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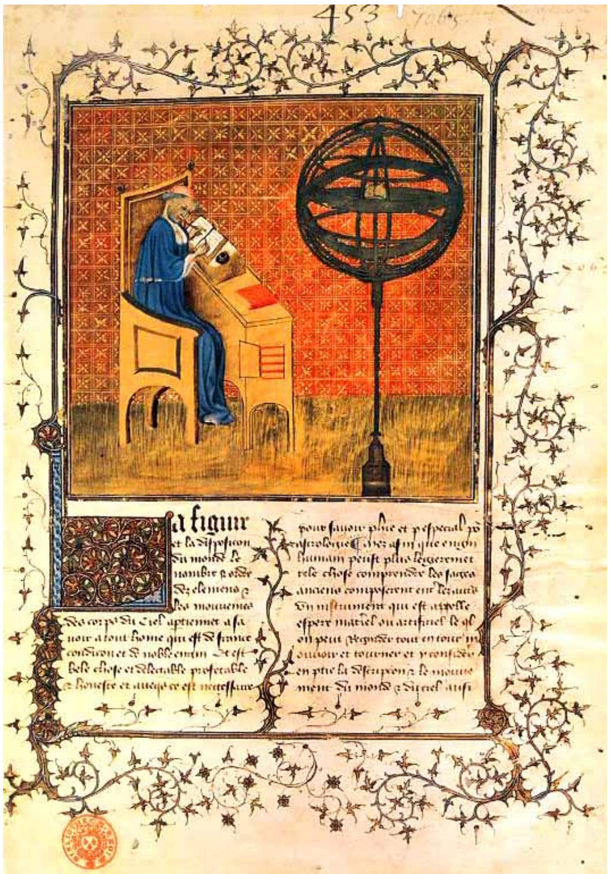

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A NOTE ON COUPLED NONLINEAR SCHRÖDINGER SYSTEMS UNDER THE EFFECT OF GENERAL NONLINEARITIES

A. POMPONIO AND S. SECCHI

Abstract. We prove the existence of radially symmetric ground–states for the system of Nonlinear Schrödinger equations

$$\begin{cases} -\Delta u + u = f(u) + \beta u v^2 & \text{in } \mathbb{R}^3, \\ -\Delta v + v = g(v) + \beta u^2 v & \text{in } \mathbb{R}^3, \end{cases}$$

under very weak assumptions on the two nonlinearities f and g. In particular, no "Ambrosetti–Rabinowitz" condition is required.

1. Introduction

In the last years, nonlinear Schrödinger systems have been widely investigated by several authors. These systems are models for different physical phenomena: the propagation in birefringent optical fibers, Kerr–like photorefractive media in optics, and Bose–Einstein condensates. Roughly speaking, two ore more semilinear Schrödinger equations like

$$-\Delta u + au = u^3 \qquad \text{in } \mathbb{R}^3 \tag{1}$$

are coupled together. Equation (1) describes the propagation of pulse in a nonlinear optical fiber, and the existence of a unique (up to translation) least energy solution has been proved. It turns out that this *ground state solution* is radially symmetric with respect to some point, positive and exponentially decaying together with its first derivatives at infinity.

Unluckily, we know (see [15]) that single-mode optical fibers are not really "single-mode", but actually bimodal due to the presence of birefringence. This birefringence can deeply influence the way an optical evolves during the propagation along the fiber. Indeed, it can occur that the linear birefringence makes a pulse split in two, while nonlinear birefringent traps them together against splitting. The evolution of two orthogonal pulse envelopes in birefringent optical fibers

