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A Stochastic Optimal Control Problem for the Heat Equation on the Halfline with Dirichlet Boundary-noise and Boundary-control

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Abstract

We consider a controlled state equation of parabolic type on the halfline $(0, +\infty)$ with boundary conditions of Dirichlet type in which the unknown is equal to the sum of the control and of a white noise in time. We study finite horizon and infinite horizon optimal control problem related by menas of backward stochastic differential equations.

1 Introduction

In this paper we study an optimal control problem for a state equation of parabolic type on the halfline $(0, +\infty)$. We stress the fact that we consider boundary conditions of Dirichlet type in which the unknown is equal to the sum of the control and of a white noise in time, namely:

$$\begin{cases} \dfrac{\partial y}{\partial s}(s,\xi) = \dfrac{\partial^2 y}{\partial \xi^2}(s,\xi) + f(s, y(s,\xi)), & s \in [t,T],\ \xi \in (0,+\infty), \\ y(t,\xi) = x(\xi), \\ y(s,0) = u_s + \dot{W}_s, \end{cases} \tag{1.1}$$

In this equation $\{W_t,\ t \geq 0\}$ is a standard real Wiener process, $y(s, \xi, \omega)$ is the unknown real-valued process and represents the state of the system; the control is given by the real-valued processes $u(s, \xi, \omega)$ acting at 0; $x : (0, +\infty) \to \mathbb{R}$.

Boundary control problems have been widely studied in the deterministic literature ([23]) and have been addressed in the stochastic case as well (see[9], [17], [20], [24]). In these works, the equation always contains noise also as a forcing term. In [8] a finite horizon optimal control problem for the stochastic heat equation with Neumann boundary conditions is treated by backward stochastic differential equations. Here we follow a similar approach but we consider the case with Dirichlet boundary conditions, and we address both the finite horizon and the infinite horizon stochastic optimal control problems. The main difficulties that we encounter in studying the control problem for the state equation with Dirichlet boundary conditions are related to the fact that the solution of equation (1.1) is not L^2-valued unlike to the case of Neumann boundary conditions. Indeed, in [5] it is shown that, if we replace Neumann by Dirichlet boundary conditions, the solution of (1.1) is well defined in a negative Sobolev space H^α, for $\alpha < -\dfrac{1}{4}$. Then in [1], see also [3], it is shown that the solution $y(t, \cdot)$ of equation (1.1) with $u = 0$ takes values in a weighted space $L^2((0, +\infty); \xi^{1+\theta} d\xi)$, nevertheless the problem was not reformulated as a stochastic evolution equation in $L^2((0,+\infty);\xi^{1+\theta}d\xi)$. The solutions are singular at the boundary, the singularity is described in [1] and [29]. The reason is that the smoothing properties of the heat equation are not strong enough to regularize a rough term such as a white noise.

In [11] equation (1.1), with $f=0$ is reformulated as an evolution equation in $L^2((0,+\infty);\xi^{1+\theta}d\xi)$ using results in [21] and in [22]. In these two papers it is shown that the Dirichlet Laplacian extends to a generator A of an analytic semigroup on $L^2((0,+\infty);\xi^{1+\theta}d\xi)$.

Here we follow [11] and in Section 2 we reformulate equation (1.1) as a stochastic evolution equation in $L^2((0,+\infty);\xi^{1+\theta}d\xi)$. Namely we rewrite it as:

$$\begin{cases} dX_s = AX_s ds + F(s,X_s)ds + BdW_s + Bu_s ds, & s\in[t,T],\\ X_t = x, \end{cases} \tag{1.2}$$

where A stands for the Laplace operator with homogeneous Dirichlet boundary conditions, which is the generator of an analytic semigroup in $L^2((0,+\infty);\xi^{1+\theta}d\xi)$ (see [21] and [22]), F is the evaluation operator corresponding to f, $B=(\lambda-A)D_\lambda$ where λ is an arbitrary positive number and D_λ is the Dirirchlet map (for more details on the abstract formulation of equation (1.1) see section 2.1).

The optimal control problem we wish to treat in this paper consists in minimizing the following finite horizon cost

$$J(t,x,u) = \mathbb{E}\int_t^T\int_0^{+\infty} \ell(s,\xi,y(s,\xi),u_s)\, d\xi\, ds + \mathbb{E}\int_0^{+\infty} \phi(\xi,y(T,\xi))\, d\xi. \tag{1.3}$$

Our purpose is not only to prove existence of optimal controls but mainly to characterize them by an optimal feedback law. To this aim first we solve (in a suitable sense) the Hamilton-Jacobi-Bellman equation; then we prove that such a solution is the value function of the control problem and allows to construct the optimal feedback law. Hamilton-Jacobi-Bellman equation can be formally written as

$$\begin{cases} \dfrac{\partial v(t,x)}{\partial t} + \mathcal{L}_t[v(t,\cdot)](x) = \Psi(t,x,\nabla v(t,x)B), & t\in[0,T],\ x\in H,\\ v(T,x) = \Phi(x). \end{cases} \tag{1.4}$$

where $\mathcal{L}_t$ is the infinitesimal generator of the Markov semigroup corresponding to the process X. We notice that $\mathcal{L}_t$ is highly degenerate, indeed $\nabla^2 f(x)$ appears only multiplied by B, and so the equation 1.4 has very poor smoothing properties.

We formulate the equation (1.4) in a mild sense, see for instance [15] and [16]. We notice that, when the state equation is linear, it is known that the semigroup $\{P_{s,t}[\,\cdot\,] : 0\le s\le t\}$ is strongly Feller, nevertheless it seems that equation (1.4) cannot be solved by a fixed point argument as, for instance, in [15] or [16], see also [8] and references therein.

We also mention here that, as it is well known, when the space is finite dimensional Hamilton-Jacobi-Bellman equations can be successfully treated using the notion of viscosity solution, see [17] for viscosity approach to boundary optimal control. The point is that, in the infinite dimensional case, very few uniqueness results are available for viscosity solutions and all of them, obtained by analytic techniques, impose strong assumptions on the operator B and on the nonlinearity Ψ, see, for instance, [17] or [30] and references within.

To solve the Hamilton-Jacobi-Bellman equation (1.4) in mild sense we follow the approach based on Forward-Backward stochastic differential equations, mainly developped, in a finite dimensional setting,in the fundamental papers [10], [27] and [28], and generalized, in infinite dimensions, in [13]. The backward stochastic differential equation is in our case

$$\begin{cases} dY_s^{t,x} = -\Psi(s, X_s^{t,x}, Z_s^{t,x})ds + Z_s^{t,x}dW_s, & s \in [t,T] \\ Y_T^{t,x} = \Phi(X_T^{t,x}) \end{cases} \tag{1.5}$$

and we need to study regular dependence of Y on the initial datum x: in order to give sense to the term $\nabla v(t,\cdot)B$ in 1.4 we have to differentiate Y in the direction $(\lambda - A)^\alpha h$.
The control problem is solved by using the probabilistic representation of the unique mild solution to equation (1.4) which also gives existence of an optimal feedback law, see Theorem 5.5.

We also treat the infinite horizon optimal control problem: minimize, over all admissible controls, the following infinite horizon cost

$$J(x,u) = \mathbb{E}\int_0^{+\infty} e^{-\mu s}\int_0^{+\infty} \ell(s,\xi,y(s,\xi),u_s)\, d\xi\, ds. \tag{1.6}$$

The controlled state y solves

$$\begin{cases} \dfrac{\partial y}{\partial s}(s,\xi) = \dfrac{\partial^2 y}{\partial \xi^2}(s,\xi) - My(s,\xi) + f(s,y(s,\xi)), & s \in [t,T],\ \xi \in (0,+\infty), \\ y(t,\xi) = x(\xi), \\ y(s,0) = u_s + \dot{W}_s, \end{cases} \tag{1.7}$$

where M has to be taken sufficiently large, see also lemma 6.1. The reason is that in the space $\mathcal{H}$ we need to treat Dirichlet boundary conditions it is not clear whether A is dissipative or not.

As in the finite horizon case we consider mild solution of the Hamilton Jacobi Bellman equation related, which this time is stationary. The main tool for solving this stationary Hamilton-Jacobi-Bellman equation is again BSDEs, where the final condition is replaced by boundedness requirements on Y, see also [14] and [19]..
As for the finite horizon case, in order to give sense to the term $\nabla v(\cdot)B$, we have to differentiate the following backward stochastic differential equation

$$dY_s^x = -\Psi(X_s^x, Z_s^x)\, ds + \mu Y_s^x\, ds + Z_s^x\, dW_s, \qquad s \geq 0, \tag{1.8}$$

where $\mu > 0$ and Ψ is the hamiltonian function defined in a classical way. To study the regularity property of equation (1.8), we use similar ideas as in [19], where differentiability with respect to x of (Y^x, Z^x), solution of an equation like (1.8) with an arbitrary $\mu > 0$, is investigated. We notice again that since we have to give sense to $\nabla_x Y^x(\lambda - A)^\alpha h$, for any $h \in \mathcal{H}$, we also need to differentiate equation (1.8) in the direction $(\lambda - A)^\alpha h$, and consequentely we have to study a BSDE with some terms unbounded in time: such a situation is not studied in [19].

The paper is structured as follows: in Section 2 we transpose the controlled state equation in the infinite dimensional framework and we study regularity properties of the solution of this (forward) state equation; in Section 3 we study the backward equation associated to the problem; in Section 4 we prove existence and uniqueness of the Hamilton-Jacobi-Bellman partial differential equation and in Section 5 we show how the previous results can be applied to perform the synthesis of the optimal control, both in a strong and weak formulation. Eventually we study the infinite horizon optimal control problem: in Section 6 we study the regularity properties of the forward-bacward equations in infinite horizon, in Section 7 we prove existence and uniquenes of the solution of the stationary Hamilton-Jacobi-Bellman equation, and in section 8 we briefly present and solve the infinite horizon optimal control problem.

2 The forward equation

In this section we introduce the "concrete" state equation, that we reformulate in an abstract sense following [11], and then we study some regularity properties.

2.1 Reformulation of the state equation

We consider the following stochastic semilinear heat equation with control and noise on the boundary:

$$\begin{cases} \dfrac{\partial y}{\partial s}(s,\xi) = \dfrac{\partial^2 y}{\partial \xi^2}(s,\xi) + f(s,y(s,\xi)), & s \in [t,T],\ \xi \in (0,+\infty), \\ y(t,\xi) = x(\xi), \\ y(s,0) = u_s + \dot{W}_s, \end{cases} \tag{2.1}$$

In this equation $\{W_t,\ t \geq 0\}$, is a standard real Wiener process; $y(s,\xi,\omega)$ is the unknown real-valued process and represents the state of the system; the control is given by the real-valued process $u(s,\xi,\omega)$ which belongs to the class of admissible controls $\mathcal{U}$, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $x : [0,+\infty) \to \mathbb{R}$.

It is our purpose to write the state equation as an evolution equation in the space $\mathcal{H} = L^2((0,+\infty);\xi^{1+\theta}d\xi)$, or in the space $L^2((0,+\infty);(\xi^{1+\theta} \wedge 1)d\xi)$, that we also denote by $\mathcal{H}$. The parameter $\theta \in (0,1)$. On equation (2.1) we assume that

Hypothesis 2.1 *1) The function $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is measurable, for every $t \in [0,T]$ the function $f(t,\cdot) : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and there exists a constant C_f such that*

$$|f(t,0)| + \left| \frac{\partial f}{\partial r}(t,r) \right| \leq C_f, \qquad t \in [0,T],\ r \in \mathbb{R}.$$

2) The initial condition $x(\cdot)$ belongs to $\mathcal{H}$.

3) The set of admissible control actions $\mathcal{U}$ is a bounded closed subset of $\mathbb{R}$.

Equation 1.1, in the case of $f = 0$, is reformulated as an evolution equation in $\mathcal{H}$ in [11] and we follow that approach. Let us denote by A the Laplacian operator with Dirichlet boundary conditions: it is proved in [22] that the strongly continuous heat semigroup generated in $L^2((0,+\infty))$ by A extends to a bounded C_0 semigroup $(e^{tA})_{t\geq 0}$ in $\mathcal{H}$ with generator still denoted by $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$. The semigroup $(e^{tA})_{t\geq 0}$ is analytic. So, for every $\beta > 0$,

$$\|(\lambda - A)^\beta e^{tA}\| \leq C_\beta t^{-\beta} \qquad \text{for all} \qquad t \geq 0. \tag{2.2}$$

Let us also introduce the Dirichlet map: for given $\lambda > 0$, let $D_\lambda : \mathbb{R} \to \mathcal{H}$ be such that, for $a \in \mathbb{R}$, $D_\lambda(a) = a\psi_\lambda$, where $\psi_\lambda : \mathbb{R} \to \mathbb{R}^+$, $\psi_\lambda : \xi \mapsto e^{-\lambda\xi}$. In the following proposition we collect some results contained in [11]. From now on $\lambda > 0$ is fixed.

Proposition 2.2 *For all $\alpha \in [0, \frac{1}{2} + \frac{\theta}{4})$, $\psi_\lambda \in D((\lambda - A)^\alpha)$, and in particular $D_\lambda \in \mathcal{L}(\mathbb{R}; D((\lambda - A)^\alpha))$. So the operator*

$$B := (\lambda - A)D_\lambda : \mathbb{R} \to \mathcal{H}^{\alpha-1} \tag{2.3}$$

is bounded, and for every $t > 0$ the operator

$$(\lambda - A)e^{tA}D_\lambda = (\lambda - A)^{1-\alpha}e^{tA}(\lambda - A)^\alpha D_\lambda : \mathbb{R} \to \mathcal{H} \tag{2.4}$$

is bounded as well. From now on let $\alpha \in (\frac{1}{2}, \frac{1}{2} + \frac{\theta}{4})$. For all $\gamma < 2\alpha - 1$, the following holds *i)* *For each $t > 0$, the operator $e^{tA}B : \mathbb{R} \to \mathcal{H}$ is bounded and the function $t \mapsto e^{tA}Ba$ is continuous $\forall\ a \in \mathbb{R}$.*

ii)

$$\int_0^T s^{-\gamma} \|e^{sA}B\|_{HS}^2 ds < +\infty \tag{2.5}$$

where $\|\cdot\|_{HS}$ stands for the Hilbert-Schmidt norm.

iii) *For every $0 \leq t < T$ the stochastic convolution*

$$W_A(s) = \int_t^s e^{(s-r)A} B dW_r, \qquad s \in [t,T] \tag{2.6}$$

is well defined, belongs to $L^2(\Omega; C([t,T],\mathcal{H}))$ and has continuous trajectories in $\mathcal{H}$.

iv) *For every $0 \leq t < T$ and $u \in L^2_{\mathcal{P}}(\Omega \times [0,T]; \mathbb{R})$*

$$I_s = \int_t^s e^{(s-r)A} B u_r dr, \qquad s \in [t,T] \tag{2.7}$$

is well defined in $L^2_{\mathcal{P}}(\Omega\times[0,T];\mathbb{R})$. Moreover $I \in L^2(\Omega; C([t,T],\mathcal{H}))$ and $\|I\|_{L^2(\Omega;C([t,T],\mathcal{H}))} \leq C\|u\|_{L^2_{\mathcal{P}}(\Omega\times[0,T];\mathbb{R})}$.

We want to rewrite equation (2.1) as en evolution equation in $\mathcal{H}$. The state will be denoted by $X_s^u = y(s,\cdot)$. Thus $\{X_s^u, s \in [t,T]\}$ is a process in $\mathcal{H}$ and the initial condition is assumed to belong to $\mathcal{H}$. Equation (2.1), in the case of $f = 0$, can now be reformulated as

$$\begin{cases} dX_s^u = AX_s^u ds + Bu_s ds + BdW_s & s \in [t,T], \\ X_t^u = x, \end{cases} \tag{2.8}$$

Definition 2.1 *An $\mathcal{H}$-valued predictable process X is called a mild solution to equation (2.8) on $[0,T]$ if*

$$\mathbb{P}\int_0^T |X_r^u|^2 dr < +\infty$$

and, for every $0 \leq t < T$, X satisfies the integral equation

$$X_s^u = e^{(s-t)A} + \int_t^s e^{(s-r)A} B u_r dr + \int_t^s e^{(s-r)A} B dW_r$$

Following [11], theorem 2.6, we state the following:

Theorem 2.3 *Assume that hypothesis 2.1 holds true, then equation (2.8) has, according to definition 2.1, a unique mild solution $X \in L^2(\Omega; C([t,T],\mathcal{H}))$. Moreover if $u = 0$ then X is a Markov process in $\mathcal{H}$.*

Next we want to give an abstract reformulation in $\mathcal{H}$ of the semilinear equation (2.1). We define $F : [0,T] \times \mathcal{H} \to \mathcal{H}$ setting for $s \in [0,T]$ and $X \in \mathcal{H}$

$$F(s,X)(\xi) = f(s, X(\xi)) \tag{2.9}$$

By hypothesis 2.1, point 1), it turns out that $F : [0,T] \times \mathcal{H} \to \mathcal{H}$ is a measurable function and

$$|F(t,0)| + |F(t,x_1) - F(t,x_2)| \leq C_f(1 + |x_1 - x_2|), \qquad t \in [0,T],\ x_1, x_2 \in \mathcal{H}.$$

Moreover, for every $t \in [0,T]$, $F(t,\cdot)$ has a Gâteaux derivative $\nabla_x F(t,x)$ at every point $x \in \mathcal{H}$, and we get that $|\nabla F(t,x)| \leq C_f$. Finally, the function $(x,h) \to \nabla F(t,x)h$ is continuous as a map $\mathcal{H} \times \mathcal{H} \to \mathbb{R}$. Note that we consider $\nabla F(t,x)$ as an element of $\mathcal{H}^*$ and we denote its action on $h \in \mathcal{H}$ by $\nabla F(t,x)h$. Equation (2.1) can now be reformulated as

$$\begin{cases} dX_s^u = AX_s^u ds + F(s, X_s^u)ds + Bu_s ds + BdW_s & s \in [t,T], \\ X_t^u = x, \end{cases} \tag{2.10}$$

