Rational Interpolants with Tension Parameters

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Abstract. In this paper we present a NURBS version of the rational interpolating spline with tension introduced in [2], and we extend our proposal to the rectangular topology case. In particular we present some rational interpolating techniques that enable us to reconstruct shape-preserving bivariate NURBS and allow us to interactively modify the resulting surface by a set of tension parameters.

§1. Introduction

Looking at the history of CAGD in industry, it appears that the need for compatible formats to exchange data among different systems led to the introduction of NURBS representation, which is even today, the current industry standard. However, even if NURBS is the most important entity in industrial applications, in all commercial software packages the weights are not used when approximating or interpolating with NURBS. This is because the desired range of values that they should attain is rather restrictive (weights should be positive, bounded away from zero and also have a reasonable upper bound in standard form). However, the main reason for the limited use of interpolating NURBS is that treating weights and control points as unknowns immediately requires the solution of a nonlinear problem. Therefore, the literature is extremely poor in the field of rational interpolation, since very few authors have tried to face all the difficulties that arise. Furthermore, as NURBS representation can only be used to a very limited extent for actual modelling purposes, the only exception might be to style an application where the weights are used as fairing or sculpting parameters.

These considerations motivated the research reported in this paper, where we propose a univariate/bivariate interpolating NURBS, whose weights are used as tension parameters. The univariate form is the NURBS version of the rational interpolating spline with tension originally proposed in [2] and [3] in order to improve classical polynomial tension methods. In fact, while the most famous interpolating splines with tension, the so-called ν-splines [5,6], are limited to the parametric case, our proposal works efficiently both in non-parametric and parametric cases. However, as our aim is simply to underline the possibility of applying an interpolating method with tension properties to non-parametric sets of points, we will limit our attention to the scalar formulation only. In the same way, when we present the extension of this proposal to the bivariate case, we address our attention to non-parametric rational interpolating techniques of a rectangular set of points, while stressing that, in comparison to the few other tension methods proposed up to now, our solution is highly flexible for geometric modelling purposes and can be implemented in a NURBS-based CAD system with a really low computational cost.

§2. Piecewise Rational Cubic Interpolant with Tension

Definition 1. *Let (x_i, F_i), $i = 0, ..., N$ (with $x_0 < x_1 < ... < x_N$) be interpolating points and let D_i, $i = 0, ..., N$, denote first derivative values defined at the knots x_i. Then a piecewise rational cubic interpolating function $c(x) \in C^1_{[x_0,x_N]}$ is defined for $x \in [x_i, x_{i+1}]$, $i = 0, ..., N-1$, by the following expression*

$$c_i(x) = F_i R^i_{0,3}(x) + G_i R^i_{1,3}(x) + H_i R^i_{2,3}(x) + F_{i+1} R^i_{3,3}(x) \qquad (1)$$

where $G_i := F_i + \frac{h_i D_i}{3w_i}$, $H_i := F_{i+1} - \frac{h_i D_{i+1}}{3w_i}$ (with $h_i = x_{i+1} - x_i$) and $R^i_{j,3}(x) = \frac{\lambda^i_j B^i_{j,3}(x)}{\sum_{k=0}^{3} \lambda^i_k B^i_{k,3}(x)}$, $j = 0, ..., 3$ are the rational basis functions defined via the positive weights $\lambda^i_0 = \lambda^i_3 := 1, \lambda^i_1 = \lambda^i_2 := w_i$ and the cubic Bernstein polynomials $B^i_{j,3}(x)$, $j = 0, ..., 3$.

Here w_i is a tension parameter for the single piece $c_i(x)$, i.e., if $w_i \to \infty$, $c_i(x)$ converges uniformly to the linear interpolant on $[x_i, x_{i+1}]$. Additionally, when all the tension parameters $w_i, i = 0, ..., N-1$, are increased, the rational spline $c(x)$ converges uniformly to a C^1 piecewise linear interpolant; these piecewise rational cubic Bézier functions can be represented as NURBS, that is on a single knot partition with triple knots, and in the special case of equal tension parameters, with double knots. Note that, assuming the derivative values D_i, $i = 1, ..., N-1$, to be degrees of freedom, and computing them by the imposition of C^2-continuity conditions, $c(x) \in C^2_{[x_0,x_N]}$, see [2,3].