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is governed (see [23, 24]) by the nonlinear Schrödinger system

$$
\begin{cases}
\mathrm{i}\dfrac{\partial\phi}{\partial t}+\dfrac{\partial^2\phi}{\partial x^2}+|\phi|^2\phi+\beta|\psi|^2\phi=0,\\[2ex]
\mathrm{i}\dfrac{\partial\psi}{\partial t}+\dfrac{\partial^2\psi}{\partial x^2}+|\psi|^2\psi+\beta|\phi|^2\psi=0,
\end{cases}
\tag{2}
$$

where β is a positive constant depending on the anisotropy of the fibers. System (2) is also important for industrial applications in fiber communications systems [11] and all-optical switching devices, see [13]. If one looks for standing wave solutions of (2), namely solutions of the form

$$
\phi(x,t)=e^{\mathrm{i}\omega_1^2 t}u(x)\quad\text{and}\quad \psi(x,t)=e^{\mathrm{i}\omega_2^2 t}v(x),
$$

then (2) becomes

$$
\begin{cases}
-\dfrac{\partial^2 u}{\partial x^2}+u=u^3+\beta v^2u & \text{in }\mathbb{R},\\[2ex]
-\dfrac{\partial^2 v}{\partial x^2}+\omega^2 v=v^3+\beta u^2v & \text{in }\mathbb{R},
\end{cases}
\tag{3}
$$

with $\omega^2=\omega_2^2/\omega_1^2$. Other physical phenomena, such as Kerr–like photorefractive media in optics, are also described by (3), see [1, 6]. As a word of caution, (3) possesses the "simple" solutions of the form $(u,0)$ and $(0,v)$, where u and v solve (1).

Problem (3), in a more general situation and also in higher dimension, has been studied in [7, 8], where smooth ground state solutions $(u,v)\neq(0,0)$ are found by concentration compactness arguments. Later on, Ambrosetti *et al.* in [2], Maia *et al.* in [21] and Sirakov in [27] deal with problem

$$
\begin{cases}
-\Delta u+u=u^3+\beta uv^2 & \text{in }\mathbb{R}^3,\\
-\Delta v+v=v^3+\beta u^2v & \text{in }\mathbb{R}^3,
\end{cases}
\tag{4}
$$

and, among other results, they prove the existence of ground state solutions of the type (u,v), with $u,v>0$, for $\beta>0$ sufficiently big. Similar problems have been treated also in [12, 16, 18, 28]. Some results in the singularly perturbed case can be found in [17, 25, 26], while the orbital stability and blow-up proprieties have been studied in [10, 22].

Although the interest lies in solutions with both non-trivial components, solutions of (4) are somehow related to solutions of the single nonlinear Schrödinger equation (1). The *nonlinearity* $g(u)=u^3$ is typical in physical models, but much more general Schrödinger equation of the form

$$
-\Delta u+au=g(u),\quad \text{in }\mathbb{R}^3,
$$

still have at least a ground state solution under general assumptions on the nonlinearity g which, for example, do not require any Ambrosetti–Rabinowitz growth condition. The crucial feature is, to summarize, that ground states are necessarily radially symmetric with respect to some point, and this knowledge recovers some compactness. We refer to the celebrated papers [5, 9] for a deep study of these scalar–field equations (see also [4, 14]).

Motivated by these remarks, we want to find ground state solutions for the system

$$\begin{cases} -\Delta u + u = f(u) + \beta u v^2 & \text{in } \mathbb{R}^3, \\ -\Delta v + v = g(v) + \beta u^2 v & \text{in } \mathbb{R}^3, \end{cases} \tag{5}$$

where $\beta \in \mathbb{R}$ and $f, g \in C(\mathbb{R}^3, \mathbb{R})$ satisfy the following assumptions:

- **(f1)** $\displaystyle\lim_{t\to 0} \frac{f(t)}{t} = 0$;
- **(f2)** $\displaystyle\lim_{t\to \infty} \frac{f(t)}{|t|^p} = 0$, for some $1 < p < 5$;
- **(f3)** there exists $T_1 > 0$ such that $\frac{1}{2}T_1^2 < F(T_1)$, where $F(t) = \int_0^t f(s)\,ds$;
- **(g1)** $\displaystyle\lim_{t\to 0} \frac{g(t)}{t} = 0$;
- **(g2)** $\displaystyle\lim_{t\to \infty} \frac{g(t)}{|t|^q} = 0$, for some $1 < q < 5$;
- **(g3)** there exists $T_2 > 0$ such that $\frac{1}{2}T_2^2 < G(T_2)$, where $G(t) = \int_0^t g(s)\,ds$.

