Uniqueness and Least Energy Property for Solutions to Strongly Competing Systems *

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Abstract

For the reaction–diffusion system of three competing species:

$$-\Delta u_i = -\kappa u_i \sum_{j\neq i} u_j, \qquad i = 1, 2, 3,$$

we prove uniqueness of the limiting configuration as $\kappa \to \infty$ on a planar domain Ω, with appropriate boundary conditions. Moreover we prove that the limiting configuration minimizes the energy associated to the system

$$E(U) = \sum_{i=1}^{3} \int_\Omega |\nabla u_i(\mathbf{x})|^2 \, d\mathbf{x}$$

among all segregated states ($u_i \cdot u_j = 0$ a.e.) with the same boundary conditions.

1 Introduction

Spatial segregation may occur in population dynamics when two or more species interact in a highly competitive way. A wide literature is devoted to this topic, mainly for the case of competition models of Lotka–Volterra type (see e.g. [1, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]). As a prototype for the study of this phenomenon, in [4] we consider the competition–diffusion system of k differential equations:

$$-\Delta u_i = -\kappa u_i \sum_{j\neq i} u_j, \qquad u_i > 0 \text{ in } \Omega,\ u_i = \varphi_i \text{ on } \partial\Omega, \qquad i = 1, \dots, k. \tag{1}$$

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Here $\Omega \subset \mathbb{R}^N$ is a bounded regular domain, and $(\varphi_1, \ldots, \varphi_k)$ is a given boundary datum (regular, non negative, and satisfying $\varphi_i \cdot \varphi_j \equiv 0$ for $i \neq j$). This system describes the stationary states of the evolution of k species diffusing and competing for resources. The internal dynamics of the populations and the diffusion coefficients are trivialized (although a wide class of internal dynamics and diffusion coefficients could be considered, without providing substantial changes to the qualitative behaviour of the model; see [2, 3, 6]), while the attention is pointed on the coefficient κ, the rate of mutual competition. As a matter of fact, it can be shown that the large interaction induces the spatial segregation of the species in the limit configuration, as $\kappa \to \infty$. Precisely, the following result has been proved by the authors in [4]:

Theorem 1.1 *The system* (1) *admits (at least) a solution* $(u_{1,\kappa}, \ldots, u_{n,\kappa}) \in (H^1(\Omega))^k$ *for every* $\kappa > 0$. *Moreover there exists* $(\bar{u}_1, \ldots, \bar{u}_k) \in (H^1(\Omega))^k$ *such that* $\bar{u}_i \cdot \bar{u}_j = 0$ *for* $i \neq j$ *and, up to subsequences,*

$$u_{i,\kappa} \to \bar{u}_i \quad \text{in } H^1, \text{ for every } i.$$

Not only the limiting configuration exhibits segregation, but also the differential structure of the model passes to the limit in the form of a system of distributional inequalities. We collect these properties introducing the functional class

$$\mathcal{S} = \left\{ U = (u_1, \ldots, u_k) \in (H^1(\Omega))^k : \begin{array}{l} u_i \geq 0,\ u_i \cdot u_j = 0 \text{ if } i \neq j, \text{ in } \Omega \\ u_i = \varphi_i \text{ on } \partial\Omega \\ -\Delta u_i \leq 0,\ -\Delta \left(u_i - \sum_{j \neq i} u_j \right) \geq 0 \end{array} \right\}.$$

In fact, we have

$$(\bar{u}_1, \ldots, \bar{u}_k) \in \mathcal{S}.$$

Thus the study of $\mathcal{S}$ provides the understanding of the segregated states induced by strong competition. In this direction, a number of regularity properties, both of the densities and of the mutual interfaces, were obtained by the authors in [2, 3, 4, 5, 6].

On the other hand, in [6] we studied the minimal energy configurations in the class of all the possible segregated states. Precisely, let us define the energy of a k–tuple of densities as

$$E(U) = \sum_{i=1}^{k} \int_\Omega |\nabla u_i(\mathbf{x})|^2 \, d\mathbf{x}.$$

Then in [6] we proved the following:

Theorem 1.2 *The problem*

$$\min \left\{ E(U) :\ u_i \in H^1(\Omega),\ u_i|_{\partial\Omega} = \varphi_i,\ u_i \geq 0,\ u_i \cdot u_j = 0 \text{ if } i \neq j \right\}. \tag{2}$$

admits a solution. In addition

(a) the minimum is unique;

