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ATOMIC DECOMPOSITIONS AND OPERATORS ON HARDY SPACES

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Abstract. This paper is essentially the second author's lecture at the CIMPA–UNESCO Argentina School 2008, Real Analysis and its Applications. It summarises large parts of the three authors' paper [MSV]. Only one proof is given. In the setting of a Euclidean space, we consider operators defined and uniformly bounded on atoms of a Hardy space H^p. The question discussed is whether such an operator must be bounded on H^p. This leads to a study of the difference between countable and finite atomic decompositions in Hardy spaces.

1. Introduction and definitions

We start by introducing the Hardy spaces in $\mathbb{R}^n$. In this paper, we shall have $0 < p \leq 1$, except when otherwise explicitly stated. We write as usual

$$\|f\|_p = \left(\int |f(x)|^p \, \mathrm{d}x \right)^{1/p}$$

for $f \in L^p(\mathbb{R}^n)$. Observe that for $0 < p < 1$, this is only a quasinorm, in the sense that there is a factor $C = C(p) > 1$ in the right-hand side of the triangle inequality.

The Hardy spaces $H^p = H^p(\mathbb{R}^n)$ have several equivalent definitions. The oldest is for $n = 1$; the space $H^p(\mathbb{R})$ can be defined as the boundary values on the real axis of the real parts of those analytic functions F in the upper half-plane which satisfy

$$\sup_{y>0} \int |F(x + iy)|^p \, \mathrm{d}x < \infty.$$

The definition by means of maximal functions works in all dimensions, in the following way. Let $\varphi \in \mathcal{S}$ be any Schwartz function with $\int \varphi \neq 0$ and write $\varphi_t(x) = t^{-n}\varphi(x/t)$ for $t > 0$. The associated maximal operator M_φ is then defined by

$$M_\varphi f(x) = \sup_{t>0} |\varphi_t * f(x)|,$$

where f can be any distribution in $\mathcal{S}'$. To define the so-called grand maximal function, one takes the supremum of this expression when φ varies over a suitable

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subset of $\mathcal{S}$. More precisely, for a fixed, large natural number N we set

$$\mathcal{B}_N = \{\varphi \in \mathcal{S} : \sup_x (1+|x|)^N |\partial^\alpha \varphi(x)| \le 1 \ \text{ for } \ \alpha \in \mathbb{N}^n, \ \ |\alpha| \le N\}.$$

Then

$$\mathcal{M}f(x) = \sup_{\varphi \in \mathcal{B}_N} M_\varphi f(x).$$

Choose again a $\varphi \in \mathcal{S}$ such that $\int \varphi \neq 0$. Given f in $\mathcal{S}'$, one can prove that $M_\varphi f$ is in L^p if and only if $\mathcal{M}f$ is in L^p, and their L^p norms are equivalent. Here $N = N(n,p)$ must be taken large enough; actually $N > 1 + n/p$ always works. One now defines H^p as the space of distributions f in $\mathcal{S}'$ such that $\mathcal{M}f$ is in L^p. For further details, we refer to [St, Ch. III].

Example. The function $f = \mathbf{1}_B$, where B is the unit ball, is not in H^p for any $0 < p \le 1$. Indeed, $M_\varphi f$ decays like $|x|^{-n}$ at infinity. But if one splits B into halves, say B_+ and B_-, and considers instead $g = \mathbf{1}_{B_+} - \mathbf{1}_{B_-}$, the decay of $M_\varphi g$ will be better, and one verifies that g is in H^p for $n/(n+1) < p \le 1$. What matters here is the vanishing of the integral, i.e., the moment of order 0, of g. This is actually an example of (a multiple of) an atom.

Definition. Given $q \in [1,\infty]$ with $p < q$, a (p,q)-atom is a function $a \in L^q$ supported in a ball B, verifying $\|a\|_q \le |B|^{1/q-1/p}$ and having vanishing moments up to order $[n(1/p-1)]$, i.e.,

$$\int f(x)\, x^\alpha \,\mathrm{d}x = 0 \qquad \text{for} \quad \alpha \in \mathbb{N}^n, \quad |\alpha| \le [n(1/p-1)].$$

Observe that each (p,∞)-atom is a (p,q)-atom, and also that for $n/(n+1) < p \le 1$ only the moment of order 0 is required to vanish.

