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Abstract

In this paper we determine the full character table of a certain split extension $H_1(q) \rtimes Sp(2,q)$ of the Heisenberg group H_1 by the odd-characteristic symplectic group $Sp(2,q)$.

Keywords: Character table, Heisenberg group, Symplectic group.

AMS subject classification: 20C15.

1 Introduction

In his paper ([Gér]) P. Gérardin constructed the Weil representations of the odd-characteristic symplectic groups using the properties of a certain split extension $H_t(q) \rtimes Sp(2t,q)$ of the Heisenberg group $H_t(q)$ of order q^{2t+1} by the symplectic group $Sp(2t,q)$. In this paper we explicitly determine the character table of this extension, in the case where $t=1$. A motivation lies in the fact that knowledge of this character table seems to be useful in the study of the restrictions to parabolic subgroups of certain unipotent characters of odd-dimensional orthogonal groups (see [DPW]).

Let V be the column vector space of dimension $2t$ over a finite field F of order q, where q is odd, and V is provided with a non-degenerate symplectic form j. Given $w \in V$, we denote by w^* the element of the dual space (we think at w^* as a row) such that $w^* w_1 = j(w,w_1)/2$. Let $H_t(q)$ be the group consisting of the matrices

$$h = h_{(w,z)} = \left(\begin{array}{c|c|c} 1 & w^* & z \\ \hline & 1 & w \\ \hline & & 1 \end{array}\right) \in Mat(2t+2, F),$$

where $w \in V$ and $z \in F$. We call this group the Heisenberg group of V. $H_t(q)$ is obviously a central extension of $(V,+)$ by $(F,+)$. Furthermore, $H_t(q)$ is a two-step nilpotent group of order q^{2t+1} whose center is isomorphic to F (cf. [Gér, Lemma 2.1]).

Let S be the symplectic group associated to the form j and, for each $s \in S$, denote by sw the image of w under the natural action of S on V. Then, the map

$h_{(w,z)} \mapsto h_{(sw,z)}$ defines an automorphism of $H_t(q)$ fixing pointwise $\mathbf{Z}(H_t(q))$. Viewed as acting on matrices, this map is the conjugation by the element $\mathbf{s} = diag(1, s, 1)$

Let us denote by G the semidirect product $H_t(q) \rtimes Sp(2t, q)$ defined by the above action of S. We want to construct the character table of G in the case where $t = 1$. So, $G = H_1(q) \rtimes Sp(2, q)$. In this case, we can write in a unique way a generic element g of G as

$$g = g_{(s,w,z)} = sh_{(w,z)} = \left(\begin{array}{c|c|c} 1 & w^* & z \\ \hline & s & sw \\ \hline & & 1 \end{array}\right),$$

where $s \in S = Sp(2, q)$ (here we identify $s \in S$ with $\mathbf{s} \in G$), $w \in V$ and $z \in F$. If $w = \begin{pmatrix} x \\ y \end{pmatrix} \in V$, then we can take as w^* the row $\frac{1}{2}(-y, x)$. Note that $|G| = q^4(q^2 - 1)$.

2 The conjugacy classes

In the sequel, we denote by (g) the conjugacy class of G containing the element g, and by $|(g)|$ the size of the conjugacy class (g). The following lemma lists the conjugacy classes of G.

Lemma 2.1. *Let $F = GF(q)$, q odd, and let $F^\times = \langle \nu \rangle$ be the multiplicative group of F. Set*

$$\mathscr{A}(z) = \left(\begin{array}{c|cc|c} 1 & & & z \\ \hline & 1 & & \\ & & 1 & \\ \hline & & & 1 \end{array}\right), \qquad \mathscr{B} = \left(\begin{array}{c|cc|c} 1 & & \frac{1}{2} & \\ \hline & 1 & & 1 \\ & & 1 & \\ \hline & & & 1 \end{array}\right),$$

$$\mathscr{C}(z) = \left(\begin{array}{c|cc|c} 1 & & & z \\ \hline & -1 & & \\ & & -1 & \\ \hline & & & 1 \end{array}\right), \qquad \mathscr{D}_k(z) = \left(\begin{array}{c|cc|c} 1 & & & z \\ \hline & \nu^k & & \\ & & \nu^{-k} & \\ \hline & & & 1 \end{array}\right),$$

$$\mathscr{E}(z) = \left(\begin{array}{c|cc|c} 1 & & & z \\ \hline & -1 & & \\ & -1 & -1 & \\ \hline & & & 1 \end{array}\right), \qquad \mathscr{F}(z) = \left(\begin{array}{c|cc|c} 1 & & & z \\ \hline & -1 & & \\ & -\nu & -1 & \\ \hline & & & 1 \end{array}\right),$$

