$$W = \mathrm{Tr}\, mM \tag{9.2.20}$$

where $M_{ij} = Q_i\tilde{Q}_j$ is the meson superfield and m is a non-degenerate $N_f \times N_f$ mass matrix. The latter can always be diagonalized by a bi-unitary transformation. For simplicity, we consider it of the form $m \equiv m \cdot \mathbf{1}$ with real positive m such that $m \ll |\Lambda|$. The generalization to different masses is straightforward.

The Witten index predicts N_c supersymmetric ground states of the theory. They are the N_c root of

$$\langle M\rangle = \left(\det m\, \Lambda^{3N_c - N_f}\right)^{\frac{1}{N_c}} \frac{1}{m} \tag{9.2.21}$$

Note that in the limit of small masses they approach the origin of the field space.

The Seiberg dual of the theory (9.2.20) is characterized by the $SU(\tilde{N} = N_f - N_c)$ magnetic gauge group, N_f magnetic quarks q_i and $\tilde{q}_i$ in the fundamental and antifundamental representation of the magnetic gauge group and N_f^2 gauge singlets M_{ij}. In the range we are considering this theory is infrared free; this means that the Kähler potential assumes the form

$$K = \frac{1}{\beta}\mathrm{Tr}(\bar{q}q + \bar{\tilde{q}}\tilde{q}) + \frac{1}{\alpha|\Lambda|^2}\mathrm{Tr}\bar{M}M + \ldots \tag{9.2.22}$$

where the two dimensionless coefficients α and β have been introduced. They are real positive numbers which carry the non-holomorphic informations of the theory, hence they cannot be exactly computed. However, the qualitative features of the model will not depend upon them.

The magnetic superpotential is

$$W_m = \frac{1}{\hat{\Lambda}}\mathrm{Tr}\tilde{q}Mq + \mathrm{Tr}\, mM \tag{9.2.23}$$

where the dimension-one coefficient $\hat{\Lambda}$ is related to the dynamical scales of the electric and magnetic theories through [101]

$$\Lambda^{3N_c-N_f}\tilde{\Lambda}^{3(N_f-N_c)-N_f} = (-)^{N_f-N_c}\hat{\Lambda}^{N_f} \tag{9.2.24}$$

The dimensionful parameters of the magnetic theory $\tilde{\Lambda}$ and $\hat{\Lambda}$ are not uniquely determined by the electric informations. Indeed, we have the freedom to rescale the magnetic quarks. This rescaling does not only change the value of the magnetic scales, but also the value of β in the Kähler potential and the relations between the electric baryons $B = Q^{N_c}$ and $\tilde{B} = \tilde{Q}^{N_c}$ and their expressions in terms of the magnetic superfields q and $\tilde{q}$. In what follows, we find convenient to set $\beta = 1$ and give the results in terms of $\tilde{\Lambda}$ and $\hat{\Lambda}$.

Let us define a new field and parameters

$$\Phi = \frac{M}{\sqrt{\alpha}\Lambda} \qquad h = \frac{\sqrt{\alpha}\Lambda}{\hat{\Lambda}} \qquad \mu^2 = -m\hat{\Lambda} \tag{9.2.25}$$

in terms of which the Kähler potential assumes the canonical form

$$K = \mathrm{Tr}\bar{q}q + \mathrm{Tr}\bar{\tilde{q}}\tilde{q} + \mathrm{Tr}\bar{\Phi}\Phi \tag{9.2.26}$$

and the tree-level superpotential becomes

$$W = h\mathrm{Tr}q\Phi\tilde{q} - h\mu^2\mathrm{Tr}\Phi \tag{9.2.27}$$

We consider the $SU(\tilde{N})$ group as a global symmetry for a while, and discuss the effects of the gauge fields below. Thus the global symmetry group and the quantum numbers assignment is given in table 9.1. Note that the superpotential is the most general one consistent with the global symmetries.

	$SU(\tilde{N})$	$SU(N_f)$	$U(1)_B$	$U(1)_R$
Φ	1	$\square \otimes \bar{\square}$	0	2
q	$\square$	$\bar{\square}$	1	0
$\tilde{q}$	$\bar{\square}$	$\square$	-1	0

Table 9.1: Matter content and quantum numbers of the low energy ISS model

The F-terms of Φ, $F_{\Phi_{ij}} \sim h\tilde{q}_j{}^c q_{ic} - h\mu^2\delta_{ij}$ (c is a $SU(\tilde{N})$ index), cannot all vanish because δ_{ij} has rank N_f while $\tilde{\phi}_j^c\phi_{ic}$ has only rank $\tilde{N} < N_f$. Thus, supersymmetry is spontaneously broken by what has been referred to as the rank-condition mechanism. There is a classical moduli space of degenerate nonsupersymmetric vacua given by, up to global symmetries,

$$\Phi = \begin{pmatrix} 0 & 0 \\ 0 & \Phi_0 \end{pmatrix} \qquad q = \begin{pmatrix} q_0 \\ 0 \end{pmatrix} \qquad \tilde{q}^T = \begin{pmatrix} \tilde{q}_0 \\ 0 \end{pmatrix} \tag{9.2.28}$$

where Φ_0 is an arbitrary $(N_f - \tilde{N}) \times (N_f - \tilde{N})$ matrix and q_0 and $\tilde{q}_0$ are $\tilde{N} \times \tilde{N}$ matrices satisfying $\tilde{q}_0 q_0 = \mu^2 \mathbf{1}$. On this moduli space the scalar potential is

$$V = (N_f - \tilde{N}) \left| h^2 \mu^4 \right| \tag{9.2.29}$$

We now consider the quantum fluctuations. To simplify matters, we take $\Phi_0 = 0$ and $q_0 = \tilde{q}_0 = \mu \mathbf{1}$. We parametrize the quantum fluctuations as

$$\Phi = \begin{pmatrix} Y & Z^T \\ \tilde{Z} & \hat{\Phi} \end{pmatrix} \qquad q = \begin{pmatrix} \mu + \frac{1}{\sqrt{2}}(\chi_+ + \chi_-) \\ \frac{1}{\sqrt{2}}(\rho_+ + \rho_-) \end{pmatrix} \qquad \tilde{q}^T = \begin{pmatrix} \mu + \frac{1}{\sqrt{2}}(\chi_+ - \chi_-) \\ \frac{1}{\sqrt{2}}(\rho_+ - \rho_-) \end{pmatrix} \tag{9.2.30}$$

where Y and $\chi_\pm$ are $\tilde{N} \times \tilde{N}$ matrices and Z, $\tilde{Z}$ and $\rho_\pm$ are $(N_f \times \tilde{N}) \times \tilde{N}$ matrices. Substituting (9.2.30) into (9.2.27) we obtain

$$\begin{aligned} W = h\mathrm{Tr} \; & \left[\frac{\mu}{\sqrt{2}} Z^T (\rho_+ - \rho_-) + \frac{\mu}{\sqrt{2}} \tilde{Z}^T (\rho_+ + \rho_-) + \frac{1}{2} (\rho_+ + \rho_-)^T \hat{\Phi} (\rho_+ - \rho_-) - \mu^2 \hat{\Phi} \right. \\ & \left. + \frac{\mu}{\sqrt{2}} Y (\chi_+ - \chi_-) + \frac{\mu}{\sqrt{2}} Y^T (\chi_+ + \chi_-) \right] + \ldots \end{aligned} \tag{9.2.31}$$

where we omitted some cubic and higher order terms in the fluctuations because they do not affect the final results. Note that (9.2.31) contains $N_f - \tilde{N}$ decoupled copies of an O'Raifeartaigh-type model. The tree-level scalar and fermion mass matrices are

$$m_0^2 = \begin{pmatrix} W^{\dagger ac} W_{cb} & W^{\dagger abc} W_c \\ W_{abc} W^{\dagger c} & W_{ac} W^{\dagger cb} \end{pmatrix} \qquad m_{1/2}^2 = \begin{pmatrix} W^{\dagger ac} W_{cb} & 0 \\ 0 & W_{ac} W^{\dagger cb} \end{pmatrix} \tag{9.2.32}$$

with $W_c \equiv \partial W / \partial q^c$ and so on. By computing the eigenvalues and eigenvectors of these mass matrices as a function of the pseudo-moduli expectation values we find that the following fields are tree-level massless:

$$\mathrm{Im}\left(\frac{\mu^*}{|\mu|} \chi_- \right), \qquad \mathrm{Re}\left(\frac{\mu^*}{|\mu|} \rho_+ \right), \qquad \mathrm{Im}\left(\frac{\mu^*}{|\mu|} \rho_- \right) \tag{9.2.33}$$

$$\hat{\Phi}, \qquad \hat{\chi} \equiv \mathrm{Re}\left(\frac{\mu^*}{|\mu|} \chi_- \right) \tag{9.2.34}$$

In the first line we indicated the Goldstone bosons of the broken global symmetries, while the second line contains the classical pseudoflat directions. The other fluctuations get tree-level masses of order $|h\mu|$. The one-loop corrections to the scalar potential (9.2.29) can be computed using the Coleman-Weinberg formula [124]

$$V_{eff}^{(1)} = \frac{1}{64\pi^2} \mathrm{STr} \mathcal{M}^4 \log \frac{\mathcal{M}^2}{\Lambda^2} \equiv \frac{1}{64\pi^2} \left(\mathrm{Tr}\, m_0^4 \log \frac{m_0^2}{\Lambda^2} - \mathrm{Tr}\, m_{1/2}^4 \log \frac{m_{1/2}^2}{\Lambda^2} \right) \tag{9.2.35}$$

and it gives the leading quantum corrections to the effective action because the superpotential coupling h is marginally irrelevant. Substituting (9.2.32) into (9.2.35) and expanding up to quadratic order in the pseudo-moduli fields we find

$$V_{eff}^{(1)} = \frac{|h^4\mu^2|(\log 4 - 1)}{8\pi^2}\left(\frac{1}{2}(N_f - \tilde{N})\mathrm{Tr}\hat{\chi}^2 + \tilde{N}\mathrm{Tr}\hat{\Phi}^\dagger\hat{\Phi}\right) + \dots \tag{9.2.36}$$

The kinetic terms for the pseudo-moduli fields are inherited from the tree-level kinetic terms of the full theory, so they are canonical and diagonal at leading order. The quadratic effective Lagrangian

$$\mathcal{L}_{eff} = \mathrm{Tr}\left|\partial_\mu\hat{\Phi}\right|^2 + \frac{1}{2}\mathrm{Tr}\left(\partial_\mu\hat{\chi}\right)^2 - V_{eff}^{(1)} + \dots \tag{9.2.37}$$

shows that the pseudo-moduli are stabilized with positive mass squared and we conclude that the vacua (9.2.28) are stable, without any tachyonic directions.

Let us resume the spectrum of the theory. We find that the nonsupersymmetric vacuum has a hierarchy of mass scales dictated by the coupling h. There are fields with tree-level masses of order $|h\mu|$. The pseudo-moduli have masses of order $|h^2\mu|$; note that they are suppressed by a loop factor. The massless spectrum contains the Goldstone bosons and the exactly massless Goldstino from supersymmetry breaking.

We now turn to the inclusion of the gauge fields. We gauge the $SU(\tilde{N})$ symmetry. Because we are restricting our analysis to the case $N_c + 1 < N_f < \frac{3}{2}N_c$, the $SU(\tilde{N})$ gauge theory has $N_f > 3\tilde{N}$ and it is IR free instead of asymptotically free: above its dynamical scale $\tilde{\Lambda}$ it is strongly coupled. For energies of order $\tilde{\Lambda}$ and above, the weakly coupled electric description is much more accurate.

Having gauged $SU(\tilde{N})$, the D-term contribution should be added to the full potential

$$V_D = \frac{g^2}{2}\sum_A\left(\mathrm{Tr}q^\dagger T_A q - \mathrm{Tr}\tilde{q}T_A\tilde{q}^\dagger\right)^2 \tag{9.2.38}$$

The D-term potential vanishes in the nonsupersymmetric vacuum, so it remains a minimum of the tree-level potential. The $SU(\tilde{N})$ gauge group is completely Higgsed in this vacuum. The $SU(\tilde{N})$ gauge fields acquire mass $g\mu$ through the super-Higgs mechanism. The traceless parts of the would-be Goldstone bosons $\mathrm{Im}\mu^*\chi_-/|\mu|$ are eaten and the traceless parts of the pseudo-moduli $\hat{\chi}$ get a positive tree-level mass $g\mu$ from the coupling with the gauge fields. Then $\hat{\Phi}$ and $\mathrm{Tr}\hat{\chi}$ only remain as classical pseudo-moduli. Along the lines of our previous discussion, we should compute the quantum effective potential for them to determine if they are stabilized or get tachyonic masses. It turns out that the effect of the gauge fields drops out in the leading order effective potential for the pseudo-moduli. The reason is that the tree-level spectrum of the massive $SU(\tilde{N})$ fields is supersymmetric, so the supertrace vanishes. This is due to the fact that the $SU(\tilde{N})$ gauge fields do not directly couple to the supersymmetry breaking fields: the D-terms vanish on the pseudo-flat space, and the non-zero expectation values of q and $\tilde{q}$, which give the $SU(\tilde{N})$ gauge fields their masses, do not couple directly to any non-zero F-terms. Thus, the leading order effective potential we already computed stabilizes the pseudo-moduli with positive squared masses. The supersymmetry breaking vacuum (9.2.28) survives to the $SU(\tilde{N})$ gauging.

We saw that the gauging does not affect the supersymmetry breaking vacua. Its effects are important elsewhere in the field space: it leads to supersymmetric vacua. This is in accordance with both the Witten index argument and the electric description of the theory, see (9.2.21). When the field Φ assumes a non-vanishing vacuum expectation value, the superpotential (9.2.27) give the flavors q and $\tilde{q}$ mass $\langle h\Phi\rangle$. Below the energy scale $\langle h\Phi\rangle$ they can be integrated out and we are left with the low energy pure Yang-Mills $SU(\tilde{N})$ theory. This theory exhibits gaugino condensation; that is, the low energy effective superpotential is

$$W_{eff} = \tilde{N}\left(\frac{h^{N_f}\det\Phi}{\tilde{\Lambda}^{N_f-3\tilde{N}}}\right)^{1/\tilde{N}} - h\mu^2\mathrm{Tr}\Phi \tag{9.2.39}$$

The first term comes from gaugino condensation. The $N_c = N_f - \tilde{N}$ supersymmetric vacua are found by the extremization of the superpotential as

$$\langle h\Phi\rangle = \tilde{\Lambda}\epsilon^{2\tilde{N}/(N_f-\tilde{N})} \tag{9.2.40}$$

where we defined $\epsilon \equiv \frac{\mu}{\tilde{\Lambda}}$. According to the electric description, when $\mu \to 0$, $\epsilon \to 0$ and the supersymmetric vacua approach the origin of the field space. Because $2\tilde{N} \ll N_f - \tilde{N}$ and $|\epsilon| \ll 1$ we have

$$|\mu| \ll |\langle h\Phi\rangle| \ll |\tilde{\Lambda}| \tag{9.2.41}$$

The first inequality guarantees the longevity of the metastable nonsupersymmetric vacua. Further, because the vacuum expectation value (9.2.40) is well below the Landau pole $\tilde{\Lambda}$ the low energy analysis is reliable.

The appearance of supersymmetric vacua is referred to as dynamical supersymmetry restoration. For $\tilde{\Lambda} \to \infty$ with μ fixed, the IR gauge coupling $g \to 0$ and the model breaks supersymmetry. For finite $\tilde{\Lambda}$ and small ϵ the supersymmetric vacua come in from infinity, due to the relevant non-perturbative effects of the IR free gauge theory. Because the infrared gauge coupling is small, these effects can be consistently computed in the low energy theory.

The discussion above agrees with the Nelson-Seiberg theorem [119]. In the ungauged theory there is a conserved $U(1)_R$ symmetry and it breaks supersymmetry. In the gauged theory, this R-symmetry becomes anomalous under the gauge group; thus, the full low energy superpotential (9.2.39) explicitly breaks the $U(1)_R$ and supersymmetric vacua appears. Nevertheless, for small expectation values the gauge coupling is small and there is an approximate accidental R-symmetry. Thus, the nonsupersymmetric vacua become only metastable in the gauged model. This shows that supersymmetric $SU(N_c)$ QCD in the IR free magnetic range exhibits metastable supersymmetry breaking.

Because the nonsupersymmetric vacua are only metastable vacua, we have to show that their lifetime is larger than our Universe. Our intention here is to show that the smallness of the parameter ϵ makes them parametrically long lived. The semi-classical decay probability is proportional to $\exp(-S)$, where S is the bounce action [125].

In order to estimate the bounce action, we need to give a qualitative picture of the potential for the scalar fields. The nonsupersymmetric vacuum is characterized by

$$\Phi = 0 \qquad q = \tilde{q}^T \begin{pmatrix} \mu \\ 0 \end{pmatrix} \qquad V = (N_f - \tilde{N}) \left| h^2 \mu^4 \right| \tag{9.2.42}$$

We write the supersymmetric one as

$$\Phi = \frac{\mu}{h} \frac{1}{\epsilon^{(N_f - 3\tilde{N})/(N_f - \tilde{N})}} \qquad q = \tilde{q} = 0 \qquad V = 0 \tag{9.2.43}$$

The bounce action is computed from the minimum path between these two vacua. The $|hq\Phi|^2$ and $|h\Phi\tilde{q}|^2$ terms in the potential provide a large potential energy cost to having both Φ and q or $\tilde{q}$ being non-zero. Thus, the most efficient path is to climb quickly from the nonsupersymmetric vacuum to a point near the local peak

$$\Phi = 0 \qquad q = \tilde{q} = 0 \qquad V_{peak} = N_f |h^2 \mu^4| \tag{9.2.44}$$

and then to take increasing Φ towards the supersymmetric minimum, keeping $q = \tilde{q} = 0$. As long as $\epsilon \ll 1$, the slope of the potential along this path is small. We approximate the full path by a triangle potential barrier. Thus, using the result [126]

$$S \sim \frac{(\Delta\Phi)^4}{\Delta V} \sim \frac{1}{\epsilon^{4(N_f - 3\tilde{N})/(N_f - \tilde{N})}} \tag{9.2.45}$$

we see that taking $\epsilon \to 0$ we can make the bounce action arbitrarily large and the decay rate of the metastable state into the supersymmetric one arbitrarily large. We conclude that the metastable vacuum can be made arbitrarily long lived.

We conclude this section by noticing the fate of the metastable states in the other windows of the supersymmetric QCD. For $N_f > 3N_c$ the electric theory is no longer asymptotically free. Its infrared dynamics is trivial and the metastable states are simply not present. For $\frac{3}{2}N_c < N_f < 3N_c$ the theory flows to a nontrivial fixed point. The magnetic dual flows to the same IR fixed point, and we could repeat the above procedure. However, in this case the infrared theory is interacting, and the low energy effective superpotential should be replaced with

$$W = (N_c - N_f) \left(\frac{\det M}{\Lambda^{3N_c - N_f}} \right)^{1/(N_f - N_c)} \tag{9.2.46}$$

where we used the electric variables. Near the origin of the field space, this superpotential scales as $M^{N_f/(N_f - N_c)}$ which is larger than M^3 and its contribution to the potential cannot be neglected. Stated in another way, the vacuum expectation value of M in this range is too close to the origin and destabilize the metastable states: the bounce action

$$S \sim \epsilon^{4 \frac{2N_f - 3N_c}{N_c}} \tag{9.2.47}$$

jumps from $S \ll 1$ to $S \gg 1$ for $\epsilon \ll 1$. Note the transition for $N_f = \frac{3}{2}N_c$. In this case the magnetic theory is IR free only because of its two-loop β-function. Here the superpotential scales like M^3 and again it cannot be neglected near the origin. As a result, the bounce action is of order $S \sim 1$.

In the next section we show how the superpotential should be deformed to overcome the problem of the instability of the nonsupersymmetric vacuum in the conformal window $\frac{3}{2}N_c < N_f < 3N_c$.

9.3 Metastable vacua in the conformal window

In the range $\frac{3}{2}N_c < N_f < 3N_c$ the asymptotically free supersymmetric $SU(N_c)$ QCD with N_f massless fundamental flavors flows to an IR interacting fixed point. If we further restrict N_f such that $N_f < 2N_c$, the electric description of the fixed point is strongly coupled, whereas its dual magnetic description is weakly coupled. If we deform the electric theory by adding a small mass for the fundamental quarks, the infrared theory is only approximately conformal and undergoes RG evolution until it exits the conformal regime at some scale [121]. At this mass scale we are left with a weakly coupled $SU(\tilde{N} = N_f - N_c)$ gauge theory which possesses both supersymmetric and metastable nonsupersymmetric vacua. We argued above that the bounce action is too small to meet the phenomenological requirement of long lifetime of the nonsupersymmetric vacuum. In the following subsection we make this statement more precise by analyzing the RG evolution of both the couplings of the theory and the bounce action.

9.3.1 A closer look to the conformal window

Consider again the $SU(N_c)$ gauge theory with N_f flavors of quarks charged under an $SU(N_f)^2$ flavor symmetry broken to $SU(N_f)$ by the superpotential

$$W = mQ\tilde{Q} \tag{9.3.48}$$

where the mass m is much smaller than the holomorphic scale of the theory Λ. In the window $N_c + 1 < N_f$ this theory admits a dual description in term of a *magnetic* gauge group $SU(\tilde{N}) = SU(N_f - N_c)$, N_f magnetic quarks q and $\tilde{q}$ and the *electric* meson $N = Q\tilde{Q}$ normalized to have mass dimension one. The dual superpotential reads

$$W_m = -h\mu^2 N + hNq\tilde{q} + \tilde{N}\left(\tilde{\Lambda}^{\tilde{b}} h^{N_f} \det N\right)^{\frac{1}{\tilde{N}}} \tag{9.3.49}$$

where we introduced the marginal coupling h and the holomorphic scale of the dual theory $\tilde{\Lambda}$, and we added the non perturbative contribution due to gaugino condensation. From now on we set $h = 1$. The holomorphic scales Λ and $\tilde{\Lambda}$ are related by a scale matching relation [101]. The one loop beta function coefficient is $\tilde{b} = 3\tilde{N} - N_f = 2N_f - 3N_c$.

In the range $N_c + 1 < N_f < 3/2N_c$, this theory has a supersymmetry breaking vacuum at $N = 0$, with non zero vev for the quarks. The supersymmetric vacuum is recovered in the large field region for N. The parametrically long distance between the two vacua guarantees the long life time of the non supersymmetric one.

