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Shock propagation and the topology of complex networks

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Abstract

In this work we propose a model in order to study shocks propagation in complex networks. We investigate the relation between shocks propagation and the underlying topological structure. We find that, under certain assumptions, shocks converge towards an asymptotic vector (distribution) which represents the configuration of equilibrium for the system and depends only on the topology of the network. By the use of numerical simulations, we compare diverse classes of networks.

1 Introduction

The recent economic and financial crisis showed how much mutual interdependence matters in determining collapses, cascades of defaults, recessions and catastrophic events in general. Complex economic systems, as well as financial institutions, are nowadays much more interconnected than they were in past, and modeling such interconnections represents an open challenge for researchers [20].

For instance, Allen and Gale [2] find that small liquidity preference in banking networks can spread by contagion, hitting all the nodes. Besides complete structures are more robust than incomplete ones.

Recently the study of complex systems by means of network theory has been widely used in order to analyze economic and financial problems. Some contributes ([4], [6], [12]) focus on the analysis of the shareholding network topology. The statistical properties of the interlocking directorates through the network of boards are analyzed in [5]. In [7] and [15] the network of Italian board interlocks is studied by using vertex centrality measures.

The vulnerability of scale-free networks as been investigated, for example, by Albert [1] and Crucitti [11]: they find that scale-free networks are robust to random attacks but are vulnerable to targeted attacks. Motter and Lai [17] show that targeted attacks in networks where loads can redistribute among nodes can

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lead to cascades of failures, especially for those networks in which the load distribution is heterogeneous. Pastor–Satorras and Vespignani [19] show that the topology of scale-free networks can favor the spreading of computer viruses, proposing a dynamical mean-field model for the spreading of infections. From the empirical point of view, Soramaki *et al.* [21] show that the attack of September 2001 dramatically modified the network topology of the FedWire interbank payments and find that the vulnerability of a particular network depends both on its structure and on the mechanism of contagion.

In this work we aim at proposing a model in order to further understand how shocks propagation in complex networks depends on the topological structure. In our model shocks propagate to the adjacent nodes (local interaction) according to a *spreading rule*. A propagation matrix accounts for this rule and we assume that it is determined by the network topology.

Studying the shocks dynamics over time, we find that, under some conditions on the propagation matrix, the shocks converge to a limit vector (distribution) representing the configuration of equilibrium for the system. Such a configuration depends only on the topological network structure.

We compare, by numerical simulation, the dynamics of the propagation on different network topological structures.

The paper is organized as follows: in Section 2 some basic definitions about networks are provided. In Sections 3 and 4 we describe analytically the propagation model, first in a general framework (Subsection 3.1) and then in the particular case of equidistribution of the shocks (Subsection 3.2). In Section 5 we propose some simulation results in order to further assess convergence properties for diverse classes of topologies. In Section 6 we draw the conclusions. The proofs of the theorems we used throughout the text are reported in the Appendix.

2 Preliminaries

We remind in this section some mathematical definition concerning matrices and graphs, that we will use later in the paper.

Let $G = (V, E)$ be a graph where V is the set of N nodes and E is a set of pairs of nodes of V ; the pair $(i, j) \in E$ is called an edge of G , and i and j are called adjacent. An edge $(i, i) \in E$ is said a loop.

When two, or more edges, connect two vertices i, j we have a multiedge. A multigraph is a graph with at least a multiedge. A simple graph is a graph without loops and multiedges.

G is undirected if $(j, i) \in E$ whenever $(i, j) \in E$, otherwise G is directed. Moreover, a weight $w_{ij} > 1$ can be eventually associated to an edge, in which case we have a weighted graph.

A subgraph $H = (V', E')$ of G is a graph such that $V' \subseteq V$ and $E' \subseteq E$. Two subgraphs of G are disjoint if they have no common vertices.

The degree k_i of a node i ($i = 1, \dots, N$) is the number of edges incident to it. The vector $\mathbf{k}^T = [k_1 \dots k_n]$ is the degree vector.

A path is a sequence of distinct adjacent vertices. An $i - j$ path is a path starting from i to j . Any shortest path joining vertices i and j is called an $i - j$ geodesic. The distance $d(i, j)$ between two vertices i and j is the length of the $i - j$ geodesic.

A graph is connected if for each pair of nodes i and j , ($i, j = 1, 2, \dots, n$), there is a path from i to j . A connected component of G is the maximal connected subgraph of G . Observe that in general $G = G_1 \cup \dots \cup G_l$, where G_1, G_2, \dots, G_l (with $l \geq 1$), are pairwise disjoint subgraphs of G representing its connected components, and G is their disjoint union. Clearly, $l = 1$ if and only if G is connected.