$$\mathscr{G}_m(z) = \left(\begin{array}{c|c|c} 1 & & z \\ \hline & \mathbf{b}^m & \\ \hline & & 1 \end{array}\right), \qquad \mathscr{H}(z) = \left(\begin{array}{c|cc|c} 1 & & & z \\ \hline & 1 & & \\ & 1 & 1 & \\ \hline & & & 1 \end{array}\right),$$

$$\mathscr{I}(z) = \left(\begin{array}{c|cc|c} 1 & & & z \\ \hline & 1 & & \\ & \nu & 1 & \\ \hline & & & 1 \end{array}\right), \qquad \mathscr{L}_m = \left(\begin{array}{c|cc|c} 1 & 0 & \frac{1}{2}\nu^m & \\ \hline & 1 & & \nu^m \\ & 1 & 1 & \nu^m \\ \hline & & & 1 \end{array}\right),$$

$$\mathscr{M}_m = \left(\begin{array}{c|ccc|c} 1 & 0 & \frac{1}{2}\nu^m & & \\ \hline & 1 & & & \nu^m \\ & \nu & & 1 & \nu^{m+1} \\ \hline & & & & 1 \end{array}\right),$$

where $z \in F$, $1 \leq k \leq \frac{q-3}{2}$, $1 \leq m \leq \frac{q-1}{2}$ and $\mathbf{b}$ is an element of order $q+1$ (a 'Singer cycle') of $Sp(2,q)$. These are elements of G, and G admits exactly q^2+5q conjugacy classes (g) with representative g, as listed in the Table below.

g	$\vert(g)\vert$	*Parameters*
$\mathscr{A}(z)$	*1*	$z \in F$
$\mathscr{B}$	$q(q^2-1)$	
$\mathscr{C}(z)$	q^2	$z \in F$
$\mathscr{D}_k(z)$	$q^3(q+1)$	$z \in F$, $1 \leq k \leq \frac{q-3}{2}$
$\mathscr{E}(z)$	$\frac{1}{2}q^2(q^2-1)$	$z \in F$
$\mathscr{F}(z)$	$\frac{1}{2}q^2(q^2-1)$	$z \in F$
$\mathscr{G}_m(z)$	$q^3(q-1)$	$z \in F$, $1 \leq m \leq \frac{q-1}{2}$
$\mathscr{H}(z)$	$\frac{1}{2}q(q^2-1)$	$z \in F$
$\mathscr{I}(z)$	$\frac{1}{2}q(q^2-1)$	$z \in F$
$\mathscr{L}_m$	$q^2(q^2-1)$	$1 \leq m \leq \frac{q-1}{2}$
$\mathscr{M}_m$	$q^2(q^2-1)$	$1 \leq m \leq \frac{q-1}{2}$

Proof. Let $g_1 = g_{(s_1,w_1,z_1)}$ and $g_2 = g_{(s_2,w_2,z_2)}$ be two generic elements of G. Then $g_1g_2g_1^{-1} = g_{(s_1s_2s_1^{-1}, s_1(w_2-w_1+s_2^{-1}w_1), z_2-(w_2+s_2^{-1}w_1+s_2w_2)^*w_1)}$. It easily follows that if g_1 is conjugate to g_2 in G, then s_1 is conjugate to s_2 in S. Moreover, if $z_1 \neq z_2$, then the elements $g_{(s_1,0,z_1)}$ and $g_{(s_2,0,z_2)}$ cannot be conjugate in G. Observe that $g_1 \in \mathbf{C}_G(g_2)$ if and only if

$$\begin{cases} s_1 \in \mathbf{C}_S(s_2) \\ w_2 + s_2^{-1}w_1 = w_1 + s_1^{-1}w_2 \\ w_1^*(s_2w_2) = w_2^*(s_1w_1) \end{cases}.$$

Let us consider the elements $\mathscr{A}(z) = g_{(1,0,z)}$, $z \in F$. It is straightforward to see that $\mathbf{Z}(G) = \mathbf{Z}(H_1(q)) = \{\mathscr{A}(z) : z \in F\} \cong (F,+)$. Therefore, each of these q elements of G forms a central class of order 1. In particular, $\mathscr{A}(0)$ is the identity of G.

Now, let us consider the element $g_{(1,w,0)} = \mathscr{B} \in H_1(q) \setminus \mathbf{Z}(H_1(q))$. Then, $|\mathbf{C}_G(\mathscr{B})| = q^3$, i.e. $|(\mathscr{B})| = q(q^2-1)$. Since

$$g_{(s_1,w_1,z_1)}\mathscr{B}g_{(s_1,w_1,z_1)}^{-1} = g_{(1,s_1w,-2w^*w_1)},$$

it turns out that the elements of $H_1(q) \setminus \mathbf{Z}(H_1(q))$ form a single conjugacy class $(\mathscr{B})$ of G.