It is easy to verify that any (p,q)-atom is in H^p. More remarkably, this has a converse, if one passes to linear combinations of atoms. Indeed, let q be as above. Then a distribution $f \in \mathcal{S}'$ is in H^p if and only if it can be written as $f = \sum_{j=1}^\infty \lambda_j a_j$, where the a_j are (p,q)-atoms and $\sum_{j=1}^\infty |\lambda_j|^p < \infty$. The sum representing f here converges in $\mathcal{S}'$, and for $p = 1$ also in L^1.

To define the quasinorm $\|f\|_{H^p}$, one can use either of the three equivalent expressions

$$\|M_\varphi f\|_p, \quad \|\mathcal{M}f\|_p \quad \text{or} \quad \inf\Big\{\Big(\sum_{j=1}^\infty |\lambda_j|^p\Big)^{1/p} : \ f = \sum_{j=1}^\infty \lambda_j a_j, \ a_j \ (p,q){-}\text{atoms}\Big\}.$$

Different admissible choices of φ and q will lead to equivalent quasinorms. In the sequel, we shall choose the third expression here. If $p = 1$, this quasinorm is a norm, otherwise not.

In Section 2 we state the results. Section 3 contains a proof of one of the results in the case $0 < p < 1$.

In the sequel, C will denote several different positive constants. The open ball of centre x and radius R is written $B(x,R)$.

2. Statement of results

Hardy spaces are often used to state so-called endpoint results for L^p boundedness of operators. In particular, many linear and sublinear operators are bounded on L^p for $1 < p < \infty$ but not on L^1. However, they often turn out to be bounded from H^1 to L^1, and this is useful, since one can in many cases start by proving this last property and then obtain the L^p boundedness by interpolation.

To prove the boundedness from H^1 to L^1, or from H^p to L^p, of an operator, a common method is to take one atom at a time. It is usually not hard to verify that (p, q)-atoms are mapped into L^p, uniformly. From this one wants to conclude the boundedness from H^p to L^p, by summing over the atomic decomposition. We shall see that without extra assumptions this conclusion is correct in some cases, but not always.

With p and q as above, we consider a linear operator T, defined on the space $H^{p,q}_{\text{fin}}$ of all finite linear combinations of (p, q)-atoms. This space is endowed with the natural quasinorm

$$\|f\|_{H^{p,q}_{\text{fin}}} = \inf\Big\{\Big(\sum_{j=1}^{N} |\lambda_j|^p\Big)^{1/p} : \ f = \sum_{j=1}^{N} \lambda_j a_j. \quad a_j \ (p, q){-}\text{atoms}. \quad N \in \mathbb{N}\Big\}.$$

Notice that $H^{p,q}_{\text{fin}}$ coincides with the space of L^q functions with compact support and vanishing moments up to order $[n(1/p - 1)]$. It is dense in H^p.

For linear operators and $q < \infty$, the above conclusion about the boundedness of operators is always correct. More precisely, one has the following extension theorem.

Theorem 1. *Let $q \in [1, \infty)$ with $p < q$. Assume that $T : H^{p,q}_{\text{fin}} \to L^p$ is a linear operator such that*

$$\sup\{\|Ta\|_p : a \text{ is a } (p, q){-}\text{atom}\} < \infty.$$

Then T has a (necessarily unique) extension to a bounded linear operator from H^p to L^p.

The case $q = 2$ of this theorem was obtained independently by Yang and Zhou [YZ1]. Theorem 1 is an immediate consequence of the following quasinorm equivalence result and the density of $H^{p,q}_{\text{fin}}$ in H^p.

Proposition 2. *Let q be as in Theorem 1. The two quasinorms $\|\cdot\|_{H^{p,q}_{\text{fin}}}$ and $\|\cdot\|_{H^p}$ are equivalent on $H^{p,q}_{\text{fin}}$.*

It is a remarkable fact that this proposition does not hold for $q = \infty$. An explicit construction of a counterexample for $q = \infty$ and $p = 1$ can be found in Meyer, Taibleson and Weiss [MTW, p. 513]. See also García-Cuerva and Rubio de Francia [GR, III.8.3]. Bownik [B] used this contruction to show that Theorem 1 fails for $q = \infty$, if $p = 1$. He also extended the counterexample to the case $0 < p < 1$.