Among particular kind of graphs we quote the complete graph and the star graph.

The *complete graph* K_N of N nodes is a graph in which every pair of vertices are adjacent. In the complete graph, each node has degree $N - 1$ and $L = \frac{N(N-1)}{2}$.

The *star graph* with N nodes $K_{N,1}$ is a graph where only one node has degree $N - 1$, being connected to all the other nodes, which have degree 1. $L = N - 1$ in this case.

A binary N -square matrix \mathbf{A} (the *adjacency matrix*) representing the adjacency relationships between the nodes of G is associated with the graph:

$$A_{ij} = \begin{cases} 1 & (i, j) \in E \\ 0 & (i, j) \notin E \end{cases}$$

If the graph is simple, the diagonal elements of \mathbf{A} are zero.

Let $\{\lambda_1, \lambda_2, \dots, \lambda_N\}$ be the spectrum of \mathbf{A} , that is: the set of its eigenvalues.

$\rho(\mathbf{A}) = \max |\lambda_i|$ is the spectral radius of \mathbf{A} .

A non-negative matrix \mathbf{Z} is reducible if there exists a permutation matrix \mathbf{M} such that

$$\mathbf{M}^T \mathbf{Z} \mathbf{M} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{0} & \mathbf{D} \end{bmatrix}$$

where \mathbf{B} and \mathbf{D} are square matrices. A matrix is irreducible if it is not reducible.

If $G = (V, E)$ is undirected, \mathbf{A} is symmetric and its eigenvalues are always real. In this case, \mathbf{A} is irreducible if and only if $G = (V, E)$ is connected.

A real square matrix \mathbf{A} is primitive if it is irreducible and has only one eigenvalue of maximum modulus (see [16]), which coincides with its spectral radius $\rho(\mathbf{A})$.

A well known sufficient and necessary condition for a non-negative matrix \mathbf{A} to be primitive is that its $r - th$ power \mathbf{A}^r is positive for some $r > 0$.

3 The model

3.1 General model

The model we propose depicts a network in which the shocks spread by means of a transition matrix \mathbf{P} . Let $G = (V, E)$ be a connected graph with associated

adjacency matrix \mathbf{A} , \mathbf{s}_t be a vector representing the shocks affecting each vertex of G at time t , $\mathbf{P} = [P_{ij}]_{i,j=1,\dots,N}$ (the propagation matrix) be a square matrix such that

$$\begin{cases} P_{ij} > 0 & (i,j) \in E \\ P_{ij} = 0 & (i,j) \notin E \end{cases}$$

so that

$$P_{ij} = 0 \Leftrightarrow A_{ij} = 0 \quad (1)$$

Usually the matrix \mathbf{P} is not symmetric, i.e. $P_{ij} \neq P_{ji}$, however the previous hypothesis implies

$$P_{ji} \neq 0 \Leftrightarrow P_{ij} \neq 0 \quad (2)$$

At time $t + 1$ each node of G receives a shock given by

$$s_{i,t+1} = \sum_j P_{ij} s_{j,t} \quad i = 1, \dots, N$$

so that the vector of the shocks at time t is

$$\mathbf{s}_t = \mathbf{P} \mathbf{s}_{t-1}$$

which implies

$$\mathbf{s}_t = \mathbf{P}^t \mathbf{s}_0$$

where \mathbf{s}_0 is the initial vector of the shocks.

A classical result of linear algebra states that the asymptotic behavior of \mathbf{s}_t depends on the spectral radius $\rho(\mathbf{P})$ ([18]):

1. $\lim_{t \rightarrow +\infty} \mathbf{s}_t = \mathbf{0}$ for every vector \mathbf{s}_0 if and only if $\rho(\mathbf{P}) < 1$;
2. if $\rho(\mathbf{P}) > 1$, $\|\mathbf{s}_t\|$ tends to infinity for some vector \mathbf{s}_0 , for every vector norm $\|\cdot\|$;
3. if $\rho(\mathbf{P}) = 1$ and \mathbf{P} is primitive, there is a unique vector \mathbf{s}_∞ such that $\lim_{t \rightarrow +\infty} \mathbf{s}_t = \mathbf{s}_\infty$ for every starting vector \mathbf{s}_0 .

We hereby study the case $\rho(\mathbf{P}) = 1$. This happens, for instance, when \mathbf{P} is a column stochastic matrix. In this case all columns sum to one, i.e. in matrix notation:

$$\mathbf{u}^T \mathbf{P} = \mathbf{u}^T \quad (3)$$

where $\mathbf{u}^T = [1, \dots, 1]$ and¹ $\rho(\mathbf{P}) = 1$.