Set $g = g_{(s,0,z)} \in \{\mathscr{C}(z), \mathscr{D}_k(z), \mathscr{E}(z), \mathscr{F}(z), \mathscr{G}_m(z)\}$. Recall (e.g., see [Dor, §38]) that S admits elements $\mathbf{b}$ of order $q+1$, the so-called 'Singer cycles'. As observed before, for different values of z and s the elements $g_{(s,0,z)}$ belong to q^2+q distinct conjugacy classes of G. Now, an element $g_{(s_1,w_1,z_1)}$ belongs to $\mathbf{C}_G(g)$ if and only if

$$\begin{cases} s_1 \in \mathbf{C}_S(s) \\ sw_1 = w_1 \end{cases} . \tag{1}$$

Since s does not have eigenvalue 1, the condition $sw_1 = w_1$ implies $w_1 = 0$. It follows that $|\mathbf{C}_G(g)| = q|\mathbf{C}_S(s)|$, and using the information about the centralizers of elements of S contained in [Dor, §38], we obtain the results listed in the statement of the lemma.

Next, let us consider elements $g = g_{(s,0,z)} \in \{\mathscr{H}(z), \mathscr{I}(z)\}$. We argue as above, but note that this time s does admit the eigenvalue 1. This implies that in (1) $w_1 = \begin{pmatrix} 0 \\ y \end{pmatrix}$, where $y \in F$. So $|\mathbf{C}_G(g)| = q^2|\mathbf{C}_S(s)| = 2q^3$, i.e. $|(g)| = \frac{q(q^2-1)}{2}$.

Finally, let us consider elements $g = g_{(s,w,0)} \in \{\mathscr{L}_m, \mathscr{M}_m\}$, where

$$s = \begin{pmatrix} 1 & 0 \\ \epsilon & 1 \end{pmatrix}, \quad w = \begin{pmatrix} \nu^m \\ 0 \end{pmatrix},$$

and $\epsilon \in \{1, \nu\}$. Easy calculations show that if $1 \le m \le \frac{q-1}{2}$ the elements g belong to distinct conjugacy classes of G. An element $g_{(s_1,w_1,z_1)}$ belongs to $\mathbf{C}_G(g)$ if and only if

$$\begin{cases} s_1 \in \mathbf{C}_S(s) = \left\{ \begin{pmatrix} a & 0 \\ c & a \end{pmatrix} : a = \pm 1, c \in F \right\} \\ w + s^{-1}w_1 = w_1 + s_1^{-1}w \\ w_1^*(sw) = w^*(s_1 w_1) \end{cases} .$$

Since the condition $w + s^{-1}w_1 = w_1 + s_1^{-1}w$ implies $a = 1$, it follows that $g_{(s_1,w_1,z_1)}$ can be chosen in q^2 different ways. Thus, $|\mathbf{C}_G(g)| = q^2$, i.e. $|(\mathscr{L}_m)| = |(\mathscr{M}_m)| = q^2(q^2-1)$.

So far, we have found q^2+5q distinct conjugacy classes, adding up to $|G|$ elements. Thus, we are done. □

3 The character table

First of all, we observe that the character table of $SL(2,q) \cong Sp(2,q) \cong G/H_1(q)$ is well-known, e.g., see [Dor, §38], to which we refer for notation and all the information needed in the sequel.

Next, note that, as $\mathbf{Z}(G) = \{\mathscr{A}(z) : z \in F\}$, for any irreducible character χ of G

$$\chi(\mathscr{C}(z)) = \frac{\chi(\mathscr{A}(z))}{\chi(1)} \chi(\mathscr{C}(0))$$

for all $z \in F$. The same holds for the classes $(\mathscr{D}_k(z))$, $(\mathscr{E}(z))$, $(\mathscr{F}(z))$, $(\mathscr{G}_m(z))$, $(\mathscr{H}(z))$ and $(\mathscr{I}(z))$. So, in the character table we only report the values of a character on $\mathscr{C}(0)$, $\mathscr{D}_k(0)$ and so on.

Since $G/H_1(q) \cong SL(2,q)$, knowledge of the character table of $SL(2,q)$ gives us by inflation $q+4$ characters: namely 1_G, η_1, η_2, ξ_1, ξ_2, θ_j $(1 \leq j \leq \frac{q-1}{2})$, ψ and χ_i $(1 \leq i \leq \frac{q-3}{2})$.