The equation (2.10) is formal. The precise meaning of the state equation is in the following

Definition 2.2 *An $\mathcal{H}$-valued predictable process X is called a mild solution to equation (2.10) on $[0,T]$ if*

$$\mathbb{P}\int_0^T |X_r^u|^2 dr < +\infty$$

and for every $0 \leq t < T$, X satisfies the integral equation

$$X_s^u = e^{(s-t)A}x + \int_t^s e^{(s-r)A}F(r, X_r^u)\, dr + \int_t^s e^{(s-r)A}Bu_r dr + \int_t^s e^{(s-r)A}BdW_r. \tag{2.11}$$

We now prove existence and uniqueness of a mild solution of equation (2.10)

Theorem 2.4 *Assume that hypothesis 2.1 holds true, then equation (2.10) has, according to definition 2.2, a unique mild solution $X \in L^2(\Omega; C([t,T],\mathcal{H}))$ and, if $u = 0$, X is a Markov process in $\mathcal{H}$. Moreover for every $p \in [1,\infty)$, $\alpha \in [0,\theta/4)$, $t \in [0,T]$ there exists a constant $c_{p,\alpha}$ such that*

$$\mathbb{E} \sup_{s\in(t,T]} (s-t)^{p\alpha} |X_s^{t,x}|^p_{D(-A)^\alpha} \leq c_{p,\alpha}(1 + |x|_{\mathcal{H}})^p. \tag{2.12}$$

Proof. We consider the Picard approximation scheme; for the sake of simplicity we consider $u = 0$ in equation (2.10) and we denote by $(X_s)_{s\in[t,T]}$ the solution. We define

$$X_s^0 = e^{(s-t)A}x,$$
$$X_s^{n+1} = e^{(s-t)A}x + \int_t^s e^{(s-r)A}F(r, X_r^n)\, dr + \int_t^s e^{(s-r)A}BdW_r, \qquad n \geq 0.$$

By induction it follows that for every $n \geq 0$, $X^n \in L^2(\Omega; C([t,T],\mathcal{H})$. Moreover, by equipping $L^2(\Omega; C([t,T],\mathcal{H})$ with the equivalent norm

$$\|Y\|^2_{\beta, L^2(\Omega;C([t,T],\mathcal{H})} = \mathbb{E} \sup_{s\in[t,T]} e^{-\beta s}|Y_s|^2_{\mathcal{H}}$$

it turns out that $(X^n)_n$ is a Cauchy sequence in $(L^2(\Omega, C([t,T],\mathcal{H})), \|\cdot\|^2_{\beta,L^2(\Omega;C([t,T],\mathcal{H})})$, whose limit is the unique mild solution to equation (2.10). Next we want to prove estimate (2.12): first we prove that the stochastic convolution defined in (2.6) belongs to $L^2(\Omega; C([t,T], D(-A)^\alpha))$. By the factorization method, see e.g. [6], p. 128, let $\gamma \in (0, \frac{1}{2})$: the stochastic convolution can be written as

$$W_A(s) = \frac{\sin \pi\gamma}{\pi} \int_t^s e^{(s-r)A}(s-r)^{\gamma-1} Y_r dr,$$

where

$$Y_r = \int_t^r (r-\sigma)^{-\gamma} e^{(r-\sigma)A} BdW_r.$$

Let us write, for $\beta \in (\frac{1}{2}, \frac{1}{2}+\frac{\theta}{4})$, $e^{(r-\sigma)A}B = (\lambda-A)^{1-\beta}e^{(r-\sigma)A}(\lambda-A)^{\beta}D_\lambda$, so that the stochastic convolution is given by

$$W_A(s) = \frac{\sin \pi\gamma}{\pi}\int_t^s e^{(s-r)A}(s-r)^{\gamma-1}(\lambda-A)^{1-\beta}\widehat{Y}_r dr,$$

where

$$\widehat{Y}_r = \int_t^r (r-\sigma)^{-\gamma}(\lambda-A)^{\beta}D_\lambda dW_r.$$

It turns out that $\widehat{Y} \in L^p(\Omega\times[t,T],\mathcal{H})$, and so, see e.g. [7] Proposition A.1.1, for $\alpha+1-\beta+\frac{1}{p} < \gamma < \frac{1}{2}$, $(\lambda-A)^\alpha W_A \in L^p(\Omega, C([t,T],\mathcal{H})$, where

$$(\lambda-A)^\alpha W_A(s) = \frac{\sin \pi\gamma}{\pi}\int_t^s e^{(s-r)A}(s-r)^{\gamma-1}(\lambda-A)^{\alpha+1-\beta}\widehat{Y}_r dr.$$

We can conclude that for $\alpha+1-\beta+\frac{1}{p} < \gamma < \frac{1}{2}$, i.e. for $\alpha < \frac{\theta}{4}$, and p sufficiently large, $W_A \in L^p(\Omega, C([t,T], D(\lambda-A)^\alpha)$. In a similar, and simpler way, if $u \neq 0$, we could treat the term

$$\int_t^s e^{(s-r)A}Bu_r dr.$$

For $a>0$ we denote by $\mathbb{K}_{a,\alpha,t}$ the Banach space of all predictable processes $X : \Omega\times(t,T] \to D(\lambda-A)^\alpha$ such that

$$|X|^p_{\mathbb{K}_{a,\alpha,t}} := \mathbb{E}\sup_{s\in(t,T]} e^{pas}(s-t)^{p\alpha}|X_s|^p_{D(\lambda-A)^\alpha} < +\infty$$

endowed with the above norm. We have just shown that $W_A \in \mathbb{K}_{a,\alpha,t}$. Moreover, for all $x \in H$,

$$\sup_{s\in(t,T]}(s-t)^\alpha|e^{(s-t)A}x|_{D(\lambda-A)^\alpha} \le c|x|.$$

Thus if we define for $X \in \mathbb{K}_{a,\alpha,t}$

$$\Lambda(X,t)(s) = \int_t^s e^{(s-r)A}F(r,X_r)\,dr + e^{(s-t)A}x + W_A(s),$$

it is immediate to prove that $\Lambda(X,t) \in \mathbb{K}_{a,\alpha,t}$. Moreover by straightforward estimates

$$|\Lambda(X^1,t)-\Lambda(X^2,t)|^p_{\mathbb{K}_{a,\alpha,t}} \le g^p(a)C_F^p|X^1-X^2|^p_{\mathbb{K}_{a,\alpha,t}}$$

where

$$g(a) = \sup_{t\in[0,T]} t^{1-\alpha}\int_0^1 (1-s)^{-\alpha}s^{-\alpha}e^{-ats}ds.$$

By the Cauchy-Schwartz inequality $g(a) \le T^{1/2-\alpha}a^{-1/2}\left(\int_0^1(1-s)^{-2\alpha}s^{-2\alpha}ds\right)^{1/2}$ thus if a is large enough $\Lambda(\cdot,t)$ is a contraction in $\mathbb{K}_{a,\alpha,t}$. The unique fixed point is clearly a mild solution of equation (2.10) and (2.12) holds. Uniqueness is an immediate consequence of the Gronwall lemma. □

It is also useful to consider the uncontrolled version of equation (2.10) namely:

$$X_s = e^{(s-t)A}x + \int_t^s e^{(s-r)A}F(r,X_r)\,dr + \int_t^s e^{(s-r)A}B\,dW_r, \qquad s\in[t,T]. \tag{2.13}$$

We will refer to (2.13) as the forward equation.

2.2 Regular dependence on initial conditions.

In this section we consider again the solution of the forward equation (2.13), i.e. of the uncontrolled state equation on the time interval $[t,T]$ with initial condition $x \in \mathcal{H}$. It will be denoted by $X_s^{t,x}$, to stress dependence on the initial data t and x. It is also convenient to extend the process $X_\cdot^{t,x}$ letting $X_s^{t,x} = x$ for $s \in [0,t]$. In a similar way, we extend also the stochastic convolution by setting $W_A(s) = 0$ for $s \in [0,t)$. From now on we assume that Hypothesis 2.1 holds.
We study the dependence of the process $\{X_s^{t,x},\ s \in [0,T]\}$ on the parameters t,x.

Proposition 2.5 *For any $p \geq 1$ the following holds:*

1. *the map $(t,x) \to X_\cdot^{t,x}$ defined on $[0,T] \times \mathcal{H}$ and with values in $L^p_{\mathcal{P}}(\Omega; C([0,T];\mathcal{H}))$ is continuous.*
2. *For every $t \in [0,T]$ the map $x \to X_\cdot^{t,x}$ has, at every point $x \in \mathcal{H}$, a Gâteaux derivative $\nabla_x X_\cdot^{t,x}$. The map $(t,x,h) \to \nabla_x X_\cdot^{t,x} h$ is continuous as a map $[0,T] \times \mathcal{H} \times \mathcal{H} \to L^p_{\mathcal{P}}(\Omega; C([0,T];\mathcal{H}))$ and, for every $h \in \mathcal{H}$, the following equation holds $\mathbb{P}$-a.s.:*

$$\nabla_x X_s^{t,x} h = e^{(s-t)A} h + \int_t^s e^{(s-\sigma)A} \nabla_x F(\sigma, X_\sigma^{t,x}) \nabla_x X_\sigma^{t,x}\, d\sigma, \qquad s \in [t,T], \tag{2.14}$$

and $\nabla_x X_s^{t,x} h = h$ for $s \in [0,t]$.

Proof. We start by proving continuity. We begin considering the stochastic convolution: we know that $\int_t^s e^{(s-r)A} B dW_r = W_A(s) \in L^p_{\mathcal{P}}(\Omega, C([0,T];\mathcal{H}))$ and we have to prove that the map $t \to \int_t^s e^{(s-r)A} B dW_r$ is continuous with values in $L^p_{\mathcal{P}}(\Omega, C([0,T];\mathcal{H}))$. Fix $t \in [0,T]$, $\beta \in (\frac{1}{2}, \frac{1}{2} + \frac{\theta}{4})$ and let $t_n \to t^+$, (in a similar way if $t_n \to t^-$)

$$\begin{aligned}
\mathbb{E} \sup_{s\in[t,T]} \left| \int_t^{s\wedge t_n} e^{(s-\sigma)} B dW_\sigma \right|_{\mathcal{H}}^p &\leq \mathbb{E} \sup_{s\in[t,T]} \left| \int_t^{s\wedge t_n} (\lambda - A)^{1-\beta} e^{(s-\sigma)A} (\lambda - A)^\beta D_\lambda dW_\sigma \right|_{\mathcal{H}}^p \\
&\leq \sup_{s\in[t,T]} \left(\int_t^{s\wedge t_n} \left| (\lambda - A)^{1-\beta} e^{(s-\sigma)} (\lambda - A)^\beta D_\lambda \right|_{\mathcal{H}}^2 d\sigma \right)^{\frac{p}{2}} \\
&\leq C \sup_{s\in[t,T]} \left(\int_t^{s\wedge t_n} (s-\sigma)^{2(1-\beta)} d\sigma \right)^{p/2} \to 0.
\end{aligned}$$

Similarly if we extend $e^{(s-t)A} x = x$ for $s < t$ then

$$\sup_{s\in[0,T]} \left| e^{(s-t_n)A} x - e^{(s-t)A} x \right| \to 0$$

as $t_n \to t$; moreover the map $x \to e^{(\cdot - t)A} x$ considered with values in $C([0,T],\mathcal{H})$ is clearly continuous in x uniformly in t.

Now let $t_n \to t^+$ and $x_n \to x$:

$$\begin{aligned}
&\mathbb{E} \sup_{s\in[0,T]} |X_s^{t_n,x_n} - X_s^{t,x}|_{\mathcal{H}}^p \\
&\leq C \sup_{s\in[0,T]} \left| e^{(s-t_n)A}x - e^{(s-t)A}x \right|_{\mathcal{H}}^p + C\mathbb{E} \sup_{s\in[0,T]} \left| \int_t^{s\wedge t_n} e^{(s-\sigma)} B dW_\sigma \right|_{\mathcal{H}}^p \\
&+ C\mathbb{E} \sup_{s\in[0,T]} \left| \int_{t_n}^s e^{(s-\sigma)A} F(\sigma, X_\sigma^{t_n,x_n}) d\sigma - \int_t^s e^{(s-\sigma)A} F(\sigma, X_\sigma^{t,x}) d\sigma \right|_{\mathcal{H}}^p \\
&\leq \epsilon(|x_n - x|_{\mathcal{H}}, |t_n - t|) + + C_{F,T} \int_{t_n}^t |X_\sigma^{t,x} - X_\sigma^{t_n,x_n}|^p d\sigma + C_T (t_n - t)^{1/p} (1 + |x|_{\mathcal{H}}^p),
\end{aligned}$$

where $\epsilon(|x_n - x|_{\mathcal{H}}, |t_n - t|) := \sup_{s\in[0,T]} \left| e^{(s-t_n)A}x - e^{(s-t)A}x \right| + \mathbb{E} \sup_{s\in[0,T]} \left| \int_t^{s\wedge t_n} e^{(s-\sigma)} B dW_\sigma \right|_{\mathcal{H}}^p$ and $C_{F,T}$ is a constant that depends on F and on T. By the Gronwall lemma

$$\mathbb{E}(\sup_{s\in[0,T]} |X_s^{t_n,x_n} - X_s^{t,x}|_{\mathcal{H}}^p \to 0$$

as $t_n \to t^+$ and $x_n \to x$.
The proof of differentiability is similar to the proof of proposition 3.1 in [8], and we omit it. □

Proposition 2.6 *For every $\alpha \in [0, 1)$ there exists a family of predictable processes $\{\Theta^\alpha(\cdot, t, x)h : h \in H, x \in H, t \in [0, T]\}$ all defined on $\Omega \times [0, T] \to H$ such that the following holds:*

1. *the map $h \to \Theta^\alpha(\cdot, t, x)h$ is linear and, if $h \in D(\lambda - A)^\alpha$, then*

$$\Theta^\alpha(s, t, x)h = \begin{cases} \left(\nabla_x X_s^{t,x} - e^{(s-t)A}\right)(\lambda - A)^\alpha h & \text{if } s \in [t, T], \\ 0 & \text{if } s \in [0, t). \end{cases} \tag{2.15}$$

2. *the map $(t, x, h) \to \Theta^\alpha(\cdot, t, x)h$ is continuous $[0, T] \times \mathcal{H} \times \mathcal{H} \to L^\infty_{\mathcal{P}}(\Omega; C([0, T]; \mathcal{H}))$.*

3. *there exists a constant $C_{\theta,\alpha}$ such that*

$$|\Theta^\alpha(\cdot, t, x)h|_{L^\infty_{\mathcal{P}}(\Omega, C([0,T];\mathcal{H}))} \leq C_{\theta,\alpha}|h| \text{ for all } t \in [0, T],\ x, h \in \mathcal{H}. \tag{2.16}$$

Proof: For fixed $t \in [0, T]$ and $x, h \in \mathcal{H}$ consider the equation:

$$\begin{aligned} \Theta^\alpha(s, t, x)h &= \int_t^s e^{(s-\sigma)A} \nabla_x F(\sigma, X_\sigma^{t,x}) \Theta^\alpha(\sigma, t, x) h d\sigma \\ &\quad + \int_t^s e^{(s-\sigma)A} \nabla_x F(\sigma, X_\sigma^{t,x}) (\lambda - A)^\alpha e^{(\sigma-t)A} h \, d\sigma. \end{aligned} \tag{2.17}$$

Notice that

$$\int_t^s \left| e^{(s-\sigma)A} \nabla_x F(\sigma, X_\sigma^{t,x}) (\lambda - A)^\alpha e^{(\sigma-t)A} h \right| d\sigma \leq C_f \int_t^s (\sigma - t)^{-\alpha} |h| d\sigma \leq c|h|$$

for a suitable constant c.

Since $\nabla_x F$ bounded it is immediate to prove that equation (2.17) has $\mathbb{P}$-almost surely a unique solution in $C([t, T]; \mathcal{H})$. Moreover extending $\Theta^\alpha(s, t, x)h = 0$ for $s < t$ and considering it as a process we have $\Theta(\cdot, t, x)h \in L^\infty_{\mathcal{P}}(\Omega, C([0, T]; \mathcal{H}))$ and $|\Theta^\alpha(\cdot, t, x)h|_{L^\infty_{\mathcal{P}}(\Omega, C([0,T];\mathcal{H}))} \leq C_\alpha|h|$.

The continuity with respect to t, x and h can be easily shown as in the proof of the previous Proposition. Moreover linearity in h is straight-forward. Finally for all $k \in D(\lambda - A)^\alpha$ setting $h = (\lambda - A)^\alpha k$ equation (2.14) can be rewritten:

$$\begin{aligned}&\Big(\nabla_x X(s,t,x)(\lambda - A)^\alpha k - e^{(s-t)A}(\lambda - A)^\alpha k\Big) = \\ &\quad + \int_t^s e^{(s-\sigma)A} \nabla_x F(\sigma, X(\sigma,t,x)) e^{(\sigma-t)A}(\lambda - A)^\alpha k d\sigma \\ &\quad + \int_t^s e^{(s-\sigma)A} \nabla_x F(\sigma, X(\sigma,t,x)) \Big(\nabla_x X(\sigma,t,x)(\lambda - A)^\alpha k - e^{(\sigma-t)A}(\lambda - A)^\alpha k\Big)\, d\sigma.\end{aligned}$$

Comparing the above equation with equation (2.17) by the Gronwall Lemma we get $\Theta^\alpha(s,t,x)k = \Big(\nabla_x X(s,t,x) - e^{(s-t)A}\Big)(\lambda - A)^\alpha k$ $\mathbb{P}$-a.s. for all $s \in [t,T]$. □

2.3 Regularity in the Malliavin sense.

In order to state the following results we need to recall some basic definitions from the Malliavin calculus, mainly to fix notation. We refer the reader to the book [25] for a detailed exposition; the paper [18] treats the extensions to Hilbert space valued random variables and processes.