Remark 1. *In the parametric formulation, this proposal has the fundamental property that the curve parameterization is good for any choice of the tension parameters, as these influence both basis functions and control points.*

Now, writing (1) in the equivalent form

$$c_i(x) = F_i \phi^i_{0,3}(x) + D_i \phi^i_{1,3}(x) + D_{i+1} \phi^i_{2,3}(x) + F_{i+1} \phi^i_{3,3}(x), \qquad (2)$$

we can see $c_i(x)$ as the rational cubic Hermite interpolant, where

$$\begin{array}{cc} \phi^i_{0,3}(x) = R^i_{0,3}(x) + R^i_{1,3}(x), & \phi^i_{1,3}(x) = \frac{h_i}{3w_i} R^i_{1,3}(x) \\ \phi^i_{2,3}(x) = -\frac{h_i}{3w_i} R^i_{2,3}(x), & \phi^i_{3,3}(x) = R^i_{2,3}(x) + R^i_{3,3}(x) \end{array} \tag{3}$$

are rational cubic Hermite basis functions.

When $w_i = 1$, $R^i_{j,3}(x)$ $j = 0, ..., 3$ become the cubic Bernstein polynomials $B^i_{j,3}(x)$ $j = 0, ..., 3$, $\phi^i_{j,3}(x)$ $j = 0, ..., 3$ become the cubic Hermite basis functions, and $c(x)$ the well-known piecewise cubic Hermite interpolant. The latter represents the only tension method implemented in commercial modelling systems. But, while it requires the changing of the modulus of the derivatives assigned at the given interpolating points in order to achieve tension effects on the resulting curve, the rational piecewise cubic Hermite interpolating technique we propose allows local/global tension effects, without changing the given data set.

§3. Bivariate Rational Interpolating Functions with Tension

In this section we introduce some methods that allow us to reconstruct the single-valued surface interpolant of a given rectangular set of points and to modify the resulting shape, either locally or globally, using the so-called tension parameters.

3.1. Composition of bicubic partially blended tension patches

The first method we present is a transfinite interpolating method. After defining a network of rational cubic Hermite interpolating functions, by assuming the values $z_{i,j}, i = 0, ..., M, j = 0, ..., N$, and the first derivatives $f^x_{i,j}$, $f^y_{i,j}$, $i = 0, ., M$, $j = 0, ..., N$, corresponding to the rectangular grid points (x_i, y_j), $i = 0, ..., M$, $j = 0, ..., N$ (with $x_0 < x_1 < ... < x_M$ and $y_0 < y_1 < ... < y_N$), we suggest blending the four intersecting rational cubics forming the boundary of each individual patch using the partially bicubic Coons technique (see [1]).

Definition 2. *The $(i, j)^{th}$ bicubic partially blended Coons patch $S_{i,j}(x, y)$, assuming values $z_{h,k}$ and derivatives $f^x_{h,k}$, $f^y_{h,k}$, $h = i, i+1$, $k = j, j+1$ corresponding to the four corners of the domain $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$, can be defined by the matrix form:*

$$S_{i,j}(x, y) = -\left[\, -1 \quad \phi^i_{0,3}(x) \quad \phi^i_{3,3}(x) \,\right] \times$$
$$\begin{bmatrix} 0 & S_{(x,y_j)} & S_{(x,y_{j+1})} \\ S_{(x_i,y)} & S_{(x_i,y_j)} & S_{(x_i,y_{j+1})} \\ S_{(x_{i+1},y)} & S_{(x_{i+1},y_j)} & S_{(x_{i+1},y_{j+1})} \end{bmatrix} \begin{bmatrix} -1 \\ \phi^j_{0,3}(y) \\ \phi^j_{3,3}(y) \end{bmatrix} \tag{4}$$

where $w_i = w_j = 1$ $\forall i, j$ so that $\phi^i_{s,3}(x)$ and $\phi^j_{s,3}(y)$, $s = 0, 3$ are the cubic Hermite blending functions and $S(x, y_j)$, $S(x, y_{j+1})$, $S(x_i, y)$, $S(x_{i+1}, y)$