Next, we construct $q-1$ distinct irreducible characters of G having degree q. Denote by λ a fixed non-trivial character of $\mathbf{Z}(G) \cong (F,+)$. Clearly, each of the q linear characters of $\mathbf{Z}(G)$ can be parametrised as λ_u $(u \in F)$, where $\lambda_u(z) = \lambda(uz)$ for all $z \in F$. In particular, $\lambda_0 = 1_{\mathbf{Z}(G)}$. We know by [Gér, Lemma 1.2] that $H_1(q)$ has exactly $q-1$ non-linear irreducible characters $\tilde{\lambda}_u$, defined as

$$\tilde{\lambda}_u(h) = \begin{cases} q\lambda_u(h) & \text{if } h \in \mathbf{Z}(H_1(q)) \\ 0 & \text{if } h \notin \mathbf{Z}(H_1(q)) \end{cases} \qquad (u \in F^\times).$$

Furthermore, by [Gér, Theorem 2.4] the characters $\tilde{\lambda}_u$ can be extended to G. We denote such extensions by ω_u $(u \in F^\times)$. The values taken by the characters ω_u on the elements of S can be found in [Sze, Proposition 2]. Namely:

g	1	$\mathscr{A}(z)$	$\mathscr{B}$	$\mathscr{C}(0)$	$\mathscr{D}_k(0)$	$\mathscr{E}(0)$	$\mathscr{F}(0)$	$\mathscr{G}_m(0)$	$\mathscr{H}(0)$	$\mathscr{I}(0)$
$\omega_u(g)$	q	$q\lambda_u(z)$	0	δ	$(-1)^k$	δ	δ	$(-1)^{m+1}$	$Q(\lambda_u)$	$-Q(\lambda_u)$

where

$$Q(\lambda) = \sum_{t \in F} \lambda(-t^2/2), \qquad Q(\lambda_u) = \sum_{t \in F} \lambda_u(-t^2/2) = \left(\frac{u}{F}\right) Q(\lambda)$$

and

$$\left(\frac{u}{F}\right) = \begin{cases} +1 & \text{if } u \text{ is a square in } F \\ -1 & \text{if } u \text{ is not a square in } F \end{cases}$$

(it turns out that $|Q(\lambda)|^2 = q$).

We are left to compute the values of the ω_u's on the classes $(\mathscr{L}_m)$ and $(\mathscr{M}_m)$. To this purpose, we compute

$$1 = (\omega_u, \omega_u)_G = \frac{q^4(q^2-1) + q^2(q^2-1)\sum_{m=1}^{\frac{q-1}{2}}(|\omega_u(\mathscr{L}_m)|^2 + |\omega_u(\mathscr{M}_m)|^2)}{q^4(q^2-1)}.$$

This implies that $\omega_u(\mathscr{L}_m) = \omega_u(\mathscr{M}_m) = 0$, for all $1 \leq m \leq \frac{q-1}{2}$.

It is easy to verify that the characters $\omega_u\eta_1$, $\omega_u\eta_2$, $\omega_u\xi_1$, $\omega_u\xi_2$, $\omega_u\theta_j$, $\omega_u\psi$ and $\omega_u\chi_i$ $(u \in F^\times)$ are pairwise distinct irreducible characters of G.

At this stage, q irreducible characters of G are still missing. We construct them as follows.

Let us consider the Sylow p-subgroup K of G consisting of the matrices of shape

$$k_{(a,x,y,z)} = \left(\begin{array}{c|cc|c} 1 & -y/2 & x/2 & z \\ \hline & 1 & a & x+ay \\ & & 1 & y \\ \hline & & & 1 \end{array}\right),$$

where $a, x, y \in F$. Define the linear characters μ_{u_1,u_2} $(u_1, u_2 \in F)$ of K setting $\mu_{u_1,u_2}(k_{(a,x,y,z)}) = \lambda_{u_1}(a)\lambda_{u_2}(y) = \lambda(u_1 a + u_2 y)$, where, as above, λ_u denotes the non-trivial linear character of $\mathbf{Z}(G)$ associated to $u \in F^\times$ (in particular, $\mu_{0,0} = 1_K$).

We consider the induced characters μ_{u_1,u_2}^G.

