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SEMICLASSICAL LIMIT FOR SCHRÖDINGER EQUATIONS WITH MAGNETIC FIELD AND HARTREE-TYPE NONLINEARITIES

SILVIA CINGOLANI, SIMONE SECCHI, AND MARCO SQUASSINA

Abstract. The semi-classical regime of standing wave solutions of a Schrödinger equation in presence of non-constant electric and magnetic potentials is studied in the case of non-local nonlinearities of Hartree type. It is show that there exists a family of solutions having multiple concentration regions which are located around the minimum points of the electric potential.

Contents

1. Introduction and main result

1.1. **Introduction.** Some years ago, Penrose derived in [26] a system of nonlinear equations by coupling the linear Schrödinger equation of quantum mechanics with Newton's gravitational law. Roughly speaking, a point mass interacts with a density of matter described by the square of the wave function that solves the Schrödinger equation. If m

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is the mass of the point, this interaction leads to the system in $\mathbb{R}^3$

$$\begin{cases} \frac{\hbar^2}{2m}\Delta\psi - V(x)\psi + U\psi = 0, \\ \Delta U + 4\pi\gamma|\psi|^2 = 0, \end{cases} \tag{1.1}$$

where ψ is the wave function, U the gravitational potential energy, V a given Schrödinger potential, $\hbar$ the Planck constant and $\gamma = Gm^2$, G being Newton's constant of gravitation. Notice that, by means of the scaling

$$\psi(x) = \frac{1}{\hbar}\frac{\hat{\psi}(x)}{\sqrt{8\pi\gamma m}}, \quad V(x) = \frac{1}{2m}\hat{V}(x), \quad U(x) = \frac{1}{2m}\hat{U}(x)$$

system (1.1) can be written, maintaining the original notations, as

$$\begin{cases} \hbar^2\Delta\psi - V(x)\psi + U\psi = 0, \\ \hbar^2\Delta U + |\psi|^2 = 0. \end{cases} \tag{1.2}$$

The second equation in (1.2) can be explicitly solved with respect to U, so that the system turns into the single nonlocal equation

$$\hbar^2\Delta\psi - V(x)\psi + \frac{1}{4\pi\hbar^2}\Big(\int_{\mathbb{R}^3}\frac{|\psi(\xi)|^2}{|x-\xi|}d\xi\Big)\psi = 0 \quad \text{in } \mathbb{R}^3. \tag{1.3}$$

The Coulomb type convolution potential $W(x) = |x|^{-1}$ in $\mathbb{R}^3$ is also involved in various physical applications such as electromagnetic waves in a Kerr medium (in nonlinear optics), surface gravity waves (in hydrodynamics) as well as ground states solutions (in quantum mechanical systems). See for instance [1] for further details and [14] for the derivation of these equations from a many-body Coulomb system.

In the present paper, we will study the semiclassical regime (namely the existence and asymptotic behavior of solutions as $\hbar \to 0$) for a more general equation having a similar structure. Taking ε in place of $\hbar$, our model will be written as

$$\Big(\frac{\varepsilon}{\mathrm{i}}\nabla - A(x)\Big)^2 u + V(x)u = \frac{1}{\varepsilon^2}\left(W * |u|^2\right)u \quad \text{in } \mathbb{R}^3, \tag{1.4}$$

in $\mathbb{R}^3$, where the convolution kernel $W : \mathbb{R}^3 \setminus \{0\} \to (0,\infty)$ is an even smooth kernel, homogeneous of degree -1 and we denote by i the imaginary unit. The choice of $W(x) = |x|^{-1}$ recovers (1.3). Equation (1.4) is equivalent to

$$\Big(\frac{1}{\mathrm{i}}\nabla - A_\varepsilon(x)\Big)^2 u + V_\varepsilon(x)u = \left(W * |u|^2\right)u \quad \text{in } \mathbb{R}^3, \tag{1.5}$$

where we have set $A_\varepsilon(x) = A(\varepsilon x)$ and $V_\varepsilon(x) = V(\varepsilon x)$.

The vector-valued field A represents a given external magnetic potential, and forces the solutions to be, in general, complex-valued (see [11] and references therein). To the best of our knowledge, in this framework, no previous result involving the electromagnetic field can be found in the literature. On the other hand, when $A \equiv 0$, it is known that solutions have a constant phase, so that it is not a restriction to look for real-valued solutions. In this simpler situation, we recall the results contained in [24, 25], stating that at fixed $\hbar = \varepsilon$ the system (1.2) can be uniquely solved by radially symmetric functions. Moreover these solutions decay exponentially fast at infinity together with their first derivatives. The mere existence of one solution can be traced backed to the paper [22].

Later on, Wei and Winter proposed in [28] a deeper study of the multi-bumps solutions to the same system, and proved an existence result that can be summarized as follows: if $k \geq 1$ and $P_1, \dots, P_k \in \mathbb{R}^3$ are given non-degenerate critical points of V (but local extrema are also included without any further requirements), then multi-bump solutions $\psi_\hbar$ exist that concentrate at these points when $\hbar \to 0$. A similar equation is also studied in [23], where multi-bump solutions are found by some finite-dimensional reduction. The main result about existence leans on some *non-degeneracy* assumption on the solutions of a limiting problem, which was actually proved in [28, Theorem 3.1] only in the particular case $W(x) = |x|^{-1}$ in $\mathbb{R}^3$. Moreover, the equation investigated in [23] cannot be deduced from a singularly perturbed problem like (1.3), because the terms do not scale coherently.

For precise references to some classical works (well-posedness, regularity, long-term behaviour) related to the nonlinear Schrödinger equation with Hartree nonlinearity for Coulomb potential and $A = 0$, we refer to [27, p.66]. We would also like to mention the work of Carles et al. [8].

1.2. **Statement of the main result.** We shall study equation (1.5) by exploiting a penalization technique which was recently developed in [12], whose main idea is searching for solutions in a suitable class of functions whose location and shape is the one expected for the solution itself. This approach seems appropriate, since it does not need very strong knowledge of the *limiting problem* (2.1) introduced in the next section. In particular, for a general convolution kernel W, we still do not know if its solutions are non-degenerate. In order to state our main result (as well as the technical lemma contained in Section 2 and 3), the following conditions will be retained:

(A1): $A : \mathbb{R}^3 \to \mathbb{R}^3$ is of class C^1.

(V1): $V : \mathbb{R}^3 \to \mathbb{R}$ is a continuous function such that

$$0 \leq V_0 = \inf_{x \in \mathbb{R}^3} V(x), \qquad \liminf_{|x| \to \infty} V(x) > 0.$$

(V2): There exist bounded disjoint open sets $O^1, \dots, O^k$ such that

$$0 < m_i = \inf_{x \in O^i} V(x) < \min_{x \in \partial O^i} V(x), \qquad i = 1, \dots, k.$$

(W): $W : \mathbb{R}^3 \setminus \{0\} \to (0, \infty)$ is a function of class C^1 such that $W(\lambda x) = \lambda^{-1} W(x)$ for any $\lambda > 0$ and $x \neq 0$.

Convolution kernels such as $W(x) = x_i^2/|x|^3$, for $x \in \mathbb{R}^3 \setminus \{0\}$ or, more generally, $W(x) = W_1(x)/W_2(x)$, for $x \in \mathbb{R}^3 \setminus \{0\}$, where W_1, W_2 are positive, even and (respectively) homogeneous of degree m and $m+1$ satisfy (W).

For each $i \in \{1, \dots, k\}$, we define

$$\mathcal{M}^i = \{x \in O^i : V(x) = m_i\}$$

and $Z = \{x \in \mathbb{R}^3 : V(x) = 0\}$ and $m = \min_{i\in\{1,\dots,k\}} m_i$. By **(V1)** we can fix $\widetilde{m} > 0$ with

$$\widetilde{m} < \min \Big\{ m, \liminf_{|x|\to\infty} V(x) \Big\}$$

and define $\tilde{V}_\varepsilon(x) = \max\{\widetilde{m}, V_\varepsilon(x)\}$. Let H_ε be the Hilbert space defined by the completion of $C_0^\infty(\mathbb{R}^3, \mathbb{C})$ under the scalar product

$$\langle u, v\rangle_\varepsilon = \Re \int_{\mathbb{R}^3} \Big(\frac{1}{\mathrm{i}}\nabla u - A_\varepsilon(x)u\Big)\overline{\Big(\frac{1}{\mathrm{i}}\nabla v - A_\varepsilon(x)v\Big)} + \tilde{V}_\varepsilon(x)u\bar{v}dx$$

and $\|\cdot\|_\varepsilon$ the associated norm.

The main result of the paper is the following

Theorem 1.1. *Suppose that **(A)**, **(V1-2)** and **(W)** hold. Then for any $\varepsilon > 0$ sufficiently small, there exists a solution $u_\varepsilon \in H_\varepsilon$ of equation (1.5) such that $|u_\varepsilon|$ has k local maximum points $x_\varepsilon^i \in O^i$ satisfying*

$$\lim_{\varepsilon\to 0} \max_{i=1,\dots,k} \mathrm{dist}(\varepsilon x_\varepsilon^i, \mathcal{M}^i) = 0,$$

and for which

$$|u_\varepsilon(x)| \le C_1 \exp\big\{ - C_2 \min_{i=1,\dots,k} |x - x_\varepsilon^i|\big\},$$

for some positive constants C_1, C_2. Moreover for any sequence $(\varepsilon_n) \subset (0, \varepsilon]$ with $\varepsilon_n \to 0$ there exists a subsequence, still denoted by (ε_n), such that for each $i \in \{1, \dots, k\}$ there exist $x^i \in \mathcal{M}^i$ with $\varepsilon_n x_{\varepsilon_n}^i \to x^i$, a constant $w_i \in \mathbb{R}$ and $U_i \in H^1(\mathbb{R}^3, \mathbb{R})$ a positive least energy solution of

$$-\Delta U_i + m_i U_i - (W * |U_i|^2)U_i = 0, \quad U_i \in H^1(\mathbb{R}^3, \mathbb{R}); \tag{1.6}$$

for which one has

$$u_{\varepsilon_n}(x) = \sum_{i=1}^{k} U_i\left(x - x_{\varepsilon_n}^i\right) \exp\left(\mathrm{i}\left(w_i + A(x^i)(x - x_{\varepsilon_n}^i)\right)\right) + K_n(x) \tag{1.7}$$

where $K_n \in H_{\varepsilon_n}$ satisfies $\|K_n\|_{H_{\varepsilon_n}} = o(1)$ as $n \to +\infty$.

The one and two dimensional cases would require a separate analysis in the construction of the penalization argument (see e.g. [5] for a detailed discussion). The study of the cases of dimensions larger than three is less interesting from the physical point of view. Moreover, having in mind the soliton dynamics as a possible further development, in dimensions $N \ge 4$ the time dependent Schrödinger equation with kernels, say, of the type $W(x) = |x|^{2-N}$ does not have global existence in time for all H^1 initial data (see e.g. [9, Remark 6.5.2, p.114]) as well as the heuristic discussion in the next section).

1.3. **A heuristic remark: multi-bump dynamics.** We could also think of Theorem 1.1 as the starting point in order to rigorously justify a multi-bump soliton dynamics for the full Schrödinger equation with an external magnetic field

$$
\begin{cases} \mathrm{i}\varepsilon\partial_t u + \frac{1}{2}\left(\frac{\varepsilon}{\mathrm{i}}\nabla - A(x)\right)^2 u + V(x)u = \frac{1}{\varepsilon^2}\left(W * |u|^2\right) u & \text{in } \mathbb{R}^3, \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^3. \end{cases} \tag{1.8}
$$

We describe in the following what we expect to hold (the question is open even for $A = 0$, see the discussion by J. Fröhlich et al. in [16]). Given $k \geq 1$ positive numbers $g_1, \dots, g_k$, if $\mathcal{E} : H^1(\mathbb{R}^3) \to \mathbb{R}$ is defined as

$$
\mathcal{E}(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{1}{2}\int_{\mathbb{R}^6} W(x-y)|u(x)|^2|u(y)|^2 dxdy,
$$

let $U_j : \mathbb{R}^3 \to \mathbb{R}$ $(j = 1, \dots, k)$ be the solutions to the minimum problems

$$
\mathcal{E}(U_j) = \min\{\mathcal{E}(u) : u \in H^1(\mathbb{R}^3),\ \|u\|_{L^2}^2 = g_j\},
$$

which solve the equations

$$
-\frac{1}{2}\Delta U_j + m_j U_j = W * |U_j|^2 U_j \qquad \text{in } \mathbb{R}^3,
$$

for some $m_j \in \mathbb{R}$. Consider now in (1.8) an initial datum of the form

$$
u_0(x) = \sum_{j=1}^{k} U_j\Big(\frac{x - x_0^j}{\varepsilon}\Big) e^{\frac{\mathrm{i}}{\varepsilon}[A(x_0^j)\cdot(x - x_0^j) + x\cdot\xi_0^j]}, \quad x \in \mathbb{R}^3,
$$

where $x_0^j \in \mathbb{R}^3$ and $\xi_0^j \in \mathbb{R}^3$ $(j = 1, \dots, k)$ are initial position and velocity for the ODE

$$
\begin{cases} \dot{x}_j(t) = \xi_j(t), \\ \dot{\xi}_j(t) = -\nabla V(x_j(t)) - \varepsilon \sum\limits_{i\neq j}^{k} m_i \nabla W(x_j(t) - x_i(t)) - \xi_j(t) \times B(x_j(t)), \\ x_j(0) = x_0^j, \quad \xi_j(0) = \xi_0^j, \qquad j = 1, \dots, k, \end{cases} \tag{1.9}
$$

with $B = \nabla \times A$. The systems can be considered as a mechanical system of k interacting particles of mass m_i subjected to an external potential as well as a mutual Newtonian type interaction. Therefore, the conjecture it that, under suitable assumptions, the following representation formula might hold

$$
u_\varepsilon(x,t) = \sum_{j=1}^{k} U_j\Big(\frac{x - x_j(t)}{\varepsilon}\Big) e^{\frac{\mathrm{i}}{\varepsilon}[A(x_j(t))\cdot(x - x_j(t)) + x\cdot\xi_j(t) + \theta_\varepsilon^j(t)]} + \omega_\varepsilon, \tag{1.10}
$$

locally in time, for certain phases $\theta_\varepsilon^i : \mathbb{R}^+ \to [0, 2\pi)$, where ω_ε is small (in a suitable sense) as $\varepsilon \to 0$, provided that the centers x_0^j in the initial data are chosen sufficiently far from each other. Now, neglecting as $\varepsilon \to 0$ the interaction term (ε-dependent)

$$
\varepsilon \sum_{i\neq j}^{k} m_i \nabla W(x_j(t) - x_i(t))
$$

in the Newtonian system (1.9) and taking

$$x_0^1, \dots, x_0^k \in \mathbb{R}^3 : \ \nabla V(x_0^j) = 0 \quad \text{and} \quad \xi_0^j = 0, \qquad \text{for all } j = 1, \dots, k,$$

then the solution of (1.9) is

$$x_j(t) = x_0^j, \ \ \xi_j(t) = 0, \qquad \text{for all } t \in [0, \infty) \text{ and } j = 1, \dots, k,$$

so that the representation formula (1.10) reduces, for $\varepsilon = \varepsilon_n \to 0$,

$$u_{\varepsilon_n}(x,t) = \sum_{j=1}^{k} U_j\Big(\frac{x - x_0^j}{\varepsilon_n}\Big) e^{\frac{\mathrm{i}}{\varepsilon_n}[A(x_0^j)\cdot(x-x_0^j)+\theta_{\varepsilon_n}^j(t)]} + \omega_{\varepsilon_n},$$

namely to formula (1.7) up to a change in the phase terms and up to replacing x with $\varepsilon_n x$ and x_0^j with $\varepsilon_n x_{\varepsilon_n}^j$ for all $j = 1, \dots, k$.

Plan of the paper.

In Section 2 we obtain several results about the structure of the solutions of the limiting problem (1.6). In particular, we study the compactness of the set of real ground states solutions and we achieve a result about the orbital stability property of these solutions for the Pekar-Choquard equation. In Section 3 we perform the penalization scheme. In particular we obtain various energy estimates in the semiclassical regime $\varepsilon \to 0$ and we get a Palais-Smale condition for the penalized functional which allows to find suitable critical points inside the concentration set. Finally we conclude the proof of Theorem 1.1.