denote, the rational cubics with tension defined on the sides of $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ using (2) with

$$\begin{array}{lccccc}
 & F_i & D_i & D_{i+1} & F_{i+1} & w_i \\
S(x, y_j) & z_{i,j} & f^x_{i,j} & f^x_{i+1,j} & z_{i+1,j} & \omega_{i,j} \\
S(x, y_{j+1}) & z_{i,j+1} & f^x_{i,j+1} & f^x_{i+1,j+1} & z_{i+1,j+1} & \omega_{i,j+1} \\
S(x_i, y) & z_{i,j} & f^y_{i,j} & f^y_{i,j+1} & z_{i,j+1} & \tau_{i,j} \\
S(x_{i+1}, y) & z_{i+1,j} & f^y_{i+1,j} & f^y_{i+1,j+1} & z_{i+1,j+1} & \tau_{i+1,j}.
\end{array}$$

Remark 2. *The composition $S(x, y)$ of the partially blended Coons patches $S_{i,j}(x, y)$, $i = 0, ..., M-1$, $j = 0, ..., N-1$, inherits all the properties of the network of boundary curves. Therefore it is characterized by a set of $M \times (N+1)$ tension parameters $\omega_{i,j}, i = 0, ..., M-1, j = 0, ..., N$, in the x direction and by a set of $(M+1) \times N$ tension parameters $\tau_{i,j}, i = 0, ..., M, j = 0, ..., N-1$ in the y direction, which ensure local/global tension effects (see Fig. 1 and Fig. 2 left for examples of local tension).*

Proposition 3. *The composition $S(x, y)$ defined above is a C^1-continuous degree-seven piecewise rational surface with shape-preserving properties when interpolating to monotonic and convex data sets.*

Proof: To prove that $S(x, y)$ is made of degree-seven rational patches, it is sufficient to make the boolean sum form (4) explicit; the result immediatly follows if we note that a rational cubic with tension can be simplified as a cubic over a quadratic (this is due to the equality of its central weights).
Since the boundary curves are C^1-continuous and the blending functions satisfy the conditions $(\phi^i_{s,3})'(x_i) = (\phi^i_{s,3})'(x_{i+1}) = 0$ and $(\phi^j_{s,3})'(y_j) = (\phi^j_{s,3})'(y_{j+1}) = 0$, $s = 0, 3$, $i = 0, ..., M-1$, $j = 0, ..., N-1$, continuity of cross-boundary derivatives is automatically satisfied.
Furthermore, from the tension and shape-preserving properties of the network of boundary curves (proved in [2,3]), the corresponding tension and shape-preserving properties on the transfinite interpolating surface $S(x, y)$ trivially follow. □

3.2. Composition of bicubically blended tension patches

The bicubic partially blended Coons patch defined above is easy to use in a design environment, since only the four boundary curves are needed. However, a more flexible composite C^1 Coons surface can be developed if, together with the boundary curves, the cross-boundary derivatives, interpolating to the tangent vectors and twist vectors assigned in the 2×2 grid points, are given.

Definition 4. *The* $(i,j)^{th}$ *bicubically blended surface patch* $S_{i,j}(x,y)$, $i = 0,...,M-1$, $j = 0,...,N-1$, *assuming the values* $z_{h,k}$ *and the derivatives* $f^x_{h,k}$, $f^y_{h,k}$, $f^{xy}_{h,k}$, *in the* 2×2 *grid points* (x_h, y_k), $h = i, i+1$, $k = j, j+1$, *can be defined by the matrix form:*

$$S_{i,j}(x,y) = -\left[-1 \quad \phi^i_{0,3}(x) \quad \phi^i_{3,3}(x) \quad \phi^i_{1,3}(x) \quad \phi^i_{2,3}(x)\right] \times \tag{5}$$