First of all, note that $(\mathscr{C}(z)) \cap K = \emptyset$ and that the same holds also for $(\mathscr{D}_k(z))$, $(\mathscr{E}(z))$, $(\mathscr{F}(z))$ and $(\mathscr{G}_m(z))$. So, the value of μ_{u_1,u_2}^G on these classes is 0, whereas the value on $\mathscr{A}(z)$ is

$$\mu_{u_1,u_2}^G(\mathscr{A}(z)) = \frac{q^4(q^2-1)}{q^4} = q^2 - 1.$$

To compute $\mu_{u_1,u_2}^G(\mathscr{B})$, we observe that if $s = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \in S$ and $g = g_{(s,w,z)}$, then $\mu_{u_1,u_2}(g\mathscr{B}g^{-1}) = \lambda_{u_1}(0)\lambda_{u_2}(s_{21}) = \lambda_{u_2}(s_{21})$. So, if $s_{21} = 0$ the matrix s can be chosen in $q(q-1)$ ways, whereas if we fix $s_{21} \neq 0$, s can be chosen in q^2 different ways. For $u_2 \neq 0$ we obtain

$$\begin{aligned} \mu_{u_1,u_2}^G(\mathscr{B}) &= \frac{q^3[q(q-1) + q^2 \sum_{s_{21}\neq 0} \lambda_{u_2}(s_{21})]}{q^4} \\ &= \frac{q^3[q(q-1) - q^2]}{q^4} = -1 \end{aligned}$$

Next, we look at the classes $(\mathscr{H}(z))$. The matrices s such that $g\mathscr{H}(z)g^{-1} \in K$ are of shape $\begin{pmatrix} s_{11} & s_{12} \\ -1/s_{12} & 0 \end{pmatrix}$. It follows that

$$\mu_{u_1,u_2}(g\mathscr{H}(z)g^{-1}) = \lambda_{u_1}(-s_{12}^2)\lambda_{u_2}(0)$$

and therefore

$$\begin{aligned} \mu_{u_1,u_2}^G(\mathscr{H}(z)) &= \frac{q^3 q \sum_{t\neq 0} \lambda_{u_1}(-t^2)}{q^4} \\ &= -1 + Q(\lambda_{2u_1}) = -1 + \left(\frac{2}{F}\right) Q(\lambda_u) \end{aligned}$$

In a similar way, one also obtains that

$$\mu_{u_1,u_2}^G(\mathscr{I}(z)) = \sum_{t\neq 0} \lambda_{u_1}(-\nu t^2).$$

In particular, for $u_1 = 0$ we get $\mu^G_{0,u_2}(\mathscr{H}(z)) = \mu^G_{0,u_2}(\mathscr{I}(z)) = q - 1$, whereas for $u_1 \neq 0$, we get $\mu^G_{u_1,u_2}(\mathscr{I}(z)) = -1 - Q(\lambda_{2u_1})$.

The value of $\mu^G_{u_1,u_2}$ on $\mathscr{L}_m$ is obtained in the same way as above: s has the same shape as in the case $\mathscr{H}(z)$, but $\mu_{u_1,u_2}(g\mathscr{L}_m g^{-1}) = \lambda_{u_1}(-s_{12}^2)\lambda_{u_2}(\frac{-\nu^m}{s_{12}})$. Thus,

$$\begin{aligned}\mu^G_{u_1,u_2}(\mathscr{L}_m) &= \frac{q^3 q \sum_{t\neq 0} \lambda_{u_1}(-t^2)\lambda_{u_2}(-\nu^m/t)}{q^4} \\ &= \sum_{t\neq 0} \lambda\Big(-\frac{u_1 t^3 + u_2\nu^m}{t}\Big).\end{aligned}$$

Similarly, in the case of $\mathscr{M}_m$, we obtain

$$\begin{aligned}\mu^G_{u_1,u_2}(\mathscr{M}_m) &= \frac{q^3 q \sum_{t\neq 0} \lambda_{u_1}(-\nu t^2)\lambda_{u_2}(-\nu^m/t)}{q^4} \\ &= \sum_{t\neq 0} \lambda\Big(-\frac{u_1 \nu t^3 + u_2\nu^m}{t}\Big).\end{aligned}$$

In particular, for $u_1 = 0$ we get $\mu^G_{0,u_2}(\mathscr{L}_m) = \mu^G_{0,u_2}(\mathscr{M}_m) = -1$.

Set $\kappa_0 = \mu^G_{0,1}$. Computing $(\kappa_0, \kappa_0)_G$, one sees that κ_0 is irreducible. Furthermore, for all $u_1, u_2 \in F^\times$, κ_0 is different from any of the $\mu^G_{u_1,u_2}$'s because

$$\kappa_0(\mathscr{H}(0)) + \kappa_0(\mathscr{I}(0)) = 2q - 2 \neq \mu^G_{u_1,u_2}(\mathscr{H}(0)) + \mu^G_{u_1,u_2}(\mathscr{I}(0)) = -2.$$

Next, we show that we can always pick $q-1$ pairwise distinct irreducible characters among the $\mu^G_{u_1,u_2}$'s. For instance, we can take as (u_1, u_2) the pairs $(1, \nu^n)$ and (ν, ν^n), where $1 \leq n \leq \frac{q-1}{2}$. Set $\kappa_{1,n} = \mu^G_{1,\nu^n}$, $\kappa_{\nu,n} = \mu^G_{\nu,\nu^n}$. We start showing that these characters are irreducible.