Main notations.

(1) i is the imaginary unit.
(2) The complex conjugate of any number $z \in \mathbb{C}$ is denoted by $\bar{z}$.
(3) The real part of a number $z \in \mathbb{C}$ is denoted by $\Re z$.
(4) The imaginary part of a number $z \in \mathbb{C}$ is denoted by $\Im z$.
(5) The symbol $\mathbb{R}^+$ (resp. $\mathbb{R}^-$) means the positive real line $[0, \infty)$ (resp. $(-\infty, 0]$).
(6) The ordinary inner product between two vectors $a, b \in \mathbb{R}^3$ is denoted by $\langle a \mid b \rangle$.
(7) The standard L^p norm of a function u is denoted by $\|u\|_{L^p}$.
(8) The standard L^∞ norm of a function u is denoted by $\|u\|_{L^\infty}$.
(9) The symbol Δ means $D_{x_1}^2 + D_{x_2}^2 + D_{x_3}^2$.
(10) The convolution $u * v$ means $(u * v)(x) = \int u(x-y)v(y)dy$.

2. Properties of the set of ground states

For any positive real number a, the limiting equation for the Hartree problem (1.4) is

$$-\Delta u + au = \big(W * |u|^2\big)\, u \qquad \text{in } \mathbb{R}^3. \tag{2.1}$$

2.1. **A Pohozaev type identity.** We now give the statement of a useful identity satisfied by solutions to problem (2.1).

Lemma 2.1. *Let $u \in H^1(\mathbb{R}^3, \mathbb{C})$ be a solution to* (2.1). *Then*

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2\,dx + \frac{3}{2}a\int_{\mathbb{R}^3} |u|^2 dx = \frac{5}{4}\int_{\mathbb{R}^3\times\mathbb{R}^3} W(x-y)|u(x)|^2|u(y)|^2 dxdy. \tag{2.2}$$

Proof. The proof is straightforward, and we include it just for the sake of completeness. We multiply equation (2.1) by $\langle x \mid \overline{\nabla u}\rangle$. Notice that

$$\Delta u\langle x \mid \overline{\nabla u}\rangle = \operatorname{div}\Big(\langle x \mid \overline{\nabla u}\rangle \nabla u - \frac{1}{2}|\nabla u|^2 x\Big), \tag{2.3}$$

$$-au\langle x \mid \overline{\nabla u}\rangle = -a\operatorname{div}\Big(\frac{1}{2}u^2 x\Big) + \frac{3}{2}au^2, \tag{2.4}$$

$$\varphi(x)u\langle x \mid \overline{\nabla u}\rangle = \operatorname{div}\Big(\frac{1}{2}u^2\varphi(x)x\Big) - \frac{1}{2}u^2\operatorname{div}(\varphi(x)x), \tag{2.5}$$

where $\varphi(x) = \int_{\mathbb{R}^3} W(x-y)|u(y)|^2 dy$. We can easily obtain that

$$\begin{aligned}\operatorname{div}(\varphi(x)x) &= \sum_{i=1}^{3}\frac{\partial}{\partial x_i}\Big(x_i\int_{\mathbb{R}^3} W(x-y)|u(y)|^2\,dy\Big)\\ &= N\int_{\mathbb{R}^3} W(x-y)|u(y)|^2\,dy + \int_{\mathbb{R}^3}\langle \nabla W(x-y) \mid x\rangle |u(y)|^2 dy.\end{aligned}$$

Summing up (2.3), (2.4) and (2.5) and integrating by parts, we reach the identity

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2\,dx + \frac{3}{2}a\int_{\mathbb{R}^3} u^2\,dx - \frac{3}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} W(x-y)|u(x)|^2|u(y)|^2\,dx\,dy&\\ -\frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3}\langle\nabla W(x-y) \mid x\rangle|u(x)|^2|u(y)|^2 dxdy = 0.&\end{aligned} \tag{2.6}$$

By exchanging x with y, we find that

$$\begin{aligned}&\int_{\mathbb{R}^3\times\mathbb{R}^3}\langle\nabla W(x-y) \mid x\rangle|u(x)|^2|u(y)|^2\,dx\,dy =\\ &\qquad -\int_{\mathbb{R}^3\times\mathbb{R}^3}\langle\nabla W(x-y) \mid y\rangle|u(x)|^2|u(y)|^2\,dx\,dy.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_{\mathbb{R}^3\times\mathbb{R}^3}\langle\nabla W(x-y) \mid x\rangle|u(x)|^2|u(y)|^2\,dx\,dy\\ &\qquad = \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3}\langle\nabla W(x-y) \mid x-y\rangle|u(x)|^2|u(y)|^2\,dx\,dy\\ &\qquad = -\frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} W(x-y)|u(x)|^2|u(y)|^2\,dx\,dy\end{aligned}$$

via Euler's identity for homogeneous functions. Plugging this into (2.6) yields (2.2). $\square$

2.2. **Orbital stability property.** In this section, we consider the Schrödinger equation

$$\begin{cases} \mathrm{i}\frac{\partial u}{\partial t} + \Delta u + W * |u|^2 u = 0 & \text{in } \mathbb{R}^3 \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}^3. \end{cases} \tag{2.7}$$

This equation is also known as Pekar-Choquard equation (see e.g. [10, 19, 21]). Consider the functionals

$$\mathcal{E}(u) = \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{4}\mathbb{D}(u), \quad J(u) = \frac{1}{2}\|\nabla u\|_{L^2}^2 + \frac{a}{2}\|u\|_{L^2}^2 - \frac{1}{4}\mathbb{D}(u),$$

where

$$\mathbb{D}(u) = \int_{\mathbb{R}^6} W(x-y)|u(x)|^2|u(y)|^2\, dxdy, \tag{2.8}$$

and let us set

$$\mathcal{M} = \big\{u \in H^1(\mathbb{R}^3, \mathbb{C}) : \|u\|_{L^2}^2 = \rho\big\},$$
$$\mathcal{N} = \big\{u \in H^1(\mathbb{R}^3, \mathbb{C}) : u \neq 0 \text{ and } J'(u)(u) = 0\big\},$$

for some positive number $\rho > 0$.

Definition 2.2. *We denote by $\mathcal{G}$ the set of ground state solutions of (2.1), that is solutions to the minimization problem*

$$\Lambda = \min_{u \in \mathcal{N}} J(u). \tag{2.9}$$

In Lemma 2.5 we will prove that a ground state solution of (2.1) can be obtained as scaling of a solution to the minimization problem

$$\Lambda = \min_{u \in \mathcal{M}} \mathcal{E}(u), \tag{2.10}$$

which is a quite useful characterization for the stability issue. We now recall two global existence results for problem (2.7) (see e.g. [9, Remark 6.5.2, p.114]).

Proposition 2.3. *Let $u_0 \in H^1(\mathbb{R}^3)$. Then problem (2.7) admits a unique global solution $u \in C^1([0, \infty), H^1(\mathbb{R}^3, \mathbb{C}))$. Moreover, the charge and the energy are conserved in time, namely*

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \qquad \mathcal{E}(u(t)) = \mathcal{E}(u_0), \tag{2.11}$$

for all $t \in [0, \infty)$.

Definition 2.4. *The set $\mathcal{G}$ of ground state solutions of (2.1) is said to be orbitally stable for the Pekar-Choquard equation (2.7) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\forall u_0 \in H^1(\mathbb{R}^3, \mathbb{C}): \quad \inf_{\phi \in \mathcal{G}} \|u_0 - \phi\|_{H^1} < \delta \quad \textit{implies that} \quad \sup_{t \geq 0} \inf_{\psi \in \mathcal{G}} \|u(t, \cdot) - \psi\|_{H^1} < \varepsilon,$$

where $u(t, \cdot)$ is the solution of (2.7) corresponding to the initial datum u_0.

Roughly speaking, the ground states are orbitally stable if any orbit starting from an initial datum u_0 close to $\mathcal{G}$ remains close to $\mathcal{G}$, uniformly in time.

In the classical orbital stability of Cazenave and Lions (see e.g. [10]) the ground states set $\mathcal{G}$ is meant as the set of minima of the functional $\mathcal{E}$ constrained to a sphere of $L^2(\mathbb{R}^3)$. In this section we just aim to show that orbital stability holds with respect to $\mathcal{G}$ as defined in Definition 2.2.

Consider the following sets:

$$K_{\mathcal{N}} = \{m \in \mathbb{R} : \text{there is } w \in \mathcal{N} \text{ with } J'(w) = 0 \text{ and } J(w) = m\},$$
$$K_{\mathcal{M}} = \{c \in \mathbb{R}^- : \text{there is } u \in \mathcal{M} \text{ with } \mathcal{E}'|_{\mathcal{M}}(u) = 0 \text{ and } \mathcal{E}(u) = c\}.$$

In the next result we establish the equivalence between minimization problems (2.9) and (2.10), namely that a suitable scaling of a solution of the first problem corresponds to a solution of the second problem with a mapping between the critical values.

Lemma 2.5. *The following minimization problems are equivalent*

$$\Lambda = \min_{u\in\mathcal{M}} \mathcal{E}(u), \qquad \Gamma = \min_{u\in\mathcal{N}} J(u), \tag{2.12}$$

for $\Lambda < 0$ and $\Lambda = \Psi(\Gamma)$, where $\Psi : K_{\mathcal{N}} \to K_{\mathcal{M}}$ is defined by

$$\Psi(m) = -\frac{1}{2}\left(\frac{3}{a\rho}\right)^{-3} m^{-2}, \quad m \in K_{\mathcal{N}}.$$

Proof. Observe that if $u \in \mathcal{M}$ is a critical point of $\mathcal{E}|_{\mathcal{M}}$ with $\mathcal{E}(u) = c < 0$, then there exists a Lagrange multiplier $\gamma \in \mathbb{R}$ such that $\mathcal{E}'(u)(u) = -\gamma\rho$, that is $\|\nabla u\|_{L^2}^2 - \mathbb{D}(u) = -\gamma\rho$. By combining this identity with $\mathbb{D}(u) = 2\|\nabla u\|_{L^2}^2 - 4c$, we obtain $-\|\nabla u\|_{L^2}^2 + 4c = -\gamma\rho$, which implies that $\gamma > 0$. The equation satisfied by u is

$$-\Delta u + \gamma u = \left(W * |u|^2\right) u \qquad \text{in } \mathbb{R}^3.$$

After trivial computations one shows that the scaling

$$w(x) = T^\lambda u(x) := \lambda^2 u(\lambda x), \qquad \lambda := \sqrt{\frac{a}{\gamma}} \tag{2.13}$$

is a solution of equation (2.1). On the contrary, if w is a nontrivial critical point of J, then choosing

$$\lambda = \rho^{-1}\|w\|_{L^2}^2, \tag{2.14}$$

the function $u = T^{1/\lambda} w$ belongs to $\mathcal{M}$ and it is a critical point of $\mathcal{E}_{|\mathcal{M}}$. Now, Let m be the value of the free functional J on w, $m = J(w)$. Then

$$\begin{aligned} m &= \frac{1}{2}\|\nabla w\|_{L^2}^2 + \frac{a}{2}\|w\|_{L^2}^2 - \frac{1}{4}\mathbb{D}(w) \\ &= \frac{1}{2}\lambda^3\|\nabla u\|_{L^2}^2 + \frac{a}{2}\lambda\|u\|_{L^2}^2 - \frac{1}{4}\lambda^3\mathbb{D}(u) \\ &= \lambda^3\mathcal{E}(u) + \frac{a}{2}\lambda\rho \\ &= c\lambda^3 + \frac{a}{2}\lambda\rho. \end{aligned} \tag{2.15}$$

Observe that, since of course $\mathbb{D}(w) = \|\nabla w\|_{L^2}^2 + a\|w\|_{L^2}^2$ and w satisfies the Pohozaev identity (2.2), we have the system

$$\begin{aligned}
&\frac{1}{2}\|\nabla w\|_{L^2}^2 + \frac{3}{2}a\|w\|_{L^2}^2 = \frac{5}{4}\big(\|\nabla w\|_{L^2}^2 + a\|w\|_{L^2}^2\big), \\
&\frac{1}{4}\|\nabla w\|_{L^2}^2 + \frac{a}{4}\|w\|_{L^2}^2 = m,
\end{aligned}$$

namely

$$\begin{aligned}
3\|\nabla w\|_{L^2}^2 - a\|w\|_{L^2}^2 = 0, \\
\|\nabla w\|_{L^2}^2 + a\|w\|_{L^2}^2 = 4m,
\end{aligned}$$

which, finally, entails

$$\|\nabla w\|_{L^2}^2 = m, \qquad \|w\|_{L^2}^2 = \frac{3}{a}m. \tag{2.16}$$

As a consequence a simple rescaling yields the value of λ, that is

$$\rho\lambda = \|w\|_{L^2}^2 = \frac{3}{a}m.$$

Replacing this value of λ back into formula (2.15), one obtains

$$m = c\Big(\frac{3m}{a\rho}\Big)^3 + \frac{3}{2}m,$$

namely

$$-\frac{1}{2}m = c\Big(\frac{3m}{a\rho}\Big)^3.$$

In conclusion, we get

$$c = \Psi(m) \stackrel{\text{def}}{=} -\frac{1}{2}\Big(\frac{3}{a\rho}\Big)^{-3} m^{-2}, \tag{2.17}$$

where the function $\Psi : \mathbb{R}^+ \to \mathbb{R}^-$ is injective. In order to prove that Ψ^{-1} is surjective, let m be a free critical value for J, namely $m = J(w)$, with w solution of equation (2.1). Then, if we consider $u = T^{1/\lambda}w(x) = \lambda^{-2}w(\lambda^{-1}x)$ with λ given by (2.14), it follows that $u \in \mathcal{M}$ is a critical point of $\mathcal{E}|_{\mathcal{M}}$ with lagrange multiplier $\gamma = a\lambda^{-2}$. By using

$$\lambda = \Big(\frac{a\rho}{3m}\Big)^{-1},$$

in light of (2.16) we have

$$\begin{aligned}
4c = \|\nabla u\|_{L^2}^2 - \gamma\rho &= \lambda^{-3}\|\nabla w\|_{L^2}^2 - a\rho\lambda^{-2} \\
&= \Big(\frac{a\rho}{3m}\Big)^3 m - a\rho\Big(\frac{3m}{a\rho}\Big)^{-2},
\end{aligned}$$

which yields $m = \Psi^{-1}(c)$, after a few computations. We are now ready to prove the assertion. Notice that by formula (2.17) we have

$$\begin{aligned}\Lambda &= \min_{u\in\mathcal{M}} \mathcal{E}(u) = \min_{u\in\mathcal{M}} c_u \\ &= \min_{v\in\mathcal{N}} \Psi(m_v) = -\max_{v\in\mathcal{N}} \frac{1}{2}\Big(\frac{3}{a\rho}\Big)^{-3} m_v^{-2} \\ &= -\frac{1}{2}\Big(\frac{3}{a\rho}\Big)^{-3}\Big(\min_{v\in\mathcal{N}} m_v\Big)^{-2} \\ &= -\frac{1}{2}\Big(\frac{3}{a\rho}\Big)^{-3}\Gamma^{-2} = \Psi(\Gamma).\end{aligned}$$

If $\hat{u} \in \mathcal{M}$ is a minimizer for Λ, that is $\Lambda = \mathcal{E}(\hat{u}) = \min_{\mathcal{M}} \mathcal{E}$, the function $\hat{w} = T^\lambda \hat{u}$ is a critical point of J with $J(\hat{w}) = \Psi^{-1}(\Lambda) = \Gamma$, so that w is a minimizer for Γ, that is $J(w) = \min_{\mathcal{N}} J$. This concludes the proof. □

Corollary 2.6. *Any ground state solution u to equation (2.1) satisfies*

$$\|u\|_{L^2}^2 = \rho, \qquad \rho = \frac{3\Gamma}{a}, \tag{2.18}$$

where Γ is defined in (2.12). Moreover, for this precise value of the radius ρ, we have

$$\min_{u\in\mathcal{M}} J(u) = \min_{u\in\mathcal{N}} J(u), \tag{2.19}$$

where $\mathcal{M} = \mathcal{M}_\rho$.

