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POINT AND INTERVAL ESTIMATION FOR SOME FINANCIAL PERFORMANCE MEASURES

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SUMMARY

We study the estimators of three financial performance measures: the Sharpe Ratio, the Mean Difference Ratio and the Mean Absolute Deviation Ratio. The analysis is performed under two sets of assumptions. Firstly, the case of independent and identically normally distributed returns is considered. After that, relaxing the normality assumption, the case of independent and identically distributed returns is investigated. In both situations, we study the bias of the estimators and we propose their bias-corrected version. The exact and the asymptotic distribution of the three estimators is derived under the assumption of i.i.d.-normal returns. Concerning the case of i.i.d. returns, the asymptotic distributions of the estimators are provided. The latter distributions are used to define exact or large sample confidence intervals for the three indices. Finally, we perform a simulation study in order to assess the efficiency of the bias corrected estimator, the coverage accuracy and the length of the asymptotic confidence intervals.

Keywords: *financial performance measure, Sharpe Ratio, Mean Difference Ratio, Mean Absolute Deviation Ratio, concentration measures, Statistical analysis of financial data*

1. INTRODUCTION

Most of the known financial performance measures are defined as the ratio between a reward measure and a risk measure. For example, the well-known Sharpe Ratio, the Mean Difference Ratio* (MD Ratio) and the Mean Absolute Deviation Ratio (MAD Ratio) have this structure. For all these three indices, the reward measure is the expected excess return (beyond some risk-free rate); while the risk measures differ among the indices. In particular, the risk is measured by the standard deviation of returns in the Sharpe Ratio, by the Gini mean difference of returns in the MD Ratio and by the mean absolute deviation of returns in the MAD Ratio.

The MD Ratio and the MAD Ratio were proposed to overcome the criticism against the use of the standard deviation as risk measure. These criticisms

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*In literature, the Mean Difference Ratio is sometimes referred to as the Gini Ratio (see, e.g., Farinelli et al., 2008). In this paper we do not use this nomenclature in order to avoid confusion with well known Gini Concentration Ratio.

are motivated by the fact that the standard deviation is the natural risk measure only when the distribution of returns is Gaussian, a fact contradicted by empirical evidence which shows that the distributions of financial returns are characterized by fatter tails than the normal distribution and slight asymmetry.

In this paper we derive Confidence Intervals (CIs) for the three mentioned indices and we study the bias of their estimators. In detail, the paper is organized as follows. In Section 2. the definitions of Sharpe Ratio, MD Ratio and MAD Ratio are provided. In Section 3. we analyze the case of independent and identically normally distributed returns (i.i.d.-normal returns) and in Section 4. we consider the case of independent and identically distributed returns (i.i.d. returns). In Section 5. we describe the design of a simulation study performed in order to assess the coverage accuracy of the large sample CIs and in order to study the bias of the estimators of the three indices. In Section 6. and 7. we discuss the main results obtained in the simulations. Finally, Section 8. is devoted to the conclusions.

In the following, we denote with F_H, μ_H and σ_H^2 the distribution function, the expectation and the variance of the random variable (r.v.) H, respectively. The Gini's mean difference of H is denoted by Δ_H and the mean absolute deviation of H is denoted by δ_H. Finally, σ_{HK} stands for the covariance between the r.v.s H and K.

2. THE SHARPE RATIO, THE MD RATIO AND THE MAD RATIO

Let X be r.v. describing the return of a risky financial activity and let Y be the r.v. representing the return of the risk free financial activity. The r.v. $D = X - Y$ describes the excess return of the risky financial activity with respect to the risk free financial activity. The Sharpe Ratio (see Sharpe, 1966 and 1994) is given by:

$$\psi^* = \frac{\mu_D}{\sqrt{\sigma_D^2}} = \frac{\mu_X - \mu_Y}{\sqrt{\sigma_X^2 + \sigma_Y^2 + 2\sigma_{XY}}} \quad . \tag{1}$$

As pointed out in Sharpe (1994), this measure can be interpreted as the expected excess return per unit of risk where the risk is measured by the standard deviation of D.

Similarly, the MD Ratio (see Shalit and Yitzhaki, 1984) and the MAD Ratio (see Konno and Yamazaki, 1991) are, respectively, given by

$$\psi_\Delta^* = \frac{\mu_D}{\Delta_D} \qquad \text{and} \qquad \psi_\delta^* = \frac{\mu_D}{\delta_D} \quad .$$

The interpretation of these two performance measures are similar to that of ψ^*.

Even if the formula (1) is the definition of ψ proposed in Sharpe (1994), in literature the Sharpe Ratio is usually defined as

$$\psi = \frac{(\mu_X - \mu_Y)}{\sigma_X} \quad .$$

That because, theoretically, the risk free financial activity has a constant return. Therefore, Y is a degenerate r.v. on μ_Y and $\psi^* \equiv \psi$. According to the most part of the literature, here we adopt this last "simplified" definition of Sharpe Ratio and we assume that the *risk free rate* μ_Y is known.

A similar simplification can be made for the MD Ratio and the MAD Ratio. In particular, it is easy to verify that, for all $a \in \mathbb{R}$, $\Delta_{H+a} = \Delta_H$ and $\delta_{H+a} = \delta_H$. If Y is degenerate on μ_Y, it follows that $\Delta_D = \Delta_X$ and $\delta_D = \delta_X$. Consequently, the simplified definitions of the MD Ratio and of the MAD Ratio are

$$\psi_\Delta = \frac{(\mu_X - \mu_Y)}{\Delta_X} \qquad \text{and} \qquad \psi_\delta = \frac{(\mu_X - \mu_Y)}{\delta_X} \quad .$$

It can be noted that, under the assumption that μ_Y is a known constant, the just introduced simplified ratios depend only on the features of X. Then, to simplify the notation, in the following we denote the risk free rate by ξ and we drop the subscript X in σ_X^2, Δ_X, δ_X, and F_X.

3. THE CASE OF I.I.D.-NORMAL RETURNS

Let X be a normal r.v. with parameters μ and σ: $X \sim \mathcal{N}(\mu, \sigma^2)$. In this case the MD Ratio and the MAD Ratio can be expressed as a scale transformation of the Sharpe Ratio. In particular, it is well-known (see Johnson, Kotz, and Balakrishnan, 1995a) that

$$\Delta = \frac{2\sigma}{\sqrt{\pi}} \qquad \text{and} \qquad \delta = \sigma\sqrt{\frac{2}{\pi}} \quad . \tag{2}$$

It then follows that

$$\psi_\Delta = \frac{\sqrt{\pi}}{2}\psi \qquad \text{and} \qquad \psi_\delta = \sqrt{\frac{\pi}{2}}\psi \quad . \tag{3}$$

Let $X_1, X_2, ..., X_n$ be an i.i.d. sample from the normal distribution with parameters μ and σ^2. In order to estimate the Sharpe Ratio it is natural to use the plug-in estimator

$$\widehat{\Psi} = \frac{(\bar{X} - \xi)}{\sqrt{S^2}}$$

where $\bar{X}$ and S^2 denote the sample mean and the unbiased sample variance, respectively:

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \ ; \qquad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 \quad .$$

By the relations (3), the r.v.

$$\widehat{\Psi}^*_\Delta = \frac{\sqrt{\pi}}{2}\widehat{\Psi}$$

is an estimator of the MD Ratio. Similarly, the r.v.

$$\widehat{\Psi}^*_\delta = \sqrt{\frac{\pi}{2}}\widehat{\Psi}$$

is an estimator of the MAD Ratio.

The properties of $\widehat{\Psi}^*_\Delta$ and $\widehat{\Psi}^*_\delta$ can be easily deduced from those of $\widehat{\Psi}$. For this reason, in the remainder of this section, we will mainly focus on $\widehat{\Psi}$.

In order to obtain the exact distribution of the estimator $\widehat{\Psi}$, it is useful to observe that

$$\sqrt{n}\widehat{\Psi} = \frac{\sqrt{n}\frac{\bar{X}-\mu}{\sigma} + \sqrt{n}\psi}{\sqrt{\left(\frac{(n-1)S^2}{\sigma^2}\right)/(n-1)}} \quad . \tag{4}$$

From expression (4), it is clear that the r.v. $\sqrt{n}\widehat{\Psi}$ is defined as the ratio of two r.vs. The r.v. in the numerator is Gaussian with mean ($\sqrt{n}\psi$) and standard deviation 1. The r.v. in the denominator is the square root of a chi square r.v. divided by its degrees of freedom. Further, $\bar{X}$ and S^2 are independent since X is normally distributed (see Mood, Graybill, and Boes, 1974, theorem 8, p. 243). As a consequence, the r.v. in the numerator and that in the denominator of (4) are independent and the distribution of $\sqrt{n}\widehat{\Psi}$ is non-central t with $(n-1)$ degrees of freedom and non-centrality parameter $\sqrt{n}\psi$ (see Johnson, Kotz, and Balakrishnan, 1995b, ch. 31, p. 508).

REMARK 1 *It is worthwhile to note that the distribution of the estimator $\widehat{\Psi}$ depends on the parameter ψ and not on the particular values of μ and σ. In detail, let $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$. Suppose that*

$$\frac{\mu_1 - \xi}{\sigma_1} = \frac{\mu_2 - \xi}{\sigma_2} = \psi^* \quad .$$

In this case the distribution of the estimators $\widehat{\Psi}$ is non-central t with $(n-1)$ d.f. and non-centrality parameter $\sqrt{n}\psi^*$ whether we sample from X_1 or X_2. In financial terms: the estimator of the Sharpe ratio has the same distribution for all the financial activities belonging on the same capital market line[†].

It is now possible to define an exact CI for ψ. In detail, let $t_{\nu,p}(a)$ be the p-quantile of a non-central t distribution with ν degrees of freedom and non-centrality parameter a. Let $\widehat{\psi}$ be an estimate of ψ. The extremes of the $(1-\alpha)$-CI for ψ, denoted by $(\psi_-; \psi_+)$, are the following (see Casella and Berger (2002), ch. 9, p. 432 and Johnson, Kotz, and Balakrishnan (1995b), ch. 31, p. 510):

$$\begin{array}{ll} \psi_+ : & t_{(n-1),\frac{\alpha}{2}}(\sqrt{n}\psi_+) = \sqrt{n}\cdot\widehat{\psi} \\ \psi_- : & t_{(n-1),1-\frac{\alpha}{2}}(\sqrt{n}\psi_-) = \sqrt{n}\cdot\widehat{\psi} \quad . \end{array} \tag{5}$$

The last two equations cannot be analytically solved but their solutions can be easily numerically computed[‡].

[†]For a definition of *capital market line*, see Sharpe (1964).

[‡]The R or Matlab programs to numerically solve equations (5) are available upon request.

The CI just introduced improves the one based on the asymptotic distribution of $\widehat{\Psi}$ obtained in Jobson and Korkie (1981). In detail, in Jobson and Korkie (1981) it is shown that

$$\sqrt{n}(\widehat{\Psi} - \psi) \overset{a}{\sim} \mathcal{N}\left(0; \frac{2+\psi^2}{2}\right) \quad . \tag{6}$$

Form expression (6) the following asymptotic $(1-\alpha)$-CI can be obtained:

$$\left(\widehat{\Psi} - z_{1-\frac{\alpha}{2}}\sqrt{\frac{2+\widehat{\Psi}^2}{2n}}\ ;\ \widehat{\Psi} - z_{1-\frac{\alpha}{2}}\sqrt{\frac{2+\widehat{\Psi}^2}{2n}}\right) \quad . \tag{7}$$

Another aspect to consider in the estimation of ψ is the bias of $\widehat{\Psi}$. In Miller and Gher (1978) it is shown that the estimator $\widehat{\Psi}$ is biased and $E(\widehat{\Psi}) = \psi \cdot d$, where

$$d = \sqrt{\frac{n-1}{2}\frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}} \quad .$$

It is possible to prove that the bias factor d is greater than 1 for all $n > 2$. Then, the estimator $\widehat{\Psi}$ tends to overestimate (underestimate) ψ when the later is positive (negative). Further, the estimator $\widehat{\Psi}_u = d^{-1}\widehat{\Psi}$ is unbiased and it is more efficient than $\widehat{\Psi}$.

In Jobson and Korkie (1981), the following, easy to calculate, approximation of the bias factor d is given[§]:

$$d \approx d_1 = \left(1 + \frac{3}{4(n-1)} + \frac{25}{32(n-1)^2}\right) \quad . \tag{8}$$

A further approximation of the bias factor d can be derived as follows. First, note that

$$d = \frac{\sqrt{2(n-1)}}{(n-2)}\left(\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\right) \quad . \tag{9}$$

In Graham, Knuth, and Patashnik (1994) (see response to problems 9.60), it is shown that

$$\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} = \sqrt{\frac{n-1}{2}}\left(1 - \frac{1}{4(n-1)} + \frac{1}{32(n-1)^2} + O(n^{-3})\right) \tag{10}$$

and, from expressions (9) and (10), it follows that

$$d \approx d_2 = \frac{n-1}{n-2}\left(1 - \frac{1}{4(n-1)} + \frac{1}{32(n-1)^2}\right) \quad . \tag{11}$$

[§]It is worthwhile to note that, in realty, the approximation of the bias factor d given in the original article Jobson and Korkie (1981) contains, probably a typos. In particular the approximation given in Jobson and Korkie (1981) can be obtained from expression (8) replacing $(n-1)$ with n: $d \approx \left(1 + 3/(4n) + 25/(32n^2)\right)$. We also point out that this approximation is sometimes recalled in the just cited uncorrect version (see, for example, Knight and Satchell, 2005). Finally, we remark that the uncorrect approximation is less accurate than the correct one.