Remark 1.1. In [5], even weaker assumptions can be found. However, the techniques used in that paper depend strongly on the maximum principle for a single equation, and do not seem to fit the framework of systems of equations.

System (5) has a variational structure, in particular solutions of (5) can be found as critical points of the functional $I : H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$I(u,v) = I_F(u) + I_G(v) - \frac{\beta}{2}\int_{\mathbb{R}^3} |u|^2|v|^2\,dx,$$

where we have set

$$I_F(u) = \frac{1}{2}\|u\|_{H^1}^2 - \int_{\mathbb{R}^3} F(u),$$
$$I_G(v) = \frac{1}{2}\|v\|_{H^1}^2 - \int_{\mathbb{R}^3} G(v).$$

We will call *ground state solution* any couple $(u,v) \neq (0,0)$ which solves (5) and minimizes the functional I among all possible nontrivial solutions. Thus we have to overcome the strong lack of compactness under our weak assumptions on f and g, and also to exclude "simple" solutions with a null component. To fix terminology, we introduce the following definition.

Definition 1.2. A solution of (5), $(u,v) \in \mathbb{H}$, $(u,v) \neq (0,0)$ will be called *scalar solution* if either $u \equiv 0$ or $v \equiv 0$; while a solution of (5), $(u,v) \in \mathbb{H}$, $(u,v) \neq (0,0)$ will be called *vector solution* if $u \neq 0$ and $v \neq 0$.

Scalar solutions for problem (5) exist by the results of [5]. Indeed, since f satisfies **(f1-3)**, there exists a (least–energy) solution $u_0 \in H^1(\mathbb{R}^3)$ for the single Schrödinger equation

$$-\Delta u + u = f(u) \qquad \text{in } \mathbb{R}^3, \tag{6}$$

and since g satisfies (**g1-3**), there exists a (least–energy) solution $v_0 \in H^1(\mathbb{R}^3)$ for

$$-\Delta v + v = g(v) \qquad \text{in } \mathbb{R}^3. \tag{7}$$

It can be checked immediately that the couples $(u_0, 0)$ and $(0, v_0)$ are non-trivial solutions of (5).

As a first step, we will prove that for any $\beta \in \mathbb{R}$ the problem (5) admits a ground state.

Theorem 1.3. *Let f and g satisfy* (**f1-3**) *and* (**g1-3**). *Then for any $\beta \in \mathbb{R}$ there exists a ground state solution of* (5). *Moreover, if $\beta > 0$, this solution is radially symmetric.*

Then we will prove that vector solutions exist whenever the coupling parameter β is sufficiently large.

Theorem 1.4. *Let f and g satisfy* (**f1-3**) *and* (**g1-3**). *Then there exists $\beta_0 > 0$ such that, for any $\beta > \beta_0$, there there exists a vector solution of* (5), *which is a ground state solution. Moreover this solution is radially symmetric.*

We want to highlight that the symmetry of the ground states is essentially part of the variational argument: since we have no Ambrosetti–Rabinowitz condition, we have to work in the space of radially symmetric H^1 functions to gain some compactness. We do no know whether vector ground states may exist without symmetry. Furthermore, the standard Nehari manifold is of no help for our general nonlinearities f and g, and we will exploit the Pohozaev manifold.

Notation

- If $r > 0$ and $x_0 \in \mathbb{R}^3$, $B_r(x_0) := \{x \in \mathbb{R}^3 : |x - x_0| < r\}$. We denote with B_r the ball of radius r centered in the origin.
- We denote with $\| \cdot \|$ the norm of $H^1(\mathbb{R}^3)$.
- We set $\mathbb{H} = H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ and, for any $(u, v) \in \mathbb{H}$, we set $\|(u, v)\|^2 = \|u\|^2 + \|v\|^2$.
- With C_i and c_i, we denote generic positive constants, which may also vary from line to line.

