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Abstract

The aim of this paper is to obtain the analytical solution to the method of moments for Zenga's model (Zenga, M. M., 2010). First, the central moments of Polisicchio's distribution are used to derive the corresponding central moments for Zenga's model. Secondly, the method of moments is applied to such central moments, and then the analytical solution of the related system is obtained. These analytical results are then compared with the numerical ones in Zenga et al. (2010a).

Keywords: central moments, mixture, central moments of a mixture, method of moments.

1 Introduction

Zenga, M. M. (2010) proposed a new three-parameter model whose characteristics can be useful for income, wealth, financial and actuarial distributions. This new model is obtained by mixturing of particular truncated Pareto distributions introduced by Polisicchio (2008) with Beta weights. The probability density function of the model (for $\mu > 0$, $\alpha > 0$ and $\theta > 0$) is given by:

$$f(x:\mu;\alpha;\theta) = \begin{cases} \dfrac{1}{2\mu B(\alpha;\theta)} \left(\dfrac{x}{\mu}\right)^{-1.5} \displaystyle\int_0^{\frac{x}{\mu}} k^{\alpha+0.5-1}(1-k)^{\theta-2}\,dk, & \text{if } 0 < x < \mu \\ \dfrac{1}{2\mu B(\alpha;\theta)} \left(\dfrac{\mu}{x}\right)^{-1.5} \displaystyle\int_0^{\frac{\mu}{x}} k^{\alpha+0.5-1}(1-k)^{\theta-2}\,dk, & \text{if } x > \mu. \end{cases} \tag{1}$$

*Although this paper arises from a collaboration of the three authors, the paragraphs 1 and 7 have to be attribuited to M. M. Zenga, the paragraphs 3 and 4 to F. Porro, and the paragraphs 2, 5 and 6 to A. Arcagni

The parameter μ corresponds to the expected value, while α and θ are shape parameters. The model has Paretian right tail and the moment of order r is finite only if $r < \alpha + 1$. As the method of moments needs the existence of the third moment, the corresponding estimates of α will be necessarily greater than 2.

Zenga et al. (2010a) inplemented the method of moments estimation for model (1) starting from the following system:

$$\begin{cases} \bar{x} &= \mu \\ m_2 &= \dfrac{\mu^2}{3} \dfrac{\theta(\theta+1)}{(\alpha-1)(\alpha+\theta)} \\ m_3' &= \dfrac{\mu^3}{5} \dfrac{1}{B(\alpha;\theta)} \{B(\alpha-2;\theta-1) - B(\alpha+3;\theta-1)\} \end{cases} \quad (2)$$

where $\bar{x}$, m_2 and m_3' are the sample mean, variance and third moment about zero, respectively. Using the second equation of the system (2), the following expression for α is easily obtained:

$$\tilde{\alpha}(\theta) = \frac{(\theta+1)\left\{-1 + \sqrt{1 + \dfrac{4}{3}\dfrac{\bar{x}^2}{m_2}\dfrac{\theta}{\theta+1}}\right\}}{2} + 1. \quad (3)$$

Rearranging the third equation using (3), we obtain a polynomial of fourth-degree, quite difficult to solve, which is here provided in its implicit form only:

$$T(\tilde{\alpha}(\theta);\theta) = 5\frac{m_3'}{\bar{x}^3}. \quad (4)$$

The estimate of θ is obtained by solving numerically equation (4) while the estimate of α can be achieved by replacing in (3) θ with the corresponding estimate.

In this paper we find the analytical solution of the method of moments, starting from a different system based on the mean, the variance and the third central moment. We will show that replacing the third equation in (2) results in a definite simplification.

The paper is organized as follows. A general result regarding the central moments of a mixture is provided in section 2. In section 3 the first four central moments of Polisicchio random variable are obtained. In section 4 the corresponding central moments of Zenga distribution are derived. In section 5 the analytical solution of the method of moments is provided. In section 6, the results are compared with the numerical solution of the method of moments, evaluated in Zenga et al. (2010a).