From here on we will restrict our study to this case. As a consequence, \mathbf{u} is a left eigenvector of \mathbf{P} associated to the eigenvalue 1, which is also the maximal eigenvalue, being equal to the spectral radius $\rho(\mathbf{P})$; hence \mathbf{u} is a principal left eigenvector of \mathbf{P} .

Equation (3) implies

$$\mathbf{u}^T \mathbf{s}_{t+1} = \mathbf{u}^T (\mathbf{P} \mathbf{s}_t) = (\mathbf{u}^T \mathbf{P}) \mathbf{s}_t = \mathbf{u}^T \mathbf{s}_t$$

¹For a well known inequality, given a matrix \mathbf{M} the spectral radius $\rho(\mathbf{M})$ is equal to the row (column) sum of \mathbf{M} when all the \mathbf{M} - row (column) sums are equal.

so that

$$\sum_{j=1}^n s_{j,t} = \mathbf{u}^T \mathbf{s}_t = \mathbf{u}^T \mathbf{s}_0 = \sum_{j=1}^n s_{j,0}$$

Thereby the sum of the shocks remains constant for every t .

When \mathbf{P} is primitive

$$\lim_{t \rightarrow \infty} \mathbf{P}^t = \frac{1}{\mathbf{u}^T \mathbf{v}} \mathbf{v} \mathbf{u}^T$$

where \mathbf{v} is a principal right eigenvector ([16]). So the shocks vector $\mathbf{s}_t = \mathbf{P}^t \mathbf{s}_0$ converges to the limit vector:

$$\mathbf{s}_\infty = \mathbf{v} \mathbf{u}^T \mathbf{s}_0 = \mathbf{v} (\mathbf{u}^T \mathbf{s}_0) = N \bar{s} \mathbf{v} \quad (4)$$

where \mathbf{v} is the principal right eigenvector of \mathbf{P} such that $\mathbf{u}^T \mathbf{v} = 1$ (from now on \mathbf{v} will refer to this unique vector when the propagation matrix is primitive) and \bar{s} is the average shock.

In our case (hypothesis 1) \mathbf{P} is irreducible if and only if the graph G is connected (see Prop. A.1 in the Appendix A) and when it is not, every connected component evolves independently because \mathbf{P} can assume a block-diagonal form after a proper permutation of the nodes. So the matrix \mathbf{P} can be assumed irreducible with no loss of generality. However it is not necessarily primitive (see Appendix B).

Equation (4) shows that the total shock $N\bar{s}$ tends to be distributed among the nodes according to the fractions determined by the components of the principal right eigenvector \mathbf{v} , which depends only on the propagation matrix \mathbf{P} . Considering \mathbf{P} as a matrix of relationships, as in Bonacich [8], a well known interpretation of the right eigenvector of \mathbf{P} is that each component v_i represents a centrality measure for the i -th node. From this point of view, for $\bar{s} \neq 0$, we can rewrite Equation 4 as follows:

$$\mathbf{v} = \frac{1}{N\bar{s}} \mathbf{s}_\infty$$

which is a normalized asymptotic equilibrium vector. Thus it is possible to interpret eigencentality as an equilibrium value.

3.2 Equidistributed shocks

We consider now the case in which the shock hitting a node is equidistributed among its adjacent nodes. In this case, the propagation matrix \mathbf{P} is necessarily stochastic, (i.e. the sum of all its column elements is equals to 1) so that everything said in the previous section still holds, but the matrix \mathbf{P} reveals more interesting properties related to the degree vector (and hence to the degree distribution).

Every adjacent node at time $t + 1$ receives from node i only a fraction $1/k_i$ of the shock $s_{i,t}$. For this reason, starting from the adjacency matrix \mathbf{A} , the

propagation matrix is defined by $\mathbf{P} = \mathbf{A}\mathbf{K}^{-1}$, where \mathbf{K} is the diagonal matrix having the node degrees on its diagonal entries:

$$P_{ij} = \begin{cases} \frac{1}{k_i} & \text{if } (i, j) \in E \\ 0 & \text{if } (i, j) \notin E \end{cases}$$

The following theorem holds (the proof is in the Appendix A):

Theorem 1 *If $\mathbf{P} = \mathbf{A}\mathbf{K}^{-1}$, then \mathbf{w} is a left eigenvector of \mathbf{P} if and only if $\mathbf{K}\mathbf{w}$ is a right eigenvector of \mathbf{P} .*

Since \mathbf{u} is a left eigenvector of \mathbf{P} (Eq. 3), Theorem 1 implies $\mathbf{k} = \mathbf{K}\mathbf{u}$ is a right eigenvector of \mathbf{P} , being \mathbf{k} the degree vector of the graph.