$$\begin{bmatrix} 0 & S_{(x,y_j)} & S_{(x,y_{j+1})} & S^y_{(x,y_j)} & S^y_{(x,y_{j+1})} \\ S_{(x_i,y)} & S_{(x_i,y_j)} & S_{(x_i,y_{j+1})} & S^y_{(x_i,y_j)} & S^y_{(x_i,y_{j+1})} \\ S_{(x_{i+1},y)} & S_{(x_{i+1},y_j)} & S_{(x_{i+1},y_{j+1})} & S^y_{(x_{i+1},y_j)} & S^y_{(x_{i+1},y_{j+1})} \\ S^x_{(x_i,y)} & S^x_{(x_i,y_j)} & S^x_{(x_i,y_{j+1})} & S^{xy}_{(x_i,y_j)} & S^{xy}_{(x_i,y_{j+1})} \\ S^x_{(x_{i+1},y)} & S^x_{(x_{i+1},y_j)} & S^x_{(x_{i+1},y_{j+1})} & S^{xy}_{(x_{i+1},y_j)} & S^{xy}_{(x_{i+1},y_{j+1})} \end{bmatrix} \begin{bmatrix} -1 \\ \phi^j_{0,3}(y) \\ \phi^j_{3,3}(y) \\ \phi^j_{1,3}(y) \\ \phi^j_{2,3}(y) \end{bmatrix},$$

where $w_i = w_j = 1\ \forall i,j$ and $S(x,y_j), S(x,y_{j+1}), S(x_i,y), S(x_{i+1},y)$ denote the rational cubics with tension defined above, interpolating to the corner data and tangent vectors, while $S^y(x,y_j)$, $S^y(x,y_{j+1})$, $S^x(x_i,y)$, $S^x(x_{i+1},y)$ denote the rational cubics with tension interpolating to tangent and twist vectors, defined by (2) with

$$\begin{array}{cccccc} & F_i & D_i & D_{i+1} & F_{i+1} & w_i \\ S^y(x,y_j) & f^y_{i,j} & f^{xy}_{i,j} & f^{xy}_{i+1,j} & f^y_{i+1,j} & \omega_{i,j} \\ S^y(x,y_{j+1}) & f^y_{i,j+1} & f^{xy}_{i,j+1} & f^{xy}_{i+1,j+1} & f^y_{i+1,j+1} & \omega_{i,j+1} \\ S^x(x_i,y) & f^x_{i,j} & f^{xy}_{i,j} & f^{xy}_{i,j+1} & f^x_{i,j+1} & \tau_{i,j} \\ S^x(x_{i+1},y) & f^x_{i+1,j} & f^{xy}_{i+1,j} & f^{xy}_{i+1,j+1} & f^x_{i+1,j+1} & \tau_{i+1,j}. \end{array}$$

Remark 3. *As asserted above, the composition* $S(x,y)$ *of the bicubically blended Coons patches* $S_{i,j}(x,y)$, $i = 0,...,M-1$, $j = 0,...,N-1$ *is a degree-seven piecewise rational surface, which inherits all the properties of the network of boundary curves (see Fig. 3 left, Fig. 4). In this case, using a network of* C^2 *boundary curves, we are able to obtain a* C^2 *surface.*

Remark 4. *While in the previous method only two of the cubic Hermite basis functions are used (as the name bicubic partially blended suggests), this time, we have to use all four cubic Hermite blending functions in both the coordinate directions to obtain a patch that incorporates cross-boundary derivatives as well. Exchanging the used cubic Hermite blending functions with the four rational cubic Hermite polynomials defined in (3), and setting all tension parameters in* x *and* y *directions, respectively equal to* $\hat{w}$ *and* $\overline{w}$, *we can achieve a global tension control of the interpolating surface in the coordinate directions (see Fig. 2 right).*

3.3. C^1-joined tension patches

The tensor-product surface of the rational cubics with tension defined in Section 2 is not useful because any one of the shape parameters controls the tension on the entire corresponding strip of the surface. The two Coons techniques proposed in Sects. 3.1 and 3.2 can describe a much richer variety of interpolating NURBS surfaces than do tensor-product surfaces and, additionally, they can ensure local tension effects. However, they cannot be represented in a bicubic degree NURBS form. Sarfraz's method [7] (which seems to be the only one presented in the literature up to now) can provide an interpolatory surface with local tension properties, but it cannot be described in a NURBS form because it uses a rational bicubic degree representation with functional weights. Moreover, its computation is too time-consuming to be useful in a geometric modelling system. These considerations prompted us to look for an alternative strategy which could generate a rational Hermite interpolating C^1 surface, with a bicubic degree NURBS representation and a local tension capability. Since a rational bicubic C^1 surface, interpolating to a given set of $(M+1)\times(N+1)$ data and derivative values, can always be viewed as a collection of $M \times N$ rational C^1 Hermite bicubic patches that are pieced together, we are able to stretch a specified patch of the original surface and, using the following conditions, to keep the C^1-continuity with the adjacent patches.