Proof. The first conclusion is an immediate consequence of the previous proof. Let us now prove that the second conclusion holds, with ρ as in (2.18). We have

$$\begin{aligned}\min_{u\in\mathcal{M}} J(u) &= \min_{u\in\mathcal{M}} \mathcal{E}(u) + \frac{a}{2}\|u\|_{L^2}^2 = \Lambda + \frac{a\rho}{2} \\ &= -\frac{1}{2}\Big(\frac{3}{a\rho}\Big)^{-3}\Gamma^{-2} + \frac{a\rho}{2} \\ &= \Gamma = \min_{u\in\mathcal{N}} J(u),\end{aligned}$$

by the definition of ρ. □

The following is the main result of the section.

Theorem 2.7. *Then the set $\mathcal{G}$ of the ground state solutions to (2.1) is orbitally stable for (2.7).*

Proof. Assume by contradiction that the assertion is false. Then we can find $\varepsilon_0 > 0$, a sequence of times $(t_n) \subset (0, \infty)$ and of initial data $(u_0^n) \subset H^1(\mathbb{R}^3, \mathbb{C})$ such that

$$\lim_{n\to\infty} \inf_{\phi\in\mathcal{G}} \|u_0^n - \phi\|_{H^1} = 0 \quad \text{and} \quad \inf_{\psi\in\mathcal{G}} \|u_n(t_n, \cdot) - \psi\|_{H^1} \ge \varepsilon_0, \tag{2.20}$$

where $u_n(t, \cdot)$ is the solution of (2.7) corresponding to the initial datum u_0^n. Taking into account (2.18) and (2.19) of Corollary 2.6, for any $\phi \in \mathcal{G}$, we have

$$\|\phi\|_{L^2}^2 = \rho_0, \qquad J(\phi) = \min_{u\in\mathcal{M}_{\rho_0}} J(u), \qquad \rho_0 = \frac{3\Gamma}{a}.$$

Therefore, considering the sequence $\Upsilon_n(x) := u_n(t_n, x)$, which is bounded in $H^1(\mathbb{R}^3, \mathbb{C})$, and recalling the conservation of charge, as $n \to \infty$, from (2.20) it follows that

$$\|\Upsilon_n\|_{L^2}^2 = \|u_n(t_n, \cdot)\|_{L^2}^2 = \|u_0^n\|_{L^2}^2 = \rho_0 + o(1).$$

Hence, there exists a sequence $(\omega_n) \subset \mathbb{R}^+$ with $\omega_n \to 1$ as $n \to \infty$ such that

$$\|\omega_n \Upsilon_n\|_{L^2}^2 = \rho_0, \quad \text{for all } n \geq 1. \tag{2.21}$$

Moreover, by the conservation of energy (2.11) and the continuity of $\mathcal{E}$, as $n \to \infty$,

$$\begin{aligned} J(\omega_n \Upsilon_n) &= \mathcal{E}(\omega_n \Upsilon_n) + \frac{a}{2}\|\omega_n \Upsilon_n\|_{L^2}^2 = \mathcal{E}(\Upsilon_n) + \frac{a}{2}\|\Upsilon_n\|_{L^2}^2 + o(1) \\ &= \mathcal{E}(u_n(t_n, \cdot)) + \frac{a}{2}\|u_0^n\|_{L^2}^2 + o(1) = \mathcal{E}(u_0^n) + \frac{a}{2}\|u_0^n\|_{L^2}^2 + o(1) \\ &= J(u_0^n) + o(1) = \min_{u \in \mathcal{M}_{\rho_0}} J(u) + o(1). \end{aligned} \tag{2.22}$$

Combining (2.21)-(2.22), it follows that $(\omega_n \Upsilon_n) \subset H^1(\mathbb{R}^3, \mathbb{C})$ is a minimizing sequence for the functional J (and also for $\mathcal{E}$) over $\mathcal{M}_{\rho_0}$. By taking into account the homogeneity property of W, following the lines of the proof of [10, Theorem IV.1], we learn that, up to a subsequence, $(\omega_n \Upsilon_n)$ converges to some function Υ_0, which thus belongs to the set $\mathcal{G}$, since by (2.21)-(2.22) and equality (2.19)

$$J(\Upsilon_0) = \min_{u \in \mathcal{M}_{\rho_0}} J(u) = \min_{u \in \mathcal{N}} J(u)$$

Evidently, this is a contradiction with (2.20), as we would have

$$\varepsilon_0 \leq \lim_{n \to \infty} \inf_{\psi \in \mathcal{G}} \|\Upsilon_n - \psi\|_{H^1} \leq \lim_{n \to \infty} \|\Upsilon_n - \Upsilon_0\|_{H^1} = 0.$$

This concludes the proof. $\square$

In the particular case $W(x) = |x|^{-1}$, due to the uniqueness of ground states up to translations and phase changes (cf. [24]), Theorem 2.7 strengthens as follows.

Corollary 2.8. *Assume that w is the unique real ground state of*

$$-\Delta w + aw = |x|^{-1} * w^2 w, \quad \text{in } \mathbb{R}^3.$$

Then for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$u_0 \in H^1(\mathbb{R}^3, \mathbb{C}) \quad \text{and} \quad \inf_{\substack{y \in \mathbb{R}^3 \\ \theta \in [0, 2\pi)}} \|u_0 - e^{\mathrm{i}\theta} w(\cdot - y)\|_{H^1} < \delta$$

implies that

$$\sup_{t \geq 0} \inf_{\substack{y \in \mathbb{R}^3 \\ \theta \in [0, 2\pi)}} \|u(t, \cdot) - e^{\mathrm{i}\theta} w(\cdot - y)\|_{H^1} < \varepsilon.$$

2.3. **Structure of least energy solutions.** We can now state the following

Lemma 2.9. *Any complex ground state solution u to (2.1) has the form*

$$u(x) = e^{\mathrm{i}\theta}|u(x)|, \quad \text{for some } \theta \in [0, 2\pi).$$

Proof. In view of Lemma 2.5 (see also Corollary 2.6), searching for ground state solutions of (2.1) is equivalent to consider the constrained minimization problem $\min_{u\in\mathcal{M}_\rho} \mathcal{E}(u)$ for a suitable value of $\rho > 0$. Then the proof is quite standard; we include a proof here for the sake of selfcontainedness. Consider

$$\sigma_{\mathbb{C}} = \inf\big\{\mathcal{E}(u) : u \in H^1(\mathbb{R}^3, \mathbb{C}),\ \|u\|^2_{L^2} = \rho\big\},$$
$$\sigma_{\mathbb{R}} = \inf\big\{\mathcal{E}(u) : u \in H^1(\mathbb{R}^3, \mathbb{R}), \|u\|^2_{L^2} = \rho\big\}.$$

It holds $\sigma_{\mathbb{C}} = \sigma_{\mathbb{R}}$. Indeed, trivially one has $\sigma_{\mathbb{C}} \le \sigma_{\mathbb{R}}$. Moreover, if $u \in H^1(\mathbb{R}^3, \mathbb{C})$, due to the well-known inequality $|\nabla|u(x)|| \le |\nabla u(x)|$ for a.e. $x \in \mathbb{R}^3$, it holds

$$\int |\nabla|u(x)||^2 dx \le \int |\nabla u(x)|^2 dx,$$

so that $\mathcal{E}(|u|) \le \mathcal{E}(u)$. In particular $\sigma_{\mathbb{R}} \le \sigma_{\mathbb{C}}$, yielding $\sigma_{\mathbb{C}} = \sigma_{\mathbb{R}}$. Let now u be a solution to $\sigma_{\mathbb{C}}$ and assume by contradiction that $\mu(\{x \in \mathbb{R}^3 : |\nabla|u|(x)| < |\nabla u(x)|\}) > 0$, where μ denotes the Lebesgue measure in $\mathbb{R}^3$. Then $\||u|\|_{L^2} = \|u\|_{L^2} = 1$, and

$$\sigma_{\mathbb{R}} \le \frac{1}{2}\int |\nabla|u||^2 dx - \frac{1}{4}\mathbb{D}(|u|) < \frac{1}{2}\int |\nabla u|^2 dx - \frac{1}{4}\mathbb{D}(u) = \sigma_{\mathbb{C}},$$

which is a contradiction, being $\sigma_{\mathbb{C}} = \sigma_{\mathbb{R}}$. Hence, we have $|\nabla|u(x)|| = |\nabla u(x)|$ for a.e. $x \in \mathbb{R}^3$. This is true if and only if $\Re u\nabla(\Im u) = \Im u\nabla(\Re u)$. In turn, if this last condition holds, we get

$$\bar{u}\nabla u = \Re u\nabla(\Re u) + \Im u\nabla(\Im u), \quad \text{a.e. in } \mathbb{R}^3,$$

which implies that $\Re(i\bar{u}(x)\nabla u(x)) = 0$ a.e. in $\mathbb{R}^3$. From the last identity one finds $\theta \in [0, 2\pi)$ such that $u = e^{\mathrm{i}\theta}|u|$, concluding the proof. □

We then get the following result about least-energy levels for the limiting problem (2.1).

Corollary 2.10. *Consider the two problems*

$$-\Delta u + au = W * |u|^2 u, \qquad u \in H^1(\mathbb{R}^3, \mathbb{R}), \tag{2.23}$$

$$-\Delta u + au = W * |u|^2 u, \qquad u \in H^1(\mathbb{R}^3, \mathbb{C}),. \tag{2.24}$$

Let E_a and E^c_a denote their least-energy levels. Then

$$E_a = E^c_a. \tag{2.25}$$

Moreover any least energy solution of (2.23) has the form $e^{i\tau}U$ where U is a positive least energy solution of (2.24) and $\tau \in \mathbb{R}$.

2.4. **Compactness of the ground states set.** In light of assumption **(W)**, there exist two positive constants C_1, C_2 such that

$$\frac{C_1}{|x|} \leq W(x) \leq \frac{C_2}{|x|}, \quad \text{for all } x \in \mathbb{R}^3 \setminus \{0\}. \tag{2.26}$$

We recall two Hardy-Littlewood-Sobolev type inequality (see e.g. [20]) in $\mathbb{R}^3$:

$$\forall u \in L^{\frac{6q}{3+2q}}(\mathbb{R}^3): \quad \big\||x|^{-1} * u^2\big\|_{L^q} \leq C\|u\|^2_{L^{\frac{6q}{3+2q}}}, \tag{2.27}$$

$$\forall u \in L^{\frac{12}{5}}(\mathbb{R}^3): \quad \int_{\mathbb{R}^6} |x-y|^{-1} u^2(y) u^2(x) dy dx \leq C\|u\|^4_{L^{\frac{12}{5}}}. \tag{2.28}$$

Notice that, taking the limit $q \to \infty$ in (2.27) yields

$$\forall u \in L^3(\mathbb{R}^3): \quad \big\||x|^{-1} * u^2\big\|_{L^\infty} \leq C\|u\|^2_{L^3}.$$

We have the following

Lemma 2.11. *There exists a positive constant C such that*

$$\forall u \in H^1(\mathbb{R}^3): \quad \mathbb{D}(u) \leq C\|u\|^3_{L^2}\, \|u\|_{H^1}.$$

Proof. By combining (2.26), (2.28) and the Gagliardo-Nirenberg inequality, we obtain

$$\mathbb{D}(u) \leq C_2 \int_{\mathbb{R}^6} |x-y|^{-1} u^2(y) u^2(x) dy dx \leq C\|u\|^4_{L^{\frac{12}{5}}} \leq C\|u\|^3_{L^2}\, \|u\|_{H^1},$$

which proves the assertion. □

Let $\mathcal{S}_a$ denote the set of (complex) least energy solutions u to equation (2.1) such that

$$|u(0)| = \max_{x\in\mathbb{R}^3} |u(x)|.$$

By Lemma 2.9, up to a constant phase change, we can assume that u is real valued.

Proposition 2.12. *For any $a > 0$ the set $\mathcal{S}_a$ is compact in $H^1(\mathbb{R}^3, \mathbb{R})$ and there exist positive constants C, σ such that $u(x) \leq C\exp(-\sigma|x|)$ for any $x \in \mathbb{R}^3$ and all $u \in \mathcal{S}_a$.*

Proof. If $J_a : H^1(\mathbb{R}^3) \to \mathbb{R}$ denotes the functional associated with (2.1),

$$J_a(u) = J(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{a}{2}\int_{\mathbb{R}^3} u^2 dx - \frac{1}{4}\mathbb{D}(u),$$

since $\mathbb{D}(u) = \|\nabla u\|^2_{L^2} + a\|u\|^2_{L^2}$ for all $u \in \mathcal{M}_a$, we have

$$m_a = L_a(u) = \frac{1}{4}\big(\|\nabla u\|^2_{L^2} + a\|u\|^2_{L^2}\big),$$

where $m_a = \min\{L_a(u) : u \neq 0 \text{ solves } (2.1)\}$. Hence, it follows that the set $\mathcal{M}_a$ is bounded in $H^1(\mathbb{R}^3, \mathbb{R})$. Moreover, $\mathcal{S}_a$ is also bounded in $L^\infty(\mathbb{R}^3)$ and $u(x) \to 0$ as $|x| \to \infty$ for any $u \in \mathcal{S}_a$. Indeed, from the Hardy-Littlewood-Sobolev inequality (2.27), for any $q \geq 3$

$$\|W * u^2\|_{L^q} \leq C\||x|^{-1} * u^2\|_{L^q} \leq C\|u\|^2_{L^{6q/(3+2q)}} \leq C\|u\|^2_{L^6} \tag{2.29}$$

it follows that $|x|^{-1} * u^2 \in L^q(\mathbb{R}^3)$ for any $q \geq 3$. Then, by Hölder inequality, we get for all m such that $3/2 < m < 6$,

$$\|W * u^2 u\|_{L^m}^m \leq C\||x|^{-1} * u^2 u\|_{L^m}^m \leq C\||x|^{-1} * u^2\|_{L^{\frac{6m}{6-m}}}^m \|u\|_{L^6}^m.$$

From equation (2.1) it follows that $u \in W^{2,m}(\mathbb{R}^3)$ for all $2 \leq m < 6$. Hence, it follows that u is a bounded function which vanishes at infinity. Actually u has further regularity. Indeed, using again the equation, the boundedness of u as well as the Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned}\|u\|_{W^{2,m}} &\leq C\|\Delta u\|_{L^m} \leq C\|u\|_{L^m} + C\|(|x|^{-1} * u^2)u\|_{L^m} \\ &\leq C + C\||x|^{-1} * u^2\|_{L^m} \leq C.\end{aligned}$$

It follows, that $u \in W^{2,m}(\mathbb{R}^3)$ for every $m > 3$. Also these summability properties implies that $\mathcal{S}_a$ is uniformly bounded in $C^{1,\alpha}(\mathbb{R}^3)$. Let us now show that the limit $u(x) \to 0$ as $|x| \to \infty$ holds uniformly for $u \in \mathcal{S}_a$. Assuming by contradiction that $u_m(x_m) \geq \sigma > 0$ along some sequences $(u_m) \subset \mathcal{S}_a$ and $(x_m) \subset \mathbb{R}^3$ with $|x_m| \to \infty$, shifting u_m as $v_m(x) = u_m(x + x_m)$ it follows that (v_m) is bounded in $H^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and it converges, up to a subsequence to a function v, weakly in $H^1(\mathbb{R}^3)$ and locally uniformly in $C(\mathbb{R}^3)$. If u denotes the weak limit of u_m, we also claim that u, v are both solutions to equation (2.1), which are nontrivial as follows from (local) uniform convergence and $u_m(0) \geq u_m(x_m) \geq \sigma$ (since 0 is a global maximum for u_m) and $v_m(0) = u_m(x_m) \geq \sigma$. To see that u, v are solutions to (2.1), set

$$\varphi_m(x) = \int_{\mathbb{R}^3} W(x-y)u_m^2(y)dy, \qquad \varphi(x) = \int_{\mathbb{R}^3} W(x-y)u^2(y)dy.$$

Let us show that $\varphi_m(x) \to \varphi(x)$, as $m \to \infty$, for any fixed $x \in \mathbb{R}^3$. Indeed, we can write $\varphi_m(x) - \varphi(x) = I_m^1(\rho) + I_m^2(\rho)$ for any $m \geq 1$ and any $\rho > 0$, where we set

$$\begin{aligned}I_m^1(\rho) &= \int_{B_\rho(0)} W(x-y)(u_m^2(y) - u^2(y))dy, \\ I_m^2(\rho) &= \int_{\mathbb{R}^3 \setminus B_\rho(0)} W(x-y)(u_m^2(y) - u^2(y))dy.\end{aligned}$$