2. The Pohozaev manifold

By (**f1-2**) and (**g1-2**), we get that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|f(t)| \le \varepsilon|t| + C_\varepsilon|t|^p, \qquad \text{for all } t \in \mathbb{R}; \tag{8}$$

$$|F(t)| \le \varepsilon|t|^2 + C_\varepsilon|t|^{p+1}, \qquad \text{for all } t \in \mathbb{R}; \tag{9}$$

$$|f(t)| \le \varepsilon|t| + C_\varepsilon|t|^5, \qquad \text{for all } t \in \mathbb{R}; \tag{10}$$

$$|F(t)| \le \varepsilon|t|^2 + C_\varepsilon|t|^6, \qquad \text{for all } t \in \mathbb{R}. \tag{11}$$

$$|g(t)| \le \varepsilon|t| + C_\varepsilon|t|^q, \qquad \text{for all } t \in \mathbb{R}; \tag{12}$$

$$|G(t)| \le \varepsilon|t|^2 + C_\varepsilon|t|^{q+1}, \qquad \text{for all } t \in \mathbb{R}; \tag{13}$$

$$|g(t)| \le \varepsilon|t| + C_\varepsilon|t|^5, \qquad \text{for all } t \in \mathbb{R}; \tag{14}$$

$$|G(t)| \le \varepsilon|t|^2 + C_\varepsilon|t|^6, \qquad \text{for all } t \in \mathbb{R}. \tag{15}$$

By [3, Lemma 3.6] and repeating the arguments of [5], it is easy to see that each solution of (5), $(u,v) \in \mathbb{H}$, satisfies the following Pohozaev identity:

$$\int_{\mathbb{R}^3} |\nabla u|^2 + |\nabla v|^2 = 6 \int_{\mathbb{R}^3} F(u) + G(v) - \frac{u^2}{2} - \frac{v^2}{2} + \frac{\beta}{2} u^2 v^2. \tag{16}$$

Therefore each non-trivial solution of (5) belongs to $\mathcal{P}$, where

$$\mathcal{P} := \{(u,v) \in \mathbb{H} \mid (u,v) \neq (0,0),\ (u,v) \text{ satisfies } (16)\ \}. \tag{17}$$

We call $\mathcal{P}$ the Pohozaev manifold associated to (5). We collect its main properties of the set $\mathcal{P}$ in the next Proposition: the proof is easy and left to the reader.

Proposition 2.1. *Define the functional $J \colon \mathbb{H} \to \mathbb{R}$ by*

$$J(u,v) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + |\nabla v|^2 - 3 \int_{\mathbb{R}^3} F(u) + G(v) - \frac{u^2}{2} - \frac{v^2}{2} + \frac{\beta}{2} u^2 v^2.$$

Then

(1) $\mathcal{P} = \{(u,v) \in \mathcal{H} \setminus \{(0,0)\} \mid J(u,v) = 0\}$*;*
(2) $\mathcal{P}$ *is a* C^1*–manifold of codimension one.*

Lemma 2.2. *There exists $C > 0$ such that $\|(u,v)\| \geq C$, for any $(u,v) \in \mathcal{P}$.*

Proof. Let $(u,v) \in \mathcal{P}$. By (9), (13) and (17), we easily get

$$\begin{aligned} \|u\|^2 + \|v\|^2 &\leq C_1 \int_{\mathbb{R}^3} |u|^{p+1} + |v|^{q+1} + u^2 v^2 \\ &\leq C_2 (\|u\|^{p+1} + \|v\|^{q+1} + \|u\|^2 \|v\|^2), \end{aligned}$$

which shows the claim. □

According to the definition of [19], we say that a sequence $\{(u_n, v_n)\}_n$ vanishes if, for all $r > 0$

$$\lim_{n \to +\infty} \sup_{\xi \in \mathbb{R}^3} \int_{B_r(\xi)} u_n^2 + v_n^2 = 0.$$

Lemma 2.3. *Any bounded sequence $\{(u_n, v_n)\}_n \subset \mathcal{P}$ does not vanish.*

Proof. Suppose by contradiction that $\{(u_n, v_n)\}_n$ vanishes, then, in particular there exists $\bar{r} > 0$ such that

$$\lim_{n \to +\infty} \sup_{\xi \in \mathbb{R}^3} \int_{B_{\bar{r}}(\xi)} u_n^2 = 0, \quad \lim_{n \to +\infty} \sup_{\xi \in \mathbb{R}^3} \int_{B_{\bar{r}}(\xi)} v_n^2 = 0.$$