Use of Mackey's formula implies that

$$(\kappa_{1,n}, \kappa_{1,n})_G = \sum_{r\in\mathcal{R}} (\mu_{1,\nu^n}, {}^r\mu_{1,\nu^n})_{K\cap {}^rK},$$

where $\mathcal{R}$ is a complete set of representatives for the double cosets of K in G. As $\mathcal{R}$ we can choose the set $\{s(\alpha), \overline{s}(\beta) \mid \alpha, \beta \in F^\times\}$, where

$$s = s(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{pmatrix}, \quad \overline{s} = \overline{s}(\beta) = \begin{pmatrix} 0 & -1/\beta \\ \beta & 0 \end{pmatrix} \in S.$$

Note that $|KsK| = q^4$ and $|K\overline{s}K| = q^5$. Since the μ_{1,ν^n}'s are linear characters, it suffices to show that for $r \neq s(1)$, the restrictions of μ_{1,ν^n} and ${}^r\mu_{1,\nu^n}$ to $K \cap {}^rK$ are distinct.

First, we look at the double cosets $K\overline{s}(\beta)K$. For all ${}^{\overline{s}}k \in K \cap {}^{\overline{s}}K$, we have $\mu_{1,\nu^n}({}^{\overline{s}}k) = \lambda_{\nu^n}(\beta x)$ and ${}^{\overline{s}}\mu_{1,\nu^n}({}^{\overline{s}}k) = \mu_{1,\nu^n}(k) = \lambda_{\nu^n}(y)$. It follows that, if $\mu_{1,\nu^n} = {}^{\overline{s}}\mu_{1,\nu^n}$, then $\lambda_{\nu^n}(\beta x) = \lambda_{\nu^n}(y)$, for all $x, y \in F$. In particular, for $x = 0$, we have $\lambda_{\nu^n}(y) = 1$ for all $y \in F$, i.e. $Ker(\lambda_{\nu^n}) = \mathbf{Z}(H_1(q))$, forcing $\nu^n = 0$, a contradiction.

Next, we look at the double cosets $Ks(\alpha)K$. For all ${}^sk \in K \cap {}^sK$, we have $\mu_{1,\nu^n}({}^sk) = \lambda_1(a\alpha^2)\lambda_{\nu^n}(\frac{y}{\alpha})$ and ${}^s\mu_{1,\nu^n}({}^sk) = \mu_{1,\nu^n}(k) = \lambda_1(a)\lambda_{\nu^n}(y)$. It follows that, if $\mu_{1,\nu^n} = {}^s\mu_{1,\nu^n}$, then $\lambda_1(a\alpha^2)\lambda_{\nu^n}(\frac{y}{\alpha}) = \lambda_1(a)\lambda_{\nu^n}(y)$, for all $a, y \in F$. In particular, for $y = 0$, we get $\lambda_{\alpha^2} = \lambda_1$, and so $\alpha = 1$. Clearly, for $\alpha = 1$, the two restrictions are the same character. This proves that the characters $\kappa_{1,n}$ are irreducible.

In the same way, we can prove that the characters $\kappa_{\nu,n}$ are also irreducible. To conclude, we are left to show that the characters $\kappa_{1,n}$ and $\kappa_{\nu,n}$ are pairwise distinct. This can be obtained proving that $(\kappa_{d,n}, \kappa_{d_1,n_1})_G = 0$, for $d, d_1 \in \{1, \nu\}$, $1 \leq n \leq \frac{q-1}{2}$ and $(d, n) \neq (d_1, n_1)$. As above, we exploit Mackey's formula. The double cosets $K\overline{s}(\beta)K$ are dealt with in the same way as before. In the case of the double cosets $Ks(\alpha)K$, for $d = d_1$ we can argue as before. In the case $(d, d_1) = (1, \nu)$, if the restrictions of μ_{1,ν^n} and $\mu_{\nu,\nu^{n_1}}$ are the same, then

$$\lambda_1(a\alpha^2)\lambda_{\nu^n}(\frac{y}{\alpha}) = \lambda_\nu(a)\lambda_{\nu^{n_1}}(y)$$

for all $a, y \in F$. In particular, for $y = 0$, we get $\lambda_{\alpha^2} = \lambda_\nu$, a contradiction, since ν is not a square in F.