As a consequence, when \mathbf{P} is primitive, the principal right eigenvector \mathbf{v} is proportional to \mathbf{k} , so it must be equal to:

$$\mathbf{v} = \frac{1}{\mathbf{u}^T \mathbf{k}} \mathbf{k} = \frac{1}{2L} \mathbf{k};$$

and (see Eq. 4) the shock propagation vector \mathbf{s}_t converges to the vector

$$\mathbf{s}_\infty = (\mathbf{u}^T \mathbf{s}_0) \mathbf{v} = \frac{N\bar{s}}{2L} \mathbf{k} = \frac{\bar{s}}{\bar{k}} \mathbf{k}, \quad (5)$$

where $\bar{k} = 2L/N$ is the average degree, which is also a network density measure. This convergence clearly shows the strong dependence on the typical parameters of the network topological structure: the number N of nodes, the number L of edges and, especially, the degree vector \mathbf{k} . The previous equation underlines also the dependence on \bar{s} , the average shock hitting the network. Also the speed of convergence depends on the degree distribution and on the topological properties of the network (see the Appendix A.3).

The equilibrium configuration in case of equidistributed shocks is analogous to the one found, for example, in the Google Page Rank (see [9]).

4 Random shocks

Suppose that the initial shock \mathbf{s}_0 is a random vector with a known probability distribution. In particular, let \mathbf{m} and $\mathbf{\Gamma}$ be its expected value and covariance matrix respectively, whose elements are:

$$m_i = E[s_{i,0}] \quad , \quad \Gamma_{ij} = \text{cov}[s_{i,0}, s_{j,0}] \quad i, j = 1, \dots, n.$$

Note that, if \mathbf{s}_0 is Gaussian, \mathbf{s}_t is Gaussian for every t . Anyway, no matter the distribution, the mean vector and the covariance matrix of \mathbf{s}_t are

$$E[\mathbf{s}_t] = \mathbf{P}^t \mathbf{m} \quad , \quad \mathbf{C}_t = [C_{ij,t}] = [\text{cov}[s_{i,t}, s_{j,t}]] = \mathbf{P}^t \mathbf{\Gamma}^T \mathbf{P}^t$$

If \mathbf{P} is primitive, as t tends to infinity, they converge to

$$E[\mathbf{s}_\infty] = \frac{\mathbf{u}^T \mathbf{m}}{\mathbf{u}^T \mathbf{k}} \mathbf{k} \quad , \quad \mathbf{C}_\infty = \frac{\sum_{i,j=1}^n \Gamma_{ij}}{\left(\sum_{i=1}^n k_i\right)^2} \mathbf{k} \mathbf{k}^T$$

Note that the correlations $\rho_{ij,t} = \frac{C_{ij,t}}{\sqrt{C_{ii,t}C_{jj,t}}}$ tend to 1 as t tends to infinity.

When every component of \mathbf{s}_0 has the same mean μ and variance σ^2 and is uncorrelated with respect to the other ones, the result is

$$E[\mathbf{s}_\infty] = \frac{\mu}{\bar{k}} \mathbf{k} \quad , \quad \mathbf{C}_\infty = \left(\frac{\sigma}{\bar{k}}\right)^2 \mathbf{k} \mathbf{k}^T.$$

4.1 Uniform distribution

We consider here the case in which every component of the initial shock vector is independent on the other ones and uniformly distributed in the interval $[a, b]$. In this case $\mathbf{m} = \mu \mathbf{u}$, with $\mu = \frac{a+b}{2}$, while $\mathbf{\Gamma}$ is diagonal with all the diagonal entries equal to $\sigma^2 = \frac{(b-a)^2}{12}$. Besides, when \mathbf{P} is primitive, the limit shock vector is $\frac{\mathbf{u}^T \mathbf{s}_0}{\mathbf{u}^T \mathbf{k}} \mathbf{k}$. The random variable $\mathbf{u}^T \mathbf{s}_0$ is the sum of N uniform variables on the same interval $[a, b]$. Actually it is very well approximated by a Gaussian variable as just as the number of the nodes exceed 10. So the limit distribution of the i -th node shock is very well approximated by a Gaussian variable with mean and standard deviation equal to

$$\frac{k_i}{\bar{k}} \mu = \frac{k_i}{\bar{k}} \frac{a+b}{2} \quad \text{and} \quad \sqrt{N} \frac{k_i}{\bar{k}} \sigma = \sqrt{\frac{N}{12}} (b-a) \frac{k_i}{\bar{k}}.$$