Then, by [20, Lemma 1.1], we infer that $u_n, v_n \to 0$ in $L^s(\mathbb{R}^3)$, for any $2 < s < 6$. Since $\{(u_n, v_n)\}_n \subset \mathcal{P}$, we have that $(u_n, v_n) \to 0$ in $\mathbb{H}$, contradicting Lemma 2.2. □

Lemma 2.4. *For any $\beta \in \mathbb{R}$, $\mathcal{P}$ is a natural constraint for the functional I.*

Proof. First we show that the manifold is nondegenerate in the following sense:

$$J'(u,v) \neq 0 \quad \text{for all } (u,v) \in \mathcal{P}.$$

By contradiction, suppose that $(u,v) \in \mathcal{P}$ and $J'(u,v) = 0$, namely (u,v) is a solution of the equation

$$\begin{cases} -\Delta u + 3u = 3f(u) + 3\beta uv^2 & \text{in } \mathbb{R}^3, \\ -\Delta v + 3v = 3g(v) + 3\beta u^2 v & \text{in } \mathbb{R}^3. \end{cases} \tag{18}$$

As a consequence, (u,v) satisfies the Pohozaev identity referred to (18), that is

$$\int_{\mathbb{R}^3} |\nabla u|^2 + |\nabla v|^2 = 18 \int_{\mathbb{R}^3} F(u) + G(v) - \frac{u^2}{2} - \frac{v^2}{2} + \frac{\beta}{2} u^2 v^2. \tag{19}$$

Since $(u,v) \in \mathcal{P}$, by (19) we get

$$2 \int_{\mathbb{R}^3} |\nabla u|^2 + |\nabla v|^2 = 0$$

and we conclude that $u = v = 0$: we get a contradiction since $(u,v) \in \mathcal{P}$.
Now we pass to prove that $\mathcal{P}$ is a natural constraint for I. Suppose that $(u,v) \in \mathcal{P}$ is a critical point of the functional $I_{|\mathcal{P}}$. Then, by Proposition 2.1, there exists $\mu \in \mathbb{R}$ such that

$$I'(u,v) = \mu J'(u,v).$$

As a consequence, (u,v) satisfies the following Pohozaev identity

$$\mu^{-1} J(u,v) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + |\nabla v|^2 - 9 \int_{\mathbb{R}^3} F(u) + G(v) - \frac{u^2}{2} - \frac{v^2}{2} + \frac{\beta}{2} u^2 v^2$$

which, since $J(u,v) = 0$, can be written

$$\mu \int_{\mathbb{R}^3} |\nabla u|^2 + |\nabla v|^2 = 0.$$

Since either $u \neq 0$ or $v \neq 0$ we deduce that $\mu = 0$, and we conclude. $\square$

We set

$$m = \inf_{(u,v) \in \mathcal{P}} I(u,v).$$

We set $\mathbb{H}_r = H^1_r(\mathbb{R}^3) \times H^1_r(\mathbb{R}^3)$: here $H^1_r(\mathbb{R}^3)$ denotes the radially symmetric functions of $H^1(\mathbb{R}^3)$.

By means of the previous lemma, we are reduced to look for a minimizer of I restricted to $\mathcal{P}$. By the well known properties of the Schwarz symmetrization, we are allowed to work on the functional space $\mathbb{H}_r$ as shown by the following

Lemma 2.5. *For any $\beta > 0$ and for any $(u,v) \in \mathcal{P}$, there exists $(\bar{u}, \bar{v}) \in \mathcal{P} \cap \mathbb{H}_r$ such that $I(\bar{u}, \bar{v}) \leq I(u,v)$.*

