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Rinaldo M. Colombo, Graziano Guerra

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tel.: +39 02 6448 5755-5758 fax: +39 02 6448 5705

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Hyperbolic Balance Laws with a Dissipative Non Local Source

Rinaldo M. Colombo
Dipartimento di Matematica
Università degli Studi di Brescia
Via Branze, 38
25123 Brescia, Italy
Rinaldo.Colombo@UniBs.it

Graziano Guerra
Dip. di Matematica e Applicazioni
Università di Milano – Bicocca
Via Bicocca degli Arcimboldi, 8
20126 Milano, Italy
Graziano.Guerra@UniMiB.it

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Abstract

This paper considers systems of balance law with a dissipative non local source. A global in time well posedness result is obtained. Estimates on the dependence of solutions from the flow and from the source term are also provided. The technique relies on a recent result on quasidifferential equations in metric spaces.

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1 Introduction and Main Result

Consider the following nonlinear system of balance laws:

$$\partial_t u + \partial_x f(u) = G(u) \tag{1.1}$$

where f is the flow of a nonlinear hyperbolic system of conservation laws and $G: \mathbf{L^1} \mapsto \mathbf{L^1}$ is a (possibly) *non local* operator. It is known, see [11, Theorem 2.1], that for small times equation (1.1) generates a Lipschitz semigroup.

When the source term is *dissipative*, the existence of solutions can be proved for all times, see [13, 14] as well as the continuous dependence, see [2, 8]. These papers all deal with local sources. Memory effects, i.e. sources non local in time, were recently considered, for instance, in [9].

Here, we deal with dissipative non local sources and we provide the well posedness of the Cauchy problem for (1.1), as well as estimates on the dependence of the solutions from f and G. The proof relies on the combination of the Standard Riemann Semigroup S generated by the conservation law $\partial_t u + \partial_x f(u) = 0$, see [5, Definition 9.1], combined through the operator splitting technique with the Euler polygonal $(t,u) \mapsto u + t\,G(u)$ generated by the ordinary differential system $\partial_t u = G(u)$.

Here, we limit our attention to right hand sides of the type

$$G(u) = g(u) + Q * u \tag{1.2}$$

where $g \in \mathbf{C^{1,1}}(\Omega;\mathbb{R}^n)$ and $Q \in \mathbf{L^1}(\mathbb{R};\mathbb{R}^n)$, the convolution being in the space variable, see [11, § 2] for several physical motivations.

Let R be the matrix whose columns are the right eigenvectors of $Df(0)$. We call the source term (1.2) *column diagonally dominant*, see [14], if there exists a $c > 0$ such that for $i = 1, \ldots, n$ the matrix $M = R^{-1} Dg(0)\, R$ satisfies

$$M_{ii} + \sum_{j=1,\, j\neq i}^{n} \left|M_{ji}\right| < -c\,, \tag{1.3}$$

see [1, formula (5)] for a coordinate independent extension of diagonal dominance.

It is well known that this dissipativity condition allows to prove the well posedness globally in time of the Cauchy problem for (1.1)–(1.2) in the case $Q = 0$ and $g\colon \mathbb{R}^n \mapsto \mathbb{R}^n$, see [2, 14]. Similar global results can be obtained by means of suitable $\mathbf{L^1}$ estimates for relevant classes of systems, see [13].

Our main result is the following.

Theorem 1.1 *Fix an open set $\Omega \subseteq \mathbb{R}^n$, with $0 \in \Omega$, and assume that*

(F) *$f \in \mathbf{C^4}(\Omega;\mathbb{R}^n)$ is such that Df is strictly hyperbolic with each characteristic field either genuinely nonlinear or linearly degenerate;*

(G) *$g \in \mathbf{C^{1,1}}(\Omega;\mathbb{R}^n)$, $g(0) = 0$, g is column diagonally dominant and $Q \in \mathbf{L^1}(\mathbb{R};\mathbb{R}^n)$.*

Then, there exist positive $\delta_1, \delta_2, \mathcal{L}, \kappa$ such that for all Q with $\|Q\|_{\mathbf{L^1}} \leq \delta_1$ there exists a global semigroup $P\colon [0,+\infty[\times \mathcal{D} \mapsto \mathcal{D}$ such that

(1) $\mathcal{D} \supseteq \left\{u \in \mathbf{L^1}(\mathbb{R};\Omega) \colon \mathrm{TV}(u) \leq \delta_2\right\}$;

(2) for all $u_o \in \mathcal{D}$ and for $t \in [0,+\infty[$, the map $(t,x) \mapsto (P_t u_o)(x)$ is a weak entropy solution to (1.1) with initial datum u_o;

(3) for $t, s \in [0,+\infty[$, $u, w \in \mathcal{D}$ and $s < t$, then

$$\begin{aligned} \|P_t u - P_t w\|_{\mathbf{L^1}} &\leq \mathcal{L} \cdot e^{-\kappa t} \cdot \|u - w\|_{\mathbf{L^1}} \\ \|P_t u - P_s u\|_{\mathbf{L^1}} &\leq \mathcal{L} \cdot \left(1 + \|u\|_{\mathbf{L^1}}\right) \cdot |t - s|\,; \end{aligned} \tag{1.4}$$

(4) if S is the SRS generated by $\partial_t u + \partial_x f(u) = 0$, then for all $u \in \mathcal{D}$,

$$\lim_{t\to 0} \frac{1}{t} \left\| P_t u - \big(S_t u + t\, G(u)\big) \right\|_{\mathbf{L^1}} = 0\,;$$

*Moreover, let $f, \tilde f$ both satisfy **(F)**, both pairs g, Q and $\tilde g, \tilde Q$ satisfy **(G)**. Denote by $P, \tilde P$ the corresponding processes and $\mathcal{D}_\delta, \tilde{\mathcal{D}}_{\tilde\delta}$ their domains. Choose $\delta, \tilde\delta$ so that $\tilde{\mathcal{D}}_{\tilde\delta} \subseteq \mathcal{D}_\delta$. Then, for all $u \in \tilde{\mathcal{D}}_{\tilde\delta}$*

$$\begin{aligned} \left\| P_t u - \tilde P_t u \right\|_{\mathbf{L^1}} \quad \leq \quad & \mathcal{L} \cdot \left\| Df - D\tilde f_2 \right\|_{\mathbf{C^0}(\Omega;\mathbb{R}^{n\times n})} \cdot t \\ & + \mathcal{L} \cdot \|g - \tilde g\|_{\mathbf{C^0}(\Omega;\mathbb{R}^n)} \cdot t \\ & + \mathcal{L} \cdot \left\| Q - \tilde Q \right\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^n)} \cdot t\,. \end{aligned} \tag{1.5}$$

Condition (4) ensures that the orbits of P are weak entropy solutions, see [11, Corollary 3.13]. Moreover, the solution yielded by P can be characterized also as *viscosity solution* in the sense of the integral inequalities in [5, § 9.2], see [11, (6) and (7) in Theorem 1.2]. Theorem 1.1 is obtained applying [10, Theorem 2.5] with $X = \mathbf{L^1}(\mathbb{R}; \bar\Omega)$, see Section 3.