For every $h \in L^2([0,T];\mathbb{R})$ we denote

$$W(h) = \int_0^T h(t)\, dW(t).$$

Given a Hilbert space K, let S_K be the set of K-valued random variables F of the form

$$F = \sum_{j=1}^m f_j(W(h_1), \ldots, W(h_n)) e_j,$$

where $h_1, \ldots, h_n \in L^2([0,T];\mathbb{R})$, $\{e_j\}$ is a basis of K and $f_1, \ldots f_m$ are infinitely differentiable functions $\mathbb{R}^n \to \mathbb{R}$ bounded together with all their derivatives. The Malliavin derivative DF of $F \in S_K$ is defined as the process $\{D_s, s \in [0,T]\}$, where

$$D_s F = \sum_{j=1}^m \sum_{k=1}^n \partial_k f_j(W(h_1), \ldots, W(h_n))\, h_k^i(s)\, e_j.$$

By ∂_k we denote the partial derivative with respect to the k-th variable. DF is a process with values in K, that we will identify with an element of $L^2(\Omega \times [0,T];K)$ with the norm:

$$\|DF\|^2_{L^2(\Omega\times[0,T];K)} = \mathbb{E}\int_0^T \|D_s F\|_K^2\, ds.$$

It is known that the operator $D : S_K \subset L^2(\Omega;K) \to L^2(\Omega \times [0,T];K)$ is closable. We denote by $\mathbb{D}^{1,2}(K)$ the domain of its closure, and use the same letter to denote D and its closure:

$$D : \mathbb{D}^{1,2}(K) \subset L^2(\Omega;K) \to L^2(\Omega \times [0,T];K).$$

The adjoint operator of D,

$$\delta :\ \text{dom}\,(\delta) \subset L^2(\Omega \times [0,T];K) \to L^2(\Omega;K),$$

is called Skorohod integral. It is known that dom(δ) contains $L^2_{\mathcal{P}}(\Omega; L^2([0,T];K))$ and the Skorohod integral of a process in this space coincides with the Itô integral. The class $\mathbb{L}^{1,2}(K)$ is also contained in $\mathrm{dom}(\delta)$, the latter being defined as the space of processes $u \in L^2(\Omega\times[0,T];K)$ such that $u_r \in \mathbb{D}^{1,2}(K)$ for a.e. $r\in[0,T]$ and there exists a measurable version of $D_s u_r$ satisfying

$$\|u\|^2_{\mathbb{L}^{1,2}(K)} = \|u\|^2_{L^2(\Omega\times[0,T];K)} + \mathbb{E}\sum_{i=1}^{2}\int_0^T\int_0^T \|D^i_s u_r\|^2_K\, dr\, ds < \infty.$$

Moreover, $\|\delta(u)\|^2_{L^2(\Omega;K)} \le \|u\|^2_{\mathbb{L}^{1,2}(K)}$. The definition of $\mathbb{L}^{1,2}(K)$ for an arbitrary Hilbert space K is entirely analogous; clearly, $\mathbb{L}^{1,2}(K)$ is isomorphic to $L^2([0,T];\mathbb{D}^{1,2}(K))$.

Finally we recall that if $F\in\mathbb{D}^{1,2}(K)$ is $\mathcal{F}_t$-adapted then $DF=0$ a.s. on $\Omega\times(t,T]$.

Now for (t,x) fixed let us consider again the process $\{X^{t,x}_s, s\in[t,T]\}$ solution of the forward equation (2.13). It will be denoted simply by $\{X_s, s\in[t,T]\}$ or even X. We still agree that $X_s = x$ for $s\in[0,t)$. We will soon prove that X belongs to $\mathbb{L}^{1,2}(\mathcal{H})$. Then it is clear that the equality $D_\sigma X_s = 0$ $\mathbb{P}$-a.s. holds for a.a. σ, t, s if $s<t$ or $\sigma > s$.

In the rest of this section we still assume that Hypothesis 2.1 holds.

Proposition 2.7 *Let $t\in[0,T]$ and $x\in\mathcal{H}$ be fixed. Then $X\in\mathbb{L}^{1,2}(\mathcal{H})$, and $\mathbb{P}$-a.s. we have, for a.a. σ, s such that $t\le\sigma\le s\le T$, and for $\beta\in(0,\frac12+\frac\theta4)$*

$$D_\sigma X_s = (\lambda - A)\, e^{(s-\sigma)A} b + \int_\sigma^s e^{(s-r)A}\nabla F(r,X_r)\, D_\sigma X_r\, dr, \tag{2.18}$$

$$|D_\sigma X_s| \le C(s-\sigma)^{\beta-1}. \tag{2.19}$$

Moreover for every $s\in[0,T]$ we have $X_s\in\mathbb{D}^{1,2}(\mathcal{H})$ and $DX_s\in L^\infty(\Omega;L^2([0,T];\mathcal{H}))$.

Finally, for every $q\in[2,\infty)$ the map $s\to X_s$ is continuous from $[0,T]$ to $L^q(\Omega;\mathcal{H})$ and the map $s\to DX_s$ is continuous from $[0,T]$ to $L^q(\Omega;L^2([0,T];\mathcal{H}))$.

Proof. For simplicity of notation we write the proof for the case $t=0$. Thus,

$$X_s = e^{sA}x + \int_0^s e^{(s-r)A}F(r,X_r)\,dr + \int_0^s e^{(s-r)A}B\,dW_r \qquad s\in[0,T]. \tag{2.20}$$

We set $J_n = n(n-A)^{-1}$ and we consider the approximating equation

$$\begin{aligned} X^n_s &= e^{sA}x + \int_0^s e^{(s-r)A}F(r,X^n_r)\,dr + \int_0^s e^{(s-r)A}J_nB\,dW_r,\\ &= e^{sA}x + \int_0^s e^{(s-r)A}F(r,X^n_r)\,dr + \int_0^s e^{(s-r)A}(\lambda-A)J_nD_\lambda\,dW_r, \qquad s\in[0,T]. \end{aligned} \tag{2.21}$$

Since $(\lambda-A)J_n$ is a linear bounded operator in $\mathcal{H}$, we can apply Proposition 3.5 of [13] and conclude that $X^n\in\mathbb{L}^{1,2}(\mathcal{H})$, and that $\mathbb{P}$-a.s. we have, for a.a. σ, s such that $0\le\sigma\le s\le T$,

$$D_\sigma X^n_s = (\lambda-A)\,e^{(s-\sigma)A}J_nD_\lambda + \int_\sigma^s e^{(s-r)A}\nabla F(r,X^n_r)\,D_\sigma X^n_r\,dr. \tag{2.22}$$

Since for $0<\beta<\frac12+\frac\theta4$

$$|(\lambda-A)\,e^{(s-\sigma)A}J_nD_\lambda| \le |(\lambda-A)^{1-\beta}e^{(s-r)A}|\,|J_n|\,|(\lambda-A)^\beta D_\lambda| \le C(s-r)^{\beta-1},$$

by the boundedness of ∇F and the Gronwall lemma it is easy to deduce that $|D_\sigma X^n_s|\le C(s-\sigma)^{\beta-1}$. In particular it follows that DX^n is bounded in the space $L^2(\Omega\times[0,T]\times[0,T];\mathcal{H})$.

Subtracting (2.21) from (2.20) and using the Lipschitz character of F we obtain

$$\mathbb{E}|X_s^n - X_s|^2 \le C \int_0^s \mathbb{E}|X_r^n - X_r|^2\, dr + C \int_0^s |(\lambda - A)^{1-\beta} e^{(s-r)A} (\lambda - A)^\beta (J_n D_\lambda - D_\lambda)|^2\, dr.$$

For $\beta > \frac{1}{2}$ the last integral can be estimated by

$$C \int_0^s (s-r)^{2\beta-2}\, dr |(J_n - I)(\lambda - A)^\beta D_\lambda|^2 \le C|(J_n - I)(\lambda - A)^\beta D_\lambda|^2,$$

and so as $n \to \infty$ it tends to zero for , by well-known properties of the operators J_n. If follows from the Gronwall lemma that $\sup_s \mathbb{E}|X_s^n - X_s|^2 \to 0$ and in particular $X^n \to X$ in $L^2(\Omega \times [0,T]; \mathcal{H})$.

The boundedness of the sequence DX^n proved before and the closedness of the operator D imply that $X \in \mathbb{L}^{1,2}(\mathcal{H})$ and that $DX^n \to DX$ weakly in the space $L^2(\Omega \times [0,T] \times [0,T]; \mathcal{H})$. Passing to the limit in (2.22) is easily justified and this proves equation (2.18). The estimate (2.19) on DX can be proved in the same way as it was done for DX^n.

We note that for any fixed $s \in [0,T]$, the estimate $|D_\sigma X_s^n| \le C(s-\sigma)^{\beta-1}$ also shows that DX_s^n is bounded in the space $L^2(\Omega \times [0,T]; \mathcal{H})$. Arguing as before we conclude that $X_s \in \mathbb{D}^{1,2}(\mathcal{H})$ for every s. The estimate (2.19) implies that $DX_s \in L^\infty(\Omega; L^2([0,T]; \mathcal{H}))$.

The continuity statement can be proved as in [8], Proposition 3.4. □

We still set $X_s = X_s^{0,x}$, for simplicity. Given a function $w : [0,T] \times \mathcal{H} \to \mathbb{R}$, we investigate the existence of the joint quadratic variation of the process $\{w(s, X_s),\ s \in [0,T]\}$ with the Brownian motion W on an interval $[0,s] \subset [0,T)$. As usual, this is defined as the limit in probability of

$$\sum_{i=1}^n (w(s_i, X_{s_i})) - w(s_{i-1}, X_{s_{i-1}}))(W_{s_i} - W_{s_{i-1}})$$

where $\{s_i\}$, $0 = s_0 < s_1 < \cdots < s_n = s$ is an arbitrary subdivision of $[0,s]$ whose mesh tends to 0. We do not require that this convergence takes place uniformly in time. This definition is easily adapted to an arbitrary interval of the form $[t,s] \subset [0,T)$. Existence of the joint quadratic variation is not trivial. Indeed, due to the occurrence of convolution type integrals in the definition of mild solution, it is not obvious that the process X is a semimartingale. Moreover, even in this case, the process $w(\cdot, X)$ might fail to be a semimartingale if w is not twice differentiable, since the Itô formula does not apply. Nevertheless, the following result holds true. Its proof could be deduced from generalization of some results obtained in [26] to the infinite-dimensional case, but we prefer to give a simpler direct proof.

Proposition 2.8 *Suppose that $w \in C([0,T) \times \mathcal{H}; \mathbb{R})$ is Gâteaux differentiable with respect to x, and that for every $s < T$ there exist constants K and m (possibly depending on s) such that*

$$|w(t,x)| \le K(1+|x|)^m, \qquad |\nabla w(t,x)| \le K(1+|x|)^m, \qquad t \in [0,s],\ x \in H. \tag{2.23}$$

Assume that for every $t \in [0,T)$, $x \in \mathcal{H}$, $\beta \in (0, \frac{1}{2} + \frac{\theta}{4})$, the linear operator $k \to \nabla w(t,x)(\lambda - A)^{1-\beta} k$ (a priori defined for $k \in D(\lambda - A)^{1-\beta}$) has an extension to a bounded linear operator $\mathcal{H} \to \mathbb{R}$, that we denote by $[\nabla w(\lambda - A)^{1-\beta}](t,x)$.

Moreover assume that the map $(t,x,k) \to [\nabla w(\lambda - A)^{1-\beta}](t,x)k$ is continuous $[0,T) \times \mathcal{H} \times \mathcal{H} \to \mathbb{R}$.

For $t \in [0,T)$, $x \in \mathcal{H}$, let $\{X_s^{t,x},\ s \in [t,T]\}$ be the solution of equation (2.13). Then the process $\{w(s, X_s^{t,x}),\ s \in [t,T]\}$ admits a joint quadratic variation process with W, on every interval $[t,s] \subset [t,T)$, given by

$$\int_t^s [\nabla w(\lambda - A)^{1-\beta}](r, X_r^{t,x})\, (\lambda - A)^\beta D_\lambda\, dr.$$

Proof. For simplicity we take $t=0$, and we write $X_s = X_s^{0,x}$, $w_s = w(s,X_s)$. It follows from the assumptions that the map $(t,x,h) \to \nabla w(t,x)h$ is also continuous on $[0,T) \times \mathcal{H} \times \mathcal{H}$. By the chain rule for the Malliavin derivative operator (see [13] for details), it follows that for every $s<T$ we have $w_s \in \mathbb{D}^{1,2}(\mathbb{R})$ and $Dw_s = \nabla w(s,X_s) DX_s$.

In order to compute the joint quadratic variation of w and W on a fixed interval $[0,s] \subset [0,T)$. we take $0 = s_0 < s_1 < \cdots < s_n = s$, a subdivision of $[0,s] \subset [0,T]$ with mesh $\delta = \max_i(s_i - s_{i-1})$. By well-known rules of Malliavin calculus (see [26], Theorem 3.2, or [18], Proposition 2.11) we have

$$(w_{s_i} - w_{s_{i-1}})(W_{s_i} - W_{s_{i-1}}) = \int_{s_{i-1}}^{s_i} D_\sigma^j (w_{s_i} - w_{s_{i-1}})\, d\sigma + \int_{s_{i-1}}^{s_i} (w_{s_i} - w_{s_{i-1}}) \hat{d}W_\sigma^j,$$

where we use the symbol $\hat{d}W$ to denote the Skorohod integral. We note that $D_\sigma w_{s_{i-1}} = 0$ for $\sigma > s_{i-1}$. Therefore setting $U_\delta(\sigma) = \sum_{i=1}^n (w_{s_i} - w_{s_{i-1}})\, 1_{(s_{i-1},s_i]}(\sigma)$ we obtain

$$\sum_{i=1}^n (w_{s_i} - w_{s_{i-1}})(W_{s_i}^j - W_{s_{i-1}}^j) = \int_0^s U_\delta(\sigma)\, \hat{d}W_\sigma^j + \sum_{i=1}^n \int_{s_{i-1}}^{s_i} \nabla w(s_i, X_{s_i})\, D_\sigma^j X_{s_i}\, d\sigma.$$

Recalling (2.18) we obtain

$$\sum_{i=1}^n (w_{s_i} - w_{s_{i-1}})(W_{s_i}^j - W_{s_{i-1}}^j) = \int_0^s U_\delta(\sigma)\, \hat{d}W_\sigma^j + \sum_{i=1}^n \int_{s_{i-1}}^{s_i} \nabla w(s_i, X_{s_i})\, e^{(s_i-\sigma)A} B\, d\sigma$$
$$+ \sum_{i=1}^n \int_{s_{i-1}}^{s_i} \nabla w(s_i, X_{s_i}) \int_\sigma^{s_i} e^{(s_i - r)A} \nabla F(r, X_r)\, D_\sigma X_r\, dr\ \ d\sigma =: I_1 + I_2 + I_3.$$

Now we let the mesh δ tend to 0. Following proposition 3.5 [8], we can prove that $I_1 \to 0$ in $L^2(\Omega,\mathbb{R})$,

$$I_2 \to \int_0^s [\nabla w (\lambda - A)^{1-\beta}](r, X_r)\, (\lambda - A)^\beta D_\lambda\, dr, \qquad \mathbb{P}-a.s.$$

and $I_3 \to 0$, $\mathbb{P}$-a.s.. □

3 The backward stochastic differential equation

We consider the following backward stochastic differential equation:

$$\begin{cases} dY_s^{t,x} = -\Psi(s, X_s^{t,x}, Z_s^{t,x})\, ds + Z_s^{t,x}\, dW_s, & s \in [0,T], \\ Y_T = \Phi(X_T^{t,x}), \end{cases} \tag{3.1}$$

for the unknown real processes $Y^{t,x}$ and $Z^{t,x}$, also denoted by Y and Z. The equation is understood in the usual way: $\mathbb{P}$-a.s.,

$$Y_s^{t,x} + \int_s^T Z_r^{t,x}\, dW_r = \Phi(X_T^{t,x}) + \int_s^T \Psi(r, X_r^{t,x}, Z_r^{t,x})\, dr, \qquad s \in [0,T], \tag{3.2}$$

but we will use the shortened notation above for equation (3.1) and similar equations to follow. In (3.1) and (3.2), $t \in [0,T]$ and $x \in \mathcal{H}$ are given and the process $X^{t,x}$ is the solution of (2.13), with the convention that $X_s^{t,x} = x$ for $s \in [0,t)$. On the generator Ψ and on the final datum Φ we make the following assumptions:

Hypothesis 3.1

1) $|\Phi(x_1) - \Phi(x_2)|_{\mathcal{H}} \le C_\Phi(1 + |x_1| + |x_2|)|x_2 - x_1|$ *for all* x_1, x_2 *in* $\mathcal{H}$.

2) There exists a constant C_ψ *such that* $|\Psi(t, x_1, z) - \Psi(t, x_2, z)| \le C_\psi(1 + |x_1| + |x_2|)|x_2 - x_1|$ *for all* x_1, x_2 *in* $\mathcal{H}$, $z \in \mathbb{R}$ *and* $t \in [0, T]$ *and* $|\Psi(s, x, z_1) - \Psi(s, x, z_2)| \le C_\psi|z_1 - z_2|$, *for every* $s \in [0, T]$, $x \in \mathcal{H}$, $z_1, z_2 \in \mathbb{R}$.

3) $\sup_{s\in[0,T]} |\Psi(s, 0, 0)| \le C_\ell$.