The metastable non supersymmetric vacua found in the magnetic free window of massive SQCD are destabilized in the conformal window $3/2N_C < N_f < 3N_C$. This fact is based on the observation that the non perturbative superpotential in (9.3.49) is not negligible in the small field region, as instead it happens in the magnetic free window.

Here we study more deeply this problem. In general, in the presence of relevant deformations the conformal regime is only approximated. If these deformation are small enough there is a large regime of scales in which the theory flows to lower energies while remaining approximatively conformal. The physical couplings vary along the RG flow because of the wave function renormalization of the fields, until the theory exits from the conformal regime. Below this scale the theory is IR free and the renormalization effects are negligible.

We study the RG properties of the ISS model in the conformal window by using a canonical basis for the fields. Flowing from a UV scale E_{UV} to an IR scale E_{IR} the fields are not canonically normalized anymore, and we have to renormalize them by the wave function renormalization $Z_i(E_{IR}, E_{UV})$, namely $\phi_i^{IR} = \sqrt{Z_i}\phi_i^{UV}$. In terms of the renormalized fields the Kähler potential is canonical. The couplings appearing in the superpotential undergo RG evolution, and are the physical couplings. In this way the coupling μ_{IR} of the IR superpotential becomes

$$\mu_{IR} = \mu_{UV} Z_N(E_{IR}, E_{UV})^{-\frac{1}{4}} \tag{9.3.50}$$

The holomorphic scale that appears in the superpotential is unphysical in the conformal window and it is defined as

$$\tilde{\Lambda} = E e^{-\frac{8\pi^2}{g_*^2 \tilde{b}}} \tag{9.3.51}$$

where E is the RG running scale, and g_* is the gauge coupling at the superconformal fixed point. In the canonical basis $\tilde{\Lambda}$ is rescaled as well during the RG conformal evolution as [109, 110, 127]

$$\tilde{\Lambda}_{IR} = \tilde{\Lambda}_{UV} \frac{E_{IR}}{E_{UV}} \tag{9.3.52}$$

In the ISS model the two possible sources of breaking of the conformal invariance are the masses of the fields at the non supersymmetric vacuum and at the supersymmetric vacuum. We define the CFT exit scale as $E_{IR} = \Lambda_c$. In this model this scale is necessarily set by the masses of the fields at the supersymmetric vacuum, which are proportional to the vev of the field N. In fact by setting

$$\Lambda_c \equiv \langle N \rangle_{susy} = \mu_{IR} \left(\frac{\mu_{IR}}{\tilde{\Lambda}_{IR}} \right)^{\frac{\tilde{b}}{N_f - \tilde{N}}} \tag{9.3.53}$$

the physical mass at this scale results

$$\mu_{IR} = \Lambda_c e^{-\frac{4\pi^2}{g*^2 \tilde{N}}} \ll \Lambda_c \tag{9.3.54}$$

Hence the assumption that $\langle N \rangle_{susy}$ stops the conformal regime is consistent. The opposite case, with $\Lambda_c \equiv \mu_{IR} \gg \langle N \rangle_{susy}$ cannot be consistently realized.

The bounce action at the scale Λ_c is [121]

$$S_B \sim \left(\frac{\mu_{IR}}{\tilde{\Lambda}_{IR}}\right)^{\frac{4\tilde{b}}{N_f-\tilde{N}}} \sim e^{\frac{16\pi^2}{g_*^2\tilde{N}}} \tag{9.3.55}$$

This bounce is not parametrically large and it depends only on the coupling constant g_* at the fixed point. In general, as we shall see in the appendix C.2, the bounce action is not RG invariant, but it runs during the RG flow. In this case S_B at the CFT exit scale only depends on the ratio of the two relevant scales in the theory which is the RG invariant coupling constant.

In general, by adding other deformations, the bounce action is not RG invariant anymore and we have to study its flow. In some cases, the lifetime of a vacuum decreases as we flow towards the infrared. In the next section, by adding new massive quarks to the ISS model, we show that long living metastable vacua exist in the conformal window.

9.3.2 Metastable vacua by adding relevant deformations

In this section we describe our proposal for realizing metastable supersymmetry breaking in the conformal window of $N = 1$ SQCD-like theories [121]. The key point is the addition of massive quarks in the dual magnetic description. This introduces a new mass scale that controls the distance in the field space of the supersymmetric vacua.

We start from the magnetic description of the ISS model of the previous section. We add a new set of massive fields p and $\tilde{p}$ charged under a new $SU(N_f^{(2)})$ flavor symmetry. We also add new bifundamental fields K and L charged under $SU(N_f^{(1)}) \times SU(N_f^{(2)})$. The added number of flavors is such that $3/2\tilde{N} < N_f^{(1)} + N_f^{(2)} < 3\tilde{N}$. The superpotential of the model is

$$W = Kp\tilde{q} + L\tilde{p}q + Nq\tilde{q} + \rho p\,\tilde{p} - \mu^2 N \tag{9.3.56}$$

and the field content is summarized in the Table 9.2. This model corresponds to the dual

	$N_f^{(1)}$	$N_f^{(2)}$	$\tilde{N}$
N	$N_f^{(1)} \otimes N_f^{(1)}$	1	1
$q+\tilde{q}$	$\bar{N}_f^{(1)} \oplus N_f^{(1)}$	1	$\tilde{N} \oplus \bar{\tilde{N}}$
$p+\tilde{p}$	1	$\bar{N_f}^{(2)} \oplus N_f^{(2)}$	$\tilde{N} \oplus \bar{\tilde{N}}$
$K+L$	$\bar{N_f}^{(1)} \oplus N_f^{(1)}$	$N_f^{(2)} \oplus \bar{N_f}^{(2)}$	1

Table 9.2: Matter content of the dual SSQCD

description of the SSQCD studied in [117], deformed by two relevant operators.

In the rest of this section we shall show that in the case $N_f^{(1)} > \tilde{N}$ there are ISS like metastable supersymmetry breaking vacua if we are near the IR free border of the conformal window, i.e. $N_f^{(1)} + N_f^{(2)} \sim 3\tilde{N}$.

We shall work in the window between the number of flavor and the number of colors

$$2\tilde{N} < N_f^{(1)} + N_f^{(2)} < 3\tilde{N} \tag{9.3.57}$$

such that the gauge group is weakly coupled and we can rely on the perturbative analysis.

The non supersymmetric vacuum

The non supersymmetric vacuum is located near the origin of the field space where the superpotential (9.3.56) can be studied perturbatively. Appropriate bounds on the parameters ρ and μ allow to neglect the non perturbative dynamics. We will see that these bounds can be consistent with the running of the coupling constants.

Tree level supersymmetry breaking is possible if

$$N_f^{(1)} > \tilde{N} \qquad \Rightarrow \qquad 2\tilde{N} > N_f^{(2)} \tag{9.3.58}$$

where the second inequality follows from (9.3.57). The equation of motion for the field N breaks supersymmetry through the rank condition mechanism. We solve the other equations of motion and we find the non supersymmetric vacuum

$$\begin{aligned} & q = \begin{pmatrix} \mu + \sigma_1 \\ \phi_1 \end{pmatrix} \quad \tilde{q} = (\, \mu + \sigma_2 \;\; \phi_2 \,) \quad N = \begin{pmatrix} \sigma_3 & \phi_3 \\ \phi_4 & X \end{pmatrix} \\ & p = \phi_5 \quad \tilde{p} = \phi_6 \quad L = (\, \phi_7 \;\; \tilde{Y} \,) \quad K = \begin{pmatrix} \phi_8 \\ Y \end{pmatrix} \end{aligned} \tag{9.3.59}$$

where we have also inserted the fluctuations around the minimum, σ_i and ϕ_i. The fields X, Y and $\tilde{Y}$ are pseudo-moduli. The infrared superpotential is

$$\begin{aligned} W_{\text{IR}} = \; & X\phi_1\phi_2 - \mu^2 X + \mu(\phi_1\phi_4 + \phi_2\phi_3) + \mu(\phi_5\phi_8 + \phi_6\phi_7) \\ & + Y\phi_2\phi_5 + \tilde{Y}\phi_1\phi_6 + \rho\phi_5\phi_6 \end{aligned} \tag{9.3.60}$$

In the limit of small ρ, this is the same superpotential studied in [128]. This superpotential corresponds to the one studied in [129] in the R symmetric limit. The fields X, Y and $\tilde{Y}$ are stabilized by one loop corrections at the origin with positive squared masses.

The supersymmetric vacuum

We derive here the low energy effective action for the field N, and we recover the supersymmetric vacuum in the large field region. The supersymmetric vacuum is characterized by a large expectation value for N. This vev gives mass to the quarks q and $\tilde{q}$ and we can integrate them out at zero vev. Also the quarks p and $\tilde{p}$ are massive and are integrated out at low energy. The scale of the low energy theory Λ_L is related to the holomorphic scale $\tilde{\Lambda}$ via the scale matching relation

$$\Lambda_L^{3\tilde{N}} = \tilde{\Lambda}^{3\tilde{N} - N_f^{(1)} - N_f^{(2)}} \det \rho \det N \tag{9.3.61}$$

The resulting low energy theory is $\mathcal{N} = 1$ SYM plus a singlet, with effective superpotential

$$W = -\mu^2 N + \tilde{N}(\tilde{\Lambda}^{3\tilde{N} - N_f^{(1)} - N_f^{(2)}} \det \rho \det N)^{1/\tilde{N}} \tag{9.3.62}$$

where the last term is the gaugino condensate. By solving the equation of motion for N we find the supersymmetric vacuum

$$\langle N\rangle_{susy} = \frac{\mu^{\frac{2\tilde{N}}{N_f^{(1)}-\tilde{N}}}}{\tilde{\Lambda}^{\frac{3\tilde{N}-N_f^{(1)}-N_f^{(2)}}{N_f^{(1)}-\tilde{N}}}\,\rho^{\frac{N_f^{(2)}}{N_f^{(1)}-\tilde{N}}}} \tag{9.3.63}$$

Lifetime

The lifetime of the non supersymmetric vacuum is controlled by the bounce action to the supersymmetric vacuum. In this case, the triangular approximation [126] is valid and the action can be approximated as $S_B \simeq (\Delta\Phi)^4/(\Delta V)$. If we estimate $\Delta\Phi \sim \langle N\rangle_{susy}$ and $\Delta V \sim \mu^4$ we obtain

$$S_B = \left(\frac{\tilde{\Lambda}}{\rho}\right)^{\frac{4N_f^{(2)}}{N_f^{(1)}-\tilde{N}}}\left(\frac{\mu}{\tilde{\Lambda}}\right)^{\frac{12\tilde{N}-4N_f^{(1)}}{N_f^{(1)}-\tilde{N}}} \tag{9.3.64}$$

This expression is not automatically very large since $\mu \ll \tilde{\Lambda}$. However, we can impose the following bound on ρ

$$\langle N\rangle_{\text{susy}} \gg \mu \quad \rightarrow \quad \rho \ll \tilde{\Lambda}\left(\frac{\mu}{\tilde{\Lambda}}\right)^{(3\tilde{N}-N_f^{(1)})/N_f^{(2)}} \tag{9.3.65}$$

If this bound is satisfied, the supersymmetric and the non supersymmetric vacua are far away apart in the field space and the non perturbative terms can be neglected at the supersymmetry breaking scale. This differs from the ISS model in the conformal window. In that case the non-perturbative effects became important at the supersymmetry breaking scale. The bounce action was proportional to the gauge coupling constant at the fixed point and it was impossible to make it parametrically long. The introduction of the new mass scale ρ allows a solution to this problem.

The bound (9.3.65) should be imposed on the IR couplings at the CFT exit scale $E_{IR} = \Lambda_c$. In this case we have a new possible source of CFT breaking, namely the relevant deformation ρ. However we look for a regime of couplings such that the CFT exit scale is set by the supersymmetric vacuum scale, i.e. $\Lambda_c = \langle N\rangle_{\text{susy}} \gg \mu_{IR}, \rho_{IR}$. The scale Λ_c is

$$\Lambda_c = \langle N\rangle_{\text{susy}} = \tilde{\Lambda}_{IR}\left(\frac{\mu_{IR}}{\tilde{\Lambda}_{IR}}\right)^{\frac{2\tilde{N}}{N_f^{(1)}-\tilde{N}}}\left(\frac{\tilde{\Lambda}_{IR}}{\rho_{IR}}\right)^{\frac{N_f^{(2)}}{N_f^{(1)}-\tilde{N}}} \tag{9.3.66}$$

At this scale we define ϵ_{IR} as the ratio between the IR masses ρ_{IR} and μ_{IR} and we demand that

$$\epsilon_{IR} = \frac{\rho_{IR}}{\mu_{IR}} \ll 1 \tag{9.3.67}$$

Rearranging (9.3.66) for μ_{IR} and ρ_{IR} we have

$$\mu_{IR} = \Lambda_c e^{-\frac{8\pi^2}{g_*^2(2\tilde{N}-N_f^{(2)})}} \epsilon_{IR}^{\frac{N_f^{(2)}}{2\tilde{N}-N_f^{(2)}}} \ll \Lambda_c$$
$$\rho_{IR} = \Lambda_c e^{-\frac{8\pi^2}{g_*^2(2\tilde{N}-N_f^{(2)})}} \epsilon_{IR}^{\frac{2\tilde{N}}{2\tilde{N}-N_f^{(2)}}} \ll \Lambda_c \tag{9.3.68}$$

This shows that requiring $\epsilon_{IR} \ll 1$ is consistent with the CFT exit scale to be $\langle N \rangle_{susy}$.

By substituting (9.3.66) and (9.3.68) in (9.3.64), the bounce action becomes

$$S_B = \frac{e^{\frac{32\pi^2}{g_*^2(2\tilde{N}-N_f^{(2)})}}}{\epsilon_{IR}^{\frac{4N_f^{(2)}}{2\tilde{N}-N_f^{(2)}}}} \tag{9.3.69}$$

and in the limit $N_f^{(2)} \to 0$ it reduces to the one computed in the (9.3.55). Here the bounce is not only proportional to a numerical factor depending on g_*^2, but there is also a parameter, relating the ratios of the physical masses ρ_{IR} and μ_{IR} at the CFT exit scale. The bounce action can be large if $\epsilon_{IR} \ll 1$, providing a parametrically large lifetime for the non supersymmetric vacuum.

Using the RG evolution equations the bound $\epsilon_{IR} \ll 1$ translates in constraints on the UV masses ρ_{UV} and μ_{UV} at the UV scale. These masses are relevant perturbations and their ratio must be small along the RG flow.

RG flow in the approximate conformal regime

The relevant coupling constants run from E_{UV} to $E_{IR} = \Lambda_c$. We require that these terms are so small in the UV to be considered as perturbations of the CFT, i.e. $\rho_{UV}, \mu_{UV} \ll \Lambda_{UV}$.

The ratio ϵ_{UV} given at the scale E_{UV} runs as the coupling constants down to Λ_c. We now study the evolution of this ratio. The requirement of long lifetime of the metastable vacuum (9.3.65) corresponds to $\epsilon_{IR} \ll 1$ and it constrains both ϵ_{UV} and the duration of the approximate conformal regime, Λ_c/E_{UV}.

The running of the relevant couplings in the conformal windows is parameterized by the equations

$$\rho_{IR} = \rho_{UV} Z_p(\Lambda_c, E_{UV})^{-1/2} Z_{\tilde{p}}(\Lambda_c, E_{UV})^{-1/2} \tag{9.3.70}$$
$$\mu_{IR} = \mu_{UV} Z_N(\Lambda_c, E_{UV})^{-1/4} \tag{9.3.71}$$

The wave function renormalization Z is obtained by integrating the equation

$$\frac{\mathrm{d}\log Z_i}{\mathrm{d}\log E} = -\gamma_i \tag{9.3.72}$$

from E_{UV} to Λ_c, where γ_i is constant in the conformal regime, and it reads

$$Z_\phi(\Lambda_c, E_{UV}) = \left(\frac{\Lambda_c}{E_{UV}}\right)^{-\gamma_{\phi_i}} \tag{9.3.73}$$

The physical couplings at the CFT exit scale are

$$\rho_{IR} = \rho_{UV}\left(\frac{\Lambda_c}{E_{UV}}\right)^{\gamma_p}, \qquad \mu_{IR} = \mu_{UV}\left(\frac{\Lambda_c}{E_{UV}}\right)^{\gamma_N/4} \tag{9.3.74}$$

where we have used the relation $\gamma_p = \gamma_{\tilde{p}}$.

The ratio ϵ evolves as

$$\epsilon_{IR} = \epsilon_{UV}\left(\frac{\Lambda_c}{E_{UV}}\right)^{\gamma_p - \gamma_N/4} \tag{9.3.75}$$

and we demand that it is $\epsilon_{IR} \ll 1$ in order to satisfy the stability constraint for the non supersymmetric vacuum. The flow from ϵ_{UV} to ϵ_{IR} depends on Λ_c/E_{UV} and on the anomalous dimensions. The precise relation between ϵ_{UV} and ϵ_{IR} is found by calculating the exact value of γ_p and γ_N. The anomalous dimensions of the fields ϕ_i are obtained from the relation $\Delta_i = 1 + \gamma_i/2$ where $\Delta_i = \frac{3}{2}R_i$. The R charges can be computed by using a-maximization.

The a-maximization procedure, defined in [130], shows that in SCFT the correct R-charge at the fixed point is found by maximizing the function

$$a_{trial}(R) = \frac{3}{32}\left(3\mathrm{Tr}R^3 - \mathrm{Tr}R\right) \tag{9.3.76}$$

The R-charges in (9.3.76) are all the non anomalous combinations of the R_0 charges under which the supersymmetry generators have charge -1 and all the other flavor symmetries commuting with the supersymmetry generators. The $\mathrm{Tr}(R^3)$ and $\mathrm{Tr}(R)$ are the coefficients of the gauge anomaly and gravitational anomaly. The R-charges that maximize (9.3.76) are the R charges appearing in the superconformal algebra.

The R charge assignment has to satisfy the anomaly free condition and the constraint that the superpotential couplings should be marginal. These conditions are

$$\tilde{N} + N_f^{(1)}(R[q]-1) + N_f^{(2)}(R[p]-1) = 0, \quad R[p]+R[q]+R[L] = 2, \quad R[N]+2R[q] = 2 \tag{9.3.77}$$

where the symmetry enforces $R[q] = R[\tilde{q}]$, $R[p] = R[\tilde{p}]$ and $R[K] = R[L]$. The a_{trial} function that has to be maximized is

$$\begin{aligned} a_{trial} = & \frac{3}{32}\Big(2N_f^{(1)}\tilde{N}\left(3(R[q]-1)^3 - R[q]+1\right) + 2N_f^{(2)}\tilde{N}\left(3(R[p]-1)^3 - R[q]+1\right) \\ & + 2N_f^{(1)}N_f^{(2)}\left(3(R[L]-1)^3 - R[L]+1\right) + N_f^{(1)\,2}\left(3(R[N]-1)^3 - R[N]+1\right) + 2\tilde{N}^2\Big) \end{aligned} \tag{9.3.78}$$

By defining $R[N] = 2y$ we have $R[q] = 1-y$. The other R charges are

$$R[p] = \frac{1}{n}(n-x+y), \quad R[L] = y + \frac{x-y}{n} \tag{9.3.79}$$

where $n = \frac{N_f^{(2)}}{N_f^{(1)}}$ and $x = \frac{\tilde{N}}{N_f^{(1)}}$. We can simplify the a maximization in terms of the only variable y, obtaining

$$y_{\max} = \frac{-3(n+n^3)+3(-1+n)^2x-3x^2+\sqrt{n^2(n^4-8n(x-1)+8n^3(x-1)+9(x-1)^4-6n^2(1+3(x-2)x))}}{3\left(1-n\left(3+n+n^2\right)+\left(-1+n^2\right)x\right)} \tag{9.3.80}$$

Once we know the anomalous dimensions and once we fix the duration of the approximate

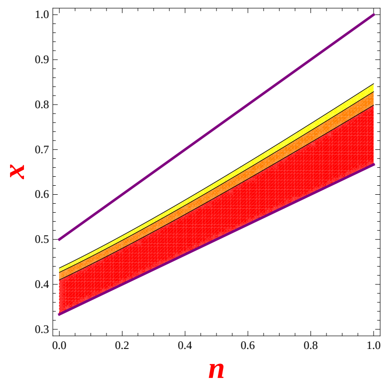

Figure 9.1: $\frac{\rho_{UV}}{\mu_{UV}} = 10^{-2}, \frac{\Lambda_c}{E_{UV}} = 10^{-4}$

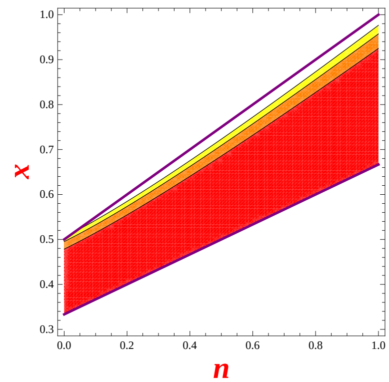

Figure 9.2: $\frac{\rho_{UV}}{\mu_{UV}} = 10^{-4}, \frac{\Lambda_c}{E_{UV}} = 10^{-4}$

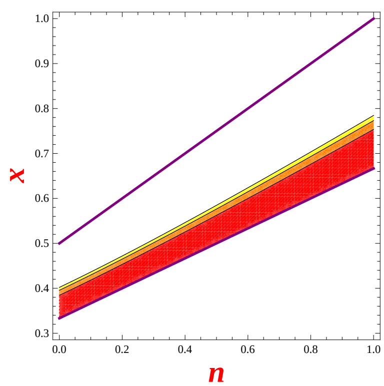

Figure 9.3: $\frac{\rho_{UV}}{\mu_{UV}} = 10^{-2}, \frac{\Lambda_c}{E_{UV}} = 10^{-6}$

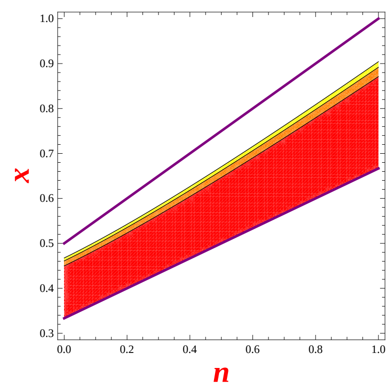

Figure 9.4: $\frac{\rho_{UV}}{\mu_{UV}} = 10^{-4}, \frac{\Lambda_c}{E_{UV}} = 10^{-6}$

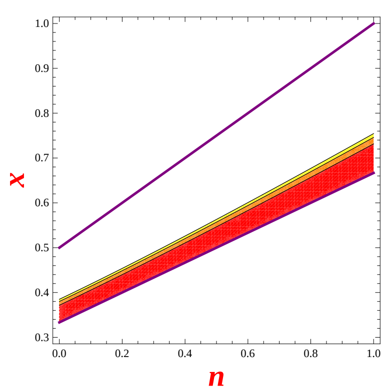

Figure 9.5: $\frac{\rho_{UV}}{\mu_{UV}} = 10^{-2}, \frac{\Lambda_c}{E_{UV}} = 10^{-8}$

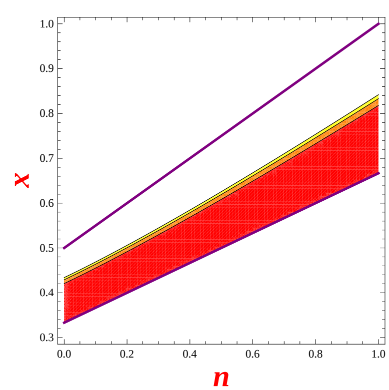

Figure 9.6: $\frac{\rho_{UV}}{\mu_{UV}} = 10^{-4}, \frac{\Lambda_c}{E_{UV}} = 10^{-8}$

conformal regime we can see what is the bound to impose on the UV ratio $\epsilon_{UV} = \rho_{UV}/\mu_{UV}$ such that

$$\epsilon_{IR} = \epsilon_{UV} \left(\frac{\Lambda_c}{E_{UV}} \right)^{\frac{3}{2n}(n-2x+2y-yn)} \ll 1 \tag{9.3.81}$$

As a result of the a-maximization procedure, we find that $\gamma_N > 0$ and $\gamma_p < 0$ in the range (9.3.57). The former is a consequence of the unitarity of the theory, while the latter cannot be explained *a priori*. Thus, from (9.3.74) we note that the coupling μ is suppressed during the RG flow, while ρ is enhanced, and (9.3.67) poses a nontrivial constraint on the coupling constants of the theory at the CFT exit scale.