2 Outline of the Proof

We sketch below the procedure used to prove Theorem 1.1. All technical details are deferred to Section 3.

Our general reference for the basic notions related to systems of conservation laws is [5]. We assume throughout that $0 \in \Omega$ and that f satisfies **(F)** in Theorem 1.1. Let $\lambda_1(u), \ldots, \lambda_n(u)$ be the n real distinct eigenvalues of $Df(u)$, indexed so that $\lambda_j(u) < \lambda_{j+1}(u)$ for all j and u. The j-th right, respectively left, eigenvector is denoted $r_j(u)$, respectively $l_j(u)$.

Let $\sigma \mapsto R_j(\sigma)(u)$, respectively $\sigma \mapsto S_j(\sigma)(u)$, be the j-rarefaction curve, respectively the j-shock curve, exiting u. If the j-th field is linearly degenerate, then the parameter σ above is the arc-length. In the genuinely nonlinear case, see [5, Definition 5.2], we choose σ so that

$$\frac{\partial \lambda_j}{\partial \sigma} \big(R_j(\sigma)(u)\big) = k_j \quad \text{and} \quad \frac{\partial \lambda_j}{\partial \sigma} \big(S_j(\sigma)(u)\big) = k_j\,,$$

where $k_1, \ldots, k_n$ are positive and such that, as in [2],

$$\frac{\partial}{\partial \sigma} \big(R_j(\sigma)(0)\big) = r_j(0)\,, \quad \big\|r_j(0)\big\| = 1\,.$$

Introduce the j-Lax curve

$$\sigma \mapsto \psi_j(\sigma)(u) = \begin{cases} R_j(\sigma)(u) & \text{if} \quad \sigma \geq 0 \\ S_j(\sigma)(u) & \text{if} \quad \sigma < 0 \end{cases}$$

and for $\boldsymbol{\sigma} \equiv (\sigma_1, \ldots, \sigma_n)$, define the map

$$\boldsymbol{\Psi}(\boldsymbol{\sigma})(u^-) = \psi_n(\sigma_n) \circ \ldots \circ \psi_1(\sigma_1)(u^-).$$

By [5, § 5.3], given any two states $u^-, u^+ \in \Omega$ sufficiently close to 0, there exists a map E such that

$$\boldsymbol{\sigma} = E(u^-, u^+) \quad \text{if and only if} \quad u^+ = \boldsymbol{\Psi}(\boldsymbol{\sigma})(u^-). \tag{2.1}$$

elementary computations show that

$$D_u E(0,u)\Big|_{u=0} = R^{-1} = \begin{bmatrix} l_1(0) \\ \vdots \\ l_n(0) \end{bmatrix}. \tag{2.2}$$

Similarly, let the map $\mathbf{S}$ and the vector $\mathbf{q} = (q_1, \ldots, q_n)$ be defined by

$$u^+ = \mathbf{S}(\mathbf{q})(u^-) = S_n(q_n) \circ \ldots \circ S_1(q_1)(u^-) \tag{2.3}$$

as the gluing of the Rankine - Hugoniot curves.

Let u be piecewise constant with finitely many jumps and assume that $\mathrm{TV}(u)$ is sufficiently small. Call $\mathcal{I}(u)$ the finite set of points where u has a jump. Let $\sigma_{x,i}$ be the strength of the i-th wave in the solution of the Riemann problem for

$$\partial_t u + \partial_x f(u) = 0 \tag{2.4}$$

with data $u(x-)$ and $u(x+)$, i.e. $(\sigma_{x,1}, \ldots, \sigma_{x,n}) = E\big(u(x-), u(x+)\big)$. Obviously if $x \notin \mathcal{I}(u)$ then $\sigma_{x,i} = 0$, for all $i = 1, \ldots, n$. As in [5, § 7.7], $\mathcal{A}(u)$ denotes the set of approaching waves in u:

$$\mathcal{A}(u) = \left\{ \begin{array}{c} \big((x,i),(y,j)\big) \in \big(\mathcal{I}(u) \times \{1, \ldots, n\}\big)^2 : \\ x < y \text{ and either } i > j \text{ or } i = j, \text{ the } i\text{-th field} \\ \text{is genuinely non linear, } \min\big\{\sigma_{x,i}, \sigma_{y,j}\big\} < 0 \end{array} \right\}$$

while the linear and the interaction potential, see [15] or [5, formula (7.99)], are

$$\mathbf{V}(u) = \sum_{x \in I(u)} \sum_{i=1}^{n} \left|\sigma_{x,i}\right| \quad \text{and} \quad \mathbf{Q}(u) = \sum_{((x,i),(y,j)) \in \mathcal{A}(u)} \left|\sigma_{x,i}\sigma_{y,j}\right|.$$

Moreover, let

$$\boldsymbol{\Upsilon}(u) = \mathbf{V}(u) + C_0 \cdot \mathbf{Q}(u) \tag{2.5}$$

where $C_0 > 0$ is the constant appearing in the functional of the wave–front tracking algorithm, see [5, Proposition 7.1]. Recall that C_0 depends only on the flow f and the upper bound of the total variation of initial data.

Finally we define

$$\begin{aligned} \mathcal{D}_\delta^* &= \left\{ v \in \mathbf{L^1}(\mathbb{R}, \Omega) : v \text{ piecewise constant and } \boldsymbol{\Upsilon}(v) < \delta \right\} \\ \mathcal{D}_\delta &= \mathrm{cl}\left\{ \mathcal{D}_\delta^* \right\} \end{aligned} \tag{2.6}$$

where the closure is in the strong $\mathbf{L^1}$–topology. Observe that $\mathcal{D}_\delta$ contains all $\mathbf{L^1}$ functions with sufficiently small total variation.

We now pass to the stability functional introduced in [7, 16, 17]. For any $\bar{v} \in \mathcal{D}_\delta^*$, denote by $\bar{\sigma}_{x,i}$ the size of the i–wave in the solution of the Riemann Problem with data $\bar{v}(x-)$ and $\bar{v}(x+)$. Then define

$$A_j^-[\bar{v}](x) = \sum_{y \le x} \left|\bar{\sigma}_{y,j}\right|, \quad A_j^+[\bar{v}](x) = \sum_{y > x} \left|\bar{\sigma}_{y,j}\right|, \qquad \text{for } j = 1, \ldots, n\,.$$

If the i-th characteristic field is linearly degenerate, then define $\mathbf{A}_i$ as

$$\mathbf{A}_i\left[\bar{v}\right](q,x) = \sum_{1 \le j < i} A_j^+\left[\bar{v}\right](x) + \sum_{i < j \le n} A_j^-\left[\bar{v}\right](x). \tag{2.7}$$

While if the i-th characteristic field is genuinely nonlinear

$$\begin{aligned} \mathbf{A}_i[\bar{v}](q,x) &= \sum_{1 \le j < i} A_j^+[\bar{v}](x) + \sum_{i < j \le n} A_j^-[\bar{v}](x) \\ &\quad + A_i^+[\bar{v}](x) \cdot \chi_{[0,+\infty[}(q) + A_i^-[\bar{v}](x) \cdot \chi_{]-\infty,0[}(q). \end{aligned} \tag{2.8}$$