4) $\Phi \in \mathcal{G}^1(\mathcal{H})$ *and for almost every* $s \in [0, T]$ *the map* $\Psi(s, \cdot, \cdot)$ *is Gâteaux differentiable on* $\mathcal{H} \times \mathbb{R}$ *and the maps* $(x, h, z) \to \nabla_x \Psi(s, x, z)h$ *and* $(x, z, \zeta) \to \nabla_z \Psi(s, x, z)\zeta$ *are continuous on* $\mathcal{H} \times \mathcal{H} \times \mathbb{R}$ *and* $\mathcal{H} \times \mathbb{R} \times \mathbb{R}$ *respectively.*

Proposition 3.2 1) *For all* $x \in \mathcal{H}$, $t \in [0, T]$ *and* $p \in [2, \infty)$ *there exists a unique pair of processes* $(Y^{t,x}, Z^{t,x})$ *with* $Y^{t,x} \in L^p_{\mathcal{P}}(\Omega, C([0, T], \mathbb{R}))$, $Z^{t,x} \in L^p_{\mathcal{P}}(\Omega, L^2([0, T], \mathbb{R}))$ *solving (3.1); in the following we denote such a solution by* $(Y^{t,x}_\cdot, Z^{t,x}_\cdot)$.

2) *The map* $(t, x) \to (Y^{t,x}_\cdot, Z^{t,x}_\cdot)$ *is continuous from* $[0, T] \times \mathcal{H}$ *to* $L^p_{\mathcal{P}}(\Omega, C([0, T], \mathbb{R})) \times L^p_{\mathcal{P}}(\Omega, L^2([0, T], \mathbb{R}))$.

3) *For all* $t \in [0, T]$ *the map* $x \to (Y^{t,x}_\cdot, Z^{t,x}_\cdot)$ *is Gâteaux differentiable as a map from* $\mathcal{H}$ *to* $L^p_{\mathcal{P}}(\Omega, C([0, T], \mathbb{R})) \times L^p_{\mathcal{P}}(\Omega, L^2([0, T], \mathbb{R}))$*; moreover the map* $(t, x, h) \to (\nabla_x Y^{t,x}_\cdot h$, $\nabla_x Z^{t,x}_\cdot h)$ *is continuous from* $[0, T] \times \mathcal{H} \times \mathcal{H}$ *to* $L^p_{\mathcal{P}}(\Omega, C([0, T], \mathbb{R})) \times L^p_{\mathcal{P}}(\Omega, L^2([0, T], \mathbb{R}))$.

4) *The following equation holds for all* $t \in [0, T]$, $x, h \in \mathcal{H}$.

$$\begin{cases} d\, \nabla_x Y^{t,x}_s h = -\nabla_x \Psi(s, X^{t,x}_s, Z^{t,x}_s)\, \nabla_x X^{t,x}_s h\, ds \\[1ex] \qquad\qquad -\nabla_z \Psi(s, X^{t,x}_s, Z^{t,x}_s)\, \nabla_x Z^{t,x}_s h\, ds + \nabla_x Z^{t,x}_s h\, dW_s, \\[1ex] \nabla_x Y^{t,x}_T h = \nabla_x \Phi(X^{t,x}_T)\, \nabla_x X^{t,x}_T h, \qquad\qquad s \in [t, T]. \end{cases} \tag{3.3}$$

Proof. The claim follows directly from Proposition 4.8 in [13], from Proposition 2.5 above and from the chain rule (in the form stated in Lemma 2.1 of [13]). □

Remark 3.3 The inequality (2.12), for $\alpha = 0$, together with the inequality (4.9) in [13], implies that there exists a constant $C_{Y,p}$ such that for all $t \in [0, T]$ and $x \in \mathcal{H}$

$$\mathbb{E} \sup_{s\in[0,T]} |Y^{t,x}_s|^p + \mathbb{E}\left(\int_0^T |Z^{t,x}_s|^2 ds\right)^{p/2} \le C_{Y,p}(1 + |x|)^{2p}.$$

Remark 3.4 $Y^{t,x}_t$ is adapted both to the σ-field $\sigma\{W_s : s \in [0, t]\}$ and to the σ-field $\sigma\{W_s - W_t : s \in [t, T]\}$. Thus $Y^{t,x}_t$ and $\nabla Y^{t,x}_t h$, $x, h \in H$ are deterministic.

Proposition 2.6 yields the following further regularity result.

Proposition 3.5 *For every* $\alpha \in [0, 1/2)$, $p \in [2, \infty)$ *there exist two families of processes*

$$\{P^\alpha(s, t, x)k : s \in [0, T]\} \text{ and } \{Q^\alpha(s, t, x)k : s \in [0, T]\}; \quad t \in [0, T),\ x \in \mathcal{H},\ k \in \mathcal{H}$$

with $P^\alpha(\cdot, t, x)k \in L^p_{\mathcal{P}}(\Omega, C([0, T], \mathbb{R}))$ *and* $Q^\alpha(\cdot, t, x)k) \in L^p_{\mathcal{P}}(\Omega, L^2([0, T], \mathbb{R}))$ *such that if* $k \in D(\lambda - A)^\alpha$, $t \in [0, T)$, $x \in \mathcal{H}$, *then* $\mathbb{P}$*-a.s.*

$$P^\alpha(s, t, x)k = \begin{cases} \nabla_x Y^{t,x}_s (\lambda - A)^\alpha k & \text{for all } s \in [t, T], \\ \nabla_x Y^{t,x}_t (\lambda - A)^\alpha k & \text{for all } s \in [0, t), \end{cases} \tag{3.4}$$

$$Q^\alpha(s,t,x)k = \begin{cases} \nabla_x Z_s^{t,x}(\lambda - A)^\alpha k & \textit{for a.e. } s \in [t,T], \\ 0 & \textit{if } s \in [0,t). \end{cases} \tag{3.5}$$

Moreover the map $(t,x,k) \to P^\alpha(\cdot,t,x)k$ and the map $(t,x,k) \to Q^\alpha(\cdot,t,x)k$ are continuous from $[0,T) \times \mathcal{H} \times \mathcal{H}$ to $L^p_{\mathcal{P}}(\Omega, C([0,T],\mathbb{R}))$ and linear with respect to k.

Finally there exists a constant $C_{\nabla Y,\alpha,p}$ such that

$$\mathbb{E} \sup_{s\in[0,T]} |P^\alpha(s,t,x)k|^p_{\mathcal{H}} + \mathbb{E}\left(\int_0^T |Q^\alpha(s,t,x)k|_{(\mathbb{R})} ds\right)^{p/2} \leq C_{\nabla Y,\alpha,p}(T-t)^{-\alpha p}(1+|x|_{\mathcal{H}})^p |k|^p_{\mathcal{H}}. \tag{3.6}$$

Proof. Let, for $t \in [0,T]$, $x \in \mathcal{H}$, $k \in D(\lambda - A)^\alpha$, $P^\alpha(\cdot,t,x)k$ and $Q^\alpha(\cdot,t,x)k$ be defined by (3.4) and (3.5) respectively.
By Proposition 3.2 the map $k \to (P^\alpha(\cdot,t,x)k, Q^\alpha(\cdot,t,x)k)$ is a bounded linear operator from $D(\lambda - A)^\alpha$ to $L^p_{\mathcal{P}}(\Omega, C([0,T],\mathbb{R})) \times L^p_{\mathcal{P}}(\Omega, L^2([0,T],\mathbb{R}))$. Moreover $(P^\alpha(\cdot,t,x)k, Q^\alpha(\cdot,t,x)k)$ solves the equation

$$\begin{cases} dP^\alpha(s,t,x)k = -1_{[t,T]}(s)\nabla_x\Psi(s, X_s^{t,x}, Z^{t,x})\, \nabla_x X_s^{t,x}(\lambda - A)^\alpha k \; ds \\ \qquad\qquad -\nabla_z \Psi(s, X_s^{t,x}, Z_s^{t,x})\; Q^\alpha(s,t,x)k \; ds + Q^\alpha(s,t,x)k \; dW_s, \\ P^\alpha(T,t,x)k = \nabla_x\Phi(X_s^{t,x})\, \nabla_x X_T^{t,x}(\lambda - A)^\alpha k, \qquad s \in [t,T]. \end{cases} \tag{3.7}$$

By (2.15) equation (3.7) can be rewritten

$$\begin{cases} dP^\alpha(s,t,x)k = \nu(s,t,x)k \; ds - \nabla_z\Psi(s, X_s^{t,x}, Z_s^{t,x}) Q^\alpha(s,t,x)k ds + Q^\alpha(s,t,x)k \; dW_s \\ P^\alpha(T,t,x)k = \eta(t,x)k, \qquad s \in [0,T], \end{cases} \tag{3.8}$$

where

$$\begin{aligned} &\nu(s,t,x)k = -1_{[t,T]}(s)\nabla_x\Psi(s, X_s^{t,x}, Z_s^{t,x})\left(\Theta^\alpha(s,t,x)k + e^{(s-t)A}(\lambda - A)^\alpha k\right), \\ &\eta(t,x)k = \nabla_x\Phi(X_T^{t,x})\left(\Theta^\alpha(T,t,x)k + e^{(T-t)A}(\lambda - A)^\alpha k\right). \end{aligned}$$

Now we choose arbitrary $k \in \mathcal{H}$ and notice that $\nu(s,t,x)k$ and $\eta(t,x)k$ can still be defined by the above formulae. Remark 5.4, and relations (2.12), with $\alpha = 0$, (2.16) yield:

$$\begin{aligned} \mathbb{E}\left(\int_0^T |\nu(s,t,x)k|^2 ds\right)^{p/2} &\leq c_1 \mathbb{E}\left(\int_t^T (1+|X_s^{t,x}|)^2 \left(|\Theta^\alpha(s,t,x)k| + (s-t)^{-\alpha}|k|\right)^2 ds\right)^{p/2} \\ &\leq c_2 \left[(T-t)^{p/2} + (T-t)^{(1-2\alpha)p/2}\right](1+|x|)^p|k|^p \leq c_3(1+|x|)^p|k|^p, \end{aligned}$$

where c_1, c_2 and c_3 are suitable constants independent on t,x,k. In the same way

$$\begin{aligned} \mathbb{E}|\eta(t,x)k|^p &\leq c_4 \mathbb{E}\Big((1+|X_T^{t,x}|)\left(|\Theta^\alpha(T,t,x)k| + (T-t)^{-\alpha}|k|\right)\Big)^p \\ &\leq c_5 (T-t)^{-p\alpha}(1+|x|)^p|k|^p. \end{aligned}$$

By Proposition 4.3 in [13], for all $k \in \mathcal{H}$ there exists a unique pair $(P^\alpha(\cdot,t,x)k, Q^\alpha(\cdot,t,x)k)$ belonging to $L^p_{\mathcal{P}}(\Omega, C([0,T],\mathbb{R})) \times L^p_{\mathcal{P}}(\Omega, L^2([0,T],\mathbb{R}))$ and solving equation (3.8); moreover (3.6) holds. The map $k \to (P^\alpha(\cdot,t,x)k, Q^\alpha(\cdot,t,x)k)$ is clearly linear, so we can conclude that the required extension exists. The proof of its continuity can be achieved as in [8], proposition 4.4. □

Corollary 3.6 *Setting $v(t,x) = Y_t^{t,x}$, we have $v \in C([0,T] \times \mathcal{H}; \mathbb{R})$ and there exists a constant C such that $|v(t,x)| \leq C(1+|x|)^2$, $t \in [0,T]$, $x \in \mathcal{H}$. Moreover v is Gâteaux differentiable with respect to x on $[0,T] \times \mathcal{H}$ and the map $(t,x,h) \to \nabla v(t,x)h$ is continuous.*

For all $\alpha \in [0,1/2)$, $t \in [0,T)$ and $x \in \mathcal{H}$ the linear operator $k \to \nabla v(t,x)(\lambda - A)^\alpha k$ - a priori defined for $k \in D(\lambda - A)^\alpha$ - has an extension to a bounded linear operator $\mathcal{H} \to \mathbb{R}$, that we denote by $[\nabla v(\lambda - A)^\alpha](t,x)$.

Finally the map $(t,x,k) \to [\nabla v(\lambda - A)^\alpha](t,x)k$ is continuous $[0,T) \times \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ and there exists $C_{\nabla v, \alpha}$ for which:

$$|[\nabla v(\lambda - A)^\alpha](t,x)k| \leq C_{\nabla v,\alpha}(T-t)^{-\alpha}(1+|x|_{\mathcal{H}})|k|_{\mathcal{H}}, \qquad t \in [0,T),\ x,k \in \mathcal{H}. \tag{3.9}$$

Proof. We recall that $Y_t^{t,x}$ is deterministic. Since the map $(t,x) \to Y^{t,x}$ is continuous with values in $L^p_{\mathcal{P}}(\Omega, C([0,T],\mathbb{R}))$, $p \geq 2$, then the map $(t,x) \to Y_t^{t,x}$ is continuous with values in $L^p(\Omega, \mathbb{R})$ and so the map $(t,x) \to Y_t^{t,x} = v(t,x)$ is continuous with values in $\mathbb{R}$.

Similarly, $\nabla_x v(t,x) = \nabla_x Y_t^{t,x}$ exists and has the required continuity properties, by Proposition 3.2.

Next we notice that $P^\alpha(t,t,x)k = \nabla_x Y_t^{t,x}(\lambda - A)^\alpha k$. The existence of the required extensions and its continuity are direct consequences of Proposition 3.5. Finally the estimate (3.9) follows from (3.6). □

Remark 3.7 It is evident by construction that the law of $Y^{t,x}$ and consequently the function v depends on the law of the Wiener process W but not on the particular probability $\mathbb{P}$ and Wiener process W we have chosen.

Corollary 3.8 *For every $t \in [0,T]$, $x \in H$ we have, $\mathbb{P}$-a.s.,*

$$Y_s^{t,x} = v(s, X_s^{t,x}), \qquad \text{for all } s \in [t,T], \tag{3.10}$$

$$Z_s^{t,x} = [\nabla v(\lambda - A)^{1-\beta}](s, X_s^{t,x})\,(\lambda - A)^\beta D_\lambda, \qquad \text{for almost all } s \in [t,T]. \tag{3.11}$$

Proof. We start from the well-known equality: for $0 \leq t \leq r \leq T$, $\mathbb{P}$-a.s.,

$$X_s^{t,x} = X_s^{r, X_r^{t,x}}, \qquad \text{for all } s \in [r,T].$$

It follows easily from the uniqueness of the backward equation (3.1) that $\mathbb{P}$-a.s.,

$$Y_s^{t,x} = Y_s^{r, X_r^{t,x}}, \qquad \text{for all } s \in [r,T].$$

Setting $s = r$ we arrive at (3.10).

To prove (3.11) we note that it follows immediately from the backward equation (3.1) that the joint quadratic variation of $\{Y_s^{t,x},\ s \in [t,T]\}$ and W on an arbitrary interval $[t,s] \subset [t,T)$ is equal to $\int_t^s Z^j\,dr$. By (3.10) the same result can be obtained by considering the joint quadratic variation of $\{v(s, X_s^{t,x}),\ s \in [t,T]\}$ and W. An application of Proposition 2.8 (whose assumptions hold true by Corollary 3.6) leads to the identity

$$\int_t^s Z_r\,dr = \int_t^s [\nabla v(\lambda - A)^{1-\beta}](r, X_r^{t,x})\,(\lambda - A)^\beta D_\lambda\,dr,$$

and (3.11) is proved. □

4 The Hamilton-Jacobi-Bellman equation

In this section the aim is to solve a second order partial differential equation, where the second order differential operator is the generator of the Markov process $\{X_s^{t,x}, s \in [t,T]\}$, solution of equation (2.13). Namely we are interested in Hamilton Jacobi Bellman equations related to a control problem that we present in the next section.

Let us consider again the solution $X_s^{t,x}$ of equation (2.13) and denote by $P_{t,s}$ its transition semigroup:

$$P_{t,s}[\phi](x) = \mathbb{E}\,\phi(X_s^{t,x}), \qquad x \in \mathcal{H},\ 0 \le t \le s \le T,$$

for any bounded measurable $\phi : \mathcal{H} \to \mathbb{R}$. We note that by the estimate (2.12) (with $\alpha = 0$) this formula is meaningful for every ϕ with polynomial growth. In the following $P_{t,s}$ will be considered as an operator acting on this class of functions.

Let us denote by $\mathcal{L}_t$ the generator of $P_{t,s}$, formally:

$$\mathcal{L}_t[\phi](x) = \frac{1}{2}\langle \nabla^2\phi(x)B, B\rangle + \langle Ax + F(t,x), \nabla\phi(x)\rangle,$$

where $\nabla\phi(x)$ and $\nabla^2\phi(x)$ are first and second Gâteaux derivatives of ϕ at the point $x \in \mathcal{H}$ (here they are identified with elements of $\mathcal{H}$ and $L(\mathcal{H})$ respectively).

The Hamilton-Jacobi-Bellman equation for the optimal control problem is

$$\begin{cases} \dfrac{\partial v(t,x)}{\partial t} + \mathcal{L}_t[v(t,\cdot)](x) = -\Psi(t,x,\nabla v(t,x)B), \qquad t \in [0,T],\ x \in \mathcal{H}, \\ v(T,x) = \Phi(x). \end{cases} \tag{4.1}$$

This is a nonlinear parabolic equation for the unknown function $v : [0,T] \times \mathcal{H} \to \mathbb{R}$. The operators $\mathcal{L}_t$ are very degenerate, since the space $\mathcal{H}$ is infinite-dimensional but the noise W is a real Wiener process.

Now we consider the variation of constants formula for (4.1):

$$v(t,x) = P_{t,T}[\Phi](x) - \int_t^T P_{t,s}[\Psi(s,\cdot,\nabla v(s,\cdot)B](x)\,ds, \qquad t \in [0,T],\ x \in \mathcal{H},$$

where we remember $B = (\lambda - A)D_\lambda$. This equality is still formal, since the term $(\lambda - A)D_\lambda$ is not defined. However with a slightly different interpretation we arrive at the following precise definition:

Definition 4.1 *Let $\beta \in [0, \frac{1}{2})$. We say that a function $v : [0,T] \times \mathcal{H} \to \mathbb{R}$ is a mild solution of the Hamilton-Jacobi-Bellman equation (4.1) if the following conditions hold:*

(i) $v \in C([0,T] \times \mathcal{H}; \mathbb{R})$ and there exist constants $C, m \ge 0$ such that $|v(t,x)| \le C\,(1 + |x|)^m$, $t \in [0,T]$, $x \in \mathcal{H}$.