It is easy to check by direct computation that the approximation (11) is more accurate than approximation (8). Moreover, both d_1 and d_2 are greater than 1 for all $n > 2$ and, in more detail, $1 < d_1 < d_2 < d$ for all $n > 2$. As a consequence, the approximately unbiased estimators $\widehat{\Psi}_{u1} = d_1^{-1}\widehat{\Psi}$ and $\widehat{\Psi}_{u2} = d_2^{-1}\widehat{\Psi}$ are more efficient than $\widehat{\Psi}$. Naturally, $\widehat{\Psi}_u$ is more efficient than $\widehat{\Psi}_{u2}$ which is more efficient than $\widehat{\Psi}_{u1}$.

Concerning the MD ratio and the MAD ratio, it is possible to observe that an exact CI for ψ_Δ and ψ_δ can be obtained by multiplying the extremes of the CI for ψ by $\frac{\sqrt{\pi}}{2}$ and by $\sqrt{\frac{\pi}{2}}$, respectively. In detail, the exact CI for ψ_δ is

$$\left(\sqrt{\frac{\pi}{2}}\psi_-; \sqrt{\frac{\pi}{2}}\psi+\right), \tag{12}$$

while the exact CI for ψ_Δ is

$$\left(\frac{\sqrt{\pi}}{2}\psi_-; \frac{\sqrt{\pi}}{2}\psi_+\right) \quad . \tag{13}$$

Analogously, large sample CIs for ψ_Δ and ψ_δ can be obtained from the CI (7):

$$\left(\widehat{\Psi}^*_\delta - z_{1-\frac{\alpha}{2}}\sqrt{\frac{\pi + \widehat{\Psi}^{*2}_\delta}{2n}}\ ;\ \widehat{\Psi}^*_\delta + z_{1-\frac{\alpha}{2}}\sqrt{\frac{\pi + \widehat{\Psi}^{*2}_\delta}{2n}}\right) \quad , \tag{14}$$

$$\left(\widehat{\Psi}^*_\Delta - z_{1-\frac{\alpha}{2}}\sqrt{\frac{\pi}{4n} + \frac{\widehat{\Psi}^{*2}_\Delta}{2n}}\ ;\ \widehat{\Psi}^*_\Delta + z_{1-\frac{\alpha}{2}}\sqrt{\frac{\pi}{4n} + \frac{\widehat{\Psi}^{*2}_\Delta}{2n}}\right) \quad . \tag{15}$$

Finally, the estimators $\widehat{\Psi}^*_{\Delta u} = d^{-1}\widehat{\Psi}^*_\Delta$ and $\widehat{\Psi}^*_{\delta u} = d^{-1}\widehat{\Psi}^*_\delta$ are unbiased and more efficient than $\widehat{\Psi}^*_\delta$ and $\widehat{\Psi}^*_\delta$, respectively. Moreover, it is possible to introduce approximately unbiased estimators for the MD ratio and the MAD ratio using the approximations of the bias factor (8) and (11). Naturally, the usefulness of the approximations d_1 and d_2 is low since, nowadays, computers manage to calculate the values of the Γ function even for quite large values of its argument.

Also for the MD ratio and the MAD ratio we can observe that the features of their estimators $\widehat{\Psi}^*_\Delta$ and $\widehat{\Psi}^*_\delta$ remains unchanged if we consider different financial activities belonging on the same capital market line.

In Section 4. we show some additional asymptotic results concerning the MD ratio and the MAD ratio under the assumption of i.i.d.-Normal returns. The just cited results will be obtained as a special case of the more general results derived under the assumption of i.i.d. returns.

4. THE CASE OF I.I.D. RETURNS

The estimators for the MD Ratio and the MAD Ratio proposed in the previous section stem from the particular relations existing among the standard deviation, the Gini Mean Difference and the Mean Absolute Deviation of a normal r.v.. In this section, we do not specify a particular parametric model for F. Consequently, the estimators of ψ_Δ and ψ_δ cannot be defined starting from the estimators of the parameters of F. Then, we consider the following plug-in estimators for ψ_δ and ψ_Δ:

$$\widehat{\Psi}_\Delta = \frac{\bar{X} - \xi}{\hat{\Delta}} \qquad \text{and} \qquad \widehat{\Psi}_\delta = \frac{\bar{X} - \xi}{\hat{\delta}}$$

where

$$\hat{\Delta} = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} |X_i - X_j| \qquad \text{and} \qquad \hat{\delta} = \frac{1}{n} \sum_{i=1}^{n} |X_i - \bar{X}| \ .$$

The exact distributions of $\widehat{\Psi}$, $\widehat{\Psi}_\Delta$, and $\widehat{\Psi}_\delta$ cannot be derived because F is unknown. However, their limit distribution can be obtained. For that purpose, it is useful to note that the MD Ratio and the MAD Ratio are strictly related to two well-known concentration measures. In more detail, $\psi_\Delta = (2G)^{-1}$ and $\psi_\delta = (2P)^{-1}$ where $G = \Delta/(2\mu)$ and $P = \delta/(2\mu)$ are the Gini concentration ratio and the Pietra concentration ratio of the r.v. $X - \xi$. Consequently, several results concerning the estimators of G and P can be used in the study of $\widehat{\Psi}_\Delta$ and $\widehat{\Psi}_\delta$.

3.1 *Limit distribution of $\widehat{\Psi}$ and CI for ψ*

In order to derive the limit distribution of $\widehat{\Psi}$ the following well known result is necessary (see Serfling (1980), p. 114):

THEOREM 1 *Let $X_1, ..., X_n$ be an i.i.d. sample from F ad assume that $E[X^4] < \infty$. It follows that*

$$\sqrt{n} \begin{bmatrix} \bar{X} - \mu \\ S^2 - \sigma^2 \end{bmatrix} \xrightarrow{d} \mathbf{NB} \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} ; \begin{bmatrix} \sigma^2 & \mu_3 \\ \mu_3 & \mu_4 - \sigma^4 \end{bmatrix} \right) \tag{16}$$

where $\mu_k = E[(X - \mu)^k]$ and **NB** *means "bivariate normal".*

Starting from expression (16), Lo (2002) derived the limit distribution of $\widehat{\Psi}$ by the so called *delta-method* (see Serfling (1980), Theorem A in section 3.3, p. 122):

$$\sqrt{n}(\widehat{\Psi} - \psi) \xrightarrow{d} \mathcal{N}(0; V) \quad \text{where} \quad V = 1 - \frac{\mu_3}{\sigma^3}\psi + \left(\frac{\mu_4}{\sigma^4} - 1\right)\frac{\psi^2}{4} \ . \tag{17}$$

The variance V can be consistently estimated by

$$\widehat{V} = 1 - \frac{\hat{\mu}_3}{S^3}\widehat{\Psi} + \left(\frac{\hat{\mu}_4}{S^4} - 1\right)\frac{\widehat{\Psi}^2}{4}$$

where $\hat{\mu}_3$ and $\hat{\mu}_4$ denote the third and the fourth sample central moment:

$$\hat{\mu}_3 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^3 \ ; \qquad \hat{\mu}_4 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^4 \ .$$

Then, an asymptotic $(1-\alpha)$-CI for ψ is

$$\left(\widehat{\Psi} - z_{1-\frac{\alpha}{2}}\sqrt{\widehat{V}/n}\, ;\, \widehat{\Psi} + z_{1-\frac{\alpha}{2}}\sqrt{\widehat{V}/n}\right) \quad . \tag{18}$$

3.2 *Limit distribution of* $\widehat{\Psi}_\Delta$ *and CI for* ψ_Δ

In order to obtain the limit distribution of $\widehat{\Psi}_\Delta$, we recall the following result, due to Hoeffding (1948).

THEOREM 2 *Let* $X_1, ..., X_n$ *be an i.i.d. sample from* F *ad assume that* $E[X^2] < \infty$. *Then*

$$\sqrt{n}\begin{bmatrix} \bar{X}-\mu \\ \hat{\Delta}-\Delta \end{bmatrix} \xrightarrow{d} \mathbf{NB}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} ; \begin{bmatrix} \sigma^2 & \gamma \\ \gamma & \zeta^2 \end{bmatrix}\right) \tag{19}$$

where $\gamma = 2(\mathcal{D}-\mu\Delta)$, $\zeta^2 = 4(\mathcal{F}-\Delta^2)$ *and*

$$\mathcal{D} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x|x-y|dF(y)dF(x), \qquad \mathcal{F} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x-y||x-z|dF(z)dF(y)dF(x)\,.$$

In Hoeffding (1948) this result is used to derive the limit distribution of the Gini Concentration Ratio. Here, starting from (19), the limit distribution of $\widehat{\Psi}_\Delta$ is obtained by the *delta-method*:

$$\sqrt{n}(\widehat{\Psi}_\Delta - \psi_\Delta) \xrightarrow{d} \mathcal{N}(0; V_\Delta) \quad \text{where} \quad V_\Delta = \frac{\sigma^2}{\Delta^2} - 2\frac{\gamma}{\Delta^2}\psi_\Delta + \frac{\zeta^2}{\Delta^2}\psi_\Delta^2 \quad . \tag{20}$$

An unbiased and consistent estimator for ζ^2 is given by (see Zenga, Polisicchio, and Greselin, 2004):

$$\hat{\zeta}^2 = \frac{4n}{(n-2)(n-3)}\left[S^2 + (n-2)\hat{\mathcal{F}} - \frac{(2n-3)}{2}\hat{\Delta}^2\right]$$

where

$$\hat{\mathcal{F}} = \frac{1}{n(n-1)(n-2)}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n}|X_i - X_j||X_i - X_l| - \frac{2S^2}{n-2} \quad .$$

An unbiased and consistent estimator for γ is given by (see Johnson, Kotz, and Balakrishnan, 1995b):

$$\hat{\gamma} = 2\left[\hat{\mathcal{D}} - \bar{X}\hat{\Delta}\right] \qquad \text{where} \qquad \hat{\mathcal{D}} = \frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n} X_i|X_i - X_j| \quad .$$

So, the variance V_Δ can be consistently estimated by

$$\widehat{V}_\Delta = \frac{S^2}{\hat{\Delta}^2} - 2\frac{\hat{\gamma}}{\hat{\Delta}^2}\psi_\Delta + \frac{\hat{\zeta}^2}{\hat{\Delta}^2}\widehat{\Psi}_\Delta^2$$

and the following asymptotic $(1-\alpha)$-CI for ψ_Δ is obtained:

$$\left(\widehat{\Psi}_\Delta - z_{1-\frac{\alpha}{2}}\sqrt{\widehat{V}_\Delta/n}\,;\,\widehat{\Psi}_\Delta + z_{1-\frac{\alpha}{2}}\sqrt{\widehat{V}_\Delta/n}\right) \quad . \tag{21}$$

REMARK 2 *Under the additional assumption that X is normally distributed, the expression of the variance V_Δ can be substantially simplified. In detail, in Polisicchio and Zini (2000) it is shown that the covariance between the sample mean and the sample Gini mean difference is null when sampling from a symmetric distribution. Further, in Zenga, Polisicchio, and Greselin (2004) the expression of the functional $\mathcal{F}$ for the normal distribution is given:*

$$\mathcal{F} = \frac{\sigma^2}{3\pi}(\pi + 6\sqrt{3}) \quad . \tag{22}$$

Thanks to expression (22) and remembering the relation between σ and Δ recalled in (2), the variance of V_Δ results:

$$V_\Delta = \frac{\pi}{4} + \left(\frac{\pi + 6\sqrt{3} - 12}{3}\right)\psi_\Delta^2 \quad .$$

Consequently, under the assumption of i.i.d.-Normal returns the following asymptotic $(1-\alpha)$-CI for ψ_Δ can be introduced:

$$\left(\widehat{\Psi}_\Delta - z_{1-\frac{\alpha}{2}}\sqrt{\frac{\pi}{4n} + \left(\frac{\pi + 6\sqrt{3} - 12}{3n}\right)\widehat{\Psi}_\Delta^2}\;;\;\widehat{\Psi}_\Delta + z_{1-\frac{\alpha}{2}}\sqrt{\frac{\pi}{4n} + \left(\frac{\pi + 6\sqrt{3} - 12}{3n}\right)\widehat{\Psi}_\Delta^2}\right). \tag{23}$$

3.3 *Limit distribution of $\widehat{\Psi}_\delta$ and CI for ψ_δ*

The limit distribution of $\widehat{\Psi}_\delta$ can be obtained, once a time, by the *delta-method* starting from the following result, due to Gastwirth (1974).

