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# More on M.M. Zenga's new three-parameter distribution for non-negative variables

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In memory of Paolo Radaelli

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**Abstract**

Recently M.M. Zenga (2010) has proposed a new three-parameter density function $f(x : \mu; \alpha; \theta)$, ($\mu > 0$; $\alpha > 0$; $\theta > 0$), for non-negative variables. The parameter $\mu$ is equal to the expectation of the distribution. The new density has positive asymmetry and Paretian right tail. For $\theta > 1$, M.M. Zenga (2010) has obtained the expressions of: the distribution function, the moments, the truncated moments, the mean deviation and Zenga's (2007a) point inequality $A(x)$ at $x = \mu$. In the present paper, as to the general case $\theta > 0$, the expressions of: the distribution function, the ordinary and truncated moments, the mean deviations and Zenga's point inequality $A(\mu)$ are obtained. These expressions are more complex than those previously obtained for $\theta > 1$ by M.M. Zenga (2010). The paper is enriched with many graphs of: the density functions ($0.5 \leq \theta \leq 1.5$), the Lorenz $L(p)$ and Zenga's $I(p)$ curves as well as the hazard and survival functions.

**Keywords:** non-negative variables, positive asymmetry, paretian right tail, beta function, Lorenz curve, Zenga's inequality curve, hazard function, survival function

## 1 Introduction

Recently M. M. Zenga (2010) has proposed a new three parameter density function $f(x : \mu; \alpha; \theta)$, ($\mu > 0$; $\alpha > 0$; $\theta > 0$), for non negative variables. The density $f(x : \mu; \alpha; \theta)$ has been obtained as a mixture of

Polisicchio's (2008) following truncated Pareto density

$$f(x:\mu;k) = \begin{cases} \frac{\sqrt{\mu}}{2} k^{0.5}(1-k)^{-1}x^{-1.5}, & \mu k \le x \le \frac{\mu}{k}; \quad \mu > 0, 0 < k < 1; \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

with a fixed $\mu > 0$ and all the values of $k$ in the interval $(0;1)$. The density on the parameter $k$ is given by the beta density

$$g(k:\alpha;\theta) = \begin{cases} \frac{k^{\alpha-1}(1-k)^{\theta-1}}{B(\alpha;\theta)}, & 0 < k < 1; \quad \theta > 0, \alpha > 0; \\ 0, & \text{otherwise,} \end{cases} \tag{2}$$

where $B(\alpha;\theta)$ is the beta function.

The parameter $\mu$ of Polisicchio's density (1) and of Zenga's density $f(x:\mu;\alpha;\theta)$ is equal to their expectations. Moreover, the density $f(x:\mu;\alpha;\theta)$ has positive asymmetry and Paretian right tail.

M.M. Zenga (2010) has obtained, in the case $\theta > 1$, the expression of the distribution function, of the moments, of the truncated moments, of the mean deviation, of the relative mean deviations and of Zenga's (2007) point inequality $A(x)$ at $x = \mu$. Many graphs of $f(x:\mu;\alpha;\theta)$ for $\theta \ge 2$ are reported in M.M. Zenga (2010) too. According to this graphs it seems that the shapes of the new density function are broader than those of the more traditional models of income distributions like Dagum's distribution.

In this paper, using a new approach, we obtain for the general case $\theta > 0$ a different expression of: the density, the truncated and ordinary moments, the mean deviation, the relative mean deviations and Zenga's point inequality measure. Obviously, we have shown that the new expressions obtained in this paper for $\theta > 0$, are equivalent to those obtained previously for $\theta > 1$ by M.M. Zenga (2010). The paper is enriched with many graphs of the density for $0.5 \le \theta \le 1.5$, of the Lorenz $L(p)$ curve and of Zenga's $I(p)$ curve, and of the hazard and survival functions.

The paper is organized as follows. Section 2 deals with notation, definitions and some useful lemma involving beta functions. Section 3 reports many graphs of the density function $f(x:\mu;\alpha;\theta)$ for $0.5 \le \theta \le 1.5$. In section 4 and 5, by the use of the moments and of the distribution function of Polisicchio's density $f(x:\mu;k)$, the moments and the distribution function of $f(x:\mu;\alpha;\theta)$ for the case $\theta > 0$ are derived. In section 6, for $\theta > 0$, the distribution function, the ordinary and the truncated moments are obtained (directly) from the density $f(x:\mu;\alpha;\theta)$. Section 7 shows the following two equalities:

a) $\int_0^1 xf(x:1;\alpha;\theta)dx = \int_1^\infty f(x:1;\alpha;\theta)dx$;

b) $\int_1^\infty xf(x:1;\alpha;\theta)dx = \int_0^1 f(x:1;\alpha;\theta)dx$.

In section 8, for $\theta > 0$, the mean deviation and the Zenga point inequality $A(\mu)$ are derived. In section 9 it is shown that the new expressions obtained for $\theta > 0$ are equivalent to those previously obtained by M.M. Zenga (2010). Section 10 reports many graphs of the Lorenz $L(p)$ curve and of the Zenga $I(p)$ curve. Many graphs of the hazard and of the survival functions are reported in section 11. Finally, section 12 is devoted to the conclusions.

## 2 Notation, Definitions and Rules on Beta Functions

**Definition 1.** *The beta function is defined by the Eulerian integral of the first kind:*

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt, \quad a > 0, b > 0. \tag{3}$$

**Definition 2.** *The gamma function is defined by the Eulerian integral of the second kind:*

$$\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}dt, \quad a > 0. \tag{4}$$

The relationship between the beta and gamma function is

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \quad a > 0, b > 0. \tag{5}$$

**Definition 3.** *The incomplete beta function is defined by*

$$IB(z:a;b) = \int_0^z t^{a-1}(1-t)^{b-1}dt, \quad 0 < z < 1; \quad a > 0. \tag{6}$$

If $b > 0$, the domain of $IB(z:a;b)$ can be extended to $0 < z \leq 1$.

**Definition 4.** *The normalized incomplete beta function is defined by*

$$B(z:a;b) = \frac{IB(z:a;b)}{B(a;b)}, \quad 0 < z \leq 1; \quad a > 0, b > 0. \tag{7}$$

**Lemma 1.** *If $a > 0$ and $0 < z < 1$,*

$$IB(z:a;b) = IB(z:a;b-1) - IB(z:a+1;b-1). \tag{8}$$

*Proof.*

$$\begin{aligned}\int_0^z t^{a-1}(1-t)^{b-1}dt &= \int_0^z t^{a-1}(1-t)^{b-1-1}(1-t)dt \\ &= \int_0^z \left\{t^{a-1}(1-t)^{b-1-1} - t^{a+1-1}(1-t)^{b-1-1}\right\} dt \\ &= IB(z:a;b-1) - IB(z:a+1;b-1)\end{aligned}$$

□

If $b > 1$, the above equality is also valid for $z = 1$.

**Lemma 2.** *If $a > 0$ and $b > 0$, then*

$$\int_0^z t^{a-1}(1-t)^{b-2}dt = \sum_{i=1}^{\infty} \int_0^z t^{(a-1+i)-1}(1-t)^{b-1}dt, \quad 0 < z < 1. \tag{9}$$

*Proof.*

$$\begin{aligned}\int_0^z t^{a-1}(1-t)^{b-2}dt &= \int_0^z t^{a-1}(1-t)^{b-1}(1-t)^{-1}dt \\ &= \int_0^z t^{a-1}(1-t)^{b-1}(1+t+t^2+t^3+...)dt \\ &= \int_0^z \sum_{i=1}^{\infty} t^{(a-1+i)-1}(1-t)^{b-1}dt \\ &= \sum_{i=1}^{\infty} \int_0^z t^{(a-1+i)-1}(1-t)^{b-1}dt.\end{aligned}$$

□

**Lemma 3.** *If $a > 0$, $b > 1$ and $0 < z \leq 1$, then*

$$\sum_{i=1}^{\infty} IB(z:a-1+i;b) = IB(z:a;b-1). \tag{10}$$

*Proof.* From lemma 1 we have

$$\begin{aligned}\sum_{i=1}^{n} IB(z:a-1+i;b) &= \sum_{i=1}^{n} \{IB(z:a-1+i;b-1) - IB(z:a+i;b-1)\} \\ &= IB(z:a;b-1) - IB(z:a+1;b-1) + \\ &+ IB(z:a+1;b-1) - IB(z:a+2;b-1) + \\ &\vdots \\ &+ IB(z:a+n-1;b-1) - IB(z:a+n;b-1) \\ &= IB(z:a;b-1) - IB(z:a+n;b-1)\end{aligned}$$

Since $\lim_{n\to\infty} IB(z : a+n; b-1) = 0$ whenever $0 < z \leq 1$, it follows that

$$\lim_{n\to\infty} \sum_{i=1}^{n} IB(z : a-1+i; b) = IB(z : a; b-1).$$

□

**Lemma 4.** *If $\alpha > 0$, $\theta > 0$, $i = 1, 2, ...,$ $r = 0, 1, 2, ...$ and $0 < x \leq 1$, then*

$$\int_0^x \left\{ t^r t^{-1.5} \int_0^t k^{(\alpha-0.5+i)-1}(1-k)^{\theta-1} dk \right\} dt = \frac{x^{r-0.5}}{r-0.5} IB(x : \alpha - 0.5 + i; \theta) + \\ - \frac{1}{r-0.5} IB(x : \alpha + r - 1 + i; \theta). \tag{11}$$

*Proof.* The integral $\int_0^x \{\cdots\} dt$ can be written as follows

$$\int_0^x \left( \int_0^t k^{(\alpha-0.5+i)-1}(1-k)^{\theta-1} dk \right) d\left( \frac{t^{r-0.5}}{r-0.5} \right) \tag{12}$$

Using integration by parts we have

$$\left| \left( \int_0^t k^{(\alpha-0.5+i)-1}(1-k)^{\theta-1} dk \right) \frac{t^{r-0.5}}{r-0.5} \right|_0^x - \int_0^x \frac{t^{r-0.5}}{r-0.5} t^{\alpha-0.5+i-1}(1-t)^{\theta-1} dt = \\ \frac{x^{r-0.5}}{r-0.5} \int_0^x k^{(\alpha-0.5+i)-1}(1-k)^{\theta-1} dk - \lim_{t\to 0} \frac{t^{r-0.5}}{r-0.5} \int_0^t k^{(\alpha-0.5+i)-1}(1-k)^{\theta-1} dk + \\ - \frac{1}{r-0.5} \int_0^x t^{\alpha+r-1+i-1}(1-t)^{\theta-1} dt$$

Since

$$\lim_{t\to 0} \frac{t^{r-0.5}}{r-0.5} \int_0^t k^{(\alpha-0.5+i)-1}(1-k)^{\theta-1} dk = 0,$$

the conclusion follows. □

**Lemma 5.** *If $\alpha > 0$, $\theta > 0$, $i = 1, 2, 3, ...,$ $r = 0, 1, 2, ...,$ $1 \leq x$ and $\alpha + 1 > r$, then*

$$\int_x^\infty \left\{ t^r t^{-1.5} \int_0^{\frac{1}{t}} k^{(\alpha-0.5+i)-1}(1-k)^{\theta-1} dk \right\} dt = \frac{1}{r-0.5} IB\left( \frac{1}{x} : \alpha - r + i; \theta \right) + \\ - \frac{x^{r-0.5}}{r-0.5} IB\left( \frac{1}{x} : \alpha - 0.5 + i; \theta \right) \tag{13}$$