5 Simulations

In this section we illustrate simulation results run for three well known classes of networks: the Erdos –Renyi (ER) random network ([13] and [14]), the Barabasi – Albert (BA) scale-free network ([3]) and the Watts – Strogatz (WS) small-world network ([22]). The ER network is obtained randomly connecting nodes with a fixed probability and leads to a homogeneous connectivity structure where the degree distribution is asymptotically Poisson distributed. The BA network, instead, follows the so called *preferential attachment* principle. The network formation starts from a kernel composed of m_0 nodes and then a new vertex connects with $m \leq m_0$ links with a probability proportional to their connectivity. For large values of N , the degree distribution is such that $Prob(k_i > k) = (m/k)^2$ and $\bar{k} = 2m$. In the WS model, nodes are originally connected in a simple lattice ring and then rewired randomly.

We set $N = 500$ for all graphs. A probability 0.1 has been used for the ER graph, while for the WS network the neighborhood has been set equal to 5 with

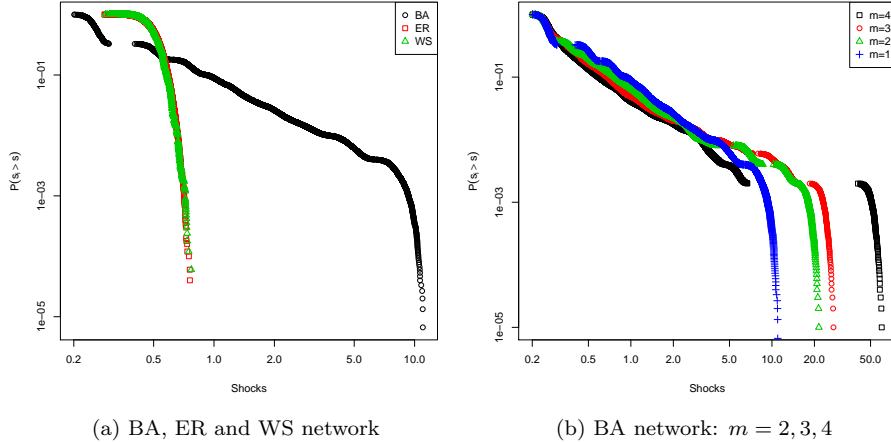


Figure 1: Log-log plot of the cumulative shock distribution ($P(s_{i,\infty} > s)$) for 100 simulations according to different network topologies (a) and for different values of m for the BA network (b).

a rewiring probability of 0.05. For the BA graph we generated four networks with $m = 1, 2, 3, 4$. Multiple edges have been removed. We assumed, for sake of simplicity, that the initial shocks vector \mathbf{s}_0 is a random variable uniformly distributed in the interval $[a, b]$. See Section 4.1 for the theoretical results. We ran 100 simulations for $a = -1, b = 2$, so that $\bar{s} = 1/2$.

The speed of convergence is quite interesting to be analyzed. We remind it depends on the second largest norm eigenvalue: nearer to 1, slower the convergence. The vector \mathbf{s}_t converges in norm² with times that are higher for the BA networks when $m = 1$ (significantly over $t = 100$) than for the WS network ($t \approx 80$) and for the ER network ($t \approx 3$). Increasing the parameter m for the BA network leads to a lower convergence time ($\approx 45, \approx 35, \approx 30$, for $m = 2, 3, 4$ respectively). In other words, simulations show that a densification of the network leads to a faster convergence.

The asymptotic cumulative shock distribution ($Prob(s_{\infty,i} > s)$) for the three topologies (BA, ER and WS) are depicted in Fig. 1a, while in Fig. 1b the cumulative shock distributions for BA networks with different values of m are reported.

There is an amplification of the maxima for more concentrated (heterogeneous) topologies (BA networks), while for the ER and WS network we observe a reduction of the extremal values. Such an absorption effect is due both to the degree distribution and to the network density.

The simulations show that the BA graph leads to concentrate a big fraction

²We assume $\|\mathbf{s}_\infty - \mathbf{s}_t\| < 10^{-2}$.

of shocks in a few nodes with high connectivity. This confirms the fact that highly-connected nodes are positively exposed to shocks.

