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# MULTIPLE SOLUTIONS FOR A HENON-LIKE EQUATION ON THE ANNULUS

MARTA CALANCHI, SIMONE SECCHI, AND ELIDE TERRANEO

ABSTRACT. For the equation  $-\Delta u = ||x| - 2|^\alpha u^p$ ,  $1 < |x| < 3$ , we prove the existence of two solutions for  $\alpha$  large, and of two additional solutions when  $p$  is close to the critical Sobolev exponent  $2^* = 2N/(N - 2)$ . In particular, a symmetry-breaking phenomenon appears when  $\alpha \rightarrow +\infty$ , showing that the least-energy solution cannot be a radial function.

## 1. INTRODUCTION

In this paper we will consider the following problem:

$$(1) \quad \begin{cases} -\Delta u = \Psi_\alpha u^{p-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $\Omega = \{x \in \mathbb{R}^N \mid 1 < |x| < 3\}$  is an annulus in  $\mathbb{R}^N$ ,  $N \geq 3$ ,  $\alpha > 0$ ,  $p > 2$  and  $\Psi_\alpha$  is the radial function

$$\Psi_\alpha(x) = \left| |x| - 2 \right|^\alpha.$$

This equation can be seen as a natural extension to the annular domain  $\Omega$  of the celebrated Hénon equation with Dirichlet boundary conditions (see [10, 12])

$$(2) \quad \begin{cases} -\Delta u = |x|^\alpha |u|^{p-1} & \text{for } |x| < 1 \\ u = 0 & \text{if } |x| = 1. \end{cases}$$

Actually, the weight function  $\Psi_\alpha$  reproduces on  $\Omega$  a similar qualitative behavior of  $|\cdot|^\alpha$  on the unit ball  $B$  of  $\mathbb{R}^N$ .

A standard reasoning shows that the infimum

$$(3) \quad \inf_{\substack{u \in H_0^1(B) \\ u \neq 0}} \frac{\int_B |\nabla u|^2 dx}{\left( \int_B |x|^\alpha |u|^p dx \right)^{2/p}}$$

is achieved for any  $2 < p < 2^*$  and any  $\alpha > 0$ . In 1982, Ni proved in [12] that the infimum

$$(4) \quad \inf_{\substack{u \in H_{0,\text{rad}}^1(B) \\ u \neq 0}} \frac{\int_B |\nabla u|^2 dx}{\left( \int_B |x|^\alpha |u|^p dx \right)^{2/p}}$$

is achieved for any  $p \in (2, 2^* + \frac{2\alpha}{N-2})$  by a function in  $H_{0,\text{rad}}^1(B)$ , the space of radial  $H_0^1(B)$  functions. Thus, radial solutions of (2) exist also for (Sobolev) supercritical exponents  $p$ . Actually, radial  $H_0^1$  elements show a power-like decay away from the origin (as a consequence of the Strauss Lemma, see [19, 1]) that combines with the

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weight  $|x|^\alpha$  and provides the compactness of the embedding  $H_{0,\text{rad}}^1(B) \subset L^p(B)$  for any  $2 < p < 2^* + \frac{2\alpha}{N-2}$ .

A natural question is whether any minimizer of (3) must be radially symmetric. This is a non-trivial question, since the weight  $|\cdot|^\alpha$  is an increasing function to which neither rearrangement arguments nor the moving plane techniques of [9] can be applied.

In their pioneering paper [17], Smets, Su and Willem proved some symmetry-breaking results for (2). They proved that minimizers of (3) (the so-called *ground-state solutions*, or least energy solutions) cannot be radial, at least for  $\alpha$  large enough. As a consequence, (2) has at least two solutions when  $\alpha$  is large.

Later on, Serra proved in [16] the existence of at least one non-radial solution to (2) in the critical case  $p = 2^*$ , and in [2] the authors proved the existence of more than one solution to the same equation also for some supercritical values of  $p$ . These solutions are non-radial and they are obtained by minimization under suitable symmetry constraints.

Quite recently, Cao and Peng proved in [6] that, for  $p$  sufficiently close to  $2^*$ , the ground-state solutions of (2) possess a unique maximum point whose distance from  $\partial B$  tends to zero as  $p \rightarrow 2^*$ .

This kind of result was improved in [15], where multibump solutions for the Hénon equation with almost critical Sobolev exponent  $p$  are found, by means of a finite-dimensional reduction. These solutions are not radial, though they are invariant under the action of suitable subgroups of  $O(N)$ , and they concentrate at boundary points of the unit ball of  $\mathbb{R}^N$  as  $p \rightarrow 2^*$ . The rôle of  $\alpha$  is however a static one.

In this paper we will prove that similar phenomena take place for problem (1) on the annulus  $\Omega$ . In Section 2, we present some estimates for the least energy radial solutions of (1) when  $p < 2^*$  is kept fixed but  $\alpha \rightarrow +\infty$ . These will lead us to a first symmetry-breaking result, stating that for  $\alpha$  sufficiently large there exist at least two solutions of (1): a global minimizer of the associated Rayleigh quotient, and a global minimizer *among radial functions*.

In Section 3, another symmetry-breaking is proved, with  $\alpha$  fixed and  $p \rightarrow 2^*$ . To show this phenomenon, we will use a decomposition lemma in the spirit of P.L. Lions' concentration and compactness theory, and inspired by [6]. It will turn out that global minimizers of the same Rayleigh quotient concentrate as  $p \rightarrow 2^*$  at precisely one point of the boundary  $\partial\Omega$ , which has two connected components. A second nonradial solution can then be found in a tricky but natural way, by minimization over functions that are "heavier" on the opposite connected component of  $\partial\Omega$ .

In Section 4, a third nonradial solution is singled out, by means of a linking argument. Roughly speaking, the previous nonradial solutions can be used to build a mountain pass level. In particular, this third solution will not be a local minimizer of the Rayleigh quotient.

Section 5 describes the behavior of ground-state solutions of (1) as  $\alpha \rightarrow +\infty$  and  $p < 2^*$  is kept fixed. Although the conclusion is not as precise as in the case  $p \rightarrow 2^*$ , we can nevertheless show that a sort of concentration near the boundary  $\partial\Omega$  still appears. For more results about asymptotic estimates for solutions of the Hénon equation with  $\alpha$  large, see [4, 5].

The paper ends with an appendix containing some technical proofs.

2. SYMMETRY BREAKING FOR  $\alpha$  LARGE

Let  $H_{0,\text{rad}}^1(\Omega)$  be the space of radially symmetric functions of  $H_0^1(\Omega)$ . With a slight but common abuse of notation, we will systematically write  $u(x) = u(|x|)$  for  $u \in H_{0,\text{rad}}^1(\Omega)$ .

Consider the minimization problem

$$(5) \quad S_{\alpha,p}^{\text{rad}} = \inf_{u \in H_{0,\text{rad}}^1(\Omega) \setminus \{0\}} R_{\alpha,p}(u),$$

where

$$(6) \quad R_{\alpha,p}(u) = \frac{\int_{\Omega} |\nabla u|^2 dx}{\left(\int_{\Omega} \Psi_{\alpha} |u|^p dx\right)^{\frac{2}{p}}}, \quad u \in H_0^1(\Omega), u \neq 0$$

is the *Rayleigh quotient* associated to (1). It is known that the minimizers of (5) can be scaled so as to become solutions of (1). Therefore, we will use freely this fact in the sequel.

Unlike the result of [12], the fact that the annulus  $\Omega$  does not contain the origin implies the existence of a radial solution of (1) for any  $p > 2$ . This is the content of the next Proposition.

**Proposition 1.** *Let  $N \geq 3$ . Then  $S_{\alpha,p}^{\text{rad}}$  is attained for every  $p > 2$  and every  $\alpha > 0$ . In particular, equation (1) possesses a radial solution for every  $p > 2$ .*

*Proof.* Observe that for any  $u \in H_{0,\text{rad}}^1(\Omega)$ , the Hölder inequality implies

$$\begin{aligned} |u(x)| &= |u(|x|) - u(1)| \leq \int_1^{|x|} |u'(t)| dt \\ &\leq \sqrt{\frac{1 - 3^{2-N}}{(N-2)\omega_{N-1}}} \|\nabla u\|_{L^2(\Omega)} \quad \text{for a.e. } x \in \Omega, \end{aligned}$$

where  $\omega_{N-1}$  is the surface measure of  $S^{N-1}$ . Therefore  $H_{0,\text{rad}}^1(\Omega)$  is continuously embedded into  $L^\infty(\Omega)$ . It follows from standard arguments that this embedding  $H_{0,\text{rad}}^1(\Omega) \subset L^q(\Omega)$  is compact for every  $q > 2$ , and this easily implies that  $S_{\alpha,p}^{\text{rad}}$  is attained. A suitable scaling gives the desired solution of (1).  $\square$

In the next Proposition, we provide an estimate of the energy  $S_{\alpha,p}^{\text{rad}}$  as  $\alpha \rightarrow \infty$ .