Proposition 5. *Let $S_{i,j}(x,y)$ and $S_{i+1,j}(x,y)$ be two degree-(m,n) rational Bézier patches respectively, with positive weights $\{\lambda_{k,l}^{i,j}\}$, $\{\lambda_{k,l}^{i+1,j}\}$ and control points $\{c_{k,l}^{i,j}\}$, $\{c_{k,l}^{i+1,j}\}$ $k=0,...,m$, $l=0,...,n$, defined over the domains $[x_i,x_{i+1}]\times[y_j,y_{j+1}]$ and $[x_{i+1},x_{i+2}]\times[y_j,y_{j+1}]$. C^1 sufficient conditions between the two adjacent C^0-continuous rational patches $S_{i,j}(x,y)$ and $S_{i+1,j}(x,y)$, along the common boundary $S_{i,j}(x_{i+1},y)$, are*

$$\begin{aligned}
\lambda_{k,1}^{i+1,j} &= -\tfrac{h_{i+1}^x}{h_i^x}\lambda_{k,n-1}^{i,j} + \tfrac{\lambda_{k,n}^{i,j}}{\lambda_{0,n}^{i,j}}\left(\tfrac{h_{i+1}^x}{h_i^x}\lambda_{0,n-1}^{i,j} + \lambda_{0,1}^{i+1,j}\right), \quad && k=1,...,m,\\
c_{k,1}^{i+1,j} &= c_{k,n}^{i,j} + \tfrac{h_{i+1}^x}{h_i^x}\tfrac{\lambda_{k,n-1}^{i,j}}{\lambda_{k,1}^{i+1,j}}\left(c_{k,n}^{i,j} - c_{k,n-1}^{i,j}\right), && k=0,...,m,
\end{aligned} \tag{6}$$

where $h_i^x = x_{i+1}-x_i$ and $\lambda_{0,1}^{i+1,j}$ is an arbitrary positive constant.

Proof: Using the notation proposed in [4], C^1-continuity between the rational patches $S_{i,j}$ and $S_{i+1,j}$ is equivalent to the condition

$$(D^x T_{i+1,j}(x,y))|_{x=x_{i+1}} = (D^x T_{i,j}(x,y))|_{x=x_{i+1}} + \sigma T_{i,j}(x_{i+1},y), \tag{7}$$

where $T_{i,j}(x,y), T_{i+1,j}(x,y)$ represent the homogeneus coordinate systems of the patches $S_{i,j}(x,y), S_{i+1,j}(x,y)$, respectively, and D^x denotes the first partial derivative along the x direction. The constant σ is a free parameter since we are considering a rational C^1-continuity between the two patches $S_{i,j}$ and $S_{i+1,j}$, while the coefficient of the term $D^x T_{i,j}$ is fixed equal to 1. This derives from considering the patches in the classical positional continuity C^0 (and not in the rational one), which requires the equality of weights and control points on the common boundary curve:

$$\begin{array}{rcll} \lambda_{k,n}^{i,j} & = & \lambda_{k,0}^{i+1,j}, & k=0,...,m, \\ c_{k,n}^{i,j} & = & c_{k,0}^{i+1,j}, & k=0,...,m. \end{array} \tag{8}$$