THEOREM 3 *Let $X_1, ..., X_n$ be an i.i.d. sample from F ad assume that $E[X^2] < \infty$. Then*

$$\sqrt{n}\begin{bmatrix} \bar{X} - \mu \\ \hat{\delta}^2 - \delta \end{bmatrix} \xrightarrow{d} \mathbf{NB}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix};\begin{bmatrix} \sigma^2 & \kappa \\ \kappa & \upsilon^2 \end{bmatrix}\right)$$

where

$$\upsilon^2 = 4p^2\sigma^2 + 4(1-2p)\int_{-\infty}^{\mu}(x-\mu)^2 dF(x) - \delta^2 \ ,$$

$$\kappa = 2p\sigma^2 - 2\int_{-\infty}^{\mu}(x-\mu)^2 dF(x) \ , \qquad \textit{and} \qquad p = F(\mu).$$

Gastwirth (1974) used this result to obtain the limit distribution of the Pietra concentration ratio. In the same way we obtain the limit distribution of $\widehat{\Psi}_\delta$:

$$\sqrt{n}(\widehat{\Psi}_\delta - \delta) \xrightarrow{d} \mathcal{N}(0; V_\delta) \quad \text{where} \quad V_\delta = \frac{\sigma^2}{\delta^2} - 2\frac{\kappa}{\delta^2}\psi_\delta + \frac{\upsilon^2}{\delta^2}\psi_\delta^2 \quad . \tag{24}$$

A consistent estimator of υ^2 is given by $\hat{\upsilon}^2 = \left(4\hat{p}^2 S^2 + 4(1-2\hat{p})\hat{\mu}_2^- - \hat{\delta}^2\right)$ where

$$\hat{p} = \frac{1}{n}\sum_{i=1}^{n} I_i\,, \quad \hat{\mu}_2^- = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 I_i \quad \text{and} \quad I_i = \begin{cases} 1\,, & X_i < \bar{X} \\ 0\,, & \text{otherwise} \end{cases} .$$

A consistent estimator of κ is given by $\hat{\kappa} = \left(2\hat{p}S^2 - 2\hat{\mu}_2^-\right)$.
Consequently,

$$\widehat{V}_\delta = \frac{S^2}{\hat{\delta}^2} - 2\frac{\hat{\kappa}}{\hat{\delta}^2}\widehat{\Psi}_\delta + \frac{\hat{\upsilon}^2}{\hat{\delta}^2}\widehat{\Psi}_\delta^2$$

is consistent for V_δ and

$$\left(\widehat{\Psi}_\delta - z_{1-\frac{\alpha}{2}}\sqrt{\widehat{V}_\delta/n}\,;\, \widehat{\Psi}_\delta + z_{1-\frac{\alpha}{2}}\sqrt{\widehat{V}_\delta/n}\right) \tag{25}$$

is an asymptotic $(1-\alpha)$-CI for ψ_δ.

REMARK 3 *Under the further assumption of symmetry of F, it follows that $p = 1/2$ and $\int_{-\infty}^{\mu}(x-\mu)^2 dF(x) = \sigma^2/2$. Then, if F is symmetric, $\kappa = 0$ and $\upsilon^2 = \sigma^2 - \delta^2$. In this case $V_\delta = \sigma^2/\delta^2 + [(\sigma^2 - \delta^2)/\delta^2]\psi_\delta^2$. Moreover, if it is assumed that F is Gaussian, thanks to expressions (2), we obtain that $V_\delta = \pi/2 + [(\pi-2)/2]\psi_\delta^2$. As a consequence, under the assumption of i.i.d.-Normal returns the following asymptotic $(1-\alpha)$-CI can be used in place of that provided in Section 3.:*

$$\left(\widehat{\Psi}_\delta - z_{1-\frac{\alpha}{2}}\sqrt{\frac{\pi + (\pi-2)\widehat{\Psi}_\delta^2}{2n}}\;;\; \widehat{\Psi}_\delta + z_{1-\frac{\alpha}{2}}\sqrt{\frac{\pi + (\pi-2)\widehat{\Psi}_\delta^2}{2n}}\right) . \tag{26}$$

REMARK 4 *Note that it is necessary to assume the existence of the fourth moment of X in order to derive the limit distribution of $\widehat{\Psi}$ while it is only necessary to assume the existence of the second moment of X to obtain the limit distributions of $\widehat{\Psi}_\Delta$ and $\widehat{\Psi}_\delta$. This observation is important since empirical evidence suggests that the distribution of the returns of many financial activities may have infinite fourth moment (especially when high frequency returns are considered as shown in Genay et al., 2001). In these cases the CIs (18) cannot be used. This is a potential limitation of ψ.*

REMARK 5 *An observation analogous to that of Remark 1 can be made also in the context of the i.i.d. returns. In detail, if two financial activities have the same Sharpe Ratio (MD Ratio or MAD Ratio) and the distributions of their returns belong to the same location-scale family, then the limit distribution of the estimators of the two Sharpe Ratio (MD Ratio or MAD Ratio) is the same. To be more clear, let X_1 and X_2 be the r.v.s describing the returns of the two financial activities and denote all the objects relating to X_i by the subscript i (e.g., ψ_i is the Sharpe Ratio of X_i, $\mathcal{F}_i$ is the functional $\mathcal{F}$ associated to the distribution of X_i, and so on). Let $X_2 \stackrel{d}{=} kX_1 + h$ so that the distributions F_1*

and F_2 belong to the same location-scale family. It can be easily shown that, if $h = (1-k)\xi$ then $\psi_1 = \psi_2$ and $V_1 = V_2$. Then, asymptotically, $\widehat{\Psi}_1$ and $\widehat{\Psi}_2$ have the same distribution. Analogously, it can be shown that $\mathcal{D}_2 = k^2(\mathcal{D}_1 - \mu_1\Delta_1)$ and $\mathcal{F}_2 = k^2\mathcal{F}_1$. As a consequence, if $h = (1-k)\xi$ then $\psi_{\Delta 1} = \psi_{\Delta 2}$ and $V_{\Delta 1} = V_{\Delta 2}$ so that $\widehat{\Psi}_{\Delta 1}$ and $\widehat{\Psi}_{\Delta 2}$ have the same limit distribution. A similar result can be obtained also for the MAD Ratio since $\upsilon_2^2 = k^2\upsilon_1^2$ and $\kappa_2 = k^2\kappa_1$.

3.4 *Bias of the estimators $\widehat{\Psi}$, $\widehat{\Psi}_\Delta$, and $\widehat{\Psi}_\delta$*

The estimators $\widehat{\Psi}$, $\widehat{\Psi}_\Delta$, and $\widehat{\Psi}_\delta$ are biased. Unlike the case of i.i.d.-Normal returns, in this context is not possible to determine exactly their bias since F is unknown. Then, we approximate the expectation of $\widehat{\Psi}$, $\widehat{\Psi}_\Delta$, and $\widehat{\Psi}_\delta$ recalling the following Taylor series approximation of the function $f(x,y) = x/y$ around the point (a,b):

$$\frac{x}{y} \approx \frac{a}{b} + \frac{1}{b}(x-a) + \frac{a}{b^2}(y-b) - \frac{1}{b^2}(x-a)(y-b) + \frac{a}{b^3}(y-b)^2 \ . \tag{27}$$

Concerning the case of the estimator $\widehat{\Psi}$, let $(a,b) = (\mu - \xi, \sigma)$ and let $\bar{X}$ and S replace x and y, respectively, in expression (27). Taking expectation on both side of the resulting formula, we obtain an approximation of $E[\widehat{\Psi}]$:

$$\begin{aligned} E\left[\frac{\bar{X}-\xi}{S}\right] &\approx E\left[\psi + \frac{\bar{X}-\mu}{\sigma} + \psi\left(\frac{S-\sigma}{\sigma}\right) - \frac{(\bar{X}-\mu)(S-\sigma)}{\sigma^2} + \psi\left(\frac{S-\sigma}{\sigma}\right)^2\right] \\ &= \psi\left(2 - \frac{E[S]}{\sigma}\right) - \frac{E[(\bar{X}-\mu)(S-\sigma)]}{\sigma^2} \ . \end{aligned} \tag{28}$$

The above expression depends on the unknown expectations $E[S]$ and $E[(\bar{X} - \mu)(S-\sigma)]$ which can be approximated using, once again, a Taylor series expansion. In detail, an approximation of $E[S]$ can be derived starting from the following Taylor approximation:

$$\sqrt{y} \approx \sqrt{b} + \frac{1}{2\sqrt{b}}(y-b) - \frac{1}{8\sqrt{b^3}}(y-b)^2 \ . \tag{29}$$

Let $b = \sigma^2$ and let S^2 replace y in (29). Taking expectation on both side of the resulting expression and remembering that (see Johnson, Kotz, and Balakrishnan, 1995b)

$$E[(S^2-\sigma^2)^2] = \frac{1}{n}\left(\mu_4 - \frac{n-3}{n-1}\sigma^4\right) \ ,$$

we obtain

$$E[S] \approx \sigma - \frac{1}{8\sigma^3}E\left[\left(S^2-\sigma^2\right)^2\right] = \sigma\left[1 - \frac{1}{8n}\left(\frac{\mu_4}{\sigma^4} - 3\right) - \frac{1}{4(n-1)}\right] \ . \tag{30}$$

In order to approximate the value of $E[(\bar{X}-\mu)(S-\sigma)]$, we first observe that

$$E[(\bar{X}-\mu)(S-\sigma)] = E[\bar{X}S] - \mu E[S] = \sigma_{\bar{X}S} \ .$$

Further, we recall the following Taylor series approximation:

$$x\sqrt{y} \approx a\sqrt{b} + \sqrt{b}(x-a) + \frac{a}{2\sqrt{b}}(y-b) + \frac{1}{2\sqrt{b}}(x-a)(y-b) - \frac{a}{8\sqrt{b^3}}(y-b)^2 \ . \quad (31)$$

Let $(a, b) = (\mu, \sigma^2)$ and let $\bar{X}$ and S^2 replace x and y, respectively, in (31). After taking expectation on both sides, the following expression is obtained:

$$E[\bar{X}S] \approx \frac{\mu_3}{2n\sigma} + \mu\left(\sigma - \frac{1}{8\sigma^3}E\left[\left(S^2-\sigma^2\right)^2\right]\right) \ .$$

Consequently

$$\sigma_{\bar{X}S} \approx \frac{\mu_3}{2n\sigma} \ . \quad (32)$$

Finally, plugging (30) and (32) into (28), we obtain

$$E\left[\widehat{\Psi}\right] \approx \psi\left[1 + \frac{1}{4(n-1)} + \frac{1}{8n}\left(\frac{\mu_4}{\sigma^4} - 3\right)\right] - \frac{1}{2n}\frac{\mu_3}{\sigma^3} \ . \quad (33)$$

In a similar way, the following approximations are derived:

$$E[\widehat{\Psi}_\Delta] \approx \psi_\Delta\left(1 + \frac{\zeta^2}{n\Delta^2}\right) - \frac{\gamma}{n\Delta^2} \ ; \qquad E[\widehat{\Psi}_\delta] \approx \psi_\delta\left(1 + \frac{\upsilon^2}{n\delta^2}\right) - \frac{\kappa}{n\delta^2} \ . \quad (34)$$

From (33) and (34) it turn out that the estimators

$$\widehat{\Psi}'_u = \left(\widehat{\Psi} + \frac{1}{2n}\frac{\hat{\mu}_3}{S^3}\right)\left[1 + \frac{1}{4(n-1)} + \frac{1}{8n}\left(\frac{\hat{\mu}_4}{S^4} - 3\right)\right]^{-1} \ , \quad (35)$$

$$\widehat{\Psi}_{\Delta u} = \left(\widehat{\Psi}_\Delta + \frac{\hat{\gamma}}{n\hat{\Delta}^2}\right)\left(1 + \frac{\hat{\zeta}^2}{n\hat{\Delta}^2}\right)^{-1}, \ \text{and} \ \widehat{\Psi}_{\delta u} = \left(\widehat{\Psi}_\delta + \frac{\hat{\kappa}}{n\hat{\delta}^2}\right)\left(1 + \frac{\hat{\upsilon}^2}{n\hat{\delta}^2}\right)^{-1} \quad (36)$$

are approximatively unbiased for ψ, ψ_Δ, and ψ_δ, respectively. However, the variability of the just introduced bias-corrected estimators could be much higher than the variability of uncorrected ones, because of the variability of the estimators S, $\hat{\mu_3}$, $\hat{\mu_4}$, $\hat{\gamma}$, $\hat{\Delta}$, $\hat{\zeta}^2$, $\hat{\kappa}$, $\hat{\delta}^2$, and $\hat{\upsilon}^2$. So, at this time, it is not possible to assert that the approximately bias-corrected estimators are more/less efficient than the plug-in estimators previously introduced.

2. DESIGN OF THE SIMULATION STUDY

In order to assess the coverage accuracy and the length of he large sample CIs and in order to compare the efficiency of the approximatively unbiased estimators with that of the plug- in estimators, we built a wide simulation study. Several scenarios are considered both concerning the case of i.i.d.-Normal returns and the case of i.i.d. returns. In each scenario, the coverage accuracy of the large sample CIs is evaluated estimating the actual coverage of the CI by the proportion of simulated CIs containing the true value of the ratio. Further, the simulated Average Length (AL) of the different CIs is computed. Finally, in order to compare the efficiency of the approximately bias-corrected estimators and the uncorrected ones, we calculate in each scenario the simulated MSE and the simulated bias of the two kind of estimators.

The differences among the scenarios considered concerns: the distribution type of the r.v. X (12 different distributions), the standard deviation σ of the r.v. X (3 different values), the value of ψ (3 different values), the sample size (4 different values), the nominal coverage (4 different values). Globally, $(12 \times 3 \times 3 \times 4 \times 4) = 1728$ scenarios are investigated.