**Proposition 2.** *Let  $p > 2$ . As  $\alpha \rightarrow \infty$ , there exist two constants  $C_1$  and  $C_2$  depending on  $p$  such that*

$$(7) \quad 0 < C_1 \leq \frac{S_{\alpha,p}^{\text{rad}}}{\alpha^{1+2/p}} \leq C_2 < +\infty.$$

Moreover, for any  $M > 2$  it is possible to choose the constants  $C_1$  and  $C_2$  independent of  $p \in (2, M]$ .

*Proof.* Fix a positive, radial function  $\psi \in C_0^\infty(\Omega)$ , and set  $\psi_\alpha(x) = \psi_\alpha(|x|) = \psi(\alpha(|x| - 3 + 3/\alpha))$ . Then

$$\begin{aligned} \int_{\Omega} |\nabla \psi_\alpha|^2 dx &= \omega_{N-1} \int_{3-\frac{2}{\alpha}}^3 (\psi'_\alpha(r))^2 r^{N-1} dr \\ &= \omega_{N-1} \int_1^3 \alpha^2 \psi'(s)^2 \left(\frac{s}{\alpha} + 3 - \frac{3}{\alpha}\right)^{N-1} \alpha^{-1} ds \\ &= \alpha \omega_{N-1} \int_1^3 \psi'(s)^2 \left(\frac{s+3\alpha-3}{\alpha s}\right)^{N-1} s^{N-1} ds \\ &\leq 3^{N-1} \alpha \int_{\Omega} |\nabla \psi|^2 dx, \quad (\text{since } 1 \leq \frac{s+3\alpha-3}{\alpha s} \leq 3) \end{aligned}$$

and

$$\int_{\Omega} \Psi_{\alpha} \psi_{\alpha}^p dx \geq \left(1 - \frac{2}{\alpha}\right)^{\alpha} \alpha^{-1} \int_{\Omega} \psi^p dx.$$

This proves that  $S_{\alpha,p}^{\text{rad}} \leq C(\alpha,p) \alpha^{1+\frac{2}{p}}$ , where

$$C(\alpha,p) = 3^{N-1} \frac{\int_{\Omega} |\nabla \psi|^2 dx}{\left(1 - \frac{2}{\alpha}\right)^{\frac{2\alpha}{p}} \left(\int_{\Omega} \psi^p(x) dx\right)^{\frac{2}{p}}} \leq C_2 \quad \text{for any } p > 2 \text{ and } \alpha > 1.$$

To check the other inequality, we will perform some scaling. Let us define the functions  $\psi_1: [1, 2] \rightarrow [1, 2]$  and  $\psi_2: [2, 3] \rightarrow [2, 3]$  as follows:

$$(8) \quad \psi_1(r) = 2 - (2-r)^{\beta}$$

$$(9) \quad \psi_2(r) = 2 + (r-2)^{\beta},$$

where  $\beta \in (0, 1)$  will be chosen hereafter. It is clear that we can obtain a piecewise  $C^1$  homeomorphism  $\psi: [1, 3] \rightarrow [1, 3]$  by gluing  $\psi_1$  and  $\psi_2$ . Now, for any radial function  $u \in H_0^1(\Omega)$ , we set  $v(\rho) = u(\psi(\rho))$  and compute:

$$\begin{aligned} \int_{\Omega} |\nabla u|^2 dx &= \omega_{N-1} \int_1^3 |u'(r)|^2 r^{N-1} dr \\ &\geq \omega_{N-1} \int_1^3 |u'(r)|^2 dr \\ &= \omega_{N-1} \left( \int_1^2 |v'(\rho)|^2 \frac{1}{\psi_1'(\rho)} d\rho + \int_2^3 |v'(\rho)|^2 \frac{1}{\psi_2'(\rho)} d\rho \right) \\ &= \omega_{N-1} \frac{1}{\beta} \left( \int_1^2 |v'(\rho)|^2 (2-\rho)^{1-\beta} d\rho \right. \\ &\quad \left. + \int_2^3 |v'(\rho)|^2 (\rho-2)^{1-\beta} d\rho \right) \\ &= \omega_{N-1} \frac{1}{\beta} \int_1^3 |v'(\rho)|^2 |\rho-2|^{1-\beta} d\rho \\ (10) \quad &\geq \omega_{N-1} \frac{1}{\beta} \int_1^3 |v'(\rho)|^2 |\rho-2| d\rho. \end{aligned}$$

Choosing  $\beta = 1/(\alpha + 1)$ ,

$$\begin{aligned} \int_{\Omega} \Psi_{\alpha}(x) |u(|x|)|^p dx &= \omega_{N-1} \int_1^3 \Psi_{\alpha}(r) |u(r)|^p r^{N-1} dr \\ &\leq 3^{N-1} \omega_{N-1} \int_1^3 \Psi_{\alpha}(r) |u(r)|^p dr \\ &= 3^{N-1} \omega_{N-1} \left( \int_1^2 \Psi_{\alpha}(\psi_1(\rho)) |v(\rho)|^p \psi_1'(\rho) d\rho \right. \\ &\quad \left. + \int_2^3 \Psi_{\alpha}(\psi_2(\rho)) |v(\rho)|^p \psi_2'(\rho) d\rho \right) \\ (11) \quad &= 3^{N-1} \omega_{N-1} \beta \int_1^3 |v(\rho)|^p d\rho. \end{aligned}$$

Therefore,

$$(12) \quad R_{\alpha,p}(u) \geq C \alpha^{1+\frac{2}{p}} \inf_{\substack{v \in H_0^1(\Omega) \\ v \neq 0}} \frac{\int_1^3 |v'(\rho)|^2 |4\rho-3| d\rho}{\left(\int_1^3 |v(\rho)|^p d\rho\right)^{2/p}}.$$

where  $C$  depends only on  $N$ . To complete the proof, we will show that the right-hand side of (12) is greater than zero. This follows from some general Hardy-type

inequality (see [13, Theorem 1.14]), but we present here an elementary proof for the sake of completeness. Indeed, given  $v \in H_{0,\text{rad}}^1(\Omega)$ , we can write for  $\rho \in [1, 2]$

$$\begin{aligned} |v(\rho)| &= |v(\rho) - v(1)| \leq \int_1^\rho |v'(t)| |2-t|^{1/2} \frac{dt}{|2-t|^{1/2}} \\ &\leq \left( \int_1^\rho |v'(t)|^2 |2-t| dt \right)^{1/2} \left( \int_1^\rho \frac{dt}{|2-t|} \right)^{1/2} \\ &\leq \left( \int_1^3 |v'(t)|^2 |2-t| dt \right)^{1/2} (-\log |2-\rho|)^{1/2}. \end{aligned}$$

Hence

$$\begin{aligned} \int_1^2 |v(\rho)|^p d\rho &\leq \left( \int_1^3 |v'(\rho)|^2 |2-\rho| d\rho \right)^{p/2} \int_1^2 (-\log(2-\rho))^{p/2} d\rho \\ &= \left( \int_1^3 |v'(\rho)|^2 |2-\rho| d\rho \right)^{p/2} \int_0^\infty t^{p/2} e^{-t} dt \\ &= \Gamma\left(\frac{p+2}{2}\right) \left( \int_1^3 |v'(\rho)|^2 |2-\rho| d\rho \right)^{p/2}, \end{aligned}$$

and in a similar way

$$\int_2^3 |v(\rho)|^p d\rho \leq \Gamma\left(\frac{p+2}{2}\right) \left( \int_1^3 |v'(\rho)|^2 |2-\rho| d\rho \right)^{p/2}.$$

Therefore

$$\int_1^3 |v'(\rho)|^2 |2-\rho| d\rho \geq \frac{1}{2^{2/p} \Gamma\left(\frac{p+2}{2}\right)^{2/p}} \left( \int_1^3 |v(\rho)|^p d\rho \right)^{\frac{2}{p}}.$$

This implies that the infimum in (12) is strictly positive and for any  $M > 2$  there exists a constant  $C_1 > 0$  such that  $2^{-2/p} \geq C_1 \Gamma\left(\frac{p+2}{2}\right)^{2/p}$  for any  $p \in (2, M]$ , since the Gamma function is positive,  $C^\infty$  and  $\Gamma\left(\frac{p+2}{2}\right) \sim (p/2)^{p/2} e^{-p/2} \sqrt{\pi p}$  for  $p \rightarrow +\infty$ . We finally collect (10) and (11) to get the desired conclusion

$$S_{\alpha,p}^{\text{rad}} \geq C_1 \alpha^{1+\frac{2}{p}}.$$

□

Set now

$$(13) \quad S_{\alpha,p} = \inf_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} R_{\alpha,p}(u).$$

It is easily proved that  $S_{\alpha,p}$  is attained by a function  $u_{\alpha,p}$  that satisfies (up to a scaling) equation (1).