The construction in [MTW] is based on the discontinuities of the function considered. This is crucial; the two results above hold also in the "bad" case $q = \infty$, if only *continuous* atoms are considered. Indeed, let $H^{p,\infty}_{\text{fin,cont}}$ be the space of finite linear combinations of continuous (p,∞)-atoms, endowed with the quasinorm $\|f\|_{H^{p,\infty}_{\text{fin,cont}}}$ defined as the infimum of the ℓ^p quasinorm of the coefficients, taken over all such atomic decompositions. Then one has the following results.

Theorem 3. *Assume that $T : H^{p,\infty}_{\text{fin,cont}} \to L^p$ is a linear operator and*

$$\sup\{\|Ta\|_p : a \text{ is a continuous } (p,\infty)\text{−atom}\} < \infty.$$

Then T has a (necessarily unique) extension to a bounded linear operator from H^p to L^p.

Proposition 4. *The two quasinorms $\|\cdot\|_{H^{p,\infty}_{\text{fin,cont}}}$ and $\|\cdot\|_{H^p}$ are equivalent on $H^{p,\infty}_{\text{fin,cont}}$.*

Observe that if T is a linear operator defined on $H^{p,\infty}_{\text{fin}}$ and uniformly bounded on (p,∞)-atoms, then by Theorem 3 its restriction to $H^{p,\infty}_{\text{fin,cont}}$ has an extension $\tilde{T}$ to a bounded operator from H^p to L^p. But $\tilde{T}$ will in general not be an extension of T, since these two operators may differ on discontinuous atoms.

In [MSV], Theorem 1 was proved for $p = 1$ also in the setting of a space of homogeneous type with infinite measure. Grafakos, Liu and Yang [GLY] then dealt with a space of homogeneous type which locally has a well-defined dimension, and there proved Theorems 1 and 3 and the propositions, for p close to 1. See also Yang and Zhou [YZ2].

These results were extended to the weighted case by Bownik, Li, Yang and Zhou [BLYZ]. It should be pointed out that most of the papers mentioned consider operators from H^p into general (quasi-)Banach spaces rather than into L^p. Recently, Ricci and Verdera [RV] found a description of the completion of the space $H^{p,\infty}_{\text{fin}}$ and of its dual space, for $0 < p \leq 1$.

3. A proof

In [MSV], the proofs of the results stated above were given for the case $p = 1$; in the case $p < 1$, the proof of Proposition 2 was only sketched and that of Proposition 4 omitted. Therefore, we shall prove the case $p < 1$ of Proposition 4 here.

Assuming thus $p < 1$, we first observe that the estimate $\|f\|_{H^p} \leq \|f\|_{H^{p,\infty}_{\text{fin,cont}}}$ is obvious. For the converse inequality, let $f \in H^{p,\infty}_{\text{fin,cont}}$ be given. This means that f is a continuous function with compact support and vanishing moments up to order $[n(1/p-1)]$. We must find a finite atomic decomposition of f, using continuous atoms and with control of the coefficients.

As in the first part of the proof of [MSV, Theorem 3.1], we assume that the support of f is contained in a ball $B = B(0,R)$, and introduce the dyadic level sets of the grand maximal function $\Omega_k = \{x : \mathcal{M}f(x) > 2^k\}$, $k \in \mathbb{Z}$. Now f is bounded, and so is $\mathcal{M}f$, since the operator $\mathcal{M}$ is bounded on L^∞. Thus Ω_k will be empty if $k > \kappa$, for some $\kappa \in \mathbb{Z}$. Following [MSV], we cover each Ω_k, $k \leq \kappa$, with Whitney cubes Q^k_i, $i = 1, 2, \ldots$. We then invoke the proof of the atomic decomposition in

H^p given in [St, Theorem III.2, p. 107] or [Sj, Thm 3.5, p. 12]. This produces a countable decomposition

$$f = \sum_{k \le \kappa} \sum_{i=1}^{\infty} \lambda_i^k a_i^k,$$

where the a_i^k are (p, ∞)-atoms and $(\sum_k \sum_i |\lambda_i^k|^p)^{1/p} \le C\|f\|_{H^p}$. The sum converges in the distribution sense. Moreover, the a_i^k are supported in balls B_i^k concentric with the Q_i^k and contained in Ω_k. As pointed out in [MSV, p. 2924], the balls B_i^k, $i = 1, 2, \ldots$, will have bounded overlap, uniformly in k, if the Whitney cubes are chosen in a suitable way. Further, one will have the bound