(b) the minimum belongs to $\mathcal{S}$*;*

(c) the minimum depends H^1*–continuously from the boundary data, endowed with the* $H^{1/2}$ *norm.*

In particular, this result shows that the unique minimal energy configuration shares with the limiting states of system (1) the common property of belonging to $\mathcal{S}$. In the case of two populations, we can say much more, indeed we know the explicit solution of both the problems. Setting Φ the harmonic extension of $\varphi_1 - \varphi_2$ on Ω, it is easy to see that the pair (Φ^+, Φ^-) achieves (2), while in [4] we proved that it is the limit configuration of *any* sequence of pairs $(u_{1,\kappa}, u_{2,\kappa})$ as $\kappa \to \infty$. As a consequence, when $k = 2$, the class $\mathcal{S}$ consists in exactly one element, the minimal one. One may wonder if this result can be extended to the case of three or more densities, case in which no explicit solution is provided. Even without uniqueness,

Problem: *is the minimal energy configuration the limiting state for the corresponding competitive system?*

When $k \geq 3$, the answer is not obvious: it is worthwhile noticing that while problem (2) has an evident variational structure, the reaction–diffusion system (1) is not variational at all (the nonlinear part is not of gradient–type). Nevertheless, the present paper provides a partial positive answer to this question: indeed we prove that, for 3 populations in the plane, the only element of $\mathcal{S}$ is the minimizer of the energy. The main result we give is:

Theorem 1.3 *Let $k = 3$, and Ω be a simply connected domain in $\mathbb{R}^2$. Then, for every admissible datum $(\varphi_1, \varphi_2, \varphi_3)$, $\mathcal{S}$ consists in exactly one element.*

This theorem, together with the results contained in [4, 6], immediately provides:

Theorem 1.4 *Let Ω be a simply connected domain in $\mathbb{R}^2$, $(u_{1,\kappa}, u_{2,\kappa}, u_{3,\kappa})$ be any solution of* (1) *and $(\bar{u}_1, \bar{u}_2, \bar{u}_3)$ be the minimizer of* (2). *Then, for every $\alpha \in (0, 1)$,*

$$\text{the whole sequence } u_{i,\kappa} \text{ tends to } \bar{u}_i \text{ in } H^1 \cap C^{0,\alpha} \text{ as } \kappa \to \infty.$$

As we already observed, this is a remarkable fact, since it shows a deep connection between the variational problem (2) and the non variational system (1).

2 Basic facts and notation

Due to the conformal invariance the problem, with no loss of generality we take

$$\Omega = B = \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| < 1\}.$$

Throughout the paper we will assume that:

- i, j, h denote integers between 1 and 3.
- $(\varphi_1, \varphi_2, \varphi_3) \in (W^{1,\infty}(\partial B))^3$ (an *admissible boundary datum*) is such that $\varphi_i \geq 0$, for every i, and $\varphi_i \cdot \varphi_j = 0$ on ∂B, for $i \neq j$. The sets $\{\varphi_i > 0\}$ are open connected arcs, and the function $\sum \varphi_i$ vanishes in exactly 3 points of ∂B (the endpoints of the supports).

With the above notation, we define the class $\mathcal{S}$ of the *segregated densities* as

$$\mathcal{S} = \left\{ U = (u_1, u_2, u_3) \in (H^1(B))^3 : \begin{array}{l} u_i \geq 0,\ u_i \cdot u_j = 0 \text{ if } i \neq j, \text{ in } B \\ u_i = \varphi_i \text{ on } \partial B \\ -\Delta u_i \leq 0,\ -\Delta \widehat{u}_i \geq 0 \end{array} \right\}, \tag{3}$$

where the *hat operator* is defined on the generic component of a triple as

$$\widehat{u}_i = u_i - \sum_{j \neq i} u_j. \tag{4}$$

In the following, with some abuse of notation, U will denote both the generic triple (u_1, u_2, u_3) in $\mathcal{S}$ and the function $\sum u_i$ in $H^1(B)$.

For any $U \in \mathcal{S}$ we define the sets (the "*supports*")

$$\omega_i = \{\mathbf{x} \in B : u_i(\mathbf{x}) > 0\}.$$

The *multiplicity* of a point $\mathbf{x} \in \overline{B}$ (with respect to U) is

$$m(\mathbf{x}) = \sharp \{i : \text{measure}\,(\omega_i \cap B(\mathbf{x}, r)) > 0 \ \forall r > 0\}\,.$$

The *interfaces* between two densities are defined as

$$\Gamma_{ij} = \partial\omega_i \cap \partial\omega_j \cap \{\mathbf{x} \in B : m(\mathbf{x}) = 2\},$$

in such a way that $\omega_i \cup \omega_j \cup \Gamma_{ij} = B \setminus \overline{\omega_h}$. The supports ω_i and ω_j are said to be *adjacent* if Γ_{ij} is not empty.

