A Piecewise Rational Quintic Hermite Interpolant for use in CAGD

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Abstract. In this paper we describe and analyze a new class of C^2 piecewise rational quintic Hermite interpolants for use in CAGD which are capable of exactly representing any conic arc of arbitrary length by using only one segment. They can also provide a variety of local/global shape parameters for intuitively sculpting free form curves without affecting the C^2 continuity inherent in the original construction.

§1. Introduction

During the last two decades there has been considerable interest in developing a C^2 solution to the interpolation problem of zeroth, first and second order derivatives at a given selection of points. This is due to the fact that such geometric properties turn out to be of primary concern in geometric modelling or computer aided design applications, e.g., in the smoothing of curves. However, the work done over the years has resulted in many C^2 interpolation methods [1, 3, 4, 5, 8, 9, 10] that cannot satisfy at the same time all the properties we have found vital or simply desirable to include in a user-oriented model, designed to be integrated in a conventional NURBS-based CAD system. For this reason, we are going to propose a new class of C^2 piecewise rational quintic Hermite interpolants that

- possesses an explicit construction, i.e., that does not involve the solution of any equations;
- provides local and intuitive sculpting parameters for fast interactive manipulation of the shape of a curve;
- is capable of representing both smooth shapes and sharp shapes, and more precisely of mixing smooth zones and sharp ones in the same curve, so that the transition between them always preserves the original C^2 continuity.

The paper is structured as follows. In Section 2 we develop and construct this new class; in Section 3 we show that our rational quintic Hermite interpolants are capable to exactly represent any conic arc of arbitrary length by using only one segment with positive weights and finite control points. Finally, in Section 4 we exploit our model for intuitively sculpting C^2-continuous free form curves by local deformations which do not compromise the original C^2 continuity. Although our proposal is not limited to parametric sets of points, but works efficiently both in the vectorial and in the scalar cases, we confine our attention here to the parametric formulation only.

§2. The Class of Piecewise Rational Quintic Hermite Interpolants

Our objective is to construct over an arbitrary non-trivial interval $[t_i, t_{i+1}] \subset \mathbb{R}$, a rational curve segment $\mathbf{c}_i(t) : [t_i, t_{i+1}] \to \mathbb{R}^\nu$, $\nu > 1$, that satisfies the following interpolation conditions at the endpoints:

$$\begin{gathered} \mathbf{c}_i(t_i) = \mathbf{F}_i, \quad \mathbf{c}_i'(t_i) = \mathbf{D}_i^{(1)}, \quad \mathbf{c}_i''(t_i) = \mathbf{D}_i^{(2)}, \\ \mathbf{c}_i''(t_{i+1}) = \mathbf{D}_{i+1}^{(2)}, \quad \mathbf{c}_i'(t_{i+1}) = \mathbf{D}_{i+1}^{(1)}, \quad \mathbf{c}_i(t_{i+1}) = \mathbf{F}_{i+1}. \end{gathered} \tag{1}$$

To fulfill our aim, we write $\mathbf{c}_i(t)$ as the rational quintic Bézier curve

$$\mathbf{c}_i(t) = \sum_{j=0}^{5} \mathbf{P}_j^i R_{j,5}^i(t) \tag{2}$$

where

$$R_{j,5}^i(t) = \frac{\mu_j^i B_{j,5}^i(t)}{\sum_{k=0}^5 \mu_k^i B_{k,5}^i(t)} \quad \text{with} \quad B_{j,5}^i(t) = \binom{5}{j} \frac{(t_{i+1}-t)^{5-j}(t-t_i)^j}{(t_{i+1}-t_i)^5} \tag{3}$$

and the control points $\mathbf{P}_j^i \in \mathbb{R}^\nu$, $j = 0, ..., 5$ have to be determined.
Two of them turn out to be quickly defined: $\mathbf{P}_0^i = \mathbf{F}_i$ and $\mathbf{P}_5^i = \mathbf{F}_{i+1}$. For the remaining four, we recall the first and second endpoint derivative formulae for rational quintic Bézier curves:

$$\mathbf{c}_i'(t_i) = \frac{5\mu_1^i(\mathbf{P}_1^i - \mathbf{P}_0^i)}{h_i \mu_0^i}, \quad \mathbf{c}_i''(t_i) = \frac{20\mu_2^i(\mathbf{P}_2^i - \mathbf{P}_0^i)}{h_i^2 \mu_0^i} - \left(\frac{5\mu_1^i}{\mu_0^i} - 1\right) \frac{10\mu_1^i(\mathbf{P}_1^i - \mathbf{P}_0^i)}{h_i^2 \mu_0^i},$$

