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On General Balance Laws with Boundary

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Abstract

This paper is devoted to general balance laws (with a possibly non local source term) with a non-characteristic boundary. Basic well posedness results are obtained, trying to provide sharp estimates. In particular, bounds tend to blow up as the boundary tends to be characteristic. New uniqueness results for the solutions to conservation and/or balance laws with boundary are also provided.

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1 Introduction

This paper is concerned with initial boundary value problems (IBVP) for systems of balance laws of the form

$$\left\{\begin{array}{lcl} \partial_t u + \partial_x f(u) = G(u) & x > & \gamma(t) \\ b\Big(u\left(t,\gamma(t)\right)\Big) = g(t) & t \geq & 0 \\ u(0,x) = u_o(x) & x \geq & \gamma(0) \end{array}\right. \tag{1.1}$$

where f is smooth, Df is strictly hyperbolic, u_o is the initial datum and G is a possibly non-local source term. The boundary γ is assumed non characteristic, i.e. ℓ characteristics point outwards and $n-\ell$ inwards. The role of b is that of letting $n-\ell$ component of u be assigned by the boundary data g. Above and in what follows, we assume that all **BV** functions are right continuous.

Systems belonging to this class were already considered in the literature. See, for instance, [7, 8] for the case with a non local source but no boundary and [12] for the case of a Temple type f.

Examples of physical models that fit into this class are found, besides in the cited references, also in [11]. There, a model describing the flow of a fluid in a simple pipeline is based on a system essentially of the form (1.1).

As is well known, preliminary to the study of (1.1), is that of the purely convective system

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 & x > \gamma(t) \\ b\Big(u\left(t,\gamma(t)\right)\Big) = g(t) & t \geq 0 \\ u(0,x) = u_o(x) & x \geq \gamma(0) \end{cases} \tag{1.2}$$

considered, for instance, in [1, 2, 12, 15, 16]. Below, we provide results on (1.2) that are not contained in these papers. In particular, the present estimates explicitly blow up as the boundary tends to be characteristic. The choice of the Glimm type functionals on which most of the proof relies is here simplified, compare for instance (4.11) below with [15, (2.10)-(2.13)] and (4.14)–(4.15) with [15, (3.5)-(3.10)].

In the homogeneous case (1.2), we also provide a uniqueness result that has no analogue in the case of Cauchy problems with no boundary. Indeed, let u solve (1.2) and assume a second boundary $\bar\gamma$ is given, such that $\bar\gamma \geq \gamma$. Along $\bar\gamma$ assign the trace of u as boundary data, i.e. let $\tilde g(t) = b\Big(u\left(t,\gamma(t)\right)\Big)$. Then, the solution to

$$\begin{cases} \partial_t \tilde u + \partial_x f(\tilde u) = 0 & x > \tilde\gamma(t) \\ b\Big(\tilde u\left(t,\tilde\gamma(t)\right)\Big) = \tilde g(t) & t \geq 0 \\ \tilde u(0,x) = u_o(x) & x \geq \tilde\gamma(0) \end{cases} \tag{1.3}$$

coincides with the restriction of u to $x \geq \bar\gamma(t)$, see Proposition 2.4. We show that an analogous result may not hold in the case of (1.1), see (3.1).

Besides, we also provide a Lipschitz estimate on the process generated by (1.2) that contains also a second order part on a generic perturbation, see *2)* in Theorem 2.2. This technical estimate, already known in less general situations, played a key role in several other results, see for instance [7, Proposition 3.10] and [3, Remark 4.1].

All what we obtain in the case of (1.2) is used in the proof of the results on (1.1). In particular, for both systems, we provide bounds on the total variation of time like curves. These estimates are optimal in the sense that they blow up as the boundary tends to be characteristic, see propositions 2.3 and 3.3.

The next section is devoted to the homogeneous problem (1.2), while Section 3 presents the results related to (1.1). The proofs are deferred to the last two sections.

2 The Purely Convective IBVP

On system (1.2) we require the following conditions:

(*f*) $f\colon \Omega \to \mathbb{R}^n$ is smooth, with $\Omega \subseteq \mathbb{R}^n$ being open, such that $Df(u)$ is strictly hyperbolic for all $u \in \Omega$, each characteristic field is either genuinely nonlinear or linearly degenerate.

Without loss of generality, we may assume that $0 \in \Omega$ and for all u in Ω, $Df(u)$ admits n real distinct eigenvalues $\lambda_1(u), \ldots, \lambda_n(u)$, ordered so that $\lambda_{i-1}(u) < \lambda_i(u)$ for all u, with right eigenvectors $r_1(u), \ldots, r_n(u)$.

(γ) $\gamma \in \mathbf{C^{0,1}}(\mathbb{R}^+;\mathbb{R})$ and, for a fixed positive c, $\lambda_\ell(u)+c \leq \dot\gamma(t) \leq \lambda_{\ell+1}(u)-c$ for a fixed $\ell \in \{1,\ldots,n-1\}$ and for all $u \in \Omega$.

(*b*) $b \in \mathbf{C^1}(\Omega;\mathbb{R}^{n-\ell})$ is such that $b(0)=0$ and

$$\det\left[Db(0)\, r_{\ell+1}(0) \quad Db(0)\, r_{\ell+2}(0) \quad \cdots \quad Db(0)\, r_n(0)\right] \neq 0\,.$$

For notational simplicity, we say below that a curve γ is *ℓ–non-characteristic* if $\gamma \in \mathbf{C^{0,1}}(\mathbb{R}^+;\mathbb{R})$, and for a fixed positive c, for all $u \in \Omega$, $\lambda_\ell(u)+c \leq \dot\gamma(t) \leq \lambda_{\ell+1}(u)-c$. This notion is more restrictive than that of a *non-resonant* curve, see [13, Chapter 14].

We define below the domain

$$\mathbb{D}_\gamma = \left\{(t,x) \in \mathbb{R}^+ \times \mathbb{R} \colon x \geq \gamma(t)\right\}$$

and extend to $[0,+\infty[\, \times \mathbb{R}$ any function defined on $\mathbb{D}_\gamma$ to vanish outside $\mathbb{D}_\gamma$.

We slightly modify the definition given in [17] of solution to (1.2) in the non characteristic case, see also [1, 2, 15] and [12, Definition 2.1]. Indeed, here we require the boundary condition to be satisfied by the solution only *almost everywhere*. This softening allows for a simpler proof without any substantial change, since we provide below a full characterization of this solution, see 1), 2) with $\omega = 0$ and 3) in Theorem 2.2.

Definition 2.1 *Let $T > 0$. A map $u = u(t,x)$ is a solution to (1.2) if*

1. $u \in \mathbf{C^0}\left([0,T];\mathbf{L^1}(\mathbb{R};\mathbb{R}^n)\right)$ *with* $u(t,x) \in \Omega$ *for a.e.* $(t,x) \in \mathbb{D}_\gamma$ *and* $u(t,x) = 0$ *otherwise;*
2. $u(0,x) = u_o(x)$ *for a.e.* $x \geq \gamma(0)$ *and* $\lim_{x\to 0+} b\left(u(t,x)\right) = g(t)$ *for a.e.* $t \geq 0$*;*
3. *for* $x > \gamma(t)$*,* u *is a weak entropy solution to* $\partial_t u + \partial_x f(u) = 0$*.*

Theorem 2.2 *Let the system (1.2) satisfy ($\boldsymbol{f}$), ($\boldsymbol{b}$), ($\boldsymbol{\gamma}$). Assume also that $g \in \mathbf{BV}(\mathbb{R}^+; \mathbb{R}^{n-\ell})$ has sufficiently small total variation. Then, there exists a family of closed domains*

$$\mathcal{D}_t \subseteq \left\{ u \in \left(\mathbf{L^1} \cap \mathbf{BV}\right)(\mathbb{R}; \Omega) : u(x) = 0 \text{ for all } x \leq \gamma(t) \right\}$$

defined for all $t \geq 0$ and containing all $\mathbf{L^1}$ functions with sufficiently small total variation that vanish to the left of $\gamma(t)$, a constant $L > 0$ and a process

$$P(t, t_o) \colon \mathcal{D}_{t_o} \to \mathcal{D}_{t_o+t}, \qquad \text{for all } t_o, t \geq 0,$$

such that

1) *for all $t_o \geq 0$ and $u \in \mathcal{D}_{t_o}$, $P(0, t_o)\, u = u$ while for all $t, s, t_o \geq 0$ and $u \in \mathcal{D}_{t_o}$, $P(t+s, t_o)\, u = P(t, t_o + s) \circ P(s, t_o) u$;*

2) *let ω be an $\mathbf{L^1}$ function with small total variation, if $(\bar{P}, \bar{\mathcal{D}}_t)$ are the process and the domain corresponding to the boundary $\bar{\gamma}(t)$ and boundary data $\bar{g}(t)$, then, for any $u \in \mathcal{D}_{t_o}$, $v \in \bar{\mathcal{D}}_{t'_o}$, we have the following Lipschitz estimate with a second order error term accounting for ω:*

$$\begin{aligned} & \left\| P(t, t_o) u - \bar{P}(t', t'_o) v - \omega \right\|_{\mathbf{L^1}} \\ \leq \quad & L \cdot \Bigg\{ \left\| u - v - \omega \right\|_{\mathbf{L^1}} + \left| t - t' \right| + \left| t_o - t'_o \right| \\ & \quad + \int_{t_o}^{t_o+t} \left\| g(\tau) - \bar{g}(\tau) \right\| d\tau + \sup_{\tau \in [t_o, t]} \left| \gamma(\tau) - \bar{\gamma}(\tau) \right| \\ & \quad + t \cdot \mathrm{TV}(\omega) \Bigg\}; \end{aligned}$$

3) *the tangent vector to P in the sense of [5, Section 5] is the map F defined at (4.6), i.e. for all $t_o \geq 0$ and $u \in \mathcal{D}_{t_o}$*

$$\lim_{t \to 0+} \frac{1}{t} \left\| F(t, t_o) u - P(t, t_o) u \right\|_{\mathbf{L^1}} = 0;$$

4) *for all $u_o \in \mathcal{D}_0$, the map $u(t, x) = \big(P(t, 0) u_o\big)(x)$ defined for $t \in [0, T]$ and $(t, x) \in \mathbb{D}_\gamma$, solves (1.2) in the sense of Definition 2.1.*

P is uniquely characterized by 1), 2) with $\omega = 0$ and 3).

The conditions *1)–3)* constitute what is the natural generalization to the present case of the definition of *Standard Riemann Semigroup*, see [6, Definition 9.1].

Remark that, in general, the Lipschitz constant L blows up as the the boundary tends to become characteristic, i.e. as $c \to 0$, see (4.18) and the next proposition. Indeed, in the proof of Theorem 2.2, we prove also the following result on the regularity of the solutions to (1.2) along non characteristic curves.

Proposition 2.3 *Fix a positive T. Let the system (1.2) satisfy the assumptions of Theorem 2.2 and call u the solution to (1.2) constructed therein. Let Γ_0, Γ_1 be $\tilde{\ell}$–non-characteristic curves, for $\tilde{\ell} \in \{1, \ldots, n-1\}$. Then, there exists a constant $\mathcal{K} > 0$ independent from T, u_o, g such that*

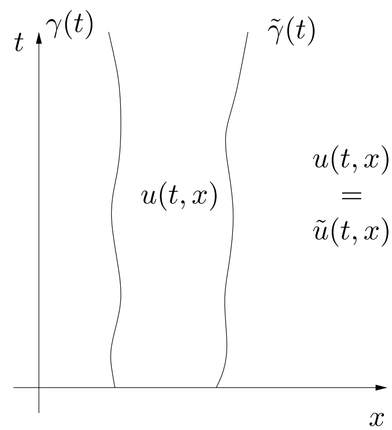
$$\int_0^T \left\| u\left(t, \Gamma_0(t)\right) - u\left(t, \Gamma_1(t)\right) \right\| dt \le \frac{\mathcal{K}}{c} \left(\mathrm{TV}(u_o) + \mathrm{TV}(g)\right) \|\Gamma_1 - \Gamma_0\|_{\mathbf{C^0}([0,T])}.$$

A uniqueness property proved in Section 4 is the following.

Proposition 2.4 *Let the system (1.2) satisfy the same assumptions of Theorem 2.2 and call u the solution to (1.2) constructed therein. Let $\tilde{\gamma} \in \mathbf{C^{0,1}}(\mathbb{R}^+; \mathbb{R})$ be any ℓ–non-characteristic curve satisfying $\tilde{\gamma}(t) \ge \gamma(t)$ for all $t \ge 0$. Define $\tilde{g}(t) = b\left(u(t, \tilde{\gamma}(t)+\right)$. Then, (1.3) also satisfies the assumptions on Theorem 2.2 and the solution $\tilde{u}$ constructed by this Theorem satisfies*

$$\tilde{u}(t,x) = u(t,x)$$

for all $x \ge \tilde{\gamma}(t)$ and $t \ge 0$.



3 The IBVP with General Source Term

To deal with the source term, for all positive δ, define

$$\mathcal{U}_\delta = \left\{ u \in \mathbf{L^1}(\mathbb{R}; \Omega) \colon \mathrm{TV}(u) \le \delta \right\}.$$

We add the following assumption on the source term of system (1.1):

(G) For a positive δ_o, $G \colon \mathcal{U}_{\delta_o} \to \mathbf{L^1}(\mathbb{R}, \mathbb{R}^n)$ is such that for suitable positive L_1, L_2

$$\begin{array}{lrcl} \forall\, u, w \in \mathcal{U}_{\delta_o} & \left\|G(u) - G(w)\right\|_{\mathbf{L^1}} & \le & L_1 \cdot \|u - w\|_{\mathbf{L^1}} \\ \forall\, u \in \mathcal{U}_{\delta_o} & \mathrm{TV}\left(G(u)\right) & \le & L_2 . \end{array}$$

The natural extension of Definition 2.1 to the present case is the following.

Definition 3.1 *Let $T > 0$. A map $u = u(t,x)$ is a solution to (1.1) if*

1. $u \in \mathbf{C^0}\left([0,T];\mathbf{L^1}(\mathbb{R};\mathbb{R}^n)\right)$ *with* $u(t,x) \in \Omega$ *for a.e.* $(t,x) \in \mathbb{D}_\gamma$ *and* $u(t,x) = 0$ *otherwise;*

2. $u(0,x) = u_o(x)$ *for a.e.* $x \geq \gamma(0)$ *and* $\lim_{x\to 0+} b\left(u(t,x)\right) = g(t)$ *for a.e.* $t \geq 0$*;*

3. *for* $x > \gamma(t)$, u *is a weak entropy solution to* $\partial_t u + \partial_x f(u) = G(u)$.

With this notation, we may now state the extension of Theorem 2.2 to the present non homogeneous case.