Proof. Let $(u,v)\in\mathcal{P}$ and set $u^*,v^*\in H^1_r(\mathbb{R}^3)$ their respective symmetrized functions. We have

$$\begin{aligned}\int_{\mathbb{R}^3}|\nabla u^*|^2+|\nabla v^*|^2 &\le \int_{\mathbb{R}^3}|\nabla u|^2+|\nabla v|^2\\ &=6\int_{\mathbb{R}^3}F(u)+G(v)-\frac{u^2}{2}-\frac{v^2}{2}+\frac{\beta}{2}u^2v^2\\ &\le 6\int_{\mathbb{R}^3}F(u^*)+G(v^*)-\frac{(u^*)^2}{2}-\frac{(v^*)^2}{2}+\frac{\beta}{2}(u^*)^2(v^*)^2.\end{aligned}$$

Hence, there exists $\bar t\in(0,1]$ such that $(\bar u,\bar v):=(u^*(\cdot/\bar t),v^*(\cdot/\bar t))\in\mathcal{P}\cap\mathbb{H}_r$ and

$$\begin{aligned}I(\bar u,\bar v)&=\frac13\int_{\mathbb{R}^3}|\nabla\bar u|^2+|\nabla\bar v|^2=\frac{\bar t}{3}\int_{\mathbb{R}^3}|\nabla u^*|^2+|\nabla v^*|^2\\ &\le\frac13\int_{\mathbb{R}^3}|\nabla u|^2+|\nabla v|^2=I(u,v).\end{aligned}$$

□

Proposition 2.6. *For any $\beta>0$, the value m is achieved as a minimum by I on $\mathcal{P}$ by $(u,v)\in\mathbb{H}_r$.*

Proof. For any $(u,v)\in\mathcal{P}$ we have

$$I(u,v)=\frac13\int_{\mathbb{R}^3}|\nabla u|^2+|\nabla v|^2\ge 0. \tag{20}$$

Let $\{(u_n,v_n)\}_n\subset\mathcal{P}$ be such that $I(u_n,v_n)\to m$. By Lemma 2.5, we can assume that $\{(u_n,v_n)\}_n\subset\mathcal{P}\cap\mathbb{H}_r$.

By (20), we infer that $\{u_n\}_n,\{v_n\}_n$ are bounded in $\mathcal{D}^{1,2}(\mathbb{R}^3)$.

Let $\varepsilon>0$ be given, and let $C_\varepsilon>0$ be the positive constant as in (8)–(15). We observe that, for any $\beta\in\mathbb{R}$, there exists a positive constant $C>0$ such that for all $x,y\in\mathbb{R}$:

$$\varepsilon(x^2+y^2)+C_\varepsilon(x^6+y^6)+\frac{\beta}{2}x^2y^2\le 2\varepsilon(x^2+y^2)+CC_\varepsilon(x^6+y^6).$$

Hence, since $\{(u_n,v_n)\}_n\subset\mathcal{P}$, by (11) and (15), we get

$$\begin{aligned}\|u_n\|^2+\|v_n\|^2&\le 6\int_{\mathbb{R}^3}\varepsilon(u_n^2+v_n^2)+C_\varepsilon(u_n^6+v_n^6)+\frac{\beta}{2}u_n^2v_n^2\\ &\le 6\int_{\mathbb{R}^3}2\varepsilon(u_n^2+v_n^2)+CC_\varepsilon(u_n^6+v_n^6).\end{aligned}$$

Since $\{u_n\}_n,\{v_n\}_n$ are bounded in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ and $\mathcal{D}^{1,2}(\mathbb{R}^3)\hookrightarrow L^6(\mathbb{R}^3)$, then $\{(u_n,v_n)\}_n$ is bounded in $\mathbb{H}$.

By Lemma 2.3 we know that $\{(u_n,v_n)\}_n$ does not vanish, namely there exist $C,r>0$, $\{\xi_n\}_n\subset\mathbb{R}^3$ such that

$$\int_{B_r(\xi_n)}u_n^2+v_n^2\ge C,\ \text{ for all } n\ge 1. \tag{21}$$

Since we are dealing with radially symmetric functions, without loss of generality, we can assume that $\xi_n = 0$, for all $n \geq 1$.
Since $\{(u_n, v_n)\}_n$ is bounded in $\mathbb{H}_r$, there exist $u, v \in H^1_r(\mathbb{R}^3)$ such that, up to a subsequence,

$$u_n \rightharpoonup u \text{ in } H^1_r(\mathbb{R}^3); \quad u_n \to u \text{ a.e. in } \mathbb{R}^3; \quad u_n \to u \text{ in } L^s(\mathbb{R}^3),\ 2 < s < 6;$$
$$v_n \rightharpoonup v \text{ in } H^1_r(\mathbb{R}^3); \quad v_n \to v \text{ a.e. in } \mathbb{R}^3; \quad v_n \to v \text{ in } L^s(\mathbb{R}^3),\ 2 < s < 6.$$