In order to prove that this solution is not radial (at least for  $\alpha$  large) we need an estimate from above of the level  $S_{\alpha,p}$ .

**Lemma 3.** *Let  $p \in (2, 2^*)$ . There exists  $\bar{\alpha}$  such that for  $\alpha \geq \bar{\alpha}$*

$$(14) \quad S_{\alpha,p} \leq C \alpha^{2-N+\frac{2N}{p}}.$$

*Proof.* Let  $\psi$  be a positive smooth function with support in the unit ball  $B$ . Following [17], let us consider  $\psi_\alpha(x) = \psi(\alpha(x - x_\alpha))$ , where  $x_\alpha = (3 - \frac{1}{\alpha}, 0, \dots, 0)$ .

Since  $\psi_\alpha$  has support in the ball  $B(x_\alpha, \frac{1}{\alpha})$ , by the change of variable  $y = \alpha(x - x_\alpha)$  we obtain:

$$\int_{\Omega} \Psi_\alpha(x) \psi_\alpha^p(x) dx = \int_{B(x_\alpha, \frac{1}{\alpha})} ||x| - 2|^\alpha \psi_\alpha^p(x) dx \geq \left(1 - \frac{2}{\alpha}\right)^\alpha \alpha^{-N} \int_B \psi^p(y) dy$$

Moreover

$$\int_{\Omega} |\nabla \psi_\alpha|^2 dx = \alpha^2 \int_{\Omega} |\nabla \psi(\alpha(x - x_\alpha))|^2 dx = \alpha^{2-N} \int_B |\nabla \psi|^2 dx,$$

so that

$$S_{\alpha,p} \leq R_{\alpha,p}(\psi_\alpha) \leq C \alpha^{2-N+\frac{2N}{p}}$$

for all  $\alpha$  sufficiently large. This proves the Lemma.  $\square$

By comparing (14) and (7), we deduce a first symmetry-breaking result.

**Theorem 4.** *Let  $p \in (2, 2^*)$ . For  $\alpha$  sufficiently large, the ground state  $u_{\alpha,p}$  is a non-radial function.*

*Proof.* From (14) and (7) it follows that  $S_{\alpha,p} < S_{\alpha,p}^{\text{rad}}$  when  $\alpha$  is large.  $\square$

### 3. SYMMETRY BREAKING AS $p \rightarrow 2^*$

In this section we consider  $\alpha$  fixed,  $p$  close to  $2^*$  and we establish the following

**Theorem 5.** *Let  $\alpha > 0$ . For  $p$  close to  $2^*$  the quotient  $R_{\alpha,p}$  has at least two non radial local minima.*

We briefly explain how the proof proceeds. Of course, we already know that any global minimizer of  $R_{\alpha,p}$  produces a first solution  $u_{\alpha,p}$ . As the Theorem 6 states, this solution concentrates at precisely one point of the boundary  $\partial\Omega$ . Since this boundary has two connected components, we will minimize  $R_{\alpha,p}$  over the set  $\Lambda$  of  $H_0^1$  functions which “concentrate” at the opposite component of the boundary. A careful estimate is proved in order to show that minimizers fall inside the interior of  $\Lambda$ .

Consider now  $u_{\alpha,p}$ . As in [6] we have a description of the profile of  $u_{\alpha,p}$  as  $p \rightarrow 2^*$ .

**Theorem 6.** *Let  $p \in (2, 2^*)$  and  $\alpha > 0$ . The minimum  $u_{\alpha,p}$  of  $R_{\alpha,p}(u)$  in  $H_0^1 \setminus \{0\}$  satisfies (passing to a subsequence) for some  $x_0 \in \partial\Omega$*

- i)  $|\nabla u_{\alpha,p}|^2 \rightarrow \mu \delta_{x_0}$  in the sense of measure as  $p \rightarrow 2^*$ ,
- ii)  $|u_{\alpha,p}|^{2^*} \rightarrow \nu \delta_{x_0}$  in the sense of measure as  $p \rightarrow 2^*$ ,

where  $\mu > 0$  and  $\nu > 0$  are such that  $\mu \geq S_{0,2^*} \nu^{2/2^*}$  and  $\delta_x$  is the Dirac mass at  $x$ .

*Proof.* Postponed to the Appendix.  $\square$

To get a second local minimizer, we will assume without loss of generality that  $u_{\alpha,p}$  concentrates at some point on the sphere  $|x| = 3$  (a similar argument holds if  $u_{\alpha,p}$  concentrates at some point  $x$  with  $|x| = 1$ ). After a rotation, we can even assume that  $u_{\alpha,p}$  concentrates at the point  $(3, 0, \dots, 0)$ .

Let

$$\Omega^- = \{x \in \mathbb{R}^N \mid 1 < |x| < 2\}, \quad \Omega^+ = \{x \in \mathbb{R}^N \mid 2 < |x| < 3\}$$

and

$$\Sigma = \left\{ u \in H_0^1 \setminus \{0\} \mid \int_{\Omega^+} |\nabla u|^2 dx = \int_{\Omega^-} |\nabla u|^2 dx \right\}.$$

Let us denote

$$T_{\alpha,p} = \inf_{u \in \Sigma} R_{\alpha,p}(u).$$



We have the following uniform estimate for  $T_{\alpha,p}$ .

**Proposition 7.** *Let  $\alpha > 0$ . There exists  $\delta > 0$  such that*

$$\liminf_{p \rightarrow 2^*} T_{\alpha,p} > S_{0,2^*} + \delta.$$

*Proof.* By standard compactness argument  $T_{\alpha,p}$  is achieved by a function  $v_{\alpha,p}$ . Moreover for any  $\alpha > 0$ , and  $2 < p < 2^*$  we have  $T_{\alpha,p} \geq S_{\alpha,p}$ . We want to prove that the inequality is strict at least for  $p \rightarrow 2^*$ . Indeed assume on the contrary that

$$\liminf_{p \rightarrow 2^*} T_{\alpha,p} = \liminf_{p \rightarrow 2^*} R_{\alpha,p}(v_{\alpha,p}) = S_{0,2^*}.$$

From the definition of  $S_{0,2^*}$  and Hölder inequality we get, for a subsequence  $p = p_k \rightarrow 2^*$

$$\begin{aligned} S_{0,2^*} &\leq \frac{\int_{\Omega} |\nabla v_{\alpha,p}|^2 dx}{\left(\int_{\Omega} |v_{\alpha,p}|^{2^*} dx\right)^{2/2^*}} \leq |\Omega|^{\frac{(2^*-p)2}{2^*p}} \frac{\int_{\Omega} |\nabla v_{\alpha,p}|^2 dx}{\left(\int_{\Omega} |v_{\alpha,p}|^p dx\right)^{2/p}} \\ &\leq |\Omega|^{\frac{(2^*-p)2}{2^*p}} \frac{\int |\nabla v_{\alpha,p}|^2 dx}{\left(\int_{\Omega} \Psi_{\alpha}(x) |v_{\alpha,p}|^p dx\right)^{2/p}} = S_{0,2^*} + o(1) \end{aligned}$$

since the weight satisfies  $\Psi_{\alpha}(x) \leq 1$ . In particular

$$\frac{\int_{\Omega} |\nabla v_{\alpha,p}|^2 dx}{\left(\int_{\Omega} |v_{\alpha,p}|^{2^*} dx\right)^{2/2^*}} \rightarrow S_{0,2^*},$$

and  $v_{\alpha,p}$  is a minimizing sequence of  $S_{0,2^*}$ .