Theorem 3.2 *Let system (1.1) satisfy (**f**), (**G**), (**b**), (**γ**). Assume also that* $g \in \mathbf{BV}(\mathbb{R}^+;\mathbb{R}^{n-\ell})$ *has sufficiently small total variation. Then, there exist positive* δ, L, T*, domains* $\hat{\mathcal{D}}_t$*, for* $t \in [0,T]$ *and maps*

$$\hat{P}(t,t_o) \colon \hat{\mathcal{D}}_{t_o} \to \hat{\mathcal{D}}_{t_o+t}$$

for $t_o, t_o + t \in [0,T]$*, such that*

i) $\hat{\mathcal{D}}_t \supseteq \left\{u \in \mathcal{U}_\delta \colon u(x) = 0 \text{ for } x < \gamma(t)\right\}$*;*

ii) for all t_o, t_1, t_2 *with* $t_o \in [0,T[$, $t_1 \in [0, T - t_o[$ *and* $t_2 \in [0, T - t_o - t_1]$, $\hat{P}(t_2, t_o + t_1) \circ \hat{P}(t_1, t_o) = \hat{P}(t_1 + t_2, t_o)$ *and* $\hat{P}(0, t_o) = \mathrm{Id}$*;*

iii) if $(\bar{P}, \bar{\mathcal{D}}_t)$ *are the process and the domains corresponding to the boundary* $\bar{\gamma}(t)$ *and boundary data* $\bar{g}(t)$*, satisfying the same assumptions above, then, for* $t_o, t'_o \in [0,T[$, $t \in [0, T-t_o]$ *and* $t' \in [0, T - t'_o]$*, for all* $u \in \hat{\mathcal{D}}_{t_o}$, $\bar{u} \in \hat{\mathcal{D}}_{t'_o}$

$$\begin{aligned} & \left\| \hat{P}(t,t_o)u - \bar{P}(t',t'_o)\bar{u} \right\|_{\mathbf{L^1}} \\ \leq \quad & L \cdot \left\{ \|u - \bar{u}\|_{\mathbf{L^1}} + \left(1 + \|u\|_{\mathbf{L^1}}\right) \left(\left|t - t'\right| + \left|t_o - t'_o\right| \right) \right. \\ & \qquad \left. + \int_{t_o}^{t_o+t} \left\| g(\tau) - \bar{g}(\tau) \right\| d\tau + \sup_{\tau \in [t_o, t]} \left| \gamma(\tau) - \bar{\gamma}(\tau) \right| \right\}; \end{aligned}$$

iv) for all $t_o \in [0,T[$, $t \in [0, T - t_o]$, $u \in \hat{\mathcal{D}}_{t_o}$ *define*

$$\hat{F}(t,t_o)u = P(t,t_o)u + t\, G(u)\, \chi_{\left[\gamma(t_o+t), +\infty\right[}$$

then

$$\lim_{t\to 0+} \frac{1}{t} \left\| \hat{P}(t,t_o)u - \hat{F}(t,t_o)u \right\|_{\mathbf{L^1}} = 0\,;$$

v) for all $u_o \in \hat{\mathcal{D}}_0$*, the map* $u(t,x) = \left(\hat{P}(t,0)u_o\right)(x)$ *defined for* $t \in [0,T]$ *and* $(t,x) \in \mathbb{D}_t$*, solves (1.1) in the sense of Definition 3.1.*

The process $\hat{P}$ is uniquely characterized by ii), iii) and iv).

Again, as remarked after Theorem 2.2, the Lipschitz constant in general blows up as $c \to 0$. The proof of this result is deferred to Section 5, it heavily relies on Theorem 2.2. Remark that it is possible to extend to the non homogeneous case also Proposition 2.3.

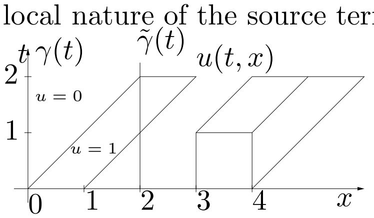
Proposition 3.3 *Let system (1.1) satisfy the same assumptions of Theorem 3.2 and call u the solution to (1.1) constructed therein. Let Γ_0, Γ_1 be $\tilde{\ell}$–non-characteristic curves, for $\tilde{\ell} \in \{1, \ldots, n-1\}$. Then, for all u_o and g, there exists a constant $\mathcal{K} > 0$ such that*

$$\int_0^T \left\| u\left(t, \Gamma_0(t)\right) - u\left(t, \Gamma_1(t)\right) \right\| dt \leq \frac{\mathcal{K}}{c} \left\|\Gamma_1 - \Gamma_0\right\|_{\mathbf{C^0}([0,T])}.$$

Remark 3.4 *Proposition 3.3 implies also that, if Γ is any ℓ–non-characteristic curve, then the map $x \to \left(\hat{P}(t,0)u\right)(x)$ is continuous in $x = \Gamma(t)$ for almost all $t \in [0,T]$. Indeed, denote $u(t,x) = \left(\hat{P}(t,0)u\right)(x)$ and compute*

$$\begin{aligned}
&\int_0^T \left| u\left(t, \Gamma(t)-\right) - u\left(t, \Gamma(t)\right) \right| dt \\
&= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^\varepsilon \int_0^T \left| u\left(t, \Gamma(t) - x\right) - u\left(t, \Gamma(t)\right) \right| dt\, dx \\
&\leq \frac{\mathcal{K}}{c} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^\varepsilon x\, dx = 0
\end{aligned}$$

Contrary to Proposition 2.3, the uniqueness result of Proposition 2.4 may not be extended to the present non homogeneous case, due to the non local nature of the source term here considered. Indeed, let



$$\begin{cases} \partial_t u + \partial_x u = \left(\int_0^1 u(t,\xi)\, d\xi\right) \chi_{[3,4]}(x) \\ u(t,0) = 0 \\ u(0,x) = \chi_{[0,1]}(x)\,. \end{cases} \tag{3.1}$$

It is immediate to verify that the assumptions of Theorem 3.2 hold. The solution u, shown above, is non zero in the delimited area above and, in particular, for $t \in [0,1]$ and $x \in [3,4]$ but it vanishes for $t \in [0,1]$ and $x = 1$. Therefore, with the same notation of Proposition 2.3, letting $\gamma(t) = 0$ and $\tilde{\gamma}(t) = 2$ we have $\tilde{g}(t) = 0$ for $t \in [0,1]$. Problem (1.3) thus admits, in the present case, only the trivial solution $u \equiv 0$, contradicting what would be the analog of Proposition 2.3 in the non homogeneous case.

4 Proofs Related to Section 2

Below, C denotes a positive constant dependent only on f, G and b whose precise value is not relevant.

This section is devoted to the homogeneous initial boundary value problem (1.2) and proves Theorem 2.2. Our general reference on the theory of conservation laws is [6].

Let $\sigma \to R_j(\sigma)(u)$, respectively $\sigma \to S_j(\sigma)(u)$, be the j-rarefaction curve, respectively the j-shock curve, exiting u. If the j-th field is linearly degenerate, then the parameter σ above is the arc-length. In the genuinely nonlinear case, see [6, Definition 5.2], we choose σ so that (see [6, formula (5,37) and Remark 5.4])

$$\frac{\partial \lambda_j}{\partial \sigma}\big(R_j(\sigma)(u)\big) = 1 \quad \text{and} \quad \frac{\partial \lambda_j}{\partial \sigma}\big(S_j(\sigma)(u)\big) = 1\,. \tag{4.1}$$

Introduce the j-Lax curve

$$\sigma \to \psi_j(\sigma)(u) = \begin{cases} R_j(\sigma)(u) & \text{if} \quad \sigma \geq 0 \\ S_j(\sigma)(u) & \text{if} \quad \sigma < 0 \end{cases}$$

and for $\boldsymbol{\sigma} \equiv (\sigma_1, \ldots, \sigma_n)$, define the map

$$\boldsymbol{\Psi}(\boldsymbol{\sigma}) = \psi_n(\sigma_n) \circ \ldots \circ \psi_1(\sigma_1)\,.$$

By $\boldsymbol{(f)}$, see [6, Paragraph 5.3], given any two states $u^-, u^+ \in \Omega$ sufficiently close to 0, there exists a $\mathbf{C^2}$ map E such that

$$\boldsymbol{\sigma} = E(u^-, u^+) \quad \text{if and only if} \quad u^+ = \boldsymbol{\Psi}(\boldsymbol{\sigma})(u^-)\,. \tag{4.2}$$

Similarly, let the map $\boldsymbol{S}$ and the vector $\boldsymbol{q} = (q_1, \ldots, q_n)$ be defined by

$$u^+ = \boldsymbol{S}(\boldsymbol{q})(u^-) \quad \text{and} \quad \boldsymbol{S}(\boldsymbol{q}) = S_n(q_n) \circ \ldots \circ S_1(q_1)\,, \tag{4.3}$$

i.e. $\boldsymbol{S}$ is the gluing of the Rankine - Hugoniot curves.

We first consider the non characteristic Riemann problem at the boundary

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 & x > \gamma(t) \\ b\Big(u\big(t, \gamma(t)\big)\Big) = g_o & t \geq 0 \\ u(0,x) = u_o & x \geq 0 \end{cases} \tag{4.4}$$

where $g_o \in \mathbb{R}^{n-\ell}$ and $u_o \in \Omega$ are constants and γ satisfies $\boldsymbol{(\gamma)}$. Then, a solution to (4.4) is constructed as in [17], see also [1, 2, 12].

Lemma 4.1 *Let (**f**), (γ) and (**b**) hold. If u_o, g_o are sufficiently small, then there exists unique E_b^σ, E_b^q of class $\mathbf{C^2}$ and states u^σ, u^q such that*

$$(\sigma_{\ell+1}, \ldots, \sigma_n) = E_b^\sigma(u_o, g_o) \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} b(u^\sigma) = g_o \text{ and} \\ \psi_n(\sigma_n) \circ \ldots \circ \psi_{\ell+1}(\sigma_{\ell+1})(u^\sigma) = u_o \,, \end{array} \right.$$

$$(q_{\ell+1}, \ldots, q_n) = E_b^q(u_o, g_o) \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} b(u^q) = g_o \text{ and} \\ S_n(q_n) \circ \ldots \circ S_{\ell+1}(q_{\ell+1})(u^q) = u_o \,. \end{array} \right.$$

Proof. We prove this statement only for the Lax curves, the results for the shock curves is proved similarly. Let $\sigma_i \to \bar\psi_i(\sigma_i)(u)$ be the inverse Lax curve, i.e.

$$\sigma \to \bar\psi_j(\sigma)(u) = \left\{ \begin{array}{lll} S_j(\sigma)(u) & \text{if} & \sigma \geq 0 \\ R_j(\sigma)(u) & \text{if} & \sigma < 0. \end{array} \right.$$

The choice (4.1) of the parameters implies that $\bar\psi_i(-\sigma_i) \circ \psi_i(\sigma_i)(u) = u$ for all small u and σ_i. Define the $\mathbf{C^2}$ function

$$G\left(\sigma_{\ell+1}, \ldots, \sigma_n, g_o, u_o\right) = b\left(\bar\psi_{\ell+1}(-\sigma_{\ell+1}) \circ \ldots \circ \bar\psi_n(-\sigma_n)(u_o)\right) - g_o \,.$$

By (***b***), G satisfies $G(0,0,0) = 0$ and

$$\det D_{\left(\sigma_{\ell+1},\ldots,\sigma_n\right)} G(0,0,0) = (-1)^{n-\ell} \det\left[Db(0)\, r_{\ell+1}(0) \cdots Db(0)\, r_n(0)\right] \neq 0 \,.$$

The Implicit Function Theorem guarantees the existence of a map $E_b^\sigma = E_b^\sigma(u_o, g_o)$ with the required properties, if $(\sigma_{\ell+1}, \ldots, \sigma_n) = E_b^\sigma(u_o, g_o)$ and $u^\sigma = \bar\psi_{\ell+1}(-\sigma_{\ell+1}) \circ \ldots \circ \bar\psi_n(-\sigma_n)(u_o)$. □

The notation introduced above allows the definition of a local flow tangent to the process generated by (1.2). Fix $t_o \geq 0$ and $u \in \mathcal{D}_{t_o}$, define $g_o = g(t_o+)$, $u_o = u\left(\gamma(t_o)+\right)$ and u^σ as in Lemma 4.1. Let

$$\tilde u(x) = \left\{ \begin{array}{lll} u^\sigma & \text{if } x < & \gamma(t_o) \\ u(x) & \text{if } x \geq & \gamma(t_o) \end{array} \right. \tag{4.5}$$

Call $\mathcal{S}$ the Standard Riemann Semigroup generated by f, see [6, Definition 9.1]. Finally, for $t \geq 0$, define the tangent vector, see [5, Section 5],

$$\left(F(t,t_o)u\right)(x) = \left\{ \begin{array}{lll} 0 & \text{if } x < & \gamma(t_o + t) \\ (\mathcal{S}_t \tilde u)(x) & \text{if } x \geq & \gamma(t_o + t) \end{array} \right. \tag{4.6}$$

We record here the following interaction estimates, see Figure 1.

Lemma 4.2 *Let* ***(f)***, ***(γ)*** *and* ***(b)*** *hold. If the following relations hold*

$$\begin{array}{rclcrcl} g^- & = & b(u^-)\,, & & u_r & = & \psi_n(\sigma_n)\circ\ldots\circ\psi_1(\sigma_1)(u^-) \\ g^+ & = & b(u^+)\,, & & u_r & = & \psi_n(\tilde\sigma_n)\circ\ldots\circ\psi_{\ell+1}(\tilde\sigma_{\ell+1})(u^+) \end{array}$$

then, we have the estimate

$$\sum_{i=\ell+1}^{n} |\tilde\sigma_i - \sigma_i| \leq C\left(\sum_{i=1}^{\ell}|\sigma_i| + \left\|g^+ - g^-\right\|\right)\,.$$

Analogously, for the shock curves, if ω *is a small vector satisfying*

$$\begin{array}{rcl} g & = & b(u), \\ \bar g & = & b(v), \end{array} \qquad \text{and} \qquad v+\omega = S_n(q_n)\circ\ldots\circ S_1(q_1)(u)$$

then, we have the estimate

$$\sum_{i=\ell+1}^{n}|q_i| \leq C\left(\sum_{i=1}^{\ell}|q_i| + \|\bar g - g\| + \|\omega\|\right)\,.$$

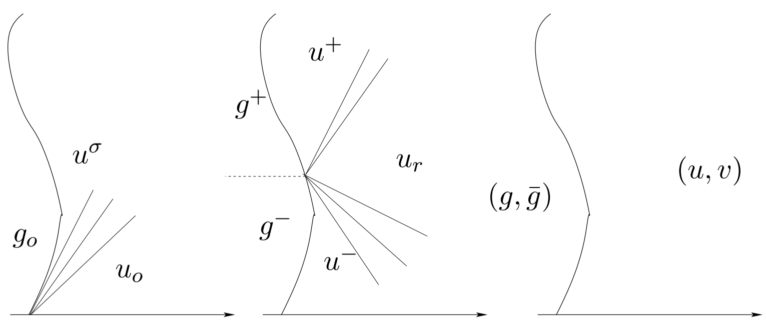


Figure 1: Interactions at the boundary

Proof. By Lemma 4.1, $(\tilde\sigma_{\ell+1},\ldots,\tilde\sigma_n) = E_b^\sigma(u_r, g^+)$ and $(\sigma_{\ell+1},\ldots,\sigma_n) = E_b^\sigma\Big(u_r, b\left(\psi_\ell(\sigma_\ell)\circ\cdots\circ\psi_1(\sigma_1)(u^-)\right)\Big)$. Therefore, the Lipschitz continuity of E_b^σ implies:

$$\begin{array}{rcl} \displaystyle\sum_{i=\ell+1}^{n}|\tilde\sigma_i - \sigma_i| & \leq & C\left(\left\|b(u^-) - b\Big(\psi_\ell(\sigma_\ell)\circ\cdots\circ\psi_1(\sigma_1)(u^-)\Big)\right\| + \left\|g^+ - g^-\right\|\right) \\ & \leq & C\left(\displaystyle\sum_{i=1}^{\ell}|\sigma_i| + \left\|g^+ - g^-\right\|\right) \end{array}$$

Concerning the shock curves, by Lemma 4.1 we can write

$$(q_{\ell+1},\ldots,q_n) \;=\; E_b^q\Big(v+\omega, b\left(S_\ell(q_\ell)\circ\cdots\circ S_1(q_1)(u)\right)\Big)\,.$$

Since $E_b^q(v,\bar g) = 0$ and $b(u) = g$, the Lipschitz continuity of E_b^q and b implies

$$\begin{aligned}
\sum_{\ell+1}^{n} |q_i| &\leq C \left\| E_b^q\Big(v+\omega, b\left(S_\ell(q_\ell)\circ\cdots\circ S_1(q_1)(u)\right)\Big) - E_b^q(v,\bar g) \right\| \\
&\leq C\left(\|\omega\| + \left\| b\left(S_\ell(q_\ell)\circ\cdots\circ S_1(q_1)(u)\right) - b(u)\right\| + \|g-\bar g\| \right) \\
&\leq C\left(\sum_{i=1}^{\ell} |q_i| + \|\omega\| + \|\bar g - g\| \right),
\end{aligned}$$

completing the proof. □

Remark 4.3 *We need below the first statement of Lemma 4.2 in the particular case of only one incident wave, i.e.* $u_r = \psi_i(\sigma_i)(u^-)$ *for some* $1 \leq i \leq \ell$.