$$\mathbf{c}_i''(t_{i+1}) = \frac{20\mu_3^i(\mathbf{P}_3^i - \mathbf{P}_5^i)}{h_i^2 \mu_5^i} - \left(\frac{5\mu_4^i}{\mu_5^i} - 1\right) \frac{10\mu_4^i(\mathbf{P}_4^i - \mathbf{P}_5^i)}{h_i^2 \mu_5^i}, \quad \mathbf{c}_i'(t_{i+1}) = \frac{5\mu_4^i(\mathbf{P}_5^i - \mathbf{P}_4^i)}{h_i \mu_5^i}$$

where $h_i = t_{i+1} - t_i$ and μ_j^i, $j = 0, ..., 5$ are the positive weights of the rational representation (2). Thus, by solving for $\mathbf{P}_1^i$, $\mathbf{P}_2^i$, $\mathbf{P}_3^i$, $\mathbf{P}_4^i$, we get

$$\mathbf{P}_1^i = \mathbf{F}_i + \frac{h_i\ \mu_0^i}{5\mu_1^i} \mathbf{D}_i^{(1)}, \quad \mathbf{P}_2^i = \mathbf{F}_i + \frac{(5\mu_1^i - \mu_0^i) h_i}{10\mu_2^i} \mathbf{D}_i^{(1)} + \frac{h_i^2\ \mu_0^i}{20\mu_2^i} \mathbf{D}_i^{(2)},$$

$$\mathbf{P}_3^i = \mathbf{F}_{i+1} - \frac{(5\mu_4^i - \mu_5^i)h_i}{10\mu_3^i}\mathbf{D}_{i+1}^{(1)} + \frac{h_i^2\ \mu_5^i}{20\mu_3^i}\mathbf{D}_{i+1}^{(2)}, \quad \mathbf{P}_4^i = \mathbf{F}_{i+1} - \frac{h_i\ \mu_5^i}{5\mu_4^i}\mathbf{D}_{i+1}^{(1)}.$$

If a sequence of interpolating data $\mathbf{F}_i$, $\mathbf{D}_i^{(1)}$, $\mathbf{D}_i^{(2)}$, $i = 0, ..., N$ is given, such a construction allows to solve the interpolation problem by a piecewise rational quintic made of pieces $\mathbf{c}_i(t)$ that join together with the so-called C^2 rational continuity (see [6]). In order to guarantee that adjacent curve segments join exactly parametrically C^0 (not just rationally C^0), we set $\mu_0^i = \mu_5^i$ and, without loss of generality, we assume them to be 1. Thus,

Definition 1. *Given the interpolating points* $\mathbf{F}_i \in \mathbb{R}^\nu$*,* $i = 0, ..., N$ *and the first and second order derivatives* $\mathbf{D}_i^{(1)}$*,* $\mathbf{D}_i^{(2)} \in \mathbb{R}^\nu$*,* $i = 0, ..., N$ *defined at the knots* t_i*,* $i = 0, ..., N$ *(with* $t_0 < t_1 < ... < t_N$*), a piecewise rational quintic Hermite interpolant* $\mathbf{c}(t) \in C^2_{[t_0,t_N]}$ *is defined for* $t \in [t_i, t_{i+1}]$*,* $i = 0, ..., N-1$ *by the expression*

$$\mathbf{c}_i(t) = \sum_{j=0}^{5} \mathbf{P}_j^i R_{j,5}^i(t) \tag{4}$$

where

$$\mathbf{P}_0^i = \mathbf{F}_i,\ \mathbf{P}_1^i = \mathbf{F}_i + \frac{h_i\mathbf{D}_i^{(1)}}{5\mu_1^i},\ \mathbf{P}_2^i = \mathbf{F}_i + \frac{(5\mu_1^i - 1)h_i\mathbf{D}_i^{(1)}}{10\mu_2^i} + \frac{h_i^2\mathbf{D}_i^{(2)}}{20\mu_2^i}, \tag{5}$$

$$\mathbf{P}_3^i = \mathbf{F}_{i+1} - \frac{(5\mu_4^i - 1)h_i\mathbf{D}_{i+1}^{(1)}}{10\mu_3^i} + \frac{h_i^2\mathbf{D}_{i+1}^{(2)}}{20\mu_3^i},\ \mathbf{P}_4^i = \mathbf{F}_{i+1} - \frac{h_i\mathbf{D}_{i+1}^{(1)}}{5\mu_4^i},\ \mathbf{P}_5^i = \mathbf{F}_{i+1}$$

and $\{R_{j,5}^i(t)\}_{j=0,...,5}$ *are the rational quintic Bézier polynomials in (3) defined by the positive weights* μ_j^i*,* $j = 0, ..., 5$ *with* $\mu_0^i \equiv \mu_5^i = 1$*.*