Below we list the principal properties of the elements of $\mathcal{S}$. We refer to [3, 6] for their proof, and for further details.

Theorem 2.1 *Let $U \in \mathcal{S}$.*

(a) $U \in W^{1,\infty}(\overline{B})$. As a consequence, every ω_h is open and $\mathbf{x} \in \omega_h$ implies $m(\mathbf{x}) = 1$.

(b) u_i is harmonic in ω_i, $u_i - u_j$ is harmonic on $B \setminus \overline{\omega_h}$ (with $h \neq i, j$). In particular

$$\lim_{\substack{\mathbf{y}\to\mathbf{x} \\ \mathbf{y}\in\omega_i}} \nabla u_i(\mathbf{y}) = - \lim_{\substack{\mathbf{y}\to\mathbf{x} \\ \mathbf{y}\in\omega_j}} \nabla u_j(\mathbf{y}) \neq 0.$$

(c) For every $\mathbf{x} \in \overline{B}$ we have $1 \le m(\mathbf{x}) \le 3$, and $m(\mathbf{x}) = 3$ for a finite number of points.

(d) Each Γ_{ij} is (either empty or) a connected arc, locally C^1, with endpoints either on ∂B or points with multiplicity 3.

(e) If $m(\mathbf{x}_0) = 3$, then $|\nabla U(\mathbf{x})| \to 0$ as $\mathbf{x} \to \mathbf{x}_0$. More precisely, we have the following asymptotic estimate:

$$U(r, \vartheta) = Cr^{3/2} \left| \cos\left(\frac{3}{2}\vartheta + \vartheta_0\right) \right| + o(r^{3/2})$$

(here (r, ϑ) denotes a system of polar coordinates around $\mathbf{x}_0$).

Remark 2.1 *Every ω_i is (pathwise) connected.* Indeed, let $\omega_i = \alpha \cup \beta$, with α and β disjoint, open, and non empty. Recall that u_i is continuous on $\overline{B}$, hence it vanishes (continuously) on $\partial\omega_i \setminus \{\varphi_i > 0\}$. Since $\{\varphi_i > 0\}$ is connected, it can not intersect both $\partial\alpha$ and $\partial\beta$ (recall that u_i is strictly positive on this set). We infer that u_i vanishes, for instance, on $\partial\beta$. But $u_i \in C(\overline{\beta})$, and it is harmonic on β. The classical maximum principle implies $u_i \equiv 0$ in β, a contradiction.

We recall that, by Theorem 1.2, $\mathcal{S}$ possesses at least one element. In the next section we prove that it is unique.

3 Uniqueness results

To start with, we prove a topological result, stating that every triple in $\mathcal{S}$ has exactly one triple point.

Lemma 3.1 *For every $U \in \mathcal{S}$ there exists exactly one point $\mathbf{a}_U \in \overline{B}$ such that $m(\mathbf{a}_U) = 3$.*

Proof: this is an easy consequence of the fact that, if $m(\mathbf{a}) = 3$, then any neighborhood of $\mathbf{a}$ contains points of every ω_i, and hence every non empty Γ_{ij} satisfies $\overline{\Gamma_{ij}} \ni \mathbf{a}$. But every Γ_{ij} is connected and starts from ∂B. ■

3.1 Uniqueness when the triple point is on the boundary

The simplest situation is when $\mathbf{a}_U$ belongs to ∂B. In this case, one Γ_{ij} is empty, and $\widehat{u}_h$ (for $h \neq i, j$) is harmonic on B.

Proposition 3.1 *Let $U, V \in \mathcal{S}$ with $a_U \in \partial B$. Then $V \equiv U$.*

Proof: the assumption implies that a_U is the common endpoint of the supports of two data, say φ_1 and φ_3, and, as a consequence, that Γ_{13} is empty. Now, $\widehat{u}_2$ is $C(\overline{B})$ and, by Theorem 2.1,(b), it is harmonic both on $B \setminus \overline{\omega_3}$ and on $B \setminus \overline{\omega_1}$. Being $\overline{\omega_1} \cap \overline{\omega_3} = \mathbf{a}_U$ we deduce that $\widehat{u}_2$ is harmonic on B. We are going to prove that $\mathbf{a}_V \equiv \mathbf{a}_U$. This will conclude the proof, indeed it will imply that also $\widehat{v}_2$ is harmonic, with the same boundary data, and thus $\widehat{u}_2 \equiv \widehat{v}_2$; but they have exactly three nodal regions, therefore they correspond to the same triple in $\mathcal{S}$.