*Proof.*

$$\begin{aligned}
\int_x^{\infty}\{\cdots\}dt &= \int_x^{\infty}\left(\int_0^{\frac{1}{t}} k^{(\alpha-0.5+i)-1}(1-k)^{\theta-1}dk\right) d\left(\frac{t^{r-0.5}}{r-0.5}\right) \\
&= \left|\left(\frac{t^{r-0.5}}{r-0.5}\right)\left(\int_0^{\frac{1}{t}} k^{(\alpha-0.5+i)-1}(1-k)^{\theta-1}dk\right)\right|_x^{\infty} + \\
&\quad -\int_x^{\infty}\frac{t^{r-0.5}}{r-0.5}\left(\frac{1}{t}\right)^{\alpha-0.5+i-1}\left(1-\frac{1}{t}\right)^{\theta-1}(-1)\left(\frac{1}{t}\right)^2 dt \\
&= \frac{1}{r-0.5}\left\{\lim_{t\to\infty} t^{r-0.5}\int_0^{\frac{1}{t}} k^{(\alpha-0.5+i)-1}(1-k)^{\theta-1}dk + \right. \\
&\qquad -x^{r-0.5}\int_0^{\frac{1}{x}} k^{\alpha-0.5+i-1}(1-k)^{\theta-1}dk + \\
&\qquad \left. +\int_x^{\infty}\left(\frac{1}{t}\right)^{\alpha-r+i+1}\left(1-\frac{1}{t}\right)^{\theta-1}dt\right\}.
\end{aligned}$$

With the substitution $y = \frac{1}{t}$ we have

$$\begin{aligned}
\int_x^{\infty}\left(\frac{1}{t}\right)^{\alpha-r+i+1}\left(1-\frac{1}{t}\right)^{\theta-1}dt &= \int_{\frac{1}{x}}^{0} y^{\alpha-r+i+1}(1-y)^{\theta-1}(-1)y^{-2}dy \\
&= \int_0^{\frac{1}{x}} y^{\alpha-r+i-1}(1-y)^{\theta-1}dy \\
&= IB\left(\frac{1}{x}:\alpha-r+i;\theta\right)
\end{aligned} \tag{14}$$

Moreover,

$$\lim_{t\to\infty} t^{r-0.5}\int_0^{\frac{1}{t}} k^{(\alpha-0.5+i)-1}(1-k)^{\theta-1}dk = 0.$$

□

## 3 Graphs of $f(x:\mu;\alpha;\theta)$ for $0.5 \leq \theta \leq 1.5$

M.M. Zenga (2010) has shown that for $\theta > 0$ the density $f(x:\mu;\alpha;\theta)$ is given by

$$f(x:\mu;\alpha;\theta) = \begin{cases} \frac{1}{2\mu B(\alpha;\theta)}\left(\frac{x}{\mu}\right)^{-1.5}\int_0^{\frac{x}{\mu}} k^{\alpha+0.5-1}(1-k)^{\theta-2}dk, & 0 < x < \mu \\ \\ \frac{1}{2\mu B(\alpha;\theta)}\left(\frac{\mu}{x}\right)^{1.5}\int_0^{\frac{\mu}{x}} k^{\alpha+0.5-1}(1-k)^{\theta-2}dk, & \mu < x. \end{cases} \tag{15}$$

Note that:

$$\lim_{x\to\mu} f(x:\mu;\alpha;\theta) = \begin{cases} \frac{B(\alpha+0.5;\theta-1)}{2\mu B(\alpha;\theta)}, & \text{if } \theta > 1 \\ \\ \infty, & \text{if } 0 < \theta \leq 1; \end{cases} \tag{16}$$

$$\lim_{x\to 0} f(x:\mu;\alpha;\theta) = \begin{cases} 0, & \text{for } \alpha > 1 \\ \\ \frac{1}{3}\frac{\theta}{\mu}, & \text{for } \alpha = 1 \\ \\ \infty, & \text{for } 0 < \alpha < 1. \end{cases} \tag{17}$$

For $\theta > 1$ the density (15) can be written as follows (M.M. Zenga, 2010)

$$f(x:\mu;\alpha;\theta) = \begin{cases} \frac{B(\alpha+0.5;\theta-1)}{2\mu B(\alpha;\theta)} \left(\frac{x}{\mu}\right)^{-1.5} B\left(\frac{x}{\mu}:\alpha+0.5;\theta-1\right), & 0 < x \leq \mu \\ \\ \frac{B(\alpha+0.5;\theta-1)}{2\mu B(\alpha;\theta)} \left(\frac{\mu}{x}\right)^{1.5} B\left(\frac{\mu}{x}:\alpha+0.5;\theta-1\right), & \mu < x. \end{cases} \tag{18}$$

Using lemma 2 the density (15) can be obtained by

$$f(x:\mu;\alpha;\theta) = \begin{cases} \frac{1}{2\mu B(\alpha;\theta)} \sum_{i=1}^{\infty} \left(\frac{x}{\mu}\right)^{-1.5} \int_0^{\frac{x}{\mu}} k^{(\alpha-0.5+i)-1}(1-k)^{\theta-1}dk, & 0 < x < \mu \\ \frac{1}{2\mu B(\alpha;\theta)} \sum_{i=1}^{\infty} \left(\frac{\mu}{x}\right)^{1.5} \int_0^{\frac{\mu}{x}} k^{(\alpha-0.5+i)-1}(1-k)^{\theta-1}dk, & \mu < x. \end{cases} \tag{19}$$

$$= \begin{cases} \frac{1}{2\mu} \sum_{i=1}^{\infty} \frac{B(\alpha-0.5+i;\theta)}{B(\alpha;\theta)} \left(\frac{x}{\mu}\right)^{-1.5} B\left(\frac{x}{\mu}:\alpha-0.5+i;\theta\right), & 0 < x < \mu \\ \frac{1}{2\mu} \sum_{i=1}^{\infty} \frac{B(\alpha-0.5+i;\theta)}{B(\alpha;\theta)} \left(\frac{\mu}{x}\right)^{1.5} B\left(\frac{\mu}{x}:\alpha-0.5+i;\theta\right), & \mu < x. \end{cases} \tag{20}$$

In M.M. Zenga (2010) are reported many graphs of $f(x:2;\alpha;\theta)$ for $\theta \geq 2$ and for $\alpha \geq 0.5$. We report now graphs of $f(x:2;\alpha;\theta)$ for some $0.5 \leq \theta \leq 1.5$ and for some $0.5 \leq \alpha \leq 3.5$.

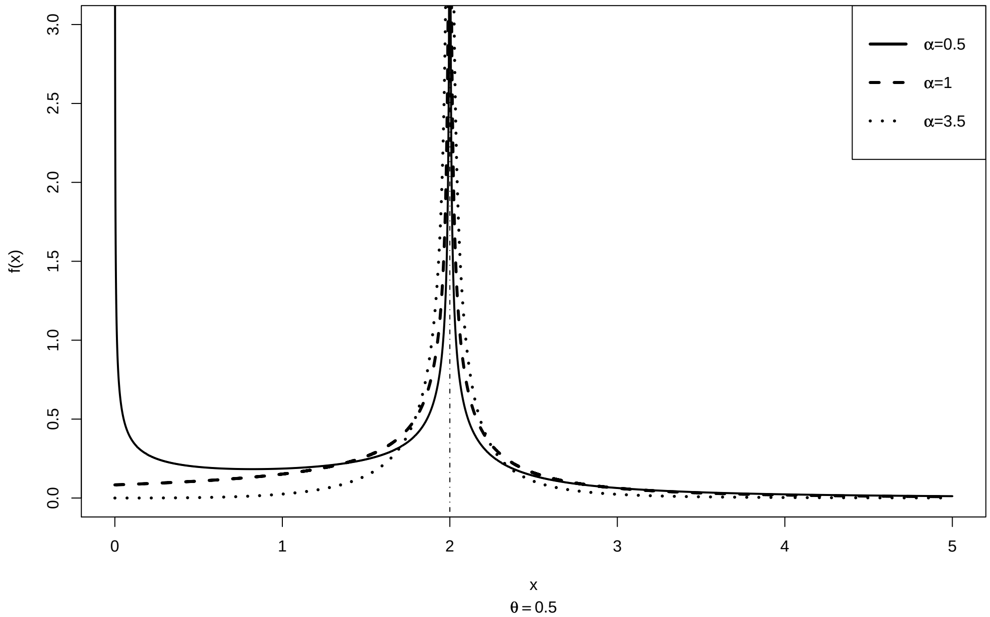


Figure 1: Graphs of $f(x:2;\alpha;0.5)$ for $\theta = 0.5$ and $\alpha = 0.5; 1; 3.5$

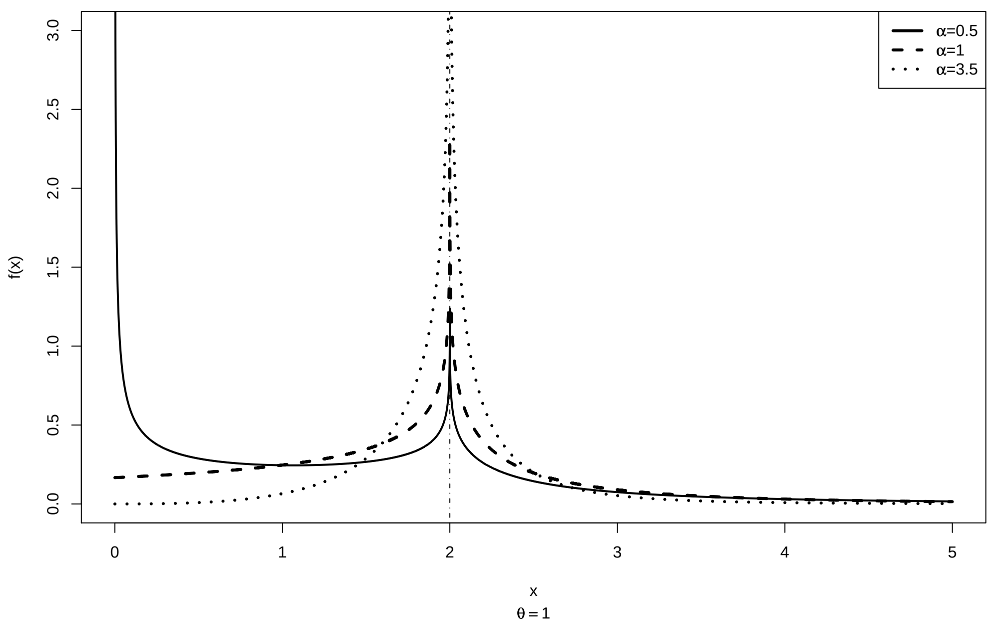


Figure 2: Graphs of $f(x:2;\alpha;1)$ for $\theta = 1$ and $\alpha = 0.5; 1; 3.5$