§3. Exact Representation of Conic Arcs of Arbitrary Length

Conic arcs play a fundamental role in CAD/CAM applications. In this section we will show that a rational quintic Hermite segment can be used for precisely representing any conic arc of arbitrary length. Thus, the piecewise interpolatory model presented in Section 2 allows us to incorporate the class of the so-called *conic section subsplines* (see [7], page 79), which contains all those smooth curves made of pieces of conic sections that pass through given points and assume prescribed derivatives. Such a model is of great use in engineering applications, e.g. in milling processes and in the construction of disk cams.

Theorem 1. *The rational quintic Hermite segment* $\mathbf{c}_0(t)$*,* $t \in [0,1]$ *interpolating the data in Tab.1 with prescribed positive weights* $\{\mu_j^0\}_{j=1,...,4}$*, allows us to exactly represent any conic arc of arbitrary length.*

	Parabolic Arc $y = ax^2$ $a \neq 0$ $\delta \in]0, +\infty[$	Hyperbolic Arc $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $b \neq 0$ $a \lessgtr 0 \; \frac{\text{left branch}}{\text{right branch}}$ $\delta \in]0, +\infty[$ $c = cosh(\delta)$ $s = sinh(\delta)$	Elliptic Arc $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $a, b \neq 0$ $\delta \in]0, \pi]$ $\rho = \frac{1}{2}(1 + tg\frac{\delta}{4})$ $\omega = 4\rho^4 - 8\rho^3 + 8\rho^2 - 4\rho + 1$
$\mathbf{F}_0 = \begin{bmatrix} f_0^x \\ f_0^y \end{bmatrix}$	$\begin{bmatrix} -\delta \\ a\delta^2 \end{bmatrix}$	$\begin{bmatrix} ac \\ -bs \end{bmatrix}$	$\begin{bmatrix} \frac{a(4\rho^4-8\rho^3+4\rho-1)}{\omega} \\ \frac{4b\rho(-2\rho^2+3\rho-1)}{\omega} \end{bmatrix}$
$\mathbf{F}_1 = \begin{bmatrix} f_1^x \\ f_1^y \end{bmatrix}$	$\begin{bmatrix} -f_0^x \\ f_0^y \end{bmatrix}$	$\begin{bmatrix} f_0^x \\ -f_0^y \end{bmatrix}$	$\begin{bmatrix} f_0^x \\ -f_0^y \end{bmatrix}$
$\mathbf{D}_0^{(1)} = \begin{bmatrix} \bar{f}_0^x \\ \bar{f}_0^y \end{bmatrix}$	$\begin{bmatrix} 2\delta \\ -4a\delta^2 \end{bmatrix}$	$\begin{bmatrix} -2as^2 \\ 2bsc \end{bmatrix}$	$\begin{bmatrix} \frac{16a\rho(-4\rho^4+10\rho^3-10\rho^2+5\rho-1)}{\omega^2} \\ \frac{4b(-8\rho^6+24\rho^5-20\rho^4+10\rho^2-6\rho+1)}{\omega^2} \end{bmatrix}$
$\mathbf{D}_1^{(1)} = \begin{bmatrix} \bar{f}_1^x \\ \bar{f}_1^y \end{bmatrix}$	$\begin{bmatrix} \bar{f}_0^x \\ -\bar{f}_0^y \end{bmatrix}$	$\begin{bmatrix} -\bar{f}_0^x \\ \bar{f}_0^y \end{bmatrix}$	$\begin{bmatrix} -\bar{f}_0^x \\ \bar{f}_0^y \end{bmatrix}$
$\mathbf{D}_0^{(2)} = \begin{bmatrix} \bar{\bar{f}}_0^x \\ \bar{\bar{f}}_0^y \end{bmatrix}$	$\begin{bmatrix} 0 \\ 8a\delta^2 \end{bmatrix}$	$\begin{bmatrix} 4as^2(2c-1) \\ 4bs(1+c-2c^2) \end{bmatrix}$	$\begin{bmatrix} \frac{16a(-12\rho^4+24\rho^3-10\rho^2-2\rho+1)}{\omega^2} \\ \frac{8b(-8\rho^5+20\rho^4+8\rho^3-32\rho^2+14\rho-1)}{\omega^2} \end{bmatrix}$
$\mathbf{D}_1^{(2)} = \begin{bmatrix} \bar{\bar{f}}_1^x \\ \bar{\bar{f}}_1^y \end{bmatrix}$	$\begin{bmatrix} \bar{\bar{f}}_0^x \\ \bar{\bar{f}}_0^y \end{bmatrix}$	$\begin{bmatrix} \bar{\bar{f}}_0^x \\ -\bar{\bar{f}}_0^y \end{bmatrix}$	$\begin{bmatrix} \bar{\bar{f}}_0^x \\ -\bar{\bar{f}}_0^y \end{bmatrix}$
$\mu_1^0 = \mu_4^0$	1	$\frac{3+2c}{5}$	$\frac{-12\rho^4+24\rho^3-16\rho^2+4\rho+1}{5\omega}$
$\mu_2^0 = \mu_3^0$	1	$\frac{2+3c}{5}$	$\frac{4\rho^4-8\rho^3+4\rho^2+1}{5\omega}$