Fix $x \in \mathbb{R}^3$ and let $\varepsilon > 0$. Choose $\rho_0 > 0$ sufficiently large that

$$I_m^2(\rho_0) \leq \int_{\mathbb{R}^3 \setminus B_{\rho_0}(0)} \frac{C}{|y| - |x|}|u_m^2(y) - u^2(y)|dy \leq \frac{C}{\rho_0 - |x|} < \frac{\varepsilon}{2}. \tag{2.30}$$

On the other hand, by the uniform local convergence of u_m to u as $m \to \infty$ and Hölder inequality, for some $1 < r < 3$

$$\begin{aligned}I_m^1(\rho_0) &\leq C\|u_m - u\|_{L^{r'}(B_{\rho_0}(0))}\Big(\int_{B_{\rho_0}(0)} \frac{1}{|x-y|^r}|u_m(y) + u(y)|^r dy\Big)^{1/r} \\ &\leq C\|u_m - u\|_{L^{r'}(B_{\rho_0}(0))}\Big(\int_{B_{\rho_0}(x)} \frac{1}{|y|^r}dy\Big)^{1/r} \\ &\leq C\|u_m - u\|_{L^{r'}(B_{\rho_0}(0))} < \frac{\varepsilon}{2}\end{aligned} \tag{2.31}$$

for all m sufficiently large, where r' denotes the conjugate exponent of r. The bound $r < 3$ ensures that the singular integral which appears in the second inequality is finite. Combining (2.30) with (2.31) concludes the proof of the pointwise convergence of φ_m to φ. Notice that (φ_m) is uniformly bounded by inequality (2.27), since $\|W * u_m^2\|_{L^\infty} \leq C\|u_m\|_{L^3}^2 \leq C$. Hence, $\varphi_m(x)u_m(x)\eta(x) \to \varphi(x)u(x)\eta(x)$ as $m \to \infty$, for all $\eta \in C_c^\infty(\mathbb{R}^3)$ with compact support K and any $x \in \mathbb{R}^3$ fixed. In turn, by the Dominated Convergence Theorem (recall that $|\varphi_m u_m \eta| \leq C \in L^1(K)$), we get

$$\lim_{m\to+\infty} \int_K \varphi_m u_m \eta \, dx = \int_K \varphi u \eta \, dx,$$

for all $\eta \in C_c^\infty(\mathbb{R}^3)$ with compact support K. This concludes the proof that u, v are nontrivial solutions to (2.1). It follows that, for any m and k,

$$\begin{aligned} J_a(u_m) = J_a(u_k) &= \frac{1}{4}\int_{\mathbb{R}^3} (|\nabla u_m|^2 + a u_m^2) dx, \\ J_a(u) &\geq J_a(z) = m_a, \\ J_a(v) &\geq J_a(z) = m_a, \end{aligned}$$

for all $z \in \mathcal{S}_a$. On the other hand, for any $R > 0$ and $m \geq 1$ with $2R \leq |x_m|$,

$$\begin{aligned} m_a = J_a(u_m) &\geq \frac{1}{4}\liminf_m \int_{B_R(0)} (|\nabla u_m|^2 + a u_m^2) dx + \frac{1}{4}\liminf_m \int_{B_R(0)} (|\nabla v_m|^2 + a v_m^2) dx \\ &\geq J_a(u) + J_a(v) - \varepsilon \geq 2m_a - o(1) \end{aligned}$$

as $R \to \infty$, which yields a contradiction for R large enough. Hence the conclusion follows. Let us now prove that

$$\lim_{|x|\to\infty} \varphi(x) = 0, \qquad \varphi(x) = \int_{\mathbb{R}^3} W(x-y) u^2(y) \, dy. \tag{2.32}$$

Taken $\rho > 0$, write

$$\varphi(x) = \int_{B_\rho(0)} W(x-y) u^2(y) \, dy + \int_{\mathbb{R}^3 \setminus B_\rho(0)} W(x-y) u^2(y) \, dy.$$

Chosen $\varepsilon > 0$, there exists $\rho_0 > 0$ such that $u^{2/3}(y) \leq \varepsilon$ for all y in $\mathbb{R}^3 \setminus B_{\rho_0}(0)$. Hence, for any $x \in \mathbb{R}^3$, we obtain

$$\varphi(x) \leq \|u\|_{L^\infty}^2 \eta_1(x) + \varepsilon \eta_2(x).$$

where

$$\eta_1(x) = \int_{B_{\rho_0}(0)} W(x-y) \, dy, \qquad \eta_2(x) = \int_{\mathbb{R}^3 \setminus B_{\rho_0}(0)} \frac{u^{4/3}(y)}{|x-y|} \, dy.$$

Notice that η_2 is bounded, since by Hardy-Littlewood-Sobolev inequality (2.27) it follows

$$\|\eta_2\|_{L^\infty} \leq \big\| |x|^{-1} * u^{4/3} \big\|_{L^\infty} \leq C\|u\|_{L^2}^{4/3},$$

since $u \in L^2(\mathbb{R}^3)$. Moreover, since $|x-y| \geq |x| - |y| \geq |x| - \rho_0$ for any $x, y \in \mathbb{R}^3$ with $|y| \leq \rho_0$ and $|x| > \rho_0$, then we have (μ is the Lebesgue measure in $\mathbb{R}^3$)

$$0 < \eta_1(x) \leq C_2 \int_{B_{\rho_0}(0)} \frac{1}{|x| - \rho_0} dy \leq C_2 \frac{\mu(B_{\rho_0}(0))}{|x| - \rho_0} \to 0, \quad \text{as } |x| \to \infty.$$

Then, in conclusion, there exists $R(\varepsilon) \geq \rho_0$ such that $\eta_1(x) \leq C\varepsilon$, as $|x| \geq R(\varepsilon)$, which yields $\varphi(x) \leq C\varepsilon$ as $|x| \geq R(\varepsilon)$, so that assertion (2.32) follows. In light of (2.32), let $R_a > 0$ such that $\varphi(x) \leq \frac{a}{2}$, for any $|x| \geq R_a$. As a consequence,

$$-\Delta u(x) + \frac{a}{2} u(x) \leq 0, \qquad \text{for } |x| \geq R_a.$$

It is thus standard to see that this yields the exponential decay ou u, with uniform decay constants in $\mathcal{S}_a$. We can finally conclude the proof. Let (u_n) be any sequence in $\mathcal{S}_a$. Up to a subsequence it follows that (u_n) converges weakly to a function u which is also a solution to equation (2.1). If $\mathbb{D}$ is the function defined in (2.8), we immediately get

$$\lim_{n\to+\infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 + a u_n^2 - \mathbb{D}(u_n) = 0 = \int_{\mathbb{R}^3} |\nabla u|^2 + a u^2 - \mathbb{D}(u). \tag{2.33}$$

Hence the desired strong convergence of (u_n) to u in $H^1(\mathbb{R}^3)$ follows once we prove that $\mathbb{D}(u_n) \to \mathbb{D}(u)$, as $n \to \infty$. In view of the uniform exponential decay of u_n it follows that $u_n \to u$ strongly in $L^{12/5}(\mathbb{R}^3)$, as $n \to \infty$. Taking into account that (u_n) is bounded in $H^1(\mathbb{R}^3)$ and that W is even, we easily get the inequality

$$|\mathbb{D}(u_n) - \mathbb{D}(u)| \leq \sqrt{\mathbb{D}(||u_n|^2 - |u|^2|^{1/2})} \sqrt{\mathbb{D}((|u_n|^2 + |u|^2)^{1/2})}, \quad n \in \mathbb{N}.$$

By Hardy-Littlewood-Sobolev inequality and Hölder's inequality, it follows that

$$|\mathbb{D}(u_n) - \mathbb{D}(u)|^2 \leq C \| \, ||u_n|^2 - |u|^2|^{1/2} \|^4_{L^{\frac{12}{5}}} \| \, (|u_n|^2 + |u|^2)^{1/2} \|^4_{L^{\frac{12}{5}}} \leq C \|u_n - u\|^2_{L^{\frac{12}{5}}}.$$

As a consequence

$$\mathbb{D}(u_n) = \mathbb{D}(u) + o(1), \quad \text{as } n \to \infty,$$

which concludes the proof in light of formula (2.33). □

3. The penalization argument

Throughout this and the following sections we shall mainly used the arguments of [12] highlighting the technical steps where the Hartree nonlinearity is involved in place of the local one. For the sake of self-containedness and for the reader's convenience we develop the arguments with all the detail.

For any set $\Omega \subset \mathbb{R}^3$ and $\varepsilon > 0$, let $\Omega_\varepsilon = \{x \in \mathbb{R}^3 : \varepsilon x \in \Omega\}$.

3.1. **Notations and framework.** The following lemmas, taken from [12] show that the norm in H_ε is locally equivalent to the standard H^1 norm.

Lemma 3.1. *Let $K \subset \mathbb{R}^3$ be an arbitrary, fixed, bounded domain. Assume that A is bounded on K and $0 < \alpha \le V \le \beta$ on K for some α, $\beta > 0$. Then, for any fixed $\varepsilon \in [0,1]$, the norm*

$$\|u\|^2_{K_\varepsilon} = \int_{K_\varepsilon} \left| \left(\frac{1}{\mathrm{i}}\nabla - A_\varepsilon(y)\right) u \right|^2 + V_\varepsilon(y)|u|^2 dy$$

is equivalent to the usual norm on $H^1(K_\varepsilon, \mathbb{C})$. Moreover these equivalences are uniform, namely there exist constants $c_1, c_2 > 0$ independent of $\varepsilon \in [0,1]$ such that

$$c_1\|u\|_{K_\varepsilon} \le \|u\|_{H^1(K_\varepsilon,\mathbb{C})} \le c_2\|u\|_{K_\varepsilon}.$$

Corollary 3.2. *Retain the setting of Lemma 3.1. Then the following facts hold.*

(i) *If K is compact, for any $\varepsilon \in (0,1]$ the norm*

$$\|u\|^2_{K} := \int_{K} \left| \left(\frac{1}{\mathrm{i}}\nabla - A_\varepsilon(y)\right) u \right|^2 + V_\varepsilon(y)|u|^2 dy$$

is uniformly equivalent to the usual norm on $H^1(K, \mathbb{C})$.

(ii) *For $A_0 \in \mathbb{R}^3$ and $b > 0$ fixed, the norm*

$$\|u\|^2 := \int_{\mathbb{R}^3} \left| \left(\frac{1}{\mathrm{i}}\nabla - A_0\right) u \right|^2 + b|u|^2 dy$$

is equivalent to the usual norm on $H^1(\mathbb{R}^3, \mathbb{C})$.

(iii) *If $(u_{\varepsilon_n}) \subset H^1(\mathbb{R}^3, \mathbb{C})$ satisfies $u_{\varepsilon_n} = 0$ on $\mathbb{R}^3 \setminus K_{\varepsilon_n}$ for any $n \in \mathbb{N}$ and $u_{\varepsilon_n} \to u$ in $H^1(\mathbb{R}^3, \mathbb{C})$ then $\|u_{\varepsilon_n} - u\|_{\varepsilon_n} \to 0$ as $n \to \infty$.*

For future reference we recall the following *Diamagnetic inequality*: for every $u \in H_\varepsilon$,

$$\left| \left(\frac{\nabla}{\mathrm{i}} - A_\varepsilon\right) u \right| \ge \big|\nabla|u|\big|, \quad \text{a.e. in } \mathbb{R}^3. \tag{3.1}$$

See [15] for a proof. As a consequence of (3.1), $|u| \in H^1(\mathbb{R}^3, \mathbb{R})$ for any $u \in H_\varepsilon$.

For any $u \in H_\varepsilon$, let us set

$$\mathcal{F}_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^3} |D^\varepsilon u|^2 + V_\varepsilon(x)|u|^2\, dx - \frac{1}{4}\int_{\mathbb{R}^3\times\mathbb{R}^3} W(x-y)|u(x)|^2|u(y)|^2 dxdy \tag{3.2}$$

where we set $D^\varepsilon = (\frac{\nabla}{\mathrm{i}} - A_\varepsilon)$. Define, for all $\varepsilon > 0$,

$$\chi_\varepsilon(y) = \begin{cases} 0 & \text{if } y \in O_\varepsilon, \\ \varepsilon^{-6/\mu} & \text{if } y \notin O_\varepsilon, \end{cases} \quad \chi^i_\varepsilon(y) = \begin{cases} 0 & \text{if } y \in (O^i)_\varepsilon, \\ \varepsilon^{-6/\mu} & \text{if } y \notin (O^i)_\varepsilon, \end{cases}$$

and

$$Q_\varepsilon(u) = \Big(\int_{\mathbb{R}^3} \chi_\varepsilon|u|^2 dx - 1\Big)_+^{\frac{5}{2}}, \quad Q^i_\varepsilon(u) = \Big(\int_{\mathbb{R}^3} \chi^i_\varepsilon|u|^2 dx - 1\Big)_+^{\frac{5}{2}}. \tag{3.3}$$

The functional Q_ε will act as a penalization to force the concentration phenomena of the solution to occur inside O. In particular, we remark that the penalization terms vanish on elements whose corresponding L^∞-norm is sufficiently small. This device was firstly introduced in [7]. Finally we define the functionals $\Gamma_\varepsilon, \Gamma_\varepsilon^1, \dots, \Gamma_\varepsilon^k : H_\varepsilon \to \mathbb{R}$ by setting

$$\Gamma_\varepsilon(u) = \mathcal{F}_\varepsilon(u) + Q_\varepsilon(u), \;\; \Gamma_\varepsilon^i(u) = \mathcal{F}_\varepsilon(u) + Q_\varepsilon^i(u), \quad i = 1, \dots, k. \tag{3.4}$$

It is easy to check, under our assumptions, and using the Diamagnetic inequality (3.1), that the functionals Γ_ε and Γ_ε^i are of class C^1 over H_ε. Hence, a critical point of $\mathcal{F}_\varepsilon$ corresponds to a solution of (1.5). To find solutions of (1.5) which *concentrate* in O as $\varepsilon \to 0$, we shall look for a critical point of Γ_ε for which Q_ε is zero.

Let

$$\mathcal{M} = \bigcup_{i=1}^{k} \mathcal{M}^i, \quad O = \bigcup_{i=1}^{k} O^i$$

and for any set $B \subset \mathbb{R}^3$ and $\alpha > 0$, $B^\delta = \{x \in \mathbb{R}^3 : \mathrm{dist}(x, B) \le \delta\}$ and set

$$\delta = \frac{1}{10} \min \left\{ \mathrm{dist}(\mathcal{M}, \mathbb{R}^3 \setminus O),\, \min_{i \neq j} \mathrm{dist}(O_i, O_j),\, \mathrm{dist}(O, Z) \right\}.$$

We fix a $\beta \in (0, \delta)$ and a cutoff $\varphi \in C_0^\infty(\mathbb{R}^3)$ such that $0 \le \varphi \le 1$, $\varphi(y) = 1$ for $|y| \le \beta$ and $\varphi(y) = 0$ for $|y| \ge 2\beta$. Also, setting $\varphi_\varepsilon(y) = \varphi(\varepsilon y)$ for each $x_i \in (\mathcal{M}^i)^\beta$ and $U_i \in \mathcal{S}_{m_i}$, we define

$$U_\varepsilon^{x_1,\dots,x_k}(y) = \sum_{i=1}^{k} e^{iA(x_i)(y - \frac{x_i}{\varepsilon})} \varphi_\varepsilon \left(y - \frac{x_i}{\varepsilon} \right) U_i \left(y - \frac{x_i}{\varepsilon} \right).$$

We will find a solution, for sufficiently small $\varepsilon > 0$, near the set

$$X_\varepsilon = \{ U_\varepsilon^{x_1 \dots, x_k}(y) : x_i \in (\mathcal{M}^i)^\beta \text{ and } U_i \in \mathcal{S}_{m_i} \text{ for each } i = 1, \dots, k \}.$$

For each $i \in \{1, \dots, k\}$ we fix an arbitrary $x_i \in \mathcal{M}^i$ and an arbitrary $U_i \in \mathcal{S}_{m_i}$ and we define

$$\mathcal{W}_\varepsilon^i(y) = e^{iA(x_i)(y - \frac{x_i}{\varepsilon})} \varphi_\varepsilon \left(y - \frac{x_i}{\varepsilon} \right) U_i \left(y - \frac{x_i}{\varepsilon} \right).$$

Setting

$$\mathcal{W}_{\varepsilon,t}^i(y) = e^{iA(x_i)(y - \frac{x_i}{\varepsilon})} \varphi_\varepsilon \left(y - \frac{x_i}{\varepsilon} \right) U_i \left(\frac{y}{t} - \frac{x_i}{\varepsilon t} \right),$$

we see that

$$\lim_{t \to 0} \|\mathcal{W}_{\varepsilon,t}^i\|_\varepsilon = 0, \qquad \Gamma_\varepsilon(\mathcal{W}_{\varepsilon,t}^i) = \mathcal{F}_\varepsilon(\mathcal{W}_{\varepsilon,t}^i), \quad t \ge 0.$$

In the next Proposition we shall show that there exists $T_i > 0$ such that $\Gamma_\varepsilon(\mathcal{W}_{\varepsilon,T_i}^i) < -2$ for any $\varepsilon > 0$ sufficiently small. Assuming this holds true, let $\gamma_\varepsilon^i(s) = \mathcal{W}_{\varepsilon,s}^i$ for $s > 0$ and $\gamma_\varepsilon^i(0) = 0$. For $s = (s_1, \dots, s_k) \in T = [0, T_1] \times \ldots \times [0, T_k]$ we define

$$\gamma_\varepsilon(s) = \sum_{i=1}^{k} \mathcal{W}_{\varepsilon,s_i}^i \quad \text{and} \quad D_\varepsilon = \max_{s \in T} \Gamma_\varepsilon(\gamma_\varepsilon(s)).$$

Finally for each $i \in \{1, \dots, k\}$, let $E_{m_i} = L_{m_i}^c(U)$ for $U \in S_{m_i}$.