We follow the nowadays classical wave front tracking algorithm, see [1, 2, 6, 12], to construct solutions to the homogeneous boundary value problem (1.2). Let $u \in \mathbf{L^1}\left(]\gamma(t),+\infty[,\mathbb{R}^n\right)$ be piecewise constant with finitely many jumps and assume that $\mathrm{TV}(u)$ is sufficiently small. Call $J(u)$ the finite set of points where u has a jump. Let $\sigma_{x,i}$ be the strength of the i-th wave in the solution of the Riemann problem for

$$\partial_t u + \partial_x f(u) = 0 \tag{4.7}$$

with data $u(x-)$ and $u(x+)$, i.e. $(\sigma_{x,1},\ldots,\sigma_{x,n}) = E\big(u(x-),u(x+)\big)$. Obviously, if $x \notin J(u)$ then $\sigma_{x,i} = 0$, for all $i = 1,\ldots,n$. In $x = \gamma(t)$ define

$$\Big(\sigma_{\gamma(t),1},\ldots,\sigma_{\gamma(t),n}\Big) = \Big(0,\ldots,0,E_b^\sigma\left(u(\gamma(t)+),g(t)\right)\Big). \tag{4.8}$$

Then, consider the Glimm functionals and potentials

$$\begin{aligned}
\boldsymbol{V}_t(u) &= K \sum_{x\geq\gamma(t)} \sum_{i=1}^{\ell} \left|\sigma_{x,i}\right| + \sum_{x\geq\gamma(t)} \sum_{i=\ell+1}^{n} \left|\sigma_{x,i}\right| \\
\boldsymbol{Q}_t(u) &= \sum_{(\sigma_{x,i},\sigma_{y,j})\in\mathcal{A}} \left|\sigma_{x,i}\sigma_{y,j}\right| \\
\boldsymbol{\Upsilon}_t(u) &= \boldsymbol{V}_t(u) + H_2\,\boldsymbol{Q}_t(u) + H_1\,\mathrm{TV}\left\{g,[t,+\infty[\right\}
\end{aligned} \tag{4.9}$$

the set $\mathcal{A}$ of approaching waves being defined as usual, see [6] and the constant K, H_1, H_2 to be defined later. As in [9], using Lemma 4.2, the Glimm functional $\boldsymbol{\Upsilon}$ can be extended in a lower semicontinuous way to all functions with small total variation in $\mathbf{L^1}(\mathbb{R};\mathbb{R}^n)$ that vanish for $x \leq \gamma(t)$. On the contrary, the interaction potential $\boldsymbol{Q}$ alone does not admit a lower semicontinuous extension, due to the presence of the boundary.

We now construct ε-approximate solutions to (1.2) by means of the classical wave front tracking technique, see [6] or [1, 2] for the case with boundary.

Let $\varepsilon > 0$ be fixed and approximate the initial and boundary data in (1.2) by means of piecewise constant functions u_o^ε and g^ε such that (see [9, formula (3.1)])

$$\begin{aligned} &\left\| u_o^\varepsilon - u_o \right\|_{\mathbf{L}^1\left([\gamma(0),+\infty[;\Omega\right)} < \varepsilon\,, \quad \left\| g^\varepsilon - g \right\|_{\mathbf{L}^\infty(\mathbb{R}^+,\mathbb{R})} < \varepsilon \\ &\left| \boldsymbol{\Upsilon}_0(u_o^\varepsilon) - \boldsymbol{\Upsilon}_0(u_o) \right| < \varepsilon\,, \quad \mathrm{TV}\left(g^\varepsilon,]t,+\infty[\right) \le \mathrm{TV}\left(g,]t,+\infty[\right) \text{ for } t \ge 0\,. \end{aligned} \tag{4.10}$$

To proceed beyond time $t = 0$, we construct an approximate solution to (1.2) by means of the *Accurate* and *Simplified* Riemann solvers, see [6, Paragraph 7.2]. Introduce the threshold parameter $\rho > 0$ to distinguish which Riemann solver is used at any interaction in $x > \gamma(t)$. Whenever an interaction occurs at (t, x) with $x > \gamma(t)$, proceed exactly as in [6, Paragraph 7.2]. Recall that the former solver splits new rarefaction waves in fans of wavelets having size at most ε, while the latter yields nonphysical waves. These waves are assigned to a fictitious $n+1$-th family and their strength is the Euclidean distance between the states on their sides.

At any interaction involving the boundary, i.e. when a wave hits the boundary as well as when the approximated boundary data changes (see Figure 1, left and center), we use the Accurate solver, independently from the size of the interaction. As usual, rarefaction waves are not further split at interactions.

Along an ε-approximate solution, for suitable constants K, H_1, H_2 all greater than 1, introduce the linear and quadratic potentials and the Glimm functional:

$$\begin{aligned} V^\varepsilon(t) &= K \sum_{x \ge \gamma(t)} \sum_{i=1}^{\ell} \left|\sigma_{x,i}\right| + \sum_{x \ge \gamma(t)} \sum_{i=\ell+1}^{n+1} \left|\sigma_{x,i}\right| \\ V_g^\varepsilon(t) &= \mathrm{TV}\left(g^\varepsilon; [t,+\infty[\right) \\ Q^\varepsilon(t) &= \sum_{(\sigma_{x,i},\sigma_{y,j}) \in \mathcal{A}} \left|\sigma_{x,i}\sigma_{y,j}\right| \\ \Upsilon^\varepsilon(t) &= V^\varepsilon(t) + H_1\, V_g^\varepsilon(t) + H_2\, Q^\varepsilon(t)\,. \end{aligned} \tag{4.11}$$

The potentials just defined differ from the ones defined in (4.9) because the non–physical waves are accounted for in a different way. They coincide at $t = 0$ because of the absence, at that time, of non-physical waves. They differ of a quantity proportional to the total size of non physical waves for $t > 0$.

As usual, changing a little the velocities of the waves, we may assume that no more than two waves σ', σ'' collide at any interaction point $(\bar t, \bar x)$. When $\bar x > \gamma(\bar t)$, the usual interaction estimates yield, for a constant $C > 0$

dependent only on f and Ω,

$$\begin{array}{rclcrcl} \Delta V^\varepsilon(\bar t) & \leq & C\,K\,\left|\sigma'\sigma''\right| & & \Delta V_g^\varepsilon(\bar t) & = & 0 \\ \Delta Q^\varepsilon(\bar t) & \leq & -\frac{1}{2}\left|\sigma'\sigma''\right| & & \Delta \Upsilon^\varepsilon(\bar t) & \leq & -\frac{H_2}{4}\left|\sigma'\sigma''\right|, \end{array}$$

as soon as $CK < H_2/4$ and δ_o is sufficiently small.

When a wave σ hits the boundary, Lemma 4.2 implies that

$$\begin{array}{rclcrcl} \Delta V^\varepsilon(\bar t) & \leq & (C-K)\,|\sigma| & & \Delta V_g^\varepsilon(\bar t) & = & 0 \\ \Delta Q^\varepsilon(\bar t) & \leq & C\,|\sigma|\,V^\varepsilon(\bar t-) & & \Delta \Upsilon^\varepsilon(\bar t) & \leq & -\frac{K}{2}|\sigma| \end{array}$$

as soon as $K > 4C$ and $\delta_o < \frac{1}{2H_2}$.

When the boundary data changes, then

$$\begin{array}{rclcrcl} \Delta V^\varepsilon(\bar t) & \leq & C\left|\Delta g^\varepsilon(\bar t)\right| & & \Delta V_g^\varepsilon(\bar t) & = & -\left|\Delta g^\varepsilon(\bar t)\right| \\ \Delta Q^\varepsilon(\bar t) & \leq & C\left|\Delta g^\varepsilon(\bar t)\right|V^\varepsilon(\bar t-) & & \Delta \Upsilon^\varepsilon(\bar t) & \leq & -\frac{H_1}{3}\left|\Delta g^\varepsilon(\bar t)\right| \end{array}$$

as soon as $H_1 > 3C$ and $\delta_o < 1/H_2$.

The above choices are consistent. Indeed, choose first H_1 and K, then H_2 and finally δ_o.

The wave front tracking approximation can be constructed for all times, indeed we show that the total number of interaction points is finite: waves of families $1,\ldots,\ell$ are created only through the Accurate solver and the use of the Accurate solver in $\left\{(t,x)\colon t>0,\, x>\gamma(t)\right\}$ leads to a uniform decrease in Υ^ε. Therefore only a finite number of waves, which can hit the boundary, is present. Since also the jumps in the boundary are finite, there are at most a finite number of points in the boundary with outgoing waves. This observation, together the argument used in the standard case, see [6], shows that the total number of interactions is finite on all the domain $\left\{(t,x)\colon t\geq 0,\, x\geq\gamma(t)\right\}$.

As in [6, Paragraph 7.3], the strength of any rarefaction, respectively nonphysical, wave is smaller than $C\varepsilon$, respectively $C\rho$. This estimate is proved simply substituting $Q(t)$ in [6, formula (7.65)] with the strictly decreasing functional $\Upsilon^\varepsilon(t)$ defined at (4.11).

As in the standard case, choosing ρ sufficiently small we prove that the total size of nonphysical waves is bounded by ε. To this aim, recall the *generation order* of a wave. Waves created at time $t=0$, as well as waves originating from jumps in the boundary data, are assigned order 1. When two waves interact in the interior $\left\{(t,x)\colon t>0,\, x>\gamma(t)\right\}$ of the domain, the usual procedure [6, Paragraph 7.3] is followed. When a wave of order k hits the boundary, all the reflected wave are assigned the same order k.

For $k\geq 1$, define

$$V_k^\varepsilon(t) \;=\; K\sum\left\{|\sigma|\colon \sigma \begin{array}{l}\text{has order} \geq k\\ \text{is of family} \leq \ell\end{array}\right\} + \sum\left\{|\sigma|\colon \sigma \begin{array}{l}\text{has order} \geq k\\ \text{is of family} > \ell\end{array}\right\}$$

$$Q_k^\varepsilon(t) \;=\; \sum \left\{ \left|\sigma\sigma'\right| : \sigma, \sigma' \begin{array}{l} \text{are approaching} \\ \text{one of them has order } \geq k \end{array} \right\}.$$

As in [6, Paragraph 7.3], for $k \geq 1$, let I_k denote the set of those interaction times at which the maximal order of the interacting waves is k. I_0 denotes the set containing $t = 0$ and all times at which there is a jump in the boundary data. On the other hand, J_k is the set of those interaction times at which a wave of order k hits the boundary. A careful examinations of the possible interaction yields the following table for $k \geq 3$:

$$\begin{array}{rcllcl} \Delta V_k^\varepsilon(t) & = & 0 & t & \in & I_0 \cup I_1 \cup \ldots \cup I_{k-2}, \\ \Delta V_k^\varepsilon(t) + H_2 \, \Delta Q_{k-1}^\varepsilon(t) & \leq & 0 & t & \in & I_{k-1} \cup I_k \cup \ldots, \\ \Delta V_k^\varepsilon(t) & = & 0 & t & \in & J_1 \cup J_2 \cup \ldots \cup J_{k-1}, \\ \Delta V_k^\varepsilon(t) & \leq & 0 & t & \in & J_k \cup J_{k+1} \cup \ldots . \end{array}$$

Denote the positive, respectively negative, part of a real number by: $[\![x]\!]_+ = \max\{0, x\}$, respectively $[\![x]\!]_- = [\![-x]\!]_+$. Therefore, similarly to [6, formula (7.69)], we get for $k \geq 3$:

$$\begin{aligned} V_k^\varepsilon(t) &\leq \sum_{0<s\leq t} [\![\Delta V_k^\varepsilon(s)]\!]_+ \\ &\leq H_2 \sum_{0<s\leq t} [\![\Delta Q_{k-1}^\varepsilon(s)]\!]_- \leq H_2 \sum_{0<s\leq t} [\![\Delta Q_{k-1}^\varepsilon(s)]\!]_+ . \end{aligned}$$

Now we need to estimate the last sum: $\widetilde{Q}_k^\varepsilon(t) = \sum_{0<s\leq t} [\![\Delta Q_k^\varepsilon(s)]\!]_+$. Observe that for $k \geq 3$:

$$\begin{array}{rcllcl} \Delta Q_k^\varepsilon(t) + \Delta \Upsilon^\varepsilon(t) \cdot V_k^\varepsilon(t-) & \leq & 0 & t & \in & I_0 \cup I_1 \cup \ldots \cup I_{k-2}, \\ \Delta Q_k^\varepsilon(t) + H_2 \, \Delta Q_{k-1}^\varepsilon(t) \cdot V^\varepsilon(t-) & \leq & 0 & t & \in & I_{k-1}, \\ \Delta Q_k^\varepsilon(t) & \leq & 0 & t & \in & I_k \cup I_{k+1} \cup \ldots, \\ \Delta Q_k^\varepsilon(t) + \Delta \Upsilon^\varepsilon(t) \cdot V_k^\varepsilon(t-) & \leq & 0 & t & \in & J_1 \cup J_2 \cup \ldots \cup J_{k-1}, \\ \Delta Q_k^\varepsilon(t) + \Delta V_k^\varepsilon(t) \cdot V^\varepsilon(t-) & \leq & 0 & t & \in & J_k \cup J_{k+1} \cup \ldots . \end{array}$$

Hence we can write

$$\begin{aligned} \widetilde{Q}_k^\varepsilon(t) \;\leq\; & \sum_{0<s\leq t} \left[[\![\Delta V_k^\varepsilon(s)]\!]_- + H_2 \, [\![\Delta Q_{k-1}^\varepsilon(s)]\!]_- \right] \sup_{0\leq\tau\leq t} V^\varepsilon(\tau) \\ & + \sum_{0<s\leq t} [\![\Delta \Upsilon^\varepsilon(s)]\!]_- \cdot \sup_{0\leq\tau\leq t} V_k^\varepsilon(\tau) \end{aligned}$$

$$
\begin{aligned}
&\le \delta \sum_{0<s\le t} \left[[\![\, \Delta V_k^\varepsilon(s) \,]\!]_+ + H_2 \, [\![\, \Delta Q_{k-1}^\varepsilon(s) \,]\!]_+ \right] + \Upsilon^\varepsilon(0) \sup_{0\le\tau\le t} V_k^\varepsilon(\tau) \\
&\le 3\delta H_2 \cdot \widetilde{Q}_{k-1}^\varepsilon(t) \,.
\end{aligned}
$$

By induction we obtain

$$
\widetilde{Q}_k^\varepsilon(t) \le (3\delta H_2)^{k-2} \, \widetilde{Q}_2^\varepsilon(t) \le (3\delta H_2)^{k-2} \, \delta \,.
$$

Therefore, if δ is sufficiently small (so that $3\delta H_2 < 1$), there exists $N_\varepsilon > 0$ such that the total size of the waves of order greater or equal to N_ε is smaller than ε:

$$
V_k^\varepsilon(t) \le H_2 \, \widetilde{Q}_k^\varepsilon(t) \le H_2 \, (3\delta H_2)^{k-2} \, \delta \le \varepsilon \,, \qquad \text{for } k \ge N_\varepsilon \,.
$$

Now we observe that the numbers of wave of a given order, is bounded by a number that depends on ε but not on the threshold ρ: indeed let M_ε the maximum total number of waves that can be generated in a solution of a Riemann problem inside the domain or at the boundary. Let $\overline{M}$ be the sum of the total number of jumps in the initial data and in the boundary data. The wave of first generation are born at $t = 0$, at the jumps in the boundary or when a wave of first generation hit the boundary. Since the waves of first generation which can hit the boundary (the ones which belong to the families $i = 1, \ldots, \ell$) are born only at $t = 0$, the total number of first generation waves is bounded by a constant $C_\varepsilon^1 = \overline{M} \cdot M_\varepsilon + \overline{M} \cdot M_\varepsilon \cdot M_\varepsilon$ not depending on the threshold. Suppose now that the number of waves of generation lower or equal to k is bounded by a constant C_ε^k not depending on ρ. The waves of order $k+1$ can be generated only when two waves of lower order interact, or when a wave of order $k+1$ hit the boundary. Since the waves of order $k+1$ which can hit the boundary can only be generated by interaction of waves of lower order, the total number of generation $k+1$ waves is bounded by $C_\varepsilon^{k+1} = C_\varepsilon^k \cdot C_\varepsilon^k \cdot M_\varepsilon + C_\varepsilon^k \cdot C_\varepsilon^k \cdot M_\varepsilon \cdot M_\varepsilon$ which do not depends on ρ. Fix now k such that $V_k^\varepsilon \le \varepsilon$. Hence also the total strength of non physical waves with order greater or equal then k is lower than ε. Then observe that the total number of non physical waves with order less than k is obviously bounded by C_ε^k. Since the strength of any single non physical wave is bounded by $C\rho$, if we choose the threshold ρ such that $C\rho \cdot C_\varepsilon^k \le \varepsilon$, we have that the total strength of non physical waves is bounded by 2ε. Finally we observe that if γ is any ℓ–non-characteristic curve, then $\mathrm{TV}\Big(u\left(\cdot, \bar\gamma(\cdot)\right)\Big)$ is uniformly bounded by a constant time $\mathrm{TV}(u_o)+\mathrm{TV}(g)$. Indeed, this property is proved following the techniques in [13, Theorem 14.4.2 and formula (14.5.19)] with our strictly decreasing functional Υ^ε.