Tab. 1. Data for exact representation of arbitrary conic arcs via the rational quintic Hermite interpolatory model.

Proof: Writing the rational quintic Hermite interpolatory model (4) by using the data given in Tab.1, we obtain an expression of $\mathbf{c}_0(t)$ whose components $x(t)$ and $y(t)$ turn out to satisfy the following canonical equations in case of parabolic, hyperbolic, or elliptic arcs, respectively:

$$y(t) = ax^2(t), \quad \frac{x^2(t)}{a^2} - \frac{y^2(t)}{b^2} = 1, \quad \frac{x^2(t)}{a^2} + \frac{y^2(t)}{b^2} = 1. \quad \square$$

Remark. The positive free parameter δ allows us to exactly represent

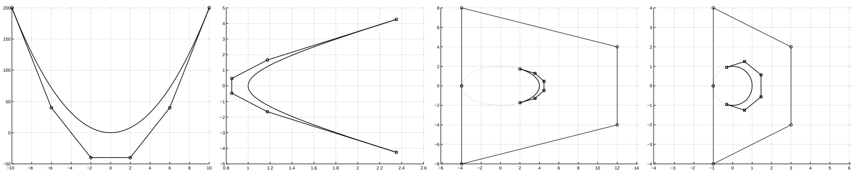

Fig. 1. Examples of arbitrary conic arcs exactly represented via a rational quintic Hermite segment: parabolic arc, hyperbolic arc, elliptic arc/full ellipse, circular arc/full circle.

conic arcs of arbitrary length. Notice that in case of elliptic arcs of amplitude 2δ, by choosing the half-angle δ equal to π, we can precisely reproduce full ellipses of radii $a,\ b \in \mathbb{R}_0$ that pass through the points $\mathbf{F}_0 \equiv \mathbf{F}_1 = (-a, 0)$ and assume the first and second order derivatives $\mathbf{D}_0^{(1)} \equiv \mathbf{D}_1^{(1)} = (0, 4b)$, $\mathbf{D}_0^{(2)} = (16a, 8b)$, $\mathbf{D}_1^{(2)} = (16a, -8b)$ (see Fig. 1). As a special case we can thus represent full circles and circular arcs just specifying the radius $r \equiv a = b$ and the half-angle $\delta \in]0, \pi]$ (see Fig. 1). We remind the reader that, as was proven in [2], the rational quintic Bézier form is the minimum degree representation that allows to obtain a full circle/ellipse using only positive weights and finite control points.

§4. Sculpting of Free Form Curves by Local/global Deformations

We now rearrange equation (4) in the equivalent cardinal form

$$\mathbf{c}_i(t) = \sum_{j=0}^{5} \mathbf{I}_j^i \phi_{j,5}^i(t)$$

where

$$\mathbf{I}_0^i = \mathbf{F}_i,\ \mathbf{I}_1^i = \mathbf{D}_i^{(1)},\ \mathbf{I}_2^i = \mathbf{D}_i^{(2)},\ \mathbf{I}_3^i = \mathbf{D}_{i+1}^{(2)},\ \mathbf{I}_4^i = \mathbf{D}_{i+1}^{(1)},\ \mathbf{I}_5^i = \mathbf{F}_{i+1}$$

and

$$\begin{aligned}
\phi_{0,5}^i(t) &= R_{0,5}^i(t) + R_{1,5}^i(t) + R_{2,5}^i(t) \\
\phi_{1,5}^i(t) &= \tfrac{h_i}{5}\left[\tfrac{1}{\mu_1^i} R_{1,5}^i(t) + \tfrac{5\mu_1^i - 1}{2\mu_2^i} R_{2,5}^i(t)\right] \\
\phi_{2,5}^i(t) &= \tfrac{h_i^2}{20\mu_2^i} R_{2,5}^i(t) \\
\phi_{3,5}^i(t) &= \tfrac{h_i^2}{20\mu_3^i} R_{3,5}^i(t) \\
\phi_{4,5}^i(t) &= -\tfrac{h_i}{5}\left[\tfrac{5\mu_4^i - 1}{2\mu_3^i} R_{3,5}^i(t) + \tfrac{1}{\mu_4^i} R_{4,5}^i(t)\right] \\
\phi_{5,5}^i(t) &= R_{3,5}^i(t) + R_{4,5}^i(t) + R_{5,5}^i(t)
\end{aligned}$$