Now choose $v, \tilde{v}$ piecewise constant in $\mathcal{D}_\delta^*$ and define the weights

$$\begin{aligned} \mathbf{W}_i[v,\tilde{v}](q,x) &= 1 + \kappa_1 \mathbf{A}_i[v](q,x) + \kappa_1 \mathbf{A}_i[\tilde{v}](-q,x) \\ &\quad + \kappa_1\kappa_2 \left(\mathbf{Q}(v) + \mathbf{Q}(\tilde{v})\right)\,. \end{aligned} \tag{2.9}$$

the constants κ_1 and κ_2 being those in [5, Chapter 8]. Define implicitly the function $\mathbf{q}(x) \equiv \left(q_1(x), \ldots, q_n(x)\right)$ by

$$\tilde{v}(x) = \mathbf{S}\left(\mathbf{q}(x)\right)\left(v(x)\right)$$

with $\mathbf{S}$ as in (2.3). The stability functional $\boldsymbol{\Phi}$ is

$$\boldsymbol{\Phi}(v,\tilde{v}) = \sum_{i=1}^{n} \int_{-\infty}^{+\infty} \left|q_i(x)\right| \cdot \mathbf{W}_i[v,\tilde{v}]\left(q_i(x), x\right)\, dx. \tag{2.10}$$

We stress that $\boldsymbol{\Phi}$ is slightly different from the functional Φ defined in [5, formula (8.6)]. Indeed, here *all* jumps in v or in $\tilde{v}$ are considered. There, on the contrary, exploiting the structure of ε-approximate front tracking solutions, see [5, Definition 7.1], in the definition of Φ the jumps due to non physical waves are neglected when defining the weights A_i and are considered as belonging to a fictitious $(n+1)$-th family in the definition [5, formula (7.54)] of Q.

Recall the following basic result in the theory of non linear systems of conservation laws.

Theorem 2.1 *Let f satisfy **(F)**. Then, there exists a positive δ_o such that the equation* (2.4) *generates for all $\delta \in]0, \delta_o[$ a Standard Riemann Semigroup (SRS) $S\colon [0,+\infty[\times \mathcal{D}_\delta \mapsto \mathcal{D}_\delta$, with Lipschitz constant L.*

We refer to [5, Chapters 7 and 8] for the proof of the above result as well as for the definition and further properties of the SRS.

Recall the following result from [12]:

Proposition 2.2 *The functionals $\mathbf{\Upsilon}$, $\mathbf{Q}$ and $\mathbf{\Phi}$ admit an $\mathbf{L^1}$ lower semi-continuous extension to all $\mathcal{D}_\delta$. Moreover,*

1. *for all $u \in \mathcal{D}_\delta$, the maps $t \mapsto \mathbf{Q}(S_t u)$ and $t \mapsto \mathbf{\Upsilon}(S_t u)$ are non increasing.*
2. *for all $u, v \in \mathcal{D}_\delta$, the map $t \mapsto \mathbf{\Phi}(S_t u, S_t v)$ is non increasing;*
3. *there exists a positive C such that for all $u \in \mathcal{D}_\delta$,*
$$\frac{1}{C}\mathrm{TV}(u) \leq \mathbf{\Upsilon}(u) \leq C\,\mathrm{TV}(u)$$
and for all $u, v \in \mathcal{D}_\delta$,
$$\frac{1}{C}\left\|u-\tilde u\right\|_{\mathbf{L^1}} \leq \mathbf{\Phi}(u,\tilde u) \leq C\left\|u-\tilde u\right\|_{\mathbf{L^1}};$$
4. *for all $u \in \mathcal{D}_\delta$,*
$$\begin{aligned}\mathbf{Q}(u) &= \liminf_{v\in\mathcal{D}_\delta^*, v\to u} \mathbf{Q}(v)\\ \mathbf{\Upsilon}(u) &= \liminf_{v\in\mathcal{D}_\delta^*, v\to u} \mathbf{\Upsilon}(v)\\ \mathbf{\Phi}(u,\tilde u) &= \liminf_{v\in\mathcal{D}_\delta^*, v\to u} \mathbf{\Phi}(v,\tilde v)\end{aligned}$$

The results in [12] also provide an explicit expression of $\mathbf{\Phi}$ in terms of *wave measures*, see [5, § 10.1]. For the properties of $\mathbf{Q}$ and $\mathbf{\Upsilon}$, see also [3, 5, 6].

Introduce the map

$$\hat F(s)u = u + s\,g(u) + s\,Q * u. \tag{2.11}$$

that satisfies the properties stated in the following lemma, whose proof is deferred to Section 3.

Lemma 2.3 *Let **(F)** and **(G)** hold. For all δ sufficiently small and all $u, \tilde u \in \mathcal{D}_\delta$,*

$$\begin{aligned}\mathbf{Q}\left(\hat F(s)u\right) &\leq \mathbf{Q}(u) + \mathcal{O}(1)\,s\,\mathbf{\Upsilon}^2(u)\\ \mathbf{\Upsilon}\left(\hat F(s)u\right) &\leq \left(1-\frac{c}{8}s\right)\mathbf{\Upsilon}(u)\\ \mathbf{\Phi}\left(\hat F(s)u, \hat F(s)\tilde u\right) &\leq \left(1-\frac{c}{4}s\right)\mathbf{\Phi}(u,\tilde u)\,.\end{aligned}$$

We now recall the basic definitions and results from [10, Section 2] that allow us to complete the proof of Theorem 1.1. In the complete metric space $X = \mathbf{L^1}(\mathbb{R}; \bar\Omega)$ with the $\mathbf{L^1}$ distance, select for a fixed M the closed domain

$$\mathcal{D}^M = \left\{ u \in \mathcal{D}_\delta \colon \mathbf{\Phi}(u,0) \leq M \right\},$$

and the local flow

$$F \colon [0,\tau] \times \mathcal{D}^M \mapsto \mathcal{D}^M \qquad\qquad F(s)u = \hat F(s) S_s u\,, \tag{2.12}$$

where S is the SRS of Theorem 2.1, τ is positive and sufficiently small. Recall that, in the present autonomous setting, a Lipschitz continuous map is a *local flow* by [10, Definition 2.1]. Note that $\mathcal{D}^M$ is invariant with respect to both $\hat F$ and S, so that the above definition makes sense.

For any positive ε, the Euler ε-polygonal generated by F is

$$F^\varepsilon(t)\, u = F(t - k\varepsilon) \circ \bigcirc_{h=0}^{k-1} F(\varepsilon)\, u \tag{2.13}$$

see [10, Definition 2.2]. Therein, the following theorem is proved in a generic complete metric space.