In detail, for each of the different distributions considered, the design of the simulation study is described below:

- sample sizes: 50, 100, 200, 400;
- number of replications: 10^4;
- nominal coverages of the large sample CIs: 0.9, 0.95, 0.975, 0.99;
- value of the (daily) risk free rate: 0.000068 (which correspond to an annual rate of return of about 2.5%);
- values of the standard deviation of X: $\sigma = 0.01$, $\sigma = 0.05$, $\sigma = 0.1$. The different values of σ are chosen coherently with the values of the standard deviation of the daily returns of the equities of the S&P 100 in the period 2005-2007.
- values of the Sharpe Ratio: $\psi_1 = 0.05$, $\psi_2 = 0.25$, $\psi_3 = 0.5$. In practice, for each value of σ, three different value of μ are considered. The first value of μ gives rise to a Sharpe Ratio of 0.05, the second gives rise to $\psi = 0.25$, finally, the third gives rise to $\psi = 0.5$. As for the values of σ, the different values of ψ are chosen on the base of the daily Sharpe Ratios of the equities in the S&P 100 calculated from the returns of the period 2005-2007. The values of ψ_δ and ψ_Δ change among the scenarios coherently with the particular value of (μ, σ) and the distribution shape.

As regards the distributions, under the hypotheses of i.i.d.-Normal returns, trivially, we evaluate the properties of the large sample CIs introduced in Section 3. and of the CIs (26) and (23), sampling from a normal distribution. Under the assumption of i.i.d. returns we investigate the features of the CIs (18), (21), and (25) and the performances of the estimators (35) and (36) sampling from the following distributions: Normal (**N**), Laplace (**L**), Student's t with 5 ($\mathbf{t}_5$) and 3 ($\mathbf{t}_3$) degrees of freedom, Skew Normal with low and high degree of positive and negative asymmetry (**SNL**$\pm$, **SNH**$\pm$), Skew t with 5 d.f. and with low and high degree of positive and negative asymmetry (**StL**$\pm$, **StH**$\pm$). The normal distribution is taken into account also under the setting of i.i.d. returns because it is a useful basis for the comment of the results obtained sampling from the others distributions. The Laplace distribution and the Student's t with 5 degrees of freedom are taken into account because they have fatter tails than the normal. In detail, the Laplace distribution has higher kurtosis than the normal distribution but it possess all the moments. On the contrary, the

Distribution	γ_1	γ_2	Distribution	γ_1	γ_2
Normal	0	0	Laplace	0	3
t with 3 d.f.	0	∞	t with 5 d.f.	0	6
Skew N. (High as.)	± 0.6670	0.5098	Skew N. (Low as.)	± 0.4538	0.3051
Skew t (High as.)	± 1.0758	8.9208	Skew t (Low as.)	± 0.5527	6.8020

TABLE 1: Values of γ_1 and γ_2 for the 12 distributions considered. The values relating the Skew t distribution are obtained by numerical integration.

Student's t distribution with 5 d.f. has fatter tails than the normal distribution and it does not possess moments of order higher than 4. The Student's t with 3 degrees of freedom is investigated in order to evaluate the impact of the non-existence of the fourth moments on the properties of the large sample CI for the Sharpe ratio (see Remark 4). Further, we chose the Skew Normal distribution (see Azzalini and Capitanio, 1999) in order to evaluate the impact of the asymmetry on the performances of the large sample CIs. Four different parameters setting of the Skew Normal distribution are taken into consideration: the first (third) setting is characterized by a low degrees of positive (negative) asymmetry (shape parameter equal to ± 2), while the second (fourth) is characterized by an higher degrees of positive (negative) asymmetry (shape parameter equal to ± 3). Finally we consider the Skew t distribution with 5 d.f. (see Azzalini and Capitanio, 2003) because this last distribution shows, at the same time, fat tails (only the moments up to order 4 exist) and asymmetry. As for the Skew Normal distribution, we investigate four different parameters setting: the first (third) is associated to a low degrees of positive (negative) asymmetry (shape parameter equal to ± 0.5), the second (fourth) to an higher degrees of positive (negative) asymmetry (shape parameter equal to ± 1). In Table 1 we give the values of the third standardized moment γ_1 (which is usually interpreted as an index of asymmetry) and the excess kurtosis γ_2 (which is commonly used in order to measure the kurtosis and it is defined as the fourth standardized moment minus 3) associated to each of the distribution considered.

6. RESULTS: I.I.D.-NORMAL RETURNS

In the following we do not give all the detailed results obtained in the simulation study because they require excessive space. Only the more interesting results are given and discussed. However, the detailed results are available upon request. We discuss first the results obtained under the assumption of i.i.d.-Normal returns and later those obtained under the i.i.d. assumption.

First, we analyze the coverage accuracy of the large sample CIs (7), (14), (15), (26), and (23). As explained in Remark 1, the features of the estimators $\widehat{\Psi}$, $\widehat{\Psi}^*_\delta$, $\widehat{\Psi}^*_\Delta$ does not depends on the value of σ but only on the the sample size n and on the true values of ψ, ψ_δ, and ψ_Δ, respectively. Simulations shows that an analogous result holds also for the estimators $\widehat{\Psi}_\delta$ and $\widehat{\Psi}_\Delta$. Then, in Table 2 we give the averages over the different values of σ of the simulated coverages. The simulated coverage of the exact CIs are also given in order to take into account of the variability due to the simulation. The main result is that the actual

100(1 − α)%	ψ_1				ψ_2				ψ_3			
	90	95	97.5	99	90	95	97.5	99	90	95	97.5	99
CIs					n=50							
(5),(12),(13)	89.99	94.97	97.49	99.00	89.99	95.03	97.53	99.02	90.01	95.00	97.49	99.03
(7),(14),(15)	89.82	94.84	97.42	98.97	89.81	94.91	97.46	98.97	89.87	94.90	97.43	99.01
(26)	89.64	94.79	97.34	98.95	89.64	94.83	97.42	98.96	89.69	94.88	97.40	99.00
(23)	89.99	94.96	97.47	99.00	89.94	95.02	97.54	99.03	89.96	95.01	97.51	99.05
					n=100							
(5),(12),(13)	89.93	94.97	97.55	99.06	89.89	94.96	97.55	99.03	89.79	94.95	97.52	99.05
(7),(14),(15)	89.85	94.93	97.51	99.04	89.80	94.91	97.51	99.01	89.75	94.90	97.50	99.02
(26)	89.75	94.84	97.50	99.03	89.70	94.86	97.44	99.01	89.73	94.91	97.42	99.03
(23)	89.91	94.96	97.58	99.07	89.90	94.97	97.53	99.03	89.82	94.99	97.54	99.06
					n=200							
(5),(12),(13)	89.91	95.01	97.56	98.98	89.87	95.04	97.55	99.00	89.80	95.05	97.57	99.01
(7),(14),(15)	89.85	94.97	97.54	98.97	89.83	95.02	97.53	98.99	89.78	95.02	97.56	99.00
(26)	89.86	94.92	97.54	98.95	89.83	95.04	97.54	98.98	89.78	95.05	97.57	99.00
(23)	89.92	95.02	97.56	98.97	89.88	95.06	97.57	98.99	89.84	95.05	97.58	99.00
					n=400							
(5),(12),(13)	89.88	95.05	97.59	99.04	89.93	95.05	97.56	99.02	89.96	95.05	97.51	99.05
(7),(14),(15)	89.87	95.03	97.58	99.03	89.92	95.04	97.55	99.02	89.94	95.04	97.50	99.06
(26)	89.86	95.00	97.59	99.01	89.91	95.01	97.55	99.03	89.93	94.98	97.49	99.05
(23)	89.88	95.05	97.59	99.03	89.93	95.03	97.55	99.04	89.97	95.01	97.53	99.05

TABLE 2: Simulated coverage of the exact and large sample CIs for the Sharpe Ratio, MAD Ratio and MD Ratio under the assumption of i.i.d.-Normal returns.

coverage of all the large sample CIs is quite similar to the simulated coverage of the exact CI in all the scenarios considered. Then, a relatively small sample size of 50 is sufficient in order to assure a good coverage accuracy for the CIs (7), (14), (15), (26), and (23).

Concerning the length of the CIs, the simulated average lengths are given in Table 3. Also in this case, the value of σ does not significantly influence the results (as suggested by the Remark 1) and, consequently, in Table 3, the averages over the different values of σ are given. In detail, the first part of Table 3 contains only the simulated average length of the exact CI for the Sharpe Ratio since the average lengths of the exact CIs for the MAD Ratio and for the MD Ratio can be obtained multiplying these latter values by $\sqrt{\pi/2}$ and $\sqrt{\pi}/2$, respectively. The length of the large sample CIs is evaluated calculating the percentage relative variation of the lengths of the large sample CIs with respect to the length of the exact ones:

$$100\left(\frac{\text{average length of the exact CI} - \text{average length of the large sample CI}}{\text{average length of the exact CI}}\right)\% \ .$$

The percentage relative variation of the CIs (7), (14), and (15) is the same because they differ only in scale. The simulations show that the large sample CIs and the exact ones have quite similar average length. Indeed, all the percentage relative variations are, in absolute value, lower than 1% and, in the most cases, they are lower than 0.1%. Concluding, under the assumption of i.i.d.-normal returns, it is possible to assert that a sample size of 50 is sufficient in order to assure that all the large sample CIs investigated approximate very well the exact CIs.

	ψ_1				ψ_2				ψ_3			
$100(1-\alpha)\%$	90	95	97.5	99	90	95	97.5	99	90	95	97.5	99
n	Exact CI for the Sharpe Ratio: (5)											
50	0.46800	0.55766	0.63773	0.73289	0.47523	0.56628	0.64759	0.74422	0.49711	0.59234	0.67740	0.77848
100	0.33001	0.39325	0.44971	0.51681	0.33503	0.39920	0.45652	0.52463	0.35018	0.41724	0.47716	0.54835
200	0.23306	0.27770	0.31757	0.36497	0.23658	0.28186	0.32238	0.37047	0.24717	0.29452	0.33685	0.38709
400	0.16469	0.19624	0.22442	0.25790	0.16715	0.19918	0.22778	0.26175	0.17460	0.20809	0.23794	0.27343
n	Large sample CIs based on the asymptotic distribution (6): (7)-(14)-(15)											
50	0.0088%	0.0088%	0.0089%	0.0090%	0.0309%	0.0310%	0.0312%	0.0316%	0.0920%	0.0921%	0.0925%	0.0929%
100	-0.0001%	0.0027%	0.0026%	0.0025%	0.0176%	0.0134%	0.0133 %	0.0134%	0.0482%	0.0440%	0.0437%	0.0437%
200	-0.0001%	-0.0014%	-0.0037%	0.0002%	0.0092%	-0.0032%	0.0107%	0.0069 %	0.0163%	0.0144%	0.0260%	0.0227%
400	0.0000%	-0.0002%	-0.0007%	-0.0014%	0.0019%	0.0054%	0.0065%	-0.0016%	-0.0013%	0.0191%	0.0074%	0.0037%
n	Large sample CI for the MAD Ratio: (26)											
50	-0.0826%	-0.0826%	-0.0826%	-0.0824%	-0.2903%	-0.2902%	-0.2900%	-0.2895%	-0.8617%	-0.8617%	-0.8613%	-0.8609%
100	-0.0472%	-0.0444%	-0.0445%	-0.0446 %	-0.2461%	-0.2503%	-0.2504 %	-0.2503%	-0.8143 %	-0.8185%	-0.8188%	-0.8188%
200	-0.0277 %	-0.0290%	-0.0312%	-0.0274%	-0.2301%	-0.2426%	-0.2286%	-0.2325%	-0.8087%	-0.8105%	-0.7988%	-0.8022%
400	-0.0179%	-0.0180%	-0.0185%	-0.0193%	-0.2243%	-0.2208%	-0.2197%	-0.2279%	-0.8042%	-0.7837%	-0.7955%	-0.7992%
n	Large sample CI for the MD Ratio: (23)											
50	0.0011%	0.0011%	0.0011%	0.0012%	0.0039%	0.0040%	0.0043%	0.0047%	0.0125%	0.0125%	0.0129%	0.0133%
100	-0.0057%	-0.0029%	-0.0030%	-0.0031%	-0.0137%	-0.0178%	-0.0179%	-0.0179%	-0.0543%	-0.0584%	-0.0587%	-0.0587%
200	-0.0038%	-0.0052%	-0.0075%	-0.0037%	-0.0239%	-0.0364%	-0.0224%	-0.0263%	-0.0985%	-0.1004%	-0.0887 %	-0.0921%
400	-0.0026%	-0.0028%	-0.0033%	-0.0041%	-0.0317%	-0.0283%	-0.0272%	-0.0353%	-0.1212%	-0.1008%	-0.1125%	-0.1169%

TABLE 3: Simulated average length of the exact CIs for the Sharpe Ratio and percentage relative variation of the average length of the large sample CIs with respect to the exact ones.