In the same way as we do in the proof of Theorem 6 for  $u_{\alpha,p}$ , we can prove that  $v_{\alpha,p}$  concentrates at precisely one point one of the boundary  $\partial\Omega$ . This contradicts the fact that  $\int_{\Omega^+} |\nabla v_{\alpha,p}|^2 dx = \int_{\Omega^-} |\nabla v_{\alpha,p}|^2 dx$ .  $\square$

Consider now the points

$$x_{0,\varepsilon} = x_0 = \left(3 - \frac{1}{|\log \varepsilon|}, 0, \dots, 0\right), \quad x_{1,\varepsilon} = x_1 = \left(1 + \frac{1}{|\log \varepsilon|}, 0, \dots, 0\right)$$

and

$$U(x) = \frac{1}{(1 + |x|)^{(N-2)/2}}.$$

We recall that  $S_{0,2^*}$  is not achieved on any proper subset of  $\mathbb{R}^N$ , and that it is independent of  $\Omega$ . However, it is known that  $S_{0,2^*}(\mathbb{R}^N)$  is achieved, and all the minimizers can be written in the form

$$\mathcal{U}_{\theta,y}(x) = \frac{1}{(\theta^2 + |x - y|^2)^{\frac{N-2}{2}}}, \quad \theta > 0, y \in \mathbb{R}^N.$$

We set

$$U_{\varepsilon}^i(x) = \varepsilon^{-\frac{N-2}{2}} U\left(\frac{x - x_i}{\sqrt{\varepsilon}}\right) = \frac{1}{(\varepsilon + |x - x_i|^2)^{\frac{N-2}{2}}},$$

and denote by  $\varphi_i$  ( $i = 0, 1$ ) two *cut-off* functions such that  $0 \leq \varphi_i \leq 1$ ,  $|\nabla \varphi_i| \leq C|\log \varepsilon|$  for some constant  $C > 0$ , and

$$\varphi_i(x) = \begin{cases} 1, & \text{if } |x - x_i| < \frac{1}{2|\log \varepsilon|} \\ 0, & \text{if } |x - x_i| \geq \frac{1}{|\log \varepsilon|}. \end{cases}$$

The following Lemma shows that the truncated functions

$$(15) \quad u_{\varepsilon}^i(x) = \varphi_i(x) U_{\varepsilon}^i(x), \quad i = 0, 1,$$

are almost minimizers for  $S_{0,2^*}$ .

**Lemma 8.** *Let  $\alpha > 0$ . There results*

$$\lim_{p \rightarrow 2^*} R_{\alpha,p}(u_\varepsilon^i) = S_{0,2^*} + K(\varepsilon)$$

with  $\lim_{\varepsilon \rightarrow 0} K(\varepsilon) = 0$ .

*Proof.* Postponed to the Appendix.  $\square$

**Remark 9.** A direct consequence of Lemma 8 is that  $S_{0,2^*} = S_{\alpha,2^*}$ . Indeed  $S_{0,2^*} \leq S_{\alpha,2^*}$  since  $\Psi_\alpha(|x|) \leq 1$ . On the other hand by Lemma 8

$$R_{\alpha,2^*}(u_\varepsilon^i) = \lim_{p \rightarrow 2^*} R_{\alpha,p}(u_\varepsilon^i) = S_{0,2^*} + K(\varepsilon).$$

Therefore  $S_{0,2^*} + K(\varepsilon) \geq S_{\alpha,2^*}$  for every  $\varepsilon > 0$ . Letting  $\varepsilon \rightarrow 0$  we conclude  $S_{0,2^*} \geq S_{\alpha,2^*}$ .

We are now ready to prove the Theorem 5.

*Proof of Theorem 5.* Let  $u_{\alpha,p}$  be a ground state solution. Let us suppose that it concentrates on the outer boundary. Consider the open subset

$$\Lambda = \left\{ u \in H_0^1(\Omega) : \int_{\Omega^-} |\nabla u|^2 dx > \int_{\Omega^+} |\nabla u|^2 dx \right\}.$$

The infimum of  $R_{\alpha,p}$  on  $\bar{\Lambda}$  is achieved. However it cannot be achieved on the boundary  $\partial\Lambda = \Sigma$ . Indeed, by Proposition 7,

$$\inf_{\Sigma} R_{\alpha,p} > S_{0,2^*} + \delta \text{ as } p \rightarrow 2^*$$

and

$$\inf_{\Lambda} R_{\alpha,p}(u) \leq R_{\alpha,p}(u_\varepsilon^1) \rightarrow S_{0,2^*} + K_1(\varepsilon) \text{ as } p \rightarrow 2^*$$

since  $u_\varepsilon^1 \in \Lambda$  for  $\varepsilon$  small enough. Then the infimum is achieved in a interior point of  $\Lambda$  and is therefore a critical point of  $R_{\alpha,p}$ .  $\square$

#### 4. EXISTENCE OF A THIRD NON-RADIAL SOLUTION

In the previous section we proved the existence of two solutions of (1) which are local minima of the Rayleigh quotient for  $p$  near  $2^*$ . One would expect another critical point of  $R_{\alpha,p}$  located in some sense between these minimum points. This is precisely the idea we are going to pursue further in the current section.

For  $\varepsilon$  small enough let  $u_\varepsilon^i = \varphi_i U_\varepsilon^i$ ,  $i \in \{0, 1\}$ , be defined as in (15). We will prove that  $R_{\alpha,p}$  has the Mountain Pass geometry.

Let us introduce the *minimax level*

$$\beta = \beta(\alpha, p) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} R_{\alpha,p}(\gamma(t)),$$

where  $\Gamma = \{\gamma \in C([0, 1], H_0^1(\Omega)) \mid \gamma(0) = u_\varepsilon^0, \gamma(1) = u_\varepsilon^1\}$  is the set of continuous paths joining  $u_\varepsilon^0$  with  $u_\varepsilon^1$ .

We begin to prove that  $\beta$  is larger, uniformly with respect to  $\varepsilon$ , than the values of the functional  $R_{\alpha,p}$  at the points  $u_\varepsilon^0$  and  $u_\varepsilon^1$ .

**Lemma 10.** *Let  $M_\varepsilon = \max\{R_{\alpha,p}(u_\varepsilon^0), R_{\alpha,p}(u_\varepsilon^1)\}$ . There exists  $\sigma > 0$  such that  $\beta \geq M_\varepsilon + \sigma$  uniformly with respect to  $\varepsilon$ .*

*Proof.* We prove that there exists  $\sigma$  such that for all  $\gamma \in \Gamma$

$$\max R_{\alpha,p}(\gamma(t)) \geq M_\varepsilon + \sigma.$$

A simple continuity argument shows that for every  $\gamma \in \Gamma$  there exists  $t_\gamma$  such that  $\gamma(t_\gamma) \in \Sigma$ , where

$$\Sigma = \left\{ u \in H_0^1 \setminus \{0\} \mid \int_{\Omega^+} |\nabla u|^2 dx = \int_{\Omega^-} |\nabla u|^2 dx \right\}.$$

Indeed the map  $t \in [0, 1] \mapsto \int_{\Omega^+} |\nabla \gamma(t)|^2 dx - \int_{\Omega^-} |\nabla \gamma(t)|^2 dx$  is continuous and it takes a negative value at  $t = 0$  and a positive value at  $t = 1$ . Now Proposition 7 implies, for  $p$  near  $2^*$  the existence of  $\delta > 0$  with

$$\max_{t \in [0,1]} R_{\alpha,p}(\gamma(t)) \geq R_{\alpha,p}(\gamma(t_\gamma)) \geq \inf_{u \in \Sigma} R_{\alpha,p}(u) \geq S_{0,2^*} + \delta.$$

On the other hand, for  $\varepsilon$  sufficiently small,

$$M_\varepsilon < S_{0,2^*} + \frac{\delta}{2}.$$

This concludes the proof.  $\square$

The previous estimate allows us to show that  $\beta$  is a critical level for  $R_{\alpha,p}$ . Therefore a further nonradial solution to (1) arises.

**Proposition 11.** *There exist  $\bar{\alpha} > 0$  and  $2 < \bar{p} < 2^*$  such that for all  $\alpha \geq \bar{\alpha}$  and  $\bar{p} \leq p < 2^*$  it results that  $\beta$  is a critical value for  $R_{\alpha,p}$  and it is attained by a non-radial function.*

*Proof.* From the previous result we can apply a deformation argument (see [1, 20]) to prove that  $\beta$  is a critical level and it is attained (since the PS condition is satisfied) by a function  $w$ . From the asymptotic estimate (7) for the radial level  $S_{\alpha,p}^{\text{rad}}$ , one has that there exists a constant  $C$  independent from  $p$  such that

$$S_{\alpha,p}^{\text{rad}} \geq C\alpha^{1+2/p}.$$

In particular

$$S_{\alpha,p}^{\text{rad}} \rightarrow +\infty \quad \text{as } \alpha \rightarrow +\infty.$$

This allows us to choose  $\alpha_0$  such that  $S_{\alpha,p}^{\text{rad}} \geq 3S_{0,2^*}$  for all  $\alpha \geq \alpha_0$ .