(ii) v is Gâteaux differentiable with respect to x on $[0,T) \times \mathcal{H}$ and the map $(t,x,h) \to \nabla v(t,x)h$ is continuous $[0,T) \times \mathcal{H} \times \mathcal{H} \to \mathbb{R}$.

(iii) For all $t \in [0,T)$ and $x \in \mathcal{H}$ the linear operator $k \to \nabla v(t,x)(\lambda - A)^{1-\beta}k$ (a priori defined for $k \in D(\lambda - A)^{1-\beta}$) has an extension to a bounded linear operator $\mathcal{H} \to \mathbb{R}$, that we denote by $[\nabla v(\lambda - A)^{1-\beta}](t,x)$.

Moreover the map $(t,x,k) \to [\nabla v(\lambda - A)^{1-\beta}](t,x)k$ is continuous $[0,T) \times \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ and there exist constants $C, m \ge 0$, $\kappa \in [0,1)$ such that

$$|[\nabla v(\lambda - A)^{1-\beta}](t,x)|_{\mathcal{H}^*} \le C(T-t)^{-\kappa}(1 + |x|)^m, \qquad t \in [0,T),\ x \in \mathcal{H}. \tag{4.2}$$

(iv) the following equality holds for every $t \in [0,T]$, $x \in \mathcal{H}$:

$$v(t,x) = P_{t,T}[\Phi](x) + \int_t^T P_{t,s}\left[\Psi\Big(s,\cdot,[\nabla v(\lambda-A)^{1-\beta}](s,\cdot)\,(\lambda-A)^{\beta}D_\lambda\Big)\right](x)\,ds. \quad (4.3)$$

We assume that Φ and Ψ satisfy hypotheses 3.1 and using the estimate (2.12) (with $\alpha = 0$) it is easy to conclude that formula (4.3) is meaningful.

Theorem 4.1 *Assume Hypotheses 2.1, 5.1 and 3.1 then there exists a unique mild solution of the Hamilton-Jacobi-Bellman equation (4.1). The solution v is given by the formula*

$$v(t,x) = Y_t^{t,x},$$

where (X,Y,Z) is the solution of the forward-backward system (2.13)-(3.2).

Proof. *Existence.* By Corollary 3.6 the solution v has the regularity properties stated in Definition 4.1. In order to verify that equality (4.3) holds we first fix $t \in [0,T]$ and $x \in \mathcal{H}$ and note that the backward equation (3.1) gives

$$Y_t^{t,x} + \int_t^T Z_s^{t,x}\,dW_s = \Phi(X_T^{t,x}) + \int_t^T \Psi\Big(s, X_s^{t,x}, Z_s^{t,x}\Big)\,ds.$$

Taking expectation we obtain

$$v(t,x) = P_{t,T}[\Phi](x) + \mathbb{E}\int_t^T \Psi\Big(s, X_s^{t,x}, Z_s^{t,x}\Big)\,ds. \quad (4.4)$$

Now we recall that by Corollary 3.8 we have

$$Z_s^{t,x} = [\nabla v(\lambda-A)^{1-\beta}](s,X_s^{t,x})\,(\lambda-A)^{\beta}D_\lambda, \qquad \mathbb{P}\text{−a.s. for a.a. } s \in [t,T].$$

It follows that

$$\mathbb{E}\int_t^T \Psi\Big(s, X_s^{t,x} Z_s^{t,x}\Big)\,ds = \int_t^T P_{t,s}\left[\Psi\Big(s,\cdot,[\nabla v(\lambda-A)^{1-\beta}](s,\cdot)\,(\lambda-A)^{\beta}D_\lambda\Big)\right](x)\,ds.$$

Comparing with (4.4) gives the required equality (4.3).

Uniqueness. Let v be a mild solution. We fix $t \in [0,T]$ and $x \in \mathcal{H}$ and look for a convenient expression for the process $v(s,X_s^{t,x})$, $s \in [t,T]$. By "standard" arguments (see e.g. [13]), by the Markov property of X the process $v(s,X_s^{t,x})$, $s \in [t,T]$ is a (real) continuous semimartingale, and, by the representation theorem for martingales, there exists $\widetilde{Z} \in L^2_{\mathcal{P}}(\Omega \times [t,T];\mathbb{R})$ such that its canonical decomposition into its continuous martingale part and its continuous finite variation part is given by

$$\begin{aligned} v(s,X_s^{t,x}) &= v(t,x) + \int_t^s \widetilde{Z}_r\,dW_r \\ &\quad + \int_t^s \Psi\Big(r, X_r^{t,x}, [\nabla v(\lambda-A)^{1-\beta}](r,X_r^{t,x})\,(\lambda-A)^{\beta}D_\lambda\Big)\,dr. \end{aligned} \quad (4.5)$$

By computing the joint quadratic variations of both sides of (4.5) we have $\mathbb{P}$-a.s., $[\nabla v(\lambda-A)^{1-\beta}](s,X_s^{t,x})\,(\lambda-A)^{\beta}D_\lambda = \widetilde{Z}_s$, so substituting into (4.5) and taking into account that $v(T,X_T^{t,x})) = \Phi(X_T^{t,x}))$ we obtain, for $s \in [t,T]$,

$$\begin{aligned} v(s,X_s^{t,x}) &+ \int_s^T [\nabla v(\lambda-A)^{1-\beta}](r,X_r^{t,x})\,(\lambda-A)^{\beta}D_\lambda\,dW_r \\ &= \Phi(X_T^{t,x}) + \int_s^T \Psi\Big(r, X_r^{t,x}, [\nabla v(\lambda-A)^{1-\beta}](r,X_r^{t,x})\,(\lambda-A)^{\beta}D_\lambda\Big)\,dr. \end{aligned}$$

Comparing with the backward equation (3.1) we note that the pairs

$$\left(Y_s^{t,x}, Z_s^{t,x}\right) \text{ and } \left(v(s, X_s^{t,x}), [\nabla v(\lambda - A)^{1-\beta}](s, X_s^{t,x})\, (\lambda - A)^{\beta} D_\lambda\right), \ s \in [t, T],$$

solve the same equation. By uniqueness, we have $Y_s^{t,x} = v(s, X_s^{t,x})$, $s \in [t, T]$, and setting $s = t$ we obtain $Y_t^{t,x} = v(t, x)$. □

5 Synthesis of the optimal control

At first we introduce a "concrete" cost functional: let $y(s, \xi)$ solution of equation (2.1). Let us consider the following cost functional

$$J(t, x, u(\cdot)) = \mathbb{E} \int_t^T \int_0^{+\infty} \ell(s, \xi, y(s, \xi), u(s))\, d\xi\, ds + \mathbb{E} \int_0^{+\infty} \phi(\xi, y(T, \xi))\, d\xi. \tag{5.1}$$

In this section we assume that the following holds:

Hypothesis 5.1 $\ell : [0, T] \times [0, +\infty) \times \mathbb{R} \times \mathcal{U} \to \mathbb{R}$ *and* $\phi : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ *are measurable. Let* $\rho(\xi) = \xi^{1+\theta}$*, or* $\rho(\xi) = \xi^{1+\theta} \wedge 1$*, depending on what weight we are considering to define the space* $\mathcal{H}$*. Assume also:*

1) there exist two constant C_1, C_2 *such that, for some* $\epsilon > 0$*, for every* $\xi \in [0, +\infty)$*,* $y_1, y_2 \in \mathbb{R}$

$$|\phi(\xi, y_1) - \phi(\xi, y_2)| \leq C_1 \frac{\sqrt{\rho(\xi)}}{(1 + \xi)^{1/2+\epsilon}} |y_1 - y_2| + C_2\, \rho(\xi)(|y_1| + |y_2|)\, |y_1 - y_2|,$$

moreover $\displaystyle\int_0^{+\infty} |\phi(\xi, 0)| d\xi < \infty$*;*

2) for every $t \in [0, T]$ *and* $\xi \in [0, +\infty)$*,* $\ell(t, \xi, \cdot, \cdot) : \mathbb{R}^2 \to \mathbb{R}$ *is continuous. Moreover there exists two constant* C_1, C_2 *such that, for some* $\epsilon > 0$*, for every* $t \in [0, T]$*,* $\xi \in [0, +\infty)$*,* $y_1, y_2 \in \mathbb{R}$*,* $u \in \mathcal{U}$*,*

$$|\ell(t, \xi, y_1, u) - \ell(t, \xi, y_2, u)| \leq C_1 \frac{\sqrt{\rho(\xi)}}{(1 + \xi)^{1/2+\epsilon}} |y_1 - y_2| + C_2\, \rho(\xi)(|y_1| + |y_2|)\, |y_1 - y_2|,$$

and for every $t \in [0, T]$

$$\int_0^{+\infty} \sup_{u \in \mathcal{U}} |\ell(t, \xi, 0, u)|\, d\xi \leq C_\ell.$$

We notice that in Hypothesis 5.1, the presence of the weight $\rho(\xi)$ is natural since we are considering as state space the weighted space $\mathcal{H}$, as well as the presence of the square integrable function $\dfrac{1}{(1 + \xi^{1/2+\epsilon})}$ since $[0, +\infty)$ is not of finite measure with any weight $\rho(\xi)$.
Further assumptions will be made on the cost functional after the following reformulation: we define

$$L(s, x, u) = \int_0^{+\infty} \ell(s, \xi, x(\xi), u)\, d\xi, \qquad \Phi(x) = \int_0^{+\infty} \phi(\xi, x(\xi))\, d\xi,$$

for $s \in [0, T]$, $x = x(\cdot) \in \mathcal{H}$, $u \in \mathcal{U}$. The functions $L : [0, T] \times \mathcal{H} \times \mathcal{U} \to \mathbb{R}$ and $\Phi : \mathcal{H} \to \mathbb{R}$ are well defined and measurable. The cost functional (5.1) can be written in the form

$$J(t, x, u(\cdot)) = \mathbb{E} \int_t^T L(s, X_s^u, u_s)\, ds + \mathbb{E}\, \Phi(X_T^u). \tag{5.2}$$

It is easy to show that the cost is finite for any admissible control $u(\cdot)$. Moreover for $s \in [0,T]$, $x \in \mathcal{H}$, $z \in \mathbb{R}$ we define the hamiltonian as

$$\Psi(s,x,z) = \inf_{u\in\mathcal{U}}\{zu + L(s,x,u)\}.$$

Since, as it is easy to check, for all $s \in [0,T]$ and all $x \in \mathcal{H}$, $L(s,x,\cdot)$ is continuous on the compact set $\mathcal{U}$ the above infimum is attained. Therefore if we define

$$\Gamma(s,x,z) = \{u \in \mathcal{U} : zu + L(s,x,u) = \Psi(s,x,z)\} \tag{5.3}$$

then $\Gamma(s,x,z) \neq \emptyset$ for every $s \in [0,T]$, every $x \in \mathcal{H}$ and every $z \in \mathbb{R}$. By [2], see Theorems 8.2.10 and 8.2.11, Γ admits a measurable selection, i.e. there exists a measurable function $\gamma : [0,T] \times \mathcal{H} \times \mathbb{R} \to \mathcal{U}$ with $\gamma(s,x,z) \in \Gamma(s,x,z)$ for every $s \in [0,T]$, every $x \in \mathcal{H}$ and every $z \in \mathbb{R}$.

Proposition 5.2 *Under Hypothesis 5.1 the following holds.*

1) $|\Phi(x_1) - \Phi(x_2)|_{\mathcal{H}} \leq C_\phi(1 + |x_1| + |x_2|)|x_2 - x_1|$ *for all* x_1, x_2 *in* $\mathcal{H}$.

2) *There exists a constant* C_ψ *such that* $|\Psi(t,x_1,z) - \Psi(t,x_2,z)| \leq C_\psi(1+|x_1|+|x_2|)|x_2-x_1|$ *for all* x_1, x_2 *in* $\mathcal{H}$, $z \in \mathbb{R}$ *and* $t \in [0,T]$.

3) *Setting* $C_{\mathcal{U}} = \sup\{|u| : u \in \mathcal{U}\}$ *we have* $|\Psi(s,x,z_1) - \Psi(s,x,z_2)| \leq C_{\mathcal{U}}\,|z_1 - z_2|$, *for every* $s \in [0,T]$, $x \in \mathcal{H}$, $z_1, z_2 \in \mathbb{R}$.

4) $\sup_{s\in[0,T]} |\Psi(s,0,0)| \leq C_\ell$.

Some of our results are based on the following assumptions:

Hypothesis 5.3 *For almost every* $\xi \in [0,+\infty)$ *the map* $\phi(\xi,\cdot)$ *is continuously differentiable on* $\mathbb{R}$. *For almost every* $s \in [0,T]$ *the map* $\Psi(s,\cdot,\cdot)$ *is Gâteaux differentiable on* $\mathcal{H} \times \mathbb{R}$ *and the maps* $(x,h,z) \to \nabla_x\Psi(s,x,z)h$ *and* $(x,z,\zeta) \to \nabla_z\Psi(s,x,z)\zeta$ *are continuous on* $\mathcal{H} \times \mathcal{H} \times \mathbb{R}$ *and* $\mathcal{H} \times \mathbb{R} \times \mathbb{R}$ *respectively.*

From this assumption and from Hypothesis 5.1 it follows easily that Φ is Gâteaux differentiable on $\mathcal{H}$ and the map $(x,h) \to \nabla\Phi(x)h$ is continuous on $\mathcal{H} \times \mathcal{H}$. Moreover it follows that Φ and Ψ satisfy hypothesis 3.1.

Remark 5.4 From Proposition 5.2 we immediately deduce the following estimates:

$$|\nabla\Phi(x)h| \leq C_\phi(1 + 2|x|)|h|, \quad |\nabla_x\Psi(t,x,z)h| \leq C_\psi(1 + 2|x|)|h|, \quad |\nabla_z\Psi(s,x,z)\zeta| \leq C_{\mathcal{U}}\,|\zeta|.$$

Hypothesis 5.3 involves conditions on the function Ψ, and not on the function ℓ that determines Ψ. However, Hypothesis 5.3 can be verified in concrete situations, see e.g. example 2.7.1 in [8].

The optimal control problem in its strong formulation is to minimize, for arbitrary $t \in [0,T]$ and $x \in \mathcal{H}$, the cost (5.2), over all admissible controls, where $\{X^u_s : s \in [t,T]\}$ solves $\mathbb{P}$-a.s.

$$\begin{aligned} X^u_s &= e^{(s-t)A}x + \int_t^s e^{(s-r)A}F(r,X^u_r)\,dr + \int_t^s (\lambda - A)^{1-\beta}e^{(s-r)A}(\lambda - A)^\beta D_\lambda\,dW_r \\ &+ \int_t^s (\lambda - A)^{1-\beta}e^{(s-r)A}(\lambda - A)^\beta D_\lambda\,u_r\,dr, \qquad s \in [t,T]. \end{aligned} \tag{5.4}$$

We will also write $X^{u,t,x}_s$ instead of X^u_s, to stress dependence on the initial data t,x. By $v : [0,T] \times \mathcal{H} \to \mathbb{R}$, we denote the mild solution of the Hamilton-Jacobi-Bellman equation (4.1).

Theorem 5.5 *Assume Hypotheses 2.1, 5.1 and 5.3. For every $t \in [0,T]$, $x \in \mathcal{H}$ and for all admissible control u we have $J(t,x,u(\cdot)) \geq v(t,x)$, and the equality $J(t,x,u(\cdot)) = v(t,x)$ holds if and only if*

$$u_s \in \Gamma\left(s, X_s^{u,t,x}, [\nabla v(\lambda - A)^{1-\beta}](s, X_s^{u,t,x})\ (\lambda - A)^\beta D_\lambda\right)$$

Proof. The proof is identical to the proof of relation (7.5) in [13, Theorem 7.2]. Just notice that in this case by (3.11) we have $Z_s^{t,x} = [\nabla v(\lambda - A)^{1-\beta}](s, X_s^{t,x})\ (\lambda - A)^\beta D_\lambda$ and the role of G in [13, Theorem 7.2] is here played by $B = (\lambda - A)D_\lambda$. □

Under the assumptions of Theorem 5.5, let us define the so called optimal feedback law:

$$u(t,x) = \gamma\Big(t, x, [\nabla v(\lambda - A)^{1-\beta}](t,x)\ (\lambda - A)^\beta D_\lambda\Big), \qquad t \in [0,T],\ x \in \mathcal{H}. \tag{5.5}$$

Assume that there exists an adapted process $\{\overline{X}_s,\ s \in [t,T]\}$ with continuous trajectories solving the so called closed loop equation: $\mathbb{P}$-a.s.

$$\begin{aligned} \overline{X}_s &= e^{(s-t)A}x_0 + \int_t^s e^{(s-r)A}F(r,\overline{X}_r)\, dr + \int_t^s (\lambda - A)^{1-\beta}e^{(s-r)A}(\lambda - A)^\beta D_\lambda\, dW_r \\ &+ \int_t^s (\lambda - A)^{1-\beta}e^{(s-r)A}(\lambda - A)^\beta D_\lambda\, u(r,\overline{X}_r)dr, \qquad s \in [t,T]. \end{aligned} \tag{5.6}$$

Then setting $\overline{u}(s) = u(s,\overline{X}_s)$ we have $J(t,x,\overline{u}(\cdot)) = v(t,x)$ and consequently the pair $(\overline{u},\overline{X})$ is optimal for the control problem. We nevertheless notice that we do not state conditions for the existence of a solution of the closed loop equation. Indeed existence is not obvious, due to the lack of regularity of the feedback law u occurring in (5.6).

However, under additional assumptions, it is also possible to solve the closed loop equation (5.6) and therefore obtain existence of an optimal control in the present strong formulation.