$$|\lambda_i^k a_i^k| \le C\, 2^k \tag{1}$$

in the support of a_i^k. A few more properties can also be seen from the construction in [St] or [Sj]. In particular, the continuity of f will imply that of each a_i^k. Moreover, each term $\lambda_i^k a_i^k$ will depend only on the restriction of f to a ball $\tilde{B}_i^k = CB_i^k$, a concentric enlargement of B_i^k. It will also depend continuously on this restriction, in the sense that

$$\sup |\lambda_i^k a_i^k| \le C \sup_{\tilde{B}_i^k} |f|. \tag{2}$$

In this estimate, one can replace f by $f - c$ for any constant c; in particular a_i^k will vanish when f is constant in $\tilde{B}_i^k$.

We next consider $\mathcal{M}f$ in the set $\overline{B(0,2R)}^c$, which is far from the support of f.

Lemma 5. *Assume that $|x|, |y| > 2R$ and $1/2 < |x|/|y| < 2$. Then there exists a positive constant C depending only on N such that*

$$\frac{1}{C}\mathcal{M}f(y) \le \mathcal{M}f(x) \le C\mathcal{M}f(y).$$

Proof. We shall estimate

$$|\varphi_t * f(x)| = t^{-n} \left| \left\langle f, \varphi\left(\frac{x - \cdot}{t}\right) \right\rangle \right|, \tag{3}$$

where $\varphi \in \mathcal{B}_N$ and $t > 0$, with an analogous expression containing y instead of x. Here the brackets $\langle \cdot, \cdot \rangle$ denote the scalar product in L^2.

Consider first the case $t \ge |x|$. Defining $\psi(z) = \varphi(z + (x - y)/t)$, we get

$$\varphi\left(\frac{x - \cdot}{t}\right) = \psi\left(\frac{y - \cdot}{t}\right).$$

Since by our assumptions $|x - y| \le C|x|$ and so $|x - y|/t \le C$, we see that $\psi \in C\mathcal{B}_N$, i.e., $\psi/C \in \mathcal{B}_N$. Thus the expression in (3) is no larger than $C\mathcal{M}f(y)$.

In the case $t < |x|$, we choose a function $\beta \in C_0^\infty$ supported in the ball $B(0, 3/2)$ and such that $\beta = 1$ in $B(0, 1)$. Since $\operatorname{supp} f \subset B(0, R)$ and $|x| > 2R$, the expression in (3) will not change if we replace the right-hand side of the scalar product by

$$\varphi\left(\frac{x - \cdot}{t}\right) \beta\left(\frac{\cdot}{|x|/2}\right).$$

We therefore define

$$\tilde{\varphi}(z) = \varphi(z)\,\beta\left(\frac{x-tz}{|x|/2}\right) \tag{4}$$

and conclude that $\varphi_t * f(x) = t^{-n}\langle f, \tilde{\varphi}((x-\cdot)/t)\rangle$. Since $t/|x| < 1$, the last factor in (4) and all its derivatives with respect to z are bounded, uniformly in t and x. It follows that $\tilde{\varphi}$ is a Schwartz function and that $\tilde{\varphi} \in C\mathcal{B}_N$. A point z in the support of $\tilde{\varphi}$ must verify $x - tz \in B(0, 3|x|/4)$, so that $t|z| > |x|/4$. Thus $\operatorname{supp}\tilde{\varphi}$ is contained in the set $\{z : |z| \geq |x|/(4t)\}$. Define now $\psi(z) = \tilde{\varphi}(z + (x-y)/t)$. We claim that $\psi \in C\mathcal{B}_N$,which would make it possible to repeat the translation argument above and again dominate the expression in (3) by $C\mathcal{M}f(y)$.

This would end the proof of the lemma, since in both the cases considered we could conclude that $\mathcal{M}f(x) \leq C\mathcal{M}f(y)$. Symmetry then gives the converse inequality.

It thus only remains to verify the claim $\psi \in C\mathcal{B}_N$. This results from the next lemma, with $r = |x|/(4t)$ and $x_0 = (x-y)/t$.