7.1 *Coverage accuracy of the large sample CIs*

First we observe that, as suggested by Remark 5, the features of the large sample CIs (18), (21), and (25) are not significantly affected by the value of σ. Then, in the following, we discuss and give the averaged coverages over the different value of σ. The simulated actual coverages of the CI for the Sharpe Ratio, the MAD Ratio , and the MD Ratio are given in Table 4, Table 5, and Table 6, respectively. The results highlight the impact of the asymmetry and fat tails on the actual coverage of the large sample CIs. In detail, as suggested by the results obtained sampling from the Student's t, the fatter the tails, the worst is the coverage accuracy. About the effect of the asymmetry, simulations show that the coverage accuracy improves when sampling from a distribution with positive γ_1. On the contrary, the coverage accuracy worsens when sampling form a distribution with negative γ_1. In the scenarios analyzed, it seems that the presence of fat tail is the element that have a greater impact on the coverage accuracy. This fact is suggested mainly by the results obtained sampling form the Skew Normal Distribution. Indeed, in these cases, we observe that the simulated coverage of the large sample CIs is quite similar to its nominal value also when $n = 50$ (a similar results is observed when sampling from the Normal distribution).

From Table 4-6, it turn out that the CI for the MAD Ratio and the MD Ratio are more accurate than the CI for the Sharpe Ratio. In details, the coverage accuracy of the CIs (25) and (21) is more robust than the coverage accuracy of the CI (18) with respect to: a) the presence of asymmetry, b) the presence of fat tails, c) the true value of the performance index.

As regard to the impact of the true value of the index on the actual coverage, we observe that, generally, the greater the true value of the performance index, the worst the coverage accuracy. However, when ψ, ψ_δ, and ψ_Δ increase, the variations in the simulated coverage of the CI (25), and (21) are relatively small while the changes in the simulated coverages of the CI (18) are greater. This effect is more evident when sampling from the Laplace and Student's t distributions suggesting that the changes int the simulated coverages are grater when the tail are fatter. In general, the CI for the MAD Ratio seem to have the better coverage accuracy (even if the differences with the CI for the MD Ratio are small).

In order to determine the minimum sample size that assure a sufficient precision of the large sample CIs we introduce the following criterion. It is well known that the t distribution approaches the Normal distribution when the d.f. increases. Further, it is common to retain that the t distribution with 30 degrees of freedom is very well approximated by the Normal distribution. As a consequence, when sampling form the Normal distribution, the large sample CI

$$(\bar{X} - z_{1-\alpha/2}\sqrt{S^2/n}; \bar{X} + z_{1-\alpha/2}\sqrt{S^2/n}) \tag{37}$$

is considered accurate if $n \geq 30$. The actual coverages of the just given CI

Sharpe Ratio		ψ_1				ψ_2				ψ_3			
$100(1-\alpha)\%$		90	95	97.5	99	90	95	97.5	99	90	95	97.5	99
dist.	n												
N	50	**89.73**	**94.74**	**97.28**	**98.86**	**89.56**	**94.63**	**97.27**	**98.82**	**89.18**	**94.35**	**97.03**	**98.74**
	100	**89.81**	**94.89**	**97.49**	**99.03**	**89.67**	**94.80**	**97.39**	**98.99**	**89.42**	**94.62**	**97.24**	**98.92**
	200	**89.86**	**94.97**	**97.51**	**98.95**	**89.78**	**94.96**	**97.50**	**98.97**	**89.64**	**94.81**	**97.47**	**98.95**
	400	**89.87**	**95.03**	**97.58**	**99.01**	**89.91**	**95.00**	**97.53**	**99.01**	**89.80**	**94.99**	**97.45**	**99.04**
L	50	88.02	93.33	96.13	97.97	87.52	92.92	95.89	97.81	86.43	92.04	95.29	97.55
	100	88.83	94.06	**96.77**	98.44	88.70	93.84	96.57	98.39	87.95	93.33	96.24	98.18
	200	**89.50**	**94.49**	**97.15**	**98.81**	**89.30**	**94.37**	**97.05**	**98.69**	**88.96**	**94.11**	**96.79**	**98.51**
	400	**89.69**	**94.78**	**97.39**	**98.86**	**89.45**	**94.73**	**97.35**	**98.88**	**89.25**	**94.48**	**97.16**	**98.79**
t_5	50	88.28	93.52	96.40	98.27	88.00	93.32	96.27	98.21	86.75	92.46	95.73	97.85
	100	88.78	**94.08**	**96.81**	**98.51**	88.47	93.85	96.66	**98.48**	87.71	93.21	96.18	98.17
	200	**89.37**	**94.53**	**97.13**	**98.71**	**89.11**	**94.25**	**97.06**	**98.68**	88.52	93.73	96.66	**98.49**
	400	**89.51**	**94.58**	**97.22**	**98.77**	**89.30**	**94.49**	**97.13**	**98.78**	88.83	**94.11**	**96.88**	**98.63**
t_3	50	86.08	91.58	94.71	97.11	83.96	90.20	93.92	96.58	79.32	86.30	90.88	94.46
	100	86.79	92.26	95.32	97.62	84.75	90.85	94.47	97.04	79.77	86.72	91.06	94.54
	200	87.99	93.20	96.07	98.07	85.62	91.52	94.83	97.27	80.54	87.35	91.64	95.07
	400	88.31	93.47	96.26	98.18	85.21	91.38	94.73	97.25	80.51	87.28	91.51	95.03
SNL+	50	**89.86**	**94.76**	**97.37**	**98.90**	**89.86**	**94.92**	**97.50**	**98.92**	**89.86**	**94.82**	**97.37**	**98.86**
	100	**89.98**	**95.17**	**97.56**	**98.99**	**90.17**	**95.15**	**97.56**	**98.97**	**90.04**	**95.01**	**97.47**	**98.91**
	200	**89.97**	**94.93**	**97.49**	**98.92**	**90.05**	**95.05**	**97.50**	**98.93**	**89.93**	**95.03**	**97.58**	**98.99**
	400	**89.66**	**94.92**	**97.42**	**98.98**	**89.74**	**94.92**	**97.45**	**98.95**	**89.80**	**94.86**	**97.45**	**98.95**
SNH+	50	**89.52**	**94.56**	**97.30**	**98.84**	**89.77**	**94.86**	**97.52**	**99.07**	**89.75**	**95.01**	**97.56**	**99.08**
	100	**90.08**	**94.97**	**97.56**	**98.95**	**90.20**	**95.26**	**97.65**	**99.03**	**90.38**	**95.25**	**97.56**	**99.10**
	200	**89.98**	**94.90**	**97.55**	**98.97**	**89.96**	**95.04**	**97.50**	**99.00**	**90.08**	**94.90**	**97.52**	**98.99**
	400	**90.08**	**94.88**	**97.38**	**99.01**	**89.97**	**94.92**	**97.41**	**99.00**	**89.96**	**94.85**	**97.45**	**98.98**
StL+	50	88.41	93.64	96.42	98.31	88.65	93.88	96.62	98.42	87.70	93.30	96.19	98.19
	100	**89.19**	**94.21**	**96.84**	**98.61**	**89.45**	**94.43**	**97.01**	**98.68**	88.41	93.84	96.57	98.47
	200	**89.43**	**94.55**	**97.04**	**98.76**	**89.52**	**94.67**	**97.16**	**98.80**	88.76	**94.19**	**96.93**	**98.67**
	400	**89.62**	**94.76**	**97.35**	**98.80**	**89.68**	**94.65**	**97.29**	**98.84**	**89.06**	**94.24**	**96.96**	**98.69**
StH+	50	88.17	93.68	96.38	98.28	**89.01**	**94.16**	**96.85**	**98.62**	88.12	93.62	96.56	98.44
	100	**89.34**	**94.26**	**96.92**	**98.54**	**89.75**	**94.67**	**97.18**	**98.77**	88.76	93.87	96.66	98.43
	200	**89.58**	**94.58**	**97.12**	**98.69**	**89.75**	**94.85**	**97.39**	**98.86**	88.85	**94.22**	**96.99**	**98.71**
	400	**89.80**	**94.70**	**97.38**	**98.90**	**89.82**	**94.74**	**97.35**	**98.90**	**89.11**	**94.34**	**96.99**	**98.69**
SNL-	50	**89.79**	**94.81**	**97.23**	**98.76**	**89.22**	**94.52**	**96.91**	**98.61**	88.49	**94.07**	96.63	98.39
	100	**89.76**	**94.89**	**97.40**	**98.96**	**89.52**	**94.58**	**97.29**	**98.89**	**89.08**	**94.24**	**97.05**	**98.75**
	200	**89.98**	**95.07**	**97.39**	**98.96**	**89.78**	**94.94**	**97.35**	**98.93**	**89.53**	**94.80**	**97.33**	**98.89**
	400	**90.07**	**95.07**	**97.60**	**99.06**	**89.98**	**95.04**	**97.61**	**99.08**	**89.81**	**94.95**	**97.65**	**99.03**
SNH-	50	**89.41**	**94.70**	**97.17**	**98.83**	88.78	**94.19**	**96.93**	**98.70**	88.23	93.52	96.47	98.42
	100	**89.60**	**94.86**	**97.48**	**98.92**	**89.48**	**94.61**	**97.30**	**98.79**	**89.01**	**94.33**	**97.01**	**98.72**
	200	**90.15**	**95.12**	**97.47**	**99.07**	**89.96**	**94.99**	**97.38**	**98.97**	**89.64**	**94.78**	**97.24**	**98.82**
	400	**90.19**	**95.01**	**97.49**	**99.05**	**89.93**	**94.92**	**97.50**	**98.98**	**89.72**	**94.76**	**97.43**	**98.99**
StL-	50	87.86	93.06	96.00	98.05	86.80	92.43	95.50	97.78	85.44	91.21	94.86	97.32
	100	**89.04**	94.01	96.73	98.47	88.20	93.57	96.31	98.33	87.06	92.57	95.72	97.98
	200	**89.14**	**94.35**	**96.93**	**98.50**	88.67	93.99	96.62	98.41	87.68	93.26	96.30	98.19
	400	**89.20**	**94.38**	**97.00**	**98.71**	88.72	**94.21**	**96.84**	**98.62**	88.25	93.63	96.57	98.42
StH-	50	87.90	93.27	96.28	98.15	86.56	92.35	95.53	97.70	84.87	90.97	94.43	97.07
	100	88.55	93.62	96.58	98.35	87.50	93.08	96.11	98.04	86.18	91.98	95.38	97.61
	200	**89.19**	**94.32**	**96.94**	**98.71**	88.32	93.77	96.56	98.39	87.31	92.92	96.04	98.11
	400	**89.48**	**94.71**	**97.16**	**98.88**	88.82	**94.18**	**97.02**	**98.72**	88.21	93.64	96.58	98.38

TABLE 4: Averages over the different values of σ of the simulated coverages of the CIs for the Sharpe Ratio.