is the rational formulation of the well-known quintic Hermite polynomials. Since $\phi_{0,5}^i(t) + \phi_{5,5}^i(t) \equiv 1$, it follows that whenever $\phi_{j,5}^i$, $j = 1, ..., 4$ vanish, the curve segment $\mathbf{c}_i(t)$ coincides with the line through $\mathbf{F}_i$, $\mathbf{F}_{i+1}$. As h_i is never zero, this is easily verified whenever μ_1^i, μ_4^i approach infinity and μ_2^i, μ_3^i approach infinity faster than μ_1^i, μ_4^i respectively. This last condition is trivially satisfied by setting $\mu_2^i = \alpha(\mu_1^i)^\beta$ and $\mu_3^i = \alpha(\mu_4^i)^\beta$, with $\alpha \geq 1$, $\beta > 1$. Here we choose $\mu_2^i = (\mu_1^i)^2$, $\mu_3^i = (\mu_4^i)^2$ and reformulate Definition 1 in the following way.

Definition 2. *Given the interpolating points* $\mathbf{F}_i \in \mathbb{R}^\nu$, $i = 0, ..., N$ *and the first and second order derivatives* $\mathbf{D}_i^{(1)}$, $\mathbf{D}_i^{(2)} \in \mathbb{R}^\nu$, $i = 0, ..., N$ *assigned at the knots* t_i, $i = 0, ..., N$ *(with* $t_0 < t_1 < ... < t_N$*), we define over the interval* $[t_i, t_{i+1}]$ *the piecewise rational quintic Hermite interpolant* $\mathbf{c}(t) \in C^2_{[t_0,t_N]}$ *by the expression*

$$\mathbf{c}_i(t) \equiv \mathbf{c}_i(t, v_i, w_i) = \sum_{j=0}^{5} \mathbf{C}_j^i R_{j,5}^i(t) \tag{6}$$

where

$$\mathbf{C}_0^i = \mathbf{F}_i,\ \mathbf{C}_1^i = \mathbf{F}_i + \tfrac{h_i \mathbf{D}_i^{(1)}}{5v_i},\ \mathbf{C}_2^i = \mathbf{F}_i + \tfrac{(5v_i-1)h_i\mathbf{D}_i^{(1)}}{10v_i^2} + \tfrac{h_i^2\mathbf{D}_i^{(2)}}{20v_i^2}, \tag{7}$$

$$\mathbf{C}_3^i = \mathbf{F}_{i+1} - \tfrac{(5w_i-1)h_i\mathbf{D}_{i+1}^{(1)}}{10w_i^2} + \tfrac{h_i^2\mathbf{D}_{i+1}^{(2)}}{20w_i^2},\ \mathbf{C}_4^i = \mathbf{F}_{i+1} - \tfrac{h_i\mathbf{D}_{i+1}^{(1)}}{5w_i},\ \mathbf{C}_5^i = \mathbf{F}_{i+1}$$

and $\{R_{j,5}^i(t)\}_{j=0,...,5}$ *are the rational quintic Bézier polynomials in (3) defined by the positive weights* $\mu_0^i = \mu_5^i := 1$, $\mu_1^i := v_i$, $\mu_2^i := v_i^2$, $\mu_3^i := w_i^2$, $\mu_4^i := w_i$.

Remark. In this way, by choosing $w_{i-1} = 2 - v_i\ \forall i = 1, ..., N-1$, the piecewise rational quintic Hermite interpolant $\mathbf{c}(t)$, represented as NURBS on a single knot-partition with internal 5-fold knots, becomes parametrically C^2-continuous at $t = t_i$ and thus can be represented on a new knot-partition with only internal 3-fold knots.