By (21), we can argue that either $u \neq 0$ or $v \neq 0$ and, moreover, since $\{(u_n, v_n)\}_n \subset \mathcal{P}$, passing to the limit, we have

$$\int_{\mathbb{R}^3} \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\nabla v|^2 + \frac{3}{2}u^2 + \frac{3}{2}v^2 \leq 3\int_{\mathbb{R}^3} F(u) + G(v) + \frac{\beta}{2}u^2v^2. \tag{22}$$

By (22), it is easy to see that there exists $\bar{t} \in (0, 1]$ such that $(\bar{u}, \bar{v}) = (u(\cdot/\bar{t}), v(\cdot/\bar{t})) \in \mathcal{P} \cap \mathbb{H}_r$. By the weak lower semicontinuity, we get

$$\begin{aligned} b \leq I(\bar{u}, \bar{v}) = \frac{\bar{t}}{3}\int_{\mathbb{R}^3} I(u, v) &\leq \frac{1}{3}\int_{\mathbb{R}^3} |\nabla u|^2 + |\nabla v|^2 \\ &\leq \liminf_{n\to+\infty} \frac{1}{3}\int_{\mathbb{R}^3} |\nabla u_n|^2 + |\nabla v_n|^2 = \liminf_{n\to+\infty} I(u_n, v_n) = b, \end{aligned}$$

hence $(\bar{u}, \bar{v})$ is a minimum of I restricted on $\mathcal{P}$ and so, by Lemma 2.4, it is a (radially symmetric) ground state solution for the problem (5). □

Finally, let us prove a lemma which will be a key point in the proof of Theorem 1.4

Lemma 2.7. *Let $u_0, v_0 \in H^1(\mathbb{R}^3)$ be two non-trivial solutions respectively of (6) and (7). Then, for any $\beta > 0$, there exists $\bar{t} > 0$ such that $(u_0(\cdot/\bar{t}), v_0(\cdot/\bar{t})) \in \mathcal{P}$.*

Proof. Since u_0 is a solution of (6), then it satisfies the following Pohozaev identity:

$$\int_{\mathbb{R}^3} |\nabla u_0|^2 + 3\int_{\mathbb{R}^3} u_0^2 = 6\int_{\mathbb{R}^3} F(u_0),$$

hence

$$\int_{\mathbb{R}^3} u_0^2 < 2\int_{\mathbb{R}^3} F(u_0). \tag{23}$$

Analogously, v_0 satisfies

$$\int_{\mathbb{R}^3} v_0^2 < 2\int_{\mathbb{R}^3} G(v_0). \tag{24}$$

We set $\gamma(t) = (u_0(\cdot/t), v_0(\cdot/t))$, with $t > 0$. We have

$$I(\gamma(t)) = \frac{t}{2}\int_{\mathbb{R}^3} |\nabla u_0|^2 + |\nabla v_0|^2 + t^3\int_{\mathbb{R}^3} \frac{u_0^2}{2} + \frac{v_0^2}{2} - F(u_0) - G(v_0) - \frac{\beta}{2}u_0^2v_0^2.$$

Since $I(\gamma(t)) > 0$ for small t and, by (23) and (24) and being $\beta > 0$, $\lim_{t\to+\infty} I(\gamma(t)) = -\infty$, there exists $\bar{t} > 0$ such that $\frac{d}{dt}I(\gamma(\bar{t})) = 0$, which implies that the couple $(u_0(\cdot/\bar{t}), v_0(\cdot/\bar{t})) \in \mathcal{P}$. □