The following lemma on the regularity of u along non-characteristic curves is of use in the sequel.

Lemma 4.4 *Fix a positive T. Let u be an ε-approximate wave front tracking solution to (1.2). Let Γ_0, Γ_1 be ℓ–non-characteristic curves. Then, there exists a constant $\mathcal{K} > 0$ independent from T such that*

$$\int_0^T \left\| u\left(t, \Gamma_0(t)\right) - u\left(t, \Gamma_1(t)\right) \right\| dt \leq \frac{\mathcal{K}}{c} \left(\mathrm{TV}(u_o) + \mathrm{TV}(g)\right) \|\Gamma_1 - \Gamma_0\|_{\mathbf{C^0}([0,T])}$$

Proof. Let Γ be an ℓ–non-characteristic curve. Consider a perturbation $\eta \in \mathbf{C^{0,1}}(\mathbb{R}^+; \mathbb{R})$ with $\|\eta\|_{\mathbf{C^0}} + \|\dot\eta\|_{\mathbf{L^\infty}}$ sufficiently small. By the above construction of ε-solutions, there exist times t_α and states u_α such that

$$u\left(t, \Gamma(t)\right) = \sum_\alpha u_\alpha \, \chi_{[t_\alpha, t_{\alpha+1}[}(t) \qquad \text{and} \qquad \Gamma(t_\alpha) = \lambda_\alpha t_\alpha + x_\alpha \tag{4.12}$$

Indeed, here $x = \lambda_\alpha t + x_\alpha$ is the equation of a discontinuity line in u crossed by Γ. If (t_α, x_α) is a point of interaction in u, then we convene that all states attained by u in a neighborhood of (t_α, x_α) appear in the sum in (4.12), possibly multiplied by the characteristic function of the empty interval.

If $\|\eta\|_{\mathbf{C^1}}$ is sufficiently small, then there exists times t'_α such that

$$u\left(t, \Gamma(t) + \eta(t)\right) = \sum_\alpha u_\alpha \chi_{\left[t'_\alpha, t'_{\alpha+1}\right[}(t) \qquad \text{and} \qquad \Gamma(t'_\alpha) + \eta(t'_\alpha) = \lambda_\alpha t'_\alpha + x_\alpha \,.$$

Subtracting term by term, we obtain

$$\begin{aligned}
\lambda_\alpha \left(t'_\alpha - t_\alpha\right) &= \left(\Gamma(t'_\alpha) - \Gamma(t_\alpha)\right) + \eta(t'_\alpha) \\
&= \int_0^1 \dot\Gamma\left(\vartheta t'_\alpha + (1-\vartheta) t_\alpha\right) \, d\vartheta \, \left(t'_\alpha - t_\alpha\right) + \eta(t'_\alpha) \,. \\
\left|t'_\alpha - t_\alpha\right| &= \left| \frac{\eta(t'_\alpha)}{\lambda_\alpha - \int_0^1 \dot\Gamma\left(\vartheta t'_\alpha + (1-\vartheta) t_\alpha\right) \, d\vartheta} \right| \\
&\leq \frac{\|\eta\|_{\mathbf{C^0}}}{c} \,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \int_0^T \left\| u\left(t, \Gamma(t)\right) - u\left(t, \Gamma(t) + \eta(t)\right) \right\| dt \\
= \; & \sum_\alpha \|u_\alpha - u_{\alpha-1}\| \left|t'_\alpha - t_\alpha\right| \;\leq\; \frac{\|\eta\|_{\mathbf{C^0}}}{c} \sum_\alpha \|u_\alpha - u_{\alpha-1}\| \\
\leq \; & \frac{\|\eta\|_{\mathbf{C^0}}}{c} \mathrm{TV}\left(u(\cdot, \Gamma(\cdot)\right) \;\leq\; \mathcal{K} \frac{\|\eta\|_{\mathbf{C^0}}}{c} \left(\mathrm{TV}(u_o) + \mathrm{TV}(g)\right)
\end{aligned} \tag{4.13}$$

proving Lipschitz continuity for η small. We pass to the general case through an interpolation argument. Introduce the map

$$\psi(\vartheta) = \int_0^T \left\| u\left(t, (1-\vartheta)\Gamma_0(t) + \vartheta \Gamma_1(t)\right) - u\left(t, \Gamma_0(t)\right) \right\| dt \,.$$

The estimates above prove that the map $\vartheta \to u\left(\cdot, (1-\vartheta)\Gamma_0(\cdot) + \vartheta\Gamma_1(\cdot)\right)$ is continuous in $\mathbf{L^1}$, hence also ψ is continuous and by (4.13) its upper right Dini derivative satisfies

$$D^+\psi(\vartheta) \leq \mathcal{K}\frac{\mathrm{TV}(u_o) + \mathrm{TV}(g)}{c}\|\Gamma_1 - \Gamma_o\|_{\mathbf{C^0}}$$

for all $\vartheta \in [0.1]$. Hence, by the theory of differential inequalities,

$$\begin{aligned}\int_0^T \left\|u\left(t, \Gamma_1(t)\right) - u\left(t, \Gamma_0(t)\right)\right\| dt &= \psi(1) - \psi(0) \\ &\leq \mathcal{K}\frac{\mathrm{TV}(u_o) + \mathrm{TV}(g)}{c}\|\Gamma_1 - \Gamma_o\|_{\mathbf{C^0}([0,T])}\end{aligned}$$

completing the proof. □

We want now to compare different solutions. Take two ε–approximate solutions u, v corresponding to the two initial data u_o, v_o and the two boundary data g and $\bar{g}$. Let ω be a piecewise constant function with the following properties: $\omega(t, \cdot)$ is an $\mathbf{L^1}$–function with small total variation, $\omega(t, x)$ has finitely many polygonal lines of discontinuity and the slope of any discontinuity line is bounded in absolute value by $\widehat{\lambda}$. The function ω does not need to have any relation with the conservation law.

Define the functions $w = v + \omega$ and $\boldsymbol{q} \equiv (q_1, \ldots, q_n)$ implicitly by

$$w(t, x) = \boldsymbol{S}\left(\boldsymbol{q}(t, x)\right)\left(u(t, x)\right)$$

with $\boldsymbol{S}$ as in (4.3). We now consider the functional

$$\begin{aligned}\Phi(u, w)(t) &= \bar{K}\sum_{i=1}^{\ell}\int_{\gamma(t)}^{+\infty}\left|q_i(t, x)\right| W_i(t, x)\, dx \\ &\quad + \sum_{i=\ell+1}^{n}\int_{\gamma(t)}^{+\infty}\left|q_i(t, x)\right| W_i(t, x)\, dx\end{aligned} \tag{4.14}$$

where $\bar{K}$ is a constant to be defined later and the weights W_i are defined setting:

$$W_i(t, x) = 1 + \kappa_1 A_i(t, x) + \kappa_2\left(\Upsilon^\varepsilon\left(u(t)\right) + \Upsilon^\varepsilon\left(v(t)\right)\right).$$

The functions A_i are defined as follows. Denote by $\sigma_{x,\kappa}$ the size of a jump (in u or v) located at x of the family κ ($\kappa = n+1$ for non physical waves). Recall that $J(u)$, respectively $J(v)$ denote the sets of all jumps in u, respectively in v, for $x > \gamma(t)$, while $\bar{J}(u)$, $\bar{J}(v)$ are the sets of the physical jumps only.

If the i-th characteristic field is linearly degenerate, we simply define

$$A_i(x) \doteq \sum\left\{\left|\sigma_{y,\kappa}\right| : y \in \bar{J}(u) \cup \bar{J}(v) \text{ and } \begin{array}{l} y < x,\ i < \kappa \leq n, \text{ or} \\ y > x,\ 1 \leq \kappa < i \end{array}\right\}$$

On the other hand, if the i-th field is genuinely nonlinear, the definition of A_i will contain an additional term, accounting for waves in u and in v of the same i-th family:

$$A_i(x) \doteq \sum\left\{\left|\sigma_{y,\kappa}\right| : y \in \bar{J}(u) \cup \bar{J}(v) \text{ and } \begin{array}{l} y < x,\ i < \kappa \leq n, \text{ or} \\ y > x,\ 1 \leq \kappa < i \end{array}\right\}$$
$$+ \begin{cases} \sum\left\{\left|\sigma_{y,i}\right| : \begin{array}{l} y \in \bar{J}(u),\ y < x \text{ or} \\ y \in \bar{J}(v),\ y > x \end{array}\right\} & \text{if } q_i(x) < 0, \\ \sum\left\{\left|\sigma_{y,i}\right| : \begin{array}{l} y \in \bar{J}(v),\ y < x \text{ or} \\ y \in \bar{J}(u),\ y > x \end{array}\right\} & \text{if } q_i(x) \geq 0. \end{cases} \tag{4.15}$$

Recall that non-physical fronts play no role in the definition of A_i. We remark that the function ω enters the definition of A_i only indirectly by influencing the sign of the scalar functions q_i. The constants κ_1, κ_2 are the same defined in [6]. We also recall that, since δ_o is chosen small enough, the weights satisfy $1 \leq W_i(t,x) \leq 2$, hence for a suitable constant $C_3 > 1$,

$$\frac{1}{C_3}\left\|w(t) - u(t)\right\|_{\mathbf{L^1}} \leq \Phi(u,w)(t) \leq C_3\left\|w(t) - u(t)\right\|_{\mathbf{L^1}}, \tag{4.16}$$

where the $\mathbf{L^1}$ norm is taken in the interval $\left]\gamma(t), +\infty\right[$.

We state now the following theorem.

Proposition 4.5 *Let the system (1.2) satisfy the assumptions of Theorem 2.2. Then, there exists a constant $\delta \in \left]0, \delta_o\right[$ such that, let u, v, ω, w be the functions previously defined, satisfying $\Upsilon^\varepsilon\left(u(t)\right)$, $\Upsilon^\varepsilon\left(v(t)\right)$, $\Upsilon^\varepsilon\left(\omega(t)\right)$, $\Upsilon^\varepsilon\left(w(t)\right) \leq \delta$, for any $t \geq 0$, then one has*

$$\begin{aligned}\Phi(u,w)(t_2) \leq\ & \Phi(u,w)(t_1) + C\varepsilon(t_2 - t_1) \\ & + C\int_{t_1}^{t_2}\left(\left\|b\left(u\left(s,\gamma(s)\right)\right) - b\left(v\left(s,\gamma(s)\right)\right)\right\| + \mathrm{TV}\left(\omega(s,\cdot)\right)\right) ds.\end{aligned}$$

An immediate consequence of the above result that is useful below is

$$\begin{aligned}\Phi(u,w)(t_2) \leq\ & \Phi(u,w)(t_1) + C\varepsilon(t_2 - t_1) \\ & + C\int_{t_1}^{t_2}\left(\left\|g(s) - \bar{g}(s)\right\| + \mathrm{TV}\left(\omega(s,\cdot)\right)\right) ds.\end{aligned} \tag{4.17}$$

Proof of Proposition 4.5. In this proof we use the main results obtained in [3, 6]. At each x define the intermediate states $U_0(x) = u(x)$, $U_1(x)$, ..., $U_n(x) = w(x)$ by setting

$$U_i(x) \doteq S_i\left(q_i(x)\right) \circ S_{i-1}\left(q_{i-1}(x)\right) \circ \cdots \circ S_1\left(q_1(x)\right)\left(u(x)\right) .$$

Moreover, call

$$\lambda_i(x) \doteq \lambda_i\left(U_{i-1}(x), U_i(x)\right)$$

the speed of the i-shock connecting $U_{i-1}(x)$ with $U_i(x)$. For notational convenience, we write $q_i^{y+} \doteq q_i(y+)$, $q_i^{y-} \doteq q_i(y-)$ and similarly for $W_i^{y\pm}$, $\lambda_i^{y\pm}$. If $y, \tilde{y}$ are two consecutive points in $J = J(u) \cup J(v) \cup J(\omega)$, then $q_i^{y+} = q_i^{\tilde{y}-}$, $W_i^{y+} = W_i^{\tilde{y}-}$, $\lambda_i^{y+} = \lambda_i^{\tilde{y}-}$. Therefore, similarly to [6, 15], outside the interaction times we can compute:

$$\begin{aligned}
\frac{d}{dt}\Phi(u,w)(t) &= \bar{K}\sum_{y\in J}\sum_{i=1}^{\ell}\left(W_i^{y+}\left|q_i^{y+}\right|(\lambda_i^{y+}-\dot{x}_y) - W_i^{y-}\left|q_i^{y-}\right|(\lambda_i^{y-}-\dot{x}_y)\right) \\
&\quad + \sum_{y\in J}\sum_{i=\ell+1}^{n}\left(W_i^{y+}\left|q_i^{y+}\right|(\lambda_i^{y+}-\dot{x}_y) - W_i^{y-}\left|q_i^{y-}\right|(\lambda_i^{y-}-\dot{x}_y)\right) \\
&\quad + \bar{K}\sum_{i=1}^{\ell} W_i^{\gamma+}\left|q_i^{\gamma+}\right|(\lambda_i^{\gamma+}-\dot{\gamma}) + \sum_{i=\ell+1}^{n} W_i^{\gamma+}\left|q_i^{\gamma+}\right|(\lambda_i^{\gamma+}-\dot{\gamma})
\end{aligned}$$

where $\dot{x}_y$ is the velocity of the discontinuity at the point y. This is because the quantities q_i vanish outside a compact set. For each jump point $y \in J$ and every $i = 1, \ldots, n$, define

$$\begin{aligned}
\bar{q}_i^{y\pm} &= \begin{cases} \bar{K} q_i^{y\pm} & \text{if } i \leq \ell \\ q_i^{y\pm} & \text{if } i \geq \ell+1 \end{cases} \\
E_{y,i} &= W_i^{y+}\left|\bar{q}_i^{y+}\right|(\lambda_i^{y+}-\dot{x}_y) - W_i^{y-}\left|\bar{q}_i^{y-}\right|(y_i^{y-}-\dot{x}_y)\,.
\end{aligned}$$

so that

$$\begin{aligned}
\frac{d}{dt}\Phi(u,w)(t) &= \sum_{y\in J}\sum_{i=1}^{n} E_{y,i} \\
&\quad + \bar{K}\sum_{i=1}^{\ell} W_i^{\gamma+}\left|q_i^{\gamma+}\right|(\lambda_i^{\gamma+}-\dot{\gamma}) + \sum_{i=\ell+1}^{n} W_i^{\gamma+}\left|q_i^{\gamma+}\right|(\lambda_i^{\gamma+}-\dot{\gamma})\,.
\end{aligned}$$