Assume by contradiction that $\mathbf{a}_V \neq \mathbf{a}_U$. Then $\mathbf{a}_U$ is a point of multiplicity 2 for V, belonging to the common boundary of $\{v_1 > 0\}$ and $\{v_3 > 0\}$. As a consequence, we can find a neighborhood $\mathcal{N}$ of $\mathbf{a}_U$ such that v_2 vanishes on $\mathcal{N}$. On the other hand, by definition of multiplicity, $\{u_2 > 0\}$ intersects $\mathcal{N}$. We infer the existence of $\bar{\mathbf{x}} \in \mathcal{N}$ such that $u_2(\bar{\mathbf{x}}) > 0$ and $v_1(\bar{\mathbf{x}}) + v_3(\bar{\mathbf{x}}) > 0$. Now, we have

$$\begin{cases} -\Delta \widehat{v}_2 \geq 0 & \text{in } B \\ \widehat{v}_2 = \widehat{\varphi}_2 & \text{on } \partial B \end{cases} \qquad \text{and} \qquad \begin{cases} -\Delta \widehat{u}_2 = 0 & \text{in } B \\ \widehat{u}_2 = \widehat{\varphi}_2 & \text{on } \partial B. \end{cases}$$

It follows that $\widehat{v}_2 - \widehat{u}_2$ is superharmonic on B and (continuously) zero on ∂B, and then it is non negative in B. But

$$(\widehat{v}_2 - \widehat{u}_2)(\bar{\mathbf{x}}) = -v_1(\bar{\mathbf{x}}) - v_3(\bar{\mathbf{x}}) - u_2(\bar{\mathbf{x}}) < 0,$$

a contradiction. ■

It remains to prove the uniqueness of the element when its triple point is in the interior of B. To this aim we are not able to proceed directly as in the previous arguments. We will start providing a sort of local uniqueness.

3.2 Interior triple point: local uniqueness

Let $U \in \mathcal{S}$ be given, with trace $(\varphi_1, \varphi_2, \varphi_3)$ and triple point $\mathbf{a} = \mathbf{a}_U$. We want to provide a local dependence between the trace and the triple point. The key point is that, if we know the triple point, we can construct an harmonic function strictly related to U: roughly speaking, the idea is to move the triple point of U to the origin via a Moebius transformation, and then to double the angle in order to obtain an even number of nodal region (compatible with an alternate sign rule). We introduce the transformation (using the complex notation: the reader will easily distinguish the index i and the imaginary unit $\imath$, that appears, by the way, only at exponent)

$$T_a : \overline{B} \longrightarrow \overline{B}, \qquad T_{\mathbf{a}}(\mathbf{z}) = \frac{\mathbf{z} + \mathbf{a}}{\bar{\mathbf{a}}\mathbf{z} + 1}.$$

It is well known that $T_{\mathbf{a}}$ is a conformal map, such that $T_{\mathbf{a}}(\partial B) = \partial B$ and $T_{\mathbf{a}}(\mathbf{0}) = \mathbf{a}$. Also the map $\mathbf{z} \to \mathbf{z}^2$ is conformal. We obtain that, if $r = |\mathbf{z}|$ and $\vartheta = \arg \mathbf{z}$, the map $T_{\mathbf{a}}(\mathbf{z}^2)$ given in coordinates by

$$\mathbf{z}=(r,\vartheta)\to(r^2,2\vartheta)\to\left(\Re\frac{r^2e^{2\imath\vartheta}+\mathbf{a}}{\bar{\mathbf{a}}r^2e^{2\imath\vartheta}+1},\Im\frac{r^2e^{2\imath\vartheta}+\mathbf{a}}{\bar{\mathbf{a}}r^2e^{2\imath\vartheta}+1}\right)=(x_1,x_2)=\mathbf{x} \tag{5}$$

is a conformal mapping. For every ω_i, the set $\{\mathbf{z}:\mathbf{z}^2\in\omega_i\}$ is made up of two open connected components, symmetric with respect to $\mathbf{0}$. We want to define a new harmonic function, having opposite sign on the two components, for every i. We set[1]