3. Proofs of Theorem 1.3 and 1.4

Proof of Theorem 1.3. If we set

$$\mathcal{S} := \{(u,v) \in \mathbb{H} \mid (u,v) \neq (0,0),\ (u,v) \text{ solves } (5)\},$$

$$b := \inf_{(u,v)\in\mathcal{S}} I(u,v) \geq 0.$$

Let $\{(u_n,v_n)\}_n \subset \mathcal{S}$ be such that $I(u_n,v_n) \to b$. By (20), we infer that $\{u_n\}_n, \{v_n\}_n$ are bounded in $\mathcal{D}^{1,2}(\mathbb{R}^3)$.
Repeating the arguments of the proof of Proposition 2.6, we can argue that $\{(u_n,v_n)\}_n$ is bounded in $\mathbb{H}_r$.
By Lemma 2.3 we know that $\{(u_n,v_n)\}_n$ does not vanish, namely there exist $C, r > 0$, $\{\xi_n\}_n \subset \mathbb{R}^3$ such that

$$\int_{B_r(\xi_n)} u_n^2 + v_n^2 \geq C, \text{ for all } n \geq 1. \tag{25}$$

Due to the invariance by translations, without loss of generality, we can assume that $\xi_n = 0$ for every n.
Since $\{(u_n,v_n)\}_n$ is bounded in $\mathbb{H}$, there exist $u, v \in H^1(\mathbb{R}^3)$ such that, up to a subsequence,

$$\begin{aligned} &u_n \rightharpoonup u \text{ in } H^1(\mathbb{R}^3); \quad u_n \to u \text{ a.e. in } \mathbb{R}^3; \quad u_n \to u \text{ in } L^s_{\mathrm{loc}}(\mathbb{R}^3),\ 1 \leq s < 6;\\ &v_n \rightharpoonup v \text{ in } H^1(\mathbb{R}^3); \quad v_n \to v \text{ a.e. in } \mathbb{R}^3; \quad v_n \to v \text{ in } L^s_{\mathrm{loc}}(\mathbb{R}^3),\ 1 \leq s < 6. \end{aligned}$$

By (25), we can argue that either $u \neq 0$ or $v \neq 0$ and then it is easy to see that $(u,v) \in \mathcal{S}$. By the weak lower semicontinuity, we get

$$\begin{aligned} b \leq I(u,v) &= \frac{1}{3}\int_{\mathbb{R}^3} |\nabla u|^2 + |\nabla v|^2 \\ &\leq \liminf_{n\to+\infty} \frac{1}{3}\int_{\mathbb{R}^3} |\nabla u_n|^2 + |\nabla v_n|^2 = \liminf_{n\to+\infty} I(u_n,v_n) = b, \end{aligned}$$

hence (u,v) is a ground state for the problem (5).
If $\beta > 0$, by Proposition 2.6, we can argue that there exists a ground state (u,v) which belongs to $\mathbb{H}_r$. □

Proof of Theorem 1.4. We will use some ideas from [21]. Let $u_0, v_0 \in H^1_r(\mathbb{R}^3)$ be two ground state solutions respectively for equation (6) and equation (7). By Lemma 2.7, we know that there exists $\bar{t} > 0$ such that $(u_0(\cdot/\bar{t}), v_0(\cdot/\bar{t})) \in \mathcal{P}$. Then, to show that any radial ground state solution $(\bar{u},\bar{v})$ is a vector solution, it is sufficient to prove that, for β positive and sufficiently large,

$$I(u_0(\cdot/\bar{t}), v_0(\cdot/\bar{t})) < \min\{I(u_0,0), I(0,v_0)\}. \tag{26}$$

Indeed, with some calculations, we have

$$I(u_0(\cdot/\bar{t}), v_0(\cdot/\bar{t})) = \frac{\left(\frac{1}{3}\int_{\mathbb{R}^3} |\nabla u_0|^2 + |\nabla v_0|^2\right)^{3/2}}{\left(2\int_{\mathbb{R}^3} F(u_0) + G(v_0) + \frac{\beta}{2}u_0^2 v_0^2 - \frac{u_0^2}{2} - \frac{v_0^2}{2}\right)^{1/2}}.$$

Then, for β positive and sufficiently large, we have (26). □

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