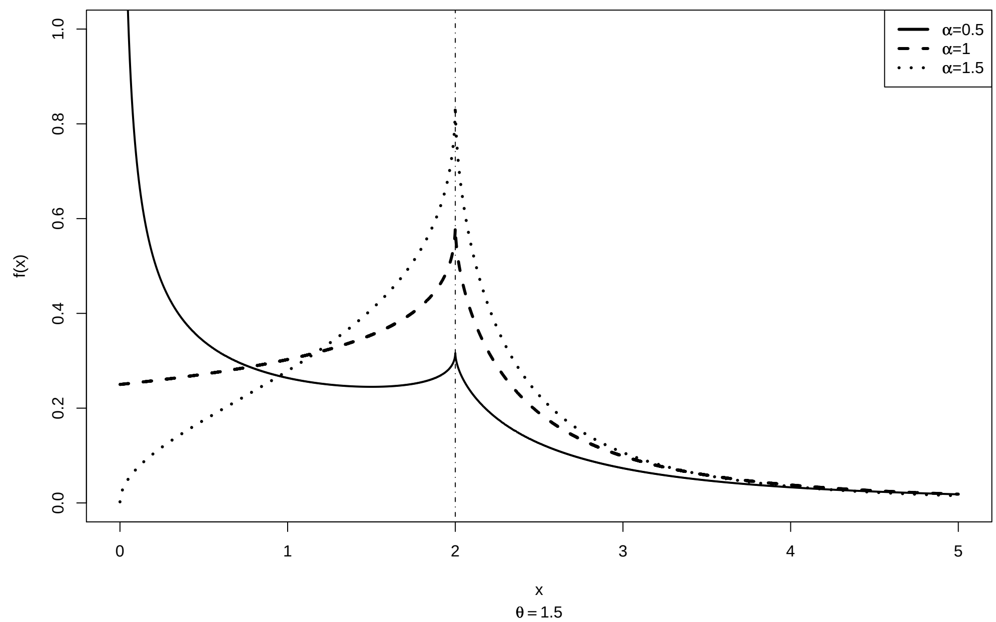


Figure 3: Graphs of $f(x:2;\alpha;1.5)$ for $\theta = 1.5$ and $\alpha = 0.5; 1; 1.5$

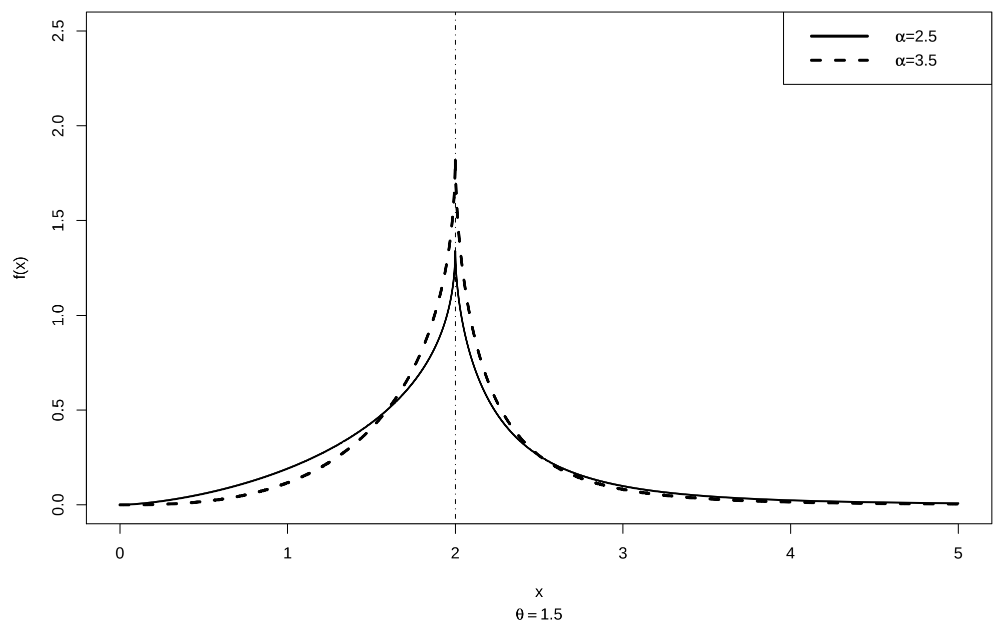


Figure 4: Graphs of $f(x:2;\alpha;1.5)$ for $\theta = 1.5$ and $\alpha = 2.5$ and 3.5

The graphs of the new density, reported in this section, ($\theta = 0.5;\ 1.0;\ 1.5$) have a behaviour quite different from those reported, for $\theta \geq 2$, in M.M. Zenga (2010).

## 4 Moments of Zenga's Distribution: general case

M.M. Zenga (2010) has shown that the $r$-th moment ($r \in \mathbb{N}$) about zero of the new three parameter density $f(x:\mu;\alpha;\theta)$ is

$$E(X^r) = \frac{\mu^r}{2r-1}\frac{1}{B(\alpha;\theta)}\left\{B(\alpha-r+1;\theta-1) - B(\alpha+r;\theta-1)\right\}, \quad \alpha+1>r, \theta>1. \tag{21}$$

In this section the moments $E(X^r)$ valid for $\theta > 0$ have been obtained using a new approach. M. Polisicchio (2008) has shown that the $r$-th moment ($r \in \mathbb{N}$) about zero of the density $f(x:\mu;k)$ is given by

$$\mu'_r = \frac{\mu^r k^{1-r}}{2r-1}\cdot\frac{1-k^{2r-1}}{1-k}, \qquad 0<k<1, \mu>0. \tag{22}$$

The relation

$$1+k+k^2+...+k^n = \frac{1-k^{n+1}}{1-k}, \tag{23}$$

is an identity in $k$ ($k \neq 1$) for every $n \in \mathbb{N}$. Using (23), the moments of Polisicchio's density $f(x:\mu;k)$ are

$$\begin{aligned}\mu'_r &= \frac{\mu^r k^{1-r}}{2r-1}(1+k+k^2+...+k^{2r-2}) \\ &= \frac{\mu^r}{2r-1}(k^{1-r}+k^{2-r}+...+k^{r-2}+k^{r-1}) \\ &= \frac{\mu^r}{2r-1}\sum_{i=1}^{2r-1}k^{i-r}\end{aligned} \tag{24}$$

The density $f(x:\mu;\alpha;\theta)$ is a mixture of $f(x:\mu;k)$ with beta weights $g(k:\alpha;\theta)$; consequently the moment $E(X^r)$ of $f(x:\mu;\alpha;\theta)$ can be obtained by

$$\begin{aligned}E(X^r) &= \int_0^1 \left(\frac{\mu^r}{2r-1}\sum_{i=1}^{2r-1}k^{i-r}\right)\frac{k^{\alpha-1}(1-k)^{\theta-1}}{B(\alpha;\theta)}dk \\ &= \frac{\mu^r}{(2r-1)B(\alpha;\theta)}\int_0^1\left(\sum_{i=1}^{2r-1}k^{(\alpha-r+i)-1}(1-k)^{\theta-1}\right)dk.\end{aligned}$$

For $(\alpha+1)>r$ the integrals

$$\int_0^1 k^{(\alpha-r+i)-1}(1-k)^{\theta-1}dk, \qquad i=1,2,...,2r-1$$

are finite. Consequently, for $(\alpha+1)>r$ and $\theta>0$, we have

$$E(X^r) = \frac{\mu^r}{(2r-1)B(\alpha;\theta)}\sum_{i=1}^{2r-1}B(\alpha-r+i;\theta). \tag{25}$$

In particular,

$$E(X) = \mu.$$

Note that the expectation of the new three parameter random variable is equal to the parameter $\mu$. For $\alpha > 1$ and $\theta > 0$, formula (25) gives

$$\begin{aligned} E(X^2) &= \frac{\mu^2}{3}\left\{\frac{B(\alpha-1;\theta)}{B(\alpha;\theta)} + \frac{B(\alpha;\theta)}{B(\alpha;\theta)} + \frac{B(\alpha+1;\theta)}{B(\alpha;\theta)}\right\} \\ &= \frac{\mu^2}{3}\left\{\frac{\alpha+\theta-1}{\alpha-1} + 1 + \frac{\alpha}{\alpha+\theta}\right\} \\ &= \frac{\mu^2}{3}\left\{\frac{\theta}{\alpha-1} + \frac{\alpha}{\alpha+\theta} + 2\right\} \end{aligned} \tag{26}$$

Consequently,

$$\begin{aligned} Var(X) &= \frac{\mu^2}{3}\left[\frac{\theta}{\alpha-1} + \frac{\alpha}{\alpha+\theta} + 2\right] - \mu^2 \frac{3}{3}\frac{(\alpha-1)(\alpha+\theta)}{(\alpha-1)(\alpha+\theta)} \\ &= \frac{\mu^2}{3}\frac{\theta(\theta+1)}{(\alpha-1)(\alpha+\theta)}. \end{aligned} \tag{27}$$

Formula (27) coincides with the expression of $Var(X)$ obtained for $\theta > 1$ by M.M. Zenga (2010). For $\alpha > 2$ and $\theta > 0$, formula (25) gives

$$\begin{aligned} E(X^3) &= \frac{\mu^3}{5}\frac{1}{B(\alpha;\theta)}\left\{B(\alpha-2;\theta) + B(\alpha-1;\theta) + B(\alpha;\theta) + B(\alpha+1;\theta) + B(\alpha+2;\theta)\right\} \\ &= \frac{\mu^3}{5}\left\{\frac{(\alpha+\theta-2)(\alpha+\theta-1)}{(\alpha-2)(\alpha-1)} + \frac{\alpha+\theta-1}{\alpha-1} + 1 + \frac{\alpha}{\alpha+\theta} + \frac{(\alpha+1)\alpha}{(\alpha+\theta+1)(\alpha+\theta)}\right\} \end{aligned} \tag{28}$$

If $\theta = 1$ and $(\alpha+1) > r$, formula (25) gives

$$E(X^r) = \frac{\mu^r}{2r-1}\sum_{i=1}^{2r-1}\frac{\alpha}{\alpha-r+i}$$

## 5 Distribution function of $f(x:\mu;\alpha;\theta)$ via distribution function of Polisicchio's random variable

M.M. Zenga (2010) has shown that - in the case $\theta > 1$ - the distribution function of the density $f(x:\mu;\alpha;\theta)$ for $0 < x \leq \mu$ is

$$F(x:\mu;\alpha;\theta) = \frac{1}{B(\alpha;\theta)}\left\{IB\left(\frac{x}{\mu};\alpha;\theta-1\right) - \left(\frac{\mu}{x}\right)^{0.5} IB\left(\frac{x}{\mu};\alpha+0.5;\theta-1\right)\right\} \tag{29}$$

In this section we give the distribution function for $\theta > 0$ and for all $0 < x$.