Note that $\bar{q}_i^{y\pm}$ is a reparametrization of the shock curve equivalent to that provided by $q_i^{y\pm}$ and that satisfies the key property, see [6, Remark 5.4],

$$\left(S_i(\bar{q}_i) \circ S_i(-\bar{q}_i)\right)(u) = u\,.$$

Therefore, the computations in [3, Section 4] and [6, Chapter 8] apply. As in [3, formula (4.13)] we thus obtain

$$\sum_{y\in J}\sum_{i=1}^{n} E_{y,i} \leq C \cdot \left(\varepsilon + \mathrm{TV}(\omega)\right)\,.$$

Concerning the term on the boundary, $(\boldsymbol{\gamma})$ implies that if $i \leq \ell$, then $\lambda_i^{\gamma+} - \dot\gamma \leq -c$. Moreover, $W_i^{\gamma+} \geq 1$. Hence, if $g^\varepsilon = b\left(u(t,\gamma(t)+\right)$, $\bar g^\varepsilon = b\left(v(t,\gamma(t)+\right)$, Lemma 4.2 implies

$$\begin{aligned}
& \bar K \sum_{i=1}^{\ell} W_i^{\gamma+} \left|q_i^{\gamma+}\right| (\lambda_i^{\gamma+} - \dot\gamma) + \sum_{i=\ell+1}^{n} W_i^{\gamma+} \left|q_i^{\gamma+}\right| (\lambda_i^{\gamma+} - \dot\gamma) \\
\leq\ & -c\bar K \sum_{i=1}^{\ell} \left|q_i^{\gamma+}\right| + C \sum_{i=\ell+1}^{n} \left|q_i^{\gamma+}\right| \\
\leq\ & -c\bar K \sum_{i=1}^{\ell} \left|q_i^{\gamma+}\right| + C \sum_{i=1}^{\ell} \left|q_i^{\gamma+}\right| + C\left(\|g^\varepsilon - \bar g^\varepsilon\| + \left\|\omega^{\gamma+}\right\|\right) \\
\leq\ & C\left(\|g^\varepsilon - \bar g^\varepsilon\| + \left\|\omega^{\gamma+}\right\|\right)
\end{aligned}$$

provided

$$\bar K > C/c \tag{4.18}$$

is sufficiently large. Therefore, reinserting the t variable, we obtain

$$\begin{aligned}
\frac{d}{dt}\Phi(u,w)(t) \ &\leq\ C\left(\varepsilon + \mathrm{TV}\left(\omega(t,\cdot)\right) + \left\|\omega(t,\gamma(t)+)\right\| + \left\|g^\varepsilon(t) - \bar g^\varepsilon(t)\right\|\right) \\
&\leq\ C\left(\varepsilon + \mathrm{TV}\left(\omega(t,\cdot)\right) + \left\|b\left(u\left(s,\gamma(s)\right)\right) - b\left(v\left(s,\gamma(s)\right)\right)\right\|\right).
\end{aligned}$$

Then, standard computations (see [6, Theorem 8.2]) show that when an interaction occurs, the possible increase in $A_i(x)$ is compensated by a decrease in Υ^ε. Therefore, the functional Φ is not increasing at interaction times. Hence, integrating the previous inequality, we obtain (4.17). □

Proposition 4.6 *Let system (1.2) satisfy the assumptions of Theorem 2.2. Then, there exists a process P satisfying* 1) *in Theorem 2.2,* 3. *in Definition 2.1 and moreover, there exists a positive L such that for all u, v, ω,*

$$\begin{aligned}
& \left\|P(t,t_o)u - \bar P(t',t_o')v - \omega\right\|_{\mathbf{L}^1} \\
\leq\ & L \cdot \left\{ \|u - v - \omega\|_{\mathbf{L}^1} + \left|t - t'\right| + \left|t_o - t_o'\right| \right. \\
& \left. \qquad + \int_{t_o}^{t_o+t} \left\|g(\tau) - \bar g(\tau)\right\| d\tau + t \cdot \mathrm{TV}(\omega) \right\}.
\end{aligned} \tag{4.19}$$

Proof. Let $\delta > 0$ be the constant of Proposition 4.5. Define

$$\mathcal{D}_t = \left\{ u \in \mathbf{L}^1(\mathbb{R};\Omega) \colon u(x) = 0 \text{ for all } x \leq \gamma(t) \text{ and } \boldsymbol{\Upsilon}_t(u) \leq \delta/2 \right\}.$$

Fix $u_o \in \mathcal{D}_o$. Approximate the initial and boundary data (u_o, g_o) as in (4.10). Since $\Upsilon^\varepsilon(0) \leq \mathbf{\Upsilon}_0\left(u^\varepsilon(0,\cdot)\right) \leq \mathbf{\Upsilon}_0(u_o) + \varepsilon < \delta/2 + \varepsilon < \delta$, we can construct the ε-approximate solutions $u^\varepsilon(t,x)$. As in [6, Section 8.3] we observe that for $0 < \varepsilon' \leq \varepsilon$, the ε'-approximate solution is also an ε-approximate solution. Therefore, we can apply (4.17) with $u^{\varepsilon'}$ in place of v with $\omega = 0$ and $g = \bar{g}$. Hence, because of (4.16), we obtain

$$\left\| u^\varepsilon(t) - u^{\varepsilon'}(t) \right\|_{\mathbf{L^1}} \leq L \cdot \left\| u_o^\varepsilon - u_o^{\varepsilon'} \right\|_{\mathbf{L^1}} + \varepsilon \cdot t\,.$$

For any $t \geq 0$, $u^\varepsilon(t)$ is a Cauchy sequence which converges to a function $u(t) \in \mathbf{L^1}(\mathbb{R};\mathbb{R}^n)$ that vanishes for $x \leq \gamma(t)$. The potential $\Upsilon^\varepsilon(t)$ defined on ε-approximate solutions is non increasing and differs from $\mathbf{\Upsilon}_t\left(u^\varepsilon(t)\right)$ due to nonphysical waves and to the different boundary conditions g^ε and g. Therefore, (4.10) and the lower semicontinuity of the total variation and of $\mathbf{\Upsilon}_t$ implies that $u(t) \in \mathcal{D}_t$ for any $t \geq 0$. We set $P(t,0)u_o = u(t)$. It is obvious that our procedure can start at any time $t_o \geq 0$, so we can define $P(t,t_o)u \in \mathcal{D}_{t+t_o}$ for any $u \in \mathcal{D}_{t_o}$.

We want to show now that the map just defined satisfies all the properties of Theorem 2.2. The Lipschitz continuity $t \to P(t,t_o)u$ is satisfied by construction. If we now consider a different initial and boundary data, say $(v,\tilde{g})$ and the same boundary curve γ, in general we have a different map $\tilde{P}$. Taking the limit in (4.17) and using (4.16) for the corresponding ε-approximations, we get that for any $\mathbf{L^1}$ function ω dependent only on x and with small total variation

$$\begin{aligned}
&\left\| P(t,t_o)u - \tilde{P}(t,t_o)v - \omega \right\|_{\mathbf{L^1}} \\
\leq\ & L \cdot \left\{ \|u - v - \omega\|_{\mathbf{L^1}} + \int_{t_o}^{t_o+t} \left\| g(\tau) - \tilde{g}(\tau) \right\| d\tau + t \cdot \mathrm{TV}(\omega) \right\}.
\end{aligned} \tag{4.20}$$

bounding the dependence from the error term ω and proving the Lipschitz continuity in g and u.

Point *3.* in Definition 2.1 is obtained by standard methods, see [6, Section 7.4].

Concerning the process property, take $u \in \mathcal{D}_0$ and consider its ε–approximation u^ε. Let $\tilde{\varepsilon} \in\,]0,\varepsilon[$ and call $\tilde{u}^{\tilde{\varepsilon}}$ be the $\tilde{\varepsilon}$-approximate solution with initial datum $u^\varepsilon(t)$ at time t. Then, if $s \geq 0$, $\tilde{u}^{\tilde{\varepsilon}}$ is also an ε-approximate solution in $[t, t+s]$. Therefore, applying (4.16) and (4.17) in the interval $[t, t+s]$, we obtain

$$\begin{aligned}
&\left\| P(t+s,0)u - P(s,t) \circ P(t,0)u \right\|_{\mathbf{L^1}} \\
=\ & \lim_{\varepsilon \to 0} \left\| u^\varepsilon(t+s) - P(s,t)u^\varepsilon(t) \right\|_{\mathbf{L^1}} \\
=\ & \lim_{\varepsilon \to 0} \lim_{\tilde{\varepsilon} \to 0} \left\| u^\varepsilon(t+s) - \tilde{u}^{\tilde{\varepsilon}}(t+s) \right\|_{\mathbf{L^1}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \lim_{\varepsilon\to 0}\lim_{\tilde\varepsilon\to 0} C\left(\int_t^{t+s}\left\|g^\varepsilon(\xi)-g^{\tilde\varepsilon}(\xi)\right\|d\xi+\varepsilon\, s\right)\\
&= 0\,.
\end{aligned}
$$

We can repeat the same argument for any initial data $t_o \geq 0$.

Concerning the dependence on the initial time t_o, take $0 \leq t_o \leq t'_o$ and $u \in \mathcal{D}_{t_o}$, $u' \in \mathcal{D}_{t'_o}$. If $0 \leq t \leq t'_o - t_o$, then obviously

$$
\left\|P(t,t_o)u - P(t,t'_o)u'\right\|_{\mathbf{L^1}} \leq C\,\left(\left|t_o - t'_o\right| + \left\|u-u'\right\|_{\mathbf{L^1}}\right)\,.
$$

If $t > t'_o - t_o$, the process property implies

$$
\begin{aligned}
&\left\|P(t,t_o)u - P(t,t'_o)u'\right\|_{\mathbf{L^1}}\\
= \;&\left\|P(t+t_o-t'_o,t'_o)\circ P(t'_o-t_o,t_o)u - P(t,t'_o)u'\right\|_{\mathbf{L^1}}\\
\leq \;& C\left\|P(t'_o-t_o,t_o)u-u'\right\|_{\mathbf{L^1}} + C\left|t_o-t'_o\right|\\
\leq \;& C\left\|u-u'\right\|_{\mathbf{L^1}} + C\left|t_o-t'_o\right|,
\end{aligned}
$$

completing the proof of (4.19). □

The following proposition extends to the present case the key properties of the Glimm functionals (4.9).

Proposition 4.7 *Let system (1.2) satisfy the assumptions of Theorem 2.2. Then, for any $u \in \mathcal{D}_0$, the map $t \to \mathbf{\Upsilon}_t\left(P(t,0)u\right)$ is non increasing for $t \geq 0$.*

Proof. Above, we showed that the map $t \to \Upsilon^\varepsilon(t)$ decreases along ε-approximate solutions. The monotonicity of $t \to \mathbf{\Upsilon}_t\left(P(t,0)u\right)$ follow passing to the limit $\varepsilon \to 0$, thanks to the lower semicontinuity proved in [4, 9], to (4.10) and to the lower semicontinuity of the total variation. □

In order to complete the proof of Theorem 2.2, we prove propositions 2.3 and 2.4 together with an auxiliary lemma.

Proof of Proposition 2.3. Let u^ε be an ε-approximate wave front tracking solution converging to u. Since the convergence is also in $\mathbf{L^1_{loc}}(\mathbb{D}_\gamma;\mathbb{R}^n)$, apply Lemma 4.4 and Lebesgue Dominated convergence Theorem to obtain:

$$
\begin{aligned}
&\int_0^T\left\|u\left(t,\Gamma_0(t)\right)-u\left(t,\Gamma_1(t)\right)\right\|dt\\
= \;&\lim_{\delta\to 0}\frac{1}{\delta}\int_0^\delta\int_0^T\left\|u\left(t,\Gamma_0(t)+x\right)-u\left(t,\Gamma_1(t)+x\right)\right\|dt\,dx\\
= \;&\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{1}{\delta}\int_0^\delta\int_0^T\left\|u^\varepsilon\left(t,\Gamma_0(t)+x\right)-u^\varepsilon\left(t,\Gamma_1(t)+x\right)\right\|dt\,dx
\end{aligned}
$$

$$\leq \;\; \mathcal{K} \cdot \frac{\mathrm{TV}(u_o) + \mathrm{TV}(g)}{c} \cdot \left\|\Gamma_1 - \Gamma_0\right\|_{\mathbf{C^0}([0,T])}$$

completing the proof. □

Lemma 4.8 *Let u^ε be an ε-approximate wave front tracking solution to (1.2) converging to u. Let Γ be an ℓ–non-characteristic curve. Then,*

$$u^\varepsilon\left(\cdot, \Gamma(\cdot)\right) \to u\left(\cdot, \Gamma(\cdot)\right) \qquad in \qquad \mathbf{L^1_{loc}}(\mathbb{R}^+; \Omega)\,.$$

Proof. By the convergence of u^ε to u in $\mathbf{L^1_{loc}}(\mathbb{D}_\gamma; \Omega)$, there exists a sequence ε_ν converging to 0 such that for a.e. x

$$u^{\varepsilon_\nu}\left(\cdot, \Gamma(\cdot) + x\right) \to u\left(\cdot, , \Gamma(\cdot) + x\right) \qquad \text{in} \qquad \mathbf{L^1_{loc}}(\mathbb{R}^+, \Omega)\,.$$

Then, for any $T > 0$ and for any x for which the convergence above holds,

$$\begin{aligned}
& \int_0^T \left\| u^{\varepsilon_\nu}\left(t, \Gamma(t)\right) - u\left(t, \Gamma(t)\right) \right\| dt \\
\leq \;\; & \int_0^T \left\| u^{\varepsilon_\nu}\left(t, \Gamma(t)\right) - u^{\varepsilon_\nu}\left(t, \Gamma(t) + x\right) \right\| dt \\
& \quad + \int_0^T \left\| u^{\varepsilon_\nu}\left(t, \Gamma(t) + x\right) - u\left(t, \Gamma(t) + x\right) \right\| dt \\
& \quad + \int_0^T \left\| u\left(t, \Gamma(t) + x\right) - u\left(t, \Gamma(t)\right) \right\| dt \\
\leq \;\; & 2\mathcal{K} \frac{\mathrm{TV}(u_o) + \mathrm{TV}(g)}{c} |x| + \int_0^T \left\| u^{\varepsilon_n}\left(t, \Gamma(t) + x\right) - u\left(t, \Gamma(t) + x\right) \right\| dt
\end{aligned}$$

where we used Lemma 4.4 and Proposition 2.3. Hence

$$\limsup_{\nu \to +\infty} \int_0^T \left\| u^{\varepsilon_\nu}\left(t, \Gamma(t)\right) - u\left(t, \Gamma(t)\right) \right\| dt \quad \leq \quad C\,|x|$$

and the final estimate follows by the arbitrariness of x, independently from the sequence ε_ν, thanks to the uniqueness of the limit u. □

Proof of Proposition 2.4. Let u^ε, respectively $\tilde{u}^\varepsilon$, be an ε-approximate wave front tracking solutions of (1.2), respectively (1.3). Apply Proposition 4.5 and use the equivalence (4.16) to obtain

$$\begin{aligned}
& \int_{\tilde{\gamma}(t)}^{+\infty} \left\| u^\varepsilon(t,x) - \tilde{u}^\varepsilon(t,x) \right\| dx \\
\leq \;\; & L \cdot \left(\int_{\tilde{\gamma}(0)}^{+\infty} \left\| u^\varepsilon(0,x) - \tilde{u}^\varepsilon(0,x) \right\| dx \right. \\
& \quad \left. + \int_0^t \left\| b\left(u^\varepsilon\left(s, \tilde{\gamma}(s)\right)\right) - b\left(\tilde{u}^\varepsilon\left(s, \tilde{\gamma}(s)\right)\right) \right\| ds \right) + C\varepsilon t
\end{aligned}$$

and the limit $\varepsilon \to 0$ completes the proof. □

Proof of Theorem 2.2. To conclude the proof of Theorem 2.2, observe first that for the ε–approximate solutions, we have

$$\left\| b\left(u^{\varepsilon}(t,\gamma(t)+)\right) - g(t) \right\| \leq \varepsilon$$

therefore as $\varepsilon \to 0$ Lemma 4.8 implies *4)*.