2 The central moments of a mixture

Let X be a continuous mixture on the support S where $g(k)$ is the mixing density function, and $f(x:k)$ the conditional density of X given k. The probability density of the mixture X is:

$$f(x) = \int_K f(x:k)g(k)\,dk$$

where K is the support of the density $g(k)$.

Let be μ'_r the r-th moment about zero and μ_r the r-th central moment of the mixture X, and let $\mu'_r(k)$ and $\mu_r(k)$ be the corrisponding moments of the conditional distribution of X given k.

By definition, the r-th central moment of the mixture X is:

$$\begin{aligned}\mu_r &= \mathbb{E}\left[(X-\mu'_1)^r\right] = \int_S (x-\mu'_1)^r f(x)\,dx \\ &= \int_S (x-\mu'_1)^r \left[\int_K f(x:k)g(k)\,dk\right] dx.\end{aligned}$$

For μ_r finite the Fubini's theorem can be applied and the integrals can be inverted:

$$\begin{aligned}\mu_r &= \int_K \left[\int_S (x-\mu'_1)^r f(x:k)\,dx\right] g(k)\,dk \\ &= \int_K \left\{\int_S \left[(x-\mu'_1(k)) + (\mu'_1(k)-\mu'_1)\right]^r f(x:k)\,dx\right\} g(k)\,dk.\end{aligned}$$

Using the binomial identity:

$$(a+b)^r = \sum_{i=0}^{r} \binom{r}{i} a^i b^{r-i},\; r \in \mathbb{N},$$

the r-th central moment of the mixture X can be written as:

$$\begin{aligned}\mu_r &= \int_K \left\{\int_S \left[\sum_{i=0}^{r} \binom{r}{i} (x-\mu'_1(k))^i (\mu'_1(k)-\mu'_1)^{r-i}\right] f(x:k)\,dx\right\} g(k)\,dk \\ &= \int_K \sum_{i=0}^{r} \binom{r}{i} \left[(\mu'_1(k)-\mu'_1)^{r-i} \int_S (x-\mu'_1(k))^i f(x:k)\,dx\right] g(k)\,dk \\ &= \sum_{i=0}^{r} \binom{r}{i} \int_K \left[(\mu'_1(k)-\mu'_1)^{r-i} \int_S (x-\mu'_1(k))^i f(x:k)\,dx\right] g(k)\,dk.\end{aligned}$$

In the last equation, the integral $\int_S (x-\mu'_1(k))^i f(x:k)\,dx$ is equal to the i-th central moment of the conditional distribution of X given k; it follows that:

$$\mu_r = \sum_{i=0}^{r} \binom{r}{i} \int_K (\mu'_1(k)-\mu'_1)^{r-i} \mu_i(k)g(k)\,dk. \tag{5}$$

If $r = 2$, the formula (5) is the known variance decomposition:

$$\begin{aligned}
\mu_2 &= \mathrm{Var}(X) = \binom{2}{0} \int_K \left(\mu_1'(k) - \mu_1'\right)^2 \mu_0(k) g(k)\, dk + \\
&+ \binom{2}{1} \int_K \left(\mu_1'(k) - \mu_1'\right) \mu_1(k) g(k)\, dk + \\
&+ \binom{2}{2} \int_K \mu_2(k) g(k)\, dk,
\end{aligned}$$

the central moment of order 0 is always equal to 1 and the central moment of order 1 is always equal to 0. Then:

$$\mu_2 = \int_K \left(\mu_1'(k) - \mu_1'\right)^2 g(k)\, dk + \int_K \mu_2(k) g(k)\, dk. \tag{6}$$

The first integral of (6) is the variance of the means (between variance) and the second one is the mean of the variances (within variance).