Now, following [4], if we derive the C^1 sufficient conditions between adjacent C^0-continuous rectangular rational Bézier patches, we obtain, for $k=0,...,m$, the explicit conditions

$$\begin{array}{r} \frac{1}{h_{i+1}^x}(\lambda_{k,1}^{i+1,j}-\lambda_{k,n}^{i,j}) = \frac{1}{h_i^x}(\lambda_{k,n}^{i,j}-\lambda_{k,n-1}^{i,j})+\sigma\lambda_{k,n}^{i,j}, \\ \frac{1}{h_{i+1}^x}(\lambda_{k,1}^{i+1,j}c_{k,1}^{i+1,j}-\lambda_{k,n}^{i,j}c_{k,n}^{i,j}) = \frac{1}{h_i^x}(\lambda_{k,n}^{i,j}c_{k,n}^{i,j}-\lambda_{k,n-1}^{i,j}c_{k,n-1}^{i,j})+\sigma\lambda_{k,n}^{i,j}c_{k,n}^{i,j}. \end{array}$$

Hence, assuming $\lambda_{0,1}^{i+1,j} = -\frac{h_{i+1}^x}{h_i^x}\lambda_{0,n-1}^{i,j} + (1+\frac{h_{i+1}^x}{h_i^x}+\sigma h_{i+1}^x)\lambda_{0,n}^{i,j}$ as a free parameter (since it depends on σ) and, substituting the resulting expression for σ in the remaining previous equations, we find (6). $\square$

Remark 5. *Note that when* $\lambda_{k,l}^{i,j} = \lambda_{k,l}^{i+1,j} = 1$ *for* $k=0,...,m,$ *and* $l=0,...,n,$ σ *must be* 0. *In this case the patches* $S_{i,j}$ *and* $S_{i+1,j}$ *are non rational and conditions (6) become classical* C^1*-continuity conditions [1].*

Since our aim is to be able to stretch a patch of a composition of $M\times N$ rational C^1 Hermite bicubic patches, keeping the interpolation and the C^1-continuity with all the patches around, from now on we will consider conditions (6) with $m=n=3$. Additionally, since rational bicubic tensor-product surfaces are a well-known class of C^1-continuous NURBS surfaces, we focalize an example of our strategy on the Hermite interpolating surface

$$S_{i,j}(x,y) = \sum_{k=0}^{3}\sum_{l=0}^{3} c_{k,l}^{i,j} R_{k,3}^i(x) R_{l,3}^j(y), \quad i=0,...,M-1 \quad j=0,...,N-1$$

with $\{\lambda_{k,l}^{i,j}\}_{k,l=0,...,3} = [1,\nu_j,\nu_j,1]^T[1,\mu_i,\mu_i,1]$. Now, since the tension parameters μ_i, $i=0,...,M-1$ and ν_j, $j=0,...,N-1$ influence the entire strips $[x_i,x_{i+1}]\times[y_0,y_N]$, $i=0,...,M-1$ and $[x_0,x_M]\times[y_j,y_{j+1}]$, $j=0,...,N-1$, respectively, in order to obtain a local tension control on a specified rational Hermite patch $S_{i,j}(x,y)$, we define two new tension parameters $\overline{\mu}_i, \overline{\nu}_j$ on the sides of its rectangular boundary. Then we recompute the patch $S_{i,j}(x,y)$ as a tensor-product, and exploit the degree of freedom characterizing conditions (6), to adjust the weights and control points of the first ring of eight patches contained in the neighbourhood of

$S_{i,j}$, in order to maintain C^1-continuity. If we consider (for simplicity of presentation) $h_i^x = h_{i+1}^x \ \forall i$, as, in our case, $\lambda_{0,3}^{i,j} = \lambda_{3,3}^{i,j} = 1$, C^1-continuity conditions (6) assume the simplified expressions

$$\begin{array}{rcll} \lambda_{k,1}^{i+1,j} & = & -\lambda_{k,2}^{i,j} + \lambda_{k,3}^{i,j}(\overline{\mu_i} + \lambda_{0,1}^{i+1,j}), & k = 1, ..., 3, \\ c_{k,1}^{i+1,j} & = & c_{k,3}^{i,j} + \frac{\lambda_{k,2}^{i,j}}{\lambda_{k,1}^{i+1,j}}(c_{k,3}^{i,j} - c_{k,2}^{i,j}), & k = 0, ..., 3, \end{array} \tag{9}$$