A further interpretation is related to the convergence time: under the model we provided, scale-free networks might be also fragile to random attacks, provided that a sufficient time passes before the asymptotic configuration is reached. Of course, since the time of convergence is significantly higher than for a random (homogeneous) network, a hub can easily spot a negative shock coming.

In the light of the recent financial crisis, this could be a possible explanation why many large financial players (hubs) have undergone severe losses, while small players at the periphery of the network did not. Moreover, the hubs were not able to foresee the negative shocks coming, and the total amount of shock they received has been amplified by their connectivity, i.e. their position in the network.

This has a further implication on systemic risk regulation, as an intuitive implication would be that higher concentration leads to more fragility not only to targeted attacks, but also to those negative random shocks that cannot be seen by hubs and propagate throughout the network.

6 Conclusions

In this article we provide a network model based on a propagation matrix capturing how the shocks spread among adjacent nodes (local interaction) throughout the whole network. Studying the diffusion of the shocks over time, we find that, under quite common conditions, the shocks converge to a limit distribution. In particular, when each node distributes its shock equally among its adjacent nodes, the limit distribution is just the rescaling of the degree distribution by a factor that depends on the network density. In the general case, the limit vector (distribution), when exists, is the principal right eigenvector of the propagation matrix, so it is related to the spectral properties of the network which depend on the underlying topological structure.

Some simulation results allows us to complete our study by confirming the analytical results and providing some additional useful information about the process of convergence according to different classes of networks.

Further research should aim to describe shock absorption capacity, leverage effects and systems where shocks are added or removed at each time step and the relation with systemic risk and vulnerability, also for policy purposes. Furthermore, it could be interesting to study such a model in different types of real-world networks (e.g. information/communication, economic and financial networks).

A Appendix

In this section we provide the proof of the theorems we have used in Section 3.

A.1 Reducibility and connection

The stochastic matrix \mathbf{P} is irreducible if and only if $G = (V, E)$ is connected.

Proof. \mathbf{P} is reducible if and only if a permutation (with representative matrix M) exists such that $M^T P M = \left[\begin{array}{c|c} B & C \\ \hline 0 & D \end{array} \right]$. Under our assumption that $P_{ij} = 0 \Leftrightarrow A_{ij} = 0$, the same permutation on A must lead to $M^T A M = \left[\begin{array}{c|c} B' & C' \\ \hline 0 & D' \end{array} \right]$, i.e., P is reducible if and only if A is reducible. Since A is a binary and symmetric matrix, A is reducible if only if $G(V, E)$ is not connected. This is evident considering that, since $M^T A M$ is symmetric, A is reducible if and only if $C' = 0$, which clearly shows that $G(V, E)$ is not connected. ■

The model can be easily extended to a disconnected graph G . In this case $G = G_1 \cup \dots \cup G_l$, where G_1, G_2, \dots, G_l (with $l \geq 1$) are the connected components of G . Every connected component can be seen as a strongly connected directed graph, and A achieves the Frobenius normal form by means of suitable row and column permutations [10]:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1k} \\ O & \mathbf{A}_2 & \cdots & \mathbf{A}_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & \mathbf{A}_k \end{bmatrix}$$

Since G is undirected, \mathbf{A} is symmetric, so $\mathbf{A}_{ij} = 0 \quad \forall i, j$, i.e., by a suitable permutation of the node indexes, \mathbf{A} is a block diagonal matrix. Besides $\mathbf{A}_1, \dots, \mathbf{A}_k$ are irreducible square matrices, each uniquely corresponding to a connected component of G .

Thus the model can be applied to each irreducible matrix \mathbf{A}_i according to the fact that shocks affecting a node $i \in G_j$ should influence just the vertices belonging to G_j . Indeed \mathbf{P} is block diagonal itself. As a result every connected component of G represents an independent system and can be studied separately with respect to the other ones.

A.2 Left and right eigenvectors of $\mathbf{P} = \mathbf{A} \mathbf{K}^{-1}$

The vector \mathbf{w} is a left eigenvector of \mathbf{P} if and only if $\mathbf{K} \mathbf{w}$ is a right eigenvector with respect to the same eigenvalue.