Equation (5) for $r = 3$ gives:

$$\begin{aligned}
\mu_3 &= \mathbb{E}\left[(X - \mu_1')^3\right] = \binom{3}{0} \int_K \left(\mu_1'(k) - \mu_1'\right)^3 \mu_0(k) g(k)\, dk + \\
&+ \binom{3}{1} \int_K \left(\mu_1'(k) - \mu_1'\right)^2 \mu_1(k) g(k)\, dk + \\
&+ \binom{3}{2} \int_K \left(\mu_1'(k) - \mu_1'\right) \mu_2(k) g(k)\, dk + \\
&+ \binom{3}{3} \int_K \mu_3(k) g(k)\, dk \\
&= \int_K \left(\mu_1'(k) - \mu_1'\right)^3 g(k)\, dk + 3 \int_K \left(\mu_1'(k) - \mu_1'\right) \mu_2(k) g(k)\, dk \\
&+ \int_K \mu_3(k) g(k)\, dk.
\end{aligned}$$

By equation (5) we can also note that:

$$\begin{aligned}
\mu_r &= \sum_{i=0}^{r-1} \binom{r}{i} \int_K \left(\mu_1'(k) - \mu_1'\right)^{r-i} \mu_i(k) g(k)\, dk + \\
&+ \binom{r}{r} \int_K \left(\mu_1'(k) - \mu_1'\right)^0 \mu_r(k) g(k)\, dk \\
&= \sum_{i=0}^{r-1} \binom{r}{i} \int_K \left(\mu_1'(k) - \mu_1'\right)^{r-i} \mu_i(k) g(k)\, dk + \int_K \mu_r(k) g(k)\, dk. \quad (7)
\end{aligned}$$

An important case is $\mu_1'(k) = \mu \; \forall k \in K$, then also $\mu_1' = \mu$, and, by equation (7) the r-th central moment of the mixture can be expressed as:

$$\mu_r = \int_K \mu_r(k) g(k)\, dk. \tag{8}$$

3 Variance, third and fourth central moment of Polisicchio's truncated Pareto density

The density of Zenga's distribution $f(x : \mu; \alpha; \theta)$ (with $\mu > 0$, $\alpha > 0$, $\theta > 0$) is obtained as a mixture of Polisicchio's truncated Pareto densities (see Polisicchio, 2008):

$$f(x : \mu; k) = \begin{cases} \frac{\sqrt{\mu}}{2} k^{0,5}(1-k)^{-1}x^{-1,5} & \mu \leq x \leq \frac{\mu}{k} \\ 0 & otherwise \end{cases} \tag{9}$$

where $\mu > 0$ and k ranges in the interval $(0, 1)$. The mixing parameter k in Zenga's distribution has a Beta density g with parameters α and θ. The parameter μ of Polisicchio's density (9) and of Zenga's density is equal to their expectation. In this section, we obtain the second, the third and the fourth central moment of Polisicchio's density.

Polisicchio (2008) proved that the $r-$th moment ($r \in \mathbb{N}$) about zero of the density (9) is given by:

$$\mu'_r = \frac{\mu^r k^{1-r}(1-k)^{2r-1}}{(2r-1)(1-k)}. \tag{10}$$

By using the relation

$$1 + k + k^2 + ... + k^n = \frac{1-k^{n+1}}{1-k}$$

Zenga, M. M. (2010) showed that the moments of Polisicchio density can be equivalently obtained by:

$$\mu'_r = \frac{\mu^r}{2r-1}\sum_{i=1}^{2r-1} k^{i-r}. \tag{11}$$

Notice that, from (11),

$$\mu'_2 = \frac{\mu^2}{3}(k^{-1} + 1 + k) = \frac{\mu^2}{3k}(1 + k + k^2).$$

Consequently the variance of the density (9) is

$$\frac{\mu^2}{3k}(1 + k + k^2) - \mu^2 = \frac{\mu^2}{3k}(1 - 2k + k^2) = \frac{\mu^2}{3k}(1-k)^2. \tag{12}$$