3.2. **Energy estimates and Palais-Smale condition.** In what follows, we set

$$E_m = \min_{i\in\{1,\dots,k\}} E_{m_i}, \qquad E = \sum_{i=1}^{k} E_{m_i}.$$

For a set $A \subset H_\varepsilon$ and $\alpha > 0$, we let $A^\alpha = \{u \in H_\varepsilon : \|u - A\|_\varepsilon \le \alpha\}$.

Proposition 3.3. *There results*

(i) $\lim_{\varepsilon\to 0} D_\varepsilon = E$,

(ii) $\limsup_{\varepsilon\to 0} \max_{s\in\partial T} \Gamma_\varepsilon(\gamma_\varepsilon(s)) \le \tilde{E} = \max\{E - E_{m_i} \mid i = 1,\dots,k\} < E$,

(iii) *for each* $d > 0$, *there exists* $\alpha > 0$ *such that for sufficiently small* $\varepsilon > 0$,

$$\Gamma_\varepsilon(\gamma_\varepsilon(s)) \ge D_\varepsilon - \alpha \textit{ implies that } \gamma_\varepsilon(s) \in X_\varepsilon^{d/2}.$$

Proof. Since $\mathrm{supp}(\gamma_\varepsilon(s)) \subset \mathcal{M}_\varepsilon^{2\beta}$ for each $s \in T$, it follows that

$$\Gamma_\varepsilon(\gamma_\varepsilon(s)) = \mathcal{F}_\varepsilon(\gamma_\varepsilon(s)) = \sum_{i=1}^{k} \mathcal{F}_\varepsilon(\gamma_\varepsilon^i(s)).$$

Arguing as in [12, Proposition 3.1], we claim that for each $i \in \{1,\dots,k\}$

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3}\left|\left(\frac{\nabla}{\mathrm{i}} - A_\varepsilon(y)\right)W_{\varepsilon,s_i}^i\right|^2 dy = s_i\int_{\mathbb{R}^3}|\nabla U_i|^2 dy. \tag{3.5}$$

Using the exponentially decay of U_i we have that, as $\varepsilon \to 0$,

$$\int_{\mathbb{R}^3} V_\varepsilon(y)|W_{\varepsilon,s_i}^i|^2 dy = \int_{\mathbb{R}^3} m_i\left|U_i\left(\frac{y}{s_i}\right)\right|^2 dy + o(1) = m_i s_i^3\int_{\mathbb{R}^3}|U_i|^2 dy + o(1), \tag{3.6}$$

and, as $\varepsilon \to 0$,

$$\begin{aligned}\int_{\mathbb{R}^6} & W(x-y)|W_{\varepsilon,s_i}^i(x)|^2|W_{\varepsilon,s_i}^i(y)|^2 dxdy \\ &= \int_{\mathbb{R}^6} W(x-y)\left|U_i\left(\frac{y}{s_i}\right)\right|^2\left|U_i\left(\frac{x}{s_i}\right)\right|^2 dxdy + o(1) \\ &= s_i^5\int_{\mathbb{R}^6} W(x-y)|U_i(x)|^2|U_i(y)|^2 dxdy + o(1).\end{aligned}$$

Thus, from the above limit and from (3.5), (3.6), we derive

$$\begin{aligned}\mathcal{F}_\varepsilon(\gamma_\varepsilon^i(s_i)) &= \frac{1}{2}\int_{\mathbb{R}^3}\left|\left(\frac{\nabla}{\mathrm{i}} - A_\varepsilon(y)\right)\gamma_\varepsilon^i(s_i)\right|^2 dy + V_\varepsilon(y)|\gamma_\varepsilon^i(s_i)|^2 dy \\ &\quad - \frac{1}{4}\int_{\mathbb{R}^6} W(x-y)|\gamma_\varepsilon^i(s_i)|^2|\gamma_\varepsilon^i(s_i)|^2 dx\, dy \\ &= \frac{s_i}{2}\int_{\mathbb{R}^3}|\nabla U_i|^2 dy + \frac{1}{2}s_i^3 m_i\int_{\mathbb{R}^3}|U_i|^2 dy \\ &\quad - \frac{1}{4}s_i^5\int_{\mathbb{R}^6} W(x-y)|U_i(x)|^2|U^i(y)|^2 dx\, dy + o(1).\end{aligned}$$

Using the Pohozaev identity (2.2) as well as the Nehari constraint, we see that

$$\mathcal{F}_\varepsilon(\gamma^i_\varepsilon(s_i)) = \left(\frac{s_i}{6} + \frac{s_i^3}{2} - \frac{2s_i^5}{6}\right) m_i \int_{\mathbb{R}^3} |U_i|^2 dy + o(1).$$

Also

$$\max_{t\in(0,\infty)} \left(\frac{t}{6} + \frac{t^3}{2} - \frac{2t^5}{6}\right) \int_{\mathbb{R}^3} |U_i|^2 dy = E_{m_i}.$$

At this point we deduce that (i) and (ii) hold. Clearly also the existence of a $T_i > 0$ such that $\Gamma_\varepsilon(W^i_{\varepsilon,T_i}) < -2$ is justified. To conclude we just observe that, setting

$$g(t) = \frac{t}{6} + \frac{t^3}{2} - \frac{2t^5}{6}$$

the derivative $g'(t)$ of $g(t)$ is positive for $t \in (0,1)$, negative for $t \in (1,+\infty)$, and vanishes at $t = 1$. We conclude by observing that $g''(1) < 0$. □

Let us define

$$\Phi^i_\varepsilon = \left\{\gamma \in C([0,T_i], H_\varepsilon) :\ \gamma(0) = \gamma^i_\varepsilon(0),\ \gamma(T_i) = \gamma^i_\varepsilon(T_i)\right\}$$

and

$$C^i_\varepsilon = \inf_{\gamma\in\Phi^i_\varepsilon} \max_{s_i\in[0,T_i]} \Gamma^i_\varepsilon(\gamma(s_i)).$$

Proposition 3.4. *For the level C^i_ε defined before, there results*

$$\liminf_{\varepsilon\to 0} C^i_\varepsilon \geq E_{m_i}.$$

In particular, $\lim_{\varepsilon\to 0} C^i_\varepsilon = E_{m_i}$.

Proof. The proof of this lemma is analogous to that of Proposition 3.2 in [12]. □

Next we define, for every $\alpha \in \mathbb{R}$, the sub-level

$$\Gamma^\alpha_\varepsilon = \{u \in H_\varepsilon :\ \Gamma_\varepsilon(u) \leq \alpha\}.$$

Proposition 3.5. *Let (ε_j) be such that $\lim_{j\to\infty} \varepsilon_j = 0$ and $(u_{\varepsilon_j}) \in X^d_{\varepsilon_j}$ such that*

$$\lim_{j\to\infty} \Gamma_{\varepsilon_j}(u_{\varepsilon_j}) \leq E \ \textit{and}\ \lim_{j\to\infty} \Gamma'_{\varepsilon_j}(u_{\varepsilon_j}) = 0. \tag{3.7}$$

Then, for sufficiently small $d > 0$, there exist, up to a subsequence, $(y^i_j) \subset \mathbb{R}^3$, $i = 1,\dots,k$, points $x^i \in \mathcal{M}^i$ (not to be confused with the points x_i already introduced), $U_i \in S_{m_i}$ such that

$$\lim_{j\to\infty} |\varepsilon_j y^i_j - x^i| = 0 \ \textit{and}\ \lim_{j\to\infty} \left\| u_{\varepsilon_j} - \sum_{i=1}^k e^{iA_\varepsilon(y^i_j)(\cdot - y^i_j)} \varphi_{\varepsilon_j}(\cdot - y^i_j) U_i(\cdot - y^i_j) \right\|_{\varepsilon_j} = 0. \tag{3.8}$$

Proof. For simplicity we write ε instead of ε_j. From Proposition 2.12, we know that the S_{m_i} are compact. Then there exist $Z_i \in S_{m_i}$ and $(x^i_\varepsilon) \subset (\mathcal{M}^i)^\beta$, $x^i \in (\mathcal{M}^i)^\beta$ for $i = 1, \dots, k$, with $x^i_\varepsilon \to x^i$ as $\varepsilon \to 0$ such that, passing to a subsequence still denoted (u_ε),

$$\left\| u_\varepsilon - \sum_{i=1}^{k} e^{iA(x^i_\varepsilon)(\cdot - \frac{x^i_\varepsilon}{\varepsilon})} \varphi_\varepsilon(\cdot - x^i_\varepsilon/\varepsilon) Z_i(\cdot - x^i_\varepsilon/\varepsilon) \right\|_\varepsilon \leq 2d \tag{3.9}$$

for small $\varepsilon > 0$. We set $u_{1,\varepsilon} = \sum_{i=1}^k \varphi_\varepsilon(\cdot - x^i_\varepsilon/\varepsilon) u_\varepsilon$ and $u_{2,\varepsilon} = u_\varepsilon - u_{1,\varepsilon}$. As a first step in the proof of the Proposition we shall prove that

$$\Gamma_\varepsilon(u_\varepsilon) \geq \Gamma_\varepsilon(u_{1,\varepsilon}) + \Gamma_\varepsilon(u_{2,\varepsilon}) + O(\varepsilon). \tag{3.10}$$

Suppose there exist $y_\varepsilon \in \bigcup_{i=1}^k B(x^i_\varepsilon/\varepsilon, 2\beta/\varepsilon) \setminus B(x^i_\varepsilon/\varepsilon, \beta/\varepsilon)$ and $R > 0$ satisfying

$$\liminf_{\varepsilon \to 0} \int_{B(y_\varepsilon, R)} |u_\varepsilon|^2 dy > 0$$

which means that

$$\liminf_{\varepsilon \to 0} \int_{B(0,R)} |v_\varepsilon|^2 dy > 0 \tag{3.11}$$

where $v_\varepsilon(y) = u_\varepsilon(y + y_\varepsilon)$. Taking a subsequence, we can assume that $\varepsilon y_\varepsilon \to x_0$ with x_0 in the closure of $\bigcup_{i=1}^k B(x^i, 2\beta) \setminus B(x^i, \beta)$. Since (3.9) holds, (v_ε) is bounded in H_ε. Thus, since $\tilde{m} > 0$, (v_ε) is bounded in $L^2(\mathbb{R}^3, \mathbb{C})$ and using the Diamagnetic inequality and the Hardy-Sobolev inequality (see also the proof of Proposition 2.12) we deduce that (v_ε) is bounded in $L^m(\mathbb{R}^3, \mathbb{C})$ for any $m < 6$. In particular, up to a subsequence, $v_\varepsilon \to \mathcal{W} \in L^m(\mathbb{R}^3, \mathbb{C})$ weakly. Also by Corollary 3.2 i), for any compact $K \subset \mathbb{R}^3$, (v_ε) is bounded in $H^1(K, \mathbb{C})$. Thus we can assume that $v_\varepsilon \to \mathcal{W}$ in $H^1(K, \mathbb{C})$ weakly for any $K \subset \mathbb{R}^3$ compact, strongly in $L^m(K, \mathbb{C})$. Because of (3.11) $\mathcal{W}$ is not the zero function. Now, since $\lim_{\varepsilon \to 0} \Gamma'_\varepsilon(u_\varepsilon) = 0$, $\mathcal{W}$ is a non-trivial solution of

$$-\Delta \mathcal{W} - \frac{2}{\mathrm{i}} A(x_0) \cdot \nabla \mathcal{W} + |A(x_0)|^2 \mathcal{W} + V(x_0) \mathcal{W} = \left(W * |\mathcal{W}|^2 \right) \mathcal{W}. \tag{3.12}$$

From (3.12) and since $\mathcal{W} \in L^m(\mathbb{R}^3, \mathbb{C})$ we readily deduce, using Corollary 3.2 ii) that $\mathcal{W} \in H^1(\mathbb{R}^3, \mathbb{C})$.

Let $\omega(y) = e^{-iA(x_0)y} \mathcal{W}(y)$. Then ω is a non trivial solution of the complex-valued equation

$$-\Delta \omega + V(x_0)\omega(y) = (W * |\omega|^2)\omega.$$

For $R > 0$ large we have

$$\int_{B(0,R)} \left| \left(\frac{\nabla}{\mathrm{i}} - A(x_0) \right) \mathcal{W} \right|^2 dy \geq \frac{1}{2} \int_{\mathbb{R}^3} \left| \left(\frac{\nabla}{\mathrm{i}} - A(x_0) \right) \mathcal{W} \right|^2 dy \tag{3.13}$$

and thus, by the weak convergence,

$$\begin{aligned}\liminf_{\varepsilon\to 0}\int_{B(y_\varepsilon,R)}|D^\varepsilon u_\varepsilon|^2dy &= \liminf_{\varepsilon\to 0}\int_{B(0,R)}\left|\left(\frac{\nabla}{\mathrm{i}}-A_\varepsilon(y+y_\varepsilon)\right)v_\varepsilon\right|^2dy\\ &\geq \int_{B(0,R)}\left|\left(\frac{\nabla}{\mathrm{i}}-A(x_0)\right)\mathcal{W}\right|^2dy\\ &\geq \frac{1}{2}\int_{\mathbb{R}^3}\left|\left(\frac{\nabla}{\mathrm{i}}-A(x_0)\right)\mathcal{W}\right|^2dy=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla\omega|^2dy. \end{aligned}\tag{3.14}$$

It follows from Lemma 2.5 that $E_a > E_b$ if $a > b$ and using Lemma 2.10 we have $L^c_{V(x_0)}(\omega) \geq E^c_{V(x_0)} = E_{V(x_0)} \geq E_m$ since $V(x_0) \geq m$. Thus from (3.14) and Lemma 2.5 we get that

$$\liminf_{\varepsilon\to 0}\int_{B(y_\varepsilon,R)}|D^\varepsilon u_\varepsilon|^2dy \geq \frac{3}{2}L^c_{V(x_0)}(\omega)\geq\frac{3}{2}E_m>0.\tag{3.15}$$

which contradicts (3.9), provided $d > 0$ is small enough. Indeed, $x_0 \neq x^i$, $\forall i \in \{1,\dots,k\}$ and the Z_i are exponentially decreasing.