where $\lambda_{0,1}^{i+1,j}$ is taken as equal to $\mu_i + \mu_{i+1} - \overline{\mu_i}$. Then, if after taking the boundary weights $\lambda_{k,1}^{i+1,j} = -\lambda_{k,2}^{i,j} + \lambda_{k,3}^{i,j}(\mu_i + \mu_{i+1})$, $\forall k = 1, ..., 3$, we repeat this computation in the directions $(i-1, j), (i, j-1), (i, j+1)$, adjusting the weight $\lambda_{1,1}^{i+1,j+1}$ as the product of $\lambda_{3,1}^{i+1,j}$ by $\lambda_{1,3}^{i,j+1}$, and the other three corner weights $\lambda_{1,2}^{i-1,j+1}$, $\lambda_{2,2}^{i-1,j-1}$, $\lambda_{2,1}^{i+1,j-1}$ in the same way, the first condition (9) is automatically satisfied for each patch around the $(i,j)^{th}$. Afterwards, by computing the corresponding control points according to the second condition (9), we are able to keep the C^1-continuity of the surface, even if we have modified the weights of the $(i,j)^{th}$ patch, causing a tension on it (see Fig. 3 right).

§4. Conclusions

The rational cubic interpolating method proposed in [2,3] has been converted into standard NURBS form and applied to obtain shape control on rectangular NURBS Hermite interpolatory surfaces. Using the techniques described, we have obtained a number of different ways of achieving shape control on these kinds of surfaces, always maintaining C^1-continuity.

In fact, comparing the following figures with Fig. 1 left (which was built with global values $\omega := \omega_{i,j} = 1 \ \forall i, j$, $\tau := \tau_{i,j} = 1 \ \forall i, j$), we can see that it is possible to apply tension:

- along a curve segment, causing the segment to tend to a straight line by increasing shape parameter $\omega_{i,j}$ for a specified couple of indices i, j (Fig. 1 right, where $\omega_{3,4} = 4$);
- along a network curve, causing the whole curve to tend to a polygon by increasing shape parameters $\omega_{0,j}, ..., \omega_{M-1,j}$ for one $j \in \{0, ..., N\}$ (Fig. 2 left shows this for $\omega_{0,4} = \cdots = \omega_{5,4} = 4$);
- along x direction (Fig. 2 right corresponds to the blending tension parameters $\hat{w} = 4, \overline{w} = 1$);
- on a single patch. This is done by applying both interval tensions at its four sides (Fig. 3 left represents the bicubically blended tension method with parameters (4,4), while Fig. 3 right the rational bicubic tensor-product with global parameters (2,2), where the C^1-join technique has been applied with parameters $\overline{\mu_3} = \overline{\nu_3} = 3$);
- on the whole surface (Fig. 4 left shows this for the global value $\omega := \omega_{i,j} = 4$, $\forall i, j$, $\tau := \tau_{i,j} = 4 \ \forall i, j$);

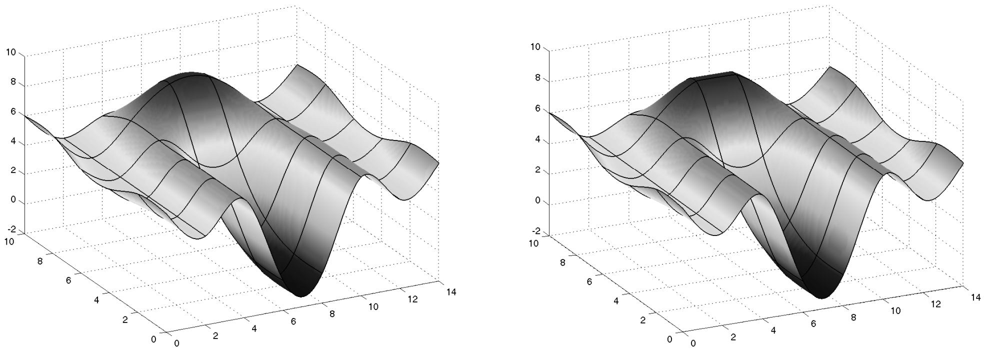

Fig. 1. left: no tension - right: tension along $S([x_3, x_4], y_4)$.