In order to obtain the third central moment, firstly it can be remarked that, for any random variable X (with finite third moment), the following holds:

$$\begin{aligned} \mathbb{E}(X-\mu)^3 &= \mathbb{E}(X^3 - 3\mu X^2 + 3\mu^2 X - \mu^3) \\ &= \mathbb{E}(X^3) - 3\mu\mathbb{E}(X^2) + 3\mu^2\mathbb{E}(X) - \mu^3 \\ &= \mathbb{E}(X^3) - 3\mu\mathbb{E}(X^2) + 2\mu^3 \end{aligned}$$

and, since $\mathbb{E}(X^2) = Var(X) + \mu^2$, it follows that:

$$\begin{aligned}\mathbb{E}(X-\mu)^3 &= \mathbb{E}(X^3) - 3\mu[Var(X)+\mu^2] + 2\mu^3 \\ &= \mathbb{E}(X^3) - 3\mu Var(X) - 3\mu^3 + 2\mu^3 \\ &= \mathbb{E}(X^3) - 3\mu Var(X) - \mu^3. \end{aligned} \tag{13}$$

By equation (11), the third moment of Polisicchio r.v. can then be obtained as:

$$\mu_3' = \frac{\mu^3}{5}(k^{-2}+k^{-1}+1+k+k^2) = \frac{\mu^3}{5k^2}(1+k+k^2+k^3+k^4) \tag{14}$$

and therefore, using (12), equation (13) gives the third central moment:

$$\begin{aligned} & \frac{\mu^3}{5k^2}(1+k+k^2+k^3+k^4) - \frac{\mu^3}{k}(1-k)^2 - \mu^3 = \\ = & \frac{\mu^3}{5k^2}[1+k+k^2+k^3+k^4-5k(1+k^2-2k)-5k^2] = \\ = & \frac{\mu^3}{5k^2}(1+k+k^2+k^3+k^4-5k+5k^3+10k^2-5k^2) = \\ = & \frac{\mu^3}{5k^2}(1-4k+6k^2-4k^3+k^4) = \\ = & \frac{\mu^3}{5k^2}(1-k)^4. \end{aligned}$$

The same approach can be used to get the fourth central moment. The following expressions hold for any random variable X (with finite fourth moment):

$$\begin{aligned}\mathbb{E}(X-\mu)^4 &= \mathbb{E}(X^4 - 4\mu X^3 + 6\mu^2 X^2 - 4\mu^3 X + \mu^4) \\ &= \mathbb{E}(X^4) - 4\mu\mathbb{E}(X^3) + 6\mu^2\mathbb{E}(X^2) - 4\mu^2\mathbb{E}(X) + \mu^4 \\ &= \mathbb{E}(X^4) - 4\mu\mathbb{E}(X^3) + 6\mu^2\mathbb{E}(X^2) - 3\mu^4 \\ &= \mathbb{E}(X^4) - 4\mu\mathbb{E}(X^3) + 6\mu^2[Var(X)+\mu^2] - 3\mu^4. \end{aligned} \tag{15}$$

By (11), the fourth moment of Polisicchio r.v. is

$$\mu_4' = \frac{\mu^4}{7}(k^{-3}+k^{-2}+k^{-1}+1+k+k^2+k^3) = \frac{\mu^4}{7k^3}(1+k+k^2+k^3+k^4+k^5+k^6)$$

and therefore, the fourth central moment of (9) is

$$\begin{aligned} & \frac{\mu^4}{7k^3}(1+k+...+k^6) - \frac{4\mu^4}{5k^2}(1+k+...+k^4) + \frac{2\mu^4}{k}(1+k+k^2) - 3\mu^4 = \\ = & \frac{\mu^4}{35k^3}\left[5(1+k...+k^6) - 28k(1+k+...+k^4) + 70k^2(1+k+k^2) - 105k^3\right] = \\ = & \frac{\mu^4}{35k^3}[5-23k+47k^2-58k^3+47k^4-23k^5+5k^6] = \\ = & \frac{\mu^4}{35k^3}(1-k)^4(5k^2-3k+5). \end{aligned}$$

4 Variance, third and fourth central moment of Zenga's density $f(x : \mu; \alpha; \theta)$

We are now ready to apply the results obtained in Section 2 to a random variable X following Zenga's distribution, which can be regarded as a mixture of conditional density (9), In the notation of Section 2, $\mu_1' = \mu_1'(k) = \mu$; hence, by applying (8) with $r = 2$:

$$\begin{aligned} Var(X) &= \int_0^1 \frac{\mu^2}{3k}(1-k)^2 \cdot g(k:\alpha;\theta)\,dk = \int_0^1 \frac{\mu^2}{3k}(1-k)^2 \cdot \frac{k^{\alpha-1}(1-k)^{\theta-1}}{B(\alpha,\theta)}\,dk = \\ &= \frac{\mu^2}{3}\frac{1}{B(\alpha,\theta)}\int_0^1 k^{(\alpha-1)-1}(1-k)^{(\theta+2)-1}\,dk. \end{aligned}$$