Finally denote by let $P^{(\gamma,g)}$ and $\mathcal{D}_t^{(\gamma,g)}$ the process and the domains corresponding to the boundary curve and data (γ, g). Fix two boundary curve and data (γ, g), $(\bar{\gamma}, \bar{g})$, two initial data $u_o \in \mathcal{D}_0^{(\gamma,g)}$, $\bar{u}_o \in \mathcal{D}_0^{(\bar{\gamma},\bar{g})}$ and define

$$\begin{array}{rclrcl}
\Gamma_0(t) & = & \min\left\{\gamma(t), \bar{\gamma}(t)\right\}, & \Gamma_1(t) & = & \max\left\{\gamma(t), \bar{\gamma}(t)\right\} \\
\tilde{u}_o(x) & = & \left\{\begin{array}{ll} 0 & \text{for } x \leq \Gamma_1(0) \\ u_o(x) & \text{for } x > \Gamma_1(0) \end{array}\right. & \tilde{\bar{u}}_o(x) & = & \left\{\begin{array}{ll} 0 & \text{for } x \leq \Gamma_1(0) \\ \bar{u}_o(x) & \text{for } x > \Gamma_1(0) \end{array}\right. \\
\tilde{g}(t) & = & b\Big(\left[P^{(\gamma,g)}(t,0)u_o\right]\left(\Gamma_1(t)\right)\Big) & \tilde{\bar{g}}(t) & = & b\Big(\left[P^{(\bar{\gamma},\bar{g})}(t,0)\bar{u}_o\right]\left(\Gamma_1(t)\right)\Big).
\end{array}$$

By Proposition 2.4 we have for $x > \Gamma_1(t)$:

$$\begin{array}{rcl}
\left[P^{(\gamma,g)}(t,0)u_o\right](x) & = & \left[P^{(\Gamma_1,\tilde{g})}(t,0)\tilde{u}_o\right](x), \\
\left[P^{(\bar{\gamma},\bar{g})}(t,0)\bar{u}_o\right](x) & = & \left[P^{(\Gamma_1,\tilde{\bar{g}})}(t,0)\tilde{\bar{u}}_o\right](x).
\end{array}$$

Applying the result for the unchanged boundary curve, we get:

$$\begin{array}{rl}
& \left\| P^{(\gamma,g)}(t,0)u_o - P^{(\bar{\gamma},\bar{g})}(t,0)\bar{u}_o \right\|_{\mathbf{L^1}} \\
= & \displaystyle\int_{\Gamma_0(t)}^{\Gamma_1(t)} \left\| P^{(\gamma,g)}(t,0)u_o - P^{(\bar{\gamma},\bar{g})}(t,0)\bar{u}_o \right\| dx \\
& \displaystyle + \int_{\Gamma_1(t)}^{+\infty} \left\| P^{(\Gamma_1,\tilde{g})}(t,0)\tilde{u}_o - P^{(\Gamma_1,\tilde{\bar{g}})}(t,0)\tilde{\bar{u}}_o \right\| dx \\
\leq & \displaystyle C\left|\Gamma_1(t) - \Gamma_0(t)\right| + C\int_0^t \left\|\tilde{g}(t) - \tilde{\bar{g}}(t)\right\| dt + C\left\|\tilde{u}_o - \tilde{\bar{u}}_o\right\|_{\mathbf{L^1}} \\
\leq & C\|\Gamma_1 - \Gamma_0\|_{\mathbf{C^0}} + C\|u_o - \bar{u}_o\|_{\mathbf{L^1}} \\
& \displaystyle + C\int_0^t \left\|\tilde{g}(t) - g(t) + g(t) - \bar{g}(t) + \bar{g}(t) - \tilde{\bar{g}}(t)\right\| dt \\
\leq & \displaystyle C\|\gamma - \bar{\gamma}\|_{\mathbf{C^0}} + C\|u_o - \bar{u}_o\|_{\mathbf{L^1}} + C\int_0^t \left\|g(t) - \bar{g}(t)\right\| dt \\
& \displaystyle + C\int_0^t \left\|\tilde{g}(t) - g(t)\right\| dt + C\int_0^t \left\|\bar{g}(t) - \tilde{\bar{g}}(t)\right\| dt.
\end{array}$$

Finally Proposition 2.3 and the Lipschitz continuity of b imply

$$\int_0^t \left\|\tilde{g}(t) - g(t)\right\| dt$$

$$
\begin{aligned}
&= \int_0^t \left\| b\left(\left[P^{(\gamma,g)}(t,0)u_o\right]\left(\Gamma_1(t)\right)\right) - b\left(\left[P^{(\gamma,g)}(t,0)u_o\right]\left(\gamma(t)\right)\right)\right\| dt \\
&\leq C\|\Gamma_1 - \gamma\|_{\mathbf{C^0}} \leq \|\bar\gamma - \gamma\|_{\mathbf{C^0}}
\end{aligned}
$$

completing the proof of *2)*, since the computations for $\bar g$ and $\tilde{\bar g}$ are identical.

We prove now the tangency condition *3)*. Fix $t_o \geq 0$, $u \in \mathcal{D}_{t_o}$ and let F be defined by (4.6) and denote by $\tilde P$ the process defined above with g replaced by $\tilde g(t_o + t) = b\left((S_t \tilde u)\left(\gamma(t_o + t)\right)\right)$ with $\tilde u$ as in (4.5). By Proposition 2.4, $F(t, t_o)u = \tilde P(t, t_o)u$. Using *2)*, we have

$$
\begin{aligned}
&\frac{1}{t}\left\|P(t,t_o)u - F(t,t_o)u\right\|_{\mathbf{L^1}} \\
\leq\ & \frac{L}{t}\int_{t_o}^{t_o+t} \left\|\tilde g(s) - g(s)\right\| ds \\
\leq\ & \frac{L}{t}\int_{t_o}^{t_o+t} \left\|\tilde g(s) - g(t_o+)\right\| ds + \frac{L}{t}\int_{t_o}^{t_o+t} \left\|g(t_o+) - g(s)\right\| ds\,.
\end{aligned}
$$

The latter term vanishes as $t \to 0$ by the definition of $g(t_o+)$. Consider now the former term. Fix a positive and sufficiently small δ so that the curve $\psi(s) = \gamma(s) + \delta(s - t_o)$ is ℓ–non-characteristic. Let $\xi \in [0,1]$.

$$
\begin{aligned}
&\frac{1}{t}\int_{t_o}^{t_o+t} \left\|\tilde g(s) - g(t_o+)\right\| ds \\
=\ & \frac{1}{t}\int_{t_o}^{t_o+t} \left\| b\left((S_{s-t_o}\tilde u)\left(\gamma(s)\right)\right) - g(t_o+)\right\| ds \\
\leq\ & \frac{1}{t}\int_{t_o}^{t_o+t} \left\| b\left((S_{s-t_o}\tilde u)\left(\gamma(s)\right)\right) - b\left((S_{s-t_o}\tilde u)\left((1-\xi)\gamma(s) + \xi\psi(s)\right)\right)\right\| ds \\
& + \frac{1}{t}\int_{t_o}^{t_o+t} \left\| b\left((S_{s-t_o}\tilde u)\left((1-\xi)\gamma(s) + \xi\psi(s)\right)\right) - g(t_o+)\right\| ds\,.
\end{aligned}
$$

By Proposition 2.3, the first term is bounded by

$$
\frac{C}{t}\left\|\gamma - \left((1-\xi)\gamma + \xi\psi\right)\right\|_{\mathbf{C^0}([t_o,t_o+t])} \leq \frac{C}{t}\|\gamma - \psi\|_{\mathbf{C^0}([t_o,t_o+t])} \leq C\delta\,.
$$

Concerning the latter term, integrate on ξ over $[0,1]$ and obtain, with the change of variable $x = (1-\xi)\gamma(s) + \xi\psi(s)$ and with u^σ as in Lemma 4.1,

$$
\begin{aligned}
&\frac{1}{t}\int_{t_o}^{t_o+t} \left\| b\left((S_{s-t_o}\tilde u)\left((1-\xi)\gamma(s) + \xi\psi(s)\right)\right) - g(t_o+)\right\| ds \\
=\ & \frac{1}{t}\int_{t_o}^{t_o+t} \frac{1}{\psi(s) - \gamma(s)}\int_{\gamma(s)}^{\psi(s)} \left\| b\left((S_{s-t_o}\tilde u)(x)\right) - b(u^\sigma)\right\| dx\, ds \\
\leq\ & \frac{C}{t\delta}\int_{t_o}^{t_o+t} \frac{1}{s - t_o}\int_{\gamma(s)}^{\psi(s)} \left\|(S_{s-t_o}\tilde u)(x) - u^\sigma\right\| dx\, ds
\end{aligned}
\tag{4.21}
$$

Following [6, Section 9.3], let $U^\sharp$ be the Lax solution to the Riemann problem

$$\left\{\begin{array}{l}\partial_t u + \partial_x f(u) = 0\\ u(0,x) = \left\{\begin{array}{lll} u^\sigma & \text{if } x & < & 0\\ u\left(\gamma(t_o)+\right) & \text{if } x & \geq & 0\,.\end{array}\right.\end{array}\right.$$

By the basic properties of the solutions to Riemann problem and the definition of ψ, for all $s \in [t_o, t]$ and $x \in [\gamma(s), \psi(s)]$, $U^\sharp(s,x) = u^\sigma$. Then,

$$\begin{aligned}(4.21) \quad &= \frac{C}{t\delta}\int_{t_o}^{t_o+t} \frac{1}{s-t_o}\int_{\gamma(s)}^{\psi(s)} \left\|(S_{s-t_o}\tilde u)(x) - U^\sharp(s,x)\right\| dx\, ds\\ &\leq \frac{C}{t\delta}\int_{t_o}^{t_o+t} \frac{1}{s-t_o}\int_{\gamma(t_o)-(s-t_o)\hat\lambda}^{\gamma(t_o)+(s-t_o)\hat\lambda} \left\|(S_{s-t_o}\tilde u)(x) - U^\sharp(s,x)\right\| dx\, ds\end{aligned}$$

By [6, formula (9.16)],

$$\lim_{s\to t_o} \frac{1}{s-t_o}\int_{\gamma(t_o)-(s-t_o)\hat\lambda}^{\gamma(t_o)+(s-t_o)\hat\lambda} \left\|(S_{s-t_o}\tilde u)(x) - U^\sharp(s,x)\right\| dx = 0$$

so that $\lim_{t\to 0}(4.21) = 0$. Collecting the various terms,

$$\limsup_{t\to 0} \frac{1}{t}\int_{t_o}^{t_o+t} \left\|\tilde g(s) - g(t_o+)\right\| ds \leq C\delta$$

and by the arbitrariness of δ, the tangency condition *3)* follows.

The characterization of P through *1)*, *2)* with $\omega = 0$ and *3)* implies its uniqueness through standard computations, see for instance [5, Section 6, Corollary 1]. □

5 The Source Term

This section is devoted to the source term, similarly to [3, 7, 8, 14, 18] but following the general metric space technique in [10], applied to $\mathbf{L^1}$ equipped with the $\mathbf{L^1}$-distance d. The key point is to show that the map

$$\check F(t,t_o)u = P(t,t_o)u + t\, G\left(P(t,t_o)u\right) \chi_{\left[\gamma(t_o+t),+\infty\right[} \tag{5.1}$$

is a local flow in the sense of [10, Definition 2.1] on suitable domains and satisfies the assumptions of [10, Theorem 2.6].

Following [9, Section 3], we modify the functional Φ in (4.14) and define $\mathbf{\Phi}_t$ on all piecewise constant functions, not necessarily ε–approximate solutions. Therefore, the definition of $\mathbf{\Phi}_t$ does not consider nonphysical waves and $\mathbf{\Phi}_0 = \Phi$ at time $t = 0$. Consider two piecewise constant functions

$u, v \in \mathbf{L^1}\left(]\gamma(t), +\infty[, \mathbb{R}^n\right)$ with finitely many jumps and assume that $\mathrm{TV}(u)$ is sufficiently small.

Define $\boldsymbol{q} \equiv (q_1, \ldots, q_n)$ implicitly by

$$v(x) = \boldsymbol{S}\left(\boldsymbol{q}(x)\right)\left(u(x)\right)$$

with $\boldsymbol{S}$ as in (4.3). We now consider the functional

$$\mathbf{\Phi}_t(u, v) = \bar{K} \sum_{i=1}^{\ell} \int_{\gamma(t)}^{+\infty} \left|q_i(x)\right| \boldsymbol{W}_i(x)\, dx + \sum_{i=\ell+1}^{n} \int_{\gamma(t)}^{+\infty} \left|q_i(x)\right| \boldsymbol{W}_i(x)\, dx$$

where $\bar{K}$ is defined in the proof of Proposition 4.5 and the weights $\boldsymbol{W}_i$ are defined setting:

$$\boldsymbol{W}_i(x) = 1 + \kappa_1 \boldsymbol{A}_i(x) + \kappa_2 \left(\mathbf{\Upsilon}_t(u) + \mathbf{\Upsilon}_t(v)\right) .$$

The functions $\boldsymbol{A}_i$ are defined as follows. Let $\sigma_{x,\kappa}$ be the strength of the κ-th wave in the solution of the Riemann problem for (4.7) in u or v located at x of the family κ. Differently from the notation in Section 4, $J(u)$, respectively $J(v)$ denote the sets of all jumps in u, respectively in v, for $x \geq \gamma(t)$. Indeed, we let $x = \gamma(t)$ in J as soon as $b\left(u\left(\gamma(t)+\right)\right) \neq g(t)$ and the waves $\sigma_{\gamma(t),k}$ are defined as in (4.8).

If the i-th characteristic field is linearly degenerate, we simply define

$$\boldsymbol{A}_i(x) \doteq \sum \left\{ \left|\sigma_{y,\kappa}\right| : y \in J(u) \cup J(v) \text{ and } \begin{array}{l} y < x,\ i < \kappa \leq n, \text{ or} \\ y > x,\ 1 \leq \kappa < i \end{array} \right\}$$

On the other hand, if the i-th field is genuinely nonlinear, the definition of $\boldsymbol{A}_i$ will contain an additional term, accounting for waves in u and in v of the same i-th family:

$$\begin{aligned} \boldsymbol{A}_i(x) \quad &\doteq \quad \sum \left\{ \left|\sigma_{y,\kappa}\right| : y \in J(u) \cup J(v) \text{ and } \begin{array}{l} y < x,\ i < \kappa \leq n, \text{ or} \\ y > x,\ 1 \leq \kappa < i \end{array} \right\} \\ &\quad + \left\{ \begin{array}{ll} \sum \left\{ \left|\sigma_{y,i}\right| : \begin{array}{l} y \in J(u),\ y < x \text{ or} \\ y \in J(v),\ y > x \end{array} \right\} & \text{if } q_i(x) < 0, \\ \sum \left\{ \left|\sigma_{y,i}\right| : \begin{array}{l} y \in J(v),\ y < x \text{ or} \\ y \in J(u),\ y > x \end{array} \right\} & \text{if } q_i(x) \geq 0. \end{array} \right. \end{aligned}$$

The constants κ_1, κ_2 are the same defined in [6, Chapter 8]. We also recall that, since δ_o is chosen small enough, the weights satisfy $1 \leq \boldsymbol{W}_i(x) \leq 2$, hence for a suitable constant $C_3 > 1$ we have

$$\frac{1}{C_3} \left\|v - u\right\|_{\mathbf{L^1}} \leq \mathbf{\Phi}_t(u, v) \leq C_3 \left\|v - u\right\|_{\mathbf{L^1}},$$

where the $\mathbf{L^1}$ norm is taken in the interval $]\gamma(t), +\infty[$.

For a fixed positive M, define

$$\hat{\mathcal{D}}_t^M = \left\{ u \in \mathbf{L^1}(\mathbb{R};\Omega) \colon \begin{array}{l} u(x) = 0 \text{ for all } x < \gamma(t) \\ \mathbf{\Upsilon}_t(u) \leq \delta - C(T-t) \\ \|u\|_{\mathbf{L^1}} \leq Me^{Ct} + Ct \end{array} \right\}$$

with $\mathbf{\Upsilon}_t$ defined in (4.9), C, δ and T to be specified below.