Since such a sequence (y_ε) does not exist, we deduce from [22, Lemma I.1] that

$$\limsup_{\varepsilon\to 0}\int_{\bigcup_{i=1}^k B(x^i_\varepsilon/\varepsilon,2\beta/\varepsilon)\setminus B(x^i_\varepsilon/\varepsilon,\beta/\varepsilon)}|u_\varepsilon|^5dy=0.\tag{3.16}$$

As a consequence, we can derive using the boundedness of $(\|u_\varepsilon\|_2)$ that

$$\begin{aligned}\lim_{\varepsilon\to 0}\Bigg\{\int_{\mathbb{R}^3\times\mathbb{R}^3}W(x-y)|u_\varepsilon(x)|^2|u_\varepsilon(y)|^2\,dx\,dy-\int_{\mathbb{R}^3\times\mathbb{R}^3}W(x-y)|u_{1,\varepsilon}(x)|^2|u_{1,\varepsilon}(y)|^2\,dx\,dy\\ -\int_{\mathbb{R}^3\times\mathbb{R}^3}W(x-y)|u_{2,\varepsilon}(x)|^2|u_{2,\varepsilon}(y)|^2\,dx\,dy\Bigg\}=0.\end{aligned}$$

At this point writing

$$\begin{aligned}\Gamma_\varepsilon(u_\varepsilon) &= \Gamma_\varepsilon(u_{1,\varepsilon})+\Gamma_\varepsilon(u_{2,\varepsilon})\\ &\quad+\sum_{i=1}^k\int_{B(x^i_\varepsilon/\varepsilon,2\beta/\varepsilon)\setminus B(x^i_\varepsilon/\varepsilon,\beta/\varepsilon)}\varphi_\varepsilon(y-x^i_\varepsilon/\varepsilon)(1-\varphi_\varepsilon(y-x^i/\varepsilon))|D^\varepsilon u_\varepsilon|^2\\ &\qquad+V_\varepsilon\varphi_\varepsilon(y-x^i_\varepsilon/\varepsilon)(1-\varphi_\varepsilon(y-x^i_\varepsilon/\varepsilon))|u_\varepsilon|^2dy\\ &\quad-\frac{1}{4}\int_{\mathbb{R}^3\times\mathbb{R}^3}W(x-y)|u_\varepsilon(x)|^2|u_\varepsilon(y)|^2\,dx\,dy\\ &\quad-\frac{1}{4}\int_{\mathbb{R}^3\times\mathbb{R}^3}W(x-y)|u_{1,\varepsilon}(x)|^2|u_{1,\varepsilon}(y)|^2\,dx\,dy\\ &\quad-\frac{1}{4}\int_{\mathbb{R}^3\times\mathbb{R}^3}W(x-y)|u_{2,\varepsilon}(x)|^2|u_{2,\varepsilon}(y)|^2\,dx\,dy+o(1),\end{aligned}$$

as $\varepsilon \to 0$ this shows that the inequality (3.10) holds. We now estimate $\Gamma_\varepsilon(u_{2,\varepsilon})$. We have

$$\begin{aligned}\Gamma_\varepsilon(u_{2,\varepsilon}) &\geq \mathcal{F}_\varepsilon(u_{2,\varepsilon}) \\ &= \frac{1}{2}\int_{\mathbb{R}^3} |D^\varepsilon u_{2,\varepsilon}|^2 + \tilde{V}_\varepsilon |u_{2,\varepsilon}|^2 dy - \frac{1}{2}\int_{\mathbb{R}^3} (\tilde{V}_\varepsilon - V_\varepsilon)|u_{2,\varepsilon}|^2 dy \\ &\quad - \frac{1}{4}\int_{\mathbb{R}^3\times\mathbb{R}^3} W(x-y)|u_{2,\varepsilon}(x)|^2 |u_{2,\varepsilon}(y)|^2\, dx\, dy \\ &\geq \frac{1}{2}\|u_{2,\varepsilon}\|_\varepsilon^2 - \frac{\tilde{m}}{2}\int_{\mathbb{R}^3\setminus O_\varepsilon^i} |u_{2,\varepsilon}|^2 dy \\ &\quad - \frac{1}{4}\int_{\mathbb{R}^3\times\mathbb{R}^3} W(x-y)|u_{2,\varepsilon}(x)|^2 |u_{2,\varepsilon}(y)|^2\, dx\, dy.\end{aligned} \tag{3.17}$$

Here we have used the fact that $\tilde{V}_\varepsilon - V_\varepsilon = 0$ on O_ε^i and $|\tilde{V}_\varepsilon - V_\varepsilon| \leq \widetilde{m}$ on $\mathbb{R}^3 \setminus O_\varepsilon^i$. For some $C > 0$,

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} W(x-y)|u_{2,\varepsilon}(x)|^2 |u_{2,\varepsilon}(y)|^2\, dx\, dy \leq C\|u_{2,\varepsilon}\|_{L^2}^3 \,\|u_{2,\varepsilon}\|_{H^1}.$$

Since (u_ε) is bounded, we see from (3.9) that $\|u_{2,\varepsilon}\|_\varepsilon \leq 4d$ for small $\varepsilon > 0$. Thus taking $d > 0$ small enough we have

$$\frac{1}{2}\|u_{2,\varepsilon}\|_\varepsilon^2 - \frac{1}{4}\int_{\mathbb{R}^3\times\mathbb{R}^3} W(x-y)|u_{2,\varepsilon}(x)|^2 |u_{2,\varepsilon}(y)|^2\, dx\, dy \geq \frac{1}{8}\|u_{2,\varepsilon}\|_\varepsilon^2. \tag{3.18}$$

Now note that $\mathcal{F}_\varepsilon$ is uniformly bounded in X_ε^d for small $\varepsilon > 0$, and such is Q_ε. This implies that for some $C > 0$,

$$\int_{\mathbb{R}^3\setminus O_\varepsilon} |u_{2,\varepsilon}|^2 dy \leq C\varepsilon^{6/\mu} \tag{3.19}$$

and from (3.17)-(3.19) we deduce that $\Gamma_\varepsilon(u_{2,\varepsilon}) \geq o(1)$.

Now for $i = 1, \dots, k$, we define $u_{1,\varepsilon}^i(y) = u_{1,\varepsilon}(y)$ for $y \in O_\varepsilon^i$, $u_{1,\varepsilon}^i(y) = 0$ for $y \notin O_\varepsilon^i$. Also we set $\mathcal{W}_\varepsilon^i(y) = u_{1,\varepsilon}^i(y + x_\varepsilon^i/\varepsilon)$. We fix an arbitrary $i \in \{1, \dots, k\}$. Arguing as before, we can assume, up to a subsequence, that $\mathcal{W}_\varepsilon^i$ converges weakly in $L^m(\mathbb{R}^3, \mathbb{C})$, $m < 6$, to a solution $\mathcal{W}^i \in H^1(\mathbb{R}^3, \mathbb{C})$ of

$$-\Delta \mathcal{W}^i - \frac{2}{\mathrm{i}} A(x^i)\cdot\nabla \mathcal{W}^i + |A(x^i)|^2 \mathcal{W}^i + V(x^i)\mathcal{W}^i = \left(W * \mathcal{W}^i\right)\mathcal{W}^i, \quad y \in \mathbb{R}^3.$$

We shall prove that $\mathcal{W}_\varepsilon^i$ tends to $\mathcal{W}^i$ strongly in H_ε. Suppose there exist $R > 0$ and a sequence (z_ε) with $z_\varepsilon \in B(x_\varepsilon^i/\varepsilon, 2\beta/\varepsilon)$ satisfying

$$\liminf_{\varepsilon\to 0} |z_\varepsilon - \varepsilon^{-1}x_\varepsilon^i| = \infty \quad \text{and} \quad \liminf_{\varepsilon\to 0}\int_{B(z_\varepsilon,R)} |u_\varepsilon^{1,i}|^2\, dy > 0.$$

We may assume that $\varepsilon z_\varepsilon \to z^i \in O^i$ as $\varepsilon \to 0$. Then $\tilde{\mathcal{W}}_\varepsilon^i(y) = \mathcal{W}_\varepsilon^i(y + z_\varepsilon)$ weakly converges in $L^m(\mathbb{R}^3, \mathbb{C})$ (for any $m < 6$) to $\tilde{\mathcal{W}}^i \in H^1(\mathbb{R}^3, \mathbb{C})$ which satisfies

$$-\Delta \tilde{\mathcal{W}}^i - \frac{2}{\mathrm{i}} A(z^i)\cdot\nabla \tilde{\mathcal{W}} + |A(z^i)|^2 \tilde{\mathcal{W}}^i + V(z^i)\tilde{\mathcal{W}}^i = \left(W * \tilde{\mathcal{W}}^i\right)\tilde{\mathcal{W}}^i, \quad y \in \mathbb{R}^3$$

and as before we get a contradiction. Then using [22, Lemma I.1] it follows that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3\times\mathbb{R}^3} W(x-y)|\mathcal{W}^i_\varepsilon(x)|^2|\mathcal{W}^i_\varepsilon(y)|^2\,dx\,dy = \int_{\mathbb{R}^3\times\mathbb{R}^3} W(x-y)|\mathcal{W}^i(x)|^2|\mathcal{W}^i(y)|^2\,dx\,dy. \tag{3.20}$$

Then from the weak convergence of $\mathcal{W}^i_\varepsilon$ to $\mathcal{W}^i\neq 0$ in $H^1(K,\mathbb{C})$ for any $K\subset\mathbb{R}^3$ compact we get, for any $i\in\{1,\dots,k\}$,

$$\begin{aligned}\limsup_{\varepsilon\to 0}\Gamma_\varepsilon(u^i_{1,\varepsilon}) &\ge \liminf_{\varepsilon\to 0}\mathcal{F}_\varepsilon(u^i_{1,\varepsilon})\\ &\ge \liminf_{\varepsilon\to 0}\frac{1}{2}\int_{B(0,R)}\left|\left(\frac{\nabla}{\mathrm{i}}-A(\varepsilon y+x^i_\varepsilon)\right)\mathcal{W}^i_\varepsilon\right|^2\\ &\quad + V(\varepsilon y+x^i_\varepsilon)|\mathcal{W}^i_\varepsilon|^2dy-\int_{\mathbb{R}^3\times\mathbb{R}^3}W(x-y)|\mathcal{W}^i_\varepsilon(x)|^2|\mathcal{W}^i_\varepsilon(y)|^2\,dx\,dy\\ &\ge \frac{1}{2}\int_{B(0,R)}\left|\left(\frac{\nabla}{\mathrm{i}}-A(x^i)\right)\mathcal{W}^i\right|^2+V(x^i)|\mathcal{W}^i|^2dy\\ &\quad -\frac{1}{4}\int_{\mathbb{R}^3\times\mathbb{R}^3}W(x-y)|\mathcal{W}^i(x)|^2|\mathcal{W}^i(y)|^2\,dx\,dy. \end{aligned}\tag{3.21}$$

Since these inequalities hold for any $R>0$ we deduce, using Lemma 2.10, that

$$\begin{aligned}\limsup_{\varepsilon\to 0}\Gamma_\varepsilon(u^i_{1,\varepsilon}) &\ge \frac{1}{2}\int_{\mathbb{R}^3}\left|\left(\frac{\nabla}{\mathrm{i}}-A(x^i)\right)\mathcal{W}^i\right|^2dy+\frac{1}{2}\int_{\mathbb{R}^3}V(x^i)|\mathcal{W}^i|^2dy\\ &\quad -\int_{\mathbb{R}^3\times\mathbb{R}^3}W(x-y)|\mathcal{W}^i(x)|^2|\mathcal{W}^i(y)|^2\,dx\,dy\\ &= \frac{1}{2}\int_{\mathbb{R}^3}|\nabla\omega^i|^2+V(x^i)|\omega^i|^2dy\\ &\quad -\frac{1}{4}\int_{\mathbb{R}^3\times\mathbb{R}^3}W(x-y)|\omega^i(x)|^2|\omega^i(y)|^2\,dxdy = L^c_{V(x^i)}(\omega^i)\ge E^c_{m_i}=E_{m_i}\end{aligned}\tag{3.22}$$

where we have set $\omega^i(y)=e^{-iA(x^i)y}\mathcal{W}^i(y)$. Now by (3.10),

$$\begin{aligned}\limsup_{\varepsilon\to 0}\Big(\Gamma_\varepsilon(u_{2,\varepsilon})+\sum_{i=1}^k\Gamma_\varepsilon(u^i_{1,\varepsilon})\Big) &= \limsup_{\varepsilon\to 0}\Big(\Gamma_\varepsilon(u_{2,\varepsilon})+\Gamma_\varepsilon(u_{1,\varepsilon})\Big)\\ &\le \limsup_{\varepsilon\to 0}\Gamma_\varepsilon(u_\varepsilon)\le E=\sum_{i=1}^k E_{m_i}.\end{aligned}\tag{3.23}$$

Thus, since $\Gamma_\varepsilon(u_{2,\varepsilon})\ge o(1)$ we deduce from (3.22)-(3.23) that, for any $i\in\{1,\dots k\}$

$$\lim_{\varepsilon\to 0}\Gamma_\varepsilon(u^i_{1,\varepsilon})=E_{m_i}. \tag{3.24}$$

Now (3.22), (3.24) implies that $L_{V(x^i)}(\omega^i)=E_{m_i}$. Recalling from [18] that $E_a>E_b$ if $a>b$ and using Lemma 2.10 we conclude that $x^i\in\mathcal{M}^i$. At this point it is clear that $W^i(y)=e^{iA(x^i)y}U_i(y-z_i)$ with $U_i\in S_{m_i}$ and $z_i\in\mathbb{R}^3$.

To establish that $W_\varepsilon^i \to W^i$ strongly in H_ε we first show that $W_\varepsilon^i \to W^i$ strongly in $L^2(\mathbb{R}^3,\mathbb{C})$. Since (W_ε^i) is bounded in H_ε the Diamagnetic inequality (3.1) immediately yields that $(|W_\varepsilon^i|)$ is bounded in $H^1(\mathbb{R}^3,\mathbb{R})$ and we can assume that $|W_\varepsilon^i| \to |W^i| = |\omega^i|$ weakly in $H^1(\mathbb{R}^3,\mathbb{R})$. Now since $L_{V(x^i)}(\omega^i) = E_{m_i}$, we get using the Diamagnetic inequality, (3.20), (3.24) and the fact that $V \geq V(x^i)$ on O^i,

$$
\begin{aligned}
&\int_{\mathbb{R}^3} |\nabla \omega^i|^2 dy + \int_{\mathbb{R}^3} m_i |\omega^i|^2 dy - 2\int_{\mathbb{R}^3\times\mathbb{R}^3} W(x-y)|\omega^i(x)|^2|\omega^i(y)|^2\, dxdy \\
&\qquad \geq \limsup_{\varepsilon\to 0} \int_{\mathbb{R}^3} \left| \left(\frac{\nabla}{\mathrm{i}} - A(\varepsilon y + x_\varepsilon^i)\right)\mathcal{W}_\varepsilon^i\right|^2 dy + \int_{\mathbb{R}^3} V(\varepsilon y + x_\varepsilon^i)|\mathcal{W}_\varepsilon^i|^2 dy \\
&\qquad\quad - 2\int_{\mathbb{R}^3\times\mathbb{R}^3} W(x-y)|\mathcal{W}_\varepsilon^i(x)|^2|\mathcal{W}_\varepsilon^i(y)|^2\, dxdy \\
&\qquad \geq \limsup_{\varepsilon\to 0} \int_{\mathbb{R}^3} \big|\nabla|\mathcal{W}_\varepsilon^i|\big|^2 dy + \int_{\mathbb{R}^3} V(x^i)|\mathcal{W}_\varepsilon^i|^2 dy \\
&\qquad - 2\int_{\mathbb{R}^3\times\mathbb{R}^3} W(x-y)|\mathcal{W}_\varepsilon^i(x)|^2|\mathcal{W}_\varepsilon^i(y)|^2\, dxdy \\
&\qquad \geq \int_{\mathbb{R}^3} \big|\nabla|\omega^i|\big|^2 dy + \int_{\mathbb{R}^3} m_i|\omega^i|^2 dy \\
&\qquad - 2\int_{\mathbb{R}^3\times\mathbb{R}^3} W(x-y)|\omega^i(x)|^2|\omega^i(y)|^2\, dxdy.
\end{aligned} \tag{3.25}
$$

But from Lemma 2.10 we know that, since $L_{V(x^i)}(\omega^i) = E_{m_i}$,

$$
\int_{\mathbb{R}^3} \big|\nabla|\omega^i|\big|^2 dy = \int_{\mathbb{R}^3} \big|\nabla\omega^i\big|^2 dy.
$$