For $\alpha > 1$ the last integral is $B(\alpha-1,\theta+2)$, therefore:

$$\begin{aligned} Var(X) &= \frac{\mu^2}{3}\cdot\frac{B(\alpha-1,\theta+2)}{B(\alpha,\theta)} \\ &= \frac{\mu^2}{3}\cdot\frac{\Gamma(\alpha-1)\Gamma(\theta+2)}{\Gamma(\alpha+\theta+1)}\cdot\frac{\Gamma(\alpha+\theta)}{\Gamma(\alpha)\Gamma(\theta)} \\ &= \frac{\mu^2}{3}\cdot\frac{\Gamma(\alpha-1)(\theta+1)\theta\Gamma(\theta)}{(\alpha+\theta)\Gamma(\alpha+\theta)}\cdot\frac{\Gamma(\alpha+\theta)}{\alpha\Gamma(\alpha-1)\Gamma(\theta)} \\ &= \frac{\mu^2}{3}\cdot\frac{\theta(\theta+1)}{(\alpha-1)(\alpha+\theta)} \end{aligned} \tag{16}$$

where:

$$\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}\,dt, \qquad a > 0$$

and:

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \qquad a > 0, b > 0.$$

Note that the expression (16) has been obtained, following another approach, in Zenga, M. M. (2010).

For the third central moment of Zenga's distribution, equation (8) with $r = 3$ gives

$$\begin{aligned} \mathbb{E}(X-\mu)^3 &= \int_0^1 \frac{\mu^3}{5k^2}(1-k)^4 \cdot g(k:\alpha;\theta)\,dk \\ &= \int_0^1 \frac{\mu^3}{5k^2}(1-k)^4 \cdot \frac{k^{\alpha-1}(1-k)^{\theta-1}}{B(\alpha,\theta)}\,dk \\ &= \frac{\mu^3}{5}\int_0^1 \frac{k^{(\alpha-2)-1}(1-k)^{(\theta+4)-1}}{B(\alpha,\theta)}\,dk. \end{aligned}$$

If $\alpha > 2$ the latter integral converges and by the definition of the beta function:

$$\begin{aligned}
\mathbb{E}(X-\mu)^3 &= \frac{\mu^3}{5}\cdot\frac{B(\alpha-2,\theta+4)}{B(\alpha,\theta)} \\
&= \frac{\mu^3}{5}\cdot\frac{\Gamma(\alpha-2)\Gamma(\theta+4)}{\Gamma(\alpha+\theta+2)}\cdot\frac{\Gamma(\alpha+\theta)}{\Gamma(\alpha)\Gamma(\theta)} \\
&= \frac{\mu^3}{5}\cdot\frac{\Gamma(\alpha-2)(\theta+3)(\theta+2)(\theta+1)\theta\Gamma(\theta)}{(\alpha+\theta+1)(\alpha+\theta)\Gamma(\alpha+\theta)}\cdot\frac{\Gamma(\alpha+\theta)}{(\alpha-1)(\alpha-2)\Gamma(\alpha-2)\Gamma(\theta)} \\
&= \frac{\mu^3}{5}\cdot\frac{\theta(\theta+1)(\theta+2)(\theta+3)}{(\alpha-1)(\alpha-2)(\alpha+\theta+1)(\alpha+\theta)}. \qquad (17)
\end{aligned}$$