Theorem 2.4 *Let $F \colon [0,\tau] \times \hat{\mathcal{D}} \mapsto \hat{\mathcal{D}}$ be a local flow that satisfies*

1. *there exists a non decreasing map $\omega \colon [0,\tau/2] \mapsto \mathbb{R}^+$ with $\int_0^{\tau/2} \frac{\omega(\xi)}{\xi}\, d\xi < +\infty$ such that for all (s,u) and all $k \in \mathbb{N}$*

$$d\Big(F(ks) \circ F(s)u, F\left((k+1)s\right) u \Big) \leq ks\, \omega(s)\,; \tag{2.14}$$

2. *there exists a positive L such that for all $\varepsilon \in [0,\tau]$ and for all $t \geq 0$*

$$d\left(F^\varepsilon(t)u, F^\varepsilon(t)\tilde u\right) \leq L \cdot d(u,\tilde u)\,. \tag{2.15}$$

Then, there exists a unique Lipschitz semigroup $P \colon [0,+\infty[\times \hat{\mathcal{D}} \mapsto \hat{\mathcal{D}}$ such that for all $u \in \hat{\mathcal{D}}$

$$\frac{1}{s}\, d\left(P_s u, F(s)u\right) \leq \frac{2L}{\ln 2} \cdot \int_0^s \frac{\omega(\xi)}{\xi}\, d\xi\,. \tag{2.16}$$

The proof of Theorem 1.1 is deferred to Section 3. It amounts to show that the above abstract result can be applied in the present setting, with $\hat{\mathcal{D}} = \mathcal{D}^M$ and F as in (2.12).

3 Technical Details

Lemma 3.1 *Let f satisfy **(F)**, Ω be a sufficiently small neighborhood of the origin; $a, b \in \mathbb{R}^n$ and $s \geq 0$ be sufficiently small. Choose $u^-, v^- \in \Omega$ and define $u^+ = u^- + sa$ and $v^+ = v^- + sb$. Then, if $\boldsymbol{\sigma}^-$, $\boldsymbol{\sigma}^+$ satisfy $v^- = \boldsymbol{\Psi}(\boldsymbol{\sigma}^-)(u^-)$ and $v^+ = \boldsymbol{\Psi}(\boldsymbol{\sigma}^+)(u^+)$,*

$$\sum_{i=1}^{n} \left|\sigma_i^+ - \sigma_i^-\right| \leq \mathcal{O}(1) \cdot s \left(\sum_{i=1}^{n} \left|\sigma_i^-\right| + \|b - a\|\right). \tag{3.1}$$

*If g satisfies **(G)**, $u^+ = u^- + s\left(a + g(u^-)\right)$ and $v^+ = v^- + s\left(b + g(v^-)\right)$, then*

$$\sum_{i=1}^{n} \left|\sigma_i^+\right| \leq \left(1 - \frac{c}{2}s\right) \sum_{i=1}^{n} \left|\sigma_i^-\right| + \mathcal{O}(1)\; s\,\|b - a\|. \tag{3.2}$$

An entirely analogous result holds with the map $\boldsymbol{\Psi}$ replaced by the gluing $\mathbf{S}$ of shock curves, i.e. $v^- = \mathbf{S}(\boldsymbol{\sigma}^-)(u^-)$ and $v^+ = \mathbf{S}(\boldsymbol{\sigma}^+)(u^+)$.

Proof. Let $\boldsymbol{\sigma}^- = \boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^+ = E\left(u + sa, \boldsymbol{\Psi}(\boldsymbol{\sigma})(u) + sb\right)$. Introduce the $\mathbf{C}^2$ map

$$\varphi(s, a, b, \boldsymbol{\sigma}) = E\left(u + sa, \boldsymbol{\Psi}(\boldsymbol{\sigma})(u) + sb\right) - \boldsymbol{\sigma}.$$

Note that $\varphi(0, a, b, \sigma) = 0$ and $\varphi(s, a, a, 0) = 0$, by [5, Lemma 2.5] we get

$$\left\|\varphi(s, a, b, \boldsymbol{\sigma})\right\| \leq \mathcal{O}(1)\, s \left(\sum_{i=1}^{n} |\sigma_i| + \|b - a\|\right)$$

proving (3.1).

To prove (3.2), introduce the functions

$$B_{ij}(a, u) = \frac{\partial^2}{\partial s\, \partial \sigma_j} E_i \left(\begin{array}{c} u + s\left(a + g(u)\right), \\ \boldsymbol{\Psi}(\boldsymbol{\sigma})(u) + s\Big(b + g\left(\boldsymbol{\Psi}(\boldsymbol{\sigma})(u)\right)\Big) \end{array} \right)\Bigg|_{\substack{\sigma=0, \\ s=0, \\ b=a,}}$$

$$\varphi(s, a, b, \sigma, u) = E \left(\begin{array}{c} u + s\left(a + g(u)\right), \\ \boldsymbol{\Psi}(\boldsymbol{\sigma})(u) + s\Big(b + g\left(\boldsymbol{\Psi}(\boldsymbol{\sigma})(u)\right)\Big) \end{array} \right) - \left(\mathrm{Id} + sB(a, u)\right) \boldsymbol{\sigma}.$$

By the $\mathbf{C}^{2,1}$ regularity of E, the $n \times n$ matrix B is a Lipschitz function of (a, u). Moreover, by (2.2)

$$\begin{aligned} B_{ij}(0, 0) &= \frac{\partial^2}{\partial s\, \partial \sigma_j} E_i \Big(0, \psi_j(\sigma_j)(0) + sg\left(\psi_j(\sigma_j)(0)\right)\Big)\Big|_{s=0, \sigma_j=0} \\ &= \frac{\partial}{\partial s} \Big(l_i(0) \cdot \left(r_i(0) + s\, Dg(0)\, r_j(0)\right)\Big)\Big|_{s=0} \\ &= l_i(0)\, Dg(0)\, r_j(0) \\ &= M_{ij}. \end{aligned}$$

Therefore, by (1.3)

$$\sum_{i=1}^{n} \left| \Big[\big(\mathrm{Id} + sB(0,0)\big)\, \boldsymbol{\sigma} \Big]_i \right| \leq (1 - cs) \sum_{i=1}^{n} |\sigma_i| \,. \tag{3.3}$$

Note that $\varphi(0,a,b,\boldsymbol{\sigma},u) = 0$, hence by the Lipschitzeanity of $D\varphi$,

$$\big\|\varphi(s,a,b,\boldsymbol{\sigma},u) - \varphi(s,a,a,\boldsymbol{\sigma},u)\big\| \leq \mathcal{O}(1)\, s \|b-a\| \,.$$

We thus have

$$\begin{aligned} \big\|\varphi(s,a,b,\boldsymbol{\sigma},u)\big\| &\leq \big\|\varphi(s,a,b,\boldsymbol{\sigma},u) - \varphi(s,a,a,\boldsymbol{\sigma},u)\big\| + \big\|\varphi(s,a,a,\boldsymbol{\sigma},u)\big\| \\ &\leq \mathcal{O}(1)\, s\|b-a\| + \big\|\varphi(s,a,a,\boldsymbol{\sigma},u)\big\| \,. \end{aligned} \tag{3.4}$$