$$\begin{aligned} w(\mathbf{z}) &= \sigma(\mathbf{z})u_i(T_{\mathbf{a}}(\mathbf{z}^2)) \qquad \text{if } \mathbf{z}\in\overline{B},\ T_{\mathbf{a}}(\mathbf{z}^2)\in\overline{\omega_i} \\ &= \sum_{i=1}^{3}\sigma(\mathbf{z})u_i(T_{\mathbf{a}}(\mathbf{z}^2)) \qquad \text{if } \mathbf{z}\in\overline{B} \end{aligned} \tag{6}$$

where σ is ±1 in such a way that w has alternate sign on the adjacent nodal regions. Then w has 6 nodal regions, it is of class C^1 (by Theorem 2.1(b) and (e))) and

$$w(-\mathbf{z})=-w(\mathbf{z}).$$

Theorem 2.1 also implies that w is harmonic. We obtain that

$$\begin{cases}\Delta w=0 & \text{in } B\\ w=\gamma_{\mathbf{a}} & \text{on } \partial B\end{cases} \tag{7}$$

where

$$\gamma_{\mathbf{a}}(\mathbf{z})=\sum_{i=1}^{3}\sigma(\vartheta)\varphi_i\left(\frac{e^{2\imath\vartheta}+\mathbf{a}}{\bar{\mathbf{a}}e^{2\imath\vartheta}+1}\right) \tag{8}$$

Clearly, also $\gamma_{\mathbf{a}}(-\mathbf{z})=-\gamma_{\mathbf{a}}(\mathbf{z})$. Observe that, given $(\varphi_1,\varphi_2,\varphi_3)$ and $\mathbf{a}$, (7) defines an unique w. With standard calculation we obtain

$$\begin{aligned}\arg\left(\frac{e^{2\imath\vartheta}+\mathbf{a}}{\bar{\mathbf{a}}e^{2\imath\vartheta}+1}\right) &= \arg\left(\frac{e^{2\imath\vartheta}+\mathbf{a}}{\bar{\mathbf{a}}e^{2\imath\vartheta}+1}\cdot\frac{e^{-2\imath\vartheta}\left(e^{2\imath\vartheta}+\mathbf{a}\right)}{\mathbf{a}e^{-2\imath\vartheta}+1}\right)= \\ &= \arg\left(e^{-2\imath\vartheta}\left(e^{2\imath\vartheta}+\mathbf{a}\right)^2\right)=-2\vartheta+2\arg\left(e^{2\imath\vartheta}+\mathbf{a}\right)\end{aligned}$$

Thus, if we set

$$\Theta_{\mathbf{a}}(\vartheta)=2\arg\left(e^{2\imath\vartheta}+\mathbf{a}\right)-2\vartheta \tag{9}$$

we can write, with the usual abuse of notation,

$$\gamma_{\mathbf{a}}(\vartheta)=\sum_{i=1}^{3}\sigma(\vartheta)\varphi_i\left(\Theta_{\mathbf{a}}(\vartheta)\right). \tag{10}$$

Remark 3.1 *Let $U,V\in\mathcal{S}$ be such that $\mathbf{a}_U=\mathbf{a}_V$. Then $U\equiv V$.* Indeed, two different triple with the same triple point should generate two different w in (7) with the same boundary condition.

The above construction allows to know wether a point in B can be the triple point of a segregated state or not. We have

Lemma 3.2 *Let $(\varphi_1,\varphi_2,\varphi_3)$ be an admissible boundary datum, and $\mathbf{a}\in B$. $\mathbf{a}$ is the triple point of an element of $\mathcal{S}$ (with datum φ_i) if and only if*

$$\nabla w(\mathbf{0})=\mathbf{0}, \tag{11}$$

where w is defined by (7), (8).

[1]We will keep writing $u_i(\mathbf{x})=u_i(re^{\imath\vartheta})=u_i(r,\vartheta)$, $\varphi_i(\mathbf{x})=\varphi_i(\vartheta)$, and so on.

Proof: if $\mathbf{a} = \mathbf{a}_U$ we have that w satisfies (6). By conformality and Theorem 2.1,(e), we obtain $\nabla w(\mathbf{0}) = \mathbf{0}$.