Although the shape of a NURBS curve can be modified by the manipulation of its weights, the possibility of controlling the shape through the classical change of the weight vector may sometimes be confusing, as the modifications of two adjacent weights are mutually cancelled, and not very effective, since the curve is forced to stay in the convex hull of its control points. For this reason we have proposed an interpolatory rational quintic spline involving two sculpting parameters per interval that, although corresponding to the weights of the rational representation, influence the control points definition and thus have large scale effects on the shape of the curve. In particular, such free parameters provide a variety of local and global shape controls, like point and interval tension effects. In order to analyze such effects on the shape of the curve, we consider here the limiting behavior of the rational piecewise interpolant whenever each shape parameter approaches infinity, where we assume that the other parameters are held constant with respect to each limit process (note that this is possible only because every weight modification do not influence its neighboring weights). Thus the following shape deformations immediately follow by inspection of equation (6).

Theorem 2 (Point Tension) *Let v_i and w_{i-1} approach infinity. Since*

$$\lim_{v_i \to \infty} \mathbf{c}_i(t) = \mathbf{F}_i = \lim_{w_{i-1} \to \infty} \mathbf{c}_{i-1}(t), \tag{8}$$

we have a point tension parameter controlling the curve tension from both right and left of the point $\mathbf{F}_i$, where the piecewise rational interpolant $\mathbf{c}(t)$ will appear to have a corner.

Proof: To prove the first (second) equality in (8) we insert the equation of $\mathbf{c}_i(t)$ ($\mathbf{c}_{i-1}(t)$), divide both its numerator and denominator by v_i^2 (w_{i-1}^2) and compute $\lim_{v_i(w_{i-1}) \to \infty}$. $\square$

While two shape parameters per interval are necessary for providing point tension effects, we are going to show now that only one shape parameter per interval is enough when interval tension is required. To this aim we assume the parameters v_i, w_i satisfy the relation $v_i = \lambda_i w_i$, with $\lambda_i \in]0, +\infty[$, and we show that the parameter w_i plays the role of interval tension parameter for the single piece $\mathbf{c}_i(t, \lambda_i, w_i)$.

Lemma 1. *Let $v_i = \lambda_i w_i$ with $\lambda_i \in]0, +\infty[$. If w_i approaches infinity, then the rational quintic Hermite interpolant $\mathbf{c}_i(t, \lambda_i, w_i)$ converges uniformly to the rational linear interpolant of $\mathbf{F}_i, \mathbf{F}_{i+1}$ with weights λ_i^2, 1:*

$$\lim_{w_i \to \infty} \; ||\mathbf{c}_i(t, \lambda_i, w_i) - \mathbf{l}_i(t, \lambda_i)|| = 0$$

where

$$\mathbf{l}_i(t, \lambda_i) = \frac{\lambda_i^2(1-\theta_i)\mathbf{F}_i + \theta_i \mathbf{F}_{i+1}}{\lambda_i^2(1-\theta_i) + \theta_i},$$

with

$$t \in [t_i, t_{i+1}], \quad \theta_i = \frac{t - t_i}{h_i} \in [0, 1].$$

Proof: By simple computations on $\mathbf{c}_i(t, \lambda_i, w_i)$ it follows that the piecewise rational quintic Hermite interpolant $\mathbf{c}_i(t, \lambda_i, w_i)$, $t \in [t_i, t_{i+1}]$ defined by (6) can be decomposed in the following way:

$$\mathbf{c}_i(t, \lambda_i, w_i) = \mathbf{l}_i(t, \lambda_i) + \mathbf{e}_i(t, \lambda_i, w_i),$$

where

$$\mathbf{l}_i(t, \lambda_i) = \frac{\lambda_i^2(1-\theta_i)\mathbf{F}_i + \theta_i \mathbf{F}_{i+1}}{\lambda_i^2(1-\theta_i) + \theta_i},$$

and

$\mathbf{f}(t)$	$\mathbf{g}(t)$	$\mathbf{h}(t)$
0	$\frac{1}{w_i^2}(\mathbf{F}_i - \mathbf{F}_{i+1})$	$\frac{1}{w_i^2}$
$\frac{h_i}{5w_i^2}\mathbf{D}_i^{(1)}$	$\frac{\lambda_i}{w_i}(\mathbf{F}_i - \mathbf{F}_{i+1}) + \frac{h_i}{5w_i^2}\mathbf{D}_i^{(1)}$	$\frac{\lambda_i}{w_i}$
$\frac{2(5\lambda_i w_i - 1)h_i\mathbf{D}_i^{(1)} + h_i^2\mathbf{D}_i^{(2)}}{20w_i^2}$	$\frac{2(5\lambda_i w_i - 1)h_i\mathbf{D}_i^{(1)} + h_i^2\mathbf{D}_i^{(2)}}{20w_i^2}$	λ_i^2
$\frac{-2(5w_i - 1)h_i\mathbf{D}_{i+1}^{(1)} + h_i^2\mathbf{D}_{i+1}^{(2)}}{20w_i^2}$	$\frac{-2(5w_i - 1)h_i\mathbf{D}_{i+1}^{(1)} + h_i^2\mathbf{D}_{i+1}^{(2)}}{20w_i^2}$	1
$\frac{1}{w_i}(\mathbf{F}_{i+1} - \mathbf{F}_i) - \frac{h_i}{5w_i^2}\mathbf{D}_{i+1}^{(1)}$	$-\frac{h_i}{5w_i^2}\mathbf{D}_{i+1}^{(1)}$	$\frac{1}{w_i}$
$\frac{1}{w_i^2}(\mathbf{F}_{i+1} - \mathbf{F}_i)$	0	$\frac{1}{w_i^2}$