Similarly, the fourth central moment of Zenga's distribution follows:

$$\begin{aligned}
\mathbb{E}(X-\mu)^4 &= \int_0^1 \frac{\mu^4}{35k^3}(1-k)^4(5k^2-3k+5)\cdot g(k:\alpha;\theta)\,dk \\
&= \int_0^1 \frac{\mu^4}{35k^3}(1-k)^4(5k^2-3k+5)\cdot\frac{k^{\alpha-1}(1-k)^{\theta-1}}{B(\alpha,\theta)}\,dk \\
&= \frac{\mu^4}{35}\int_0^1 (1-k)^4(5k^{-1}-3k^{-2}+5k^{-3})\cdot\frac{k^{\alpha-1}(1-k)^{\theta-1}}{B(\alpha,\theta)}\,dk \\
&= \frac{\mu^4}{35}\left[5\int_0^1 \frac{k^{(\alpha-1)-1}(1-k)^{(\theta+4)-1}}{B(\alpha,\theta)}\,dk - 3\int_0^1 \frac{k^{(\alpha-2)-1}(1-k)^{(\theta+4)-1}}{B(\alpha,\theta)}\,dk + \right. \\
&\quad \left. +5\int_0^1 \frac{k^{(\alpha-3)-1}(1-k)^{(\theta+4)-1}}{B(\alpha,\theta)}\,dk\right]. \qquad (18)
\end{aligned}$$

If $\alpha > 3$ all the integrals in (18) converge; therefore:

$$\begin{aligned}
\mathbb{E}(X-\mu)^4 &= \frac{\mu^4}{35}\left[5\frac{B(\alpha-1,\theta+4)}{B(\alpha,\theta)} - 3\frac{B(\alpha-2,\theta+4)}{B(\alpha,\theta)} + 5\frac{B(\alpha-3,\theta+4)}{B(\alpha,\theta)}\right] \\
&= \frac{\mu^4}{35B(\alpha,\theta)}\left[5B(\alpha-1,\theta+4) - 3B(\alpha-2,\theta+4) + 5B(\alpha-3,\theta+4)\right].
\end{aligned}$$

5 Method of moments

Let $(x_1, ..., x_n)$ be the determination of a random sample from the density $f(x:\mu;\alpha;\theta)$. Let

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i,$$

$$m_2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2,$$

and

$$m_3 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^3$$

be the mean, the variance and the third central moment of the sample, respectively. As we shown in the previous sections, when X has the density (1),

$$\begin{cases} \mathbb{E}(X) &= \mu \\ Var(X) &= \dfrac{\mu^2}{3}\cdot\dfrac{\theta(\theta+1)}{(\alpha-1)(\alpha+\theta)} \\ \mathbb{E}(X-\mu)^3 &= \dfrac{\mu^3}{5}\cdot\dfrac{\theta(\theta+1)(\theta+2)(\theta+3)}{(\alpha-1)(\alpha-2)(\alpha+\theta+1)(\alpha+\theta)} \\ &= Var(X)\cdot\dfrac{3\mu(\theta+3)(\theta+2)}{5(\alpha+\theta+1)(\alpha-2)}. \end{cases}$$

Then, according to the method of moments,

$$\begin{cases} \mu = \bar{x} \\ \dfrac{\bar{x}^2}{3}\cdot\dfrac{\theta(\theta+1)}{(\alpha-1)(\alpha+\theta)} = m_2 \\ m_2\cdot\dfrac{3}{5}\cdot\bar{x}\cdot\dfrac{(\theta+3)(\theta+2)}{(\alpha+\theta+1)(\alpha-2)} = m_3. \end{cases} \tag{19}$$

The estimate of μ is hence $\hat{\mu} = \bar{x}$, while the estimates of α and θ are obtained as solutions of the system:

$$\begin{cases} \dfrac{\bar{x}^2}{3}\cdot\theta(\theta+1) = m_2\cdot(\alpha-1)(\alpha+\theta) \\ m_2\cdot\dfrac{3}{5}\cdot\bar{x}\cdot(\theta+3)(\theta+2) = m_3\cdot(\alpha+\theta+1)(\alpha-2) \end{cases}$$