In conclusion, the desired character table of G can be described as follows:

	1	$\mathscr{A}(z)$	$\mathscr{B}$	$\mathscr{C}(0)$	$\mathscr{D}_k(0)$
1_G	1	1	1	1	1
η_1	$\frac{q-1}{2}$	$\frac{q-1}{2}$	$\frac{q-1}{2}$	$\frac{-\delta(q-1)}{2}$	0
η_2	$\frac{q-1}{2}$	$\frac{q-1}{2}$	$\frac{q-1}{2}$	$\frac{-\delta(q-1)}{2}$	0
ξ_1	$\frac{q+1}{2}$	$\frac{q+1}{2}$	$\frac{q+1}{2}$	$\frac{\delta(q+1)}{2}$	$(-1)^k$
ξ_2	$\frac{q+1}{2}$	$\frac{q+1}{2}$	$\frac{q+1}{2}$	$\frac{\delta(q+1)}{2}$	$(-1)^k$
θ_j	$q-1$	$q-1$	$q-1$	$(-1)^j(q-1)$	0
ψ	q	q	q	q	1
χ_i	$q+1$	$q+1$	$q+1$	$(-1)^i(q+1)$	$\rho^{ik} + \rho^{-ik}$
κ_0	q^2-1	q^2-1	-1	0	0
$\kappa_{1,n}$	q^2-1	q^2-1	-1	0	0
$\kappa_{\nu,n}$	q^2-1	q^2-1	-1	0	0
ω_u	q	$q\lambda_u(z)$	0	δ	$(-1)^k$
$\omega_u\eta_1$	$\frac{q(q-1)}{2}$	$\frac{q(q-1)\lambda_u(z)}{2}$	0	$-\frac{(q-1)}{2}$	0
$\omega_u\eta_2$	$\frac{q(q-1)}{2}$	$\frac{q(q-1)\lambda_u(z)}{2}$	0	$-\frac{(q-1)}{2}$	0
$\omega_u\xi_1$	$\frac{q(q+1)}{2}$	$\frac{q(q+1)\lambda_u(z)}{2}$	0	$\frac{(q+1)}{2}$	1
$\omega_u\xi_2$	$\frac{q(q+1)}{2}$	$\frac{q(q+1)\lambda_u(z)}{2}$	0	$\frac{(q+1)}{2}$	1
$\omega_u\theta_j$	$q(q-1)$	$q(q-1)\lambda_u(z)$	0	$(-1)^j\delta(q-1)$	0
$\omega_u\psi$	q^2	$q^2\lambda_u(z)$	0	δq	$(-1)^k$
$\omega_u\chi_i$	$q(q+1)$	$q(q+1)\lambda_u(z)$	0	$(-1)^i\delta(q+1)$	$(-1)^k(\rho^{ik} + \rho^{-ik})$

	$\mathscr{E}(0)$	$\mathscr{F}(0)$	$\mathscr{G}_m(0)$	$\mathscr{H}(0)$
1_G	1	1	1	1
η_1	$\frac{-\delta(-1+\sqrt{\delta q})}{2}$	$\frac{-\delta(-1-\sqrt{\delta q})}{2}$	$(-1)^{m+1}$	$\frac{(-1+\sqrt{\delta q})}{2}$
η_2	$\frac{-\delta(-1-\sqrt{\delta q})}{2}$	$\frac{-\delta(-1+\sqrt{\delta q})}{2}$	$(-1)^{m+1}$	$\frac{(-1-\sqrt{\delta q})}{2}$
ξ_1	$\frac{\delta(1+\sqrt{\delta q})}{2}$	$\frac{\delta(1-\sqrt{\delta q})}{2}$	0	$\frac{(1+\sqrt{\delta q})}{2}$
ξ_2	$\frac{\delta(1-\sqrt{\delta q})}{2}$	$\frac{\delta(1+\sqrt{\delta q})}{2}$	0	$\frac{(1-\sqrt{\delta q})}{2}$
θ_j	$(-1)^{j+1}$	$(-1)^{j+1}$	$-(\sigma^{jm}+\sigma^{-jm})$	-1
ψ	0	0	-1	0
χ_i	$(-1)^i$	$(-1)^i$	0	1
κ_0	0	0	0	$q-1$
$\kappa_{1,n}$	0	0	0	$-1+(\frac{2}{F})Q(\lambda)$
$\kappa_{\nu,n}$	0	0	0	$-1-(\frac{2}{F})Q(\lambda)$
ω_u	δ	δ	$(-1)^{m+1}$	$Q(\lambda_u)$
$\omega_u\eta_1$	$\frac{1-\sqrt{\delta q}}{2}$	$\frac{1+\sqrt{\delta q}}{2}$	1	$\frac{(-1+\sqrt{\delta q})Q(\lambda_u)}{2}$
$\omega_u\eta_2$	$\frac{1+\sqrt{\delta q}}{2}$	$\frac{1-\sqrt{\delta q}}{2}$	1	$\frac{(-1-\sqrt{\delta q})Q(\lambda_u)}{2}$
$\omega_u\xi_1$	$\frac{1+\sqrt{\delta q}}{2}$	$\frac{1-\sqrt{\delta q}}{2}$	0	$\frac{(1+\sqrt{\delta q})Q(\lambda_u)}{2}$
$\omega_u\xi_2$	$\frac{1-\sqrt{\delta q}}{2}$	$\frac{1+\sqrt{\delta q}}{2}$	0	$\frac{(1-\sqrt{\delta q})Q(\lambda_u)}{2}$
$\omega_u\theta_j$	$(-1)^{j+1}\delta$	$(-1)^{j+1}\delta$	$(-1)^m(\sigma^{jm}+\sigma^{-jm})$	$-Q(\lambda_u)$
$\omega_u\psi$	0	0	$(-1)^m$	0
$\omega_u\chi_i$	$(-1)^i\delta$	$(-1)^i\delta$	0	$Q(\lambda_u)$