Tab. 2. Bézier coefficients of the quintic polynomials $\mathbf{f}(t),\ \mathbf{g}(t),\ \mathbf{h}(t)$.

$$\mathbf{e}_i(t, \lambda_i, w_i) = \frac{\mathbf{f}(t)\lambda_i^2 B_{2,5}^i(\theta_i) + \mathbf{g}(t) B_{3,5}^i(\theta_i)}{\mathbf{h}(t)(\lambda_i^2 B_{2,5}^i(\theta_i) + B_{3,5}^i(\theta_i))},$$

with $\mathbf{f}(t)$, $\mathbf{g}(t)$, $\mathbf{h}(t)$ being quintic Bézier polynomials defined by the coefficients in Tab.2. Thus, since $\lim_{w_i \to \infty} \ ||\mathbf{e}_i(t, \lambda_i, w_i)|| = 0$, it trivially follows that $\lim_{w_i \to \infty} \ ||\mathbf{c}_i(t, \lambda_i, w_i) - \mathbf{l}_i(t, \lambda_i)|| = 0$. $\square$

Theorem 3 (Interval Tension) *Let $v_i = \lambda_i w_i$, with $\lambda_i \in]0, +\infty[$. If w_i approaches infinity, then the rational quintic Hermite interpolant $\mathbf{c}_i(t, \lambda_i, w_i)$ is pulled towards the line segment through $\mathbf{F}_i$ and $\mathbf{F}_{i+1}$.*

Proof: By standard NURBS theory it follows that when $\nu > 1$, whatever we choose $\lambda_i \in]0, +\infty[$, the rational linear interpolant $\mathbf{l}_i(t, \lambda_i)$ coincides with the line segment through $\mathbf{F}_i$ and $\mathbf{F}_{i+1}$. $\square$

As a consequence, when all the interval tension parameters approach infinity, the piecewise rational linear interpolant $\mathbf{c}(t)$ is pulled towards the polyline through the points $\mathbf{F}_i$, but practically is *never* a piecewise linear interpolant because the parameterization is C^2 here, whereas it is only C^0 for linear interpolants.

Corollary 1 (Global Tension) *Let $\mathbf{l}(t), t \in [t_0, t_N]$ denote the piecewise rational linear interpolant defined $\forall t \in [t_i, t_{i+1}]$ by $\mathbf{l}_i(t, \lambda_i)$. Suppose $w_i \equiv w$ and $v_i = \lambda_i w$, with $\lambda_i \in]0, +\infty[,\ \forall i = 0, \cdots, N-1$. Then the piecewise rational quintic Hermite interpolant $\mathbf{c}(t) \in C^2[t_0, t_N]$ converges uniformly to $\mathbf{l}(t)$ as w approaches infinity, i.e. $\lim_{w \to \infty} \|\mathbf{c}(t) - \mathbf{l}(t)\| = 0\ \forall t \in [t_0, t_N]$ and thus $\mathbf{c}(t)$ is pulled towards the polyline through the points $\mathbf{F}_i$.*

Proof: The result follows by applying Theorem 3 over each interval $[t_i, t_{i+1}]$. $\square$

While we have just shown that for sufficiently big values of the interval tension parameters we are able to introduce straight line segments into a given piecewise curve, we are going to show now that for sufficiently small values of the parameters we can pull out a bump or push in an indentation in a curve segment (see Figs. 2, 3).