In the Figures 9.1-9.6 we have plotted some region of the ranks x and n by fixing ϵ_{UV} and Λ_c/E_{UV}. The colored part of the figures represent the allowed region, where all the constraints are satisfied. We also plotted two lines delimiting the weakly coupled regime of the conformal window ($2\tilde{N} > (N_f^{(1)} + N_f^{(2)})$) and the IR free window ($3\tilde{N} < (N_f^{(1)} + N_f^{(2)})$).

From the figures we see that smaller values of the ratio ϵ_{UV} guarantees that the running can be longer in the CFT window. The red region shaded in the figures, near $N_f^{(1)} + N_f^{(2)} = 3\tilde{N}$, is filled also if the running is extended over a large regime of scales. At the lower edge of this region the anomalous dimensions are close to zero, the UV hierarchy imposed on the relevant deformations is preserved during the flow, and $\epsilon_{UV} \sim \epsilon_{IR}$. As we approach the strongly coupled region of the conformal window the anomalous dimensions get larger. In this case ϵ_{IR} approaches to one and we represented this behavior by changing the color of the shaded region from red to orange and then to yellow. The white part of the figures represents the region in which $\epsilon_{IR} > 1$.

In conclusion we have found regions in the parameter space where the theory possesses metastable non supersymmetric vacua. The RG flow analysis gives non trivial constraints on the relevant deformations and on the duration of the approximate conformal regime.

9.3.3 General strategy

We discuss here the generalization of the mechanism of supersymmetry breaking in SCFTs deformed by relevant operators. As in SSQCD, the lifetime of the metastable vacuum can be long in the conformal window of other models, with opportune choices of the parameters. Consider a $SU(N_c)$ gauge theory with $N_f^{(1)}$ flavors of quarks in the magnetic IR free window and with a metastable supersymmetry breaking vacuum in the dual phase. In the magnetic phase a new set of $N_f^{(2)}$ massive quarks must be added to reach the conformal window. If there is some gauge invariant operator $\mathcal{O}$ that hits the unitary bounds, $R(\mathcal{O}) < 2/3$, it is necessary to add other singlets and also marginal couplings in the superpotential between the quarks and these new singlets. The mass term for the new quarks is a relevant perturbation which grows in the infrared, and it has to be very small with respect to the other scales of the theory, down to the CFT exit scale. This mass term modifies the non perturbative superpotential and the supersymmetric vacuum, which sets the CFT exit scale. One must inspect a regime of couplings such that the supersymmetric vacuum is far away in the field space. This regime corresponds to a bound on the parameters of the theory, which have to be consistent with the RG running of the physical coupling constants. In the canonical basis the running of the physical couplings can be absorbed in the superpotential by the wave function renormalization of the fields. If there is a relevant operator $\Delta W = \eta \mathcal{O}$, with classical dimension $\dim(\mathcal{O}) = d$, the physical coupling η runs from the UV scale E_{UV} to the IR scale E_{IR} as

$$\eta(E_{IR}) = \eta(E_{UV}) Z_{\mathcal{O}}(E_{IR}, E_{UV})^{-\frac{1}{2}} = \eta(E_{UV}) \left(\frac{E_{IR}}{E_{UV}}\right)^{\gamma/2} \tag{9.3.82}$$

We require that the running in this approximate conformal regime stops at the energy scale Λ_c set by the masses at the supersymmetric vacuum. The bounds on the parameters that ensure the stability of the metastable vacuum have to hold at this IR CFT exit scale. The equation (9.3.82) translates these bounds in some requirements on the UV deformations. The metastable vacua have long lifetime if there is some regime of UV couplings in which the stability requirements are satisfied in the weakly coupled conformal window.

Here we have shown that in SSQCD there are some regions in the conformal window in which a large hierarchy among the couplings allows the existence of long living metastable vacua. We expect other models with this behavior.

9.3.4 Discussion

We discussed the realization of the ISS mechanism in the conformal window of SQCD-like theory. In [120] the metastable vacua disappeared if $3/2N_c < N_f < 3N_c$ because the non perturbative dynamics was not negligible in the small field region, and this destabilized the non supersymmetric vacua.

We have reformulated this problem in terms of the RG flow from the UV cut-off of the theory down to the CFT exit scale. In the ISS model the CFT exit scale and the supersymmetry breaking scale are proportional because of the equation of motion of the meson. Their ratio depends only on the gauge coupling constant at the fixed point. The bounce action is proportional to this ratio and cannot be parametrically long.

This behavior suggests a mechanism to evade the problem and to build models with long living metastable vacua in the conformal window of SQCD-like theories. A richer structure of relevant deformations than in the ISS model is necessary. Metastable vacua with a long lifetime can exist if the bounce action at the CFT exit scale depends on the relevant deformations and it is not RG invariant. We have studied this mechanism in an explicit model, the SSQCD, and we have found that in this case, by adding a new mass term for some of the quarks, the bounce action has a parametrical dependence on the relevant couplings. The RG flow of these couplings for different regimes of scales sets the desired regions of UV parameter that gives a large bounce action in the IR. We restricted the analysis to a region of ranks in which the model is interacting but weakly coupled, and the perturbative analysis at the non supersymmetric state is applicable. It is possible to extend this example to other SCFT theories as we explained in Section 9.3.3.

It would be interesting to find some dynamical mechanism to explain the hierarchy among the different relevant perturbations, that are necessary for the stability of the metastable vacua. For example in the appendix we see that in quiver gauge theories the mass of the new quarks can be generated with a stringy instanton as in [131, 132]. The supersymmetry breaking metastable vacua that we have found in the conformal window might be used in conformally sequestered scenarios, along the lines of [133]. Another application is the study of Yukawa interactions along the lines of [134, 135]. Superconformal field theories naturally explain the suppression of the Yukawa couplings if some of the gauge singlet fields are identified with the $T_i = 10_i$ and $F_i = \bar{5}_i$ generations of the $SU(5)$ GUT group. Here we have shown that supersymmetry breaking in superconformal sectors is viable. It is in principle possible to build a supersymmetry breaking SCFT where some of the generation of the MSSM are gauge singlets, marginally interacting with the fundamentals of the SCFT group. In this case the Yukawa arising from these generations can be suppressed as in [134, 135]. Since supersymmetry is broken one can imagine a mechanism of flavor blind mediation, like gauge mediation, to generate the soft masses for the rest of the multiplets of the MSSM. Closely related ideas has recently appeared in [127] and [136].

9.4 Supersymmetry breaking in three dimensions

In three dimensions the mechanisms of supersymmetry breaking have been still rather unexplored. In a recent paper [68] the authors have shown that a mechanism analog to the ISS takes place in three dimensional massive SQCD with CS or YM gauge theories. The low energy dynamics is controlled by a Wess-Zumino (WZ) model. In four dimensions WZ models have been useful laboratories for supersymmetry breaking, playing a crucial role in the ISS mechanism.

In this section we analyze supersymmetry breaking in three dimensional WZ models [69]. The WZ models studied in [68] had relevant couplings and the quantum corrections could be computed only after the addiction of an explicit R-symmetry breaking deformation. On the contrary, a different solution to the problem of the computation of quantum corrections in three-dimensional WZ models is given by preserving an $SO(2)_R \simeq U(1)_R$ R-symmetry and by adding only marginal deformations to the superpotential. The non supersymmetric vacua turn out to be only metastable, since the marginal couplings induce a runaway behavior in the scalar potential. A property of these models is that R-symmetry is spontaneously broken in the non supersymmetric vacua. As a general result it seems that in three dimensions R-symmetry needs to be broken (explicitly or spontaneously) for the validity of the perturbative expansion.

We first review the model of [68] and the problems of the perturbative approach. Then we present a model with marginal couplings and long lifetime metastable vacua, and we study the general behavior of WZ models with marginal couplings. The regime of validity of the perturbative approximation in models with relevant coupling is also discussed.

9.4.1 Effective potential in $3D$ WZ models

While a systematic study of supersymmetry breaking mechanisms in $3+1$ dimensions has been done, in $2+1$ dimension such an analysis still lacks. A recent step towards the comprehension of supersymmetry breaking in $2+1$ dimensions has been done in [68]. In this section we briefly review their model and results.
The theory is a WZ model with canonical Kähler potential

$$K = \mathrm{Tr}\left(M^\dagger M + q_i^\dagger q^i + \tilde{q}_i^\dagger \tilde{q}^i\right) \tag{9.4.83}$$

	q	$\tilde{q}$	M
$U(N)$	N	$\overline{N}$	1
$U(N_F)$	$\overline{N}_F$	N_F	N_F^2

Table 9.3: Representation of the fields in the model of [68]

and superpotential

$$W = hqM\tilde{q} + h\mathrm{Tr}\left(\frac{1}{2}\epsilon\mu M^2 - \mu^2 M\right) \tag{9.4.84}$$

with an $U(N) \times U(N_F)$ global symmetry. The representations of the matrix valued chiral superfields q, $\tilde{q}$ and M are given in Table 9.3. All the three dimensional couplings and fields in (9.4.84) have mass dimension 1/2, except ϵ which is dimensionless. The model (9.4.84) has supersymmetric vacua labeled by $k = 0, \ldots, N$. At given k the expectation values of the chiral fields in the supersymmetric vacuum is

$$M = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\mu}{\epsilon}\mathbf{1}_{N_F-k} \end{pmatrix} \quad q\tilde{q} = \begin{pmatrix} \mu^2\mathbf{1}_k & 0 \\ 0 & 0 \end{pmatrix} \tag{9.4.85}$$

Moreover this model also has metastable vacua, in which the combination of the tree level and one loop scalar potential stabilizes the fields. In the analysis of [68] the authors studied the case of different values of k. Here we only refer to the simplified case $k = N$. The vacuum is

$$M = \begin{pmatrix} 0 & 0 \\ 0 & X\mathbf{1}_{N_F-N} \end{pmatrix} \quad q\tilde{q} = \begin{pmatrix} \mu^2\mathbf{1}_N & 0 \\ 0 & 0 \end{pmatrix} \tag{9.4.86}$$

where the field X is a pseudo-modulus, stabilized, in this case, by the one loop effective potential. This potential is given by the Coleman-Weinberg formula, that in three dimensions is [68]

$$V_{eff}^{(1)} = -\frac{1}{12\pi^2}\mathrm{STr}|\mathcal{M}|^3 \equiv -\frac{1}{12\pi^2}\mathrm{Tr}\left(|\mathcal{M}_B|^3 - |\mathcal{M}_F|^3\right) \tag{9.4.87}$$

The cubic dependence on the bosonic and fermionic mass matrices $\mathcal{M}_B$ and $\mathcal{M}_F$ can be eliminated by expressing (9.4.87) as

$$V_{eff}^{(1)} = -\frac{1}{6\pi^2}\mathrm{STr}\int_0^\infty \frac{v^4}{v^2 + \mathcal{M}^2}\mathrm{d}v \tag{9.4.88}$$

In appendix C.4 we observe that (9.4.88) can be generalized to every dimension.
The superpotential that is necessary to calculate the one loop corrections for the WZ model (9.4.84) simplifies by expanding the fields around (9.4.86). The fluctuations of the fields can be organized in two sectors, respectively called ϕ_i and σ_i. The former represents the fluctuation necessary for the one loop corrections of the field X, while the latter parameterizes the supersymmetric fields that do not contribute to the one loop effective potential. We have

$$q = \begin{pmatrix} \mu + \sigma_1 \\ \phi_1 \end{pmatrix} \quad \tilde{q}^T = \begin{pmatrix} \mu + \sigma_2 \\ \phi_2 \end{pmatrix} \quad M = \begin{pmatrix} \sigma_3 & \phi_3 \\ \phi_4 & X \end{pmatrix} \tag{9.4.89}$$

The one loop CW is calculated by inserting (9.4.89) in the superpotential (9.4.84). There are $N_F(N_F - N)$ copies of WZ models with superpotential

$$W = \frac{1}{2}h\mu\epsilon X^2 - h\mu^2 X + h\mu\epsilon\phi_3\phi_4 + hX\phi_1\phi_2 + h\mu(\phi_1\phi_3 + \phi_2\phi_4) \tag{9.4.90}$$

The tree level potential and the one loop corrections calculated from (9.4.90) give raise to a non supersymmetric vacuum at

$$\phi_i = 0 \qquad X \simeq \frac{\epsilon\mu}{b} \tag{9.4.91}$$

where $b = \frac{(3-2\sqrt{2})h}{4\pi\mu}$ is a dimensionless parameter. Thereafter we use a notation that makes clear the relevancy of the cubic three-dimensional couplings, by rewriting (9.4.90) as

$$W = \frac{1}{2}\epsilon\, m\, X^2 - fX + \epsilon\, m\, \phi_3\phi_4 + \frac{m^2}{f}X\phi_1\phi_2 + m(\phi_1\phi_3 + \phi_2\phi_4) \tag{9.4.92}$$

where we have defined

$$f \equiv h\mu^2 \qquad\qquad m \equiv h\mu \tag{9.4.93}$$

The new parameters of the theory,, f and m,respectively have mass dimension 3/2 and 1. The X field vacuum expectation value is proportional to $\epsilon f/(bm)$ and the expansion of the one-loop potential near the origin is possible if the R-symmetry breaking parameter ϵ satisfies

$$\epsilon \ll b \tag{9.4.94}$$

which expresses the condition $X \ll \mu$ of [68].

The perturbative expansion is valid if higher orders in the loop expansion are negligible. This last condition is satisfied when the relevant coupling is small at the mass scale of the chiral fields

$$h^2 \ll hX \iff \frac{m^4}{f^2} \ll \frac{m^2}{f}X \tag{9.4.95}$$

This requirement imposes a lower bound on the R-breaking parameter ϵ

$$b \gg \epsilon \gg b\frac{m^3}{f^2} \tag{9.4.96}$$

and by using the definition of $b = \frac{3-2\sqrt{2}}{4\pi}\frac{m^3}{f^2}$

$$\frac{3-2\sqrt{2}}{4\pi}\frac{m^3}{f^2} \gg \epsilon \gg \frac{3-2\sqrt{2}}{4\pi}\frac{m^6}{f^4} \tag{9.4.97}$$

The parameter ϵ cannot approach zero. In fact, in this case the theory becomes strongly coupled and the effective potential cannot be evaluated perturbatively.

9.4.2 Three dimensional WZ models with marginal couplings

Relevant couplings do not complete the renormalizable interactions of a three dimensional superpotential. In fact quartic marginal terms can be also added to a WZ model. Here we study supersymmetry breaking in a renormalizable WZ model with quartic marginal couplings and no trilinear interactions. We show that supersymmetry is broken at tree level and the perturbative approximation is valid without any explicit R-symmetry breaking. The three dimensional $\mathcal{N} = 2$ superpotential is

$$W = -fX + hX^2\phi_1^2 + \mu\phi_1\phi_2 \tag{9.4.98}$$

and the classical scalar potential is

$$V_{\text{tree}} = |2hX\phi_1^2 - f|^2 + |2hX^2\phi_1 + \mu\phi_2|^2 + |\mu\phi_1|^2 \tag{9.4.99}$$

The chiral superfields have R-charges

$$R(X) = 2, \quad R(\phi_1) = -1, \quad R(\phi_2) = 3 \tag{9.4.100}$$

The F-terms of the fields X, ϕ_1 and ϕ_2 cannot be solved simultaneously and supersymmetry is broken at tree level. We study the theory around the classical vacuum $\langle\phi_1\rangle = \langle\phi_2\rangle = 0$ and arbitrary $\langle X\rangle$. Stability of supersymmetry breaking requires the computation of the one loop effective potential for the X field. The squared masses of the scalar components of the fields ϕ_1 and ϕ_2 read

$$m_{1,2}^2 = \mu^2 + 2h\langle|X|\rangle\left(h\langle|X|\rangle^3 + \eta f + \sigma\sqrt{f^2 + 2\eta f h\langle|X|\rangle^3 + h^2\langle|X|\rangle^6 + \langle|X|\rangle^2\mu^2}\right) \tag{9.4.101}$$

where $\langle|X|\rangle$ is the vacuum expectation value of the field X and η and σ are ± 1. These masses are positive for

$$\langle|X|\rangle < \frac{\mu^2}{4fh} \tag{9.4.102}$$

In this regime the pseudo-moduli space is tachyon free and classically stable. Outside this region there is a runaway in the scalar potential. The squared masses of the fermions in the superfields ϕ_1 and ϕ_2 are

$$m_{1,2}^2 = \mu^2 + 2h\langle|X|\rangle\left(h\langle|X|\rangle^3 + \sigma\sqrt{h^2\langle|X|\rangle^6 + \langle|X|\rangle^2\mu^2}\right) \tag{9.4.103}$$

The two real combinations of the fermions of X and $X^\dagger$ are the two goldstinos of the $\mathcal{N} = 2 \to \mathcal{N} = 0$ supersymmetry breaking.

The one loop effective potential is computed with the CW formula (9.4.88). At small $\frac{X}{\sqrt{\mu}}$ the field X has a negative squared mass

$$m_{X=0}^2 \sim -\frac{f^2h^2}{\mu} \tag{9.4.104}$$

and the origin is unstable.

A (meta)stable vacuum is found if there is a minimum such that (9.4.102) is satisfied. As long as the dimensionless parameter $\frac{f^2}{\mu^3}$ is small the effective potential has a minimum at

$$\langle X \rangle \simeq \sqrt{\frac{\sqrt{2}\mu}{h}} \tag{9.4.105}$$

This imposes a bound on the coupling constant

$$h < \frac{\mu^3}{16\sqrt{2}f^2} \tag{9.4.106}$$

since the scalar potential has to be tachyon free.

When (9.4.102) or (9.4.106) are saturated the classical scalar potential (9.4.99) has a runaway behavior. Indeed, if we parameterize the fields by their R-charges (9.4.100), we have

$$X = \frac{f}{2h\mu}e^{2\alpha}, \qquad \phi_1 = \sqrt{\mu}e^{-\alpha}, \qquad \phi_2 = -\frac{f^2}{2h\sqrt{\mu^5}}e^{3\alpha} \tag{9.4.107}$$

and we get $F_X = F_{\phi_1} = 0$ and $F_{\phi_2} \to 0$ as $\alpha \to \infty$.

Lifetime

The decay rate of the non-supersymmetric state is proportional to the semi-classical decay probability. This probability is proportional to e^{-S_B}, where S_B is the bounce action. Here the lifetime of the metastable vacuum is estimated from the bounce action of a triangular potential barrier, since the two vacua are well separated in field space and the maximum is approximately in the middle of them.

The computation is similar to [126], but in this case we have to deal with a three dimensional theory. In the appendix C.4 we compute the bounce action for a triangular potential barrier in three dimensions. We found that it is

$$S_B \sim \sqrt{\frac{(\Delta\Phi)^6}{\Delta V}} \tag{9.4.108}$$

In our case we estimate the behavior of the potential barrier by using the evolution of the scalar potential along the field X. The non supersymmetric minimum has been found in (9.4.105) and the potential is $V_{min} = |f|^2$.
The one loop scalar potential plotted in Figure 9.7 is always increasing between the metastable minimum and (9.4.102). When (9.4.102) is saturated there is a classical runaway direction, and the local maximum of the potential can be estimated to be at $\langle X \rangle = \frac{\mu^2}{4hf}$, where the potential is $V_{max} \sim 2|f|^2$. After this maximum the potential starts to decrease and the field X acquires large values. There is not a local minimum, nevertheless the lifetime of the non supersymmetric state can be estimated as in [137, 128]. Indeed, by using the parametrization (9.4.107) of the fields along the runaway the scalar potential has the same value as V_{min} for $\langle X_{\mathrm{R}} \rangle \sim \frac{\mu^2}{2hf}$.