On the other hand, let w be defined by (7) and let $\nabla w(\mathbf{0}) = \mathbf{0}$. We can write the Fourier expansion of $\gamma_{\mathbf{a}}$

$$\gamma_{\mathbf{a}}(\vartheta) = \frac{A_0}{2} + \sum_{n=1}^{+\infty} \left[A_n \cos n\vartheta + B_n \sin n\vartheta\right],$$

and, since $\gamma_{\mathbf{a}}$ is odd (by (8)), we immediately obtain $A_{2m} = B_{2m} = 0$. By standard separation of variables we infer $w = \sum_{m=0}^{+\infty} \left[A_{2m+1} \cos\left((2m+1)\vartheta\right) + B_{2m+1} \sin\left((2m+1)\vartheta\right)\right] r^{2m+1}$. Finally, by (11), we obtain $A_1 = B_1 = 0$ and

$$w(r, \vartheta) = \sum_{n=1}^{+\infty} \left[A_{2n+1} \cos\left((2n+1)\vartheta\right) + B_{2n+1} \sin\left((2n+1)\vartheta\right)\right] r^{2n+1}.$$

Moreover, we have that

$$A_3^2 + B_3^2 \neq 0; \tag{12}$$

indeed, if not there would be $2k$ arcs (where k is the index of the first non zero Fourier component), starting from $\mathbf{0}$, on which w vanishes. Since an harmonic function can not admit closed level lines, this contradicts the fact that $\gamma_{\mathbf{a}}$ has exactly six zeroes (remember (8), and the fact that $(\varphi_1, \varphi_2, \varphi_3)$ is an admissible datum). Now, w is odd, so $|w|$ is even. Therefore we can invert the conformal map (5) on the half ball, obtaining a non negative function with exactly three nodal region. It is not difficult, now, to prove that this function generates an element of $\mathcal{S}$, with datum (φ_i) and triple point $\mathbf{a}$. ■

Now that we have characterized, for a given datum, the possible triple points, we can state the local dependence of these points from the data.

Proposition 3.2 *Let $(\varphi_1, \varphi_2, \varphi_3)$ be an admissible boundary datum, and $\mathbf{a}_\varphi \in B$, in such a way that* (11) *holds. Then there exist $\varepsilon > 0$, $\delta > 0$ such that, for every (ψ_1, ψ_2, ψ_3) admissible datum with $\|\varphi_i - \psi_i\|_{W^{1,\infty}} < \varepsilon$ there exists exactly one $\mathbf{a}_\psi$ satisfying* (11) *with datum (ψ_i) and such that $|\mathbf{a}_\psi - \mathbf{a}_\varphi| < \delta$.*

Proof: without loss of generality (using the continuity of the fixed transformation $T_{\mathbf{a}_\varphi}$) we can assume that $\mathbf{a}_\varphi \equiv \mathbf{0}$.

We want to apply the implicit function theorem to the map

$$\begin{array}{ccc} \left(W^{1,\infty}(B)\right)^3 \times B & \longrightarrow & \mathbb{R}^2 \\ (\varphi_1, \varphi_2, \varphi_3, \mathbf{a}) & \longmapsto & \nabla w(\mathbf{0}) \end{array}$$

in order to locally solve equation (11) for $\mathbf{a}$ (recall that the dependence of w from $(\varphi_1, \varphi_2, \varphi_3)$ and $\mathbf{a}$ is given by (7), (8)). To this aim, the only non trivial thing to show is that

$$\text{the } 2 \times 2 \text{ jacobian matrix} \qquad \partial_{(a_1, a_2)} \nabla w(\mathbf{0})\big|_{\mathbf{a}=\mathbf{0}} \qquad \text{is invertible.}$$

Using the Poisson's formula we can write

$$w(\mathbf{x}) = \frac{1 - |\mathbf{x}|^2}{2\pi} \int_{\partial B} \frac{\gamma_{\mathbf{a}}(\mathbf{y})}{|\mathbf{y} - \mathbf{x}|^2}\, d_{\mathbf{y}} s$$

that implies

$$\nabla w(\mathbf{x}) = -\frac{\mathbf{x}}{\pi} \int_{\partial B} \frac{\gamma_{\mathbf{a}}(\mathbf{y})}{|\mathbf{y} - \mathbf{x}|^2}\, d_{\mathbf{y}} s + \frac{1 - |\mathbf{x}|^2}{\pi} \int_{\partial B} \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{y} - \mathbf{x}|^4}\, \gamma_{\mathbf{a}}(\mathbf{y})\, d_{\mathbf{y}} s$$

and

$$\nabla w(\mathbf{0}) = \frac{1}{\pi} \int_{\partial B} \mathbf{y}\, \gamma_{\mathbf{a}}(\mathbf{y})\, d_{\mathbf{y}} s.$$