Lemma 5.1 *For all $t_o \in [0,T]$, $t > 0$ sufficiently small and $u, \tilde{u} \in \mathcal{D}_{t_o}$,*

$$\begin{aligned} \mathbf{\Upsilon}_{t_o+t}\left(\check{F}(t,t_o)u\right) &\leq \mathbf{\Upsilon}_{t_o}(u) + C\,t \\ \mathbf{\Phi}_{t_o+t}\left(\check{F}(t,t_o)u, \check{F}(t,t_o)\tilde{u}\right) &\leq (1+Ct)\,\mathbf{\Phi}_{t_o}(u,\tilde{u})\,. \end{aligned} \tag{5.2}$$

The proof is as that of [7, Lemma 3.6 and Corollary 3.7], see also [8, Lemma 2.3].

Corollary 5.2 *For t small, $\check{F}$ in (5.1) satisfies $\check{F}(t,t_o)\hat{\mathcal{D}}_{t_o}^M \subseteq \hat{\mathcal{D}}_{t_o+t}^M$.*

Proof. The bound on $\mathbf{\Upsilon}_t$ is a direct consequence of Lemma 5.1. Concerning the estimate on the $\mathbf{L^1}$ norm, for $u \in \hat{\mathcal{D}}_{t_o}^M$, compute:

$$\begin{aligned} & \left\|\check{F}(t,t_o)u\right\|_{\mathbf{L^1}} \\ = & \left\|P(t,t_o)u + tG\left(P(t,t_o)u\right)\chi_{[\gamma(t_o+t),+\infty[}\right\|_{\mathbf{L^1}} \\ \leq & \left\|P(t,t_o)u - u\right\|_{\mathbf{L^1}} + \|u\|_{\mathbf{L^1}} + t\left\|G\left(P(t,t_o)u\right) - G(0)\right\|_{\mathbf{L^1}} + t\left\|G(0)\right\|_{\mathbf{L^1}} \\ \leq & Ct + \|u\|_{\mathbf{L^1}} + Ct\left(\|u\|_{\mathbf{L^1}} + Ct\right) + Ct \\ \leq & (1+Ct)\,\|u\|_{L1} + Ct \\ \leq & (1+Ct)\left(Me^{Ct_o} + Ct_o\right) + Ct \\ \leq & Me^{C(t_o+t)} + C(t_o+t) \end{aligned}$$

hence $\check{F}(t,t_o)u$ is in $\hat{\mathcal{D}}_{t_o+t}^M$. □

In what follows, relying on [10, Condition **(D)**], we consider $\check{F}$ as defined on the domains $\hat{\mathcal{D}}_{t_o}^M$ and not on a single domain, as in [10, Definition 2.1].

Proposition 5.3 *The map $\check{F}$ defined in (5.1) is $\mathbf{L^1}$ Lipschitz continuous, satisfies $\check{F}(0,t_o)\,u = u$ for any $(t_o,u) \in \left\{(\tau,w) \colon \tau \in [0,T],\ w \in \hat{\mathcal{D}}_\tau^M\right\}$ and there exist positive $\mathcal{L}$, independent from M, such that for $t_o, t_o' \in [0,T]$, $t \in [0,T-t_o]$, $t' \in [0,T-t_o']$, $u \in \hat{\mathcal{D}}_{t_o}^M$, $u' \in \hat{\mathcal{D}}_{t_o'}^M$*

$$\left\|\check{F}(t',t_o')u' - \check{F}(t,t_o)u\right\|_{\mathbf{L^1}} \leq \mathcal{L}\left(\left\|u'-u\right\|_{\mathbf{L^1}} + \left(1 + \|u\|_{\mathbf{L^1}}\right)\left|t'-t\right| + \left|t_o'-t_o\right|\right).$$

Proof. Compute:

$$
\begin{aligned}
& \left\| \check F(t',t_o')u' - \check F(t,t_o)u \right\|_{\mathbf{L^1}} \\
\leq\ & \left\| P(t',t_o')u' - P(t,t_o)u \right\|_{\mathbf{L^1}} + \left| t'-t \right| \left\| G\left(P(t',t_o')u'\right) \chi_{[\gamma(t_o'+t'),+\infty[} \right\|_{\mathbf{L^1}} \\
& + t \left\| G\left(P(t',t_o')u'\right) \chi_{[\gamma(t_o'+t'),+\infty[} - G\left(P(t,t_o)u\right) \chi_{[\gamma(t_o+t),+\infty[} \right\|_{\mathbf{L^1}} \\
\leq\ & \left\| P(t',t_o')u' - P(t,t_o)u \right\|_{\mathbf{L^1}} \\
& + \left| t'-t \right| \left(\left\| G\left(P(t',t_o')u' - G(0)\right) \right\|_{\mathbf{L^1}} + \left\| G(0) \right\|_{\mathbf{L^1}} \right) \\
& + t \left\| G\left(P(t',t_o')u'\right) - G\left(P(t,t_o)u\right) \right\|_{\mathbf{L^1}} \\
& + t \left\| G\left(P(t,t_o)u\right) \left(\chi_{[\gamma(t_o'+t'),+\infty[} - \chi_{[\gamma(t_o+t),+\infty[} \right) \right\|_{\mathbf{L^1}} \\
\leq\ & (1+Ct) \left\| P(t',t_o')u' - P(t,t_o)u \right\|_{\mathbf{L^1}} + \left| t'-t \right| \left\| G(0) \right\|_{\mathbf{L^1}} \\
& + C \left| t'-t \right| \left\| P(t',t_o')u' \right\|_{\mathbf{L^1}} + C\, t \left| \gamma(t_o'+t') - \gamma(t_o+t) \right| \\
\leq\ & C \left(\left| t'-t \right| + \left| t_o'-t_o \right| + \left\| u'-u \right\|_{\mathbf{L^1}} \right) \\
& + C \left| t'-t \right| \left(\left\| P(t',t_o')u' - u' \right\|_{\mathbf{L^1}} + \left\| u' \right\|_{\mathbf{L^1}} \right) \\
\leq\ & C \left(\left(1 + \left\| u' \right\|_{\mathbf{L^1}} \right) \left| t'-t \right| + \left| t_o'-t_o \right| + \left\| u'-u \right\|_{\mathbf{L^1}} \right)
\end{aligned}
$$

completing the proof. □

Recall [10, Definition 2.3]: an Euler ε-polygonal is

$$
\check F^{\varepsilon}(t,t_o)\, u = \check F(t-k\varepsilon, t_o+k\varepsilon) \circ \bigodot_{h=0}^{k-1} \check F(\varepsilon, t_o+h\varepsilon)\, u \qquad (5.3)
$$

for $k=[t/\varepsilon]$. Above and in what follows, we denote the recursive composition $\bigodot_{i=1}^{n} f_i = f_1 \circ f_2 \circ \ldots \circ f_n$. Here, $[\,\cdot\,]$ stands for the integer part, i.e. for $s \in \mathbb{R}$, $[s] = \max\{k \in \mathbb{Z} \colon k \leq s\}$.

The hypotheses to apply [10, Theorem 2.6] are satisfied.

Proposition 5.4 *The local flow $\check F$ in (5.1) is such that there exist*

1. *a positive constant C such that for all $t_o \in [0,T]$ and all $u \in \hat{\mathcal{D}}_{t_o}^M$*

$$
d\left(\check F(k\tau, t_o+\tau) \circ \check F(\tau,t_o)u, \check F\left((k+1)\tau, t_o\right) u \right) \leq C\, k\, \tau^2
$$

whenever $k \in \mathbb{N}$, $(k+1)\tau, \tau \in [0, T-t_o]$;

2. a positive constant L such that

$$d\left(\check{F}^\varepsilon(t,t_o)u, \check{F}^\varepsilon(t,t_o)w\right) \leq L \cdot d(u,w)$$

whenever $\varepsilon \in]0,\delta]$, $u,w \in \hat{\mathcal{D}}^M_{t_o}$, $t \geq 0$ and $t_o, t_o+t \in [0,T]$.

Note that *1.* states that [10, 1. in Theorem 2.6] is satisfied with $\omega(t) = Ct$.

Proof. To prove *1.*, the key property is *2)* in Theorem 2.2, see also [11, Proposition 4.9].

$$\begin{aligned}
& \check{F}(k\tau, t_o+\tau) \circ \check{F}(\tau,t_o)u - \check{F}\left((k+1)\tau, t_o\right)u \\
= \;& P(k\tau, t_o+\tau)\left(P(\tau,t_o)u + \tau G\left(P(\tau,t_o)u\right)\chi_{[\gamma(t_o+\tau),+\infty[}\right) \\
& + k\tau\, G\left(P(k\tau,t_o+\tau)\left(P(\tau,t_o)u + \tau G\left(P(\tau,t_o)u\right)\chi_{[\gamma(t_o+\tau),+\infty[}\right)\right) \cdot \\
& \quad \cdot \chi_{[\gamma(t_o+(k+1)\tau),+\infty[} \\
& - P\left((k+1)\tau,t_o\right)u \\
& - (k+1)\tau\, G\left(P\left((k+1)\tau,t_o\right)u\right)\chi_{[\gamma(t_o+(k+1)\tau),+\infty[} \\
= \;& P(k\tau, t_o+\tau)\left(P(\tau,t_o)u + \tau G\left(P(\tau,t_o)u\right)\chi_{[\gamma(t_o+\tau),+\infty[}\right) \\
& - P(k\tau,t_o+\tau)\circ P(\tau,t_o)u - \tau\, G\left(P\left((k+1)\tau,t_o\right)u\right)\chi_{[\gamma(t_o+(k+1)\tau),+\infty[} \\
& + k\tau\Big[G\left(P(k\tau,t_o+\tau)\left(P(\tau,t_o)u + \tau G\left(P(\tau,t_o)u\right)\chi_{[\gamma(t_o+\tau),+\infty[}\right)\right) \\
& \quad - G\left(P(k\tau,t_o+\tau)\circ P(\tau,t_o)u\right)\Big]\chi_{[\gamma(t_o+(k+1)\tau),+\infty[}\,.
\end{aligned}$$

Using *2)* in Theorem 2.2 in the first two lines with $t = t' = k\tau$, $t_o = t'_o$ for $t_o+\tau$, $v = P(\tau,t_o)u$, $\omega = \tau G\chi$ and in the latter two lines **(G)**, *2)* in Theorem 2.2 with $\omega = 0$. We thus get

$$\begin{aligned}
& d\left(\check{F}(k\tau,t_o+\tau)\circ\check{F}(\tau,t_o)u, \check{F}\left((k+1)\tau,t_o\right)u\right) \\
= \;& \left\|\check{F}(k\tau,t_o+\tau)\circ\check{F}(\tau,t_o)u - \check{F}\left((k+1)\tau,t_o\right)u\right\|_{\mathbf{L^1}} \\
\leq \;& C\tau\left\| G\left(P(\tau,t_o)u\right)\chi_{[\gamma(t_o+\tau),+\infty[} \right. \\
& \qquad \left. - G\left(P\left((k+1)\tau,t_o\right)u\right)\chi_{[\gamma(t_o+(k+1)\tau),+\infty[}\right\|_{\mathbf{L^1}} \\
& + C\,k\tau^2\left\|G\left(P(\tau,t_o)u\right)\right\|_{\mathbf{L^1}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \; C\tau\left(k\tau \left\|G\right\|_{\mathbf{L}^\infty} \left\|\dot\gamma\right\|_{\mathbf{L}^\infty} + C\, k\tau\right) + C\left(\left\|G(0)\right\|_{\mathbf{L}^1} + 1 + M\right) k\tau^2 \\
&\leq \; C\,(1+M)\, k\tau^2 \,.
\end{aligned}
$$

The bound *2.* is a direct consequence of the equivalence (4.16) and (5.2) in Lemma 5.1, see also [11, Proposition 4.9] and [7, formula (3.1)]. □

Proof of *i), ii)* and *iv)* in Theorem 3.2. By [10, Theorem 2.5], for any M, the local flow $\check F$ generates a Lipschitz process $\hat P$ on $\mathcal{D}_t^M$. By the characterization of $\hat P$ as limit of Euler polygonals, it follows that $\hat P$ is uniquely defined on all

$$
\hat{\mathcal{D}}_t = \bigcup_{M>0} \hat{\mathcal{D}}_t^M = \left\{ u \in \mathbf{L}^1(\mathbb{R};\Omega) \colon \begin{array}{l} u(x) = 0 \text{ for all } x < \gamma(t) \\ \mathbf{\Upsilon}_t(u) \leq \delta - C(T-t) \end{array} \right\}.
$$

Hence, $\hat P$ satisfies *ii)* in Theorem 3.2 and *i)* holds.

To prove *iv)*, note that $\frac{1}{t}\left\|\hat F(t.t_o)u - \check F(t,t_o)u\right\|_{\mathbf{L}^1} \to 0$ as $t \to 0$, for all $u \in \hat{\mathcal{D}}_{t_o}$ and apply [10, *c)* in Theorem 2.5]. □

For any $N \in \mathbb{N}$, define the operator $\Pi_N \colon \mathbf{L}^1(\mathbb{R};\mathbb{R}^n) \to \mathbf{PC}(\mathbb{R};\mathbb{R}^n)$ by

$$
\Pi_N(u) = N \sum_{k=-1-N^2}^{-1+N^2} \int_{k/N}^{(k+1)/N} u(\xi)\, d\xi \; \chi_{]k/N,(k+1)/N]} \,.
$$

Lemma 5.5 *Π_N is a linear operator with norm 1. Moreover,* $\mathrm{TV}\,(\Pi_N u) \leq 2\mathrm{TV}(u)$ *and for all $u \in \mathbf{L}^1(\mathbb{R};\mathbb{R}^n) \cap \mathbf{BV}(\mathbb{R};\mathbb{R}^n)$, $\Pi_N u \to u$ in $\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)$.*

For the proof, see [7, Lemma 3.4].

Proof of Proposition 3.3. Set for simplicity $t_o = 0$. Let $\varepsilon, \tilde\varepsilon > 0$ and $N \in \mathbb{N}$ be fixed. Consider an $\tilde\varepsilon$-approximate wave front tracking solution $u^{\varepsilon,\tilde\varepsilon,N} = u^{\varepsilon,\tilde\varepsilon,N}(t,x)$ to (1.2) on the time interval $[0,\varepsilon[$. Define it at time $t = \varepsilon$ setting

$$
u^{\varepsilon,\tilde\varepsilon,N}(\varepsilon,x) = u^{\varepsilon,\tilde\varepsilon,N}(\varepsilon-,x) + \varepsilon\chi_{\left[\gamma(\varepsilon),+\infty\right[}(x)\left(\Pi_N G\left(u^{\varepsilon,\tilde\varepsilon,N}(\varepsilon-)\right)\right)(x)\,.
$$

Extend $u^{\varepsilon,\tilde\varepsilon,N}$ recursively on $[0,T]$. Note that

$$
\lim_{\tilde\varepsilon\to 0}\lim_{N\to+\infty} u^{\varepsilon,\tilde\varepsilon,N}(t) = \check F^\varepsilon(t,0)u_o
$$

where $\check F^\varepsilon$ is defined in (5.3) and u_o is the initial datum in (1.1). Note that this is the usual operator splitting algorithm.

Given any curve $\tilde{\ell}$–non–characteristic curve Γ with support in $\mathbb{D}_\gamma$, define

$$\begin{aligned}\Xi^\varepsilon(t) &= \check{K}\left(\sum_{x\geq\Gamma(t)}\sum_{i=1}^{\tilde{\ell}}\left|\sigma_{x,i}\right| + \sum_{\gamma(t)\leq x\leq\Gamma(t)}\sum_{i=\tilde{\ell}+1}^{n+1}\left|\sigma_{x,i}\right| + \hat{K}\,\Upsilon^\varepsilon(t)\right) \\ &\quad +\mathrm{TV}\left(u^{\varepsilon,\tilde{\varepsilon},N}\left(\cdot,\Gamma(\cdot)\right);[0,t]\right)\end{aligned} \tag{5.4}$$

for suitable positive $\hat{K},\check{K}$.