M. Polisicchio (2008) has shown that the distribution function of the density $f(x:\mu;k)$ is

$$P(X \leq x) = \frac{1}{1-k}\left(1-\left(\frac{x}{\mu}\right)^{-0.5} k^{0.5}\right), \qquad \mu k \leq x \leq \frac{\mu}{k}. \tag{30}$$

It will be useful to use formula (30) for $k\mu \leq x \leq \mu$, while for $\mu \leq x \leq \frac{\mu}{k}$ we will use the equivalent expression

$$\begin{aligned}
[1-P(X \leq x)] &= 1-\left\{\frac{1}{1-k}\left[1-\left(\frac{\mu}{x}\right)^{0.5} k^{0.5}\right]\right\} \\
&= 1-\left\{\frac{1}{1-k}-\left(\frac{\mu}{x}\right)^{0.5}\frac{k^{0.5}}{1-k}\right\} \\
&= \frac{-k}{1-k}+\left(\frac{\mu}{x}\right)^{0.5}\frac{k^{0.5}}{1-k}.
\end{aligned} \tag{31}$$

Utilising the relation $\frac{1}{1-k}=\sum_{i=1}^{\infty} k^{i-1}$ we have

$$\begin{aligned}
P(X \leq x) &= \sum_{i=1}^{\infty} k^{i-1}-\left(\frac{x}{\mu}\right)^{-0.5} k^{0.5}\sum_{i=1}^{\infty} k^{i-1} \\
&= \sum_{i=1}^{\infty} k^{i-1}-\left(\frac{x}{\mu}\right)^{-0.5}\sum_{i=1}^{\infty} k^{i-0.5}, \qquad \mu k \leq x \leq \mu
\end{aligned} \tag{32}$$

$$[1-P(X \leq x)] = \left(\frac{\mu}{x}\right)^{0.5}\sum_{i=1}^{\infty} k^{i-0.5}-\sum_{i=1}^{\infty} k^{i}, \qquad \mu \leq x < \frac{\mu}{k} \tag{33}$$

If we want to utilize (32) and (33) to obtain the mathematical expression of the cumulative probability of the random variable with density $f(x:\mu;\alpha;\theta)$, we have to remember that (see M.M. Zenga (2010))

a) if $0 < x \leq \mu$, then $0 < k \leq \frac{x}{\mu}$;

b) if $\mu \leq x$, then $0 < k \leq \frac{\mu}{x}$.

From (32) and a) we have, for $0 < x \leq \mu$ and $\theta > 0$,

$$\begin{aligned}
F(x:\mu;\alpha;\theta) &= \int_0^{\frac{x}{\mu}}\left\{\sum_{i=1}^{\infty} k^{i-1}-\left(\frac{x}{\mu}\right)^{-0.5}\sum_{i=1}^{\infty} k^{i-0.5}\right\}\cdot\frac{k^{\alpha-1}(1-k)^{\theta-1}}{B(\alpha;\theta)}dk \\
&= \frac{1}{B(\alpha;\theta)}\int_0^{\frac{x}{\mu}}\left\{\sum_{i=1}^{\infty} k^{\alpha+i-1-1}(1-k)^{\theta-1}-\left(\frac{x}{\mu}\right)^{-0.5}\sum_{i=1}^{\infty} k^{\alpha+i-0.5-1}(1-k)^{\theta-1}\right\}dk \\
&= \frac{1}{B(\alpha;\theta)}\left\{\sum_{i=1}^{\infty} IB\left(\frac{x}{\mu};\alpha+i-1;\theta\right)-\left(\frac{x}{\mu}\right)^{-0.5}\sum_{i=1}^{\infty} IB\left(\frac{x}{\mu};\alpha+i-0.5;\theta\right)\right\}
\end{aligned} \tag{34}$$

From (33) and b) we obtain for $\mu < x$ and $\theta > 0$,

$$
\begin{aligned}
[1 - F(x:\mu;\alpha;\theta)] &= \int_0^{\frac{\mu}{x}} \left\{ \sum_{i=1}^{\infty} \left(\frac{\mu}{x}\right)^{0.5} k^{i-0.5} - \sum_{i=1}^{\infty} k^i \right\} \frac{k^{\alpha-1}(1-k)^{\theta-1}}{B(\alpha;\theta)} dk \\
&= \frac{1}{B(\alpha;\theta)} \int_0^{\frac{\mu}{x}} \left\{ \sum_{i=1}^{\infty} \left(\frac{\mu}{x}\right)^{0.5} k^{\alpha+i-0.5-1}(1-k)^{\theta-1} - \sum_{i=1}^{\infty} k^{\alpha+i-1}(1-k)^{\theta-1} \right\} dk \\
&= \frac{1}{B(\alpha;\theta)} \left\{ \sum_{i=1}^{\infty} \left(\frac{\mu}{x}\right)^{0.5} \int_0^{\frac{\mu}{x}} k^{\alpha+i-0.5-1}(1-k)^{\theta-1} dk - \sum_{i=1}^{\infty} \int_0^{\frac{\mu}{x}} k^{\alpha+i-1}(1-k)^{\theta-1} dk \right\} \\
&= \frac{1}{B(\alpha;\theta)} \left\{ \left(\frac{\mu}{x}\right)^{0.5} \sum_{i=1}^{\infty} IB\left(\frac{\mu}{x} : \alpha+i-0.5;\theta\right) - \sum_{i=1}^{\infty} IB\left(\frac{\mu}{x} : \alpha+i;\theta\right) \right\}
\end{aligned} \tag{35}
$$

## 6 Distribution function, Moments and Truncated Moments obtained (directly) from the density $f(x:\mu;\alpha;\theta)$: general case $\theta > 0$

It is now useful to report the following

**Lemma 6.** *Let $X$ be a random variable with density $f_X(x:1;\alpha;\theta)$, where $f_X(x:1;\alpha;\theta)$ is the density (15) for $\mu = 1$. Let $Y = \mu \cdot X$, $\mu > 0$. Then, $Y$ has density $f_Y(y:\mu;\alpha;\theta)$, where $f_Y(y:\mu;\alpha;\theta)$ is the density (15).*

For the proof see M.M. Zenga (2010).

From Lemma (6) and for simplicity's sake it is useful to put $\mu = 1$. In this case the density given by (19) is

$$
f(x:1;\alpha;\theta) = \begin{cases} \frac{1}{2B(\alpha;\theta)} \sum_{i=1}^{\infty} x^{-1.5} \int_0^x k^{(\alpha-0.5+i)-1}(1-k)^{\theta-1} dk, & 0 < x < 1 \\ \\ \frac{1}{2B(\alpha;\theta)} \sum_{i=1}^{\infty} x^{-1.5} \int_0^{\frac{1}{x}} k^{(\alpha-0.5+i)-1}(1-k)^{\theta-1} dk, & 1 < x. \end{cases} \tag{36}
$$

From (36) we obtain (for $0 < x \leq 1$)

$$
\begin{aligned}
F(x:1;\alpha;\theta) &= \int_0^x \left\{ \frac{1}{2B(\alpha;\theta)} \sum_{i=1}^{\infty} t^{-1.5} \int_0^t k^{(\alpha-0.5+i)-1}(1-k)^{\theta-1} dk \right\} dt \\
&= \frac{1}{2B(\alpha;\theta)} \sum_{i=1}^{\infty} \int_0^x \left\{ t^{-1.5} \int_0^t k^{\alpha-0.5+i-1}(1-k)^{\theta-1} dk \right\} dt
\end{aligned} \tag{37}
$$

Using lemma 4 (for $r = 0$) we get

$$\begin{aligned} F(x:1;\alpha;\theta) &= \frac{1}{2B(\alpha;\theta)}\sum_{i=1}^{\infty}\left[\frac{x^{-0.5}}{-0.5}IB(x:\alpha-0,5+i;\theta)-\frac{1}{-0.5}IB(x:\alpha-1+i;\theta)\right] \\ &= \frac{1}{B(\alpha;\theta)}\sum_{i=1}^{\infty}\left[IB(x:\alpha-1+i;\theta)-x^{-0.5}IB(x:\alpha-0.5+i;\theta)\right]. \end{aligned} \quad (38)$$

From (36) we obtain (for $1 \leq x$)

$$\begin{aligned} [1-F(x:1;\alpha;\theta)] &= \int_x^{\infty}\left\{\frac{1}{2B(\alpha;\theta)}\sum_{i=1}^{\infty}t^{-1.5}\int_0^{\frac{1}{t}}k^{(\alpha-0.5+i)-1}(1-k)^{\theta-1}dk\right\}dt \\ &= \frac{1}{2B(\alpha;\theta)}\sum_{i=1}^{\infty}\int_x^{\infty}\left\{t^{-1.5}\int_0^{\frac{1}{t}}k^{(\alpha-0.5+i)-1}(1-k)^{\theta-1}dk\right\}dt \end{aligned} \quad (39)$$

Using lemma 5 (for $r = 0$) we get

$$[1-F(x:1;\alpha;\theta)] = \frac{1}{B(\alpha;\theta)}\sum_{i=1}^{\infty}\left[x^{-0.5}IB\left(\frac{1}{x}:\alpha-0.5+i;\theta\right)-IB\left(\frac{1}{x}:\alpha+i;\theta\right)\right] \quad (40)$$

If $\mu \neq 1$, from lemma 6 derives that the distribution function for $0 < x \leq \mu$ is given by

$$F(x:\mu;\alpha;\theta) = \frac{1}{B(\alpha;\theta)}\sum_{i=1}^{\infty}\left[IB\left(\frac{x}{\mu}:\alpha-1+i;\theta\right)-\left(\frac{x}{\mu}\right)^{-0.5}IB\left(\frac{x}{\mu}:\alpha-0.5+i;\theta\right)\right]; \quad (41)$$

while for $\mu \leq x$ is

$$[1-F(x:\mu;\alpha;\theta)] = \frac{1}{B(\alpha;\theta)}\sum_{i=1}^{\infty}\left[\left(\frac{\mu}{x}\right)^{0.5}IB\left(\frac{\mu}{x}:\alpha-0.5+i;\theta\right)-IB\left(\frac{\mu}{x}:\alpha+i;\theta\right)\right]; \quad (42)$$

Note that the expressions of the distribution function obtained in this section are equal to those obtained in another way in section 5.

*Moments and truncated moments at $x = \mu$.* We have shown in section 4 that for $(\alpha + 1) > r$ and $\theta > 0$, the $r$-th moment about zero of the density $f(x : \mu; \alpha; \theta)$ is

$$E(X^r) = \frac{\mu^r}{(2r-1)B(\alpha;\theta)}\sum_{i=1}^{2r-1}B(\alpha-r+i;\theta).$$

It is possible to obtain the moments using directly the density $f(x : \mu; \alpha; \theta)$. For simplicity's sake we consider $\mu = 1$. Thus

$$\begin{aligned} E(X^r) &= \int_0^1\left\{\frac{t^r t^{-1.5}}{2B(\alpha;\theta)}\sum_{i=1}^{\infty}\int_0^t k^{\alpha-0.5+i-1}(1-k)^{\theta-1}dk\right\}dt+ \\ &+\int_1^{\infty}\left\{\frac{t^r t^{-1.5}}{2B(\alpha;\theta)}\sum_{i=1}^{\infty}\int_0^{\frac{1}{t}}k^{\alpha-0.5+i-1}(1-k)^{\theta-1}dk\right\}dt \end{aligned} \quad (43)$$

Using lemma 4 and lemma 5 we obtain

$$E(X^r) = \frac{1}{2B(\alpha;\theta)} \cdot \frac{1}{r-0.5} \sum_{i=1}^{\infty} \left[B(\alpha-0.5+i;\theta) - B(\alpha+r-1+i;\theta)\right] \tag{44}$$

$$+\frac{1}{2B(\alpha;\theta)} \cdot \frac{1}{r-0.5} \sum_{i=1}^{\infty} \left[B(\alpha-r+i;\theta) - B(\alpha-0.5+i;\theta)\right] \tag{45}$$

It follows that

$$E(X^r) = \frac{1}{2B(\alpha;\theta)} \cdot \frac{1}{r-0.5} \sum_{i=1}^{\infty} \left[B(\alpha-r+i;\theta) - B(\alpha+r-1+i;\theta)\right]$$

$$= \frac{1}{2B(\alpha;\theta)} \cdot \frac{1}{r-0.5} \sum_{i=1}^{2r-1} B(\alpha-r+i;\theta)$$

For $\mu \neq 1$, the $r$-th moment is

$$E(X^r) = \frac{\mu^r}{2B(\alpha;\theta)} \cdot \frac{1}{r-0.5} \sum_{i=1}^{2r-1} B(\alpha-r+i;\theta) \tag{46}$$

Formula (46) is equal - obviously - to formula (25) obtained in another way. Nevertheless, the new approach is more informative because the $r$-th moment can be split as follows:

$$E(X^r) = E(X^r|X \leq \mu) \cdot F(\mu:\mu;\alpha;\theta) + E(X^r|X > \mu) \cdot \left[1 - F(\mu:\mu;\alpha;\theta)\right],$$

where

$$E(X^r|X \leq \mu) = \frac{\mu^r \int_0^1 x^r f(x:1;\alpha;\theta)dx}{F(1:1;\alpha;\theta)} \tag{47}$$

and

$$E(X^r|X > \mu) = \frac{\mu^r \int_1^\infty x^r f(x:1;\alpha;\theta)dx}{1 - F(1:1;\alpha;\theta)}, \tag{48}$$

Obviously, $\int_0^1 x^r f(x:1;\alpha;\theta)dx$ is given by (44), while $\int_1^\infty x^r f(x:1;\alpha;\theta)dx$ is given by (45).