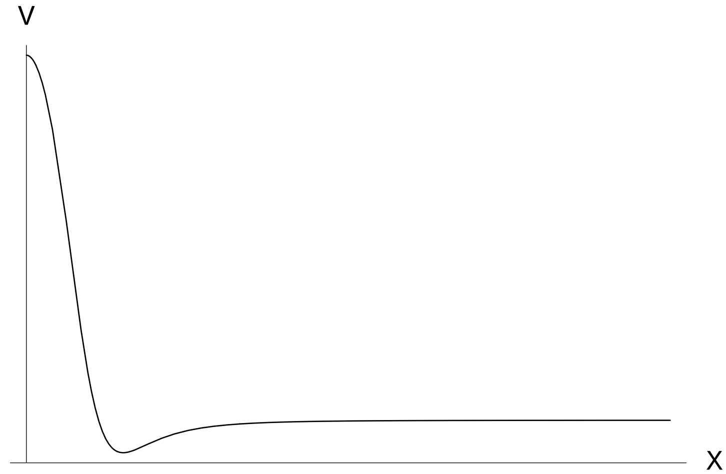

Figure 9.7: One loop scalar potential for the one-loop validity region $X < \frac{\mu^2}{4hf}$. Over this value, we find a classical runaway. At the origin the pseudo-modulus has negative squared mass. The potential is plotted for $\mu = 1,\ f = h = 0.1$

In the regime $\frac{f^2}{\mu^3} \ll h \ll 1$ the barrier is approximated to be triangular, and the gradient of the potential is constant. The non supersymmetric state is near the origin of the moduli space, and the bounce action is

$$S_B \sim \sqrt{\frac{\langle X_{\rm R}\rangle^6}{V_{\rm min}}} \sim \sqrt{\frac{1}{h}\left(\frac{\mu^3}{f^2}\right)^4} \gg \left(\frac{\mu^3}{f^2}\right)^2 \gg 1 \tag{9.4.109}$$

9.4.3 The general case

In this section we consider the class of models with a single pseudo-modulus X which marginally couples to n chiral superfields ϕ_i. As shown in [137] for the class of four-dimensional renormalizable and R-symmetric models, many general features are worked out by R-symmetry considerations. For the three-dimensional case, we find some interesting features concerning such models. The perturbative expansion is reliable under the weak condition that the coupling constants are small numbers, i.e. one can use the one-loop approximation and made higher loop corrections suppressed. The origin of the moduli space is a local maximum of the one-loop potential, and the pseudo-modulus acquires a negative squared mass. Finally the scalar tree-level potential exhibits runaway directions for every choice of the couplings.

To deal with renormalizable three-dimensional WZ models, we consider only canonical Kähler potential, and superpotential of the type

$$W = -fX + \frac{1}{2}(M^{ij} + X^2\, N^{ij})\, \phi_i\, \phi_j \tag{9.4.110}$$

in which R-symmetry imposes the conditions

$$M^{ij} \neq 0 \Rightarrow R(\phi_i) + R(\phi_j) = 2 \qquad\qquad N^{ij} \neq 0 \Rightarrow R(\phi_i) + R(\phi_j) = -2 \tag{9.4.111}$$

The conditions (9.4.111) could not be sufficient to uniquely fix the R-charges. However it is clear that a basis exists in which there are both charges greater than two and charges lower than two.

In a basis where the fields with the same R-charge are grouped together, the M matrix is written in the form

$$M = \begin{pmatrix} & & & & M_1 \\ & & & M_2 & \\ & & \cdot & & \\ & M_2^T & \cdot & & \\ M_1^T & & & & \end{pmatrix} \tag{9.4.112}$$

and similar for the N matrix. The scalar potential of this model can be written as

$$V_S = \left|-f + XN^{ij}\phi_i\phi_j\right|^2 + \left|M^{ij}\phi_j + X^2N^{ij}\phi_j\right|^2 \tag{9.4.113}$$

which we assume to have a one-dimensional space of extrema given by

$$\phi_i = 0 \qquad\qquad X \text{ arbitrary} \qquad\qquad V_S = |f|^2 \tag{9.4.114}$$

For general couplings, there can be other extrema and in particular some lower local minima away from of the origin of ϕ's. Furthermore for some choices of the coupling constants at least some of (9.4.114) are saddle points. Here we work under the hypothesis that this is not the case.

We show now that the effective potential always has a local maximum at the origin of the pseudo-moduli space:

$$V_{eff}(X) = V_0 + m_X^2|X|^2 + \mathcal{O}(X^3) \tag{9.4.115}$$

and the X field acquires a negative squared mass. We derive a general formula for m_X^2 in the one-loop approximation by using the Coleman-Weinberg formula

$$\begin{aligned} V_{eff}^{(1)} &= -\frac{1}{12\pi}\mathrm{STr}|\mathcal{M}|^3 \\ &= -\frac{1}{6\pi^2}\mathrm{Tr}\int_0^\Lambda dv\, v^4\left(\frac{1}{v^2+\mathcal{M}_B^2} - \frac{1}{v^2+\mathcal{M}_F^2}\right) \end{aligned} \tag{9.4.116}$$

where $\mathcal{M}_B^2$ and $\mathcal{M}_F^2$ are, respectively, the squared mass matrices of the bosonic and fermionic components of the superfields of the theory

$$\begin{aligned} \mathcal{M}_B^2 &= (\hat{M} + X^2\hat{N})^2 - 2fX\hat{N} \\ \mathcal{M}_F^2 &= (\hat{M} + X^2\hat{N})^2 \end{aligned} \tag{9.4.117}$$

where we have defined

$$\hat{M} \equiv \begin{pmatrix} 0 & M^\dagger \\ M & 0 \end{pmatrix} \qquad\qquad \hat{N} \equiv \begin{pmatrix} 0 & N^\dagger \\ N & 0 \end{pmatrix} \tag{9.4.118}$$

which take a form analogous to (9.4.112).

Substituting the mass formulas into the Coleman-Weinberg potential (9.4.116) and expanding up to second order in the field X we find

$$\begin{aligned} V_{eff}^{(1)} &= -\frac{1}{6\pi^2}\mathrm{Tr}\int_0^\Lambda dv\, v^4\, \frac{1}{v^2+\hat{M}^2}\left(\frac{1}{1+\frac{X^2\{\hat{M},\hat{N}\}-2fX\hat{N}}{v^2+\hat{M}^2}}-\frac{1}{1+\frac{X^2\{\hat{M},\hat{N}\}}{v^2+\hat{M}^2}}\right)+\ldots \\ &= -\frac{2f^2X^2}{3\pi^2}\mathrm{Tr}\int_0^\Lambda dv\, v^4\, \frac{1}{v^2+\hat{M}^2}\,\frac{1}{v^2+\hat{M}^2}\,\hat{N}\,\frac{1}{v^2+\hat{M}^2}\,\hat{N}+\mathcal{O}(X^3) \\ &= -\frac{f^2X^2}{2\pi^2}\mathrm{Tr}\int_0^\Lambda dv\, v^2\, \frac{1}{v^2+\hat{M}^2}\,\hat{N}\,\frac{1}{v^2+\hat{M}^2}\,\hat{N}+\mathcal{O}(X^3) \end{aligned} \tag{9.4.119}$$

where the last step follows after an integration by parts. From the previous formula we note that the origin of moduli space is always a local maximum, i.e. the pseudo-modulus always acquires a negative squared mass at one-loop level. The vacuum cannot be at the origin, and we have to find it at $X \neq 0$, where R-symmetry is spontaneously broken. The existence of this vacuum is not guaranteed by (9.4.119), but it depends on the couplings in (9.4.118)

We show now that the models in (9.4.110) have a runaway direction. In four-dimensional theories if the R-charges of the superfields are both greater than two and lower than two then the potential exhibits runaway [138]. In three-dimensional renormalizable theories (9.4.110), conditions (9.4.111) state there are always both superfields with R-charge greater than two and superfields with R-charge lower than two. We parametrize the fields by their R-charge $R(\phi_i) \equiv R_i$ as

$$\begin{aligned} \phi_i &= c_i\, e^{R_i\alpha} \\ X &= c_X\, e^{2\alpha} \end{aligned} \tag{9.4.120}$$

The runaway behavior of the potential is analyzed by looking at the R-charges of F-terms. The F-terms with R charges lower or equal to zero can be solved. All the non vanishing F terms have charge greater than zero. They vanish only if $\alpha \to \infty$. By (9.4.111), the F-terms are

$$F_X = -f + XN^{ij}\phi_i\phi_j = -f + N^{ij}c_Xc_ic_j \tag{9.4.121}$$

$$F_{\phi_i} = M^{ij}\phi_j + X^2N^{ik}\phi_k = \left(\sum_j M^{ij}c_j + \sum_k N^{ik}c_X^2c_k\right)\, e^{(2-R_i)\alpha} \tag{9.4.122}$$

We distinguish three possibilities. The first one is that the field ϕ_i couples both to the matrices M and N. In this case, the equation (9.4.122) fixes the relative coefficients of the fields. The second possibility is that the field ϕ_i does not couple to the matrix N but it has more than an entry in the matrix M. In this case we fix the relative coefficients. The last possibility is that ϕ_i only couples once to M. In this case we set $c_j = 0$.

The discussion above does not solve all the $F_{\phi_i} = 0$ in (9.4.122). Some of the F-terms with R-charge greater than zero have not been set to zero yet. They vanish when $\alpha \to \infty$, which implies a runaway behavior in the directions parameterized by some fields in (9.4.120).

9.4.4 Relevant couplings

In three dimensions there exist WZ models with relevant deformations that can be perturbatively studied without the addition of explicit R-symmetry breaking deformations. Even if R-symmetry is not explicitly broken in three dimensions, quantum non supersymmetric vacua can appear out of the origin of the moduli space. The vevs at which the vacua are found set the spontaneous R-symmetry breaking scale which plays the same role as the ϵ deformation in [68].

The models which exhibit explicit or spontaneous R-symmetry breaking can be perturbatively studied in three dimensions. The former case has been analyzed in [68]. We treat here the latter case. Assuming that the R-symmetry breaking vacuum is near the origin, we require that the pseudo-modulus acquires a negative squared mass at the origin of moduli space, i.e.there is a vacuum of the quantum theory which spontaneously breaks the R-symmetry. This happens if not all the R-charges take the values $R = 0$ and $R = 2$. This result was shown in four dimensions in [137] and can be analogously demonstrated in three dimensions. We classify these models in two subclasses.

In the first class we identify the models without runaway behavior, i.e. all the charges are lower or equal to two. There can be a regime of couplings in which supersymmetry is broken in non R-symmetric vacua. The vev of the field that breaks R-symmetry introduce a scale which bounds the perturbative window for the relevant couplings.

In the second class, that we consider in the following, there are runaway models. They have R-charges lower and higher than two. Under the assumption of a hierarchy on the mass scales, we distinguish two possibilities. Some of these models flow in the infrared to models with only marginal couplings, that have been treated in sections 9.4.2 and 9.4.3. The other possibility is that the effective descriptions of these theories share marginal and relevant terms in the superpotential. In both cases a perturbative regime is allowed.

A model with relevant couplings

Consider the superpotential

$$\begin{aligned} W &= \lambda X \phi_1 \phi_2 - f X + \mu \phi_1 \phi_3 + \mu \phi_2 \phi_4 \\ &\quad + m \phi_1 \phi_5 + m \phi_3 \phi_5 \end{aligned} \tag{9.4.123}$$

The first line corresponds to the ISS low energy superpotential, while the second line of (9.4.123) asymmetrizes the behavior of ϕ_1 and ϕ_2. If the mass term m is higher than the other scales of the theory ($m^2 \gg \mu^2$ and $m^2 \gg f^{4/3}$) we can integrate out the second line of (9.4.123), and obtain

$$W = h X^2 \phi_2^2 - f X + \mu \phi_2 \phi_4 \tag{9.4.124}$$

where $h = \frac{\lambda}{4\mu}$ is a marginal coupling. The perturbative analysis of this model is now possible, with the only requirement that $h \ll 1$.

The superpotential (9.4.124) is identical to (9.4.98). This example shows that in three dimensions models with cubic couplings in the UV can flow to theories with quartic terms in the IR, which are perturbatively accessible.

Models with relevant and marginal couplings

A model with relevant couplings can flow to a model with both marginal and relevant couplings. For example if we take the superpotential

$$W = -fX + \lambda_5 X\phi_1\phi_5 - \frac{m}{2}\phi_5^2 + \mu\phi_1\phi_2 + \lambda X\phi_3^2 + \mu\phi_3\phi_4 \tag{9.4.125}$$

and we study this model in the regime $m^2 \gg \mu^2$, $m^2 \gg f^{4/3}$, we can integrate the field ϕ_5 out. The effective theory becomes

$$W = -fX + hX^2\phi_1^2 + \mu\phi_1\phi_2 + \lambda X\phi_3^2 + \mu\phi_3\phi_4 \tag{9.4.126}$$

where $h = \frac{\lambda_5^2}{m}$. This model preserves R-symmetry and the charges of the fields are

$$R(X) = 2 \quad R(\phi_1) = -1 \quad R(\phi_2) = 3 \quad R(\phi_3) = 0 \quad R(\phi_4) = 2 \tag{9.4.127}$$

As before this theory has a runaway behavior in the large field region, and the fields are parametrized as

$$X = \frac{f}{2h\mu}e^{2\alpha}, \qquad \phi_2 = \sqrt{\mu}e^{-\alpha}, \qquad \phi_4 = -\frac{f^2}{2h\sqrt{\mu^5}}e^{3\alpha} \qquad \phi_3 = 0 \quad \phi_4 = 0 \tag{9.4.128}$$

Near the origin the classical equations of motion break supersymmetry at tree level at $\phi_i = 0$. The field X is a classical pseudo-modulus whose stability has to be studied perturbatively. The pseudo-moduli space is stable if $|\langle X\rangle| < \frac{\mu^2}{4fh}$ and $\frac{\lambda}{\sqrt{\mu}} < \frac{\mu^{3/2}}{2f}$.
We study the effective potential by expanding it in the dimensionless parameter $\frac{f^2}{\mu^3}$, finding

$$V_{eff}^{(1)}(X) = -\frac{3f^2\lambda^2(\lambda^2X^2 + 2\mu^2)}{2(\lambda^2X^2 + \mu^2)^{3/2}} - \frac{6f^2h^2X^2(h^2X^4 + 2\mu^2)}{(h^2X^4 + \mu^2)^{3/2}} \tag{9.4.129}$$

This perturbative analysis holds if the coupling λ is small at the mass scale of the chiral field ϕ_3

$$\lambda^2 \ll \lambda X \tag{9.4.130}$$

This requirement imposes that the field X cannot be fixed at the origin, and R-symmetry has to be broken in the non supersymmetric vacuum. The coupling λ has to be small, and we can expand the potential in the dimensionless parameter $\frac{\lambda}{\sqrt{\mu}}$. At the lowest order we found that a minimum exists and it is

$$X \sim 2^{1/4}\left(\frac{\mu^2}{h^2} - \frac{9\sqrt{3}\lambda^2\mu^2}{15\sqrt{3}h^2\lambda^4 + 4h^4\mu^2}\right)^{1/4} \tag{9.4.131}$$

Inserting (9.4.131) in (9.4.130) we find the condition under which the one loop approximation is valid. In this range we found a (meta)stable vacuum at non zero vev for the pseudo-modulus.

We have shown that metastable supersymmetry breaking vacua in three dimensional WZ models are generic. Relevant couplings potentially invalidate the perturbative approximation. Nevertheless, as we have seen, this problem is removed by the addiction of marginal couplings.

Our study may be useful for the analysis of spontaneous supersymmetry breaking in $3D$ gauge theories. This issue has been investigated in [139, 140, 141] as a consequence of brane dynamics. A preliminary step towards the study of supersymmetry breaking in the dual field theory living on the branes appeared in [68], where the three dimensional ISS mechanism has been discovered for Yang-Mills and Chern-Simons gauge theories. The first class has been deeply studied in [142]. The second class has become more important in the last years, because of its relation with the AdS_4/CFT_3 correspondence. It would be interesting to generalize the three dimensional ISS mechanism of [68] in theories which admit a Seiberg-like dual description [143, 144, 145, 146].

Another interesting aspect, that needs a further analysis, is the role of R-symmetry. In four dimensions supersymmetry breaking is strongly connected with R-symmetry and its spontaneous breaking [119]. Spontaneous R-symmetry breaking is a sufficient condition for supersymmetry breaking. In three dimensions similar results hold but the condition seems stronger. In fact a perturbative analysis of supersymmetry breaking is possible only if R-symmetry is broken.

Conclusions

Appendices

Appendix A

Mathematical tools

A.1 Group theory conventions

In this Appendix we list the main identities we used in the calculations of color traces.

In the fundamental representation the generators are $N \times N$ unitary matrices T^A, $A = 0, \cdots, N^2 - 1$, where $T^0 = \frac{1}{\sqrt{N}}$, whereas T^a are the $SU(N)$ generators. Their normalization is fixed by

$$\mathrm{Tr}(T^A T^B) = \delta^{AB} \tag{A.1.1}$$

The algebra of generators reads

$$[T^A, T^B] = i\, f^{ABC} T^C \tag{A.1.2}$$

where f^{ABC} are the structure constants given by

$$f^{abc} = -i\mathrm{Tr}(T^a[T^b, T^c]) \qquad , \qquad f^{0AB} = 0 \tag{A.1.3}$$

We also introduce

$$d^{abc} = \mathrm{Tr}(T^a\{T^b, T^c\}) \qquad , \qquad d^{0AB} = \frac{2}{\sqrt{N}}\delta^{AB} \tag{A.1.4}$$

Useful relations are:

$$\mathrm{Tr}\left(T^A T^B T^C\right) = \frac{1}{2}\left(if^{ABC} + d^{ABC}\right) \tag{A.1.5}$$

$$\mathrm{Tr}\left(T^A T^B T^C T^D\right) = \frac{1}{4}\left(if^{ABE} + d^{ABE}\right)\left(if^{ECD} + d^{ECD}\right) \tag{A.1.6}$$

Given two scalar objects $M \equiv M^A T^A$ and $N \equiv N^A T^A$ in the adjoint representation of the gauge group, we have the general identity

$$\begin{aligned}[M, N]_* &= \frac{1}{2}\{T^A, T^B\}[M^A, N^B]_* \;+\; \frac{1}{2}[T^A, T^B]\{M^A, N^B\}_* \\ &= \frac{1}{2}d^{ABC}[M^A, N^B]_* T^C \;+\; \frac{i}{2}f^{ABC}\{M^A, N^B\}_* T^C \end{aligned} \tag{A.1.7}$$

A.2 Useful integrals

We list here the results for the momentum loop integrals. As stated in the main text, all the divergent contributions are expressed in terms of a tadpole integral T and a self-energy S which in dimensional regularization ($n = 4 - 2\epsilon$) are

$$\mathcal{T} \equiv \int d^4q \; \frac{1}{q^2 + m\overline{m}} = -\frac{m\overline{m}}{(4\pi)^2}\;\frac{1}{\epsilon} + \mathcal{O}(1) \tag{A.2.8}$$

$$\mathcal{S} \equiv \int d^4q \; \frac{1}{((q-p)^2 + m\overline{m})(q^2 + m\overline{m})} = \frac{1}{(4\pi)^2}\;\frac{1}{\epsilon} + \mathcal{O}(1) \tag{A.2.9}$$

Other one–loop divergent integrals are obtained in terms of $\mathcal{T}$ and $\mathcal{S}$ through the following identities

$$\int d^4q \; \frac{q_{\alpha\dot{\alpha}}}{((q-p)^2 + m\overline{m})(q^2 + m\overline{m})} = \frac{1}{2} p_{\alpha\dot{\alpha}}\;\mathcal{S} \tag{A.2.10}$$

$$\begin{aligned}\int d^4q \; \frac{q_{\alpha\dot{\alpha}}q_{\beta\dot{\beta}}}{((q-p)^2 + m\overline{m})(q^2 + m\overline{m})} &= \\ \frac{1}{3}C_{\alpha\beta}C_{\dot{\alpha}\dot{\beta}}\left[\mathcal{T} - \frac{1}{2}(p^2 + 4m\overline{m})\mathcal{S}\right] \; &+ \; \frac{1}{3}\frac{p_{\alpha\dot{\alpha}}p_{\beta\dot{\beta}}}{p^2}\left[\mathcal{T} + (p^2 + m\overline{m})\mathcal{S}\right]\end{aligned} \tag{A.2.11}$$

$$\int d^4q \; \frac{q^2}{((q-p)^2 + m\overline{m})(q^2 + m\overline{m})} = \mathcal{T} - m\overline{m}\mathcal{S} \tag{A.2.12}$$

$$\int d^4q \frac{q_{\alpha\dot{\alpha}}q_{\beta\dot{\beta}}}{(q^2 + m\overline{m})((q+p)^2 + m\overline{m})((q+r)^2 + m\overline{m})} \sim \frac{1}{2}C_{\alpha\beta}C_{\dot{\alpha}\dot{\beta}}\;\mathcal{S} \tag{A.2.13}$$

$$\begin{aligned}\int d^4q \frac{q_{\alpha\dot{\alpha}}q_{\beta\dot{\beta}}q_{\gamma\dot{\gamma}}q_{\rho\dot{\rho}}}{(q^2 + m\overline{m})((q+p)^2 + m\overline{m})((q+r)^2 + m\overline{m})((q+s)^2 + m\overline{m})} &\sim \\ \frac{1}{6}\mathcal{S}\big(C_{\alpha\beta}C_{\dot{\alpha}\dot{\beta}}C_{\gamma\rho}C_{\dot{\gamma}\dot{\rho}} + C_{\alpha\gamma}C_{\dot{\alpha}\dot{\gamma}}C_{\beta\rho}C_{\dot{\beta}\dot{\rho}} + C_{\alpha\rho}C_{\dot{\alpha}\dot{\rho}}C_{\beta\gamma}C_{\dot{\beta}\dot{\gamma}}\big)&\end{aligned} \tag{A.2.14}$$

In the massless case ($m = \overline{m} = 0$) the tadpole $\mathcal{T}$ vanishes due to a complete cancellation between the UV and the IR divergence. Consequently, the results for the self-energy type integrals can be obtained from (A.2.9 - A.2.14) by setting $\mathcal{T} \sim 0$ and $m = \overline{m} = 0$.

A.3 Notations and conventions in three-dimensional field theories

In this Appendix we list our conventions about three-dimensional $N = 2$ supersymmetric field theories.

The world-volume metric is $g^{\mu\nu} = diag(-1, +1, +1)$ with index range $\mu = 0, 1, 2$. We use Dirac matrices $(\gamma^\mu)_\alpha{}^\beta = (i\sigma^2, \sigma^1, \sigma^3)$ satisfying $\gamma^\mu\gamma^\nu = g^{\mu\nu} + \epsilon^{\mu\nu\rho}\gamma_\rho$.