	$\mathscr{I}(0)$	$\mathscr{L}_m$	$\mathscr{M}_m$
1_G	1	1	1
η_1	$\frac{(-1-\sqrt{\delta q})}{2}$	$\frac{(-1+\sqrt{\delta q})}{2}$	$\frac{(-1-\sqrt{\delta q})}{2}$
η_2	$\frac{(-1+\sqrt{\delta q})}{2}$	$\frac{(-1-\sqrt{\delta q})}{2}$	$\frac{(-1+\sqrt{\delta q})}{2}$
ξ_1	$\frac{(1-\sqrt{\delta q})}{2}$	$\frac{(1+\sqrt{\delta q})}{2}$	$\frac{(1-\sqrt{\delta q})}{2}$
ξ_2	$\frac{(1+\sqrt{\delta q})}{2}$	$\frac{(1-\sqrt{\delta q})}{2}$	$\frac{(1+\sqrt{\delta q})}{2}$
θ_j	-1	-1	-1
ψ	0	0	0
χ_i	1	1	1
κ_0	$q-1$	-1	-1
$\kappa_{1,n}$	$-1-(\frac{2}{F})Q(\lambda)$	$\sum_{t\in F^\times}\lambda(-\frac{t^3+\nu^{n+m}}{t})$	$\sum_{t\in F^\times}\lambda(-\frac{\nu t^3+\nu^{n+m}}{t})$
$\kappa_{\nu,n}$	$-1+(\frac{2}{F})Q(\lambda)$	$\sum_{t\in F^\times}\lambda(-\frac{\nu t^3+\nu^{n+m}}{t})$	$\sum_{t\in F^\times}\lambda(-\frac{\nu^2 t^3+\nu^{n+m}}{t})$
ω_u	$-Q(\lambda_u)$	0	0
$\omega_u\eta_1$	$\frac{(1+\sqrt{\delta q})Q(\lambda_u)}{2}$	0	0
$\omega_u\eta_2$	$\frac{(1-\sqrt{\delta q})Q(\lambda_u)}{2}$	0	0
$\omega_u\xi_1$	$\frac{(-1+\sqrt{\delta q})Q(\lambda_u)}{2}$	0	0
$\omega_u\xi_2$	$\frac{(-1-\sqrt{\delta q})Q(\lambda_u)}{2}$	0	0
$\omega_u\theta_j$	$Q(\lambda_u)$	0	0
$\omega_u\psi$	0	0	0
$\omega_u\chi_i$	$-Q(\lambda_u)$	0	0

Notations. $1\le i,k\le \frac{q-3}{2}$, $1\le j,m,n\le \frac{q-1}{2}$. $\delta=(-1)^{\frac{q-1}{2}}$, $\rho=e^{\frac{2\pi\imath}{q-1}}$, $\sigma=e^{\frac{2\pi\imath}{q+1}}$. $F=GF(q)$, $F^\times=\langle\nu\rangle$, $u\in F^\times$. λ is a (fixed) non-trivial character of $\mathbf{Z}(G)$. λ_u is the linear character of $\mathbf{Z}(G)$ defined by $\lambda_u(z) = \lambda(uz)$ for all $z \in \mathbf{Z}(G)$.

$$Q(\lambda) = \sum_{t\in F} \lambda(-t^2/2), \qquad Q(\lambda_u) = \left(\frac{u}{F}\right) Q(\lambda).$$

For all $\chi \in Irr(G)$, we have (but this is omitted from the Table)

$$\chi(\mathscr{C}(z)) = \frac{\chi(\mathscr{A}(z))}{\chi(\mathscr{A}(0))} \chi(\mathscr{C}(0)),$$

and likewise for the other conjugacy classes.

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