We choose the parametrization $\mathbf{y} = (\cos\vartheta, \sin\vartheta)$. Taking into account (10) we obtain

$$\nabla w(\mathbf{0}) = \frac{1}{\pi} \left(\int_0^{2\pi} \sum_{i=1}^{3} \sigma(\vartheta)\varphi_i \left(\Theta_{\mathbf{a}}(\vartheta)\right) \cos\vartheta \, d\vartheta \, , \int_0^{2\pi} \sum_{i=1}^{3} \sigma(\vartheta)\varphi_i \left(\Theta_{\mathbf{a}}(\vartheta)\right) \sin\vartheta \, d\vartheta \right).$$

Now, differentiating (9) we infer that

$$\nabla_{\mathbf{a}} \Theta_{\mathbf{a}}(\vartheta)\big|_{\mathbf{a}=\mathbf{0}} = \left(-2\sin 2\vartheta, 2\cos 2\vartheta\right).$$

Since $\Theta_{\mathbf{0}}(\vartheta) = 2\vartheta$, we obtain that $\partial_{(a_1,a_2)} \nabla w(\mathbf{0})\big|_{\mathbf{a}=\mathbf{0}}$ is equal to

$$\frac{2}{\pi} \begin{pmatrix} -\displaystyle\int_0^{2\pi} \sum_{i=1}^{3} \sigma(\vartheta)\varphi_i' \left(2\vartheta\right) \sin 2\vartheta \cos\vartheta \, d\vartheta & \displaystyle\int_0^{2\pi} \sum_{i=1}^{3} \sigma(\vartheta)\varphi_i' \left(2\vartheta\right) \cos 2\vartheta \cos\vartheta \, d\vartheta \\ -\displaystyle\int_0^{2\pi} \sum_{i=1}^{3} \sigma(\vartheta)\varphi_i' \left(2\vartheta\right) \sin 2\vartheta \sin\vartheta \, d\vartheta & \displaystyle\int_0^{2\pi} \sum_{i=1}^{3} \sigma(\vartheta)\varphi_i' \left(2\vartheta\right) \cos 2\vartheta \sin\vartheta \, d\vartheta \end{pmatrix}. \tag{13}$$

Let us compute (10) and the Fourier expansion in the proof of Lemma 3.2 when $\mathbf{a} = 0$. We have

$$\sum_{i=1}^{3} \sigma(\vartheta)\varphi_i \left(2\vartheta\right) = \gamma_{\mathbf{0}}(\vartheta) = \sum_{n=1}^{+\infty} \left[A_{2n+1} \cos\left((2n+1)\vartheta\right) + B_{2n+1} \sin\left((2n+1)\vartheta\right)\right]$$

that implies

$$\sum_{i=1}^{3} \sigma(\vartheta)\varphi_i' \left(2\vartheta\right) = \frac{1}{2} \sum_{n=1}^{+\infty} (2n+1) \left[-A_{2n+1} \sin\left((2n+1)\vartheta\right) + B_{2n+1} \cos\left((2n+1)\vartheta\right)\right].$$

This (and Werner formulas) allows to compute the first term of (13):

$$\int_0^{2\pi} \sum_{i=1}^{3} \sigma(\vartheta)\varphi_i' \left(2\vartheta\right) \sin 2\vartheta \cos\vartheta \, d\vartheta =$$

$$= \frac{1}{4} \int_0^{2\pi} \sum_{n=1}^{+\infty} (2n+1) \left[-A_{2n+1} \sin\left(2n+1\right)\vartheta + B_{2n+1} \cos\left(2n+1\right)\vartheta\right] \left[\sin 3\vartheta - \sin\vartheta\right] \, d\vartheta =$$

$$= \frac{1}{4} \int_0^{2\pi} -3A_3 \sin^2 3\vartheta \, d\vartheta = -\frac{3\pi}{4} A_3.$$

Analogous calculations provide

$$\partial_{(a_1,a_2)} \nabla w(\mathbf{0})\big|_{\mathbf{a}=\mathbf{0}} = \frac{3}{2} \begin{pmatrix} A_3 & B_3 \\ -B_3 & A_3 \end{pmatrix}.$$

But we know (see (12)) that $A_3^2 + B_3^2 \neq 0$. Therefore we have that the jacobian matrix is invertible, concluding the proof. ∎