## 7 Unexpected equalities

**Theorem 1.** *For the r.v. with density $f(x:1;\alpha;\theta)$ the following relations are correct:*

a) $\int_0^1 x^1 f(x:1;\alpha;\theta)dx = \int_1^\infty f(x:1;\alpha;\theta)dx$;

b) $\int_1^\infty x^1 f(x:1;\alpha;\theta)dx = \int_0^1 f(x:1;\alpha;\theta)dx$.

*Proof.* For $r = 1$ the formula in (44) gives

$$\int_0^1 x^1 f(x:1;\alpha;\theta)dx = \frac{1}{B(\alpha;\theta)}\sum_{i=1}^{\infty}[B(\alpha-0.5+i;\theta)-B(\alpha+i;\theta)].$$

For $x = 1$ formula (40) gives

$$\int_1^{\infty} f(x:1;\alpha;\theta)dx = \frac{1}{B(\alpha;\theta)}\sum_{i=1}^{\infty}[B(\alpha-0.5+i;\theta)-B(\alpha+i;\theta)].$$

Now, formula (45) gives

$$\int_1^{\infty} x^1 f(x:1;\alpha;\theta)dx = \frac{1}{B(\alpha;\theta)}\sum_{i=1}^{\infty}[B(\alpha-1+i;\theta)-B(\alpha-0.5+i;\theta)],$$

and for $x = 1$ formula (38) gives

$$\int_0^1 f(x:1;\alpha;\theta) = \frac{1}{B(\alpha;\theta)}\sum_{i=1}^{\infty}[B(\alpha-1+i;\theta)-B(\alpha-0.5+i;\theta)]$$

□

# 8 Mean deviation, relative mean deviation of G. Pietra and Zenga's Point inequality $A(\mu)$

The mean deviation of a non negative continuous r.v. $X$ with expectation $\mu$ and density $f(x)$ is given by

$$E(|X-\mu|) = \mu F(\mu) - \int_0^{\mu} xf(x)dx + \int_{\mu}^{\infty} xf(x)dx - \mu(1-F(\mu))$$

For the non negative r.v. at hand it is evident from section 7 that

a) $\int_0^{\mu} xf(x:\mu;\alpha;\theta)dx = \mu\int_0^1 xf(x:1;\alpha;\theta)dx = \mu[1-F(1:1;\alpha;\theta)]$

b) $\int_{\mu}^{\infty} xf(x:\mu;\alpha;\theta)dx = \mu\int_1^{\infty} xf(x:1;\alpha;\theta)dx = \mu F(1:1;\alpha;\theta)$

Consequently,

$$\begin{aligned}E(|X-\mu|) &= \mu F(1:1;\alpha;\theta) - \mu[1-F(1:1;\alpha;\theta)] + \mu F(1:1;\alpha;\theta) - \mu[1-F(1:1;\alpha;\theta)]\\ &= 2\mu[2F(1:1;\alpha;\theta)-1]\end{aligned}$$

The relative mean deviation P of G. Pietra for the r.v. with density $f(x:\mu;\alpha;\theta)$ is

$$P = \frac{E(|X-\mu|)}{2\mu} = 2F(1:1;\alpha;\theta) - 1.$$

Zenga's point inequality (M.M. Zenga, 2007a) at $x = \mu$ is

$$A(\mu) = 1 - \frac{E(X|X \le \mu)}{E(X|X > \mu)}.$$

Consequently, for the r.v. with density $f(x : \mu; \alpha; \theta)$ we have, from section 8, that

$$A(\mu) = 1 - \frac{\frac{\mu[1-F(1:1;\alpha;\theta)]}{F(1:1;\alpha;\theta)}}{\frac{\mu F(1:1;\alpha;\theta)}{[1-F(1:1;\alpha;\theta)]}} = 1 - \left\{\frac{1 - F(1:1;\alpha;\theta)}{F(1:1;\alpha;\theta)}\right\}^2.$$

# 9 Equivalence among the expressions obtained for the general case ($\theta > 0$) and those previously obtained (for $\theta > 1$)

**Theorem 2.** *If $\theta > 1$ and $(\alpha + 1) > r$, then*

$$\sum_{i=1}^{2r-1} B(\alpha - r + i; \theta) = B(\alpha - r + 1; \theta - 1) - B(\alpha + r; \theta - 1)$$

*Proof.* By hypothesis $(\alpha + i - r) > 0$ for $i = 1, 2, ..., 2r - 1$. Thus from lemma 1

$$\begin{aligned}
\sum_{i=1}^{2r-1} B(\alpha - r + i; \theta) &= \sum_{i=1}^{2r-1} \{B(\alpha - r + i; \theta - 1) - B(\alpha - r + i + 1; \theta - 1)\} \\
&= B(\alpha - r + 1; \theta - 1) - B(\alpha - r + 2; \theta - 1) + \\
&\quad + B(\alpha - r + 2; \theta - 1) - B(\alpha - r + 3; \theta - 1) + \\
&\quad\ \vdots \\
&\quad + B(\alpha - r + 2r - 1; \theta - 1) - B(\alpha - r + 2r - 1 + 1; \theta - 1) \\
&= B(\alpha - r + 1; \theta - 1) - B(\alpha + r; \theta - 1).
\end{aligned}$$

□

From theorem 1 it follows that in the case $\theta > 1$ formula (25) obtained for $E(X^r)$ in this paper ($\theta > 0$) coincides with the expression of $E(X^r)$ furnished by M.M. Zenga (2010) for $\theta > 1$.

**Theorem 3.** *If $\theta > 1$, the expression of the distribution function - for $0 < x < \mu$ - given by (34) is equivalent to the expression given by (29).*

*Proof.* For $\theta > 1$ it derives from lemma 3 that

$$\begin{array}{ll} \text{a)} & \sum_{i=1}^{\infty} IB\left(\frac{x}{\mu} : \alpha + i - 1; \theta\right) = IB\left(\frac{x}{\mu} : \alpha; \theta - 1\right); \\ \text{b)} & \sum_{i=1}^{\infty} IB\left(\frac{x}{\mu} : \alpha + i - 0.5; \theta\right) = IB\left(\frac{x}{\mu} : \alpha + 0.5; \theta - 1\right); \end{array} \tag{49}$$

□

**Theorem 4.** *If $\theta > 1$, the expression of the distribution function - for $\mu < x$ - given by (35) is equivalent to*

$$[1 - F(x : \mu; \alpha; \theta)] = \frac{1}{B(\alpha; \theta)} \left\{ \left(\frac{\mu}{x}\right)^{0.5} IB\left(\frac{\mu}{x} : \alpha + 0.5; \theta - 1\right) - IB\left(\frac{\mu}{x} : \alpha + 1; \theta - 1\right) \right\}. \tag{50}$$

*Proof.* The proof is obtained using lemma 3. □

**Theorem 5.** *If $\theta > 1$, the expression of the truncated moment given in this paper for $\theta > 0$ by (47) and (48) are equivalent to the corresponding expressions obtained by M.M. Zenga (2010) for $\theta > 1$.*

*Proof.* If $\theta > 1$, it derives, using lemma 3 that:

$$\begin{array}{ll} \text{a)} & \sum_{i=1}^{\infty} B\left(\alpha - 0.5 + i; \theta\right) = B\left(\alpha + 0.5; \theta - 1\right); \\ \text{b)} & \sum_{i=1}^{\infty} B\left(\alpha + r - 1 + i; \theta\right) = B\left(\alpha + r; \theta - 1\right); \end{array}$$

this is enough to prove the equivalence - see the expression in (44) - for $E(X^r|X \leq \mu)$. Now, using lemma 3 it derives that

$$\sum_{i=1}^{\infty} B\left(\alpha - r + i; \theta\right) = B\left(\alpha - r + 1; \theta - 1\right).$$

Consequently we have proved - see the expression in (45) - the equivalence for $E(X^r|\mu < X)$ too. □

## 10 Point inequality curves

In this section we obtain for the density $f(x : \mu; \alpha; \theta)$ the graphs of the Lorenz curve and of Zenga's point inequality measure (M.M. Zenga, 2007a). These point measures are invariant to scale transformation, so it is enough to consider the case $f(x : 1; \alpha; \theta)$. To obtain the graphs of these point measures we need to evaluate the first incomplete moment $H(x)$ and the distribution function $F(x)$. From section 6 we know that

$$F(x:1;\alpha;\theta)=\begin{cases} F_1(x:1;\alpha;\theta) & =\frac{1}{B(\alpha;\theta)}\sum_{i=1}^{\infty}\{IB(x:\alpha+i-1;\theta)+ \\ & \qquad -x^{-0.5}IB(x:\alpha+i-0.5;\theta)\}, \qquad 0<x\leq 1 \\ & \\ F_2(x:1;\alpha;\theta) & =1-\frac{1}{B(\alpha;\theta)}\sum_{i=1}^{\infty}\left\{x^{-0.5}IB\left(\frac{1}{x}:\alpha+i-0.5;\theta\right)+\right. \\ & \qquad \left.-IB\left(\frac{1}{x}:\alpha+i;\theta\right)\right\}, \qquad 1<x. \end{cases}$$

The first incomplete moment of the density $f(x:1;\alpha;\theta)$ is given by

$$H(x:1;\alpha;\theta)=\int_0^x tf(t:1;\alpha;\theta)dt.$$

Using, for $r=1$, lemma 4 and 5 we get

$$H(x:1;\alpha;\theta)=\begin{cases} H_1(x:1;\alpha;\theta) & =\frac{1}{B(\alpha;\theta)}\sum_{i=1}^{\infty}[x^{0.5}IB(x:\alpha+i-0.5;\theta)+ \\ & \qquad -IB(x:\alpha+i;\theta)], \qquad 0<x\leq 1 \\ & \\ H_2(x:1;\alpha;\theta) & =1-\frac{1}{B(\alpha;\theta)}\sum_{i=1}^{\infty}\left[IB\left(\frac{1}{x}:\alpha+i-1;\theta\right)+\right. \\ & \qquad \left.+x^{0.5}IB\left(\frac{1}{x}:\alpha+i-0.5;\theta\right)\right], \qquad 1<x. \end{cases}$$

The graphs of the Lorenz curve can be obtained putting in abscissa $F(x)$ and in ordinate $H(x)$, for a sufficient number of values of $x$.