Define  $\zeta \in \Gamma$  by  $\zeta(t) = tu_\varepsilon^1 + (1-t)u_\varepsilon^0$  for all  $t \in [0, 1]$ , and let  $\tau \in [0, 1]$  be such that  $R_{\alpha,p}(\zeta(\tau)) = \max_{t \in [0,1]} R_{\alpha,p}(\zeta(t))$ .

Since  $u_\varepsilon^1$  and  $u_\varepsilon^0$  have disjoint supports one has, for  $\varepsilon$  sufficiently small,

$$\begin{aligned} R_{\alpha,p}(w) &= \beta \leq R_{\alpha,p}(\zeta(\tau)) = \frac{\int_{\Omega} |\nabla(\tau u_\varepsilon^1 + (1-\tau)u_\varepsilon^0)|^2 dx}{\left(\int_{\Omega} \Psi_\alpha |\tau u_\varepsilon^1 + (1-\tau)u_\varepsilon^0|^p dx\right)^{2/p}} \\ &= \frac{\int_{\Omega} \tau^2 |\nabla u_\varepsilon^1|^2 dx + \int_{\Omega} (1-\tau)^2 |\nabla u_\varepsilon^0|^2 dx}{\left(\tau^p \int_{\Omega} \Psi_\alpha |u_\varepsilon^1|^p dx + (1-\tau)^p \int_{\Omega} \Psi_\alpha |u_\varepsilon^0|^p dx\right)^{2/p}} \\ &\leq \frac{\tau^2 \int_{\Omega} |\nabla u_\varepsilon^1|^2 dx}{\left(\tau^p \int_{\Omega} \Psi_\alpha |u_\varepsilon^1|^p dx\right)^{2/p}} + \frac{(1-\tau)^2 \int_{\Omega} |\nabla u_\varepsilon^0|^2 dx}{\left((1-\tau)^p \int_{\Omega} \Psi_\alpha |u_\varepsilon^0|^p dx\right)^{2/p}} \\ &= R_{\alpha,p}(u_\varepsilon^0) + R_{\alpha,p}(u_\varepsilon^1) \leq 2M_\varepsilon < 3S_{0,2^*} \leq S_{\alpha,p}^{\text{rad}}. \end{aligned}$$

This concludes the proof.  $\square$

5. BEHAVIOUR OF THE GROUND-STATE SOLUTIONS FOR  $\alpha$  LARGE

This section is devoted to the analysis a ground state solution as  $\alpha \rightarrow +\infty$ . Even in this case this solution tends to “concentrate” at the boundary  $\partial\Omega$ . However, this concentration is much weaker than the concentration as  $p \rightarrow 2^*$ .

We use the notation  $C(r_1, r_2) = \{x \in \mathbb{R}^N \mid r_1 < |x| < r_2\}$ . Let  $\delta$  be sufficiently small (say  $\delta < \frac{1}{2}$ ) and  $\phi$  be a smooth cut-off function such that  $0 \leq \phi \leq 1$  with

$$(16) \quad \phi(x) = \begin{cases} 1, & x \in C(1, 1 + \delta) \cup C(3 - \delta, 3) \\ 0, & x \in C(2 - \delta, 2 + \delta) \end{cases}$$

From now on, since  $p \in (2, 2^*)$  is fixed we denote the ground state solution of problem (1)  $u_{\alpha,p}$  with  $u_\alpha$ .

**Proposition 12.** *Let  $u_\alpha$  be such that  $R_{\alpha,p}(u_\alpha) = S_{\alpha,p}$ . If  $\phi$  is as in (16), then*

$$(17) \quad R_{\alpha,p}(\phi u_\alpha) = S_{\alpha,p} + o(S_{\alpha,p}) \quad \text{as } \alpha \rightarrow +\infty.$$

*Proof.* It is not restrictive, by the homogeneity of  $R_{\alpha,p}$ , to assume  $\int_\Omega |\nabla u_\alpha|^2 dx = 1$ . We split the proof into two steps.

**Step 1.** We claim that

$$(18) \quad \int_\Omega \Psi_\alpha(u_\alpha \phi)^p dx = \int_\Omega \Psi_\alpha u_\alpha^p dx + o\left(\int_\Omega \Psi_\alpha u_\alpha^p dx\right)$$

Indeed, suppose on the contrary that

$$\limsup_{\alpha \rightarrow \infty} \frac{\int_\Omega \Psi_\alpha u_\alpha^p (1 - \phi^p) dx}{\int_\Omega \Psi_\alpha u_\alpha^p dx} = \beta > 0$$

This implies that, up to some subsequence,

$$\frac{\int_\Omega \Psi_\alpha u_\alpha^p (1 - \phi^p) dx}{\int_\Omega \Psi_\alpha u_\alpha^p dx} > \beta/2 > 0$$

Since  $1 - \phi^p \equiv 0$  on  $C(1, 1 + \delta) \cup C(3 - \delta, 3)$  we have

$$\begin{aligned} \int_\Omega \Psi_\alpha u_\alpha^p (1 - \phi^p) dx &= \int_{C(1+\delta, 3-\delta)} \Psi_\alpha u_\alpha^p (1 - \phi^p) dx \\ &\leq (1 - \delta)^\alpha \int_\Omega u_\alpha^p (1 - \phi^p) dx \leq (1 - \delta)^\alpha \int_\Omega u_\alpha^p dx. \end{aligned}$$

Therefore

$$\int_\Omega u_\alpha^p dx \geq (1 - \delta)^{-\alpha} \int_\Omega \Psi_\alpha u_\alpha^p (1 - \phi^p) dx$$

Now

$$\frac{\int_\Omega u_\alpha^p dx}{\int_\Omega \Psi_\alpha u_\alpha^p dx} \geq (1 - \delta)^{-\alpha} \frac{\int_\Omega \Psi_\alpha u_\alpha^p (1 - \phi^p) dx}{\int_\Omega \Psi_\alpha u_\alpha^p dx} \geq (1 - \delta)^{-\alpha} \frac{\beta}{2}.$$

Since  $S_{\alpha,p}^{p/2} = (\int_\Omega \Psi_\alpha u_\alpha^p dx)^{-1}$  the last inequality can be written as

$$S_{\alpha,p}^{p/2} \geq \frac{\beta}{2} \frac{(1 - \delta)^{-\alpha}}{\int_\Omega u_\alpha^p dx} \geq \frac{\beta}{2} (1 - \delta)^{-\alpha} S_{0,p}^{p/2},$$

where

$$S_{0,p} = \inf_{u \neq 0} \frac{\int_\Omega |\nabla u|^2 dx}{(\int_\Omega u^p dx)^{2/p}}$$

On the other hand from (14) one has the estimate

$$S_{\alpha,p}^{p/2} \leq C \alpha^{p - \frac{N}{2}p + N},$$

which gives a contradiction for  $\alpha$  large. This proves (18).

**Step 2.** Now we prove that

$$(19) \quad \int_{\Omega} |\nabla u_{\alpha} \phi|^2 dx = \int_{\Omega} |\nabla u_{\alpha}|^2 dx + o(1) = 1 + o(1).$$

It is not difficult to prove that  $u_{\alpha}$  satisfies the problem

$$(20) \quad \begin{cases} -\Delta u_{\alpha} = S_{\alpha,p}^{p/2} \Psi_{\alpha} u_{\alpha}^{p-1} & \text{in } \Omega, \\ u_{\alpha} > 0 & \text{in } \Omega, \\ u_{\alpha} = 0 & \text{on } \partial\Omega, \end{cases}$$

Since  $\|\nabla u_{\alpha}\|_2 = 1$ , up to subsequences, we have that, as  $\alpha \rightarrow \infty$ ,

$$u_{\alpha} \rightarrow u \quad \text{weakly in } H_0^1(\Omega), \text{ strongly in } L^q(\Omega), \text{ and a.e.}$$

We now prove that  $u = 0$ . Indeed, multiplying equation (20) by a smooth function  $\psi$  with  $\text{supp } \psi \subset\subset \Omega$  and integrate, we obtain

$$\int_{\Omega} \nabla u_{\alpha} \nabla \psi dx = \int_{\Omega} S_{\alpha,p}^{p/2} \Psi_{\alpha} u_{\alpha}^{p-1} \psi dx \rightarrow 0, \quad \alpha \rightarrow +\infty$$

since, by (14),  $S_{\alpha,p}^{p/2} \Psi_{\alpha} \rightarrow 0$  uniformly on  $\text{supp } \psi$  and  $u_{\alpha}$  is uniformly bounded in  $L^q$  for  $1 \leq q < 2^*$ . This implies that  $u = 0$ .