Theorem 4 (Interval Warping) *Progressively decreasing the shape parameters v_i and w_i towards zero, the rational quintic Hermite interpolant $\mathbf{c}_i(t)$ produces a looser and looser curve segment.*

Proof: If we look at the behavior of the control points $\mathbf{C}_j^i$, $j = 1, 2, 3, 4$, and hence of the Bernstein-Bézier convex hull, when the shape parameters v_i and w_i progressively decrease towards zero, it is a simple matter to see that we will get a looser and looser curve. $\square$

Corollary 2 (Global Warping) *Let v_i and w_i progressively decrease towards zero $\forall i = 0, \cdots, N-1$. Then the piecewise rational quintic Hermite interpolant $\mathbf{c}(t) \in C^2[t_0, t_N]$ progressively becomes looser and looser over each interval $[t_i, t_{i+1}]$.*

Proof: We apply Theorem 4 over each interval $[t_i, t_{i+1}]$. $\square$

§5. Conclusions and Future Work

In this paper we have presented a new class of piecewise rational quintic Hermite interpolants which provide a variety of local and global shape parameters for intuitively sculpting free-form curves, without affecting the C^2 continuity inherent in the original construction.
In addition, the proposed model is capable of producing either a sharp C^2-interpolation or a smooth C^2-interpolation, that is, although it always produces C^2-interpolants, it enables the creation of a variety of shape effects like angular points, sharp edges, bumps and indentations (see Figs. 2, 3). The ability to incorporate exact conic arcs of arbitrary length and to mix smooth curve segments, sharp corners and flat pieces in an unrestricted way, makes the piecewise rational quintic Hermite interpolant model a candidate of choice for many applications.

Our next step will be to show that using a non-linear optimization procedure for determining the sculpting parameters, we can use the proposed class also for approximating any trigonometric curve with the desired order of precision.

Acknowledgments. This work has been supported by FIRB 2002.

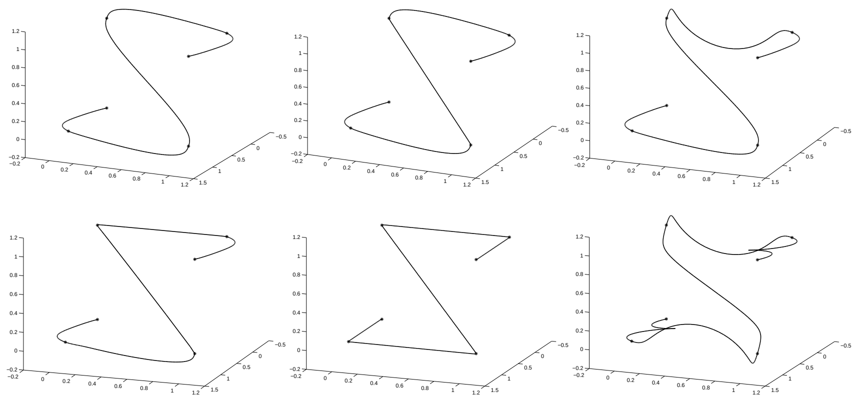

Fig. 2. Examples of local/global deformations on an open 3D curve: no tension ($v_i = w_i = 1,\ i = 0, ..., 4$), interval tension ($v_2 = w_2 = 50$), interval warping ($v_3 = w_3 = 0.001$), point tension ($v_3 = w_2 = 50$), global tension ($v_i = w_i = 50,\ i = 0, ..., 4$), global warping ($v_i = w_i = 0.001,\ i = 0, ..., 4$).

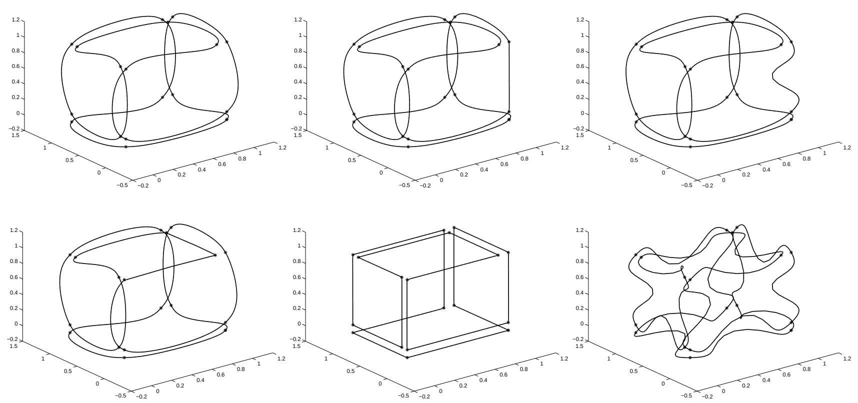

Fig. 3. Examples of local/global deformations on a closed 3D curve: no tension ($v_i = w_i = 1,\ i = 0, ..., 18$), interval tension ($v_3 = w_3 = 100$), interval warping ($v_3 = w_3 = 0.001$), point tension ($v_7 = w_6 = 60$), global tension ($v_i = w_i = 100,\ i = 0, ..., 18$), global warping ($v_i = w_i = 0.001,\ i = 0, ..., 18$).

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