The fermionic coordinates of $N = 2$ superspace are two real two-component spinors $\theta_i, i = 1, 2$ which we combine into a complex two spinor

$$\theta^\alpha = \frac{1}{\sqrt{2}}(\theta_1^\alpha + i\theta_2^\alpha) \qquad , \qquad \bar{\theta}^\alpha = \frac{1}{\sqrt{2}}(\theta_1^\alpha - i\theta_2^\alpha) \tag{A.3.15}$$

Indices are raised and lowered according to $\theta^\alpha = C^{\alpha\beta}\theta_\beta$, $\theta_\alpha = \theta^\beta C_{\beta\alpha}$, with $C^{12} = -C_{12} = i$. We have

$$\theta_\alpha\theta_\beta = C_{\beta\alpha}\theta^2 \qquad , \qquad \theta^\alpha\theta^\beta = C^{\beta\alpha}\theta^2 \tag{A.3.16}$$

and likewise for $\bar{\theta}$ and derivatives.

Supercovariant derivatives and susy generators are

$$\begin{aligned} D_\alpha = \partial_\alpha + \frac{i}{2}\bar{\theta}^\beta\partial_{\alpha\beta} = \frac{1}{\sqrt{2}}(D_\alpha^1 - iD_\alpha^2) \qquad &, \qquad \bar{D}_\alpha = \bar{\partial}_\alpha + \frac{i}{2}\theta^\beta\partial_{\alpha\beta} = \frac{1}{\sqrt{2}}(D_\alpha^1 + iD_\alpha^2) \\ Q_\alpha = i(\partial_\alpha - \frac{i}{2}\bar{\theta}^\beta\partial_{\alpha\beta}) \qquad &, \qquad \bar{Q}_\alpha = i(\bar{\partial}_\alpha - \frac{i}{2}\theta^\beta\partial_{\alpha\beta}) \end{aligned} \tag{A.3.17}$$

with the only non-trivial anti-commutators

$$\{D_\alpha, \bar{D}_\beta\} = i\partial_{\alpha\beta} \qquad , \qquad \{Q_\alpha, \bar{Q}_\beta\} = i\partial_{\alpha\beta} \tag{A.3.18}$$

We use the following conventions for integration

$$d^2\theta \equiv \frac{1}{2}d\theta^\alpha d\theta_\alpha \qquad , \qquad d^2\bar{\theta} \equiv \frac{1}{2}d\bar{\theta}^\alpha d\bar{\theta}_\alpha \qquad , \qquad d^4\theta \equiv d^2\theta\, d^2\bar{\theta} \tag{A.3.19}$$

such that

$$\int d^2\theta\,\theta^2 = -1 \qquad \int d^2\bar{\theta}\,\bar{\theta}^2 = -1 \qquad \int d^4\theta\,\theta^2\bar{\theta}^2 = 1 \tag{A.3.20}$$

The components of a chiral and an anti-chiral superfield, $Z(x_L, \theta)$ and $\bar{Z}(x_R, \bar{\theta})$, are a complex boson ϕ, a complex two-component fermion ψ and a complex auxiliary scalar F. Their component expansions are given by

$$\begin{aligned} Z &= \phi(x_L) + \theta^\alpha\psi_\alpha(x_L) - \theta^2\, F(x_L) \\ \bar{Z} &= \bar{\phi}(x_R) + \bar{\theta}^\alpha\bar{\psi}_\alpha(x_R) - \bar{\theta}^2\, \bar{F}(x_R) \end{aligned} \tag{A.3.21}$$

where

$$\begin{aligned} x_L^\mu &= x^\mu + i\theta\gamma^\mu\bar{\theta} \\ x_R^\mu &= x^\mu - i\theta\gamma^\mu\bar{\theta} \end{aligned} \tag{A.3.22}$$

The components of the vector superfield $V(x, \theta, \bar{\theta})$ in Wess-Zumino gauge ($V| = D_\alpha V| = D^2 V| = 0$) are the gauge field $A_{\alpha\beta}$, a complex two-component fermion λ_α, a real scalar σ and an auxiliary scalar D, such that

$$V = i\,\theta^\alpha\bar{\theta}_\alpha\,\sigma(x) + \theta^\alpha\bar{\theta}^\beta\,A_{\alpha\beta}(x) - \theta^2\,\bar{\theta}^\alpha\bar{\lambda}_\alpha(x) - \bar{\theta}^2\,\theta^\alpha\lambda_\alpha(x) + \theta^2\,\bar{\theta}^2\,D(x)\,. \tag{A.3.23}$$

For $SU(N)$ we use the $N \times N$ hermitian matrix generators T^a $(a = 1, \ldots, N^2 - 1)$ and for $U(N)$

$$T^A = (T^0, T^a) \quad \text{with} \quad T^0 = \frac{1}{\sqrt{N}} \tag{A.3.24}$$

The generators are normalized as $\mathrm{Tr}T^A T^B = \delta^{AB}$.

Completeness implies

$$U(N): \qquad \mathrm{Tr}AT^A\mathrm{Tr}BT^A = \mathrm{Tr}AB \quad , \quad \mathrm{Tr}AT^ABT^A = \mathrm{Tr}A\mathrm{Tr}B$$

$$\begin{aligned} SU(N): \qquad & \mathrm{Tr}AT^a\mathrm{Tr}BT^a = \mathrm{Tr}AB - \frac{1}{N}\mathrm{Tr}A\mathrm{Tr}B \\ & \mathrm{Tr}AT^aBT^a = \mathrm{Tr}A\mathrm{Tr}B - \frac{1}{N}\mathrm{Tr}AB \end{aligned} \tag{A.3.25}$$

Useful integrals for computing Feynman diagrams in momentum space and dimensional regularization ($d = 3 - 2\epsilon$) are, at one loop

$$\int \frac{d^3k}{(2\pi)^3} \, \frac{1}{k^2(k-p)^2} = \frac{1}{8}\frac{1}{|p|} \equiv B_0(p) \tag{A.3.26}$$

$$\int \frac{d^3k}{(2\pi)^3} \, \frac{k_{\alpha\beta}}{k^2(k-p)^2} = \frac{1}{2}\, p_{\alpha\beta}\, B_0(p) \tag{A.3.27}$$

and at two loops

$$F(p) \equiv \int \frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3} \, \frac{1}{k^2\, q^2\, (p-k-q)^2} = \frac{\Gamma(\epsilon)}{64\pi^2} \sim \frac{1}{64\pi^2}\,\frac{1}{\epsilon} \tag{A.3.28}$$

Appendix B

Feynman rules

B.1 Feynman rules for the general action (4.3.124)

In this Appendix we apply the NAC background field method to the action (4.3.124) and derive the Feynman rules necessary for calculations of Section 4.3.

Gauge sector

We first concentrate on the gauge sector. We work in gauge antichiral representation [13] for covariant derivatives and perform the quantum–background splitting according to

$$\nabla_\alpha = \boldsymbol{\nabla}_\alpha = D_\alpha \qquad , \qquad \overline{\nabla}_{\dot\alpha} = e_*^V * \overline{\boldsymbol{\nabla}}_{\dot\alpha} * e_*^{-V} = e_*^V * e_*^U * \bar{D}_{\dot\alpha}\; e_*^{-U} * e_*^{-V} \tag{B.1.1}$$

The derivatives transform covariantly with respect to quantum transformations

$$\begin{aligned} e_*^V \to e_*^{i\overline{\Lambda}} * e_*^V * e_*^{-i\Lambda} \qquad &, \qquad e_*^U \to e_*^U \\ \nabla_A \to e_*^{i\overline{\Lambda}} * \nabla_A * e_*^{-i\overline{\Lambda}} \qquad &, \qquad \boldsymbol{\nabla}_A \to \boldsymbol{\nabla}_A \end{aligned} \tag{B.1.2}$$

with background covariantly (anti)chiral parameters, $\boldsymbol{\nabla}_\alpha\overline{\Lambda} = \overline{\boldsymbol{\nabla}}_{\dot\alpha}\Lambda = 0$, and background transformations

$$\begin{aligned} e_*^V \to e_*^{i\overline{\lambda}} * e_*^V * e_*^{-i\overline{\lambda}} \qquad &, \qquad e_*^U \to e_*^{i\overline{\lambda}} * e_*^U * e_*^{-i\lambda} \\ \nabla_A \to e_*^{i\overline{\lambda}} * \nabla_A * e_*^{-i\overline{\lambda}} \qquad &, \qquad \boldsymbol{\nabla}_A \to e_*^{i\overline{\lambda}} * \boldsymbol{\nabla}_A * e_*^{-i\overline{\lambda}} \end{aligned} \tag{B.1.3}$$

with ordinary (anti)chiral parameters $\bar{D}_{\dot\alpha}\lambda = D_\alpha\overline{\lambda} = 0$.

The classical action

$$S_{gauge} = \frac{1}{2g^2}\int d^4x d^2\bar\theta\; \overline{W}^{\dot\alpha}\overline{W}_{\dot\alpha} \tag{B.1.4}$$

for the gauge field strength defined in eq. (4.1.13) is invariant under gauge transformations (B.1.2) and (B.1.3). Background field quantization consists in performing gauge–fixing which explicitly breaks the (B.1.2) gauge invariance while preserving manifest invariance of the effective action and correlation functions under (B.1.3). Choosing as in the ordinary case the gauge–fixing functions as $f = \overline{\boldsymbol{\nabla}}^2 * V$, $\overline{f} = \boldsymbol{\nabla}^2 * V$ the resulting gauge–fixed action has exactly the same structure as in the ordinary case [13] with products promoted to star products [14, 18]. In Feynman gauge it reads

$$
\begin{aligned}
&S_{gauge} + S_{GF} + S_{gh} = \\
&\quad -\frac{1}{2g^2}\int d^4x d^4\theta \left[e_*^V * \overline{\boldsymbol{\nabla}}^{\dot\alpha} * e_*^{-V} * D^2 (e_*^V * \overline{\boldsymbol{\nabla}}_{\dot\alpha} * e_*^{-V}) \; + \; V * (\overline{\boldsymbol{\nabla}}^2 D^2 + D^2 \overline{\boldsymbol{\nabla}}^2) * V \right] \\
&\quad + \int d^4x d^4\theta \left[\overline{c}' c - c' \overline{c} + ... + \overline{b} b \right]
\end{aligned}
\tag{B.1.5}
$$

where ghosts are background covariantly (anti)chiral superfields and dots stand for higher order interaction terms.

As discussed in details in Ref. [14, 18] and reviewed in Section 4.2, in the nonanticommutative case the $SU(N)$ and $U(1)$ parts of the gauge superfield have different kinetic terms. It turns out that the most convenient choice for the gauge–fixing action is still the one above, namely

$$
S_{GF} = -\frac{1}{g^2 \alpha} \int d^4x d^4\theta \; \mathrm{Tr} \left(\overline{\boldsymbol{\nabla}}^2 V \boldsymbol{\nabla}^2 V \right)
\tag{B.1.6}
$$

which combined with the original kinetic terms give rise to the quadratic operators

$$
\frac{1}{2g^2} V^a \left[\bar{D}^{\dot\alpha} D^2 \bar{D}_{\dot\alpha} - \alpha^{-1} \left(D^2 \bar{D}^2 + \bar{D}^2 D^2 \right) \right] V^a
\tag{B.1.7}
$$

$$
\frac{1}{2\tilde{g}^2} V^0 \left[\bar{D}^{\dot\alpha} D^2 \bar{D}_{\dot\alpha} - \tilde{\alpha}^{-1} \left(D^2 \bar{D}^2 + \bar{D}^2 D^2 \right) \right] V^0
\tag{B.1.8}
$$

where

$$
\tilde{g}^2 = \frac{g^2 g_0^2}{g_0^2 + g^2} \qquad\qquad \tilde{\alpha}^2 = \alpha \frac{g^2}{\tilde{g}^2}
\tag{B.1.9}
$$

Choosing the Feynman gauge ($\alpha = 1$) we obtain the propagators

$$
\begin{aligned}
\langle V^a V^b \rangle &= g^2 \left(\frac{1}{\hat{\Box}} \right)^{ab} \\
\langle V^0 V^0 \rangle &= g^2 \left\{ \frac{1}{\tilde{\Box}} \left[1 + \left(\frac{g^2}{g^2 + g_0^2} \right) \overline{\boldsymbol{\nabla}}^{\dot\alpha} * \boldsymbol{\nabla}^2 * \overline{\boldsymbol{\nabla}}_{\dot\alpha} * \frac{1}{\tilde{\Box}} \right] \right\}^{00}
\end{aligned}
\tag{B.1.10}
$$

where $\hat{\Box}, \tilde{\Box}$ are defined by

$$
\hat{\Box} = \Box_{cov} - i \widetilde{\mathbf{W}}^{\alpha} * \boldsymbol{\nabla}_\alpha - i \overline{\mathbf{W}}^{\dot\alpha} * \overline{\boldsymbol{\nabla}}_{\dot\alpha} \qquad , \qquad \Box_{cov} = \frac{1}{2} \overline{\boldsymbol{\nabla}}^{\alpha\dot\alpha} * \overline{\boldsymbol{\nabla}}_{\alpha\dot\alpha}
\tag{B.1.11}
$$

$$
\tilde{\Box} = \boldsymbol{\nabla}^2 * \overline{\boldsymbol{\nabla}}^2 + \overline{\boldsymbol{\nabla}}^2 * \boldsymbol{\nabla}^2 - \overline{\boldsymbol{\nabla}}^{\dot\alpha} * \boldsymbol{\nabla}^2 * \overline{\boldsymbol{\nabla}}_{\dot\alpha} = \Box_{cov} - i \widetilde{\mathbf{W}}^{\alpha} * \boldsymbol{\nabla}_\alpha + \frac{i}{2} \left(\overline{\boldsymbol{\nabla}}^{\dot\alpha} * \overline{W}_{\dot\alpha} \right)
\tag{B.1.12}
$$

in terms of $\Box_{cov}$. On a generic superfield in the adjoint representation of $SU(N)\otimes U(1)$ we have

$$\begin{aligned}
(\Box_{cov} * \phi)^A &= \left(\frac{1}{2}\bar{\nabla}^{\alpha\dot{\alpha}} * \bar{\nabla}_{\alpha\dot{\alpha}} * \phi\right)^A \\
&= \left(\Box\phi - i[\overline{\Gamma}^{\alpha\dot{\alpha}}, \partial_{\alpha\dot{\alpha}}\phi]_* - \frac{i}{2}[(\partial^{\alpha\dot{\alpha}}\overline{\Gamma}_{\alpha\dot{\alpha}}), \phi]_* - \frac{1}{2}[\overline{\Gamma}^{\alpha\dot{\alpha}}, [\overline{\Gamma}_{\alpha\dot{\alpha}}, \phi]_*]_*\right)^A \\
&\equiv \Box^{AB}_{cov} * \phi^B
\end{aligned} \tag{B.1.13}$$

Using the general NAC rule

$$[F, G]^A_* = \frac{1}{2} i f^{ABC}\{F^B, G^C\}_* + \frac{1}{2} d^{ABC}[F^B, G^C]_* \tag{B.1.14}$$

valid for any couple of field functions in the adjoint representation of the gauge group, and expanding the $*$–product we find

$$\begin{aligned}
\Box^{AB}_{cov} &= \Box\, \delta^{AB} + f^{ACB}\overline{\Gamma}^{C\,\alpha\dot{\alpha}}\partial_{\alpha\dot{\alpha}} + i d^{ACB}\mathcal{F}^{\alpha\beta}(\partial_\alpha\overline{\Gamma}^{C\,\gamma\dot{\gamma}})\partial_\beta\partial_{\gamma\dot{\gamma}} - \frac{1}{2} f^{ACB}\mathcal{F}^2(\partial^2\overline{\Gamma}^{C\,\alpha\dot{\alpha}})\partial^2\partial_{\alpha\dot{\alpha}} \\
&\quad + \cdots
\end{aligned} \tag{B.1.15}$$

Only the first two terms in (B.1.13) have been explicitly indicated. The rest can be treated in a similar manner.

The $\frac{1}{\hat{\Box}}$ and $\frac{1}{\tilde{\tilde{\Box}}}$ propagators can be expanded in powers of the background fields. We formally write

$$\begin{aligned}
\frac{1}{\hat{\Box}} &= \frac{1}{\Box_{cov}} + \frac{1}{\Box_{cov}} * \left(i\tilde{W}^\alpha\nabla_\alpha + i\overline{W}^{\dot{\alpha}} * \bar{\nabla}_{\dot{\alpha}}\right) * \frac{1}{\hat{\Box}} \\
\frac{1}{\tilde{\tilde{\Box}}} &= \frac{1}{\Box_{cov}} + \frac{1}{\Box_{cov}} * \left(i\tilde{W}^\alpha\nabla_\alpha - \frac{i}{2}(\bar{\nabla}^{\dot{\alpha}} * \overline{W}_{\dot{\alpha}})\right) * \frac{1}{\tilde{\tilde{\Box}}}
\end{aligned} \tag{B.1.16}$$

Expanding the right hand side we obtain terms proportional to $\tilde{W}_\alpha, \overline{W}_{\dot{\alpha}}$ and terms proportional to the bosonic connections coming from $1/\Box_{cov}$. As follows from dimensional considerations and confirmed by direct inspection, terms proportional to the field strengths never enter divergent diagrams as long as we focus on contributions linear in the NAC parameter. Therefore, at this stage we can neglect them. Using the expansion (B.1.15) we then find

$$\left(\frac{1}{\hat{\Box}}\right)^{ab}, \left(\frac{1}{\tilde{\tilde{\Box}}}\right)^{00} \rightarrow \left(\frac{1}{\Box_{cov}}\right)^{AB} \tag{B.1.17}$$

$$\begin{aligned}
&\simeq \frac{1}{\Box}\, \delta^{AB} - \frac{1}{\Box} f^{ACB}\, \overline{\Gamma}^{C\,\alpha\dot{\alpha}}\, \partial_{\alpha\dot{\alpha}}\frac{1}{\Box} - \frac{1}{2}\frac{1}{\Box}\, f^{ACD} f^{DEB}\, \overline{\Gamma}^{C\,\alpha\dot{\alpha}}\overline{\Gamma}^E_{\alpha\dot{\alpha}}\, \frac{1}{\Box} \\
&\quad - \frac{1}{\Box} i d^{ACB}\, \mathcal{F}^{\alpha\beta}(\partial_\alpha\overline{\Gamma}^{C\,\gamma\dot{\gamma}})\, \partial_\beta\partial_{\gamma\dot{\gamma}}\frac{1}{\Box} + \frac{1}{2}\,\frac{1}{\Box}\, f^{ACB}\mathcal{F}^2(\partial^2\overline{\Gamma}^{C\,\alpha\dot{\alpha}})\, \partial^2\partial_{\alpha\dot{\alpha}}\, \frac{1}{\Box} + \cdots
\end{aligned}$$

In this expression we recognize the ordinary bare propagator $1/\Box$ plus a number of gauge interaction vertices.

Further interactions come from the expansion of the remaining terms in (4.2.77) or (4.2.78). Their explicit expression can be found in Appendix E of [18].

Matter sector

We now derive propagators and interaction vertices for the action $S_{matter} + S_{\overline{\Gamma}} + S_{\overline{W}}$ in (4.3.124). Since we are primarily interested in computing divergent contributions linear in the NAC parameter, we restrict our analysis to Feynman rules which contribute to this kind of terms. In particular, we do not take into account vertices proportional to $\mathcal{F}^2$.

We first concentrate on the calculation of the chiral propagators. As given in eq. (4.3.125) the full covariant scalar quadratic term is

$$\int d^4x d^4\theta \, \left\{ \mathrm{Tr} \left(\bar{\Phi}\Phi \right) + \frac{\kappa - 1}{N} \mathrm{Tr}\bar{\Phi}\mathrm{Tr}\Phi \right\} \tag{B.1.18}$$

which can be expanded in terms of the background covariantly (anti)chiral fields (4.3.81) as

$$\begin{aligned} &\int d^4x d^4\theta \, \left\{ \mathrm{Tr}(\bar{\mathbf{\Phi}} * e^V * \mathbf{\Phi} * e^{-V}) + \frac{\kappa - 1}{N} \mathrm{Tr}(\bar{\mathbf{\Phi}})\mathrm{Tr}(e^V * \mathbf{\Phi} * e^{-V}) \right\} \\ &= \int d^4x d^4\theta \, \left\{ \mathrm{Tr} \left(\bar{\mathbf{\Phi}}\mathbf{\Phi} + \bar{\mathbf{\Phi}}[V, \mathbf{\Phi}]_* + \frac{1}{2}\bar{\mathbf{\Phi}}[V, [V, \mathbf{\Phi}]_*]_* + \dots \right) \right. \\ &\qquad \left. + \frac{\kappa - 1}{N} \mathrm{Tr}(\bar{\mathbf{\Phi}})\mathrm{Tr} \left(\mathbf{\Phi} + [V, \mathbf{\Phi}]_* + \frac{1}{2}[V, [V, \mathbf{\Phi}]_*]_* + \dots \right) \right\} \end{aligned} \tag{B.1.19}$$

We perform the quantum-background splitting

$$\mathbf{\Phi} \to \mathbf{\Phi} + \mathbf{\Phi}_q, \bar{\mathbf{\Phi}} \to \bar{\mathbf{\Phi}} + \bar{\mathbf{\Phi}}_q \tag{B.1.20}$$

and concentrate on the evaluation of the quadratic functional integral

$$\int \mathcal{D}\mathbf{\Phi}_q \mathcal{D}\bar{\mathbf{\Phi}}_q \; e^{\int d^4x d^4\theta \, \left\{ \mathrm{Tr}\left(\bar{\Phi}_q \Phi_q \right) + \frac{\kappa - 1}{N} \mathrm{Tr}\bar{\Phi}_q \mathrm{Tr}\Phi_q \right\}} \tag{B.1.21}$$

In order to deal with a simpler integral we make the change of variables

$$\mathbf{\Phi}_q^A \to \mathbf{\Phi}_q'^A = (\mathbf{\Phi}_q^a, \kappa_1 \mathbf{\Phi}_q^0) \qquad , \qquad \bar{\mathbf{\Phi}}_q^A \to \bar{\mathbf{\Phi}}_q'^A = (\bar{\mathbf{\Phi}}_q^a, \kappa_2 \bar{\mathbf{\Phi}}_q^0) \tag{B.1.22}$$

where κ_1 and κ_2 are two arbitrary constants satisfying $\kappa_1\kappa_2 = \kappa$. The functional integral (B.1.21) then takes the standard form

$$\int \mathcal{D}\mathbf{\Phi}_q' \mathcal{D}\bar{\mathbf{\Phi}}_q' \; e^{\int d^4x d^4\theta \; \mathrm{Tr}\bar{\mathbf{\Phi}}_q' \mathbf{\Phi}_q'} \tag{B.1.23}$$

We stress that the redefinition (B.1.22) in terms of two independent couplings is admissible because we are working in Euclidean space where chiral and antichiral fields are not related by complex conjugation.