MAD Ratio		ψ_1				ψ_2				ψ_3			
$100(1-\alpha)\%$		90	95	97.5	99	90	95	97.5	99	90	95	97.5	99
dist.	n												
N	50	**89.80**	**94.81**	**97.35**	**98.90**	**89.56**	**94.65**	**97.29**	**98.85**	**89.21**	**94.46**	**97.09**	**98.77**
	100	**89.83**	**94.92**	**97.48**	**99.04**	**89.67**	**94.78**	**97.39**	**98.99**	**89.54**	**94.65**	**97.27**	**98.93**
	200	**89.88**	**94.99**	**97.54**	**98.96**	**89.81**	**95.01**	**97.50**	**98.98**	**89.77**	**94.97**	**97.47**	**98.98**
	400	**89.87**	**95.02**	**97.58**	**99.01**	**89.91**	**95.01**	**97.51**	**99.00**	**89.90**	**94.90**	**97.46**	**99.03**
L	50	87.31	92.69	95.59	97.57	86.89	92.35	95.33	97.46	86.76	92.31	95.34	97.53
	100	88.45	93.70	96.46	98.23	88.50	93.60	96.36	98.17	88.42	93.55	96.31	98.17
	200	**89.25**	**94.31**	**96.96**	**98.67**	**89.20**	**94.15**	**96.92**	**98.59**	**89.15**	**94.16**	**96.80**	**98.55**
	400	**89.58**	**94.67**	**97.29**	**98.82**	**89.44**	**94.61**	**97.24**	**98.81**	**89.38**	**94.55**	**97.21**	**98.83**
$\mathbf{t_5}$	50	88.36	93.61	96.43	98.32	88.16	93.49	96.34	98.20	87.93	93.36	96.29	98.21
	100	88.95	**94.25**	**96.95**	**98.58**	88.85	94.05	**96.81**	**98.57**	88.68	93.96	96.74	**98.52**
	200	**89.51**	**94.64**	**97.23**	**98.77**	**89.38**	**94.51**	**97.25**	**98.79**	**89.30**	**94.41**	**97.09**	**98.76**
	400	**89.63**	**94.64**	**97.27**	**98.84**	**89.59**	**94.67**	**97.24**	**98.84**	**89.54**	**94.58**	**97.27**	**98.81**
$\mathbf{t_3}$	50	86.20	91.79	94.66	96.96	85.81	91.40	94.60	96.88	86.14	91.88	95.16	97.41
	100	87.52	92.86	95.81	97.95	87.08	92.70	95.69	97.75	87.03	92.77	95.92	97.89
	200	88.78	93.94	96.66	**98.50**	88.55	93.69	96.54	98.38	88.50	93.54	96.54	98.38
	400	**89.30**	**94.35**	**96.95**	**98.66**	**89.09**	**94.29**	**96.88**	**98.51**	**89.01**	**94.14**	**96.76**	**98.51**
SNL+	50	**89.39**	**94.42**	**97.11**	**98.69**	**89.29**	**94.52**	**97.13**	**98.69**	**89.23**	**94.26**	**96.86**	**98.57**
	100	**89.84**	**94.89**	**97.39**	**98.87**	**89.95**	**94.90**	**97.34**	**98.84**	**89.74**	**94.74**	**97.26**	**98.75**
	200	**89.97**	**94.85**	**97.37**	**98.84**	**89.98**	**94.91**	**97.42**	**98.88**	**89.85**	**94.85**	**97.47**	**98.94**
	400	**89.58**	**94.81**	**97.36**	**98.91**	**89.70**	**94.90**	**97.34**	**98.92**	**89.75**	**94.80**	**97.35**	**98.91**
SNH+	50	88.84	94.03	**96.76**	**98.53**	**89.16**	**94.28**	**96.98**	**98.70**	**89.08**	**94.35**	**96.95**	**98.65**
	100	**89.85**	**94.71**	**97.28**	**98.80**	**89.92**	**94.80**	**97.43**	**98.86**	**89.97**	**94.84**	**97.28**	**98.92**
	200	**89.69**	**94.73**	**97.36**	**98.88**	**89.72**	**94.84**	**97.37**	**98.90**	**89.89**	**94.72**	**97.36**	**98.87**
	400	**89.97**	**94.79**	**97.25**	**98.93**	**89.86**	**94.88**	**97.31**	**98.94**	**89.88**	**94.74**	**97.37**	**98.93**
StL+	50	88.33	93.61	96.43	98.24	88.26	93.64	96.41	98.28	88.42	93.68	96.49	98.40
	100	**89.27**	**94.27**	**96.90**	**98.66**	**89.25**	**94.26**	**96.94**	**98.63**	**89.05**	**94.23**	**96.95**	**98.68**
	200	**89.50**	**94.60**	**97.07**	**98.77**	**89.50**	**94.63**	**97.11**	**98.77**	**89.62**	**94.63**	**97.10**	**98.78**
	400	**89.75**	**94.83**	**97.33**	**98.86**	**89.74**	**94.86**	**97.33**	**98.87**	**89.67**	**94.75**	**97.41**	**98.84**
StH+	50	87.92	93.56	96.29	98.23	88.23	93.70	96.40	98.29	88.62	93.96	**96.76**	**98.55**
	100	**89.30**	**94.27**	**96.95**	**98.59**	**89.30**	**94.42**	**96.98**	**98.61**	**89.35**	**94.48**	**97.09**	**98.70**
	200	**89.67**	**94.59**	**97.14**	**98.72**	**89.67**	**94.60**	**97.17**	**98.76**	**89.74**	**94.73**	**97.29**	**98.85**
	400	**89.78**	**94.71**	**97.35**	**98.89**	**89.71**	**94.75**	**97.41**	**98.93**	**89.77**	**94.89**	**97.31**	**98.93**
SNL-	50	**89.89**	**94.93**	**97.33**	**98.83**	**89.40**	**94.68**	**97.11**	**98.74**	**89.05**	**94.30**	**96.86**	**98.58**
	100	**89.79**	**94.95**	**97.39**	**98.95**	**89.59**	**94.69**	**97.31**	**98.83**	**89.45**	**94.43**	**97.11**	**98.75**
	200	**89.95**	**95.07**	**97.39**	**98.98**	**89.90**	**95.00**	**97.44**	**98.93**	**89.74**	**94.85**	**97.47**	**98.91**
	400	**90.07**	**95.06**	**97.59**	**99.07**	**90.02**	**95.09**	**97.56**	**99.05**	**89.98**	**95.02**	**97.57**	**99.03**
SNH-	50	**89.56**	**94.81**	**97.31**	**98.92**	**89.13**	**94.34**	**97.10**	**98.74**	88.79	94.00	**96.80**	**98.56**
	100	**89.69**	**94.90**	**97.47**	**98.92**	**89.62**	**94.74**	**97.38**	**98.79**	**89.42**	**94.51**	**97.23**	**98.78**
	200	**90.16**	**95.14**	**97.48**	**99.07**	**90.00**	**95.00**	**97.44**	**98.96**	**89.75**	**94.86**	**97.30**	**98.86**
	400	**90.18**	**95.04**	**97.50**	**99.05**	**89.95**	**94.95**	**97.56**	**99.02**	**89.75**	**94.88**	**97.47**	**98.94**
StL-	50	87.95	93.18	96.11	98.09	87.47	92.96	95.92	97.92	87.20	92.77	95.76	97.90
	100	**89.28**	**94.23**	**96.91**	**98.53**	88.90	**94.07**	**96.74**	**98.49**	88.54	93.78	96.55	98.43
	200	**89.37**	**94.57**	**97.11**	**98.68**	**89.28**	**94.31**	**96.96**	**98.62**	**89.07**	**94.21**	**96.93**	**98.54**
	400	**89.34**	**94.60**	**97.14**	**98.78**	**89.24**	**94.48**	**97.11**	**98.82**	**89.24**	**94.33**	**97.11**	**98.75**
StH-	50	88.15	93.52	96.39	98.29	87.61	93.04	96.05	98.03	87.30	92.72	95.75	97.86
	100	88.82	93.89	**96.79**	**98.49**	88.64	93.78	96.62	98.39	88.24	93.61	96.42	98.34
	200	**89.38**	**94.53**	**97.16**	**98.81**	**89.19**	**94.34**	**97.12**	**98.74**	88.90	**94.26**	**96.91**	**98.64**
	400	**89.74**	**94.85**	**97.29**	**98.97**	**89.43**	**94.71**	**97.26**	**98.92**	**89.34**	**94.67**	**97.25**	**98.85**

TABLE 5: Averages over the different values of σ of the simulated coverages of the CIs for the MAD Ratio.

MD Ratio		ψ_1				ψ_2				ψ_3			
$100(1-\alpha)\%$		90	95	97.5	99	90	95	97.5	99	90	95	97.5	99
dist.	n												
N	50	**89.77**	**94.79**	**97.33**	**98.91**	**89.67**	**94.76**	**97.34**	**98.86**	**89.40**	**94.57**	**97.18**	**98.81**
	100	**89.84**	**94.90**	**97.50**	**99.05**	**89.71**	**94.86**	**97.40**	**99.00**	**89.58**	**94.71**	**97.28**	**98.94**
	200	**89.87**	**94.99**	**97.53**	**98.96**	**89.85**	**94.98**	**97.54**	**98.98**	**89.72**	**94.93**	**97.50**	**98.98**
	400	**89.86**	**95.02**	**97.58**	**99.01**	**89.91**	**95.03**	**97.53**	**99.02**	**89.95**	**94.97**	**97.48**	**99.05**
L	50	88.06	93.39	96.15	97.96	87.85	93.15	96.02	97.92	87.58	92.93	95.88	97.90
	100	88.88	**94.10**	**96.79**	98.46	**88.96**	94.02	96.69	98.43	88.71	93.88	96.62	98.40
	200	**89.50**	**94.52**	**97.16**	**98.82**	**89.45**	**94.40**	**97.11**	**98.73**	**89.41**	**94.37**	**96.96**	**98.66**
	400	**89.73**	**94.80**	**97.39**	**98.86**	**89.55**	**94.76**	**97.38**	**98.89**	**89.41**	**94.66**	**97.29**	**98.90**
t_5	50	88.46	93.71	96.54	98.35	88.43	93.66	96.51	98.34	88.15	93.57	96.44	98.31
	100	**88.98**	**94.26**	**96.97**	**98.62**	88.88	**94.17**	**96.87**	**98.64**	88.77	94.03	**96.78**	**98.57**
	200	**89.50**	**94.66**	**97.24**	**98.78**	**89.41**	**94.48**	**97.26**	**98.75**	**89.36**	**94.45**	**97.08**	**98.75**
	400	**89.64**	**94.66**	**97.27**	**98.83**	**89.60**	**94.66**	**97.23**	**98.84**	**89.49**	**94.58**	**97.21**	**98.82**
t_3	50	86.61	92.13	95.12	97.37	86.33	92.04	95.17	97.31	86.15	92.02	95.22	97.39
	100	87.67	93.07	96.02	98.08	87.21	92.99	95.82	97.92	87.07	92.69	95.83	97.89
	200	88.83	93.98	96.68	**98.53**	88.62	93.74	96.58	98.38	88.28	93.42	96.48	98.31
	400	**89.29**	**94.32**	**96.90**	**98.65**	**88.99**	**94.14**	**96.85**	**98.51**	88.66	94.02	96.66	**98.49**
SNL+	50	**89.88**	**94.79**	**97.44**	**98.88**	**89.67**	**94.68**	**97.44**	**98.83**	**89.57**	**94.66**	**97.13**	**98.83**
	100	**89.99**	**95.13**	**97.60**	**98.98**	**90.10**	**95.11**	**97.47**	**98.97**	**89.99**	**94.88**	**97.34**	**98.95**
	200	**90.01**	**94.94**	**97.46**	**98.91**	**90.03**	**95.06**	**97.39**	**98.93**	**89.98**	**94.96**	**97.47**	**98.94**
	400	**89.68**	**94.90**	**97.42**	**98.97**	**89.79**	**94.86**	**97.39**	**98.93**	**89.88**	**94.80**	**97.38**	**98.90**
SNH+	50	**89.56**	**94.59**	**97.31**	**98.89**	**89.57**	**94.69**	**97.39**	**98.98**	**89.46**	**94.66**	**97.43**	**98.92**
	100	**90.04**	**94.99**	**97.58**	**98.96**	**90.16**	**95.12**	**97.60**	**99.01**	**90.17**	**95.03**	**97.55**	**99.06**
	200	**89.94**	**94.91**	**97.56**	**98.97**	**89.92**	**94.95**	**97.50**	**98.98**	**89.89**	**94.94**	**97.50**	**98.98**
	400	**90.10**	**94.88**	**97.38**	**98.98**	**90.07**	**94.95**	**97.41**	**98.96**	**89.89**	**94.89**	**97.49**	**99.00**
StL+	50	88.45	93.76	96.49	98.34	88.64	93.90	96.68	98.43	88.79	**94.10**	96.73	**98.53**
	100	**89.30**	**94.29**	**96.96**	**98.67**	**89.48**	**94.30**	**96.99**	**98.68**	**89.37**	**94.29**	**97.07**	**98.75**
	200	**89.53**	**94.63**	**97.10**	**98.80**	**89.62**	**94.65**	**97.14**	**98.81**	**89.66**	**94.73**	**97.20**	**98.85**
	400	**89.74**	**94.84**	**97.37**	**98.85**	**89.77**	**94.78**	**97.36**	**98.86**	**89.71**	**94.77**	**97.34**	**98.89**
StH+	50	88.13	93.67	96.45	98.31	88.70	94.04	96.68	98.46	**89.02**	**94.39**	**97.09**	**98.66**
	100	**89.37**	**94.35**	**96.97**	**98.58**	**89.53**	**94.53**	**97.07**	**98.66**	**89.50**	**94.73**	**97.17**	**98.69**
	200	**89.64**	**94.61**	**97.14**	**98.72**	**89.74**	**94.68**	**97.22**	**98.84**	**89.70**	**94.78**	**97.31**	**98.89**
	400	**89.82**	**94.71**	**97.33**	**98.89**	**89.81**	**94.80**	**97.40**	**98.98**	**90.00**	**94.95**	**97.33**	**98.98**
SNL-	50	**89.84**	**94.82**	**97.31**	**98.83**	**89.37**	**94.70**	**97.10**	**98.73**	**89.16**	**94.39**	**96.86**	**98.55**
	100	**89.79**	**94.91**	**97.41**	**98.95**	**89.67**	**94.71**	**97.35**	**98.88**	**89.48**	**94.44**	**97.17**	**98.84**
	200	**89.98**	**95.08**	**97.40**	**98.98**	**89.81**	**95.01**	**97.41**	**98.95**	**89.77**	**94.83**	**97.42**	**98.96**
	400	**90.06**	**95.09**	**97.59**	**99.05**	**90.03**	**95.07**	**97.57**	**99.08**	**89.96**	**94.97**	**97.63**	**99.06**
SNH-	50	**89.52**	**94.77**	**97.21**	**98.87**	**89.18**	**94.34**	**97.08**	**98.77**	88.81	93.99	**96.75**	**98.62**
	100	**89.62**	**94.89**	**97.47**	**98.92**	**89.56**	**94.71**	**97.36**	**98.82**	**89.37**	**94.60**	**97.25**	**98.78**
	200	**90.13**	**95.10**	**97.45**	**99.06**	**89.88**	**95.05**	**97.43**	**98.98**	**89.80**	**94.91**	**97.35**	**98.88**
	400	**90.19**	**95.04**	**97.48**	**99.04**	**90.04**	**95.00**	**97.51**	**98.98**	**89.78**	**94.93**	**97.48**	**98.95**
StL-	50	88.09	93.22	96.19	98.15	87.53	93.09	96.05	98.11	87.24	92.89	95.90	97.94
	100	**89.29**	**94.26**	**96.93**	**98.56**	88.94	**94.12**	96.73	**98.53**	88.51	93.77	96.53	98.44
	200	**89.41**	**94.56**	**97.10**	**98.65**	**89.38**	**94.32**	**96.96**	**98.63**	**88.99**	**94.20**	**96.93**	**98.56**
	400	**89.32**	**94.56**	**97.14**	**98.77**	**89.27**	**94.53**	**97.08**	**98.80**	**89.09**	**94.32**	**97.09**	**98.74**
StH-	50	88.19	93.59	96.49	98.31	87.66	93.23	96.14	98.13	87.37	92.85	95.82	97.91
	100	88.82	93.86	**96.81**	98.48	88.60	93.71	96.64	98.44	88.19	93.61	96.47	98.36
	200	**89.40**	**94.51**	**97.13**	**98.80**	**89.14**	**94.35**	**97.08**	**98.73**	88.93	**94.17**	**96.94**	**98.64**
	400	**89.73**	**94.86**	**97.25**	**98.98**	**89.49**	**94.71**	**97.30**	**98.91**	**89.30**	**94.62**	**97.20**	**98.82**

TABLE 6: Averages over the different values of σ of the simulated coverages of the CIs for the MD Ratio.