Now we estimate the difference

$$(21) \quad \left| \int_{\Omega} |\nabla u_{\alpha}|^2 dx - \int_{\Omega} |\nabla(u_{\alpha} \phi)|^2 dx \right| \leq \\ \leq \int_{\Omega} |\nabla u_{\alpha}|^2 (1 - \phi^2) dx + \int_{\Omega} |\nabla \phi|^2 u_{\alpha}^2 dx + 2 \left| \int_{\Omega} \nabla u_{\alpha} \nabla \phi u_{\alpha} \phi dx \right|$$

The last terms tend to zero thanks to the strong convergence in  $L^q$  for all  $q \in [1, 2^*)$ . In order to estimate the term  $\int_{\Omega} |\nabla u_{\alpha}|^2 (1 - \phi^2) dx$ , we multiply (20) by  $u_{\alpha}(1 - \phi^2) = u_{\alpha} \eta$  and integrate. Since  $\text{supp } \eta = \text{supp}(1 - \phi^2) \subset\subset \Omega$  we have

$$\int_{\Omega} \nabla u_{\alpha} \nabla (\eta u_{\alpha}) dx = \int_{\Omega} S_{\alpha,p}^{p/2} \Psi_{\alpha} u_{\alpha}^p \eta dx,$$

so that

$$\left| \int_{\Omega} |\nabla u_{\alpha}|^2 \eta dx \right| \leq \left| \int_{\Omega} u_{\alpha} \nabla \eta \nabla u_{\alpha} dx \right| + \left| \int_{\Omega} S_{\alpha,p}^{p/2} \Psi_{\alpha} u_{\alpha}^p \eta dx \right| \\ \leq \|\nabla \eta\|_{\infty} \int_{\text{supp } \eta} |\nabla u_{\alpha} u_{\alpha}| dx + \left| \int_{\text{supp } \eta} S_{\alpha,p}^{p/2} \Psi_{\alpha} u_{\alpha}^p \eta dx \right| \rightarrow 0.$$

□

In proposition 12 we proved that the infimum of the Rayleigh quotient  $R_{\alpha,p}$  is essentially attained by the function  $\phi u_{\alpha}$ . Thanks to the definition of  $\phi$ , we can decompose  $\phi u_{\alpha} = u_{\alpha,1} + u_{\alpha,2}$ , where  $u_{\alpha,1}$  vanishes in  $C(2 - \delta, 3)$  and  $u_{\alpha,2}$  vanishes in  $C(1, 2 + \delta)$ . The following proposition is the main step in order to prove that the function  $\phi u_{\alpha}$  concentrates at one of the boundaries.

**Proposition 13.** *Let  $\phi u_{\alpha} = u_{\alpha,1} + u_{\alpha,2}$ , where  $\text{supp } u_{\alpha,1} \subset C(1, 2 - \delta)$  and  $\text{supp } u_{\alpha,2} \subset C(2 + \delta, 3)$ . Then*

$$(22) \quad \text{either } \lim_{\alpha \rightarrow +\infty} \frac{\int_{\Omega} \Psi_{\alpha} u_{\alpha,1}^p dx}{\int_{\Omega} \Psi_{\alpha} u_{\alpha,2}^p dx} = 0 \quad \text{or} \quad \lim_{\alpha \rightarrow +\infty} \frac{\int_{\Omega} \Psi_{\alpha} u_{\alpha,2}^p dx}{\int_{\Omega} \Psi_{\alpha} u_{\alpha,1}^p dx} = 0.$$

*Proof.* By the definition of  $u_{\alpha,1}$  and  $u_{\alpha,2}$  we have

$$(23) \quad R_{\alpha,p}(\phi u_{\alpha}) = \frac{\int_{\Omega} |\nabla u_{\alpha,1}|^2 dx + \int_{\Omega} |\nabla u_{\alpha,2}|^2 dx}{\left( \int_{\Omega} \Psi_{\alpha} u_{\alpha,1}^p dx + \int_{\Omega} \Psi_{\alpha} u_{\alpha,2}^p dx \right)^{\frac{2}{p}}}.$$

Let us define  $\lambda_\alpha$  as  $\int_\Omega \Psi_\alpha u_{\alpha,1}^p = \lambda_\alpha \int_\Omega \Psi_\alpha u_{\alpha,2}^p$ . Since  $u_\alpha$  is a positive solution and it depends continuously on the parameter  $\alpha$ , the function  $\lambda_\alpha$  is continuous in  $\alpha$  and  $\lambda_\alpha > 0$ . We obtain the following identity:

$$\begin{aligned}
R_{\alpha,p}(\phi u_\alpha) &= \frac{\int_\Omega |\nabla u_{\alpha,1}|^2 dx + \int_\Omega |\nabla u_{\alpha,2}|^2 dx}{(\lambda_\alpha \int_\Omega \Psi_\alpha u_{\alpha,2}^p dx + \int_\Omega \Psi_\alpha u_{\alpha,1}^p dx)^{2/p}} \\
&= \frac{\int_\Omega |\nabla u_{\alpha,1}|^2 dx}{(\lambda_\alpha + 1)^{2/p} (\int_\Omega \Psi_\alpha u_{\alpha,2}^p dx)^{2/p}} + \frac{\int_\Omega |\nabla u_{\alpha,2}|^2 dx}{(\lambda_\alpha + 1)^{2/p} (\int_\Omega \Psi_\alpha u_{\alpha,1}^p dx)^{2/p}} \\
(24) \quad &= \frac{\lambda_\alpha^{\frac{2}{p}} \int_\Omega |\nabla u_{\alpha,1}|^2 dx}{(\lambda_\alpha + 1)^{\frac{2}{p}} (\int_\Omega \Psi_\alpha u_{\alpha,1}^p dx)^{2/p}} + \frac{\int_\Omega |\nabla u_{\alpha,2}|^2 dx}{(\lambda_\alpha + 1)^{\frac{2}{p}} (\int_\Omega \Psi_\alpha u_{\alpha,2}^p dx)^{2/p}}.
\end{aligned}$$

By the definition of  $S_{\alpha,p}$  each quotient  $R_{\alpha,p}(u_{\alpha,1})$  and  $R_{\alpha,p}(u_{\alpha,2})$  in the last term is greater than or equal to  $S_{\alpha,p}$ . Therefore by Proposition 12 and equation (24) one obtains

$$(25) \quad S_{\alpha,p} + o(S_{\alpha,p}) \geq \frac{1 + \lambda_\alpha^{\frac{2}{p}}}{(\lambda_\alpha + 1)^{\frac{2}{p}}} S_{\alpha,p}.$$

We notice that the function  $f(x) = \frac{1+x^{2/p}}{(x+1)^{2/p}}$  is strictly greater than 1 for every  $x > 0$ ,  $f(0) = 1$  and  $f(x) \rightarrow 1$  as  $x \rightarrow +\infty$ . Moreover it is increasing in  $[0, 1]$  and decreasing in  $[1, +\infty)$  and  $\max_{x>0} f(x) = f(1) = 2^{1-2/p}$ . Let us denote  $L = \limsup_{\alpha \rightarrow +\infty} \lambda_\alpha$  and  $l = \liminf_{\alpha \rightarrow +\infty} \lambda_\alpha$ . By the inequality (25) it is easy to see that either  $l = L = +\infty$  or  $L = l = 0$ . In fact if let  $\alpha \rightarrow +\infty$  in (25), we obtain that  $1 \geq \frac{1+l^{2/p}}{(l+1)^{2/p}}$  and  $1 \geq \frac{1+L^{2/p}}{(L+1)^{2/p}}$  and so either  $l = 0$  and  $L = +\infty$ , or  $l = +\infty$  and  $L = 0$ . Let us prove that the case  $L = +\infty$  and  $l = 0$  cannot happen. Let us suppose by contradiction that  $L = +\infty$  and  $l = 0$ . Then the continuity of  $\lambda_\alpha$  implies that there exists a sequence such that  $\lim_{n \rightarrow +\infty} \lambda_{\alpha_n} = 1$ . If we evaluate the inequality (25) on this sequence  $\lambda_{\alpha_n}$  we obtain the contradiction  $1 \geq 2^{1-2/p}$ . So we have proved that either  $L = l = +\infty$  or  $L = l = 0$  and this concludes the proof of the proposition.  $\square$

**Corollary 14.** *With the notation of Proposition 13 one has*

$$(26) \quad \text{either } \lim_{\alpha \rightarrow +\infty} \frac{\int_\Omega |\nabla u_{\alpha,1}|^2 dx}{\int_\Omega |\nabla u_{\alpha,2}|^2 dx} = 0 \quad \text{or} \quad \lim_{\alpha \rightarrow +\infty} \frac{\int_\Omega |\nabla u_{\alpha,2}|^2 dx}{\int_\Omega |\nabla u_{\alpha,1}|^2 dx} = 0.$$