Adding source terms

$$\begin{aligned} &\mathrm{Tr} \int d^4x d^2\theta \; j\mathbf{\Phi}_q' + \mathrm{Tr} \int d^4x d^2\bar{\theta} \; \bar{\mathbf{\Phi}}_q' \bar{j} \\ &= \mathrm{Tr} \int d^4x d^4\theta \; \left(j * \frac{1}{\Box_+} * \mathbf{\nabla}^2 \mathbf{\Phi}_q' + \bar{\mathbf{\Phi}}_q' * \frac{1}{\Box_-} * \overline{\mathbf{\nabla}}^2 * \bar{j} \right) \end{aligned} \tag{B.1.24}$$

with $\Box_\pm$ defined in (4.3.84), and taking into account the complete action the quantum partition function reads

$$\mathbf{Z}(j,\bar{j}) = e^{S_{int}(\frac{\delta}{\delta j},\frac{\delta}{\delta \bar{j}})} \int \mathcal{D}\mathbf{\Phi}'_q \mathcal{D}\bar{\mathbf{\Phi}}'_q \exp \mathrm{Tr} \int d^4x d^4\theta \, \Big[\bar{\mathbf{\Phi}}'_q \mathbf{\Phi}'_q + j * \frac{1}{\Box_+} * \nabla^2 \mathbf{\Phi}'_q + \bar{\mathbf{\Phi}}'_q * \frac{1}{\Box_-} * \overline{\nabla}^2 * \bar{j} \Big] \tag{B.1.25}$$

Here S_{int} contains all gauge–scalar fields interaction vertices in (B.1.19) plus interactions coming from the rest of terms in $S_{matter} + S_{\overline{\Gamma}} + S_{\overline{W}}$.

We can perform the Gaussian integral in (B.1.25) by standard techniques, obtaining the NAC generalization of the usual superspace expression [13]

$$\mathbf{Z} = \Delta * e^{S_{int}(\frac{\delta}{\delta j},\frac{\delta}{\delta \bar{j}})} \exp\left(- \int d^4x d^4\theta \; j * \frac{1}{\Box_-} * \bar{j} \right) \tag{B.1.26}$$

where Δ is the functional determinant

$$\Delta = \int \mathcal{D}\mathbf{\Phi}'_q \mathcal{D}\bar{\mathbf{\Phi}}'_q \; \exp \mathrm{Tr} \int d^4x d^4\theta \; \bar{\mathbf{\Phi}}'_q \mathbf{\Phi}'_q \tag{B.1.27}$$

which contributes to the gauge effective action [14, 18].

From the expression (B.1.26) we can read the covariant propagators for prime superfields

$$\langle \mathbf{\Phi}'^A_q \bar{\mathbf{\Phi}}'^B_q \rangle = - \left(\frac{1}{\Box_-} \right)^{AB} \tag{B.1.28}$$

which, in terms of the original $\Phi, \bar{\Phi}$ superfields gives

$$\langle \mathbf{\Phi}^a_q \bar{\mathbf{\Phi}}^b_q \rangle = - \left(\frac{1}{\Box_-} \right)^{ab} \tag{B.1.29}$$

$$\langle \mathbf{\Phi}^0_q \bar{\mathbf{\Phi}}^b_q \rangle = - \frac{1}{\kappa_2} \left(\frac{1}{\Box_-} \right)^{0b} \tag{B.1.30}$$

$$\langle \mathbf{\Phi}^a_q \bar{\mathbf{\Phi}}^0_q \rangle = - \frac{1}{\kappa_1} \left(\frac{1}{\Box_-} \right)^{a0} \tag{B.1.31}$$

$$\langle \mathbf{\Phi}^0_q \bar{\mathbf{\Phi}}^0_q \rangle = - \frac{1}{\kappa} \left(\frac{1}{\Box_-} \right)^{00} \tag{B.1.32}$$

The expansion of the scalar covariant propagators can be performed following a prescription similar to the one used for the gauge propagator. We can formally write

$$\frac{1}{\Box_-} = \frac{1}{\Box_{cov}} + \frac{1}{\Box_{cov}} * \left(i\overline{W}^{\dot{\alpha}} * \bar{\nabla}_{\dot{\alpha}} + \frac{i}{2} (\bar{\nabla}^{\dot{\alpha}} * \overline{W}_{\dot{\alpha}}) \right) * \frac{1}{\Box_-} \tag{B.1.33}$$

Since terms proportional to the field strengths never enter divergent diagrams linear in $\mathcal{F}^{\alpha\beta}$, we can neglect them and write

$$\left(\frac{1}{\Box_-}\right)^{AB} \to \left(\frac{1}{\Box_{cov}}\right)^{AB} \tag{B.1.34}$$

$$\simeq \frac{1}{\Box}\,\delta^{AB} - \frac{1}{\Box} f^{ACB}\,\overline{\Gamma}^{C\,\alpha\dot\alpha}\,\partial_{\alpha\dot\alpha}\frac{1}{\Box} - \frac{1}{2}\frac{1}{\Box}\,f^{ACD} f^{DEB}\,\overline{\Gamma}^{C\,\alpha\dot\alpha}\overline{\Gamma}^{E}_{\alpha\dot\alpha}\,\frac{1}{\Box}$$

$$-\frac{1}{\Box} i d^{ACB}\,\mathcal{F}^{\alpha\beta}(\partial_\alpha\overline{\Gamma}^{C\,\gamma\dot\gamma})\,\partial_\beta\partial_{\gamma\dot\gamma}\frac{1}{\Box} + \frac{1}{2}\,\frac{1}{\Box}\,f^{ACB}\mathcal{F}^2(\partial^2\overline{\Gamma}^{C\,\alpha\dot\alpha})\,\partial^2\partial_{\alpha\dot\alpha}\,\frac{1}{\Box}\;+\;\cdots$$

The first term is diagonal in the color indices and gives the ordinary bare propagator. The rest provides interaction vertices between scalars and gauge superfields.

From the expansion (B.1.34) it is clear that the mixed propagators (B.1.30, B.1.31) are always proportional to the NAC parameter, according to the fact that in the $N = 1$ limit they need vanish. It follows that the dependence on the κ_1 and κ_2 couplings is peculiar of the NAC theory, whereas in the ordinary limit only their product κ survives.

Additional interaction terms are contained in S_{int} and arise from the background field expansion of the full action $S_{matter} + S_{\overline{\Gamma}} + S_{\overline{W}}$. We now describe the correct way to obtain such vertices concentrating only on the ones at most linear in $\mathcal{F}^{\alpha\beta}$.

We begin by considering S_{matter}. From the quadratic action $\int d^4x d^4\theta\ \mathrm{Tr}\bar\Phi\Phi$, after the expansion (B.1.19) and the shift (B.1.20) we obtain (B.1a,B.1b)–type vertices in Fig. B.1 where V is quantum and $\mathbf{\Phi}$ and/or $\bar{\mathbf{\Phi}}$ are background. Expanding the $*$–products ordinary vertices plus vertices proportional to $\mathcal{F}^{\alpha\beta}$ and $\mathcal{F}^2$ arise.

We then consider the $(\kappa - 1)$ terms in (4.3.125)

$$\frac{\kappa-1}{N}\int d^4x d^4\theta\,\Big[\mathrm{Tr}\bar\Phi * \mathrm{Tr}\Phi \tag{B.1.35}$$

$$+2i\bar\theta^2\mathcal{F}^{\alpha\beta}\mathrm{Tr}(\overline{\Gamma}_\alpha^{\ \dot\alpha} * \bar\Phi) * \mathrm{Tr}(\partial_{\beta\dot\alpha}\Phi) + 2i\bar\theta^2\mathcal{F}^{\alpha\beta}\mathrm{Tr}(\overline{\Gamma}_\alpha^{\ \dot\alpha} * \Phi) * \mathrm{Tr}(\partial_{\beta\dot\alpha}\bar\Phi)\Big]$$

We expand the (anti)chiral superfields as

$$\Phi \to \mathbf{\Phi} + \mathbf{\Phi}_q + [V, \mathbf{\Phi} + \mathbf{\Phi}_q]_* + \frac{1}{2}[V,[V,\mathbf{\Phi}+\mathbf{\Phi}_q]_*]_* \quad , \quad \bar\Phi \to \bar{\mathbf{\Phi}} + \bar{\mathbf{\Phi}}_q \tag{B.1.36}$$

and, at the same order in V, the gauge connection as

$$\overline{\Gamma}_{\alpha\dot\alpha} \to \overline{\Gamma}_{\alpha\dot\alpha} - \nabla_\alpha\left[\overline{\mathbf{\nabla}}_{\dot\alpha}, V\right]_* + \frac{1}{2}\nabla_\alpha\left[\left[\overline{\mathbf{\nabla}}_{\dot\alpha}, V\right]_*, V\right]_* \tag{B.1.37}$$

Collecting the various terms we generate (B.1c,B.1d)–vertices in Fig. B.1 with background gauge connections and quantum matter plus (B.1e,B.1f,B.1g)–vertices with quantum gauge and $\mathbf{\Phi}$ or $\bar{\mathbf{\Phi}}$ background.

As a nontrivial example, we derive in details the contributions (3e,3f,3g). Forgetting for a while the superspace integration and the overall coupling constant and writing $\partial_\alpha = \nabla_\alpha - i\bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha}$, from the first term in (B.1.35) we have

$$\mathrm{Tr}([V,\mathbf{\Phi}]_*)\,\mathrm{Tr}\bar{\mathbf{\Phi}} \to -\mathcal{F}^{\alpha\beta}\,\mathrm{Tr}([\partial_\alpha V, \partial_\beta\mathbf{\Phi}])\,\mathrm{Tr}\bar{\mathbf{\Phi}}$$

$$\to -2i\mathcal{F}^{\alpha\beta}\,\bar\theta^{\dot\alpha}\,\mathrm{Tr}(V\nabla_\alpha\mathbf{\Phi})\,\mathrm{Tr}\partial_{\beta\dot\alpha}\bar{\mathbf{\Phi}} - 2\mathcal{F}^{\alpha\beta}\bar\theta^2\,\mathrm{Tr}(\partial_\alpha^{\ \dot\alpha}V\,\mathbf{\Phi})\,\mathrm{Tr}\partial_{\beta\dot\alpha}\bar{\mathbf{\Phi}} \tag{B.1.38}$$

where superspace total derivatives have been neglected and $\mathbf{\Phi}, \bar{\mathbf{\Phi}}$ stand for either quantum or background.

Using the expansion (B.1.37) the second term in (B.1.35) gives a contribution of the form

$$2i\mathcal{F}^{\alpha\beta}\,\bar{\theta}^2\mathrm{Tr}(\overline{\Gamma}_\alpha{}^{\dot{\alpha}}\bar{\mathbf{\Phi}})\,\mathrm{Tr}(\partial_{\beta\dot{\alpha}}\mathbf{\Phi}) \rightarrow -2i\mathcal{F}^{\alpha\beta}\,\bar{\theta}^2\mathrm{Tr}([\bar{\nabla}^{\dot{\alpha}}, V]\,\bar{\mathbf{\Phi}})\,\mathrm{Tr}(\partial_{\beta\dot{\alpha}}\,\nabla_\alpha\mathbf{\Phi}) \tag{B.1.39}$$

Similarly, the third term in (B.1.35) gives

$$\begin{aligned}
&2i\mathcal{F}^{\alpha\beta}\bar{\theta}^2\mathrm{Tr}(\overline{\Gamma}_\alpha{}^{\dot{\alpha}}\mathbf{\Phi})\,\mathrm{Tr}(\partial_{\beta\dot{\alpha}}\bar{\mathbf{\Phi}}) \rightarrow 2i\mathcal{F}^{\alpha\beta}\bar{\theta}^2\Big\{\mathrm{Tr}(\overline{\Gamma}_\alpha{}^{\dot{\alpha}}[V,\mathbf{\Phi}])\,\mathrm{Tr}(\partial_{\beta\dot{\alpha}}\bar{\mathbf{\Phi}})\\
&-\mathrm{Tr}(\nabla_\alpha\overline{D}^{\dot{\alpha}}V\,\mathbf{\Phi})\,\mathrm{Tr}(\partial_{\beta\dot{\alpha}}\bar{\mathbf{\Phi}}) + \mathrm{Tr}([V,\overline{\Gamma}_\alpha{}^{\dot{\alpha}}]\mathbf{\Phi})\,\mathrm{Tr}(\partial_{\beta\dot{\alpha}}\bar{\mathbf{\Phi}}) - i\mathrm{Tr}([\overline{\Gamma}^{\dot{\alpha}}, \nabla_\alpha V]\mathbf{\Phi})\,\mathrm{Tr}(\partial_{\beta\dot{\alpha}}\bar{\mathbf{\Phi}})\Big\}\\
&= 2\mathcal{F}^{\alpha\beta}\bar{\theta}^2\mathrm{Tr}(\partial_\alpha{}^{\dot{\alpha}}V\,\mathbf{\Phi})\,\mathrm{Tr}(\partial_{\beta\dot{\alpha}}\bar{\mathbf{\Phi}}) + 2i\mathcal{F}^{\alpha\beta}\,\bar{\theta}^2\,\mathrm{Tr}([\bar{\nabla}^{\dot{\alpha}}, \nabla_\alpha V]\,\mathbf{\Phi})\,\mathrm{Tr}(\partial_{\beta\dot{\alpha}}\bar{\mathbf{\Phi}})
\end{aligned} \tag{B.1.40}$$

Summing the three contributions a nontrivial cancellation occurs between the second term in (B.1.38) and the first term in (B.1.40) and we are left with

$$\begin{aligned}
\frac{\kappa-1}{N}\int d^4x d^4\theta\,\Big\{&-2i\mathcal{F}^{\alpha\beta}\,\bar{\theta}^{\dot{\alpha}}\,\mathrm{Tr}(V\nabla_\alpha\mathbf{\Phi})\,\mathrm{Tr}\partial_{\beta\dot{\alpha}}\bar{\mathbf{\Phi}} - 2i\mathcal{F}^{\alpha\beta}\,\bar{\theta}^2\mathrm{Tr}([\bar{\nabla}^{\dot{\alpha}}, V]\,\bar{\mathbf{\Phi}})\,\mathrm{Tr}(\partial_{\beta\dot{\alpha}}\,\nabla_\alpha\mathbf{\Phi})\\
&+ 2i\mathcal{F}^{\alpha\beta}\,\bar{\theta}^2\,\mathrm{Tr}([\bar{\nabla}^{\dot{\alpha}}, \nabla_\alpha V]\,\mathbf{\Phi})\,\mathrm{Tr}(\partial_{\beta\dot{\alpha}}\bar{\mathbf{\Phi}})\Big\}
\end{aligned} \tag{B.1.41}$$

which correspond to the three vertices (B.1e, B.1f, B.1g).

The rest of terms in the S_{matter} can be easily treated by the shift (B.1.36). Neglecting $\mathcal{F}^2$ contributions only the superpotential and the $\tilde{h}_3$ term survive and lead to pure matter vertices of the form (B.1h, B.1i, B.1j, B.1k, B.1l) and the mixed vertex (B.1m).

We now turn to $S_{\overline{\Gamma}}$ and briefly sketch the quantization of t_j vertices. At linear order in the NAC parameter we can forget the $*$–product in the commutators of t_3, t_4, t_5 terms. We perform the shift (B.1.36) on the (anti)chirals and (B.1.37) on the connection. In particular, for the gauge invariant linear combination appearing in t_3, t_4, t_5 terms we have

$$\partial_{\beta\dot{\alpha}}\overline{\Gamma}_\alpha{}^{\dot{\alpha}} - \frac{i}{2}[\overline{\Gamma}_{\beta\dot{\alpha}}, \overline{\Gamma}_\alpha{}^{\dot{\alpha}}] \;\longrightarrow\; \partial_{\beta\dot{\alpha}}\overline{\Gamma}_\alpha{}^{\dot{\alpha}} - \frac{i}{2}[\overline{\Gamma}_{\beta\dot{\alpha}}, \overline{\Gamma}_\alpha{}^{\dot{\alpha}}] - \overline{\boldsymbol{\nabla}}_{\beta\dot{\alpha}}\boldsymbol{\nabla}_\alpha\overline{\boldsymbol{\nabla}}^{\dot{\alpha}}\,V \tag{B.1.42}$$

Collecting only the contributions which may contribute at one–loop we produce the (B.1n) vertex in Fig. B.1 where matter is quantum and (B.1o, B.1p) vertices where Φ or $\bar{\Phi}$ are quantum. We note that they all exhibit a gauge–invariant background dependence.

B.2 Feynman rules for the abelian actions (4.4.146) and (4.4.147)

In this Appendix we collect all one–loop Feynman rules obtained from the actions (4.4.146, 4.4.147) by applying the generalized background field method developed in [14, 18, 22] for NAC super Yang–Mills theories with chiral matter in a *real* representation of the gauge group.

Gauge sector

We specialize the previous formulas to the $U_*(1)$ case.

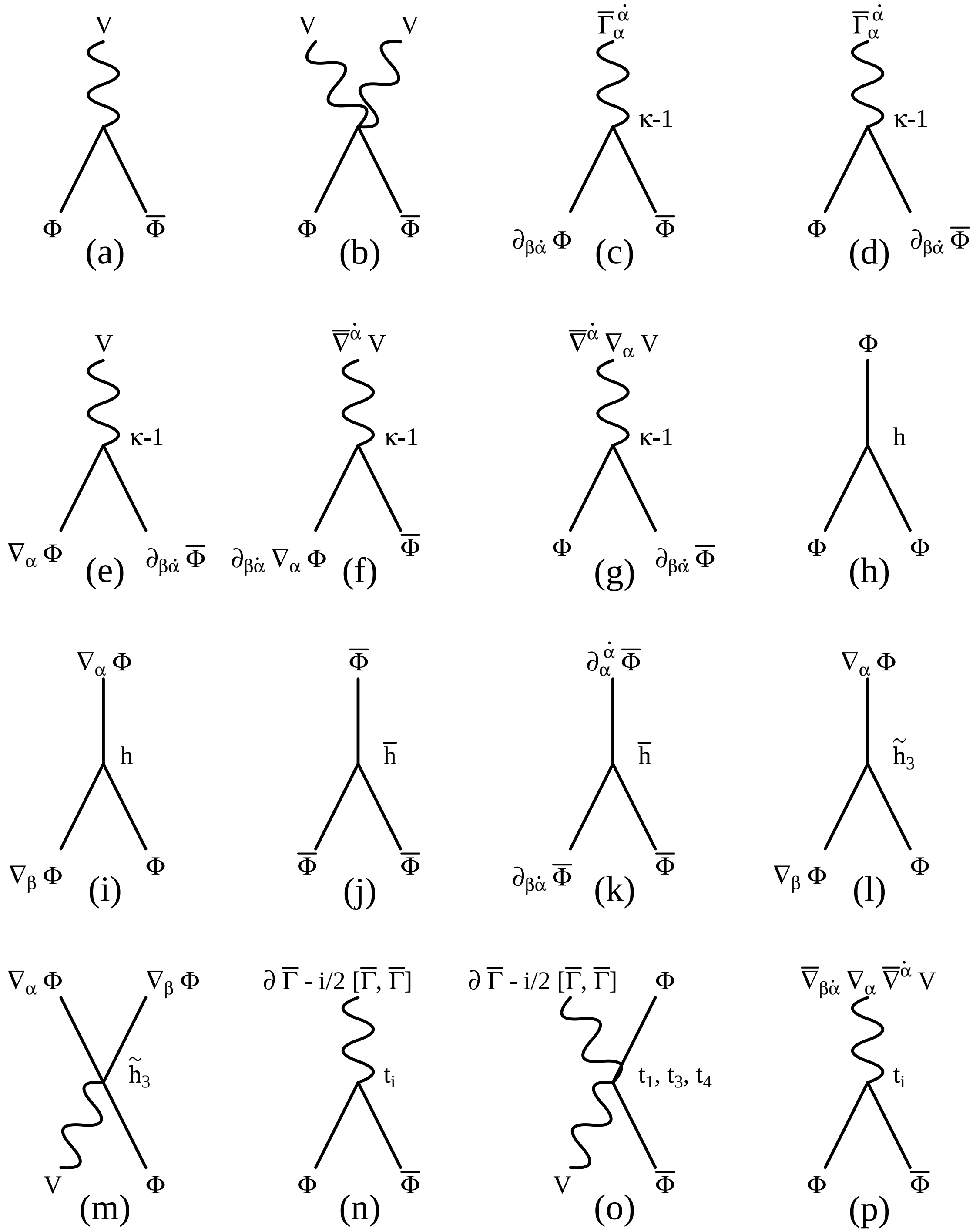

Figure B.1: Vertices from the action (4.3.124) at most linear in the NAC parameter $\mathcal{F}^{\alpha\beta}$. The (a,b,h,j)–vertices are order zero in $\bar{\theta}$, the (e)–vertex is proportional to $\bar{\theta}^{\dot{\alpha}}$ whereas the remaining vertices are all proportional to $\bar{\theta}^2$.

Working out the quadratic part of the action from (B.1.5) we find

$$S + S_{GF} \to -\frac{1}{2g^2}\int d^4x d^4\theta \; V * \hat{\Box} * V \tag{B.2.43}$$

where $\hat{\Box}$ has been defined in (B.1.11). We find convenient to rescale the gauge field as

$$V \to gV \tag{B.2.44}$$

Therefore, from the rescaled action we determine the covariant propagator

$$\langle V(z)V(z')\rangle \;=\; \frac{1}{\hat{\Box}}\delta^{(8)}(z-z') \tag{B.2.45}$$

where $z \equiv (x^{\alpha\dot{\alpha}}, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$.