Thus we deduce from (3.25) that

$$
\int_{\mathbb{R}^3} V(\varepsilon y + x_\varepsilon^i)|\mathcal{W}_\varepsilon^i|^2 dy \to \int_{\mathbb{R}^3} V(x^i)|\mathcal{W}^i|^2 dy. \tag{3.26}
$$

Thus, since $V \geq V(x^i)$ on O^i, we deduce that

$$
\mathcal{W}_\varepsilon^i \to \mathcal{W}^i \text{ strongly in } L^2(\mathbb{R}^3,\mathbb{C}). \tag{3.27}
$$

From (3.27) we easily get that

$$
\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3} \left|\left(\frac{\nabla}{\mathrm{i}} - A(\varepsilon y + x_\varepsilon^i)\right)\mathcal{W}_\varepsilon^i\right|^2 - \left|\left(\frac{\nabla}{\mathrm{i}} - A(x^i)\right)\mathcal{W}_\varepsilon^i\right|^2 dy = 0. \tag{3.28}
$$

Now, using (3.20), (3.25) and (3.26), we see from (3.28) that

$$
\begin{aligned}
(3.29)\quad & \int_{\mathbb{R}^3}\left|\left(\frac{\nabla}{\mathrm{i}}-A(x^i)\right)\mathcal{W}^i\right|^2 dy+\int_{\mathbb{R}^3}V(x^i)|\mathcal{W}^i|^2dy \\
&\geq \limsup_{\varepsilon\to 0}\int_{\mathbb{R}^3}\left|\left(\frac{\nabla}{\mathrm{i}}-A(\varepsilon y+x^i_\varepsilon)\right)\mathcal{W}^i_\varepsilon\right|^2 dy+\int_{\mathbb{R}^3}V(\varepsilon y+x^i_\varepsilon)|\mathcal{W}^i_\varepsilon|^2dy \\
&\qquad\geq \limsup_{\varepsilon\to 0}\int_{\mathbb{R}^3}\left|\left(\frac{\nabla}{\mathrm{i}}-A(x^i)\right)\mathcal{W}^i_\varepsilon\right|^2 dy+\int_{\mathbb{R}^3}V(x^i)|\mathcal{W}^i_\varepsilon|^2\,dy.
\end{aligned}
$$

At this point and using Corollary 3.2 ii) we have established the strong convergence $W^i_\varepsilon\to W^i$ in $H^1(\mathbb{R}^3,\mathbb{C})$. Thus we have

$$u^i_{1,\varepsilon}=e^{iA(x^i)(\cdot-x^i_\varepsilon/\varepsilon)}U_i(\cdot-x^i_\varepsilon/\varepsilon-z_i)+o(1)$$

strongly in $H^1(\mathbb{R}^3,\mathbb{C})$. Now setting $y^i_\varepsilon=x^i_\varepsilon/\varepsilon+z_i$ and changing U_i to $e^{iA(x^i)z_i}U_i$ we get that

$$u^i_{1,\varepsilon}=e^{iA(x^i)(\cdot-y^i_\varepsilon)}U_i(\cdot-y^i_\varepsilon)+o(1)$$

strongly in $H^1(\mathbb{R}^3,\mathbb{C})$. Finally using the exponential decay of U_i and ∇U_i we have

$$u^i_{1,\varepsilon}=e^{iA_\varepsilon(y^i_\varepsilon)(\cdot-y^i_\varepsilon)}\varphi_\varepsilon(\cdot-y^i_\varepsilon)U_i(\cdot-y^i_\varepsilon)+o(1).$$

From Corollary 3.2 iii) we deduce that this convergence also holds in H_ε and thus

$$u_{1,\varepsilon}=\sum_{i=1}^k u^i_{1,\varepsilon}=\sum_{i=1}^k e^{iA_\varepsilon(y^i_\varepsilon)(\cdot-y^i_\varepsilon)}\varphi_\varepsilon(\cdot-y^i_\varepsilon)U_i(\cdot-y^i_\varepsilon)+o(1)$$

strongly in H_ε. To conclude the proof of the Proposition, it suffices to show that $u_{2,\varepsilon}\to 0$ in H_ε. Since $E\geq \lim_{\varepsilon\to0}\Gamma_\varepsilon(u_\varepsilon)$ and $\lim_{\varepsilon\to0}\Gamma_\varepsilon(u_{1,\varepsilon})=E$ we deduce, using (3.10) that $\lim_{\varepsilon\to0}\Gamma_\varepsilon(u_{2,\varepsilon})=0$. Now from (3.17)-(3.19) we get that $u_{2,\varepsilon}\to0$ in H_ε. □

3.3. **Critical points of the penalized functional.** We first state the following

Proposition 3.6. *For sufficiently small $d>0$, there exist constants $\omega>0$ and $\varepsilon_0>0$ such that $|\Gamma'_\varepsilon(u)|\geq\omega$ for $u\in\Gamma_\varepsilon^{D_\varepsilon}\cap(X^d_\varepsilon\setminus X^{d/2}_\varepsilon)$ and $\varepsilon\in(0,\varepsilon_0)$.*

Proof. By contradiction, we suppose that for $d>0$ sufficiently small such that Proposition 3.5 applies, there exist (ε_j) with $\lim_{j\to\infty}\varepsilon_j=0$ and a sequence (u_{ε_j}) with $u_{\varepsilon_j}\in X^d_{\varepsilon_j}\setminus X^{d/2}_{\varepsilon_j}$ satisfying $\lim_{j\to\infty}\Gamma_{\varepsilon_j}(u_{\varepsilon_j})\leq E$ and $\lim_{j\to\infty}\Gamma'_{\varepsilon_j}(u_{\varepsilon_j})=0$. By Proposition 3.5, there exist $(y^i_{\varepsilon_j})\subset\mathbb{R}^3$, $i=1,\dots,k$, $x^i\in\mathcal{M}^i$, $U_i\in S_{m_i}$ such that

$$\lim_{\varepsilon_j\to0}|\varepsilon_j y^i_{\varepsilon_j}-x^i|=0,$$

$$\lim_{\varepsilon_j\to0}\left\|u_{\varepsilon_j}-\sum_{i=1}^k e^{iA_{\varepsilon_j}(y^i_{\varepsilon_j})(\cdot-y^i_{\varepsilon_j})}\varphi_{\varepsilon_j}(\cdot-y^i_{\varepsilon_j})U_i(\cdot-y^i_{\varepsilon_j})\right\|_{\varepsilon_j}=0.$$

By definition of X_{ε_j} we see that $\lim_{\varepsilon_j\to0}\mathrm{dist}(u_{\varepsilon_j},X_{\varepsilon_j})=0$. This contradicts that $u_{\varepsilon_j}\notin X^{d/2}_{\varepsilon_j}$ and completes the proof. □

From now on we fix a $d>0$ such that Proposition 3.6 holds.

Proposition 3.7. *For sufficiently small fixed $\varepsilon > 0$, Γ_ε has a critical point $u_\varepsilon \in X_\varepsilon^d \cap \Gamma_\varepsilon^{D_\varepsilon}$.*

Proof. We can take $R_0 > 0$ sufficiently large so that $O \subset B(0, R_0)$ and $\gamma_\varepsilon(s) \in H_0^1(B(0, R/\varepsilon))$ for any $s \in T$, $R > R_0$ and sufficiently small $\varepsilon > 0$.

We notice that by Proposition 3.3 (iii), there exists $\alpha \in (0, E - \tilde{E})$ such that for sufficiently small $\varepsilon > 0$,

$$\Gamma_\varepsilon(\gamma_\varepsilon(s)) \geq D_\varepsilon - \alpha \quad \Longrightarrow \quad \gamma_\varepsilon(s) \in X_\varepsilon^{d/2} \cap H_0^1(B(0, R/\varepsilon)).$$

We begin to show that for sufficiently small fixed $\varepsilon > 0$, and $R > R_0$, there exists a sequence $(u_n^R) \subset X_\varepsilon^{d/2} \cap \Gamma_\varepsilon^{D_\varepsilon} \cap H_0^1(B(0, R/\varepsilon))$ such that $\Gamma'(u_n^R) \to 0$ in $H_0^1(B(0, R/\varepsilon))$ as $n \to +\infty$.

Arguing by contradiction, we suppose that for sufficiently small $\varepsilon > 0$, there exists $a_R(\varepsilon) > 0$ such that $|\Gamma_\varepsilon'(u)| \geq a_R(\varepsilon)$ on $X_\varepsilon^d \cap \Gamma_\varepsilon^{D_\varepsilon} \cap H_0^1(B(0, R/\varepsilon))$. In what follows any $u \in H_0^1(B(0, R/\varepsilon))$ will be regarded as an element in H_ε by defining $u = 0$ in $\mathbb{R}^3 \setminus B(0, R/\varepsilon)$. Note from Proposition 3.6 that there exists $\omega > 0$, independent of $\varepsilon > 0$, such that $|\Gamma_\varepsilon'(u)| \geq \omega$ for $u \in \Gamma_\varepsilon^{D_\varepsilon} \cap (X_\varepsilon^d \setminus X_\varepsilon^{d/2})$. Thus, by a deformation argument in $H_0^1(B(0, R/\varepsilon))$, starting from γ_ε, for sufficiently small $\varepsilon > 0$ there exists a $\mu \in (0, \alpha)$ and a path $\gamma \in C([0, T], H_\varepsilon)$ satisfying

$$\gamma(s) = \gamma_\varepsilon(s) \quad \text{for } \gamma_\varepsilon(s) \in \Gamma_\varepsilon^{D_\varepsilon - \alpha}, \quad \gamma(s) \in X_\varepsilon^d \quad \text{for } \gamma_\varepsilon(s) \notin \Gamma_\varepsilon^{D_\varepsilon - \alpha}$$

and

$$\Gamma_\varepsilon(\gamma(s)) < D_\varepsilon - \mu, \quad s \in T. \tag{3.30}$$

Let $\psi \in C_0^\infty(\mathbb{R}^3)$ be such that $\psi(y) = 1$ for $y \in O^\delta$, $\psi(y) = 0$ for $y \notin O^{2\delta}$, $\psi(y) \in [0, 1]$ and $|\nabla\psi| \leq 2/\delta$. For $\gamma(s) \in X_\varepsilon^d$, we define $\gamma_1(s) = \psi_\varepsilon\gamma(s)$ and $\gamma_2(s) = (1 - \psi_\varepsilon)\gamma(s)$ where $\psi_\varepsilon(y) = \psi(\varepsilon y)$. The dependence on ε will be understood in the notation for γ_1 and γ_2. Note that

$$\begin{aligned}
\Gamma_\varepsilon(\gamma(s)) &= \Gamma_\varepsilon(\gamma_1(s)) + \Gamma_\varepsilon(\gamma_2(s)) + \int_{\mathbb{R}^3} \big(\psi_\varepsilon(1 - \psi_\varepsilon)|D^\varepsilon\gamma(s)|^2 + V_\varepsilon\psi_\varepsilon(1 - \psi_\varepsilon)|\gamma(s)|^2\big)dy \\
&\quad + Q_\varepsilon(\gamma(s)) - Q_\varepsilon(\gamma_1(s)) - Q_\varepsilon(\gamma_2(s)) \\
&\quad - \frac{1}{4}\int_{\mathbb{R}^3\times\mathbb{R}^3} W(x - y)\big(|\gamma(s)(x)|^2|\gamma(s)(y)|^2 - |\gamma_1(s)(x)|^2|\gamma_1(s)(y)|^2 \\
&\qquad - |\gamma_1(s)(x)|^2|\gamma_2(s)(y)|^2\big)dx\,dy + o(1).
\end{aligned}$$

Since for $A, B \geq 0$, $(A + B - 1)_+ \geq (A - 1)_+ + (B - 1)_+$, it follows that

$$\begin{aligned}
Q_\varepsilon(\gamma(s)) &= \Big(\int_{\mathbb{R}^3} \chi_\varepsilon|\gamma_1(s) + \gamma_2(s)|^2 dy - 1\Big)_+^{\frac{5}{2}} \\
&\geq \Big(\int_{\mathbb{R}^3} \chi_\varepsilon|\gamma_1(s)|^2 dy + \int_{\mathbb{R}^3} \chi_\varepsilon|\gamma_2(s)|^2 dy - 1\Big)_+^{\frac{5}{2}} \\
&\geq \Big(\int_{\mathbb{R}^3} \chi_\varepsilon|\gamma_1(s)|^2 dy - 1\Big)_+^{\frac{5}{2}} + \Big(\int_{\mathbb{R}^3} \chi_\varepsilon|\gamma_2(s)|^2 dy - 1\Big)_+^{\frac{5}{2}} \\
&= Q_\varepsilon(\gamma_1(s)) + Q_\varepsilon(\gamma_2(s)).
\end{aligned}$$

Now, as in the derivation of (3.19), using the fact that $Q_\varepsilon(\gamma(s))$ is uniformly bounded with respect to ε, we have, for some $C>0$

$$\int_{\mathbb{R}^3\setminus O_\varepsilon} |\gamma(s)|^2 dy \le C\varepsilon^{6/\mu}. \tag{3.31}$$

Since W is even, we have

$$\begin{aligned}
&\int\limits_{\mathbb{R}^3\times\mathbb{R}^3} W(x-y)\big(|\gamma(s)(x)|^2|\gamma(s)(y)|^2-|\gamma_1(s)(x)|^2|\gamma_1(s)(y)|^2-|\gamma_1(s)(x)|^2|\gamma_2(s)(y)|^2\big)dxdy\\
&= 2\int_{O_\varepsilon^\delta} dy \int_{\mathbb{R}^3\setminus O_\varepsilon^{2\delta}} W(x-y)|\gamma(s)(x)|^2|\gamma(s)(y)|^2dx\\
&+ 2\int_{O_\varepsilon^\delta} dy \int_{O_\varepsilon^{2\delta}\setminus O_\varepsilon^\delta} W(x-y)|\gamma(s)(x)|^2|\gamma(s)(y)|^2dx\\
&+ 2\int_{O_\varepsilon^{2\delta}\setminus O_\varepsilon^\delta} dy \int_{O_\varepsilon^{2\delta}\setminus O_\varepsilon^\delta} W(x-y)|\gamma(s)(x)|^2|\gamma(s)(y)|^2dx\\
&+ 2\int_{\mathbb{R}^3\setminus O_\varepsilon^{2\delta}} dy \int_{O_\varepsilon^{2\delta}\setminus O_\varepsilon^\delta} W(x-y)|\gamma(s)(x)|^2|\gamma(s)(y)|^2dx\\
&= 2\int_{O_\varepsilon^\delta} dy \int_{\mathbb{R}^3\setminus O_\varepsilon^\delta} W(x-y)|\gamma_2(s)(x)|^2|\gamma_1(s)(y)|^2dx\\
&\qquad + 2\int_{\mathbb{R}^3\setminus O_\varepsilon^\delta} dy \int_{O_\varepsilon^{2\delta}\setminus O_\varepsilon^\delta} W(x-y)|\gamma(s)(x)|^2|\gamma(s)(y)|^2dx
\end{aligned}$$

From (3.31) we deduce that

$$\lim_{\varepsilon\to0}\int_{O_\varepsilon^\delta} dy \int_{\mathbb{R}^3\setminus O_\varepsilon^\delta} W(x-y)|\gamma_2(s)(x)|^2|\gamma_1(s)(y)|^2dx = 0 \tag{3.32}$$

and

$$\lim_{\varepsilon\to0}\int_{\mathbb{R}^3\setminus O_\varepsilon^\delta} dy \int_{O_\varepsilon^{2\delta}\setminus O_\varepsilon^\delta} W(x-y)|\gamma_2(s)(x)|^2|\gamma_1(s)(y)|^2dx = 0 \tag{3.33}$$

From (3.32) and (3.33) we have (recall that γ_1 and γ_2 depend on ε)

$$\int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \big|W(x-y)\big(|\gamma(s)(x)|^2|\gamma(s)(y)|^2-|\gamma_1(s)(x)|^2|\gamma_1(s)(y)|^2-|\gamma_1(s)(x)|^2|\gamma_2(s)(y)|^2\big)\big|dxdy = o(1).$$

Thus, we see that, as $\varepsilon\to0$,

$$\Gamma_\varepsilon(\gamma(s)) \ge \Gamma_\varepsilon(\gamma_1(s)) + \Gamma_\varepsilon(\gamma_2(s)) + o(1).$$

Also

$$\Gamma_\varepsilon(\gamma_2(s)) \ge -\frac{1}{4}\int_{(\mathbb{R}^3\setminus O_\varepsilon)\times(\mathbb{R}^3\setminus O_\varepsilon)} W(x-y)|\gamma_2(s)(x)|^2|\gamma_2(s)(y)|^2\,dx\,dy \ge o(1).$$

Therefore it follows that

$$\Gamma_\varepsilon(\gamma(s)) \geq \Gamma_\varepsilon(\gamma_1(s)) + o(1). \tag{3.34}$$

For $i = 1, \dots, k$, we define

$$\gamma_1^i(s)(y) = \begin{cases} \gamma_1(s)(y) & \text{for } y \in (O^i)_\varepsilon^{2\delta} \\ 0 & \text{for } y \notin (O^i)_\varepsilon^{2\delta}. \end{cases}$$

Note that $(A_1 + \cdots + A_n - 1)_+ \geq \sum_{i=1}^n (A_i - 1)_+$ for $A_1, \dots, A_n \geq 0$. Then we see that

$$\Gamma_\varepsilon(\gamma_1(s)) \geq \sum_{i=1}^k \Gamma_\varepsilon(\gamma_1^i(s)) = \sum_{i=1}^k \Gamma_\varepsilon^i(\gamma_1^i(s)). \tag{3.35}$$

From Proposition 3.3 (ii) and since $0 < \alpha < E - \tilde{E}$ we get that $\gamma_1^i \in \Phi_\varepsilon^i$, for all $i \in \{1, \dots, k\}$. Thus by Proposition 3.4 in [13], Proposition 3.4, and (3.35) we deduce that, as $\varepsilon \to 0$,

$$\max_{s \in T} \Gamma_\varepsilon(\gamma(s)) \geq E + o(1).$$

Since $\limsup_{\varepsilon \to 0} D_\varepsilon \leq E$ this contradicts (3.30).