We now reformulate the optimal control problem in the weak sense, following the approach of [12]. The main advantage is that we will be able to solve the closed loop equation, and hence to find an optimal control, although the feedback law $\underline{u}$ is non-smooth.
We still assume we are given the functions f, ℓ, ϕ, the corresponding functions F, Ψ, L, Φ satisfying Hypotheses 2.1, 5.1 and 5.3, and the set $\mathcal{U}$ as in the previous sections. We also assume that initial data $t \in [0,T]$ and $x \in \mathcal{H}$ are given. We call $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W)$ an *admissible set-up*, or simply a set-up, if $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space with a right-continuous and $\mathbb{P}$-complete filtration $\{\mathcal{F}_t,\ t \in [0,T]\}$, and $\{W_t,\ t \in [0,T]\}$ is a standard, real valued, $\mathcal{F}_t$-Wiener process.

An *admissible control system* (a.c.s.) is defined as $\mathbb{U} = (\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W, u, X^u)$ where:

- $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W)$ is an admissible set-up;
- $u : \Omega \times [0,T] \to \mathbb{R}$ is an $(\mathcal{F}_t)$-predictable process with values in $\mathcal{U}$;
- $\{X_s^u,\ s \in [t,T]\}$ is an $(\mathcal{F}_t)$-adapted continuous process with values in $\mathcal{H}$, mild solution of the state equation (5.4) with initial condition $X_t^u = x$.

By Proposition 2.5, on an arbitrary set-up the process X^u is uniquely determined by u and x, up to indistinguishability. To every a.c.s. we associate the cost $J(t,x,\mathbb{U})$ given by the right-hand side of (5.2). Although formally the same, it is important to note that now the cost is a functional of the a.c.s., and not a functional of u alone. Our purpose is to minimize the functional $J(t,x,\mathbb{U})$ over all a.c.s. for fixed initial data t, x.

Theorem 5.6 *Assume Hypotheses 2.1, 5.1 and 5.3. For every $t \in [0,T]$, $x \in H$, the infimum of $J(t,x,\mathbb{U})$ over all a.c.s. is equal to $v(t,x)$. Moreover there exists an a.c.s. $\mathbb{U} = (\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W, u, X^u)$ for which $J(t,x,\mathbb{U}) = v(t,x)$ and the feedback law*

$$u_s = u(s, X_s^u), \qquad \mathbb{P} - \text{a.s. for a.a. } s \in [t,T],$$

is verified by u and X^u. Finally, the optimal trajectory X^u is a weak solution of the closed loop equation.

Proof. We notice that the closed loop equation (5.6) always admits a solution in the weak sense by an application of the Girsanov theorem. We can apply Theorem 5.5 and obtain all the required conclusions. □

6 The forward-backward stochastic differential equations in the infinite horizon case

Eventually we solve the infinite horizon control problem, that we briefly present. We consider the following infinite horizon cost, with a discount $\mu > 0$,

$$J(x, u(\cdot)) = \mathbb{E} \int_0^{+\infty} e^{-\mu s} \int_0^{+\infty} \ell(\xi, y(s,\xi), u(s))\, d\xi\, ds. \tag{6.1}$$

that we minimize over all admissible controls. The process y solves the equation

$$\begin{cases} \dfrac{\partial y}{\partial s}(s,\xi) = \dfrac{\partial^2 y}{\partial \xi^2}(s,\xi) - My(s,\xi) + f(y(s,\xi)), & s \geq 0 \ \xi \in (0,+\infty), \\ y(0,\xi) = x(\xi), \\ y(s,0) = u(s) + \dot{W}_s, \end{cases} \tag{6.2}$$

where $M > 0$ has to be chosen sufficiently large. Equation (6.2) can be reformulated in $\mathcal{H}$ as

$$\begin{cases} dX_s^u = (A - MI)X_s^u ds + F(X_s^u) ds + Bu_s ds + BdW_s & s \geq 0, \\ X_0^u = x, \end{cases} \tag{6.3}$$

Also we consider its uncontrolled version, that is, in mild form,

$$X_s = e^{s(A-MI)}x + \int_0^s e^{(s-r)(A-MI)} F(X_r^x)\, dr + \int_0^s e^{(s-r)(A-MI)} B\, dW_r, \qquad s \geq 0. \tag{6.4}$$

By theorem 2.4, for every $T > 0$, in $[0,T]$ this equation admits a unique mild solution, satisfying for every $p \in [1, +\infty)$, $\alpha \in [0, \theta/4)$,

$$\mathbb{E} \sup_{s \in (0,T]} s^{p\alpha} |X_s^x|^p_{D(-A)^\alpha} \leq c_{p,\alpha}(1 + |x|_{\mathcal{H}})^p. \tag{6.5}$$

where $c_{p,\alpha}$ is a constant. Moreover X^x is continuous and Gâteaux differentiable with respect to the initial datum x, see proposition 2.5, and for every $s \in [0,T]$, we can build the processes $\Theta^\alpha(\cdot, x)h$ following proposition 2.6. Moreover X^x admits the Malliavin derivative in every interval $[0,T]$, see proposition 2.7. In the next lemma we prove that under our assumptions the derivative $\nabla_x X_\cdot^x$ and the process $\Theta^\alpha(\cdot, x)h$ are uniformly bounded in time.

Lemma 6.1 *Assume that hypothesis 2.1 holds true and that in equation (6.4) M is sufficiently large (to be chosen in the following proof), then there exists a constant $C > 0$ such that $|\nabla_x X_t^x| + |\Theta^\alpha(t,x)h| \leq C|h|$ for every $t > 0$ and every $x, h \in \mathcal{H}$.*

Proof. We already know that, see proposition 2.5, the map $x \to X^x_\cdot$ has, at every point $x \in \mathcal{H}$, in every direction $h \in \mathcal{H}$, a Gâteaux derivative $\nabla_x X^x_\cdot h$ and the map $x \to \nabla_x X^x_\cdot$ belongs to $\mathcal{G}^1(\mathcal{H}, L^p_{\mathcal{P}}(\Omega; C([0,T];\mathcal{H}))$ and, for every direction $h \in \mathcal{H}$, the following equation holds $\mathbb{P}$-a.s.:

$$\nabla_x X^x_t h = e^{t(A-MI)}h + \int_0^t e^{(t-s)(A-MI)}\nabla_x F(X^x_s)\nabla_x X^x_s \, ds \qquad t \geq 0. \tag{6.6}$$

Since by [22], theorem 2.5, there exists $C > 0$, independent on f, such that for every $f \in \mathcal{H}$

$$|e^{tA}f|_{\mathcal{H}} \leq C|f|_{\mathcal{H}}, \qquad t \geq 0,$$

then

$$|e^{t(A-MI)}f|_{\mathcal{H}} \leq Ce^{-Mt}|f|_{\mathcal{H}}, \qquad t \geq 0.$$

So, by equation (6.6), we can deduce that

$$\begin{aligned}|\nabla_x X^x_t h| &\leq |e^{t(A-MI)}h| + |\int_0^t e^{(t-s)(A-MI)}\nabla_x F(X^x_s)\nabla_x X^x_s \, ds| \\ &\leq Ce^{-Mt}|h| + C_f C \sup_{0\leq s\leq t}|\nabla_x X^x_s h| \int_0^t e^{-M(t-s)}ds \\ &\leq C\left[|h| + \frac{C_f}{M}\sup_{0\leq s\leq t}|X^x_s|\right] \leq C\left[|h| + \frac{C_f}{M}\sup_{s\geq 0}|\nabla_x X^x_s|\right]\end{aligned}$$

The previous inequality holds true for every $t > 0$, so we get

$$\sup_{t\geq 0}|\nabla_x X^x_t h| \leq C\left[|h| + \frac{C_f}{M}\sup_{t\geq 0}|X^x_t|\right],$$

and assuming that $M > C_f \times C$ we obtain that $|\nabla_x X^x_s h| \leq C|h|$ for some constant $C > 0$ and for every $x, h \in \mathcal{H}$.
For what concerns Θ^α, we already know that they satisfy and equation like 2.17, with $A - MI$ in the place of A. We also remark that for every $\gamma > 0$, $|e^{(tA)B}| \leq M_\gamma t^{-\gamma}$, so it follows that

$$\begin{aligned}|\Theta^\alpha(t,x)h| &\leq C \times C_f \sup_{0\leq s\leq t}|\Theta^\alpha(s,x)h| \int_0^t e^{-M(t-s)}ds + M_\alpha C \times C_f|h| \int_0^t e^{-M(t-s)}s^{-\alpha}ds \\ &\leq C \times C_f\left[\frac{1}{M}\sup_{0\leq s\leq t}|\Theta^\alpha(s,x)h + \frac{1}{1-\alpha}M_\alpha|h| + \frac{M_\alpha}{M}|h|\chi_{[1,+\infty)}(t)\right].\end{aligned}$$

As for $\nabla_x X$, this inequality holds true for every $t > 0$ and again if $M > C_f \times C$ we obtain that $|\nabla_x X^x_s h \leq C|h|$ for some constant $C > 0$ and for every $x, h \in \mathcal{H}$ □

From now on, and in equations (6.2), (6.3) and (6.4), we take $M > C \times C_f$.

We need to notice that proposition 2.8 can be easily adequated to the case of a function w not depending on time, so we can state the following result about the joint quadratic variation of the process $u(X.)$ with W, where $u is a suitable function$

Proposition 6.2 *Suppose that $w \in C(\mathcal{H};\mathbb{R})$ is Gâteaux differentiable and that there exist constants K and m such that*

$$|w(x)| \leq K(1+|x|)^m, \qquad |\nabla w(x)| \leq K(1+|x|)^m, \qquad x \in H.$$

Assume that for every $x \in \mathcal{H}$, $\beta \in (0, \frac{1}{2} + \frac{\theta}{4})$, the linear operator $k \to \nabla w(x)(\lambda - A)^{1-\beta} k$ (a priori defined for $k \in D(\lambda - A)^{1-\beta}$) has an extension to a bounded linear operator $\mathcal{H} \to \mathbb{R}$, that we denote by $[\nabla w(\lambda - A)^{1-\beta}](x)$. Moreover assume that the map $(x, k) \to [\nabla w(\lambda - A)^{1-\beta}](x)k$ is continuous $\mathcal{H} \times \mathcal{H} \to \mathbb{R}$.
For $x \in \mathcal{H}$, let $\{X_s^x,\ s > 0\}$ be the solution of equation (6.4). Then the process $\{w(X_s^x),\ s > 0\}$ admits a joint quadratic variation process with W, on every interval $[t, s] \subset [0, +\infty)$, given by

$$\int_t^s [\nabla w(\lambda - A)^{1-\beta}](X_r^x)\,(\lambda - A)^\beta D_\lambda\, dr.$$

In order to solve the infinite horizon control problem, we consider the following backward stochastic differential equation:

$$dY_s^x = -\Psi(X_s^x, Z_s^x)\, ds + \mu Y_s^x\, ds + Z_s^x\, dW_s, \qquad s \geq 0, \tag{6.7}$$

for the unknown real processes Y^x, Z^x, also denoted by Y a nd Z. The equation is understood in the usual way: $\mathbb{P}$-a.s., for every $T > 0$,

$$Y_s^x + \int_s^T Z_r^x\, dW_r = Y_T^x + \int_s^T (\Psi(X_r^x, Z_r^x) - \mu Y_r^x)\, dr, \qquad s \geq 0. \tag{6.8}$$

We make the following assumptions:

Hypothesis 6.3 *i) The function $\Psi : \mathcal{H} \times \mathbb{R} \to \mathbb{R}$ is continuous in x and uniformly Lipschitz continuous in z that is $|\Psi(x, z_1) - \Psi(x, z_2)| \leq K|z_1 - z_2|$*

ii) $\sup_{x \in \mathcal{H}} |\Psi(x, 0)| := M < +\infty$

iii) $\mu > 0$.

We can state the following result on existence and uniqueness of a solution (Y, Z) of equation (6.7).

Proposition 6.4 *Assume hypotheses 2.1 and 6.3,*

i) For any $x \in \mathcal{H}$ equation (6.7) admits a unique solution (Y^x, Z^x) such that Y^x is a continuous process bounded by M/μ, and $Z \in L^2_{\mathcal{P},loc}((0, +\infty), \mathbb{R})$ with $\mathbb{E} \int_0^{+\infty} e^{-2\mu s} |Z_s|^2 ds < \infty$. The solution is unique in the class of processes such that Y^x is continuous and bounded and $Z^x \in L^2_{\mathcal{P},loc}((0, +\infty), \mathbb{R})$.

ii) Denoting by $(Y^{n,x}, Z^{n,x})$ the solution to the following, finite horizon, BSDE

$$Y_s^{n,x} + \int_s^n Z_r^{n,x}\, dW_r = + \int_s^n (\Psi(X_r^x, Z_r^{n,x}) - \mu Y_r^{n,x})\, dr, \tag{6.9}$$

then $|Y_s^{n,x}| \leq \frac{M}{\mu}$ and the following convergence rate holds:

$$|Y_s^{n,x} - Y_s^x| \leq \frac{M}{\mu} e^{-\mu(n-s)}. \tag{6.10}$$

Moreover

$$\mathbb{E} \int_0^{+\infty} e^{-2\lambda s} |Z_s^{n,x} - Z_s^x| ds \to 0. \tag{6.11}$$

iii) For all $T > 0$ and $p \geq 1$, the map $x \to (Y^x|_{[0,T]}, Z^x|_{[0,T]})$ is continuous from $\mathcal{H}$ to $L^p_{\mathcal{P}}(\Omega, C([0,T], \mathbb{R}) \times L^p_{\mathcal{P}}(\Omega, L^2([0,T], \mathbb{R})$

Proof. The proof follows the proof of proposition 3.2 in [19], in the case of ψ not depending on Y. □

We assume moreover the following:

Hypothesis 6.5 *$\Psi \in \mathcal{G}^1(\mathcal{H} \times \mathbb{R})$ and $\nabla_x \Psi(x,z) \leq c$ for every $x \in \mathcal{H}$, $z \in \mathbb{R}$, and for some constant $c > 0$. $\big(\nabla_z \Psi(x,z) \leq c$ is also bounded as a consequence of hypothesis 6.3, point i) $\big)$.*

We can state the following theorem:

Theorem 6.6 *Assume that hypotheses 2.1, 6.3 and 6.5 hold true. Then the map $x \to Y^x_0 \in \mathcal{G}^1(\mathcal{H}, \mathbb{R})$ and $|Y^x_0 + \nabla Y^x_0| \leq C$.*

Proof. We follow the proof of Theorem 3.1 in [19]. In that theorem, it was assumed that the operator $A + \nabla_x F(x)$ is dissipative. This is used in order to prove, see Lemma 3.1 in [19], that $|\nabla_x X^x_s h \leq C|h|$ for some constant $C > 0$ and for every $x, h \in \mathcal{H}$. In the present situation, we already know that, see lemma 6.1, $|\nabla_x X^x_s h \leq C|h|$ for some constant $C > 0$ and for every $x, h \in \mathcal{H}$.
The proof now follows exactly from the proof of theorem 3.1 in [19]. □

Next we have to prove a further regularity result, similar to the one stated in proposition 3.5. To this aim, we need to adapt the results in [4].

Lemma 6.7 *Let us consider the following BSDE on an infinite horizon,*

$$Y_t = Y_T - \int_t^T Z_r dW_r + \int_t^T (f_1(r, Z_r) + f_2(r) - \mu Y_r)\, dr, \tag{6.12}$$

where $\mu > 0$, $f_1 : \Omega \times [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ and $\forall z \in \mathbb{R}$ the process $(f_1(t,z))_{t \geq 0}$ is predictable, and the process $(f_2(t))_{t \geq 0}$ is predictable. Moreover assume that:

i) f_1 is uniformly lipschit continuous in z with lipschitz constant K: $\forall\, t > 0$, $\forall\, z_1, z_2 \in \mathbb{R}$

$$|f_1(t, z_1) - f_1(t, z_2)| \leq K|z_1 - z_2|, \qquad \mathbb{P} - a.s.,$$

and $f_1(t, 0)$ is bounded.

ii) there exists a constant $M > 0$ and a function $g \in L^1([0,1], \mathbb{R})$, such that for every $t \geq 0$,

$$|f_2(t)| \leq |g(t)| \chi_{[0,1]}(t) + M.$$

Then

i) there exists a solution (Y, Z) to equation (6.12) such that Y is a continuous, predictable process, bounded by a constant $C > 0$ and $Z \in L^2_{\mathcal{P},loc}([0, +\infty), \mathbb{R})$ and the solution is unique in such class of processes. Moreover $\int_0^{+\infty} e^{-2\mu s} |Z_s|^2 < +\infty$.

ii) Denoting by (Y^n, Z^n) the unique solution of the BSDE

$$Y^n_t = -\int_t^n Z^n_r dW_r + \int_t^n (f_1(r, Z^n_r) + f_2(r) - \mu Y^n_r)\, dr, \tag{6.13}$$

then $|Y_t^n| \leq C$ *and the following convergence rate holds:*

$$|Y_t^n - Y_t| \leq Ce^{-\mu(n-t)}$$

and moreover

$$\int_0^{+\infty} e^{-2\mu t}|Z_t^n - Z_t|^2 dt \to 0.$$

Proof. Let us consider (Y^n, Z^n) solution to equation (6.13). We set

$$f_3(r) := \begin{cases} \dfrac{f_1(r, Z_r^n) - f_1(r, 0)}{|Z_r^n|^2} Z_r^n & \text{if } Z_r^n \neq 0 \\ 0 & \text{otherwise .} \end{cases}$$

By our assumptions on f_1, f_3 is bounded and so by the Girsanov theorem, there exists a probabilty measure $\tilde{\mathbb{P}}$, equivalent to the original one $\mathbb{P}$, such that

$$\left\{ \tilde{W}_t = -\int_0^t f_3(r)dr + W_t, t \geq 0 \right\}$$

is a Brownian motion. So in $(\Omega, \mathcal{F}, \tilde{\mathbb{P}})$ equation (6.13) can be rewritten as

$$Y_t^n = -\int_t^n Z_r^n d\tilde{W}_r + \int_t^n (f_1(r, 0) + f_2(r) - \mu Y_r^n)\, dr,$$