$(1-\alpha)$	0.9	0.95	0.975	0.99
actual coverage(a_α)	0.8896	0.9407	0.9674	0.9848
$\epsilon_\alpha = \lvert(1-\alpha) - a_\alpha\rvert$	0.0104	0.0093	0.0076	0.0052
$(1-\alpha) + \epsilon_\alpha$	0.9104	0.9593	0.9826	0.9952

TABLE 7: Comparison between the nominal and actual coverage probabilities of the large sample CI (37).

Index	Sharpe Ratio			MAD Ratio			MD Ratio		
distribution	ψ_1	ψ_2	ψ_3	$\psi_{\delta 1}$	$\psi_{\delta 2}$	$\psi_{\delta 3}$	$\psi_{\Delta 1}$	$\psi_{\Delta 2}$	$\psi_{\Delta 3}$
N	50	50	50	50	50	50	50	50	50
L	200	200	200	200	200	200	100	200	200
T5	100	200	400	100	100	200	100	100	200
T3	>400	≫400	≫400	400	400	400	400	400	>400
SNL+	50	50	50	50	50	50	50	50	50
SNLH+	50	50	50	50	50	50	50	50	50
STL+	100	100	200	100	100	100	100	100	100
STH+	100	50	200	100	100	100	100	100	50
SNL-	50	50	100	50	50	50	50	50	50
SNH-	50	50	100	50	50	100	50	50	100
STL-	200	400	>400	100	100	200	100	100	200
STH-	200	400	>400	200	200	200	200	200	200

TABLE 8: Minimum sample sizes assuring a sufficient precision of the large sample CIs (18), (25) and (21).

when $n = 30$ are given in Table 7 and, effectively, they are quite similar to their nominal values.

Then, we think it is reasonable to take the values given in Table 7 as a benchmark and, in the following, we assert that the simulated coverage of a large sample CI is sufficiently close to its nominal value $(1-\alpha)$ if it belongs to the interval $(1-\alpha-\epsilon_\alpha; 1-\alpha+\epsilon_\alpha)$ where ϵ_α is defined in Table 7.

The simulated coverages belonging to the just defined interval are written in bold in Table 4-6. The application of the above criterion leads to the minimum sample sizes given in Table 8.

It can be noted that, with the exception of the Student's t with 3 df, a sample size of 200 is always sufficient in order to assure a good coverage accuracy of the CIs for the MAD Ratio and for the MD Ratio. Differently, a sample size of 400 is not always sufficient for the CIs for the Sharpe Ratio as it can be seen in the case of the Skew t distribution with negative γ_1. In the scenario of the Student's t with 3 df, as suggested by Remark 4, the coverage accuracy of the large sample CI (18) is quite bad while $n = 400$ turn out to be sufficient for the CI (25) and for the CI (21) with low and intermediate value of ψ_Δ. Finally, it is worthwhile to note that the scenarios concerning the Skew t distribution are the more realistic because the empirical distributions of the financial activities are generally characterized by the presence of fat tails and asymmetry. Furthermore, these are the cases in which the performances of the large sample CIs (25) and (21) are quite better than the performances of the CI (18).

7.2 *Length of the large sample CIs*

Once again, we observe that the value of σ does not impact significantly on the Average Length of the CIs. Consequently, the values given in Table 9 and Table 10 are averages calculated over the different values of σ. In detail, in Table 9 the Average Lengths (ALs) obtained sampling from the Normal distribution are given. The ALs obtained under the others scenarios are evaluated by the percentage relative variation with respect to the lengths obtained sampling from the Normal:

$$100\left(\frac{\text{AL sampling form the Normal} - \text{AL sampling from another distribution}}{\text{AL sampling form the Normal}}\right)\% \ .$$

The values of the above index are given in Table 10. Observe that, in Table 10, the values of $(1-\alpha)$ are not specified since the length of the CIs are proportional to the quantiles of the standard normal distribution. As a consequence, the percentage relative variations of the length are the same for all the values of $(1-\alpha)$.

Simulations show that the features of the ALs of the CIs (25) and (21) are quite different from those of the CI (18). In detail, as expected, the CIs (25) and (21) tends to be larger if the parent distributions has tails heavier than the Normal. On the contrary, the CI (18) tends to be larger if the parent distributions has tails heavier than the Normal only for intermediate and large values of ψ. About the effect of asymmetry, evidence from the Skew Normal distribution suggests that the AL of CIs (18), (25), and (21) increases (decreases) when the parent distribution becomes more negatively (positively) asymmetric. When sampling from the Skew t distribution (which exhibits both asymmetry and fat tails) we observe that, in a sense, the effect of the asymmetry prevails on the effect of the fat tails in the case of the CI for the Sharpe Ratio. For the other two indices the effect of the fat tails prevails. In detail, when sampling form the **StL+** and **StH+**, the CI (18) tends to be shorter than in the Normal case also for intermediate and large values of ψ even if the presence of fat tails should increase its AL. On the contrary, when sampling form the **StL+** and **StH+**, the CI (25) and (21) are larger than in the Normal case.

7.3 *Evaluation of the performances of the bias corrected estimators*

In order to compare the bias and the efficiency of the plug-in estimators with that of the approximatively unbiased ones, we compute, for each scenario, the simulated bias and MSE of the estimator $\widehat{\Psi}$, $\widehat{\Psi}_\delta$, $\widehat{\Psi}_\Delta$, $\widehat{\Psi}'_u$, $\widehat{\Psi}_{\delta u}$, and $\widehat{\Psi}_{\Delta u}$. Evidence shows that the impact of the value of σ on the bias and on the MSE is negligible. Consequently, in Table 11 and Table 12 we give the average over the values of σ of the simulated bias an MSE. In order to ease comparisons, in the just cited tables, the following gain/loss indices are given:

$$100\left(\frac{\text{bias of the plug-in est.} - \text{bias of the approx. unbiased est.}}{\text{true value of the ratio}}\right)\%$$

Sharpe Ratio	ψ_1				ψ_2				ψ_3			
$100(1-\alpha)\%$	90	95	97.5	99	90	95	97.5	99	90	95	97.5	99
n												
50	0.4675	0.5571	0.6371	0.7321	0.4734	0.5641	0.6451	0.7414	0.4917	0.5858	0.6700	0.7699
100	0.3299	0.3931	0.4496	0.5166	0.3345	0.3986	0.4558	0.5238	0.3483	0.4151	0.4747	0.5455
200	0.2330	0.2777	0.3176	0.3649	0.2364	0.2817	0.3221	0.3702	0.2465	0.2938	0.3359	0.3861
400	0.1647	0.1962	0.2244	0.2579	0.1671	0.1991	0.2277	0.2616	0.1743	0.2077	0.2376	0.2730
MAD Ratio												
50	0.5902	0.7032	0.8042	0.9242	0.5977	0.7122	0.8144	0.9360	0.6262	0.7462	0.8533	0.9806
100	0.4148	0.4943	0.5653	0.6496	0.4211	0.5017	0.5738	0.6594	0.4418	0.5264	0.6020	0.6918
200	0.2926	0.3486	0.3987	0.4582	0.2973	0.3542	0.4051	0.4655	0.3121	0.3719	0.4253	0.4887
400	0.2066	0.2462	0.2815	0.3235	0.2099	0.2502	0.2861	0.3288	0.2204	0.2627	0.3004	0.3452
MD Ratio												
50	0.4127	0.4918	0.5624	0.6464	0.4189	0.4992	0.5709	0.6560	0.4379	0.5218	0.5967	0.6857
100	0.2918	0.3476	0.3976	0.4569	0.2962	0.3529	0.4036	0.4638	0.3096	0.3690	0.4219	0.4849
200	0.2063	0.2458	0.2811	0.3231	0.2094	0.2496	0.2854	0.3280	0.2190	0.2609	0.2984	0.3429
400	0.1459	0.1738	0.1988	0.2284	0.1481	0.1764	0.2018	0.2319	0.1548	0.1844	0.2109	0.2424

TABLE 9: Averages over the different values of σ of the simulated AL of the confidence interval (18) when sampling from the Normal distribution.

$$100\left(\frac{\text{MSE of the plug-in est.} - \text{MSE of the approx. unbiased est.}}{\text{MSE of the plug-in est.}}\right)\%$$

Simulations show that the approximatively unbiased estimator generally have a lower bias than the plug-in estimators. However, the gain/loss indices in Table 11 show that the bias reduction is negligible. Furthermore, Table 12 shows that the increase in the variability of the approximatively unbiased estimators (due to the estimation of the bias factors) over-compensates the bias reduction so that the MSE of the plug-in estimators turn out to be (a little) lower than MSE of the approximatively unbiased ones. Concluding, in the scenarios considered, the plug-in estimator can be considered more efficient than the approximatively unbiased ones.

7. CONCLUSION

In this paper we study some inferential aspects of the Sharpe Ratio, the MD Ratio and the MAD Ratio.

Under the assumption of i.i.d.-normal returns, we obtain an exact CI for each ratios and we study the coverage accuracy of the large sample confidence intervals defined on the basis of the results in Jobson and Korkie (1981), Gastwirth (1974), and Hoeffding (1948). In a brief simulation study, we observe that all the just cited large sample CIs approximate very well the exact ones starting from the relively small sample size of 50. Moreover, after citing the results concerning the bias of the plug-in estimator for the Sharpe ratio given in Miller and Gher (1978), we improve the approximation of the bias factor given in Jobson and Korkie (1981).

Under the assumption if i.i.d.-returns, we introduce a large sample CIs for each ratio and we investigate their coverage accuracy in a wide simulation study. We obtain that the large sample CIs for the MD Ratio and for the MAD Ratio