*Proof.* By virtue of Proposition 13 we may assume that  $\lambda_\alpha = \frac{\int_\Omega \Psi_\alpha u_{\alpha,1}^p dx}{\int_\Omega \Psi_\alpha u_{\alpha,2}^p dx} \rightarrow 0$ , up to subsequences. Suppose that  $\limsup_{\alpha \rightarrow \infty} \xi_\alpha > 0$ . Up to subsequences,  $\xi_\alpha > \xi > 0$  for some  $\xi$ . Therefore we have

$$\begin{aligned}
S_{\alpha,p} + o(S_{\alpha,p}) &= \frac{\int_\Omega |\nabla u_{\alpha,1}|^2 dx + \int_\Omega |\nabla u_{\alpha,2}|^2 dx}{(\int_\Omega \Psi_\alpha u_{\alpha,1}^p dx + \int_\Omega \Psi_\alpha u_{\alpha,2}^p dx)^{\frac{2}{p}}} = \frac{(1 + \xi_\alpha) \int_\Omega |\nabla u_{\alpha,2}|^2 dx}{(\int_\Omega \Psi_\alpha u_{\alpha,2}^p dx)^{\frac{2}{p}} (1 + \lambda_\alpha)} \\
&\geq R_{\alpha,p}(u_{\alpha,2}) \frac{1 + \xi}{1 + o(1)} \geq (1 + \xi) S_{\alpha,p} + o(S_{\alpha,p}),
\end{aligned}$$

which is a contradiction. Hence

$$\xi_\alpha = \frac{\int_\Omega |\nabla u_{\alpha,1}|^2 dx}{\int_\Omega |\nabla u_{\alpha,2}|^2 dx} \rightarrow 0.$$

$\square$

**Remark 15.** An immediate consequence of the previous results is that in particular

$$(27) \quad \lim_{\alpha \rightarrow +\infty} \int_{\Omega} |\nabla u_{\alpha,1}|^2 dx = 0 \quad \text{or} \quad \lim_{\alpha \rightarrow +\infty} \int_{\Omega} |\nabla u_{\alpha,2}|^2 dx = 0,$$

that is the ground state solution concentrates at *only one* component of the boundary.

## 6. APPENDIX

We present here the proofs of some result used in the third section. They appear in [6] in the case of the ball. Although the proofs for the annulus follow the same lines, we repeat the details for the sake of completeness.

*Proof of Lemma 8.* By a straightforward computation we have the following estimates of Brezis-Nirenberg type (see [3], [6])

$$\|u_{\varepsilon}^i\|_p^2 = \|U\|_p^2 \varepsilon^{\frac{n}{p} - (n-2)} + CK_1(\varepsilon) \|U\|_p^{2-p} \varepsilon^{\frac{(n-2)p}{2} - \frac{n}{2} + \frac{n}{p} - (n-2)}$$

where  $K_1(\varepsilon) = C|\log \varepsilon|^{(n-2)p-n}$  and

$$(28) \quad \|\nabla u_{\varepsilon}^i\|_2^2 = \|\nabla U\|_2^2 \varepsilon^{-\frac{n-2}{2}} + \begin{cases} C|\log \varepsilon|^{n-2} + o(|\log \varepsilon|^{n-2}), & n \geq 5 \\ C|\log \varepsilon|^2 (\log(2|\log \varepsilon|)) + O(|\log \varepsilon|^2), & n = 4 \\ C|\log \varepsilon|^2 + o(|\log \varepsilon|^2), & n = 3. \end{cases}$$

Moreover if  $x_0 = (3 - \frac{1}{|\log \varepsilon|}, 0, \dots, 0)$

$$\begin{aligned} \int_{\Omega} \Psi_{\alpha}(|x|) |u_{\varepsilon}^0(x)|^p dx &\geq \left| \left( 3 - \frac{2}{|\log \varepsilon|} \right) - 2 \right|^{\alpha} \int_{\Omega} |u_{\varepsilon}^0(x)|^p dx \\ &= \left( 1 - \frac{2}{|\log \varepsilon|} \right)^{\alpha} \int_{\Omega} |u_{\varepsilon}^0(x)|^p dx \end{aligned}$$

and if  $x_1 = (1 + \frac{1}{|\log \varepsilon|}, 0, \dots, 0)$  we have

$$\begin{aligned} \int_{\Omega} \Psi_{\alpha}(|x|) |u_{\varepsilon}^1(x)|^p dx &\geq \left| \left( 1 + \frac{2}{|\log \varepsilon|} \right) - 2 \right|^{\alpha} \int_{\Omega} |u_{\varepsilon}^1(x)|^p dx \\ &= \left( 1 - \frac{2}{|\log \varepsilon|} \right)^{\alpha} \int_{\Omega} |u_{\varepsilon}^1(x)|^p dx. \end{aligned}$$

Therefore for  $n \geq 5$  we get

$$(29) \quad \begin{aligned} &\lim_{p \rightarrow 2^*} \frac{\int_{\Omega} |\nabla u_{\varepsilon}^i|^2 dx}{\left( \int_{\Omega} \Psi_{\alpha}(x) |u_{\varepsilon}^i(x)|^p dx \right)^{2/p}} \\ &\leq \lim_{p \rightarrow 2^*} \frac{1}{\left( 1 - \frac{2}{|\log \varepsilon|} \right)^{2\alpha/p}} \frac{\|\nabla U\|_2^2 \varepsilon^{-\frac{(n-2)}{2}} + C|\log \varepsilon|^{n-2} + o(|\log \varepsilon|^{n-2})}{\|U\|_p^2 \varepsilon^{\frac{n}{p} - (n-2)} + CK_1(\varepsilon) \|U\|_p^{2-p} \varepsilon^{\frac{(n-2)p}{2} - \frac{n}{2} + \frac{n}{p} - (n-2)}} \\ &= \frac{1}{\left( 1 - \frac{2}{|\log \varepsilon|} \right)^{2\alpha/2^*}} \frac{\|\nabla U\|_2^2 \varepsilon^{-\frac{(n-2)}{2}} + C|\log \varepsilon|^{n-2} + o(|\log \varepsilon|^{n-2})}{\|U\|_{2^*}^2 \varepsilon^{-\frac{n-2}{2}} + C|\log \varepsilon|^n} \\ &= \frac{1}{\left( 1 - \frac{2}{|\log \varepsilon|} \right)^{2\alpha/2^*}} \frac{\|\nabla U\|_2^2 + C(\varepsilon^{\frac{1}{2}} |\log \varepsilon|)^{n-2} + o(|\log \varepsilon|^{n-2} \varepsilon^{\frac{n-2}{2}})}{\|U\|_{2^*}^2 + C(\varepsilon^{\frac{1}{2}} |\log \varepsilon|)^n} \\ &= \frac{\|\nabla U\|_2^2}{\|U\|_{2^*}^2} + K(\varepsilon). \end{aligned}$$

On the other hand

$$\frac{\int_{\Omega} |\nabla u_{\varepsilon}^i|^2 dx}{\left(\int_{\Omega} \Psi_{\alpha} |u_{\varepsilon}^i|^p dx\right)^{\frac{2}{p}}} \geq \frac{\int_{\Omega} |\nabla u_{\varepsilon}^i|^2 dx}{\left(\int_{\Omega} |u_{\varepsilon}^i|^p dx\right)^{\frac{2}{p}}}$$

and

$$\lim_{p \rightarrow 2^*} \frac{\int_{\Omega} |\nabla u_{\varepsilon}^i|^2 dx}{\left(\int_{\Omega} \Psi_{\alpha} |u_{\varepsilon}^i|^p dx\right)^{\frac{2}{p}}} \geq \frac{\|\nabla U\|_2^2}{\|U\|_{2^*}^2} + K(\varepsilon).$$

Similar estimates hold for  $n = 3$  and  $n = 4$ .  $\square$

*Proof of Theorem 6.* We split the proof in two steps.