Since $f_1(\cdot, 0)$ is bounded and f_2 is integrable near 0 and bounded otherwise, by the Gronwall lemma it follows that for every $t \in [0, n]$

$$|Y_t^n| \leq Ce^{-\mu(n-t)} \tilde{\mathbb{E}}^{\mathcal{F}_t} \int_t^n (|f_1(r, 0)| + |f_2(r)| dr$$

where C is a constant independent on n. By setting Y as the pointwise limit of Y^n we get that Y is bounded. By applying Itô formula to $e^{-2\mu t}|Y_t^n|^2$ it follows that

$$\int_0^{+\infty} e^{-2\mu t}|Z_t^n|^2 dt < +\infty.$$

Now let us define $\tilde{Y}_t^n := Y_t^n - Y_t$ and $\tilde{Z}_t^n := Z_t^n - Z_t$. $(\tilde{Y}_t^n, \tilde{Z}_t^n)$ solve, for $t \in [0, n]$, the following BSDE:

$$\tilde{Y}_t^n = -Y_n - \int_t^n \tilde{Z}_r^n dW_r + \int_t^n (f_1(r, Z_r^n) - f_1(r, Z_r))\, dr - \int_t^n \mu \tilde{Y}_r^n dr \tag{6.14}$$

We also set

$$F_r := \begin{cases} \dfrac{f_1(r, Z_r^n) - f_1(r, Z_r)}{|Z_r^n - Z_r|^2}(Z_r^n - Z_r) & \text{if } Z_r^n - Z_r \neq 0 \\ 0 & \text{otherwise .} \end{cases}$$

By the lipschitz assumptions on f_1, F is bounded and so by the Girsanov theorem, there exists a probabilty measure $\bar{\mathbb{P}}$, equivalent to the original one $\mathbb{P}$, such that

$$\left\{ \bar{W}_t = -\int_0^t F_r dr + W_t, t \geq 0 \right\}$$

is a Brownian motion. In $(\Omega, \mathcal{F}, \bar{P})$ equation (6.14) can be rewritten as

$$\bar{Y}_t^n = -Y_n - \int_t^n \tilde{Z}_r^n d\tilde{W}_r - \int_t^n \mu \tilde{Y}_r^n dr$$

So the following rate of convergence holds true:

$$|Y_t - Y_t^n| \le Ce^{-\mu(n-t)},$$

where C is a constant not depending on n. By applying Itô formula to $e^{-2\mu t}|\tilde{Y}_t^n|^2$ it follows that

$$\int_0^{+\infty} e^{-2\mu t}|Z_t^n - Z_t|^2 dt \to 0.$$

□

Theorem 6.8 *For every $\alpha \in [0, 1/2)$, $p \in [2, \infty)$ there exist two functions $P^\alpha(x)k$ and $Q^\alpha(x)k$, $x \in \mathcal{H}$, $k \in \mathcal{H}$ such that if $k \in D(\lambda - A)^\alpha$, $x \in \mathcal{H}$, then*

$$P^\alpha(x)k = \nabla_x Y_0^x (\lambda - A)^\alpha k \tag{6.15}$$

and

$$Q^\alpha(x)k = \nabla_x Z_0^x (\lambda - A)^\alpha k \tag{6.16}$$

Moreover the map $(x, k) \to P^\alpha(x)k$ is continuous from $\mathcal{H}$ to $\mathbb{R}$ and linear with respect to k.

Finally there exists a constant $C_{\nabla Y,\alpha,p}$ such that

$$|P^\alpha(x)k| \le C_{\nabla Y,\alpha}|k|_{\mathcal{H}}. \tag{6.17}$$

Proof. For $x \in \mathcal{H}$ and $k \in D(\lambda - A)^\alpha$, let $P^\alpha(x)k$ and $Q^\alpha(x)k$ be defined by (6.15) and (6.16) respectively. By Theorem 6.6 the map $k \to (P^\alpha(x)k, Q^\alpha(x)k)$ is a bounded linear operator from $D(\lambda - A)^\alpha$ to $\mathbb{R} \times \mathbb{R}$.
Let us introduce the pair of processes $(P^\alpha(\cdot, x)k, Q^\alpha(\cdot, x)k)$ solution of the following BSDE

$$\begin{aligned} P^\alpha(t, x)k = P^\alpha(T, x)k + \int_t^T \nabla_x \Psi(X_s^x, Z_s^x) \left(\Theta^\alpha(s, x)k + (\lambda - A)^\alpha e^{sA} k\right) ds \qquad (6.18)\\ - \int_0^t \mu P^\alpha(s, x)k\, ds + \int_t^T [\nabla_z \Psi(X_s^y, Z_s^x) Q^\alpha(s, x)k]\, ds - \int_t^T Q^\alpha(s, x)k\, dW_s, \qquad t \ge 0. \end{aligned}$$

Equation (6.18) admits a unique bounded solution by applying lemma 6.7.
Moreover, let us define the processes $(P^{\alpha,n}(\cdot, x)k, Q^{\alpha,n}(\cdot, x)k$ solution of the equation

$$\begin{aligned} P^{\alpha,n}(t, x)k = \int_t^n \nabla_x \Psi(X_s^x, Z_s^{n,x}) \left(\Theta^\alpha(s, x)k + (\lambda - A)^\alpha e^{sA} k\right) ds - \int_t^n \mu P^{\alpha,n}(s, x)\, ds \qquad (6.19)\\ + \int_t^n \nabla_z \Psi(X_s^x, Z_s^{n,x})\, Q^{\alpha,n}(s, x)k\, ds + \int_t^n Q^{\alpha,n}(s, x)k\, dW_s, \qquad t \ge 0 \end{aligned}$$

We notice that equation (6.19) is obtained by formally deriving equation (6.9) in the direction $(\lambda - A)^\alpha k$.
Equation (6.19) can be rewritten

$$\begin{aligned} P^{\alpha,n}(t, x)k = \int_t^n \nu^n(s, x)k\, ds - \int_t^n \mu P^{\alpha,n}(s, x)k\, ds \qquad (6.20)\\ + \int_t^n \nabla_z \Psi(X_s^x, Z_s^{n,x})\, Q^{\alpha,n}(s, x)k\, ds + \int_t^n Q^{\alpha,n}(s, x)k\, dW_s, \qquad 0 \le t \le n, \end{aligned}$$

where

$$\nu^n(s, x)k = \nabla_x \Psi(X_s^x, Z_s^{n,x}) \left(\Theta^\alpha(s, x)k + (\lambda - A)^\alpha e^{sA} k\right), \qquad t \in [0, n].$$

Now we choose arbitrary $k \in \mathcal{H}$ and notice that $\nu^n(s,x)k$ can still be defined by the above formulae. Hypothesis 6.5 and relation (2.16) yield:

$$\mathbb{E}\left(\int_0^n |\nu^n(s,x)k|^2 ds\right)^{p/2} \leq c_1\mathbb{E}\left(\int_0^n \left(|\Theta^\alpha(s,x)k| + s^{-\alpha}|k|_{\mathcal{H}}\right)^2 ds\right)^{p/2}$$
$$c_2\left[n^{p/2} + n^{(1-2\alpha)p/2}\right]|k|^p \leq c_3|k|^p,$$

where c_1, c_2 and c_3 are suitable constants independent on t, x, k. By Proposition 4.3 in [13], for all $k \in \mathcal{H}$ there exists a unique pair $((P^{\alpha,n}(\cdot,x)k, Q^{\alpha,n}(\cdot,x)k)$ belonging to $L^p_{\mathcal{P}}(\Omega, C([0,n],\mathbb{R})) \times L^p_{\mathcal{P}}(\Omega, L^2([0,n],\mathbb{R}))$ and solving equation (6.20). Moreover, by applying proposition 3.5, we get that $(x,k) \to P^{\alpha,n}(\cdot,x)k$ and the map $(x,k) \to Q^{\alpha,n}(\cdot,x)k$ are continuous from $[0,n) \times \mathcal{H} \times \mathcal{H}$ to $L^p_{\mathcal{P}}(\Omega, C([0,n],\mathbb{R}))$ and linear with respect to k. Finally there exists a constant $C_{\nabla Y,\alpha,p,n}$ such that

$$\mathbb{E}\sup_{s\in[0,n]} |P^{\alpha,n}(s,x)k|^p_{\mathcal{H}} + \mathbb{E}\left(\int_0^n |Q^{\alpha,n}(s,x)k|_{(\mathbb{R})} ds\right)^{p/2} \leq C_{\nabla Y,\alpha,p,n} n^{-\alpha p}(1+|x|_{\mathcal{H}})^p |k|^p_{\mathcal{H}}. \quad (6.21)$$

Moreover we want to prove that $P^{\alpha,n}(\cdot,x)k$ is a bounded process, uniformly in n. To this aim let $k \in D(\lambda - A)^\alpha$ and let $x, y \in \mathcal{H}$ such that $x - y = (\lambda - A)^\alpha k$.
Let us also define $\tilde{Y}^n_t = Y^{n,x}_t - Y^{n,y}_t$ and $\tilde{Z}^n_t = Z^{n,x}_t - Z^{n,y}_t$. So the pair $(\tilde{Y}^n, \tilde{Z}^n)$ solves the following backward stochastic differential equation:

$$\tilde{Y}^n_t = \int_t^n [\Psi(X^x_s, Z^{n,x}_s) - \Psi(X^y_s, Z^{n,y}_s)]\, ds - \int_t^n \mu\tilde{Y}^n_s\, ds + \int_t^n \tilde{Z}^n_s\, dW_s, \qquad t \in [0,n],$$

that we can also write as

$$\tilde{Y}^n_t = \int_t^n [\Psi(X^x_s, Z^{n,x}_s) - \Psi(X^y_s, Z^{n,x}_s)]\, ds - \int_t^n \mu\tilde{Y}^n_s\, ds$$
$$+ \int_t^n \left[\frac{\Psi(X^y_s, Z^{n,x}_s) - \Psi(X^y_s, Z^{n,y}_s)}{|Z^{n,x}_s - Z^{n,y}_s|^2}(Z^{n,x}_s - Z^{n,y}_s)\right]\tilde{Z}^n_s ds + \int_t^n \tilde{Z}^n_s\, dW_s, \qquad t \in [0,n].$$

Since Ψ is uniformly lipschitz with respect to z,

$$\left|\frac{\Psi(X^y_s, Z^{n,x}_s) - \Psi(X^y_s, Z^{n,y}_s)}{|Z^{n,x}_s - Z^{n,y}_s|^2}(Z^{n,x}_s - Z^{n,y}_s)\right| \leq L.$$

So, as in [4], lemma 3.1, by the Girsanov theorem there exists a probability measure $\tilde{\mathbb{P}}$ such that in $(\Omega, \mathcal{F}, \tilde{\mathbb{P}})$ the process $(\tilde{W}_t)_{t\in[0,n]}$ defined by

$$\tilde{W}_t = W_t + \int_0^t \frac{\Psi(X^y_s, Z^{n,x}_s) - \Psi(X^y_s, Z^{n,y}_s)}{|Z^{n,x}_s - Z^{n,y}_s|^2}(Z^{n,x}_s - Z^{n,y}_s)\, ds$$

is a real brownian motion. In this probability space, $(\tilde{Y}^n, \tilde{Z}^n)$ solve the following backward stochastic differential equation:

$$\tilde{Y}^n_t = \int_t^n [\Psi(X^x_s, Z^{n,x}_s) - \Psi(X^y_s, Z^{n,x}_s)]\, ds - \int_t^n \mu\tilde{Y}^n_s\, ds + \int_t^n \tilde{Z}^n_s\, d\tilde{W}_s, \qquad t \in [0,n],$$

Taking the conditional expectation in the previous equation, we get that

$$\tilde{Y}^n_t = \tilde{\mathbb{E}}^{\mathcal{F}_t}\int_t^n |\Psi(X^x_s, Z^{n,x}_s) - \Psi(X^y_s, Z^{n,x}_s)| ds - \mu\tilde{\mathbb{E}}^{\mathcal{F}_t}\int_t^n \tilde{Y}^n_s ds.$$

By lemma 6.1, and since Ψ is lipschitz with respect to x, we get

$$
\begin{aligned}
|\tilde{Y}_t^n| &\leq \tilde{\mathbb{E}}^{\mathcal{F}_t} \int_t^n e^{\mu(t-s)} |\Psi(X_s^x, Z_s^{n,x}) - \Psi(X_s^y, Z_s^{n,x})| ds \\
&\leq L \int_t^n e^{\mu(t-s)} |e^{sA}(\lambda - A)^\alpha k + \Theta^\alpha(s,x)k| ds \\
&\leq CL \int_t^n e^{\mu(t-s)} \left(s^{-\alpha} + 1\right) |k|_{\mathcal{H}} ds \leq C|k|_{\mathcal{H}},
\end{aligned}
$$

where C is a constant that may change its value from line to line, and that does not depend on n. So we have

$$
\sup_{t\in[0,n]} |\tilde{Y}_t^n| \leq C|k|_{\mathcal{H}},
$$

and consequently we get that for every $x, k \in \mathcal{H}$,

$$
\sup_{t\in[0,n]} |P^{\alpha,n}(t,x)k| \leq C|k|_{\mathcal{H}}.
$$

Then, again as in [4], by applying the Itô formula to $e^{-2\mu t}|P^{\alpha,n}(t,x)k|^2$, we get

$$
\begin{aligned}
de^{-2\mu t}|P^{\alpha,n}(t,x)k|^2 &= -2\mu e^{-2\mu t}|P^{\alpha,n}(t,x)k|^2 dt + 2\mu e^{-2\mu t}|P^{\alpha,n}(t,x)k|^2 dt \\
&\quad - 2e^{-2\mu t}\nu^n(t,x)kP^{\alpha,n}(t,x)k dt + e^{-2\mu t}|Q^{\alpha,n}(t,x)k|^2 dt \\
&\quad - 2e^{-2\mu t}Q^{\alpha,n}(t,x)kP^{\alpha,n}(t,x)k dW_t - 2e^{-2\mu t}\nabla_z\Psi(X_t^x, Z_t^{n,x})Q^{\alpha,n}(t,x)kP^{\alpha,n}(t,x)k dt.
\end{aligned}
$$

By taking expectation, we get

$$
\begin{aligned}
\mathbb{E}e^{-2\mu t}|P^{\alpha,n}(t,x)k|^2 &= \mathbb{E}\int_t^n e^{-2\mu s}\nu^n(s,x)P^{\alpha,n}(s,x)k ds - \int_t^n e^{-2\mu s}|Q^{\alpha,n}(s,x)k|^2 ds \\
&\quad + 2\int_t^n e^{-2\mu s}\nabla_z\Psi(X_s^x, Z_s^{n,x})Q^{\alpha,n}(s,x)kP^{\alpha,n}(s,x)k ds
\end{aligned}
$$

By Young inequality and since $P^{\alpha,n}(\ x)k$ is a uniformly bounded process we get that

$$
\mathbb{E}\int_0^{+\infty} e^{-2\mu t}\left(|P^{\alpha,n}(t,x)k|^2 + |Q^{\alpha,n}(t,x)k|^2\right) dt < +\infty.
$$

Now our proof substantially follows the proof of theorem 3.1 in [19]. Let $\mathcal{M}^{2,-2\mu}$ be the Hilbert space of all couples of real valued, $(\mathcal{F}_t)_{t\geq 0}$-adapted processes (y, z), such that

$$
|(y,z)|^2_{\mathcal{M}^{2,-2\mu}} = \mathbb{E}\int_0^{+\infty} e^{-2\mu t}\left(|y_t|^2 + |z_t|^2\right) dt < +\infty.
$$

Fixed $x, k \in \mathcal{H}$, there exists a subsequence of $(P^{\alpha,n}(\cdot,x)k, Q^{\alpha,n}(\cdot,x)k, P^{\alpha,n}(0,x)k)$, which we still denote by itself, such that $(P^{\alpha,n}(\cdot,x)k, Q^{\alpha,n}(\cdot,x)k, P^{\alpha,n}(0,x)k)$ converges weakly in $\mathcal{M}^{2,-2\mu}$ to $(U^{1,\alpha}(\cdot,x)k, V^{1,\alpha}(\cdot,x)k, \xi(x,k))$.
Next we define

$$
\begin{aligned}
U^{2,\alpha}(t,x)k = \xi(x,k) &- \int_0^t \nabla_x\Psi(X_s^x, Z_s^x)\left(\Theta^\alpha(s,x)k + e^{sA}(\lambda - A)^\alpha k\right) ds - \int_0^t \mu U^{1,\alpha}(s,x)k\, ds \\
&- \int_0^t \left[\nabla_z\Psi(X_s^y, Z_s^x)V^{1,\alpha}(s,x)k\right] ds + \int_0^t V^{1,\alpha}(s,x)k\, dW_s, \qquad t \geq 0
\end{aligned}
\tag{6.22}
$$

where (Y, Z) is the unique bounded solution of equation (6.7). Let us rewrite, for $t \in [0, n]$, equation (6.19) as

$$P^{\alpha,n}(t,x)k = P^{\alpha,n}(0,x)k - \int_t^n \nabla_x \Psi(X_s^x, Z_s^{n,x}) \left(\Theta^\alpha(s,x)k + e^{sA}(\lambda - A)^\alpha k\right) \, ds$$
$$+ \int_0^t \mu P^{\alpha,n}(s,x) \, ds + \int_0^t \nabla_z \Psi(X_s^x, Z_s^{n,x}) \, Q^{\alpha,n}(s,x)k \, ds + \int_t^n Q^{\alpha,n}(s,x)k \, dW_s.$$

As in [19], theorem 3.1, we can deduce that $P^{\alpha,n}(\cdot,x)k$ coverges weakly to $U^{2,\alpha}(\cdot,x)k$ in $L^2_{\mathcal{P}}([0,T];\mathbb{R})$. Moreover, by lemma 6.7, $(U^{2,\alpha}(\cdot,x)k, V^{1,\alpha}(\cdot,x)k)$ is the unique bounded solution to equation

$$U^{2,\alpha}(t,x)k = U^{2,\alpha}(0,x)k - \int_0^t \nabla_x \Psi(X_s^x, Z_s^x) \left(\Theta^\alpha(s,x)k + e^{sA}(\lambda - A)^\alpha k\right) ds + \int_0^t \mu U^{2,\alpha}(s,x)k \, ds$$
$$- \int_0^t \left[\nabla_z \Psi(X_s^y, Z_s^x) V^{1,\alpha}(s,x)k\right] ds + \int_0^t V^{1,\alpha}(s,x)k \, dW_s, \qquad t \geq 0,$$

so we also have $(U^{2,\alpha}(\cdot,x)k, V^{1,\alpha}(\cdot,x)k) = (P^\alpha(\cdot,x)k, Q^\alpha(\cdot,x)k)$, where (P^α, Q^α) solve BSDE (6.18), and in particular $U^{2,\alpha}(0,x)k = \xi(x)k$ is the limit of $P^{\alpha,n}(0,x)k$ along the original sequence.