The lower mean $\bar{\mu}(x)$ and $\overset{+}{\mu}(x)$ are respectively given by:

$$\begin{aligned} \bar{\mu}(x) &= \frac{1}{F(x:1;\alpha;\theta)}\int_0^x tf(x:1;\alpha;\theta)dt=\frac{H(x:1;\alpha;\theta)}{F(x:1;\alpha;\theta)} \\ \overset{+}{\mu}(x) &= \frac{1}{1-F(x:1;\alpha;\theta)}\int_x^{\infty} tf(x:1;\alpha;\theta)dt=\frac{1-H(x:1;\alpha;\theta)}{1-F(x:1;\alpha;\theta)} \end{aligned} \tag{51}$$

In the previous expression we have utilised the relation

$$E(X)=\int_0^x tf(t:1;\alpha;\theta)dt+\int_x^{\infty} tf(t:1;\alpha;\theta)dt=1.$$

Zenga's point inequality is given by (M.M. Zenga, 2007)

$$A(x)=1-\frac{\bar{\mu}(x)}{\overset{+}{\mu}(x)}.$$

Thus, from (51) we get (M.M. Zenga, 2007a)

$$\begin{aligned} A(x) &= 1-\frac{H(x:1;\alpha;\theta)}{F(x:1;\alpha;\theta)}\cdot\frac{1-F(x:1;\alpha;\theta)}{1-H(x:1;\alpha;\theta)} \\ &= \frac{F(x:1;\alpha;\theta)-H(x:1;\alpha;\theta)}{F(x:1;\alpha;\theta)[1-H(x:1;\alpha;\theta)]} \end{aligned} \tag{52}$$

It derives that the graphs of Zenga's point inequality can be obtained putting in abscissa $F(x:1;\alpha;\theta)$ and $A(x)$ on the ordinate axis for a sufficient number of values of $x$.

In the case $\theta > 1$, we know from section 9 that

$$F(x:1;\alpha;\theta) = \begin{cases} F_1(x:1;\alpha;\theta) & = \frac{1}{B(\alpha;\theta)} IB(x:\alpha;\theta-1) \\ & \quad -\frac{x^{-0.5}}{B(\alpha;\theta)} IB(x:\alpha+0.5;\theta-1), \qquad 0 < x \leq 1 \\ & \\ F_2(x:1;\alpha;\theta) & = 1 - \frac{x^{-0.5}}{B(\alpha;\theta)} IB\left(\frac{1}{x}:\alpha+0.5;\theta-1\right) \\ & \quad +\frac{1}{B(\alpha;\theta)} IB\left(\frac{1}{x}:\alpha+1;\theta-1\right), \qquad 1 < x. \end{cases}$$

In the case $\theta > 1$, using lemma 3, we have:

for $0 < x < 1$

$$\begin{aligned} \sum_{i=1}^{\infty} IB(x:\alpha+i-0.5;\theta) &= IB(x:\alpha+0.5;\theta-1) \\ \sum_{i=1}^{\infty} IB(x:\alpha+i;\theta) &= IB(x:\alpha+1;\theta-1) \end{aligned}$$

and for $x > 1$

$$\begin{aligned} \sum_{i=1}^{\infty} IB\left(\frac{1}{x}:\alpha+i-1;\theta\right) &= IB\left(\frac{1}{x}:\alpha;\theta-1\right) \\ \sum_{i=1}^{\infty} IB\left(\frac{1}{x}:\alpha+i-0.5;\theta\right) &= IB\left(\frac{1}{x}:\alpha+0.5;\theta-1\right). \end{aligned}$$

Consequently, in the case $\theta > 1$ we have

$$H(x:1;\alpha;\theta) = \begin{cases} H_1(x:1;\alpha;\theta) & = \frac{x^{0.5}}{B(\alpha;\theta)} IB(x:\alpha+0.5;\theta-1) \\ & \quad -\frac{1}{B(\alpha;\theta)} IB(x:\alpha+1;\theta-1), \qquad 0 < x \leq 1 \\ & \\ H_2(x:1;\alpha;\theta) & = 1 - \frac{1}{B(\alpha;\theta)} IB\left(\frac{1}{x}:\alpha;\theta-1\right) \\ & \quad +\frac{x^{0.5}}{B(\alpha;\theta)} IB\left(\frac{1}{x}:\alpha+0.5;\theta-1\right), \quad 1 < x. \end{cases}$$

We report now graphs of the Lorenz curve $L(p)$ and the Zenga curve $I(p)$ - where $p = F(x)$ - for many values of the parameters $\alpha$ and $\theta$.

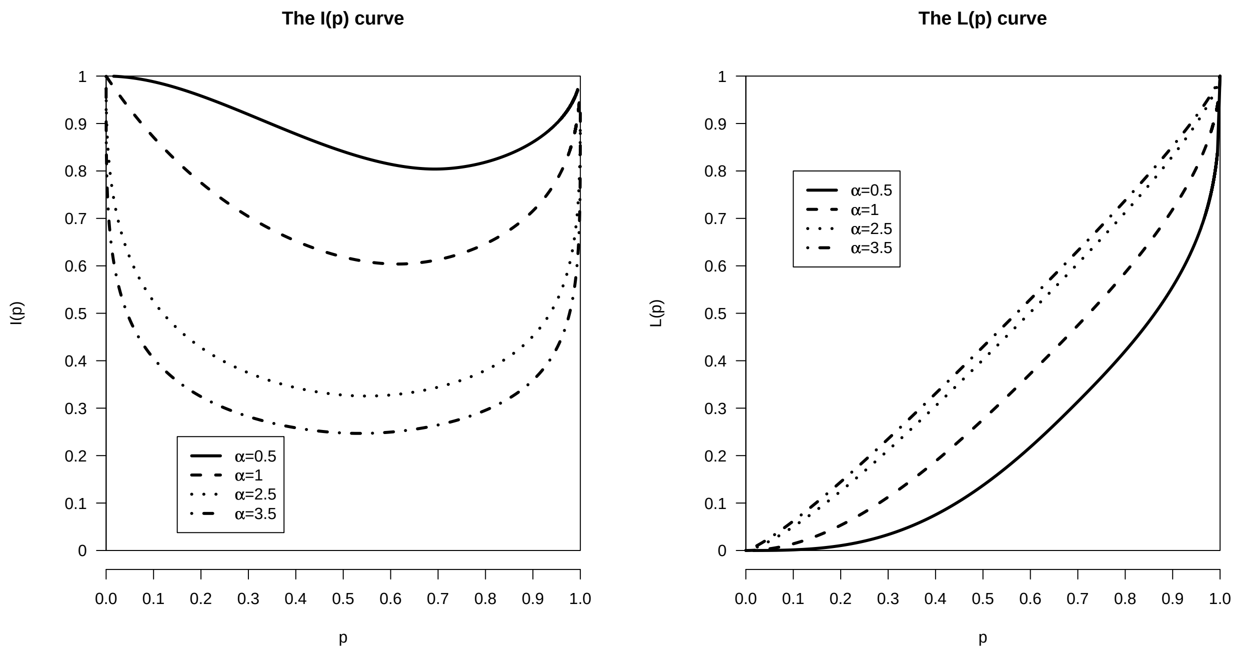


Figure 5: Graphs of $I(p)$ and $L(p)$ for $\theta = 1$ and $\alpha = 0.5; 1; 2.5; 3.5$

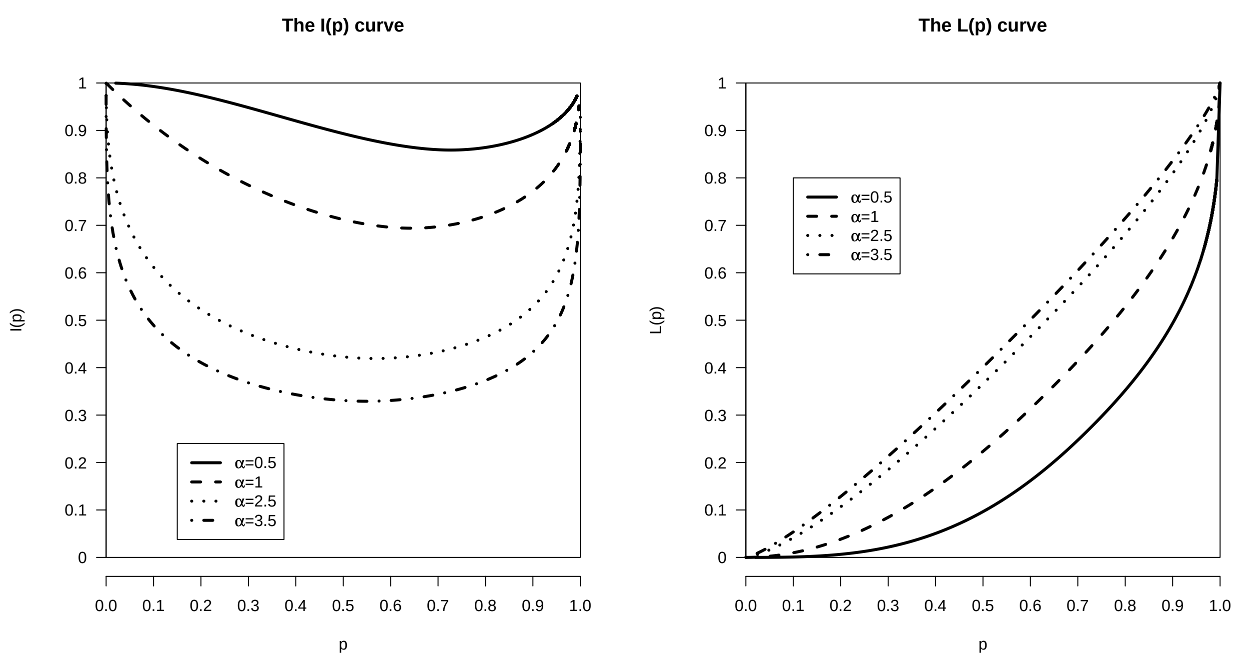


Figure 6: Graphs of $I(p)$ and $L(p)$ for $\theta = 1.5$ and $\alpha = 0.5; 1; 2.5; 3.5$