Computations similar to those above allow to prove that $\Xi^\varepsilon(t+)\leq\Xi^\varepsilon(t-)$ for all $t\notin\varepsilon\mathbb{N}$. Indeed, when a wave crosses Γ, the increase in $\mathrm{TV}(u^{\varepsilon,\bar{\varepsilon},n})$ is compensated by the decrease in the first term on the right hand side of (5.4).

At times $t\in\varepsilon\mathbb{N}$, $\Xi^\varepsilon(t)+\leq\Xi^\varepsilon(t-)+C\varepsilon$. Therefore, for all $t\in[0,T]$, $\Xi^\varepsilon(t)\leq\Xi^\varepsilon(0)+Ct$. By (5.4), we get that there exists a C dependent only on u_o and $\tilde{\ell}$ such that

$$\mathrm{TV}\left(u^{\varepsilon,\tilde{\varepsilon},N}\left(\cdot,\Gamma(\cdot)\right);[0,t]\right)\leq C\,.$$

Consider now two $\tilde{\ell}$–non-characteristic curves Γ_1,Γ_2 with support in $\mathbb{D}_\gamma$. The same steps in the proof of Lemma 4.4 lead to

$$\int_0^T\left\|u^{\varepsilon,\tilde{\varepsilon},N}\left(t,\Gamma_1(t)+\right)-u^{\varepsilon,\tilde{\varepsilon},N}\left(t,\Gamma_2(t)+\right)\right\|dt\leq\frac{C}{c}\cdot\|\Gamma_2-\Gamma_1\|_{\mathbf{C^0}}\,.$$

Let now $\tilde{\varepsilon}\to0$ and $N\to+\infty$, with the same technique of Proposition 2.3 we obtain

$$\int_0^T\left\|\left[\check{F}^\varepsilon(t,0)u\right]\!\left(\Gamma_1(t)+\right)-\left[\check{F}^\varepsilon(t,0)u\right]\!\left(\Gamma_2(t)+\right)\right\|dt\leq\frac{C}{c}\|\Gamma_2-\Gamma_1\|_{\mathbf{C^0}} \tag{5.5}$$

with a constant C that now depends also on T and on L_2 in **(G)**. Let now also $\varepsilon\to0$ and, by the $\mathbf{L^1_{loc}}(\mathbb{R}^2;\Omega)$ convergence of the Euler polygonals, obtain as in Proposition 2.3 that

$$\int_0^T\left\|\left(\hat{P}(t,0)u\right)\left(\Gamma_1(t)+\right)-\left(\hat{P}(t,0)u\right)\left(\Gamma_2(t)+\right)\right\|dt\leq\frac{C}{c}\|\Gamma_2-\Gamma_1\|_{\mathbf{C^0}} \tag{5.6}$$

completing the proof. □

Proof of *iii)* and *v)* in Theorem 3.2. The Lipschitz continuity upon the initial data is a consequence of [10, *b)* in Theorem 2.5], thanks to Proposition 5.4. The dependence of the Lipschitz constant for the variable t on the $\mathbf{L^1}$ norm of the initial data is shown in Proposition 5.3.

The Lipschitz conditions (5.5) and (5.6) allow to prove the $\mathbf{L^1}\left([0,T];\Omega\right)$ convergence of the traces as in Lemma 4.8:

$$\left(\check{F}^\varepsilon(\cdot,0)u\right)\left(\Gamma(\cdot)+\right)\to\left(P(\cdot,0)u\right)\left(\Gamma(\cdot)+\right)\quad\text{ in }\quad\mathbf{L^1}([0,T];\Omega)\,.$$

The map $t \to \check{F}^{\varepsilon}(t,0)u$ satisfies for a.e. t the boundary condition, hence the same does the solution $t \to P(t,0)u$, proving 2. in Definition 3.1. Condition 3. in the same definition is proved using the tangency condition 3), as in [7, Corollary 3.14].

We are left to prove the Lipschitz dependence from the boundary and the boundary data. To this aim, introduce two boundaries γ and $\bar\gamma$, with $\gamma \le \bar\gamma$ and boundary data g, $\bar g$. Let $\hat{\mathcal{D}}_t$, $\hat{\bar{\mathcal{D}}}_t$, $\hat P^{g,\gamma}(t,t_o)$ and $\hat P^{\bar g,\bar\gamma}(t,t_o)$ the corresponding domains and processes. We need to prove that for any $u \in \hat{\mathcal{D}}_0 \cap \hat{\bar{\mathcal{D}}}_0$ (and therefore $u(x)=0$ for $x \le \bar\gamma(0)$):

$$\left\|\hat P^{g,\gamma}(t,0)u - \hat P^{\bar g,\bar\gamma}(t,0)u\right\|_{\mathbf{L^1}(\mathbb{R})} \le C\left[\|\gamma-\bar\gamma\|_{\mathbf{C^0}([0,t])} + \int_0^t \left\|g(\tau)-\bar g(\tau)\right\| d\tau\right].$$

Note first that

$$\begin{aligned}&\left\|\hat P^{g,\gamma}(t,0)u - \hat P^{\bar g,\bar\gamma}(t,0)u\right\|_{\mathbf{L^1}(\mathbb{R})}\\ &\qquad \le C\,\|\gamma-\bar\gamma\|_{\mathbf{C^0}([0,t])} + \left\|\hat P^{g,\gamma}(t,0)u - \hat P^{\bar g,\bar\gamma}(t,0)u\right\|_{\mathbf{L^1}(I_t)}\end{aligned}$$

where $I_t = \left[\bar\gamma(t), +\infty\right[$. Hence, we consider below only the latter term in the right hand side above. Introduce the linear projector $\pi_t v = v\,\chi_{I_t}$ and denote $w(\tau) = \hat P^{g,\gamma}(\tau,0)u$. Then, applying [6, Theorem 2.9] to the process $\hat P^{\bar g,\bar\gamma}$ and to the Lipschitz curve $\tau \to \pi_\tau w(\tau)$, using the tangency condition, we compute

$$\begin{aligned}&\left\|\hat P^{g,\gamma}(t,0)u - \hat P^{\bar g,\bar\gamma}(t,0)u\right\|_{\mathbf{L^1}(I_t)}\\ &\le L\int_0^t \liminf_{\varepsilon\to 0} \frac{\left\|\pi_{\tau+\varepsilon} w(\tau+\varepsilon) - \hat P^{\bar g,\bar\gamma}(\varepsilon,\tau)\left(\pi_\tau w(\tau)\right)\right\|_{\mathbf{L^1}(I_{\tau+\varepsilon})}}{\varepsilon} d\tau\\ &\le L\int_0^t \liminf_{\varepsilon\to 0} \frac{\left\|\hat P^{g,\gamma}(\varepsilon,\tau) w(\tau) - \hat P^{\bar g,\bar\gamma}(\varepsilon,\tau)\left(\pi_\tau w(\tau)\right)\right\|_{\mathbf{L^1}(I_{\tau+\varepsilon})}}{\varepsilon} d\tau\\ &\le L\int_0^t \liminf_{\varepsilon\to 0} \frac{\left\|P^{g,\gamma}(\varepsilon,\tau)\left(w(\tau)\right) - P^{\bar g,\bar\gamma}(\varepsilon,\tau)\left(\pi_\tau w(\tau)\right)\right\|_{\mathbf{L^1}(I_{\tau+\varepsilon})}}{\varepsilon} d\tau\\ &+ L\int_0^t \left\|G\left(w(\tau)\right) - G\left(\pi_\tau w(\tau)\right)\right\|_{\mathbf{L^1}(\mathbb{R})} d\tau\,.\end{aligned}$$

For the term deriving from the source, we use the $\mathbf{L^1}$ Lipschitz continuity of G to estimate:

$$\begin{aligned}\int_0^t \left\|G\left(w(\tau)\right) - G\left(\pi_\tau w(\tau)\right)\right\|_{\mathbf{L^1}(\mathbb{R})} d\tau &\le C\int_0^t \left\|w(\tau) - \left(\pi_\tau w(\tau)\right)\right\|_{\mathbf{L^1}(\mathbb{R})} d\tau\\ &\le C\,T\,\|\gamma-\bar\gamma\|_{\mathbf{C^0}([0,t])}.\end{aligned}$$

Concerning the other term, denote by $F^{g_o,\gamma}(t,t_o)u$ the tangent vector defined in (4.6). Here, we explicitly denote the dependence of the tangent vector on the curve γ and on the pointwise boundary data $g_o = g(t_o)$. By 3) in Theorem 2.2, the curve $\eta \to P^{\bar g,\bar\gamma}(\eta,\tau)\left(\pi_\tau w(\tau)\right)$ is first order tangent to $\eta \to F^{\bar g(\tau),\bar\gamma}(\eta,\tau)\left(\pi_\tau w(\tau)\right)$, while $\eta \to P^{g,\gamma}(\eta,\tau)w(\tau)$ is first order tangent to $\eta \to F^{g(\tau),\gamma}(\eta,\tau)\left(w(\tau)\right)$. Because of the finite propagation speed, the two tangent vectors coincide in the interval $[\bar\gamma(\tau)+\eta\hat\lambda,+\infty[$. Therefore,

$$\begin{aligned}
&\int_0^t \liminf_{\varepsilon\to 0} \frac{\left\| P^{g,\gamma}(\varepsilon,\tau)w(\tau) - P^{\bar g,\bar\gamma}(\varepsilon,\tau)\left(\pi_\tau w(\tau)\right)\right\|_{\mathbf{L^1}(I_{\tau+\varepsilon})}}{\varepsilon} d\tau \\
&= \int_0^t \liminf_{\varepsilon\to 0} \frac{\left\| F^{g(\tau),\gamma}(\varepsilon,\tau)w(\tau) - F^{\bar g(\tau),\bar\gamma}(\varepsilon,\tau)\left(\pi_\tau w(\tau)\right)\right\|_{\mathbf{L^1}(I_{\tau+\varepsilon})}}{\varepsilon} d\tau \\
&= \int_0^t \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_{\bar\gamma(\tau+\varepsilon)}^{\bar\gamma(\tau)+\varepsilon\hat\lambda} \left\| \left(F^{g(\tau),\gamma}(\varepsilon,\tau)w(\tau)\right)(x) \right. \\
&\qquad\qquad \left. - \left(F^{\bar g(\tau),\bar\gamma}(\varepsilon,\tau)\left(\pi_\tau w(\tau)\right)\right)(x)\right\| dx d\tau\,.
\end{aligned}$$

Referring to Lemma 4.1, introduce the quantities

$$\begin{aligned}
w_r^\tau &= w\left(\tau,\bar\gamma(\tau)\right),\\
b(w^{\bar\sigma,\tau}) &= \bar g(\tau),\\
w_r^\tau &= \psi_n(\bar\sigma_n)\circ\ldots\circ\psi_{\ell+1}(\sigma_{\ell+1})(w^{\bar\sigma,\tau}).\\
\tilde w^\tau(x) &= \begin{cases} w(\tau,x) & \text{for } x \geq \gamma(\tau),\\ w\left(\tau,\gamma(\tau)\right) & \text{for } x < \gamma(\tau),\end{cases}\\
\tilde{\tilde w}^\tau(x) &= \begin{cases} w(\tau,x) & \text{for } x \geq \bar\gamma(\tau),\\ w^{\bar\sigma,\tau} & \text{for } x < \bar\gamma(\tau)\end{cases}
\end{aligned}$$

By formulæ (4.5)–(4.6) and since the boundary condition is satisfied for almost all τ, that is $b\Big(w\left(\tau,\gamma(\tau)\right)\Big) = g(\tau)$, one has for $x \geq \bar\gamma(\tau+\varepsilon)$

$$\begin{aligned}
\Big(F^{g(\tau),\gamma}(\varepsilon,\tau)w(\tau)\Big)(x) &= \left(\mathcal{S}_\varepsilon \tilde w^\tau\right)(x)\\
\Big(F^{\bar g(\tau),\bar\gamma}(\varepsilon,\tau)\left(\pi_\tau w(\tau)\right)\Big)(x) &= \left(\mathcal{S}_\varepsilon \tilde{\tilde w}^\tau\right)(x)
\end{aligned}$$

where $\mathcal{S}$ is the purely convective Standard Riemann Semigroup without boundary generated by f [6, Definition 9.1].

Denote by $U_\tau^\sharp$ and $\bar U_\tau^\sharp$ the solutions to the two Riemann problems:

$$\begin{cases} u_t + f(u)_x = 0\\ u(0,x) = \begin{cases} \tilde w^\tau(\bar\gamma(\tau)-) & \text{for } x<0\\ \tilde w^\tau\left(\bar\gamma(\tau)\right) & \text{for } x>0\end{cases}\end{cases} \qquad \begin{cases} u_t + f(u)_x = 0\\ u(0,x) = \begin{cases} \tilde{\tilde w}^\tau(\bar\gamma(\tau)-) & \text{for } x<0\\ \tilde{\tilde w}^\tau\left(\bar\gamma(\tau)\right) & \text{for } x>0\end{cases}\end{cases}$$

Formula [6, (9.16)] implies that

$$\begin{aligned}
&\int_0^t \liminf_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\bar\gamma(\tau+\varepsilon)}^{\bar\gamma(\tau)+\varepsilon\hat\lambda}\left\|\left(\mathcal{S}_\varepsilon\tilde w^\tau\right)(x)-\left(\mathcal{S}_\varepsilon\tilde w^\tau\right)(x)\right\| dxd\tau \\
&\qquad\le \int_0^t \liminf_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\bar\gamma(\tau+\varepsilon)-\bar\gamma(\tau)}^{\varepsilon\hat\lambda}\left\|U_\tau^\sharp(\varepsilon,x)-\bar U_\tau^\sharp(\varepsilon,x)\right\| dxd\tau
\end{aligned}$$

By Remark 3.4, for almost all τ such that $\gamma(\tau)<\bar\gamma(\tau)$ one has $\tilde w^\tau(\bar\gamma(\tau)-)=\tilde w^\tau(\bar\gamma(\tau)-)=w_r^\tau$, therefore $U_\tau^\sharp(\varepsilon,x)\equiv w_r$. While for almost all τ such that $\gamma(\tau)=\bar\gamma(\tau)$, the boundary condition implies $\tilde w^\tau(\bar\gamma(\tau)-)=\tilde w^\tau\big(\gamma(\tau)\big)=w_r^\tau$ therefore we have again $U_\tau^\sharp(\varepsilon,x)\equiv w_r^\tau$. We compute, for almost all τ

$$\begin{aligned}
&\left\|U_\tau^\sharp(\varepsilon,x)-\bar U_\tau^\sharp(\varepsilon,x)\right\|\le C\left\|E_b^\sigma\left(w_r^\tau,\bar g(\tau)\right)\right\| \\
&\qquad = C\left\|E_b^\sigma\left(w_r^\tau,\bar g(\tau)\right)-E_b^\sigma\left(w_r^\tau,b(w_r^\tau)\right)\right\|\le C\left\|\bar g(\tau)-b\left(w_r^\tau\right)\right\|.
\end{aligned}$$

Finally we compute, using Proposition 3.3,

$$\begin{aligned}
&\int_0^t \liminf_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\bar\gamma(\tau+\varepsilon)-\bar\gamma(\tau)}^{\varepsilon\hat\lambda}\left\|U_\tau^\sharp(\varepsilon,x)-\bar U_\tau^\sharp(\varepsilon,x)\right\| dxd\tau \\
&\le C\int_0^t\left\|\bar g(\tau)-b\left(w_r^\tau\right)\right\| d\tau \\
&\le C\int_0^t\left\|\bar g(\tau)-g(\tau)\right\| d\tau + C\int_0^t\left\|b\Big(w\left(\tau,\gamma(\tau)\right)\Big)-b\Big(w\left(\tau,\bar\gamma(\tau)\right)\Big)\right\| d\tau \\
&\le C\int_0^t\left\|\bar g(\tau)-g(\tau)\right\| d\tau + \left\|\gamma-\bar\gamma\right\|_{\mathbf{C^0}}.
\end{aligned}$$

The general case of two non ordered curves follows immediately by the triangle inequality. □

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