Now we have to prove that the map $x \to U^{2,\alpha}(0,x)k$ is continuous. Let us consider $(U^{n,\alpha}, V^{n,\alpha})$ the unique solution of equation

$$U^{n,\alpha}(t,x)k = \int_t^n \nabla_x \Psi(X_s^x, Z_s^x) \left(\Theta^\alpha(s,x)k + e^{sA}(\lambda - A)^\alpha k\right) ds - \int_t^n \mu U^{n,\alpha}(s,x)k \, ds$$
$$+ \int_t^n \left[\nabla_z \Psi(X_s^y, Z_s^x) V^{n,\alpha}(s,x)k\right] ds - \int_t^n V^{n,\alpha}(s,x)k \, dW_s, \qquad t \geq 0.$$

By proposition 3.5, the map $x \to U^{n,\alpha}(0,x)k$ is continuous, and by arguments similar to the ones used before $U^{n,\alpha}(\cdot,x)k$ is a uniformly bounded process. In the probability space $(\Omega, \mathcal{F}, \hat{\mathbb{P}})$ where $\hat{\mathbb{P}}$ is a probability measure, equivalent to $\mathbb{P}$, such that the process

$$\left\{\hat{W}_t := -\int_0^t \nabla_z \psi(X_s^y, Z_s^x) ds + W_t, \qquad t \geq 0.\right\}$$

is a Brownian motion, $(e^{-\mu t} U^{n,\alpha}(t,x)k - U^{2,\alpha}(t,x)k), e^{-\mu t}(V^{n,\alpha}(t,x)k - V^{1,\alpha}(t,x)k)_{t \in [0,n]}$ solve the following BSDE,

$$\begin{cases} de^{-\mu t}(U^{n,\alpha}(t,x)k - U^{2,\alpha}(t,x)k) = e^{-\mu t}(V^{n,\alpha}(t,x)k - V^{1,\alpha}(t,x)k) dW_t, & t \in [0,n], \\ e^{-\mu n}(U^{n,\alpha}(n,x)k - U^{2,\alpha}(n,x)k) = e^{-\mu n} U^{n,\alpha}(n,x)k \end{cases}$$

We already know that $U^{n,\alpha}(n,x)k$ is uniformly bounded with respect to n, so the following rate of convergence holds true:

$$|U^{n,\alpha}(t,x)k - U^{2,\alpha}(t,x)k| \leq C e^{-\mu(n-t)} |k|,$$

where $C > 0$ is a constant that does not depend on n. So, if we take $(x_j)_{j \geq 1}, x \in \mathcal{H}$ such that $x_j \to x$, then, by the triangular inequality,

$$|U^{2,\alpha}(0,x_j)k - U^{2,\alpha}(0,x)k| \leq 2C e^{-\mu(n-t)} |k| + |U^{n\alpha}(0,x_j)k - U^{n,\alpha}(0,x)k|$$

and, by arguments similar to the ones used in proposition 3.5, the map $x \to U^{n,\alpha}(0,x)k$ is continuous. So we can conclude that the map $x \to P^\alpha(x)k$ is continuous from $\mathcal{H}$ to $\mathbb{R}$, linear with respect to k and there exists a constant $C > 0$ such that $|P^\alpha(x)k| \leq C|k|_{\mathcal{H}}$, so the proof is concluded. □

Corollary 6.9 *Setting $v(x) = Y^x$, we have $v \in C(\mathcal{H};\mathbb{R})$ and there exists a constant C such that $|v(x)| \leq C\,(1+|x|)^2$, $x \in \mathcal{H}$. Moreover v is Gâteaux differentiable and the map $(x,h) \to \nabla v(x)h$ is continuous.*

For all $\alpha \in [0,1/2)$ and $x \in \mathcal{H}$ the linear operator $k \to \nabla v(x)(\lambda - A)^\alpha k$ - a priori defined for $k \in D(\lambda - A)^\alpha$ - has an extension to a bounded linear operator $\mathcal{H} \to \mathbb{R}$, that we denote by $[\nabla v(\lambda - A)^\alpha](x)$.

Finally the map $(x,k) \to [\nabla v(\lambda - A)^\alpha](x)k$ is continuous $\mathcal{H} \times \mathcal{H} \to \mathbb{R}$ and there exists a constant $C > 0$ such that:

$$|[\nabla v(\lambda - A)^\alpha](x)k| \leq C|k|_{\mathcal{H}}, \qquad x,k \in H. \tag{6.23}$$

Proof. We recall that Y_0^x is deterministic. Continuity of v follows from the fact that, for every $T > 0$, the map $x \to Y_0^x$ is continuous with values in $L^p_{\mathcal{P}}(\Omega, C([0,+\infty],\mathbb{R}))$, $p \geq 2$.
Similarly, $\nabla_x v(x) = \nabla_x Y_0^x$ exists and has the required continuity properties, by Proposition 6.6. Next we notice that $P^\alpha(x)k = \nabla_x Y_0^x(\lambda - A)^\alpha k$. The existence of the required extensions and its continuity are direct consequences of Proposition 6.8. Finally the estimate (6.23) follows from (6.17). □

Remark 6.10 It is evident by construction that the law of Y^x and consequently the function v depends on the law of the Wiener process W but not on the particular probability $\mathbb{P}$ and Wiener process W we have chosen.

Corollary 6.11 *For every $t \geq 0$, $x \in \mathcal{H}$ we have, $\mathbb{P}$-a.s.,*

$$Y_s^x = v(X_s^x), \qquad \text{for all } s \geq 0, \tag{6.24}$$

$$Z_s^x = [\nabla v(\lambda - A)^{1-\beta}](X_s^x)\,(\lambda - A)^\beta D_\lambda, \qquad \text{for almost all } s \geq 0. \tag{6.25}$$

Proof. We start from the well-known equality: for $t \geq 0$, $\mathbb{P}$-a.s.,

$$X_s^x = X_s^{r,X_r^x}, \qquad \text{for all } s \geq r.$$

It follows easily from the uniqueness of the backward equation (3.1) that $\mathbb{P}$-a.s.,

$$Y_s^x = Y_s^{r,X_r^{t,x}}, \qquad \text{for all } s \geq r.$$

Setting $s = r$ we arrive at (6.24).
To prove (6.25) we note that it follows immediately from the backward equation (6.7), see also (6.8), that the joint quadratic variation of $\{Y_s^x,\ s \geq 0$ and W on an arbitrary interval $[t,s] \subset [0,+\infty)$ is equal to $\int_t^s Z_r\,dr$. By (6.24) the same result can be obtained by considering the joint quadratic variation of $\{v(X_s^x),\ s \geq 0\}$ and W. An application of Proposition 2.8 and remark 6.2 (whose assumptions hold true by Corollary 6.9) leads to the identity

$$\int_t^s Z_r\,dr = \int_t^s [\nabla v(\lambda - A)^{1-\beta}](X_r^x)\,(\lambda - A)^\beta D_\lambda\,dr,$$

and (6.25) is proved. □

7 The stationary Hamilton-Jacobi-Bellman equation

In this section the aim is to solve a second order partial differential equation, where the second order differential operator is the generator of the Markov process X_s^x, $s \geq 0$, solution of equation (6.4). We denote by P_s its transition semigroup:

$$P_s[\phi](x) = \mathbb{E}\,\phi(X_s^x), \qquad x \in \mathcal{H},\ s \geq 0,$$

for any bounded measurable $\phi : \mathcal{H} \to \mathbb{R}$. As for the finite horizon case, P_s will be considered as an operator acting on this class of functions.

Let us denote by $\mathcal{L}$ the generator of P_s, formally:

$$\mathcal{L}[\phi](x) = \frac{1}{2}\langle \nabla^2\phi(x)B, B\rangle + \langle Ax + F(x), \nabla\phi(x)\rangle,$$

where $\nabla\phi(x)$ and $\nabla^2\phi(x)$ are first and second Gâteaux derivatives of ϕ at the point $x \in \mathcal{H}$ (here they are identified with elements of $\mathcal{H}$ and $L(\mathcal{H})$ respectively).

The stationary Hamilton-Jacobi-Bellman equation that we are going to study is

$$\mathcal{L}[v](x) = \mu v(x) - \Psi(x, \nabla v(t,x)B). \tag{7.1}$$

We consider, for every $T > 0$, the variation of constants formula for (7.1):

$$v(x) = e^{-\mu T} P_T[u](x) - \int_0^T e^{-\mu s} P_s[\Psi(\cdot, \nabla v(\cdot)B](x)\, ds, x \in \mathcal{H},$$

where we recall that $B = (\lambda - A)D_\lambda$. This equality is still formal, since the term $(\lambda - A)D_\lambda$ is not defined. However with a slightly different interpretation we arrive at the following precise definition:

Definition 7.1 *Let $\beta \in [0, \frac{1}{2})$. We say that a function $v : \mathcal{H} \to \mathbb{R}$ is a mild solution of the Hamilton-Jacobi-Bellman equation (7.1) if the following conditions hold:*

(i) $v \in C(\mathcal{H}; \mathbb{R})$, is Gâteaux differentiable and the map $(x, h) \to \nabla v(x)h$ is continuous $\mathcal{H} \times \mathcal{H} \to \mathbb{R}$.

(iii) For all $x \in \mathcal{H}$ the linear operator $k \to \nabla v(x)(\lambda - A)^{1-\beta}k$ (a priori defined for $k \in D(\lambda - A)^{1-\beta}$) has an extension to a bounded linear operator $\mathcal{H} \to \mathbb{R}$, that we denote by $[\nabla v(\lambda - A)^{1-\beta}](x)$. Moreover the map $(x, k) \to [\nabla v(\lambda - A)^{1-\beta}](x)k$ is continuous $\mathcal{H} \times \mathcal{H} \to \mathbb{R}$ and there exist constants $C, m \geq 0$, $\kappa \in [0, 1)$ such that

$$|[\nabla v(\lambda - A)^{1-\beta}](x)|_{\mathcal{H}^*} \leq C, \qquad x \in \mathcal{H}. \tag{7.2}$$

(iv) the following equality holds for every $x \in \mathcal{H}$:

$$v(x) = e^{-\mu T} P_T[u](x) - \int_0^T e^{-\mu s} P_s[\Psi(\cdot, [\nabla v(\lambda - A)^{1-\beta}](\cdot)\ (\lambda - A)^\beta D_\lambda)](x)\, ds, \tag{7.3}$$

Theorem 7.1 *Assume Hypotheses 2.1, 6.3, 6.5 and that in equation (6.4) M is taken sufficiently large (see also lemma 6.1). Then there exists a unique mild solution of the stationary Hamilton-Jacobi-Bellman equation (7.1). The solution v is given by the formula*

$$v(x) = Y_0^x,$$

where (X, Y, Z) is the solution of the forward-backward system (6.4)-(6.8).

Proof. The proof is similar to the proof of theorem 6.1 in [14], noticing, as in [19], that we can find a mild solution for every $\lambda > 0$, and noticing, as in the finite horizon case, see also theorem 4.1, that $\nabla v(x)G(x)$ is replaced by $[\nabla v(\lambda - A)^{1-\beta}](x)\ (\lambda - A)^\beta D_\lambda$.

□

8 Synthesis of the optimal control: the infinite horizon case

Let us consider the cost functional (6.1), and we make the following assumptions:

Hypothesis 8.1 *$\ell : [0, +\infty) \times \mathbb{R} \times \mathcal{U} \to \mathbb{R}$ is continuous and there exists $C > 0$ and $g \in L^1([0, +\infty))$ such that*

$$|l(\xi, x, u)| \leq Cg(\xi), \quad \textit{for every} \quad \xi \in [0, +\infty), x \in \mathbb{R}, u \in \mathcal{U}.$$

Moreover there exists $C, \epsilon > 0$ such that

$$|l(\xi, x_1, u) - l(\xi, x_2, u)| \leq C \frac{|x_1 - x_2|}{(1+\xi)^{\frac{1+\epsilon}{2}}} \sqrt{\rho(\xi)} \quad \textit{for every} \quad \xi \in [0, +\infty), x_1, x_2 \in \mathbb{R}, u \in \mathcal{U}.$$

In this section we assume that Hypothesis 8.1 holds. We briefly reformulate the cost (6.1) in an abstract form. We define

$$L(x, u) = \int_0^{+\infty} \ell(s, \xi, x(\xi), u)\, d\xi, \qquad x = x(\cdot) \in \mathcal{H}, u \in \mathcal{U},$$

and so $L : \mathcal{H} \times \mathcal{U} \to \mathbb{R}$ is well defined and measurable and the cost functional (6.1) can be written in the form

$$J(x, u(\cdot)) = \mathbb{E} \int_0^{+\infty} e^{-\mu s} L(X_s^u, u_s)\, ds. \tag{8.1}$$

Moreover for $x \in \mathcal{H}$, $z \in \mathbb{R}$ we define the hamiltonian:

$$\Psi(x, z) = \inf_{u \in \mathcal{U}} \{zu + L(x, u)\},$$

where zu denotes the scalar product in $\mathbb{R}$. We notice that setting $C_{\mathcal{U}} = \sup\{|u| : u \in \mathcal{U}\}$ we have $|\Psi(x, z_1) - \Psi(x, z_2)| \leq C_{\mathcal{U}} |z_1 - z_2|$, for every $x \in \mathcal{H}$, $z_1, z_2 \in \mathbb{R}$. Moreover we assume that the hamiltonian Ψ satisfies hypothesis 6.5.
Analougsly to the infinite horizon case, if we define

$$\Gamma(x, z) = \{u \in \mathcal{U} : zu + L(x, u) = \Psi(x, z)\} \tag{8.2}$$

then $\Gamma(x, z) \neq \emptyset$ for every $x \in \mathcal{H}$ and every $z \in \mathbb{R}$ and so it admits a measurable selection, $\gamma : \mathcal{H} \times \mathbb{R} \to \mathcal{U}$ with γ measurable and $\gamma(x, z) \in \Gamma(x, z)$ for every $x \in \mathcal{H}$ and every $z \in \mathbb{R}$.

We now reformulate the optimal control problem in the weak sense, following the approach of [12]. As in section 6.3, $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W)$ is an *admissible set-up*, and $\mathbb{U} = (\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W, u, X^u)$ is an *admissible control system* (a.c.s.) if:

- $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W)$ is an admissible set-up;
- $u : \Omega \times [0, +\infty) \to \mathbb{R}$ is an $(\mathcal{F}_t)$-predictable process with values in $\mathcal{U}$;
- $\{X_t^u,\, t \in [0, +\infty)\}$ is an $(\mathcal{F}_t)$-adapted continuous process with values in $\mathcal{H}$, mild solution of the state equation (6.3) with initial condition $X_0^u = x$.

For $x \in \mathcal{H}$ we wish to minimize the cost (8.1):

$$J(x, U(\cdot)) = \mathbb{E} \int_0^{+\infty} e^{-\mu s} L(X_s^u, u(s))\, ds \tag{8.3}$$

over all admissible control systems.

We recall that by $v : \mathcal{H} \to \mathbb{R}$, we denote the mild solution of the Hamilton-Jacobi-Bellman equation (7.1).

Theorem 8.2 *Assume Hypotheses 2.1, 8.1 and hat Ψ satisfies hypothesis 6.5. For every $x \in \mathcal{H}$ and for all admissible control systems U we have $J(t, x, U)) \geq v(x)$, and the equality holds if and only if*

$$u_s \in \Gamma\Big(, X_s^u, [\nabla v(\lambda - A)^{1-\beta}](s, X_s^u)\ (\lambda - A)^\beta D_\lambda\Big)$$

Moreover, if

$$u(x) = \gamma\Big(x, [\nabla v(\lambda - A)^{1-\beta}](x)\ (\lambda - A)^\beta D_\lambda\Big), \qquad x \in \mathcal{H},$$

then there exists an adapted process $\{\overline{X}_s,\ s \geq 0\}$ with continuous trajectories solving the closed loop equation: $\mathbb{P}$-a.s.

$$\begin{aligned} \overline{X}_s &= e^{sA} x_0 + \int_0^s e^{(s-r)A} F(r, \overline{X}_r)\, dr + \int_t^s (\lambda - A)^{1-\beta} e^{(s-r)A} (\lambda - A)^\beta D_\lambda\, dW_r \\ &+ \int_0^s (\lambda - A)^{1-\beta} e^{(s-r)A} (\lambda - A)^\beta D_\lambda\, u(r, \overline{X}_r) dr, \qquad s \geq 0, \end{aligned} \tag{8.4}$$

and $(\overline{X}_s, \gamma(\overline{X}_s, [\nabla v(\lambda - A)^{1-\beta}](\overline{X}_s)\ (\lambda - A)^\beta D_\lambda))$ is an optimal pair.

Proof. The proof is similar to the proof of Theorem 5.1 in [19]. Just notice that in this case by (6.25) we have $Z_s^x = [\nabla v(\lambda - A)^{1-\beta}](X_s^x)\ (\lambda - A)^\beta D_\lambda$ and the role of G in [19], Theorem 5.1 is here played by $B = (\lambda - A) D_\lambda$. □

Remark 8.3 *We notice that the techniques used to treat the stationary Hamilton Jacobi Bellman equation and the infinite horizon optimal control problem can be applied to the case of boundary conditions of Neumann type in the state equation, i.e to a state equation like the one studied in [8] but considered for every $t > 0$.*

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