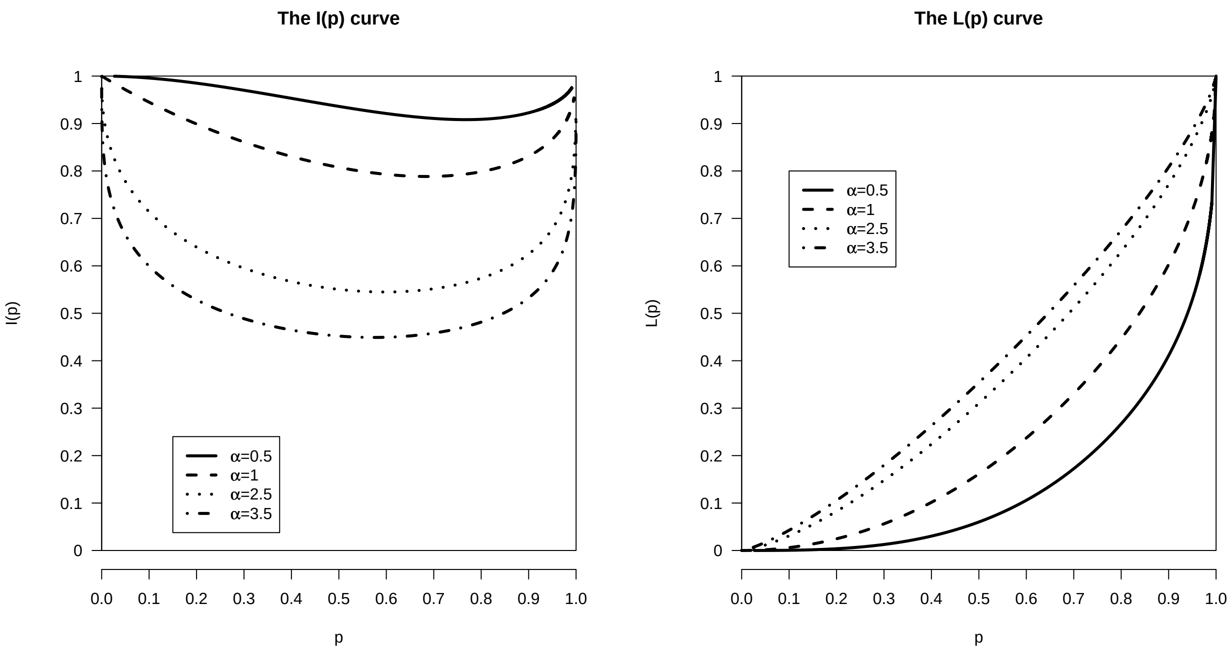


Figure 7: Graphs of $I(p)$ and $L(p)$ for $\theta = 2.5$ and $\alpha = 0.5; 1; 2.5; 3.5$

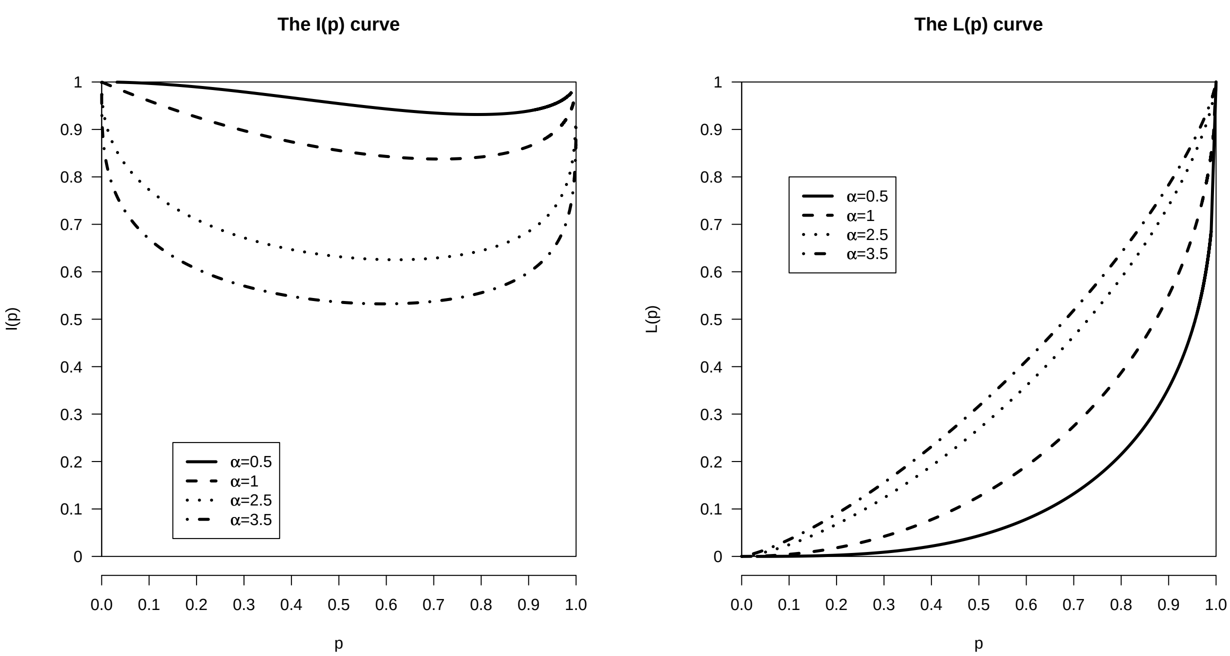


Figure 8: Graphs of $I(p)$ and $L(p)$ for $\theta = 3.5$ and $\alpha = 0.5; 1; 2.5; 3.5$

From the graphs here reported, it seems that $\alpha$ and $\theta$ are inverse parameters of the point inequality measures $I(p)$ and $L(p)$. Moreover, the shapes of some $I(p)$ graphs ($\theta = 2.5; 3.5$ and $\alpha = 2.5; 3.5$) are similar to those obtained on some real distributions by M.M. Zenga (2007a, 2007b), by P. Radaelli (2008, 2010), by F. Greselin, M. Puri and R. Zitikis 2009), and by W. Maffenini and M. Polisicchio (2010).

## 11 Hazard and Survival functions

The new three parameter denstiy function $f(x : \mu; \alpha; \theta)$ can be utilized to represent the failure time distributions. For these type of distributions it is important to consider the hazard and survival functions.

Let $X$ be a continuous non negative random variable with density function $f(x)$ and distribution function $F(x) = \int_0^x f(t)dt$, $x > 0$, the the survival function $G(x)$ and the hazard function $h(x)$ are defined as follows:

$$G(x) = 1 - F(x), \qquad x > 0 \tag{53}$$

$$h(x) = \frac{f(x)}{G(x)}, \qquad x > 0 \tag{54}$$

For simplicity's sake we can consider only the case $\mu = 1$. Then, utilizing the expressions of $f(x : 1; \alpha; \theta)$ and of $F(x : 1; \alpha; \theta)$ reported in the previous sections, we have got many graphs of the hazard function $h(x : 1; \alpha; \theta)$ and of the survival function $G(x : 1; \alpha; \theta)$.