By the definition of φ and the choice of $B_{ij}(a,u)$,

$$\begin{aligned} \varphi(0,a,a,\boldsymbol{\sigma},u) &= 0 \\ \varphi(s,a,a,0,u) &= 0 \\ \frac{\partial^2}{\partial s\, \partial \sigma_j} \varphi(s,a,b,\boldsymbol{\sigma},u) \Big|_{\substack{\boldsymbol{\sigma}=0,\\ s=0,\\ b=a}} &= 0 \end{aligned}$$

using again [5, Lemma 2.5], we obtain

$$\big\|\varphi(s,a,a,\boldsymbol{\sigma},u)\big\| \leq \mathcal{O}(1)\, s \sum_{i=1}^{n} |\sigma_i| \left(s + \sum_{i=1}^{n} |\sigma_i| \right) \,. \tag{3.5}$$

Finally,

$$\begin{aligned} & E\Big(u + s\,\big(a + g(u)\big)\,, \boldsymbol{\Psi}(\boldsymbol{\sigma})(u) + s\,\Big(b + g\,\big(\boldsymbol{\Psi}(\boldsymbol{\sigma})(u)\big)\Big)\Big) \\ = \;& \varphi(s,a,b,\boldsymbol{\sigma},u) + \big(\mathrm{Id} + sB(a,u)\big)\, \boldsymbol{\sigma} \\ = \;& \big(\mathrm{Id} + sB(0,0)\big)\, \boldsymbol{\sigma} + \varphi(s,a,b,\boldsymbol{\sigma},u) + s\,\big(B(a,u) - B(0,0)\big)\, \boldsymbol{\sigma} \,. \end{aligned}$$

Apply now (3.3), (3.4), (3.5) and the Lipschitzeanity of B

$$\begin{aligned} \sum_{i=1}^{n} \left|\sigma_i^+\right| &= \sum_{i=1}^{n} \left| E_i \left(\begin{array}{c} u + s\,\big(a + g(u)\big)\,, \\ \boldsymbol{\Psi}(\boldsymbol{\sigma})(u) + s\,\Big(b + g\,\big(\boldsymbol{\Psi}(\boldsymbol{\sigma})(u)\big)\Big) \end{array} \right) \right| \\ &\leq \sum_{i=1}^{n} \left| \Big[\big(\mathrm{Id} + sB(0,0)\big)\, \boldsymbol{\sigma} \Big]_i \right| + \mathcal{O}(1)\, s\, \|b-a\| \\ &\quad + \big\|\varphi(s,a,a,\boldsymbol{\sigma},u)\big\| + s\, \big\|B(a,u) - B(0,0)\big\|\, \|\boldsymbol{\sigma}\| \end{aligned}$$

$$
\begin{aligned}
&\leq \quad (1-cs)\sum_{i=1}^{n}|\sigma_i| + \mathcal{O}(1)\, s\|b-a\| \\
&\qquad + \mathcal{O}(1)\, s\sum_{i=1}^{n}|\sigma_i|\left(s+\sum_{i=1}^{n}|\sigma_i| + \|a\| + \|u\|\right) \\
&\leq \quad \left(1-\frac{c}{2}s\right)\sum_{i=1}^{n}|\sigma_i| + \mathcal{O}(1)\, s\|b-a\|
\end{aligned}
$$

provided s, $\sum_{i=1}^{n}|\sigma_i|$, $\|a\|$ and $\|u\|$ are sufficiently small. □

Introduce for $N \in \mathbb{N}$ the projection

$$
\Pi_N(u) = N \sum_{k=-1-N^2}^{-1+N^2} \int_{k/N}^{(k+1)/N} u(\xi)\, d\xi\ \chi_{]k/N,(k+1)/N]}
$$

as in [11, § 3.2]. Note that $\Pi_N u$ is piecewise constant. For later use, we introduce the following approximated local flows, see [10, Definition 2.1], generated by the source term

$$
\hat{F}_N(s)u = u + s\, g(u) + s\, \Pi_N(Q * v).
$$

Lemma 3.2 *Let **(F)** and **(G)** hold. If δ and $\|Q\|_{\mathbf{L^1}}$ are sufficiently small, then for all $v \in \mathcal{D}_\delta^*$*

$$
\mathbf{V}\left(\hat{F}_N(s)v\right) \leq \left(1-\frac{c}{4}s\right)\mathbf{V}(v)\,.
$$

Proof. Denote $w = \Pi_N(Q*v)$ and introduce the piecewise constant function $v' = v + sg(v) + sw$. Let $\sigma_{x,i}$, respectively $\sigma'_{x,i}$ be the jumps in v, respectively v'. Denote also $\Delta v(x) = v(x+) - v(x-)$ and similarly Δw, $\Delta v'$.

Apply (3.2) at any $x \in \mathbb{R}$, with $a = w(x-)$ and $b = w(x+)$ to obtain

$$
\begin{aligned}
\mathbf{V}(v') &= \sum_{x\in\mathbb{R}}\sum_{i=1}^{n}\left|\sigma'_{x,i}\right| \\
&\leq \left(1-\frac{c}{2}s\right)\sum_{x\in\mathbb{R}}\sum_{i=1}^{n}\left|\sigma_{x,i}\right| + \mathcal{O}(1)\, s\sum_{x\in\mathbb{R}}\left\|\Delta w(x)\right\| \\
&\leq \left(1-\frac{c}{2}s\right)\mathbf{V}(v) + \mathcal{O}(1)\, s\mathrm{TV}\left(\Pi_N(Q*v)\right) \\
&\leq \left(1-\frac{c}{2}s\right)\mathbf{V}(v) + \mathcal{O}(1)\, s\mathrm{TV}(Q*v) \\
&\leq \left(1-\frac{c}{2}s\right)\mathbf{V}(v) + \mathcal{O}(1)\, s\|Q\|_{\mathbf{L^1}}\mathbf{V}(v) \\
&\leq \left(1-\frac{c}{4}s\right)\mathbf{V}(v)
\end{aligned}
$$

provided $\|Q\|_{\mathbf{L^1}}$ is sufficiently small. $\square$

Proof of Lemma 2.3. By [12, Lemma 4.2], it is possible to choose a sequence $v^\nu \in \mathcal{D}^*_\delta$ such that $v^\nu \to u$ in $\mathbf{L^1}$, $\mathbf{Q}(v^\nu) \to \mathbf{Q}(u)$ and $\mathbf{\Upsilon}(v^\nu) \to \mathbf{\Upsilon}(u)$. By [11, Proposition 1.1], we may apply [11, Corollary 3.5] and use the estimate [11, (3.5) in Lemma 3.6] in the case $L_3 = 0$, $G(u) = g(u) + \Pi_N(Q * u)$. Note that the latter map is piecewise constant whenever u is. We thus obtain

$$
\begin{aligned}
\mathbf{Q}\left(\hat{F}(s)u\right) &\leq \liminf_{N\to+\infty} \mathbf{Q}\left(\hat{F}_N(s)u\right) \\
&\leq \liminf_{N\to+\infty} \liminf_{\nu\to+\infty} \mathbf{Q}\left(\hat{F}_N(s)v^\nu\right) \\
&\leq \liminf_{\nu\to+\infty} \left(\mathbf{Q}(v^\nu) + \mathcal{O}(1)\, s\, \mathbf{V}^2(v^\nu)\right) \\
&\leq \liminf_{\nu\to+\infty} \left(\mathbf{Q}(v^\nu) + \mathcal{O}(1)\, s\, \mathbf{\Upsilon}^2(v^\nu)\right) \\
&= \mathbf{Q}(u) + \mathcal{O}(1)\, s\, \mathbf{\Upsilon}^2(u)
\end{aligned}
$$

proving the former estimate.