Expanding this expression in powers of the background fields it turns out that the covariant propagator contains an infinite number of background–quantum interaction vertices. Precisely, we write

$$\frac{1}{\hat{\Box}} \simeq \frac{1}{\Box_{cov}} + \frac{1}{\Box_{cov}} * \left(i\widetilde{\mathbf{W}}^\alpha * \boldsymbol{\nabla}_\alpha + i\overline{\mathbf{W}}^{\dot{\alpha}} * \overline{\boldsymbol{\nabla}}_{\dot{\alpha}} \right) * \frac{1}{\Box_{cov}} + \cdots \tag{B.2.46}$$

and further expand $1/\Box_{cov}$. Since by direct inspection one can easily realize that terms proportional to $\widetilde{\mathbf{W}}^\alpha$ and $\overline{\mathbf{W}}^{\dot{\alpha}}$ never enter one–loop divergent diagrams, we approximate

$$\frac{1}{\hat{\Box}} \simeq \frac{1}{\Box_{cov}} \tag{B.2.47}$$

and study in detail its expansion.

As in (B.1.13), the action of $\Box_{cov}$ on a generic superfield in the adjoint representation of the gauge group reads

$$\begin{aligned} \Box_{cov} * \phi &\equiv \frac{1}{2}[\bar{\nabla}^{\alpha\dot{\alpha}}, [\bar{\nabla}_{\alpha\dot{\alpha}}, \phi]_*]_* \\ &= \Box\phi - i[\overline{\Gamma}^{\alpha\dot{\alpha}}, \partial_{\alpha\dot{\alpha}}\phi]_* - \frac{i}{2}\left[(\partial^{\alpha\dot{\alpha}}\overline{\Gamma}_{\alpha\dot{\alpha}}), \phi\right]_* - \frac{1}{2}\left[\overline{\Gamma}^{\alpha\dot{\alpha}}, \left[\overline{\Gamma}_{\alpha\dot{\alpha}}, \phi\right]_*\right]_* \end{aligned} \tag{B.2.48}$$

where $\Box = \frac{1}{2}\partial^{\alpha\dot{\alpha}}\partial_{\alpha\dot{\alpha}}$ is the ordinary scalar kinetic term and we specialized on the $U_*(1)$ case.

Expanding the $*$–products and neglecting terms which never enter our calculations we find

$$\Box_{cov} = \Box + 2i\mathcal{F}^{\alpha\beta}(\partial_\alpha\overline{\Gamma}^{\gamma\dot{\gamma}})\partial_\beta\partial_{\gamma\dot{\gamma}} - \mathcal{F}^2(\partial^\alpha\overline{\Gamma}^{\gamma\dot{\gamma}})\,(\partial^2\overline{\Gamma}_{\gamma\dot{\gamma}})\,\partial_\alpha + \mathcal{F}^2(\partial^\alpha\overline{\Gamma}^{\gamma\dot{\gamma}})\,(\partial_\alpha\overline{\Gamma}_{\gamma\dot{\gamma}})\,\partial^2 + ... \tag{B.2.49}$$

Inverting this expression we finally have

$$\begin{aligned} \frac{1}{\Box_{cov}} &= \frac{1}{\Box} \\ &- \frac{1}{\Box}2i\;\mathcal{F}^{\alpha\beta}(\partial_\alpha\overline{\Gamma}^{\gamma\dot{\gamma}})\;\partial_\beta\partial_{\gamma\dot{\gamma}}\frac{1}{\Box} - \frac{1}{\Box}4\;\mathcal{F}^{\alpha\beta}(\partial_\alpha\overline{\Gamma}^{\gamma\dot{\gamma}})\;\partial_\beta\partial_{\gamma\dot{\gamma}}\frac{1}{\Box}\;\mathcal{F}^{\eta\rho}(\partial_\eta\overline{\Gamma}^{\sigma\dot{\sigma}})\;\partial_\rho\partial_{\sigma\dot{\sigma}}\frac{1}{\Box} \\ &+ \frac{1}{\Box}\mathcal{F}^2\,(\partial^\alpha\overline{\Gamma}^{\gamma\dot{\gamma}})\,(\partial^2\overline{\Gamma}_{\gamma\dot{\gamma}})\,\partial_\alpha\frac{1}{\Box} - \frac{1}{\Box}\mathcal{F}^2\,(\partial^\alpha\overline{\Gamma}^{\gamma\dot{\gamma}})\,(\partial_\alpha\overline{\Gamma}_{\gamma\dot{\gamma}})\,\partial^2\frac{1}{\Box} + \ldots \end{aligned} \tag{B.2.50}$$

Here we recognize the ordinary bare propagator $1/\Box$ plus a number of gauge interaction vertices. We note that all the interactions are proportional to the NAC parameter, as a peculiar feature of the $U_*(1)$ theory.

Matter sector

In background field method we define *full* (anti)chiral superfields in the adjoint representation of the gauge group as

$$\overline{\Phi} = \overline{\mathbf{\Phi}} \qquad , \qquad \Phi = e_*^V * \mathbf{\Phi} * e_*^{-V} = e_*^V * (e_*^U * \phi * e_*^{-U}) * e_*^{-V} \tag{B.2.51}$$

where $\mathbf{\Phi} \equiv e_*^U * \phi * e_*^{-U}$ and $\overline{\mathbf{\Phi}}$ are *background* covariantly (anti)chirals.

Under both quantum (B.1.2) and background (B.1.3) transformations the full (anti)chiral superfields transform covariantly with parameters $\overline{\Lambda}$ and $\overline{\lambda}$, respectively.

Under quantum transformations background covariantly (anti)chiral fields transform as $\mathbf{\Phi}' = e_*^{i\Lambda} * \mathbf{\Phi} * e_*^{-i\Lambda}$, $\overline{\mathbf{\Phi}}' = e_*^{i\overline{\Lambda}} * \overline{\mathbf{\Phi}} * e_*^{-i\overline{\Lambda}}$. Under background transformations they both transform covariantly with parameter $\overline{\lambda}$, $\mathbf{\Phi}' = e_*^{i\overline{\lambda}} * \mathbf{\Phi} * e_*^{-i\overline{\lambda}}$, $\overline{\mathbf{\Phi}}' = e_*^{i\overline{\lambda}} * \overline{\mathbf{\Phi}} * e_*^{-i\overline{\lambda}}$.

Focusing the discussion on the $U_*(1)$ gauge group we now derive propagators and interaction vertices for matter in the actions (4.4.146, 4.4.147) where we have performed the rescaling (B.2.44). Since one–loop divergent contributions are at most quadratic in the NAC parameter, we list only Feynman rules entering these kinds of terms.

We split the actions (4.4.146, 4.4.147) according to

$$S \equiv S_{gauge} + S_{matter} = S_{gauge} + \int d^4x d^4\theta \ \bar{\Phi} * \Phi + S_{int} \tag{B.2.52}$$

where S_{gauge} is given in (B.1.5) and S_{int} is the rest of the matter actions in (4.4.146, 4.4.147) subtracted by the quadratic part.

We concentrate on S_{matter}. Its quantization proceeds as usual. We first expand the full covariant quadratic action in terms of background covariantly (anti)chiral fields (see (B.2.51))

$$\begin{aligned}&\int d^4x d^4\theta \ \ \bar{\mathbf{\Phi}} * e^{gV} * \mathbf{\Phi} * e^{-gV} \\ &= \int d^4x d^4\theta \ \left\{ \bar{\mathbf{\Phi}}\mathbf{\Phi} + g\bar{\mathbf{\Phi}}[V, \mathbf{\Phi}]_* + \frac{g^2}{2}\bar{\mathbf{\Phi}}[V,[V,\mathbf{\Phi}]_*]_* + \dots \right\}\end{aligned} \tag{B.2.53}$$

The first term in this expansion is the kinetic term for background covariantly (anti)chiral fields. In particular, ghosts fall in this category so the same procedure can be applied to the action (4.1.28), as well. The remaining terms give rise to ordinary interactions with the quantum field V.

We perform the quantum-background splitting

$$\mathbf{\Phi} \rightarrow \mathbf{\Phi} + \mathbf{\Phi}_q \qquad , \qquad \bar{\mathbf{\Phi}} \rightarrow \bar{\mathbf{\Phi}} + \bar{\mathbf{\Phi}}_q \tag{B.2.54}$$

which allows to rewrite

$$S_{matter} = \int d^4x d^4\theta \ \bar{\mathbf{\Phi}}_q \mathbf{\Phi}_q + S'_{int} \tag{B.2.55}$$

where S'_{int} collects all the interaction vertices coming from S_{int} after the splitting (B.2.54) plus the extra interactions from (B.2.53).

Adding source terms

$$\int d^4x d^2\theta \; j\mathbf{\Phi}_q + \int d^4x d^2\bar\theta \; \bar{\mathbf{\Phi}}_q \bar{j} \tag{B.2.56}$$

and performing the gaussian integral in $\mathbf{\Phi}_q, \bar{\mathbf{\Phi}}_q$, the quantum partition function reads

$$\mathbf{Z}[j,\bar{j}] = \Delta_* * e^{S'_{int}(\frac{\delta}{\delta j},\frac{\delta}{\delta \bar{j}})} \exp\left[-\frac{1}{2}\int d^4x d^4\theta \left(j * \frac{1}{\Box_-} * \bar{j} + \bar{j} * \frac{1}{\Box_+} * j\right)\right] \tag{B.2.57}$$

where $\Box_\pm$ are defined in (4.3.84) and Δ_* is the functional determinant

$$\Delta_* = \int \mathcal{D}\mathbf{\Phi}_q \mathcal{D}\bar{\mathbf{\Phi}}_q \; \exp \int d^4x d^4\theta \; \bar{\mathbf{\Phi}}_q \mathbf{\Phi}_q \tag{B.2.58}$$

From the generating functional (B.2.57) we have two types of perturbative contributions, one from the expansion of Δ_* and one from the expansion of $\exp(S'_{int})$.

As explained in [13, 14, 18], Δ_* provides an additional, one–loop contribution to the gauge effective action coming from matter/ghost loops. The corresponding Feynman rules can be worked out by applying the "doubling trick" procedure [13, 14, 18]. As a result, one–loop Feynman rules are obtained which can be formally read from the following effective action

$$\int d^4x d^4\theta \; \mathrm{Tr}\left\{\bar\xi \Box \xi + \frac{1}{2}\left[\bar\xi D^2(\overline{\mathbf{\nabla}}^2 - \overline{D}^2)\xi \;+\; \bar\xi(\Box_- - \Box)\xi\right]\right\} \tag{B.2.59}$$

where ξ , $\bar\xi$ are *unconstrained* quantum fields with ordinary scalar propagator

$$\langle \xi(z)\bar\xi(z')\rangle = -\frac{1}{\Box}\delta^{(8)}(z-z') \tag{B.2.60}$$

and the first vertex must appear once, and only once, in a one-loop diagram.

The second type of contributions come from the expansion of $\exp(S'_{int})$ in (B.2.57). The covariant matter propagators in this case are

$$\begin{aligned}\langle \mathbf{\Phi}(z)\bar{\mathbf{\Phi}}(z')\rangle &= -\frac{1}{\Box_-}\delta^{(8)}(z-z') \\ \langle \bar{\mathbf{\Phi}}(z)\mathbf{\Phi}(z')\rangle &= -\frac{1}{\Box_+}\delta^{(8)}(z-z')\end{aligned} \tag{B.2.61}$$

which can be expanded according to

$$\begin{aligned}\frac{1}{\Box_-} &\simeq \frac{1}{\Box_{cov}} + \frac{1}{\Box_{cov}} * \left(i\bar{W}^{\dot\alpha} * \bar\nabla_{\dot\alpha} + \frac{i}{2}(\bar\nabla^{\dot\alpha} * \bar{W}_{\dot\alpha})\right) * \frac{1}{\Box_{cov}} + \cdots \\ \frac{1}{\Box_+} &\simeq \frac{1}{\Box_{cov}} + \frac{1}{\Box_{cov}} * \left(i\widetilde{W}^{\alpha} * \nabla_{\alpha} + \frac{i}{2}(\nabla^{\alpha} * \widetilde{W}_{\alpha})\right) * \frac{1}{\Box_{cov}} + \cdots\end{aligned} \tag{B.2.62}$$

and contain an infinite number of interaction vertices between background gauge fields and quantum matter fields. As explained in the text, at one–loop divergent contributions arise only from the $\frac{1}{\Box_{cov}}$ part of the propagators. Therefore, we will set

$$\frac{1}{\Box_\pm} \simeq \frac{1}{\Box_{cov}} \tag{B.2.63}$$

and further expand it as done in (B.2.50).

Interaction vertices are obtained by working out the actual expression of S'_{int} after the background–quantum splitting (B.2.54). We list only the ones which effectively enter the evaluation of divergences. To keep the discussion more general we consider the three–flavor case. The one–flavor vertices are then easily obtained by dropping flavor indices and neglecting terms that, without flavors, vanish for symmetry reasons.

We begin by considering the contributions (B.2.53) coming from the quadratic action. The only contributing vertex is $(B.2a)$ in Fig. B.2 where V is quantum and $\mathbf{\Phi}$ or $\bar{\mathbf{\Phi}}$ are background. We then consider the t_1, t_2, t_3 interaction terms in (4.4.147). Because of the presence of a $\bar\theta^2$ the $*$–products are actually ordinary products. The quantization proceeds by performing the splitting (B.2.54) on the (anti)chirals and expanding the connections and the field strength as follows

$$\begin{aligned}
&\overline{\Gamma}_{\alpha\dot\alpha} \to -\nabla_\alpha e^{-V}\overline{\boldsymbol{\nabla}}_{\dot\alpha} e^V \to \overline{\Gamma}_{\alpha\dot\alpha} - \nabla_\alpha \left[\overline{\boldsymbol{\nabla}}_{\dot\alpha}, V\right]_* + \frac{1}{2}\nabla_\alpha\left[\left[\overline{\boldsymbol{\nabla}}_{\dot\alpha}, V\right]_*, V\right]_* \\
&\partial_{\beta\dot\alpha}\overline{\Gamma}_\alpha{}^{\dot\alpha} \longrightarrow \partial_{\beta\dot\alpha}\overline{\Gamma}_\alpha{}^{\dot\alpha} - \partial_{\beta\dot\alpha}\nabla_\alpha\left[\overline{\boldsymbol{\nabla}}^{\dot\alpha}, V\right]_* \\
&\overline{W}_{\dot\alpha} \to -i\nabla^2 e^{-V}\overline{\boldsymbol{\nabla}}_{\dot\alpha} e^V \to \overline{W}_{\dot\alpha} - i\nabla^2\left[\overline{\boldsymbol{\nabla}}_{\dot\alpha}, V\right]_* + \frac{i}{2}\nabla^2\left[\left[\overline{\boldsymbol{\nabla}}_{\dot\alpha}, V\right]_*, V\right]_*
\end{aligned} \tag{B.2.64}$$

Collecting only the contributions which may contribute at one–loop we obtain vertices $(B.2b, B.2d)$ where gauge is only background and vertex $(B.2c)$ where $\mathbf{\Phi}$ or $\bar{\mathbf{\Phi}}$ are background. We note that they all exhibit a gauge–invariant background dependence. We then turn to the pure matter interaction terms. By splitting (anti)chiral superfields we find vertices $(B.2f - B.2m)$.

Collecting all the results, the explicit expressions for the vertices are

$$
\begin{array}{ll}
(B.2a) & -2ig\,\bar\theta^{\dot\alpha}\mathcal{F}^{\alpha\beta}V(\partial_\alpha\Phi_i)\partial_{\beta\dot\alpha}\bar\Phi^i \\
(B.2b) & it_1\,\bar\theta^2\mathcal{F}^{\alpha\beta}(\partial_\alpha{}^{\dot\alpha}\bar\Gamma_{\beta\dot\alpha})\Phi_i\bar\Phi^i \\
(B.2c) & -igt_1\,\bar\theta^2\mathcal{F}^{\alpha\beta}(\partial_{\alpha\dot\alpha}D_\beta\bar D^{\dot\alpha}V)\Phi_i\bar\Phi^i \\
(B.2d) & t_2\,\bar\theta^2\mathcal{F}^2\bar\Gamma^{\alpha\dot\alpha}\bar\Gamma_{\alpha\dot\alpha}\bar\Phi_1\bar\Phi_2\bar\Phi_3 \\
(B.2e) & t_3\,\bar\theta^2\mathcal{F}^2\overline{W}^{\dot\alpha}\overline{W}_{\dot\alpha}\Phi_i\bar\Phi^i \\
(B.2f) & h_{12}\,\Phi_1\Phi_2\Phi_3-(h_1+h_2)\,\mathcal{F}^{\alpha\beta}\partial_\alpha\Phi_1\partial_\beta\Phi_2\Phi_3-\frac{1}{2}h_{12}\,\mathcal{F}^2\partial^2\Phi_1\partial^2\Phi_2\Phi_3 \\
(B.2g) & \bar h_{12}\,\bar\Phi_1\bar\Phi_2\bar\Phi_3-(\bar h_1+\bar h_2)\,\mathcal{F}^{\alpha\beta}\partial_\alpha\bar\Phi_1\partial_\beta\bar\Phi_2\bar\Phi_3 \\
(B.2h) & \tilde h_3\,\bar\theta^2\mathcal{F}^{\alpha\beta}\nabla_\alpha\Phi_1\nabla_\beta\Phi_2\Phi_3+\tilde h_3\,\bar\theta^2\mathcal{F}^2\nabla^2\Phi_1\nabla^2\Phi_2\Phi_3 \\
(B.2i) & h_3\,\bar\theta^2\mathcal{F}^2\nabla^2\Phi_1\nabla^2\Phi_2\Phi_3 \\
(B.2l) & h_4^{(=)}\,\bar\theta^2\mathcal{F}^2\nabla^2\Phi_i\Phi_i\bar\Phi^i\bar\Phi^i \quad ; \quad h_4^{(\neq)}\,\bar\theta^2\mathcal{F}^2\nabla^2\Phi_i\Phi_j\bar\Phi^i\bar\Phi^j \quad i<j \\
(B.2m) & h_5\,\bar\theta^2\mathcal{F}^2\Phi_i\bar\Phi^i\bar\Phi_1\bar\Phi_2\bar\Phi_3
\end{array}
\tag{B.2.65}
$$

We have not explicitly indicated background or quantum matter fields since it should be clear from the context. For instance, $\Phi_i\bar\Phi^i$ stands for $\Phi_i\bar\Phi^i_q$ or $(\Phi_i)_q\bar\Phi^i$.

We note that all vertices containing quantum gauge fields are at least of order $\mathcal{F}^{\alpha\beta}$. Hence vertices with quantum gauge fields and order $\mathcal{F}^2$ could be only employed in tadpole diagrams which vanish in dimensional regularization. This is the reason why in vertices $(B.2d, B.2e)$ we take gauge fields to be only background.

The expressions for the vertices of the one–flavor case can be obtained from the previous ones by dropping flavor indices and setting

$$
\begin{array}{lll}
h_1=-h_2=h/2 & , & \bar h_1=-\bar h_2=\bar h/2 \\
h_4^{(=)}=h_4 & , & h_4^{(\neq)}=0
\end{array}
\tag{B.2.66}
$$

Moreover, we need take into account extra symmetry factors that arise when specifying quantum or background matter. For instance, the term Φ^3 in $(B.2f)$ would give rise to $3\Phi^2\Phi_q$. The vertex $(B.2h)$ is absent for trivial symmetry reasons.

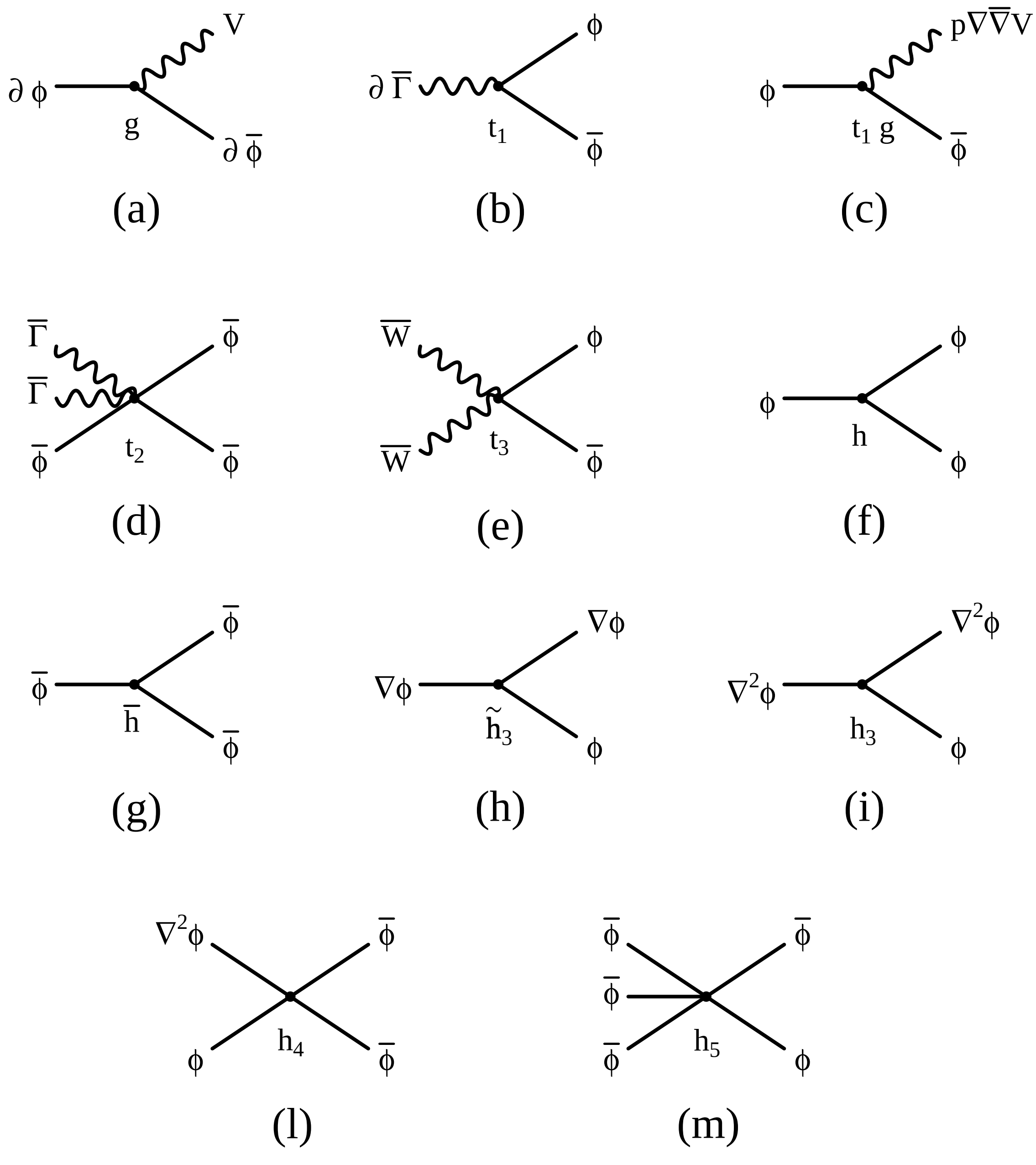

Figure B.2: Vertices from the actions (4.4.146, 4.3.88).