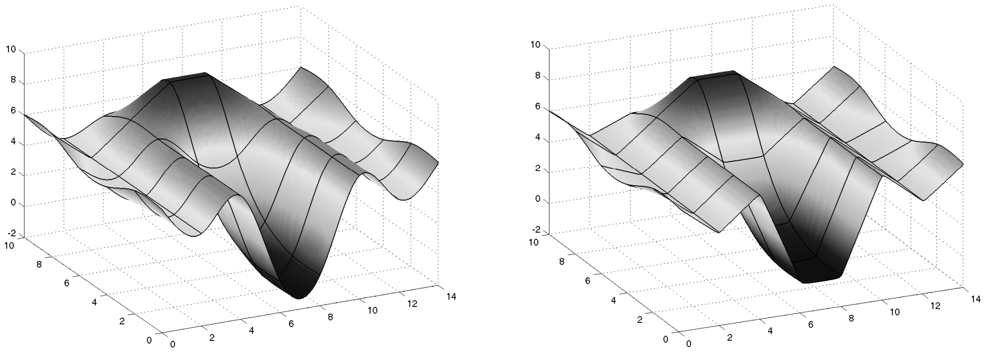

Fig. 2. left: tension along $S(x, y_4)$ - right: tension along x direction.

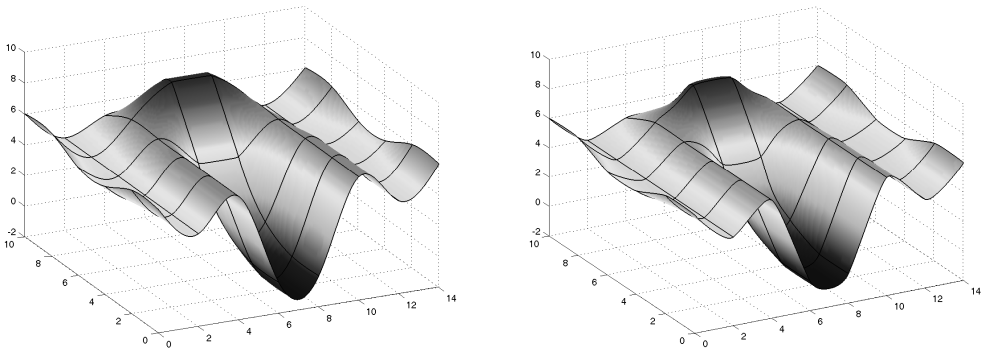

Fig. 3. left: tension on patch $S_{3,3}$ - right: tension on patch $S_{3,3}$.

- everywhere except on a specified patch (Fig. 4 right shows this using a global tension (4,4) and a local tension (1,1) on patch $S_{3,3}$).

§5. Future Work

We intend to extend these methods to the parametric case in order to be able to apply the rational G^1-continuity conditions (instead of the C^1-continuity introduced in 3.3) which will give us extra degrees of freedom.

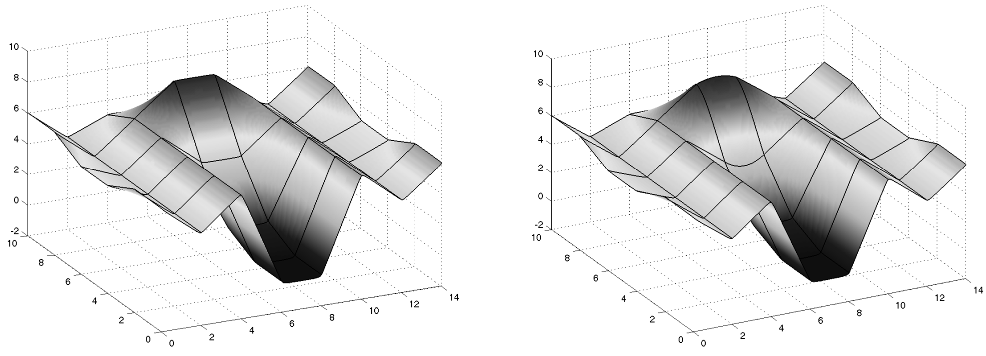

Fig. 4. left: global tension - right: global tension except on patch $S_{3,3}$.

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