To prove the latter one, use the same sequence v^ν, Lemma 3.2 and follow an analogous argument based on the results in [11], to obtain

$$
\begin{aligned}
&\mathbf{\Upsilon}\left(\hat{F}(s)u\right) \\
\leq\ & \liminf_{N\to+\infty} \liminf_{\nu\to+\infty} \mathbf{\Upsilon}\left(\hat{F}_N(s)v^\nu\right) \\
\leq\ & \liminf_{\nu\to+\infty} \left(\left(1 - \frac{c}{4}s\right) \mathbf{V}(v^\nu) + C_0 \mathbf{Q}(v^\nu) + \mathcal{O}(1)\, s \mathbf{V}^2(v^\nu)\right) \\
\leq\ & \liminf_{\nu\to+\infty} \left(\left(1 - \frac{c}{8}s\right) \mathbf{\Upsilon}(v^\nu) + \frac{c}{8}s\left(-\mathbf{V}(v^\nu) + C_0 \mathbf{Q}(v^\nu) + \mathcal{O}(1)\, \mathbf{V}^2(v^\nu)\right)\right) \\
\leq\ & \liminf_{\nu\to+\infty} \left(\left(1 - \frac{c}{8}s\right) \mathbf{\Upsilon}(v^\nu) + \frac{c}{8}s\, \mathbf{V}(v^\nu)\left(-1 + \mathcal{O}(1)\, \mathbf{V}(v^\nu)\right)\right) \\
\leq\ & \liminf_{\nu\to+\infty} \left(1 - \frac{c}{8}s\right) \mathbf{\Upsilon}(v^\nu) \\
\leq\ & \left(1 - \frac{c}{8}s\right) \mathbf{\Upsilon}(u)
\end{aligned}
$$

for a sufficiently small δ.

We now pass to the estimate on $\mathbf{\Phi}$. By [12, Lemma 4.5], we may choose two sequences of piecewise constant functions $v^\nu, \tilde{v}^\nu \in \mathcal{D}^*_\delta$ such that

$$
v^\nu \to u \quad \text{and} \quad \tilde{v}^\nu \to \tilde{u} \quad \text{in } \mathbf{L^1}, \qquad \mathbf{\Phi}(v^\nu, \tilde{v}^\nu) \to \mathbf{\Phi}(u, \tilde{u})\,.
$$

Define implicitly the functions $\mathbf{q}^{\pm}(x)$ by $\hat{F}_N(s)(\tilde{v}^\nu) = \mathbf{S}(\mathbf{q}^+)\left(\hat{F}_N(s)v^\nu\right)$ and $\tilde{v}^\nu = \mathbf{S}(\mathbf{q}^-)(v^\nu)$. Moreover, with reference to (2.9), let

$$\begin{aligned}\mathbf{W}_i^+(x) &= \mathbf{W}_i[\hat{F}_N(s)v^\nu, \hat{F}_N(s)\tilde{v}^\nu]\left(\mathbf{q}_i^+(x), x\right)\\ \mathbf{W}_i^-(x) &= \mathbf{W}_i[v^\nu, \tilde{v}^\nu]\left(\mathbf{q}_i^-(x), x\right)\end{aligned}$$

and compute

$$\begin{aligned}&\mathbf{\Phi}\left(\hat{F}_N(s)v^\nu, \hat{F}_N(s)\tilde{v}^\nu\right) - \mathbf{\Phi}(v^\nu, \tilde{v}^\nu)\\ =\ & \int_{\mathbb{R}} \sum_{i=1}^n \left(\left|q_i^+(x)\right|\mathbf{W}_i^+(x) - \left|q_i^-(x)\right|\mathbf{W}_i^-(x)\right)\,dx\\ =\ & \int_{\mathbb{R}} \mathcal{R}(x)\,dx \end{aligned} \tag{3.6}$$

where the integrand $\mathcal{R}(x)$ is estimated splitting it as follows:

$$\begin{aligned}\mathcal{R}(x) &= I(x) + II(x) + \widetilde{II}(x) + III(x) + \widetilde{III}(x)\\ I(x) &= \sum_{i=1}^n \left|q_i^+(x)\right| - \sum_{i=1}^n \left|q_i^-(x)\right|\\ II(x) &= \kappa_1 \sum_{i=1}^n \Bigg(\left|q_i^+(x)\right|\mathbf{A}_i[\hat{F}_N(s)v^\nu]\left(q_i^+(x), x\right)\\ &\qquad\qquad -\left|q_i^-(x)\right|\mathbf{A}_i[v^\nu]\left(q_i^-(x), x\right)\Bigg)\\ \widetilde{II}(x) &= \kappa_1 \sum_{i=1}^n \Bigg(\left|q_i^+(x)\right|\mathbf{A}_i[\hat{F}_N(s)\tilde{v}^\nu]\left(-q_i^+(x), x\right)\\ &\qquad\qquad -\left|q_i^-(x)\right|\mathbf{A}_i[\tilde{v}^\nu]\left(-q_i^-(x), x\right)\Bigg)\\ III(x) &= \kappa_1\,\kappa_2 \left(\sum_{i=1}^n \left|q_i^+(x)\right|\mathbf{Q}\left(\hat{F}_N(s)v^\nu\right) - \sum_{i=1}^n \left|q_i^-(x)\right|\mathbf{Q}\left(v^\nu\right)\right)\\ \widetilde{III}(x) &= \kappa_1\,\kappa_2 \left(\sum_{i=1}^n \left|q_i^+(x)\right|\mathbf{Q}\left(\hat{F}_N(s)\tilde{v}^\nu\right) - \sum_{i=1}^n \left|q_i^-(x)\right|\mathbf{Q}\left(\tilde{v}^\nu\right)\right)\end{aligned}$$

We now show that $\int_{\mathbb{R}} I(x)\,dx$ is strictly negative and controls the growth in the other terms. By Lemma 3.1, applied to shock curves instead of Lax curves and with $a = a(x) = \left(\Pi_N Q * v^\nu\right)(x)$, $b = b(x) = \left(\Pi_N Q * \tilde{v}^\nu\right)(x)$

$$I(x) \leq -\frac{c}{2} s \sum_{i=1}^n \left|q_i(x)\right| + \mathcal{O}(1)\,s\,\left\|b(x) - a(x)\right\|$$