Proof. Let \mathbf{w} be a left eigenvector of \mathbf{P} with respect to the eigenvalue λ . Since $\mathbf{A} = \mathbf{P} \mathbf{K}$, we have the following chain of identities:

$$\begin{aligned} \mathbf{w}^T \mathbf{P} = \lambda \mathbf{w}^T &\Leftrightarrow \mathbf{w}^T \mathbf{A} \mathbf{K}^{-1} = \lambda \mathbf{w}^T \Leftrightarrow \mathbf{w}^T \mathbf{A} = \lambda \mathbf{w}^T \mathbf{K} \Leftrightarrow \\ &\Leftrightarrow \mathbf{w}^T \mathbf{A}^T = \lambda \mathbf{w}^T \mathbf{K}^T \Leftrightarrow (\mathbf{A} \mathbf{w})^T = \lambda (\mathbf{K} \mathbf{w})^T \Leftrightarrow \\ &\Leftrightarrow \mathbf{A} \mathbf{w} = \lambda \mathbf{K} \mathbf{w} \Leftrightarrow \mathbf{P} (\mathbf{K} \mathbf{w}) = \lambda (\mathbf{K} \mathbf{w}) \end{aligned}$$

so that $\mathbf{K}\mathbf{w}$ is a right eigenvector of \mathbf{P} . ■

A.3 Eigenvalues and eigenvectors of $\mathbf{P} = \mathbf{A}\mathbf{K}^{-1}$

Reminding that a symmetric matrix is diagonalizable by a unitary matrix (whose columns are the eigenvectors) and has real eigenvalues, we prove the following properties.

Theorem 2 \mathbf{P} has the same eigenvalues of the symmetric matrix $\mathbf{K}^{-\frac{1}{2}}\mathbf{A}\mathbf{K}^{-\frac{1}{2}}$.

Proof. λ is an eigenvalue of \mathbf{P} if and only if $\det(\mathbf{P} - \lambda\mathbf{I}) = 0$. Since \mathbf{K} is invertible, the following identities hold

$$\begin{aligned} \det(\mathbf{P} - \lambda\mathbf{I}) &= \det(\mathbf{A}\mathbf{K}^{-1} - \lambda\mathbf{I}) = \det\left(\mathbf{K}^{\frac{1}{2}}\mathbf{K}^{-\frac{1}{2}}\mathbf{A}\mathbf{K}^{-\frac{1}{2}}\mathbf{K}^{-\frac{1}{2}} - \lambda\mathbf{K}^{\frac{1}{2}}\mathbf{K}^{-\frac{1}{2}}\right) = \\ &= \det\left(\mathbf{K}^{\frac{1}{2}}\right) \cdot \det\left(\mathbf{K}^{-\frac{1}{2}}\mathbf{A} - \lambda\mathbf{I}\right) \cdot \det\left(\mathbf{K}^{-\frac{1}{2}}\right) \end{aligned}$$

Then

$$\det(\mathbf{P} - \lambda\mathbf{I}) = 0 \quad \Leftrightarrow \quad \det\left(\mathbf{K}^{-\frac{1}{2}}\mathbf{A}\mathbf{K}^{-\frac{1}{2}} - \lambda\mathbf{I}\right) = 0$$

So λ is eigenvalue of $\mathbf{K}^{-\frac{1}{2}}\mathbf{A}\mathbf{K}^{-\frac{1}{2}}$. ■

Theorem 3 \mathbf{P} is diagonalizable and has real eigenvalues.

Proof. $\mathbf{B} = \mathbf{K}^{-\frac{1}{2}}\mathbf{A}\mathbf{K}^{-\frac{1}{2}}$ is symmetric, then there is an unitary matrix \mathbf{H} such that $\mathbf{H}^T\mathbf{B}\mathbf{H} = \mathbf{\Lambda}$. Besides the diagonal elements of $\mathbf{\Lambda}$ are real. We have already proven that the diagonal elements of $\mathbf{\Lambda}$ are the eigenvalues of \mathbf{B} , so they are the eigenvalues of \mathbf{P} . The columns of \mathbf{H} are the eigenvectors of \mathbf{B} (both right and left). Then the columns of $\mathbf{R} = \mathbf{K}^{\frac{1}{2}}\mathbf{H}$ are the right eigenvectors of \mathbf{P} :

$$\mathbf{P}\mathbf{R} = \mathbf{A}\mathbf{K}^{-1}\mathbf{K}^{\frac{1}{2}}\mathbf{H} = \mathbf{K}^{\frac{1}{2}}\mathbf{H}\mathbf{H}^T\mathbf{K}^{-\frac{1}{2}}\mathbf{A}\mathbf{K}^{-\frac{1}{2}}\mathbf{H} = \mathbf{K}^{\frac{1}{2}}\mathbf{H}\mathbf{H}^T\mathbf{B}\mathbf{H} = \mathbf{R}\mathbf{\Lambda}$$

For Theorem 1, the columns of $\mathbf{K}^{-\frac{1}{2}}\mathbf{H}$ are the corresponding left eigenvectors. ■

B Two benchmark networks

We consider here two “extreme” network structures that will be used as benchmarks for comparison: the star graph $K_{N,1}$ and the complete graph K_N .