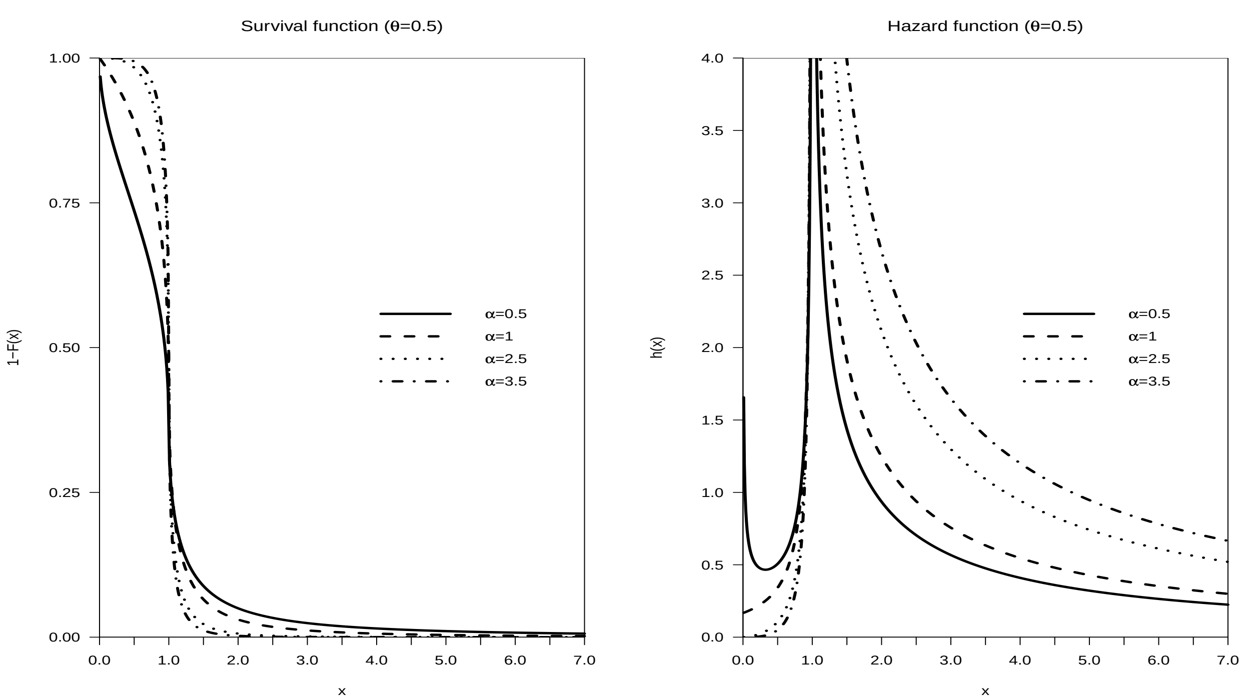

Figure 9: Graphs of the survival and hazard curves

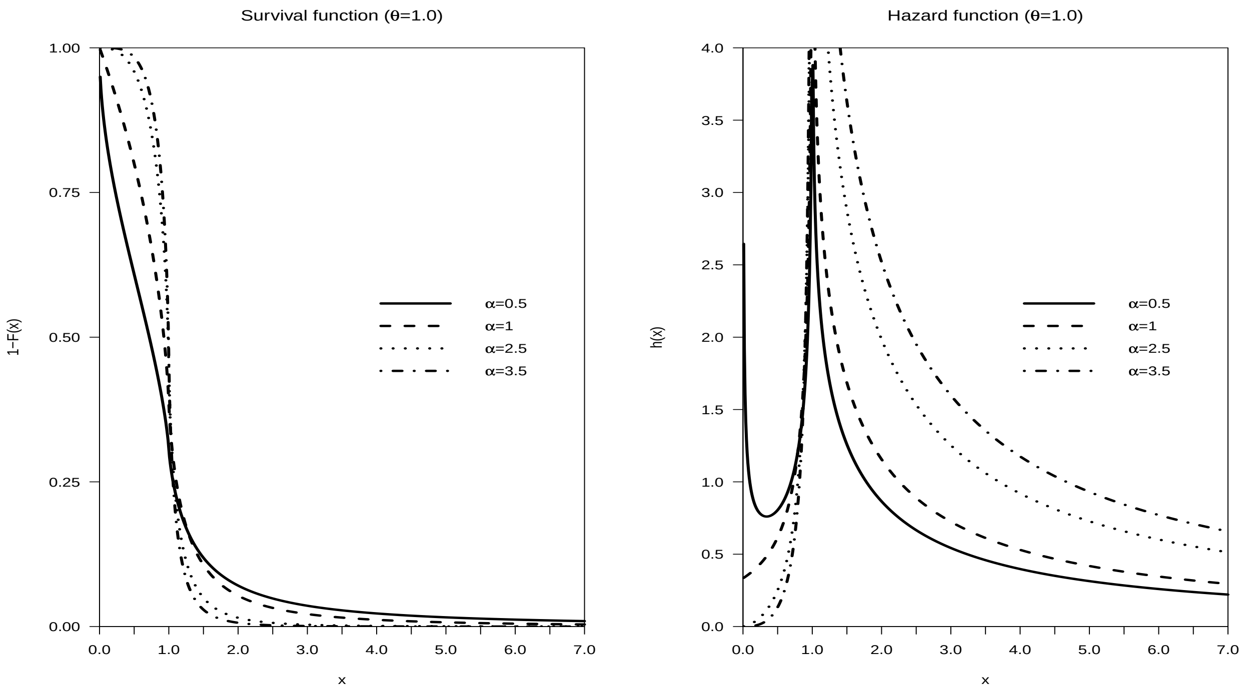


Figure 10: Graphs of the survival and hazard curves

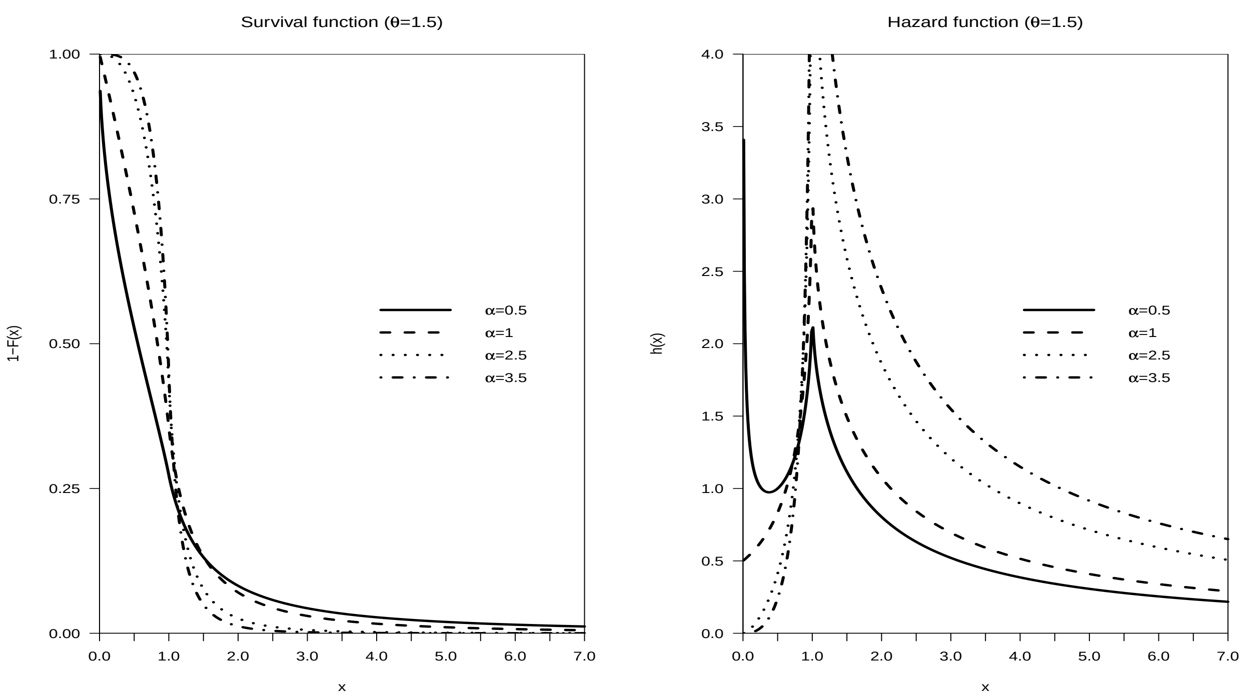


Figure 11: Graphs of the survival and hazard curves

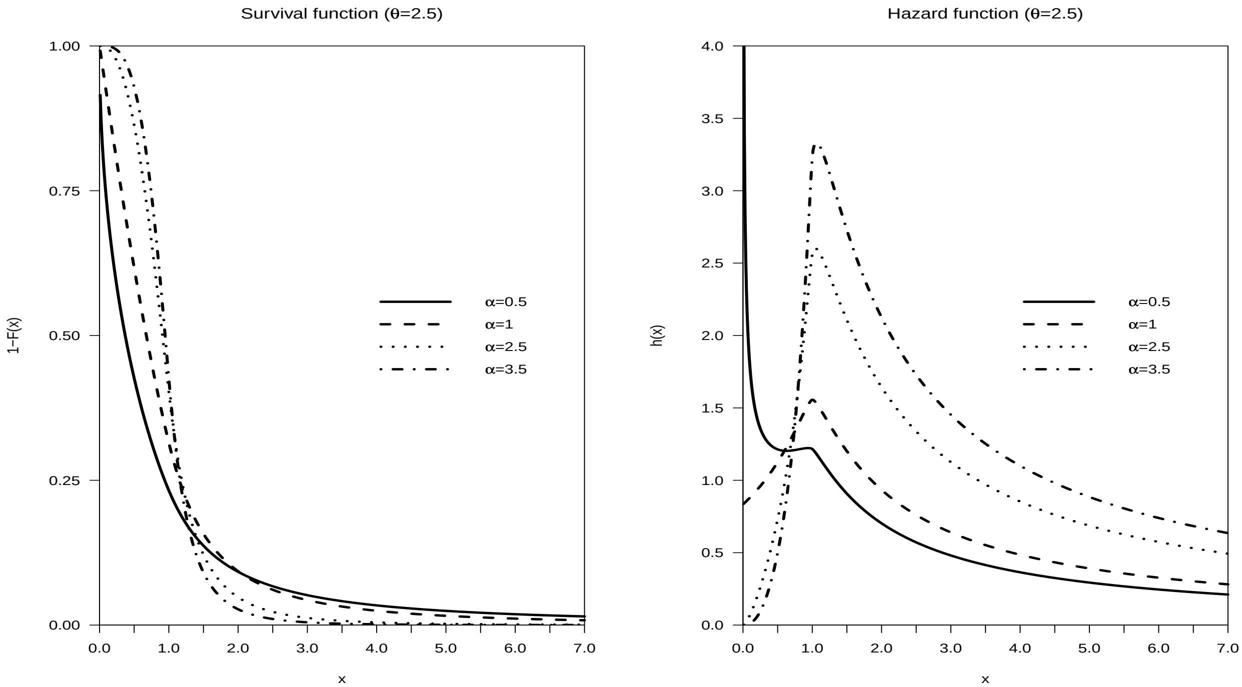


Figure 12: Graphs of the survival and hazard curves

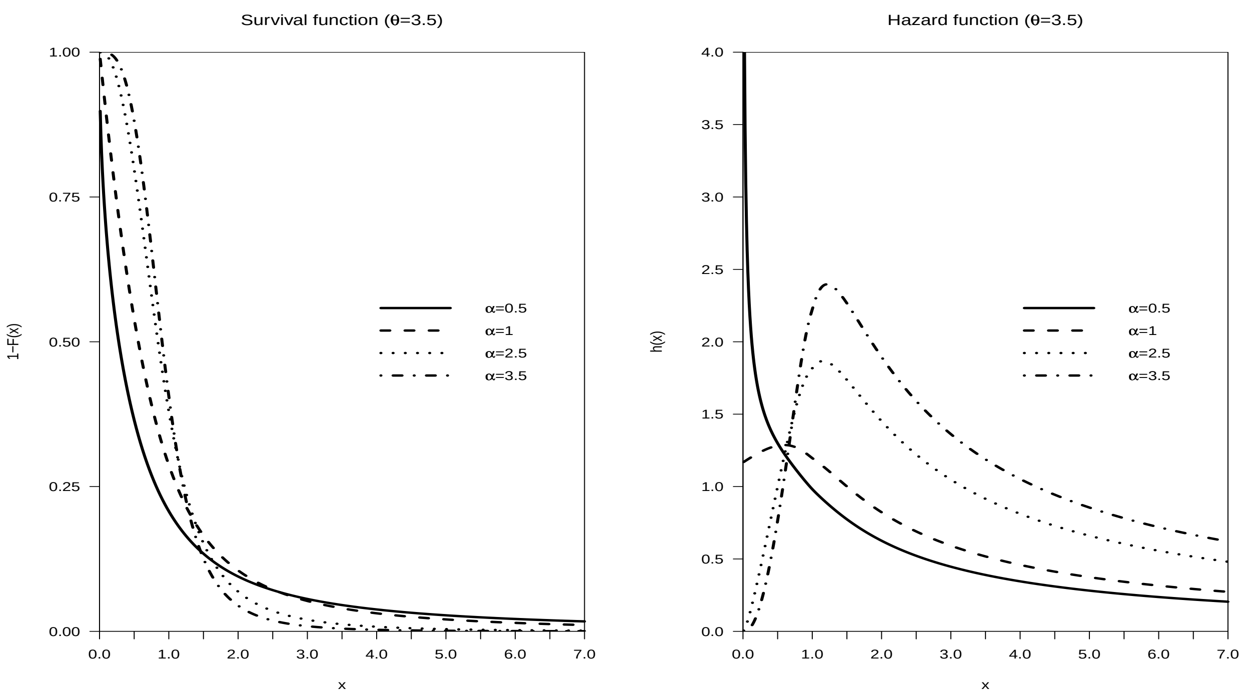


Figure 13: Graphs of the survival and hazard curves

From the graphs of the $G(x)$ we see that, for a fixed $\theta$, $G(x)$ is decreasing in $\alpha$ for large values of $x$, while $G(x)$ is increasing in $\alpha$ for small values of $x$. From the graphs of the hazard function we note that, for small values of $x$, the behaviour of $h(x)$ is "similar" to that of $f(x)$, while for large values of $x$, $h(x)$ is a decreasing function. Moreover, for large values of $x$, $h(x)$ is increasing in $\alpha$.

## 12 Conclusions

M.M. Zenga (2010) proposed a new three parameter density function $f(x : \mu; \alpha; \theta)$, $(\mu > 0; \alpha > 0; \theta > 0)$ for non negative randon variables $X$. For $\theta > 1$, M.M. Zenga (2010) has obtained the expressions of: the distribution function, the moments, the truncated moments, the mean deviation and M.M. Zenga's (2007) point inequalit $A(x)$ at $x = \mu$. Many graphs of $f(x : \mu; \alpha; \theta)$ for $\theta \geq 2$ are reported in M.M. Zenga (2010) too. According to this paper we observe that for this new distribution:

a) the parameter $\mu$ is equal to the expectation;

b) the right tail is Paretian and the asymmetry of the density is positve;

c) the shapes of the density function $f(x : \mu; \alpha; \theta)$ are broader than those of the more traditional models used for the distribution of income by size such as Dagum's distribution.

In this paper by using a new approach, we obtain for the general case $\theta > 0$ different expressions of: the density, the truncated and ordinary moments, the mean deviation and M. Zenga's point inequality. We have shown that the new expressions obtained in this paper for $\theta > 0$ are equivalent to those obtained previously for $\theta > 1$ by M.M. Zenga (2010).

The graphs of the density function for $0.5 \leq \theta \leq 1.5$ behave quite differently from those obtained for $\theta \geq 2$ (M. Zenga, 2010). These results confirm that the new density has very "flexible" shapes. The graphs reported in this paper of the Lorenz curve $L(p)$, of the Zenga curve $I(p)$ and of the hazard and survival functions suggest that the new density can be utilized to represent income, wealth, financial and actuarial, as well as failure time distributions.

Some first exploratory applications of the new density on real income distributions encourage to investigate on the estimation of the parameters of the new density. Actually, some estimates of the three parameters have been obtained by utilizing the moment method, the maximum likelihood method and some other non traditional methods. Some of these results can be found in F. Nicolussi (2009). Nevertheless, other investigations are required to analyze:

- in detail the influence of the parameters $\alpha$ and $\theta$ on the inequality curves and on the shapes of the density $f(x : \mu; \alpha; \theta)$;
- the estimation of the parameters;
- the fitting of the new model on a wide set of empirical distributions.

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