Passing to the second addend, consider each term in the sum defining it:

$$\begin{aligned}
& \left|q_i^+(x)\right| \mathbf{A}_i[\hat{F}_N(s)v^\nu]\left(q_i^+(x),x\right) - \left|q_i^-(x)\right| \mathbf{A}_i[v^\nu]\left(q_i^-(x),x\right) \\
\leq \; & \left|q_i^+(x) - q_i^-(x)\right| \mathbf{A}_i[\hat{F}_N(s)v^\nu]\left(q_i^+(x),x\right) \qquad (3.7) \\
& + \left|q_i^-(x)\right| \left| \mathbf{A}_i[\hat{F}_N(s)v^\nu]\left(q_i^+(x),x\right) - \mathbf{A}_i[v^\nu]\left(q_i^-(x),x\right) \right| \qquad (3.8)
\end{aligned}$$

By Lemma 3.1, applied to shock curves,

$$\begin{aligned}
& \left|q_i^+(x) - q_i^-(x)\right| \\
\leq \; & \mathcal{O}(1)\, s \Bigg(\left\| g\left(v^\nu(x)\right) + \Pi_N Q * v^\nu(x) - g\left(\tilde{v}^\nu(x)\right) - \Pi_N Q * \tilde{v}^\nu(x) \right\| \\
& \qquad + \sum_{i=1}^n \left|q_i^-(x)\right| \Bigg) \\
\leq \; & \mathcal{O}(1)\, s \left(\sum_{i=1}^n \left|q_i^-(x)\right| + \left\|b(x) - a(x)\right\| \right). \qquad (3.9)
\end{aligned}$$

Note that $\mathbf{A}_i[\hat{F}_i(s)v^\nu](x) \leq \mathcal{O}(1)\,\delta$, hence the first term in the right hand side of (3.7) is bounded by

$$\left|q_i^+(x) - q_i^-(x)\right| \mathbf{A}_i[\hat{F}_N(s)v^\nu]\left(q_i^+(x),x\right) \leq \mathcal{O}(1)\, s\delta \left(\sum_{i=1}^n \left|q_i^-(x)\right| + \left\|b(x) - a(x)\right\| \right).$$

Concerning the second term (3.8), if $q_i^+(x) \cdot q_i^-(x) < 0$, then

$$\begin{aligned}
\left|q_i^-(x)\right| \; & \leq \; \left|q_i^+(x) - q_i^-(x)\right| \\
& \leq \; \mathcal{O}(1)\, s \left(\sum_{i=1}^n \left|q_i^-(x)\right| + \left\|b(x) - a(x)\right\| \right).
\end{aligned}$$

Moreover, since $\left|\mathbf{A}_i[\hat{F}_i(s)v^\nu](x) - \mathbf{A}_i[v^\nu](x)\right| \leq \mathcal{O}(1)\,\delta$, we get

$$\begin{aligned}
& \left|q_i^-(x)\right| \left| \mathbf{A}_i[\hat{F}_N(s)v^\nu]\left(q_i^+(x),x\right) - \mathbf{A}_i[v^\nu]\left(q_i^-(x),x\right) \right| \\
\leq \; & \mathcal{O}(1)\, s\,\delta \left(\sum_{i=1}^n \left|q_i^-(x)\right| + \left\|b(x) - a(x)\right\| \right).
\end{aligned}$$

On the other hand, if $q_i^+(x) \cdot q_i^-(x) > 0$, by (2.7) or (2.8), a wave $\sigma'_{y,j}$ is counted in the sum defining $\mathbf{A}_i[\hat{F}_N(s)v^\nu]\left(q_i^+(x),x\right)$ if and only the wave $\sigma_{y,j}$

is counted in the sum defining $\mathbf{A}_i[v^\nu]\left(q_i^-(x),x\right)$. Therefore, by Lemma 3.1,

$$\begin{aligned}
&\left|\mathbf{A}_i[\hat{F}_N(s)v^\nu]\left(q_i^+(x),x\right)-\mathbf{A}_i[v^\nu]\left(-q_i^+(x),x\right)\right| \\
\leq\ & \sum_{x\in\mathbb{R}}\sum_{i=1}^{n}\left|\sigma'_{x,i}-\sigma_{x,i}\right| \\
\leq\ & \mathcal{O}(1)\,s\sum_{x\in\mathbb{R}}\left(\sum_{i=1}^{n}\left|\sigma_{x,i}\right|+\left\|\Delta a(x)\right\|\right) \\
\leq\ & \mathcal{O}(1)\,s\left(\mathbf{V}(v^\nu)+\mathrm{TV}(\Pi_N Q*\tilde{v}^\nu)\right) \\
\leq\ & \mathcal{O}(1)\ s\,\delta\,.
\end{aligned}$$

The second addend $II(x)$ is thus bounded as

$$II(x)\leq\mathcal{O}(1)\ s\,\delta\left(\sum_{i=1}^{n}\left|q_i^-(x)\right|+\left\|b(x)-a(x)\right\|\right)\,.$$

The term $\widetilde{II}(x)$ can be treated repeating the same procedure

Consider now each term in the sum defining $III(x)$, proceed as in (3.7)–(3.8), using (3.9) and similarly to the second part of Lemma 2.3,

$$\begin{aligned}
&\left(\left|q_i^+(x)\right|\mathbf{Q}\left(\hat{F}_N(s)v^\nu\right)-\left|q_i^-(x)\right|\mathbf{Q}\left(v^\nu\right)\right) \\
\leq\ & \left|q_i^+(x)-q_i^-(x)\right|\mathbf{Q}\left(\hat{F}_N(s)v^\nu\right)+\left|q_i^-(x)\right|\left(\mathbf{Q}\left(\hat{F}_N(s)v^\nu\right)-\mathbf{Q}(v^\nu)\right) \\
\leq\ & \mathcal{O}(1)\,s\delta\left(\sum_{i=1}^{n}\left|q_i^-(x)\right|+\left\|b(x)-a(x)\right\|\right)+\mathcal{O}(1)\,s\delta\left|q_i^-(x)\right| \\
\leq\ & \mathcal{O}(1)\,s\delta\left(\sum_{i=1}^{n}\left|q_i^-(x)\right|+\left\|b(x)-a(x)\right\|\right)\,.
\end{aligned}$$

Obviously, $\widetilde{III}(x)$ is treated analogously. Thus, $\mathcal{R}$ in (3.6) is bounded as:

$$\mathcal{R}(x)\leq-\frac{c}{2}s\sum_{i=1}^{n}q_i(x)+\mathcal{O}(1)\,s\,\delta\sum_{i=1}^{n}\left|q_i(x)\right|+\mathcal{O}(1)\,s\left\|b(x)-a(x)\right\|\,.$$