Appendix C

Details on supersymmetry breaking computations

C.1 The bounce action for a triangular barrier in four dimensions

As explained in the last part of this Thesis, a supersymmetric theory generically has both supersymmetric as well as nonsupersymmetric vacua. The former are the true, zero potential energy, vacua. The latter are metastable states which are unstable to decay into the supersymmetric ones. The lifetime of the false vacuum and its phenomenological applications strongly relies on the shape of the potential and the parameters of the theory and it is difficult to compute without some simplifying assumptions [125]. In the case of SQCD and its extensions, the potential can be well approximated with a triangular barrier as represented in figure C.1. The tunneling rate is estimated by using the trajectory of minimal energy connecting the two vacuum states. This is the bounce action. It is defined as the difference between the tunneling configuration and the metastable vacuum in the Euclidean action.

In the ISS model (and its extensions) the bounce action corresponds to the motion in the field space parametrized by the lowest component of a chiral multiplet. It is a one-dimensional motion in this space: The pseudomodulus evolves along the classical pseudo-flat direction until it reaches a local maximum of the one-loop potential; then, the motion takes place in the squark direction.

Therefore, it is sufficient to to estimate the bounce action for a single scalar field in four dimensions, from the false vacuum ϕ_F to the true vacuum ϕ_T. Because the motion is one-dimensional, the bounce action takes the form

$$S_E[\phi] = 2\pi^2 \int_0^\infty r^3 dr \left(\frac{1}{2}\dot{\phi}^2 + V(\phi) \right) \tag{C.1.1}$$

where the dot denotes differentiation with respect to r. The equation of motion is

$$\ddot{\phi} + \frac{3}{r}\dot{\phi} = V'(\phi) \tag{C.1.2}$$

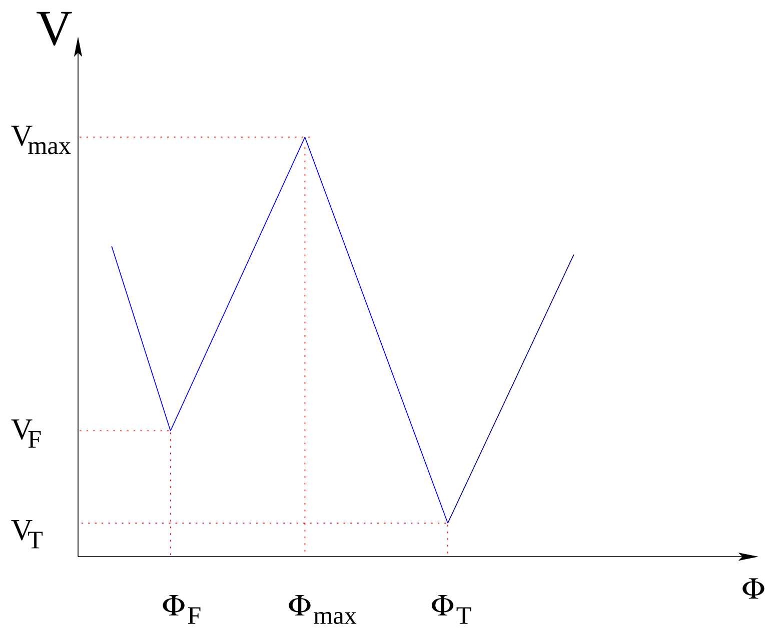

Figure C.1: Triangular potential barrier

where the prime denotes differentiation with respect to the field ϕ. The solution to this equation has to satisfy appropriate boundary conditions

$$\lim_{r\to\infty}\phi(r) = \phi_F \qquad\qquad \dot{\phi}(R_F) = 0 \tag{C.1.3}$$

which states that the field at large radius approaches the configuration of the false vacuum and that the equation of motion makes sense at the false vacuum. Once the solution is found, the bounce action B is computed as the difference between the action computed with the solution of (C.1.2) and the one computed with ϕ_F:

$$B = S_E[\phi(r)] - S_E[\phi_F] \tag{C.1.4}$$

It is then evident that for an arbitrary potential one has to look for a numerical solution. In the case of the triangular barrier the potential only depends upon the height and the width of the barrier itself. Moreover, it is helpful to define from them the gradient of the potential $V'(\phi)$

$$\begin{aligned}\lambda_F &= \frac{V_{max}-V_F}{\phi_{max}-\phi_F} \equiv \frac{\Delta V_F}{\Delta\phi_F}\\ \lambda_T &= -\frac{V_{max}-V_T}{\phi_T-\phi_{max}} \equiv -\frac{\Delta V_T}{\Delta\phi_T}\end{aligned} \tag{C.1.5}$$

The solution to (C.1.2) is found on either side of the barrier, and the two solutions are then matched at some radius $r + R_{max}$ to be determined.

Now we determine the boundary conditions. The field ϕ reaches the false vacuum ϕ_F at some finite radius R_F and stays there. Then

$$\phi(R_F) = \phi_F \qquad \dot{\phi}(R_F) = 0 \tag{C.1.6}$$

For the second boundary condition we have two possibilities. The first one is that the initial value ϕ_0 of the field at $r = 0$ is smaller than ϕ_T. In this case

$$\phi(0) = \phi_0 \qquad \dot{\phi}(0) = 0 \tag{C.1.7}$$

This case only occours for a certain range of ΔV and $\Delta\phi$. Otherwise the field remains close to ϕ_T until a radius R_0 at which it starts to evolve under its equation of motion. In this case we impose

$$\begin{aligned} \phi(r) &= \phi_T \qquad \text{for} \quad 0 < r < R_T \\ \phi(R_T) &= \phi_T \qquad \dot{\phi}(R_T) = 0 \end{aligned} \tag{C.1.8}$$

Let us first analyze conditions (C.1.7). The equation of motion has two solutions at the two sides of the barrier

$$\begin{aligned} \phi_R(r) &= \phi_0 - \frac{\lambda_T r^2}{8} \qquad &\text{for} \quad 0 < r < R_{max} \\ \phi_L(r) &= \phi^F_{max} + \frac{\lambda_F}{8r^2}\left(r^2 - R_F^2\right)^2 \qquad &\text{for} \quad R_{max} < r < R_F \end{aligned} \tag{C.1.9}$$

Matching the derivatives at R_{max} we find that R_F can be expressed in terms of R_{max}

$$R_F^4 = (1 + c)R_{max}^4 \tag{C.1.10}$$

where $c = -\lambda_T/\lambda_F$ only depends on the parameters of the potential. Matching the field values at R_{max} yelds an expression for R_{max}:

$$\begin{aligned} \phi_0 &= \phi_{max} + \frac{\lambda_T}{8} R_{max}^2 \\ \Delta\phi_F &= \frac{\lambda_F(\sqrt{1+c} - 1)^2}{8} R_{max}^2 \end{aligned} \tag{C.1.11}$$

All the unknown are determined. Integrating from $r = 0$ to $r = R_F$ we found the bounce action

$$B = \frac{32\pi^2}{3} \frac{1 + c}{(\sqrt{1+c} - 1)^4} \frac{\Delta\phi_F^4}{\Delta V_F} \tag{C.1.12}$$

For the second case the field remains at the true vacuum for some range of the euclidean radius before rolling. The solutions to the equation of motion are

$$\begin{aligned} \phi_R(r) &= \phi_T \qquad &\text{for} \quad 0 < r < R_T \\ \phi_R(r) &= \phi_T - \frac{\lambda_T}{8r^2}\left(r^2 - R_T^2\right)^2 \qquad &\text{for} \quad R_T < r < R_{max} \\ \phi_L(r) &= \phi_F - \frac{\lambda_F}{8r^2}\left(r^2 - R_F^2\right)^2 \qquad &\text{for} \quad R_{max} < r < R_F \end{aligned} \tag{C.1.13}$$

We have three unknowns to determine: R_T, R_{max} and R_F. Matching the derivatives at the top of the barrier we find

$$R_F^4 - R_{max}^4 = c(R_{max}^4 - R_0^4) \tag{C.1.14}$$

while matching the field values at R_{max}

$$\begin{aligned} \Delta\phi_T &= \frac{\lambda_T}{8R_{max}^2}\left(R_{max}^2 - R_T^2\right)^2 \\ \Delta\phi_F &= \frac{\lambda_F}{8R_{max}^2}\left(R_{max}^2 - R_T^2\right)^2 \end{aligned} \tag{C.1.15}$$

In order to express the unknowns in terms of the parameters of the potential, we define

$$\beta_F = \sqrt{\frac{8\Delta\phi_F}{\lambda_F}} \qquad \beta_T = \sqrt{\frac{8\Delta\phi_T}{\lambda_T}} \tag{C.1.16}$$

so that

$$R_T^2 = R_{max}^2 - \beta_T R_{max} \qquad R_F^2 = R_{max}^2 + \beta_F R_{max} \tag{C.1.17}$$

and

$$R_{max} = \frac{1}{2}\frac{\beta_F^2 + c\beta_T^2}{c\beta_F^2 - \beta_T^2} \tag{C.1.18}$$

Integrating the solution we obtain the bounce action

$$B = \frac{1}{96\pi^2}\lambda_F^2 R_{max}^3\left(-\beta_F^3 + 3c\beta_F^2\beta_T + 3\beta_F\beta_T^2 - c^2\beta_T^3\right) \tag{C.1.19}$$

If

$$\left(\frac{\Delta V_F}{\Delta V_T}\right)^{1/2} = \frac{2\Delta\phi_T}{\Delta\phi_T - \Delta\phi_F} \tag{C.1.20}$$

then (C.1.12) coinides with (C.1.19) and the bounce action reduces to

$$B = \frac{2\pi^2}{3}\frac{\left(\Delta\phi_F^2 - \Delta\phi_T^2\right)^2}{\Delta V_F} \tag{C.1.21}$$

C.2 The renormalization of the bounce action

In section 9.3.2 we analyzed the bounce action at the CFT exit scale. We distinguished the infrared bounce action $S_{B,IR}$ from $S_{B,UV}$, the action evaluated at the UV scale. Indeed in a supersymmetric field theory in the holomorphic basis the bounce action is obtained from the Lagrangian

$$\mathcal{L} = Z_\phi\dot{\phi}^2 + Z_\phi^{-1}V(\phi) \tag{C.2.22}$$

and we have

$$S_{B,IR} = S_{B,UV} Z_\phi^3 \tag{C.2.23}$$

Hence the bounce action undergoes non trivial renormalization. Here we show that our analysis, performed in the canonical basis, is consistent with (C.2.23), both for the ISS model and for the model in Section 9.3.2.

The ISS bounce action in the UV is

$$S_{B,UV} = \left(\frac{\mu_{UV}}{\tilde{\Lambda}_{UV}}\right)^{\frac{4\tilde{b}}{N_f-\tilde{N}}} \tag{C.2.24}$$

In the IR this action is renormalized because of the wave function renormalization of the fields. In section 9.3.2 we computed the action in the canonical basis and renormalization effects have been absorbed into the couplings. From (C.2.23) the IR renormalized action $S_{B,IR}$ is

$$S_{B,IR} = S_{B,UV} Z_N^3 \tag{C.2.25}$$

where the wave function renormalization is

$$Z_M = \left(\frac{E_{IR}}{E_{UV}}\right)^{-\gamma_N} \tag{C.2.26}$$

We now compute $S_{B,IR}$ and show that indeed it is (C.2.25). The coupling μ_{IR} and the scale $\tilde{\Lambda}_{IR}$ are given as functions of their UV values

$$\mu_{IR} = \mu_{UV} Z_N^{-1/4}, \qquad \tilde{\Lambda}_{IR} = \tilde{\Lambda}_{UV}\frac{E_{IR}}{E_{UV}} = \tilde{\Lambda}_{UV} Z_N^{-1/\gamma_N} \tag{C.2.27}$$

By substiting on the l.h.s. of (C.2.25) we have

$$S_{B,IR} = \left(\frac{\mu_{IR}}{\tilde{\Lambda}_{IR}}\right)^{\frac{4\tilde{b}}{N_f-\tilde{N}}} = \left(\frac{\mu_{UV} Z_N^{-1/4}}{\tilde{\Lambda}_{UV} Z_N^{-1/\gamma_N}}\right)^{\frac{4\tilde{b}}{N_f-\tilde{N}}} = S_{B,UV} Z_N^3 \tag{C.2.28}$$

where the last equality is obtained by substituting $\tilde{b} = 2N_f - 3N_c$ and $\gamma_N = 2\tilde{b}/N_f$. Nevertheless the bounce action in SQCD at the CFT exit scale results RG invariant. This is because the mass scales of the theory are related by the equation of motion of N. The relation between these scales is proportional to the gauge coupling which is constant during the running in the conformal window. For this reason the lifetime of the metastable vacuum cannot be parametrically large in SQCD.

In the model discussed in section 9.3.2 instead the bounce action depends non trivially on the relevant deformations

$$S_{B,IR} = \left(\frac{\tilde{\Lambda}_{IR}}{\rho_{IR}}\right)^{\frac{4N_f^{(2)}}{N_f^{(1)}-\tilde{N}}} \left(\frac{\mu_{IR}}{\tilde{\Lambda}_{IR}}\right)^{\frac{12\tilde{N}-4N_f^{(1)}}{N_f^{(1)}-\tilde{N}}} \tag{C.2.29}$$

The UV bounce action has the same expression but in term of the UV couplings and scale. The IR coupling and scale are related to the UV values as

$$\mu_{IR} = \mu_{UV} Z_N^{-1/4} \qquad \rho_{IR} = \rho_{UV} Z_N^{-\frac{\gamma_p}{\gamma_N}} \qquad \tilde{\Lambda}_{IR} = \tilde{\Lambda}_{UV} Z_N^{-\frac{1}{\gamma_N}} \tag{C.2.30}$$

The infrared bounce action is then

$$S_{B,IR} = S_{B,UV} Z_N^A \tag{C.2.31}$$

where

$$A = \frac{4N_f^{(2)}(\gamma_p - 1)}{(N_f^{(1)} - \tilde{N})\gamma_N} + \frac{(3\tilde{N} - N_f^{(1)})(4 - \gamma_N)}{(N_f^{(1)} - \tilde{N})\gamma_N} = 3 \tag{C.2.32}$$

The last equality can be obtained by substituting the relations $\gamma_{\phi_i} = 3R[\phi_i] - 2$, with $R[N] = 2y$ and $R[p] = (n - x + y)/n$. Hence we verified the general result (C.2.23) concerning the renormalization of the bounce action.

C.3 The bounce action for a triangular barrier in three dimensions

In this appendix we calculate the bounce action B for a triangular barrier in three dimensions (Figure C.1). The bounce action is the difference between the tunneling configuration and the metastable vacuum in the euclidean action. The tunneling rate of the metastable state is then given by $\Gamma = e^{-B}$

Following [126] we reduce to the case of a single scalar field with only a false vacuum ϕ_F decaying to the true vacuum, ϕ_T. The tunneling action is

$$S_E[\phi] = 4\pi \int_0^\infty r^2 dr \left(\frac{1}{2}\dot{\phi}^2 + V(\phi) \right) \tag{C.3.33}$$

where $\phi(r)$ is the tunneling solution, function of the Euclidean radius r. Solving the equation of motion for the ϕ field and imposing the boundary conditions

$$\begin{aligned} &\lim_{r\to\infty} \phi(r) = \phi_F \\ &\dot{\phi}(R_F) = 0 \end{aligned} \tag{C.3.34}$$

the bounce action is given by

$$B = S_E[\phi(r)] - S_E[\phi_F] \tag{C.3.35}$$

where we have subtracted the action for the field sitting at the false vacuum ϕ_F.

It is then helpful to define the gradient of the potential $V'(\phi)$ in terms of the parameters at the extremal points,

$$\begin{aligned} \lambda_F &= \tfrac{V_{max}-V_F}{\phi_{max}-\phi_F} \equiv \frac{\Delta V_F}{\Delta\phi_F} \\ \lambda_T &= -\tfrac{V_{max}-V_T}{\phi_T-\phi_{max}} \equiv -\frac{\Delta V_T}{\Delta\phi_T} \end{aligned} \tag{C.3.36}$$

where the first has positive sign and the second has negative one.

The solution of the equation of motion at the two side of the triangular barrier are solved by imposing the boundary conditions, and a matching condition at some radius $r + R_{max}$, that has to be determinate.

The choice of the boundary condition proceeds as follows. Firstly the field ϕ reaches the false vacuum ϕ_F at a finite radius R_F and stays there. This imposes

$$\begin{aligned} \phi(R_F) &= \phi_F \\ \dot{\phi}(R_F) &= 0 \end{aligned} \tag{C.3.37}$$

For the second boundary condition we work in the simplified situation such that the field has initial value $\phi_0 < \phi_T$ at radius $r = 0$. In this way we imposes the conditions

$$\begin{aligned} \phi(0) &= \phi_0 \\ \dot{\phi}(0) &= 0 \end{aligned} \tag{C.3.38}$$

Solving the equations of motion we have two solution at the two sides of the barrier

$$\phi_R(r) = \phi_0 - \frac{\lambda_T r^2}{6} \qquad \text{for} \qquad 0 < r < R_{max} \tag{C.3.39}$$

$$\phi_L(r) = \phi^F_{max} - \frac{\lambda_F R_F^2}{2} + \frac{\lambda_F R_F^3}{3r} + \frac{\lambda_F}{6} r^2 \qquad \text{for} \quad R_{max} < r < R_F \tag{C.3.40}$$

By matching the derivatives at R_{max} we are able to express R_F as a function of R_{max}

$$R_F^3 = (1+c) R_{max}^3 \tag{C.3.41}$$

where $c = -\lambda_T/\lambda_F$. Out of the value of the field at R_{max} we get two useful relations

$$\begin{aligned} \phi_0 &= \phi_{max} + \frac{\lambda_T}{6} R_{max}^2 \\ \Delta\phi_F &= \frac{\lambda_F(3 + 2c - 3(1+c)^{\frac{2}{3}})}{6} R_{max}^2 \end{aligned} \tag{C.3.42}$$

The bounce action B can be evaluated by integrating the equation (C.3.35) from $r = 0$ to $r = R_F$. We found

$$B = \frac{16\sqrt{6}\pi}{5} \frac{1+c}{(3+2c-3(1+c)^{2/3})^{3/2}} \sqrt{\frac{\Delta\phi_F^6}{\Delta V_F}} \tag{C.3.43}$$

There is a second possibility that holds if $\phi_0 > \phi_T$. In this case the field is closed to ϕ_0 from $R = 0$ until a radius R_0, and then it evolves outside. In this case the boundary conditions (C.3.38) become

$$\begin{aligned} \phi(r) &= \phi_T \qquad 0 < r < R_T \\ \phi(R_T) &= \phi_T \\ \dot{\phi}(R_T) &= 0 \end{aligned} \tag{C.3.44}$$

In this case the equations of motion are solved by

$$\begin{aligned} &\phi_R(r) = \phi_T && \text{for} \quad 0 < r < R_T \\ &\phi_R(r) = \phi_T + \frac{\lambda_T R_T^2}{2} - \frac{\lambda_T R_T^3}{3r} - \frac{\lambda_T r^2}{6} && \text{for} \quad R_T < r < R_{max} \\ &\phi_L(r) = \phi_F - \frac{\lambda_F R_F^2}{2} + \frac{\lambda_F R_F^3}{3r} + \frac{\lambda_F r^2}{6} && \text{for} \quad R_{max} < r < R_F \end{aligned} \tag{C.3.45}$$

By integrating these solution we can write the bounce action as

$$B = \frac{8\pi}{15} \lambda_F (R_F^3 \Delta\phi_T - c R_T^3 \Delta\phi_F) \tag{C.3.46}$$

where the relations among the unknowns and the parameters of the potential are

$$R_{max}^3 (1 + c) = R_F^3 + c R_T^3$$

$$\Delta\phi_T = \frac{(R_T - R_{max})^2 (2R_T + R_{max}) \lambda_T}{6 R_{max}} \tag{C.3.47}$$

$$\Delta\phi_F = \frac{(R_F - R_{max})^2 (2R_F + R_{max}) \lambda_F}{6 R_{max}} \tag{C.3.48}$$

We conclude by observing that in the limit $R_T = 0$ (C.3.46) coincides with (C.3.43).

C.4 Coleman-Weinberg formula in various dimensions

The CW formula for the one-loop superpotential

$$V_{eff}^{(1)} = \frac{1}{2} \,\mathrm{STr} \int \frac{d^d p}{(2\pi)^d} \ln(p^2 + m^2) \tag{C.4.49}$$

is not always straightforward to compute, since the theory can contain many fields, and one has to diagonalize the squared mass matrices. Some property of the models with metastable vacua can be analyzed without evaluating the eigenvalues of the squared mass matrices of component fields of the theory. To this purpose, we generalize a formula previously given for four-dimensional theories [137] to work in any dimension. Indeed, writing (C.4.49) in spherical coordinates and integrating by parts, we have

$$\begin{aligned} V_{eff}^{(1)} &= \frac{\pi^{d/2}}{\Gamma(d/2)} \,\mathrm{STr} \int \frac{dp}{(2\pi)^d} \, p^{d-1} \ln(p^2 + m^2) \\ &= -\frac{1}{d} \frac{1}{2^{d-1} \pi^{d/2} \Gamma(d/2)} \,\mathrm{STr} \int dp \, \frac{p^{d+1}}{p^2 + m^2} \end{aligned} \tag{C.4.50}$$

where $A_d = 2\pi^{d/2}/\Gamma(d/2)$ is the d-dimensional spherical surface. By substituting $d = 3$ we recover (9.4.88).

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