3.3 Interior triple point: global uniqueness

Proof of Theorem 1.3: by Theorem 1.2 we know that at least one element M in $\mathcal{S}$ exists. Assume by contradiction that there exists another element $U \in \mathcal{S}$ with $U \neq M$ (that is, U is not the minimal one). By Proposition 3.1 we have that $\mathbf{a}_U \in B$. Again, without loss of generality (using the transformation $T_{\mathbf{a}_U}$) we can assume that $\mathbf{a}_U \equiv \mathbf{0}$. For $r > 0$ we define

$$U_t(\mathbf{x}) = \frac{1}{t^{3/2}} U(t\mathbf{x}),$$

and we observe that

$$U_t \in \mathcal{S}_t := \left\{ V = (v_1, v_2, v_3) : \begin{array}{l} v_i \geq 0,\ v_i \cdot v_j = 0,\ -\Delta v_i \leq 0,\ -\Delta \widehat{v}_i \geq 0 \\ v_i(\mathbf{x}) = \dfrac{1}{t^{3/2}} u_i(t\mathbf{x}) \text{ for } |\mathbf{x}| = 1 \end{array} \right\}.$$

Now, it is possible to show that U_t has limit as $t \to 0$. Indeed, by Theorem 2.1,(e) we can write

$$U(r, \vartheta) = C r^{3/2} \left| \cos\left(\frac{3}{2}\vartheta + \vartheta_0\right) \right| + o(r^{3/2}) \qquad \text{as } r \to 0,$$

and hence

$$U_t(r, \vartheta) = \frac{1}{t^{3/2}} U(tr, \vartheta) = C r^{3/2} \left| \cos\left(\frac{3}{2}\vartheta + \vartheta_0\right) \right| + \frac{o(t^{3/2} r^{3/2})}{t^{3/2}}$$

tends to

$$U_0(r, \vartheta) = C r^{3/2} \left| \cos\left(\frac{3}{2}\vartheta + \vartheta_0\right) \right|.$$

Again U_0 belongs to

$$\mathcal{S}_0 := \left\{ V = (v_1, v_2, v_3) : \begin{array}{l} v_i \geq 0,\ v_i \cdot v_j = 0,\ -\Delta v_i \leq 0,\ -\Delta \widehat{v}_i \geq 0 \\ v_i(1, \vartheta) = \left| \cos\left(\dfrac{3}{2}\vartheta + \vartheta_0\right) \right|,\ \dfrac{2i}{3}\pi \leq \vartheta \leq \dfrac{2(i+1)}{3}\pi \end{array} \right\}.$$

So we have a continuous path U_t in $W^{1,\infty}(\overline{B})$ connecting U and U_0. Let us denote with M_t the minimal of E in $\mathcal{S}_t$. While $U \not\equiv M$ by assumption, it is worthwhile noticing that $U_0 \equiv M_0$ is minimal. Indeed, the datum of $\mathcal{S}_0$ is symmetric and hence, by uniqueness of the minimal (Theorem 1.2,(a)), its triple point must be the origin; this implies (Remark 3.1) that U_0 is the minimal solution. Let

$$\bar{t} = \sup\{t^* \geq 0 : U_t \equiv M_t \text{ for every } t \in [0, t^*]\}.$$

By continuity of E we immediately see that $U_{t^*} \equiv M_{t^*}$. On the other hand, we can find a sequence $\varepsilon_n > 0$ such that

$$U_{t^*+\varepsilon_n} \not\equiv M_{t^*+\varepsilon_n}.$$

By Theorem 1.2,(c), we have that

$$M_{t^*+\varepsilon_n} \to M_{t^*} \text{ a.e.,} \qquad \text{that implies } \mathbf{a}_{M_{t^*+\varepsilon_n}} \to \mathbf{a}_{M_{t^*}}.$$

On the other hand, since $U_{t^*} \equiv M_{t^*}$, we have by construction that

$$U_{t^*+\varepsilon_n} \to M_{t^*} \text{ in } W^{1,\infty}, \qquad \text{that implies } \mathbf{a}_{U_{t^*+\varepsilon_n}} \to \mathbf{a}_{M_{t^*}}.$$

We infer that both $U_{t^*+\varepsilon_n}$ and $M_{t^*+\varepsilon_n}$ belong to $\mathcal{S}_{t^*+\varepsilon_n}$, and the distance between $\mathbf{a}_{U_{t^*+\varepsilon_n}}$ and $\mathbf{a}_{M_{t^*+\varepsilon_n}}$ is arbitrary small. This contradicts Proposition 3.2, and concludes the proof. ■

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