	Simmetric-heavy tailed distributions									Global average		
	Laplace			t with 5 d.f.			t with 3 d.f.					
Sharpe Ratio	ψ_1	ψ_2	ψ_3	ψ_1	ψ_2	ψ_3	ψ_1	ψ_2	ψ_3	ψ_1	ψ_2	ψ_3
n												
50	-1.79	-0.45	3.29	-1.62	-0.63	2.25	-4.01	-1.22	6.56	-1.11	-0.27	2.12
100	-0.92	0.67	5.07	-0.99	0.44	4.44	-3.04	1.05	12.42	-0.72	0.45	3.80
200	-0.44	1.38	6.33	-0.54	1.36	6.50	-2.07	4.09	20.05	-0.43	1.16	5.56
400	-0.18	1.80	7.11	-0.27	2.02	8.16	-1.48	7.36	29.13	-0.25	1.85	7.38
MAD Ratio												
50	7.90	9.36	13.54	4.92	5.69	7.97	9.97	12.50	19.83	4.27	4.95	7.01
100	10.27	11.70	15.63	6.59	7.38	9.63	14.96	17.34	24.44	5.68	6.34	8.33
200	11.49	12.88	16.68	7.49	8.32	10.61	18.41	21.02	28.28	6.50	7.18	9.16
400	12.18	13.61	17.41	8.00	8.84	11.15	20.83	23.49	30.88	7.02	7.74	9.78
MD Ratio												
50	3.54	5.11	9.41	2.32	3.30	6.06	5.45	8.39	16.37	2.52	3.35	5.67
100	4.97	6.53	10.79	3.62	4.68	7.65	9.20	12.23	20.79	3.58	4.42	6.86
200	5.68	7.26	11.55	4.37	5.53	8.70	11.98	15.39	24.71	4.23	5.13	7.70
400	6.06	7.68	12.02	4.80	5.99	9.26	14.03	17.65	27.58	4.66	5.62	8.32
	Distributions with positive or negative asimmetry											
	SNL+			**SNH+**			**SNL-**			**SNH-**		
Sharpe Ratio	ψ_1	ψ_2	ψ_3	ψ_1	ψ_2	ψ_3	ψ_1	ψ_2	ψ_3	ψ_1	ψ_2	ψ_3
50	-0.90	-4.58	-8.41	-1.14	-6.76	-12.70	1.03	4.82	8.95	1.74	7.31	13.31
100	-0.98	-5.02	-9.14	-1.37	-7.36	-13.57	1.13	5.23	9.67	1.74	7.75	14.20
200	-1.06	-5.22	-9.43	-1.50	-7.69	-14.03	1.13	5.40	10.03	1.69	7.92	14.62
400	-1.09	-5.29	-9.52	-1.57	-7.83	-14.20	1.13	5.48	10.20	1.69	8.06	14.90
MAD Ratio												
50	-0.57	-4.97	-9.47	-0.84	-7.68	-14.78	1.72	6.14	10.73	2.50	9.15	15.96
100	-0.51	-5.10	-9.77	-1.04	-8.02	-15.24	1.84	6.39	11.10	2.55	9.35	16.33
200	-0.55	-5.16	-9.84	-1.11	-8.17	-15.46	1.86	6.46	11.21	2.45	9.28	16.27
400	-0.56	-5.17	-9.84	-1.16	-8.21	-15.49	1.83	6.45	11.23	2.45	9.35	16.42
MD Ratio												
50	-0.40	-4.52	-8.79	-0.12	-6.57	-13.42	1.77	5.94	10.32	3.11	9.35	15.80
100	-0.36	-4.74	-9.29	-0.25	-6.95	-14.08	1.89	6.23	10.78	3.18	9.65	16.36
200	-0.39	-4.84	-9.46	-0.30	-7.13	-14.41	1.91	6.33	10.98	3.13	9.70	16.50
400	-0.41	-4.87	-9.50	-0.34	-7.20	-14.52	1.90	6.36	11.05	3.15	9.80	16.68
	Distributions with positive or negative asimmetry and heavy tails											
	StL+			**StH+**			**StL-**			**StH-**		
Sharpe Ratio	ψ_1	ψ_2	ψ_3	ψ_1	ψ_2	ψ_3	ψ_1	ψ_2	ψ_3	ψ_1	ψ_2	ψ_3
50	-2.24	-3.99	-3.83	-2.74	-7.35	-10.01	-0.78	3.02	8.70	0.26	6.82	15.26
100	-1.82	-3.77	-2.90	-2.57	-7.81	-9.95	-0.01	4.80	12.05	0.95	9.03	19.50
200	-1.55	-3.56	-1.88	-2.47	-8.09	-9.35	0.46	6.17	14.93	1.59	11.00	23.35
400	-1.39	-3.24	-0.56	-2.42	-8.10	-8.35	0.89	7.44	17.52	2.00	12.64	26.77
MAD Ratio												
50	4.24	1.71	0.77	3.83	-2.18	-6.54	5.97	10.04	15.56	7.35	14.74	23.54
100	5.79	2.97	1.69	5.24	-1.23	-6.09	7.72	12.07	17.81	9.02	16.87	26.06
200	6.64	3.66	2.22	6.17	-0.59	-5.74	8.62	13.12	19.01	10.00	18.12	27.56
400	7.21	4.21	2.77	6.62	-0.24	-5.50	9.19	13.84	19.88	10.64	18.98	28.65
MD Ratio												
50	1.80	-0.17	-0.32	1.79	-3.35	-6.68	3.40	7.37	13.08	5.02	11.97	20.58
100	3.02	0.75	0.42	2.90	-2.80	-6.56	4.78	9.13	15.32	6.39	13.97	23.30
200	3.71	1.27	0.87	3.64	-2.40	-6.38	5.55	10.13	16.64	7.28	15.25	25.05
400	4.16	1.70	1.37	4.02	-2.16	-6.20	6.05	10.84	17.61	7.84	16.09	26.22

TABLE 10: Percentage relative variation of the AL of the large sample CIs with respect to the ALs obtained sampling form the Normal distribution.

		Natual Est.s			Appr. Unbiased Est.s			Gain/Loss		
Sharpe Ratio		ψ_1	ψ_2	ψ_3	ψ_1	ψ_2	ψ_3	ψ_1	ψ_2	ψ_3
n	50	0.00182	0.00990	0.01999	0.00143	0.00796	0.01612	0.79%	0.78%	0.78%
	100	0.00141	0.00603	0.01181	0.00116	0.00470	0.00912	0.50%	0.53%	0.54%
	200	0.00049	0.00325	0.00669	0.00033	0.00244	0.00507	0.33%	0.33%	0.33%
	400	0.00040	0.00203	0.00406	0.00032	0.00159	0.00319	0.17%	0.17%	0.17%
MAD Ratio		$\psi_{\delta 1}$	$\psi_{\delta 2}$	$\psi_{\delta 3}$	$\psi_{\delta 1}$	$\psi_{\delta 2}$	$\psi_{\delta 3}$	$\psi_{\delta 1}$	$\psi_{\delta 2}$	$\psi_{\delta 3}$
n	50	0.00135	0.00799	0.01629	0.00086	0.00288	0.00540	0.72%	1.52%	1.62%
	100	0.00115	0.00438	0.00842	0.00075	0.00168	0.00284	0.60%	0.80%	0.83%
	200	0.00020	0.00187	0.00397	-0.00004	0.00051	0.00119	0.23%	0.41%	0.41%
	400	0.00028	0.00112	0.00218	0.00014	0.00040	0.00071	0.21%	0.22%	0.22%
MD Ratio		$\psi_{\Delta 1}$	$\psi_{\Delta 2}$	$\psi_{\Delta 3}$	$\psi_{\Delta 1}$	$\psi_{\Delta 2}$	$\psi_{\Delta 3}$	$\psi_{\Delta 1}$	$\psi_{\Delta 2}$	$\psi_{\Delta 3}$
n	50	0.00065	0.00392	0.00800	-0.00008	0.00028	0.00072	1.24%	1.57%	1.57%
	100	0.00064	0.00224	0.00424	0.00027	0.00036	0.00047	0.81%	0.81%	0.81%
	200	0.00002	0.00087	0.00192	-0.00017	-0.00012	-0.00006	-0.33%	0.32%	0.40%
	400	0.00011	0.00054	0.00108	0.00001	0.00004	0.00008	0.20%	0.21%	0.22%

TABLE 11: Comparison between the bias of the plug-in estimators and that of the approximately unbiased ones.

		Natual Est.s			Appr. Unbiased Est.s			Gain/Loss		
Sharpe Ratio		ψ_1	ψ_2	ψ_3	ψ_1	ψ_2	ψ_3	ψ_1	ψ_2	ψ_3
n	50	0.0217	0.0234	0.0286	0.0219	0.0236	0.0288	1.10%	0.94%	0.56%
	100	0.0111	0.0120	0.0149	0.0112	0.0121	0.0150	0.98%	0.92%	0.78%
	200	0.0059	0.0064	0.0080	0.0059	0.0065	0.0081	0.89%	0.90%	0.92%
	400	0.0031	0.0034	0.0044	0.0031	0.0035	0.0044	0.88%	0.94%	1.06%
MAD Ratio		$\psi_{\delta 1}$	$\psi_{\delta 2}$	$\psi_{\delta 3}$	$\psi_{\delta 1}$	$\psi_{\delta 2}$	$\psi_{\delta 3}$	$\psi_{\delta 1}$	$\psi_{\delta 2}$	$\psi_{\delta 3}$
n	50	0.0382	0.0404	0.0473	0.0381	0.0402	0.0468	-0.22%	-0.41%	-0.91%
	100	0.0198	0.0209	0.0243	0.0198	0.0209	0.0243	0.36%	0.23%	-0.06%
	200	0.0106	0.0112	0.0130	0.0107	0.0112	0.0131	0.52%	0.43%	0.19%
	400	0.0057	0.0060	0.0070	0.0057	0.0060	0.0070	0.60%	0.48%	0.20%
MD Ratio		$\psi_{\Delta 1}$	$\psi_{\Delta 2}$	$\psi_{\Delta 3}$	$\psi_{\Delta 1}$	$\psi_{\Delta 2}$	$\psi_{\Delta 3}$	$\psi_{\Delta 1}$	$\psi_{\Delta 2}$	$\psi_{\Delta 3}$
n	50	0.0180	0.0191	0.0223	0.0180	0.0190	0.0222	-0.32%	-0.37%	-0.51%
	100	0.0094	0.0099	0.0115	0.0094	0.0099	0.0116	0.17%	0.13%	0.06%
	200	0.0050	0.0053	0.0062	0.0050	0.0053	0.0062	0.28%	0.26%	0.19%
	400	0.0027	0.0028	0.0033	0.0027	0.0028	0.0033	0.31%	0.26%	0.14%

TABLE 12: Comparison between the MSE of the plug-in estimators and that of the approximately unbiased ones.

have quite better features than the CI for the Sharpe Ratio. In detail, we highlight that the CIs for the MD Ratio and for the MAD Ratio are more robust with respect to the presence of asymmetry and fat tails in the parent distribution, especially when the later does not posses all the moments (such as the Student's t or the Skew t). Since the only difference between the three indicators is the variability measure used to quantify the risk, the differences in the performances of the large sample CIs can be attributed to the different features of the estimators S^2, $\hat{\delta}$ and $\hat{\Delta}$ and to their different covariance with the sample mean $\bar{X}$. In detail, the evidence we obtain can be partially motivated by the fact that some desirable properties of S^2 are based on stronger assumption than those necessary in the case of $\hat{\Delta}$ and $\hat{\delta}$. For example, $\hat{\Delta}$ and $\hat{\delta}$ are weakly consistent (i.e. they converge in probability to the true value of Δ and δ, respectively) if the parent distribution possess the first moment; while the existence of the second moment is required for the weak consistency of S^2. Moreover, $\hat{\Delta}$ and $\hat{\delta}$ are asymptotically normally distributed if the parent distribution possess second moment (as shown in Theorems 2 and 3). On the contrary, the existence of the fourth moment is required for the asymptotic normality of S^2 (see Theorem 1). For these reasons, as shown in Polisicchio (2006), when the parent distribution have fat tails, it could be more appropriate to measure the variability using Δ or δ because the properties of their estimators become better than those of S^2 as the weight of the tails increases[¶]. Moreover, as for the asymptotic normality of S^2, the CI for the Sharpe Ratio here considered is based on the theoretical assumption of existence of the fourth moment of the parent distribution. Indeed, its coverage accuracy dramatically worsens when sampling from the Student's t with 3 degrees of freedom, which does not possess moments of order greater than 2. This is a potential limitation of the Sharpe Ratio since empirical evidence suggests that the distribution of the returns of many financial activities may have infinite fourth moment (especially when high frequency returns are considered as shown in Genay et al., 2001). On the contrary, the CIs for the MAD Ratio and for the MD Ratio (which are based on the less stringent theoretical assumption of existence of the second moment) have a good coverage starting from the sample size of 400 when sampling from the Student's t with 3 degrees of freedom. Another interesting observation is that the actual coverage probability of the CI for the Sharpe Ratio decreases significantly when the true value of the ratio increases while the coverage probability of the CIs for the MAD Ratio and for the MD Ratio is more stable with respect to variation of the true value of the corresponding ratio. In general, with the exception of the Student's t with 3 d.f., a sample size of 200 is sufficient to reach a good coverage accuracy of the CIs for the MAD Ratio and for the MD Ratio in all the scenarios

[¶]Furthermore, as shown in Polisicchio (2006) and in Johnson, Kotz, and Balakrishnan (1995a) (ch. 13, p. 136), even if we sample from a normal distribution in order to estimates σ, the estimators based on $\hat{\delta}$ and $\hat{\Delta}$ given by

$$\hat{\sigma}_{\Delta} = \hat{\Delta}\frac{\sqrt{\pi}}{2} \qquad \text{and} \qquad \hat{\sigma}_{\delta} = \hat{\delta}\sqrt{\frac{\pi}{2}} \ ,$$

have an efficiency very close to that of S^2 which is, indeed, based on the conjoint sufficient statistics $(\sum_{i=1}^{n} X_i \ ; \ \sum_{i=1}^{n} X_i^2)$ for $(\mu \ ; \ \sigma^2)$.

considered. Concerning the confidence interval for the Sharpe Ratio, a sample size of 400 is still inadequate when sampling from the Skew t distribution with negative asymmetry, which are, indeed, scenarios of practical interest.

Another aspect we investigate under the assumption of i.i.d.-returns, is the bias of the plug-in estimators. In particular, we approximate the bias of the three estimators and we introduce an approximately unbiased estimators for the three indices. In the simulation study we compare the efficiency of the plug-in estimators with that of the approximately unbiased ones. We obtain that the plug-in estimators are, in general, more efficient than the the bias-corrected estimators.

RIASSUNTO

In questo lavoro si studiano le proprietà degli stimatori di tre misure di performance delle attività finanziarie note come Sharpe Ratio, MAD Ratio e MD Ratio. L'analisi viene svolta in due particolari contesti. In primo luogo si ipotizza che i rendimenti siano indipendentemente ed identicamente normalmente distribuiti (n.i.i.d.). In seguito, si rilassa l'ipotesi di normalità e si ipotizza che i rendimenti siano indipendentemente ed identicamente distribuiti (i.i.d.). In entrambi i contesti si studia la distorsione degli stimatori dei tre indici di performance e si propone una correzione della distorsione. Nel caso dei rendimenti n.i.i.d. si ottiene la distribuzione esatta dei tre stimatori e si richiama la loro distribuzione asintotica. Nel contesto dei rendimenti i.i.d., si deriva la distribuzione asintotica dei tre stimatori. Le distribuzioni asintotiche ed esatte appena menzionate vengono utilizzate per costruire intervalli di confidenza esatti o asintotici per le tre misure di performance. In ultimo, si svolge uno studio di simulazioni finalizzato a valutare l'efficienza degli stimatori corretti e la copertura effettiva degli intervalli di confidenza asintotici.

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