Now let (u_n^R) be a Palais-Smale sequence corresponding to a fixed small $\varepsilon > 0$. Since (u_n^R) is bounded in $H_0^1(B(0, R/\epsilon))$, and by Corollary 3.2, we have that, up to subsequence,u_n^R converges strongly to u^R in $H_0^1(B(0, R/\epsilon))$. We observe that u^R is a critical point of Γ_ε on $H_0^1(B(0, R/\epsilon))$, and it solves

$$\begin{aligned} &\left(\frac{1}{\mathrm{i}}\nabla - A_\varepsilon\right)^2 u^R + V_\varepsilon u^R \\ &\qquad = \left(W * |u^R|^2\right) u^R - 5\Big(\int \chi_\varepsilon |u^R|^2 dy - 1\Big)_+^{\frac{3}{2}} \chi_\varepsilon u^R \ \text{ in } \ B(0, R/\epsilon). \end{aligned} \tag{3.36}$$

Exploiting Kato's inequality,

$$\Delta |u^R| \geq -\Re\left(\frac{\bar{u^R}}{|u^R|}\left(\frac{\nabla}{\mathrm{i}} - A_\varepsilon\right)^2 u^R\right)$$

we obtain

$$\Delta |u^R| \geq V_\varepsilon |u^R| - \left(W * |u^R|^2\right) |u^R| + 5\Big(\int \chi_\varepsilon |u^R|^2 dy - 1\Big)_+^{\frac{3}{2}} \chi_\varepsilon |u^R| \ \text{ in } \ \mathbb{R}^3. \tag{3.37}$$

Moreover by Moser iteration it follows that $\|u^R\|_{L^\infty}$ is bounded. Since $(Q_\epsilon(u^R))$ is uniformly bounded for $\epsilon > 0$ small, we derive that $\left(W * |u^R|^2\right) |u^R| \leq \frac{1}{2} V_\varepsilon |u^R(y)|$ if $|y| \geq 2R_0$. Applying a comparison principle we derive that

$$|u^R(y)| \leq C \exp(-(|y| - 2R_0)) \tag{3.38}$$

for some $C > 0$ independent of $R > R_0$. Therefore as (u^R) is bounded in H_ϵ we may assume that it weakly converges to some u_ϵ in H_ϵ as $R \to +\infty$. Since u^R is a solution of

(3.36), we see from (3.38) that (u^R) converges strongly to $u_\epsilon \in X_\epsilon \cap \Gamma_\epsilon^{D_\epsilon}$ and it solves

$$\left(\frac{1}{\mathrm{i}}\nabla - A_\varepsilon\right)^2 u_\varepsilon + V_\varepsilon u_\varepsilon = \left(W * |u_\varepsilon|^2\right) u_\varepsilon - 5\left(\int \chi_\varepsilon |u_\varepsilon|^2 dy - 1\right)_+^{\frac{3}{2}} \chi_\varepsilon u_\varepsilon \text{ in } \mathbb{R}^3. \tag{3.39}$$

□

3.4. **Proof for the main result.** We see from Proposition 3.7 that there exists $\varepsilon_0 > 0$ such that, for $\varepsilon \in (0, \varepsilon_0)$, Γ_ε has a critical point $u_\varepsilon \in X_\varepsilon^d \cap \Gamma_\varepsilon^{D_\varepsilon}$. Exploiting Kato's inequality

$$\Delta|u_\varepsilon| \geq -\Re\left(\frac{\bar{u_\varepsilon}}{|u_\varepsilon|}\left(\frac{\nabla}{\mathrm{i}} - A_\varepsilon\right)^2 u_\varepsilon\right)$$

we obtain

$$\Delta|u_\varepsilon| \geq V_\varepsilon|u_\varepsilon| - \left(W * |u_\varepsilon|^2\right)|u_\varepsilon| + 5\left(\int \chi_\varepsilon |u_\varepsilon|^2 dy - 1\right)_+^{\frac{3}{2}} \chi_\varepsilon |u_\varepsilon| \quad \text{in} \quad \mathbb{R}^3. \tag{3.40}$$

Moreover, by (2.29) and the subsequent bootstrap arguments, we deduce that $u_\varepsilon \in L^q(\mathbb{R}^3)$ for any $q > 2$. Hence a Moser iteration scheme shows that $(\|u_\varepsilon\|_{L^\infty})$ is bounded. Now by Proposition 3.5, we see that

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^3 \setminus (\mathcal{M}^{2\beta})_\varepsilon} |D^\varepsilon u_\varepsilon|^2 + \tilde{V}_\varepsilon |u_\varepsilon|^2 dy = 0,$$

and thus, by elliptic estimates (see [17]), we obtain that

$$\lim_{\varepsilon \to 0} \|u_\varepsilon\|_{L^\infty(\mathbb{R}^3 \setminus (\mathcal{M}^{2\beta})_\varepsilon)} = 0. \tag{3.41}$$

This gives the following decay estimate for u_ε on $\mathbb{R}^3 \setminus (\mathcal{M}^{2\beta})_\varepsilon \cup (Z^\beta)_\varepsilon$

$$|u_\varepsilon(x)| \leq C \exp(-c\,\mathrm{dist}(x, (\mathcal{M}^{2\beta})_\varepsilon \cup (Z^\beta)_\varepsilon)) \tag{3.42}$$

for some constants $C, c > 0$. Indeed from (3.41) we see that

$$\lim_{\varepsilon \to 0} \|W * |u_\varepsilon|^2\|_{L^\infty(\mathbb{R}^3 \setminus (\mathcal{M}^{2\beta})_\varepsilon \cup (Z^\beta)_\varepsilon)} = 0.$$

Also $\inf\{V_\varepsilon(y) : y \notin (\mathcal{M}^{2\beta})_\varepsilon \cup (Z^\beta)_\varepsilon\} > 0$. Thus, we obtain the decay estimate (3.42) by applying standard comparison principles to (3.40).

If $Z \neq \emptyset$ we shall need, in addition, an estimate for $|u_\varepsilon|$ on $(Z^{2\beta})_\varepsilon$. Let $\{H^i\}_{i \in I}$ be the connected components of $\mathrm{int}(Z^{3\delta})$ for some index set I. Note that $Z \subset \bigcup_{i \in I} H^i$ and Z is compact. Thus, the set I is finite. For each $i \in I$, let (ϕ^i, λ_1^i) be a pair of first positive eigenfunction and eigenvalue of $-\Delta$ on $(H^i)_\varepsilon$ with Dirichlet boundary condition. From now we fix an arbitrary $i \in I$. By using the fact that $(Q_\varepsilon(u_\varepsilon))$ is bounded we see that for some constant $C > 0$

$$\|u_\varepsilon\|_{L^3((H^i)_\varepsilon)} \leq C\varepsilon^{3/\mu}. \tag{3.43}$$

Thus, from the Hardy-Littlewood-Sobolev inequality we have, for some $C > 0$

$$\|W * |u_\varepsilon|^2\|_{L^\infty((H^i)_\varepsilon)} \leq C\|u_\varepsilon\|^2_{L^3((H^i)_\varepsilon)} \leq C\varepsilon^6.$$

Denote $\phi^i_\varepsilon(y) = \phi^i(\varepsilon y)$. Then, for sufficiently small $\varepsilon > 0$, we deduce that for $y \in \mathrm{int}((H^i)_\varepsilon)$,

$$\Delta\phi^i_\varepsilon(y) - V_\varepsilon(x)\phi^i_\varepsilon(y) + W * |u_\varepsilon(y)|^2\phi^i_\varepsilon(y) \le \Big(C\varepsilon^6 - \lambda_1\varepsilon^2\Big)\phi^i_\varepsilon \le 0. \tag{3.44}$$

Now, since $\mathrm{dist}(\partial(Z^{2\beta})_\varepsilon, (Z^\beta)_\varepsilon) = \beta/\varepsilon$, we see from (3.42) that for some constants $C, c > 0$,

$$\|u_\varepsilon\|_{L^\infty(\partial(Z^{2\beta})_\varepsilon)} \le C\exp(-c/\varepsilon). \tag{3.45}$$

We normalize ϕ^i requiring that

$$\inf_{y\in(H^i)_\varepsilon\cap\partial(Z^{2\delta})_\varepsilon} \phi^i_\varepsilon(y) = C\exp(-c/\varepsilon) \tag{3.46}$$

for the same C, $c > 0$ as in (3.45). Then, we see that for some $\kappa > 0$,

$$\phi^i_\varepsilon(y) \le \kappa C\exp(-c/\varepsilon), \quad y \in (H^i)_\varepsilon \cap (Z^{2\beta})_\varepsilon.$$

Now we deduce, using (3.43), (3.44), (3.45), (3.46) that for each $i \in I$, $|u_\varepsilon| \le \phi^i_\varepsilon$ on $(H^i)_\varepsilon \cap (Z^{2\beta})_\varepsilon$. Therefore

$$|u_\varepsilon(y)| \le C\exp(-c/\varepsilon), \quad \text{on } (Z^{2\delta})_\varepsilon \tag{3.47}$$

for some $C, c > 0$. Now (3.42) and (3.47) implies that $Q_\varepsilon(u_\varepsilon) = 0$ for $\varepsilon > 0$ sufficiently small and thus u_ε satisfies (1.5). Now using Propositions 2.12 and 3.5, we readily deduce that the properties of u_ε given in Theorem 1.1 hold. Here, in (1.7) we also use Lemma 2.10. □

References

[1] M.J. Ablowitz, B. Prinari, A.D. Trubatch, *Discrete and continuous nonlinear Schrödinger systems*, London Mathematical Society, Lecture Note Series, **302** Cambridge University Press, 2004.

[2] J. Byeon, L. Jeanjean, *Standing waves for nonlinear Schrödinger equations with a general nonlinearity*, Arch. Ration. Mech. Anal. **185** (2007), 185-200.

[3] J. Byeon, L. Jeanjean, *Erratum: Standing waves for nonlinear Schrödinger equations with a general nonlinearity*, Arch. Ration. Mech. Anal. DOI 10.1007/s00205-006-0019-3.

[4] J. Byeon, L. Jeanjean, *Multi-peak standing waves for nonlinear Schrödinger equations with a general nonlinearity*, Discrete Cont. Dyn. Systems **19** (2007), 255–269.

[5] J. Byeon, L. Jeanjean, K. Tanaka, *Standing waves for nonlinear Schrödinger equations with a general nonlinearity: one and two dimensional cases*, Comm. Partial Differential Equations **33** No. 6 (2008), 1113–1136.

[6] J. Byeon, Z.-Q. Wang, *Standing waves with critical frequency for nonlinear Schrödinger equations*, Arch. Rat. Mech. Anal. **165** (2002), 295–316.

[7] J. Byeon, Z.-Q. Wang *Standing waves with critical frequency for nonlinear Schrödinger equations II*, Calc. Var. Partial Differ. Equ. **18** (2003), 207–219.

[8] R. Carles, N. Mauser, H. Stimming, *(Semi)classical limit of the Hartree equation with harmonic potential*, SIAM J. Appl. Math. **66** (2005), 29–56.

[9] T. Cazenave, *An introduction to nonlinear Schrödinger equations*, Courant lecture notes, AMS.

[10] T. Cazenave, P.L. Lions, *Orbital stability of standing waves for some nonlinear Schrödinger equations*, Commun. Math. Phys. **85** (1982), 549-561.

[11] S. Cingolani, S. Secchi, *Semiclassical states for NLS equations with magnetic potentials having polynomial growths*, J. Math. Phys. **46** (2005), 19pp.

[12] S. Cingolani, L. Jeanjean, S. Secchi, *Multi-peak solutions for magnetic NLS equations without non–degeneracy conditions*, ESAIM-COCV (2009).

[13] V. Coti Zelati, P.H. Rabinowitz, *Homoclinic orbits for second order Hamiltonian systems possessing superquadratic potentials*, J. Amer. Math. Soc., **4** (1991), 693–727.

[14] L. Erdös, H.-T. Yau, *Derivation of the nonlinear Schrödinger equation from a many body Coulomb system*, Adv. Theor. Math. Phys. **5** (2001), 1169–1205.

[15] M.J. Esteban, P.L. Lions, *Stationary solutions of nonlinear Schrödinger equations with an external magnetic field*, in PDE and Calculus of Variations, in honor of E. De Giorgi, Birkhäuser, 1990.

[16] J. Fröhlich, T.-P. Tsai, H.-T. Yau, *On the point-particle (Newtonian) limit of the non-linear Hartree equation*, Comm. Math. Phys. **225** (2002), 223-274.

[17] D. Gilbarg, N. Trudinger. Elliptic partial differential equations of the second order. Springer–Verlag, 1983.

[18] L. Jeanjean, K. Tanaka, *A remark on least energy solutions in* $\mathbb{R}^N$, Proc. Amer. Math. Soc. **131** (2003), 2399–2408.

[19] E.H. Lieb, *Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation*, Studies in Appl. Math. **57** (1976/77), 93–105.

[20] E.H. Lieb, M. Loss, Analysis. AMS, Providence, 1997.

[21] P.L. Lions, *The Choquard equation and related questions*, Nonlinear Anal. **4** (1980), 1063–1072.

[22] P.L. Lions, *The concentration–compactness principle in the calculus of variations. The locally compact case. Part II*, Annales Inst. H. Poincaré Anal. Nonlin. **1** (1984), 223–283.

[23] M. Macrì, M. Nolasco, *Stationary solutions for the non–linear Hartree equations with a slowly varying potential*, preprint (2008).

[24] I.M. Moroz, P. Tod, *An analytical approach to the Schödinger-Newton equations*, Nonlinearity **12** (1999), 201–216.

[25] I.M. Moroz, R. Penrose, P. Tod, *Spherically-symmetric solutions of the Schrödinger-Newton equations*, Classical Quantum Gravity **15** (1998), 2733–2742.

[26] R. Penrose, *Quantum computation, entanglement and state reduction*, R. Soc. Lond. Phys. Trans. Ser. A, Math. Phys. Eng. Sci. **356** (1998), 1927–1939.

[27] P.L. Sulem, C. Sulem, *The nonlinear Schrödinger equation. Self-focusing and wave collapse*, Applied Mathematical Sciences, **139**. Springer-Verlag, New York, 1999.

[28] J. Wei, M. Winter, *Strongly interacting bumps for the Schrödinger–Newton equation*, J. Math.Phys. **50** (2009), 012905.

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