$$\begin{cases} \dfrac{\bar{x}^2}{3}\cdot\theta(\theta+1) = m_2\cdot(\alpha^2+\alpha\theta-\alpha-\theta) \\ m_2\cdot\dfrac{3}{5}\cdot\bar{x}\cdot(\theta+3)(\theta+2) = m_3\cdot\left[\alpha^2-2\alpha+(\theta+1)\cdot\alpha-2(\theta+1)\right] \end{cases}$$

$$\begin{cases} \dfrac{\bar{x}^2}{3}\cdot\theta(\theta+1) = m_2\cdot\left[\alpha^2+\alpha(\theta-1)-\theta\right] \\ m_2\cdot\dfrac{3}{5}\cdot\bar{x}\cdot(\theta+3)(\theta+2) = m_3\cdot\left[\alpha^2+\alpha(\theta+1-2)-2(\theta+1)\right] \end{cases}$$

It follows that:

$$\begin{cases} \dfrac{\bar{x}^2}{3m_2}\theta(\theta+1) = \alpha^2+\alpha(\theta-1)-\theta \\ \\ \dfrac{3}{5}\cdot\bar{x}\cdot\dfrac{m_2}{m_3}(\theta+3)(\theta+2) = \alpha^2+\alpha(\theta-1)-2(\theta+1) \end{cases}$$

$$\begin{cases} \dfrac{\bar{x}^2}{3m_2}\theta(\theta+1)+\theta=\alpha^2+\alpha(\theta-1) \\ \\ \dfrac{3\bar{x}}{5}\cdot\dfrac{m_2}{m_3}(\theta+3)(\theta+2)+2(\theta+1)=\alpha^2+\alpha(\theta-1) \end{cases}$$

and therefore

$$\begin{cases} \dfrac{\bar{x}^2}{3m_2}\theta(\theta+1)+\theta=\alpha^2+\alpha(\theta-1) \\ \\ \dfrac{\bar{x}^2}{3m_2}\theta(\theta+1)+\theta=\dfrac{3}{5}\cdot\bar{x}\cdot\dfrac{m_2}{m_3}(\theta+3)(\theta+2)+2\theta+2. \end{cases} \tag{20}$$

From the second equation of (20) it derives that

$$\frac{1}{3}\cdot\frac{\bar{x}^2}{m_2}(\theta^2+\theta)+\theta=\frac{3}{5}\cdot\bar{x}\cdot\frac{m_2}{m_3}(\theta^2+5\theta+6)+2\theta+2$$

$$\theta^2\left[\frac{1}{3}\frac{\bar{x}^2}{m_2}-\frac{3}{5}\bar{x}\frac{m_2}{m_3}\right]+\theta\left[\frac{1}{3}\frac{\bar{x}^2}{m_2}-3\frac{\bar{x}\,m_2}{m_3}-1\right]-\left[\frac{18}{5}\frac{\bar{x}\,m_2}{m_3}+2\right]=0 \tag{21}$$

then the estimate of θ is:

$$\hat{\theta}=\frac{-\left[\frac{1}{3}\frac{\bar{x}^2}{m_2}-3\frac{\bar{x}\,m_2}{m_3}-1\right]+\sqrt{\left[\frac{1}{3}\frac{\bar{x}^2}{m_2}-3\frac{\bar{x}\,m_2}{m_3}-1\right]^2+4\left[\frac{1}{3}\frac{\bar{x}^2}{m_2}-\frac{3}{5}\bar{x}\frac{m_2}{m_3}\right]\left[\frac{18}{5}\frac{\bar{x}\,m_2}{m_3}+2\right]}}{2\left[\frac{1}{3}\frac{\bar{x}^2}{m_2}-\frac{3}{5}\bar{x}\frac{m_2}{m_3}\right]},$$

since the other solution of (21) is unacceptable.

Combining the last equation and the first equation of system (20) we obtain the estimate of α, as a function of $\hat{\theta}$:

$$\hat{\alpha}=\frac{-(\hat{\theta}-1)+\sqrt{\left(\hat{\theta}-1\right)^2+4\left[\frac{1}{3}\frac{\bar{x}^2}{m_2}\hat{\theta}\left(\hat{\theta}+1\right)+\hat{\theta}\right]}}{2},$$

since the other solution is not acceptable, being negative.