**Step 1.** We claim that,

$$(30) \quad \lim_{p \rightarrow 2^*} \frac{\int_{\Omega} |\nabla u_{\alpha,p}|^2 dx}{\left(\int_{\Omega} \Psi_{\alpha} |u_{\alpha,p}|^p dx\right)^{2/p}} = S_{0,2^*} \quad \text{and} \quad \lim_{p \rightarrow 2^*} \frac{\int_{\Omega} |\nabla u_{\alpha,p}|^2 dx}{\left(\int_{\Omega} |u_{\alpha,p}|^{2^*} dx\right)^{2/2^*}} = S_{0,2^*}.$$

Indeed from the definition of  $S_{0,2^*}$  and Hölder inequality we get:

$$S_{0,2^*} \leq \frac{\int_{\Omega} |\nabla u_{p,\alpha}|^2 dx}{\left(\int_{\Omega} |u_{p,\alpha}|^{2^*} dx\right)^{2/2^*}} \leq |\Omega|^{\frac{(2^*-p)^2}{2^*p}} \frac{\int_{\Omega} |\nabla u_{p,\alpha}|^2 dx}{\left(\int_{\Omega} |u_{p,\alpha}|^p dx\right)^{2/p}}.$$

Then, since the weight satisfies  $\Psi_{\alpha}(x) \leq 1$ ,

$$\frac{\int_{\Omega} |\nabla u_{p,\alpha}|^2 dx}{\left(\int_{\Omega} |u_{p,\alpha}|^p dx\right)^{2/p}} \leq \frac{\int_{\Omega} |\nabla u_{p,\alpha}|^2 dx}{\left(\int_{\Omega} \Psi_{\alpha} |u_{p,\alpha}|^p dx\right)^{2/p}} \leq \frac{\int_{\Omega} |\nabla u_{\varepsilon}^i|^2 dx}{\left(\int_{\Omega} \Psi_{\alpha} |u_{\varepsilon}^i|^p dx\right)^{2/p}},$$

where in the last inequality we used the fact that  $u_{\alpha,p}$  is a minimum. Thanks to Lemma 8 the right hand side of the inequalities goes to  $S_{0,2^*}$  and this proves the claim.

**Step 2.** Without loss of generality we set  $\|u_{p,\alpha}\|_{2^*} = 1$ . From the previous step for any subsequences  $p_k \rightarrow 2^*$  as  $k \rightarrow +\infty$   $\int_{\Omega} |\nabla u_{\alpha,p_k}|^2 dx \rightarrow S_{0,2^*}$ . So the subsequence  $u_{p_k}$  is bounded in  $H_0^1$ . By the concentration compactness principle (see [11]), there exist nonnegative measures  $\mu$  and  $\nu$  on  $\mathbb{R}^N$ , a function  $u \in H_0^1$  and an at most countable set  $J$  such that as  $k \rightarrow +\infty$

$$u_{p_k} \rightarrow u \quad \text{weakly in } H_0^1;$$

$$|\nabla u_{p_k}|^2 \rightarrow \mu \quad \text{in the sense of measure on } \mathbb{R}^N;$$

$$|u_{p_k}|^{2^*} \rightarrow \nu \quad \text{in the sense of measure on } \mathbb{R}^N;$$

and

$$\nu = |u|^{2^*} + \sum_{j \in J} \nu_j \delta_{x_j}$$

$$\mu \geq |\nabla u|^2 + \sum_{j \in J} \mu_j \delta_{x_j}$$

$$\mu_j \geq S_{0,2^*} \nu_j^{2/2^*}$$

where  $x_j \in \mathbb{R}^N$ ,  $\delta_{x_j}$  is the Dirac measure at  $x_j$  and  $\mu_j, \nu_j$  are positive constants. The convergence in the sense of measure reads as follows: for any  $\varphi \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ , as  $k \rightarrow +\infty$

$$\int_{\mathbb{R}^N} \varphi |\nabla u_{p_k}|^2 dx \rightarrow \int_{\mathbb{R}^N} \varphi d\mu$$

and

$$\int_{\mathbb{R}^N} \varphi |u_{p_k}|^{2^*} dx \rightarrow \int_{\mathbb{R}^N} \varphi d\nu.$$



When  $\varphi(x) \equiv 1$  we get

$$\int_{\mathbb{R}^N} |\nabla u_{p_k}|^2 dx \rightarrow \int_{\mathbb{R}^N} d\mu = \mu(\mathbb{R}^N)$$

and

$$\int_{\mathbb{R}^N} |u_{p_k}|^{2^*} dx \rightarrow \int_{\mathbb{R}^N} d\nu = \nu(\mathbb{R}^N).$$

Now we prove that  $J$  is not empty. Suppose on the contrary that  $J$  is empty then

$$\nu = |u|^{2^*}, \quad \int_{\mathbb{R}^N} |u_{p_k}|^{2^*} dx \rightarrow \int_{\mathbb{R}^N} |u|^{2^*} dx$$

and

$$S_{0,2^*} = \lim_{k \rightarrow +\infty} \int_{\Omega} |\nabla u_{p_k}|^2 dx = \mu(\mathbb{R}^N) \geq \int_{\Omega} |\nabla u|^2 dx.$$

Finally since  $\frac{\int_{\Omega} |\nabla u|^2 dx}{(\int_{\Omega} |u|^{2^*} dx)^{2/2^*}} \geq S_{0,2^*}$  we have that  $S_{0,2^*} = \int_{\Omega} |\nabla u|^2 dx$ . But this is impossible since the critical Sobolev constant  $S_{0,2^*}$  is not achieved in any bounded domain.

Moreover we have that  $u \equiv 0$ . Assume on the contrary that  $u \not\equiv 0$ , so that  $0 < \sum_{j \in J} \nu_j < 1$ . By

$$\begin{aligned} S_{0,2^*} &= \lim_{k \rightarrow +\infty} \|\nabla u_{p_k}\|_2^2 \geq \int_{\Omega} |\nabla u|^2 dx + \sum_{j \in J} S_{0,2^*} \nu_j^{2/2^*}, \\ 1 &= \nu(\mathbb{R}^N) = \int_{\Omega} |u|^{2^*} dx + \sum_{j \in J} \nu_j \end{aligned}$$

we have

$$\int_{\Omega} |\nabla u|^2 dx \leq S_{0,2^*} - \sum_{j \in J} S_{0,2^*} \nu_j^{2/2^*} < S_{0,2^*} (1 - \sum_{j \in J} \nu_j)^{2/2^*} = S_{0,2^*} \left( \int_{\Omega} |u|^{2^*} dx \right)^{2/2^*},$$

which is impossible. Hence  $u \equiv 0$ . Finally we prove that  $J$  is a single point set. Assume on the contrary that  $J$  contains at least two points. Since  $u = 0$  we have

$$S_{0,2^*} = \lim_{k \rightarrow +\infty} \|\nabla u_{p_k}\|_2^2 \geq \sum_{j \in J} S_{0,2^*} \nu_j^{2/2^*}$$

and

$$1 = \nu(\mathbb{R}^N) = \sum_{j \in J} \nu_j.$$

So

$$0 \leq S_{0,2^*} - \sum_{j \in J} S_{0,2^*} \nu_j^{2/2^*} < S_{0,2^*} (1 - \sum_{j \in J} \nu_j) = 0$$

which is impossible. Finally we prove that  $x_0 \in \partial\Omega$ . Suppose on the contrary that  $x_0 \in \Omega$ . Then there exists  $c \in (0, 1)$  such that  $\text{dist}(x_0, \partial\Omega) = c + r, r > 0$ . Therefore thanks to the concentration property of the solution we have

$$\int_{\Omega \setminus B(x_0, r)} \Psi_{\alpha} |u_{p_k}|^{p_k} dx \rightarrow 0$$

so that, by (30),

$$\begin{aligned}
S_{0,2^*} &= \lim_{k \rightarrow +\infty} \frac{\int_{\Omega} |\nabla u_{p_k}|^2 dx}{\left(\int_{\Omega} \Psi_{\alpha} |u_{p_k}|^{p_k} dx\right)^{2/p_k}} \\
&= \lim_{k \rightarrow +\infty} \frac{\int_{\Omega} |\nabla u_{p_k}|^2 dx}{\left(\int_{\Omega \setminus B(x_0, r)} \Psi_{\alpha} |u_{p_k}|^{p_k} dx + \int_{B(x_0, r)} \Psi_{\alpha} |u_{p_k}|^{p_k} dx\right)^{2/p_k}} \\
&\geq \lim_{k \rightarrow +\infty} \frac{\int_{\Omega} |\nabla u_{p_k}|^2 dx}{(1-c)^{2\alpha/p_k} \left(\int_{B(x_0, r)} |u_{p_k}|^{p_k} dx\right)^{2/p_k}} \\
&\geq \frac{1}{(1-c)^{2\alpha/2^*}} \lim_{k \rightarrow +\infty} \frac{\int_{\Omega} |\nabla u_{p_k}|^2 dx}{\left(\int_{\Omega} |u_{p_k}|^{p_k} dx\right)^{2/p_k}} = \frac{1}{(1-c)^{2\alpha/2^*}} S_{0,2^*}
\end{aligned}$$

which is impossible since  $(1-c)^{2\alpha/2^*} < 1$ .  $\square$

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