Any propagation matrix related to a **star graph** is never primitive and leads to a periodic dynamics. If node 1 is the center of the network (i.e., the node with degree $N - 1$), while the other ones are the peripheral nodes, the propagation matrix must have the following form:

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & \dots & 1 \\ w_2 & 0 & \dots & 0 \\ w_3 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ w_N & 0 & \dots & 0 \end{bmatrix}$$

with $\sum_{j=2}^N w_j = 1$ (in the case of equidistributed shocks $w_j = \frac{1}{N-1}$ for any j). Its spectrum is $\{-1, 0, 1\}$. A basis of right eigenvectors is

$$\begin{pmatrix} 1 \\ w_2 \\ w_3 \\ \dots \\ w_n \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ \dots \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ \dots \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \dots \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -w_2 \\ -w_3 \\ \dots \\ -w_n \end{pmatrix}$$

So \mathbf{P} is not primitive, because the spectral radius is not strictly dominant. Indeed

$$\mathbf{P}^2 = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & w_2 & \dots & w_2 \\ 0 & w_3 & \dots & w_3 \\ \dots & \dots & \dots & \dots \\ 0 & w_N & \dots & w_N \end{bmatrix}, \quad \mathbf{P}^3 = \begin{bmatrix} 0 & 1 & \dots & 1 \\ w_2 & 0 & \dots & 0 \\ w_3 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ w_N & 0 & \dots & 0 \end{bmatrix} = \mathbf{P}$$

So that:

$$\mathbf{P}^t = \begin{cases} \mathbf{P} & t = 2h + 1 \\ \mathbf{P}^2 & t = 2h \end{cases}, \quad h \in \mathbf{N}$$

In other words, every node is subjected to a periodic shock (with period 2). The shock of the central node is $s_{1,0}$ (its original shock) or $\sum_{j=2}^N s_{j,0}$ (the sum of the original shocks of all the peripheral nodes); while the shock of a peripheral node j (with $j = 2, \dots, N$) is either $s_{j,0}$ (its original shock) or $w_j \sum_{k=2}^N s_{k,0}$ (a fraction w_j of the sum of the original shocks of all the peripheral nodes).

In case of independent shocks, the central node, even though its initial shock has mean μ and standard deviation σ , must then face a much bigger and more uncertain shock since it is distributed with mean $(N-1)\mu$ and standard deviation $\sqrt{N-1}\sigma$. Similarly the peripheral node j must face a shock distributed with mean $w_j(N-1)\mu$ and standard deviation $w_j\sqrt{N-1}\sigma$ which, in case of equidistributed shocks ($w_j = \frac{1}{N-1}$), does not significantly differ from the distribution of its initial shock (it is exactly the same in case the initial shock is Gaussian).

The **complete graph** has the following adjacency matrix:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 0 \end{bmatrix} = \mathbf{U} - \mathbf{I}$$

where \mathbf{U} is the matrix with all entries equal to 1 and \mathbf{I} is the identity matrix. In the case of equidistributed shocks $\mathbf{P} = \frac{1}{N-1}\mathbf{A}$. Since $U^t = N^{t-1}U$,

$$\mathbf{P}^t = \frac{1}{(N-1)^t} \left(\frac{(N-1)^t - (-1)^t}{N} \mathbf{U} + (-1)^t \mathbf{I} \right) = \frac{1}{N} \mathbf{U} + \frac{(-1)^t}{(N-1)^t} \left(\mathbf{I} - \frac{1}{N} \mathbf{U} \right)$$

The last form clearly shows the exponential convergence for large t and implies:

$$s_{i,t} = \bar{s} + \frac{(-1)^t}{(N-1)^t} (s_{i,0} - \bar{s}) \quad , \quad \bar{s} = \frac{1}{N} \sum_{j=1}^N s_{j,0}$$

As expected, because of the symmetry, all the shocks tend to the average of the initial shocks.³ The discrepancies from the average shock decrease exponentially, with smaller and smaller oscillations. Moreover, the convergence is fast when N is large.

In case of independent shocks, every node is burdened by a shock with constant expected value equal to μ and decreasing standard deviation $\sqrt{1 + (N-1)^{1-2t}} \frac{\sigma}{\sqrt{N}}$.

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³It is interesting to note that if each node kept (loop) a fraction of the shock equal to the one it transfers, so that $\mathbf{A} = \mathbf{U}$, the shock would immediately converge to the average.

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