As above remarked, the reported analytical solutions make sense only under the restrictions $\hat{\alpha}>2$, and $\hat{\theta}>0$.

6 A comparison with the numerical solution

The previous results were used to estimate the parameters of Zenga's model for 35 different datasets. The same datasets in Zenga et al. (2010a) were selected, in order to compare the analytical solutions of the method of moments with the numerical ones. The obtained results are reported in Table 1.

Dataset	$\hat{\mu} = \bar{x}$	$\hat{\alpha}$	$\hat{\theta}$
Swiss 2000 Household income			
unweighted obs.	6764.3426	2.2639	3.3526
weighted obs.	6546.9679	2.2884	3.4979
Swiss 2001 Household income			
unweighted obs.	6706.4123	2.4945	2.4638
weighted obs.	6459.2145	2.5142	2.6711
Swiss 2002 Household income			
unweighted obs.	6915.4791	2.8680	3.1782
weighted obs.	6577.1703	3.3184	4.0603
Swiss 2003 Household income			
unweighted obs.	6785.9832	4.8198	5.7336
weighted obs.	6495.3267	4.9341	6.4201
Swiss 2004 Household income			
unweighted obs.	6559.8534	3.9073	4.4095
weighted obs.	6274.9484	4.0224	4.9375
Swiss 2005 Household income			
unweighted obs.	6783.7519	4.0210	4.8195
weighted obs.	6495.5388	4.1595	5.3924
US 2000 Household income			
unweighted obs.	56452.6973	3.7480	11.0614
weighted obs.	57312.9470	3.7931	11.1456
US 2008 Household income			
unweighted obs.	82460.2103	3.9922	10.7071
weighted obs.	79648.7342	3.9492	10.6612
Italy 2006			
Household income, unweighted obs.	31918.9279	2.4447	4.0653
Household income, weighted obs.	31813.3998	2.3782	4.2303
Equivalent Household income, unweighted obs.	19121.4372	2.1939	3.6394
Equivalent Household income, weighted obs.	19020.5880	2.1804	3.8017
Individual income obs.	18502.6989	2.2678	4.6497
Italy 2006 North West Macro Region			
Household income, unweighted obs.	34693.1135	2.5007	3.6521
Household income, weighted obs.	36194.6888	2.3933	4.3200
Equivalent Household income, unweighted obs.	21572.3309	2.7120	3.0018
Equivalent Household income, weighted obs.	22095.4823	2.5281	3.1886
Italy 2006 South Macro Region			
Household income, unweighted obs.	24550.2504	2.4047	4.2064
Household income, weighted obs.	23949.3840	2.3457	4.5969
Equivalent Household income, unweighted obs.	13666.2131	2.6086	3.6902
Equivalent Household income, weighted obs.	13154.5524	2.6245	3.5601
Italy 2006 Center Macro Region			
Household income, unweighted obs.	35595.8391	2.3365	4.2729
Household income, weighted obs.	35886.1715	2.3138	4.3195
Equivalent Household income, unweighted obs.	21543.3705	2.1880	5.9061
Equivalent Household income, weighted obs.	22248.2242	2.1900	6.4766
UK 1999/2000 Gross income			
unweighted obs.	445.8539	2.8227	5.6448
weighted obs.	461.9370	2.8373	5.7045

Table 1: Analitical solutions of Method of Moments for 35 different datasets

In most cases the analytical and the numerical solutions can be considered as equivalent, since their difference is less than 10^{-4}. Such a difference is mainly due to the iterative algorithm used in Zenga et al. (2010a) to solve the third equation of the system of the method of moments. An approximation of the exact solution is hence obtained.

7 Conclusions

In this work we showed how to find the analytical solution of the method of moments to estimate the parameters of Zenga's model, by using central moments. We compared our analytical results with the numerical ones in Zenga et al. (2010a). From this comparison, it arises that the two approaches (the analytical and the numerical) are coherent, since the differences of their estimates are negligible. However, the analytical approach allows a faster and easier application of the method of moments, whose results can be also used as starting point of other iterative procedures (see for example Thisted, 1988).

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