Integrating and using the Lipschitzeanity of Π_N, a standard inequality on the convolution and the bound $\|v^\nu-\tilde{v}^\nu\|_{\mathbf{L^1}}\leq\mathcal{O}(1)\int_{\mathbb{R}}\sum_{i=1}^{n}\left|q_i^-(x)\right|dx$ yield

$$\mathbf{\Phi}\left(\hat{F}_N(s)v^\nu,\hat{F}_N(s)\tilde{v}^\nu\right)-\mathbf{\Phi}(v^\nu,\tilde{v}^\nu)$$

$$
\begin{aligned}
&\leq \left(-\frac{c}{2}+\mathcal{O}(1)\,\delta\right) s\int_{\mathbb{R}}\sum_{i=1}^{n}\left|q_i^-(x)\right|dx \\
&\qquad +\mathcal{O}(1)\,s\|\Pi_N Q*v^\nu-\Pi_N Q*\tilde{v}^n\|_{\mathbf{L^1}} \\
&\leq \left(-\frac{c}{2}+\mathcal{O}(1)\,\delta\right) s\int_{\mathbb{R}}\sum_{i=1}^{n}\left|q_i^-(x)\right|dx \\
&\qquad +\mathcal{O}(1)\,s\|Q\|_{\mathbf{L^1}}\int_{\mathbb{R}}\sum_{i=1}^{n}\left|q_i^-(x)\right|dx \\
&\leq \left(-\frac{c}{2}+\mathcal{O}(1)\left(\delta+\|Q\|_{\mathbf{L^1}}\right)\right) s\int_{\mathbb{R}}\sum_{i=1}^{n}\left|q_i^-(x)\right|dx \\
&\leq -\frac{c}{4}s\int_{\mathbb{R}}\sum_{i=1}^{n}\left|q_i^-(x)\right|dx \\
&\leq -\frac{c}{4}\,s\,\mathbf{\Phi}(v^\nu,\tilde{v}^\nu)\,,
\end{aligned}
$$

for $\delta+\|Q\|_{\mathbf{L^1}}$ sufficiently small. The proof is completed by means of the lower semicontinuity of $\mathbf{\Phi}$:

$$
\begin{aligned}
\mathbf{\Phi}\left(\hat{F}(s)v,\hat{F}(s)\tilde{v}\right) &\leq \liminf_{N\to+\infty}\liminf_{\nu\to+\infty}\mathbf{\Phi}\left(\hat{F}_N(s)v^\nu,\hat{F}_N(s)\tilde{v}^\nu\right) \\
&\leq \lim_{\nu\to+\infty}\left(1-\frac{c}{4}s\right)\mathbf{\Phi}\left(v^\nu,\tilde{v}^\nu\right) \\
&= \left(1-\frac{c}{4}s\right)\mathbf{\Phi}\left(v,\tilde{v}\right).
\end{aligned}
$$

□

Proof of Theorem 1.1. Let $\delta>0$ be so small that lemmas 3.2 and 2.3 hold. First, we show that Theorem 2.4 can be applied with $\hat{\mathcal{D}}=\mathcal{D}^M$ and F as in (2.12).

For $u\in\mathcal{D}$, the map G defined in (1.2) is $\mathbf{L^1}$-bounded, $\mathbf{L^1}$-Lipschitz and $\mathrm{TV}\left(G(u)\right)$ is uniformly bounded. The Lipschitz constant of F with respect to time can be estimated as follows:

$$
\begin{aligned}
&\left\|F(s)u-F(s')u\right\|_{\mathbf{L^1}} \\
=\;&\left\|S_s u+sG(S_s u)-S_{s'}u-s'G(S_{s'}u)\right\|_{\mathbf{L^1}} \\
\leq\;& L\left|s-s'\right|+\left|s-s'\right|\left\|G(S_s u)\right\|_{\mathbf{L^1}}+s'\left\|G(S_s u)-G(S_{s'}u)\right\|_{\mathbf{L^1}} \\
\leq\;& \mathcal{O}(1)\left|s-s'\right|\left(1+\|u\|_{\mathbf{L^1}}\right) \qquad (3.10)
\end{aligned}
$$

hence F is Lipschitz in t uniformly in $u\in\mathcal{D}^M$, by the boundedness of $\mathcal{D}^M$ in $\mathbf{L^1}$. The Lipschitzeanity in u is straightforward.

F satisfies condition 1, indeed

$$\begin{array}{rl} & \left\| F(ks)F(s)u - F\left((k+1)s\right) u \right\|_{\mathbf{L^1}} \\ \leq & \left\| S_{ks}\left(S_s u + sG(S_s u)\right) - S_{ks}S_s u - sG(S_{ks}S_s u)\right\|_{\mathbf{L^1}} \\ & +ks \left\| G\left(S_{ks}\left(S_s u + sG(S_s u)\right)\right) - G(S_{ks}S_s u)\right\| . \end{array}$$

Apply [11, Proposition 3.10] to the first term:

$$\begin{array}{rl} & \left\| S_{ks}\left(S_s u + sG(S_s u)\right) - S_{ks}S_s u - sG(S_{ks}S_s u)\right\|_{\mathbf{L^1}} \\ \leq & \mathcal{O}(1) \left\| sG(S_s u) - sG(S_{ks}S_s u)\right\|_{\mathbf{L^1}} + \mathcal{O}(1)\ ks\,\mathrm{TV}\left(sG(S_{ks}S_s u)\right) \\ \leq & \mathcal{O}(1)\ ks\, s\,. \end{array}$$

The second term is of the same order by the Lipschitzeanity of G and of the SRS. Therefore, condition 1 is proved with $\omega(s) = s$.

F satisfies the stability condition 2. Indeed, by Lemma 2.3

$$\mathbf{\Phi}\left(F^\varepsilon(t)u, F^\varepsilon(t)\tilde{u}\right) \leq e^{-(c/4)t}\, \mathbf{\Phi}(u, \tilde{u})\,.$$

By 3 in Proposition 2.2,

$$\left\| F^\varepsilon(t)u - F^\varepsilon(t)\tilde{u}\right\|_{\mathbf{L^1}} \leq \mathcal{L}\, e^{-(c/4)t} \left\| u - \tilde{u}\right\|_{\mathbf{L^1}} \tag{3.11}$$

with $\mathcal{L}$ independent from ε and M.

Applying Theorem 2.4, we obtain for all $u \in \mathcal{D}^M$ the strong $\mathbf{L^1}$ convergence $F^\varepsilon(t)u \to P_t u$, P being the unique $\mathbf{L^1}$-Lipschitz semigroup satisfying (2.16). Moreover, passing to the limit $\varepsilon \to 0+$ in (3.11), we obtain the first estimate in (1.4), with $\mathcal{L}$ independent from M. Therefore, letting $M \to +\infty$, we may uniquely extend P to all $\mathcal{D}_\delta$, keeping the validity of the first estimate in (1.4).

Fix $u \in \mathcal{D}_\delta$ and let $M_u = \mathbf{\Phi}(u, 0)$, so that $u \in \mathcal{D}^{M_u}$. By (3.10) and [10, Lemma 2.3],

$$\begin{array}{rcl} \left\| F^\varepsilon(t)u - F^\varepsilon(s)u\right\|_{\mathbf{L^1}} & \leq & \mathcal{O}(1)\ (1 + M_u)\, |t-s| \\ & \leq & \mathcal{O}(1)\ \left(1 + \|u\|_{\mathbf{L^1}}\right)\ |t-s| \end{array}$$

and passing to the limit $\varepsilon \to 0$ we obtain the second estimate in (1.4).

Condition (4) is a direct consequence of (2.16) and it ensures that the orbits of P are weak entropy solutions, see [11, Corollary 3.13].

Finally, (1.5) is obtained as in [11, formula (1.7)], see also [4]. $\square$

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