Lemma 6. *If $\eta \in \mathcal{B}_N$ is supported in $B(0,r)^c$ for some $r > 0$ and $|x_0| < Ar$ with $A > 0$, then the translate $\eta(\cdot + x_0)$ is in $C\mathcal{B}_N$, for some $C = C(A,N)$.*

Proof. We must verify that

$$(1+|x|)^N|\partial^\alpha \eta(x+x_0)| \leq C \tag{5}$$

for $|\alpha| \leq N$. When $|x| > 2Ar$, we have $(1+|x|)^N \leq C(1+|x+x_0|)^N$, and (5) follows from the fact that $\eta \in \mathcal{B}_N$. For $|x| \leq 2Ar$, one can use the estimate $(1+r)^N|\partial^\alpha\eta(x)| \leq 1$, which is for all x a consequence of the assumed properties of η. This ends the proof of the two lemmata. □

Lemma 5 implies that $\mathcal{M}f(x) \leq CR^{-n/p}\|f\|_{H^p}$ for $|x| > 2R$. Indeed, if this were false for some x, the conclusion of the lemma would force $\mathcal{M}f$ to be large in the ring $\{y : |x| < |y| < 2|x|\}$, and this would contradict the fact that $\|\mathcal{M}f\|_p \leq C\|f\|_{H^p}$. It follows that $\Omega_k \subset \overline{B(0,2R)}$ if $k > k'$, for some k' satisfying $C^{-1}R^{-n/p}\|f\|_{H^p} \leq 2^{k'} \leq CR^{-n/p}\|f\|_{H^p}$.

Now define

$$h = \sum_{k\leq k'}\sum_i \lambda_i^k\, a_i^k \qquad \text{and} \qquad \ell = \sum_{k'<k\leq\kappa}\sum_i \lambda_i^k\, a_i^k, \tag{6}$$

so that $f = h + \ell$. Since both f and ℓ are supported in $\overline{B(0,2R)}$, so is h.

We first consider ℓ. Let $\varepsilon > 0$. The uniform continuity of f implies that there exists a $\delta > 0$ such that

$$|x-y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

We split ℓ as $\ell = \ell_1^\varepsilon + \ell_2^\varepsilon$, where ℓ_2^ε is that part of the sum defining ℓ involving only those atoms a_i^k for which $\operatorname{diam}\tilde{B}_i^k < \delta$. Notice that the remaining part ℓ_1^ε will then be a finite sum. Since the atoms are continuous, ℓ_1^ε will be a continuous function. The estimate (2), with f replaced by $f - f(y)$ for some point $y \in \tilde{B}_i^k$, implies that each term occurring in the sum defining ℓ_2^ε satisfies $|\lambda_i^k\, a_i^k| < C\varepsilon$. As i varies, the supports of the a_i^k have bounded overlap, and the sum defining ℓ_2^ε involves only

$\kappa - k'$ values of k. It follows that $|\ell_2^\varepsilon| \le C(\kappa - k')\varepsilon$. This means that one can write ℓ as the sum of one continuous term and one which is uniformly arbitrarily small. Hence, ℓ is continuous, and so is $h = f - \ell$.

To find the required finite atomic decomposition of f, we use again the splitting $\ell = \ell_1^\varepsilon + \ell_2^\varepsilon$. The part ℓ_1^ε is a finite sum of multiples of (p, ∞)-atoms, and the ℓ^p quasinorm of the corresponding coefficients is no larger than that of all the coefficients λ_i^k, which is controlled by $\|f\|_{H^p}$. The part ℓ_2^ε is supported in $\overline{B(0, 2R)}$. It satisfies the moment conditions, since it is given by a sum which converges in $\mathcal{S}'$ and whose terms are supported in a fixed ball and verify the moment conditions. By choosing ε small, we can make its L^∞ norm small and thus make $\ell_2^\varepsilon/\|f\|_{H^p}$ into a (p, ∞)-atom.

As for h, we know that it is a continuous function supported in $\overline{B(0, 2R)}$. Since $h = f - \ell_1^\varepsilon - \ell_2^\varepsilon$, we see that it also satisfies the moment conditions. The bound (1) and the bounded overlap for varying i imply that

$$|h| \le C \sum_{k \le k'} 2^k \le C 2^{k'} \le C R^{-n/p} \|f\|_{H^p}$$

at all points. This means that h is a multiple of a (p, ∞)-atom, with a coefficient no larger than $C\|f\|_{H^p}$. Together with the splitting of ℓ just discussed, this gives the desired finite atomic decomposition of f, and the norm of f in $H^{p,\infty}_{\text{fin,cont}}$ is controlled by $C\|f\|_{H^p}$. The proof is complete.

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