Nodal minimal partitions in dimension 3

B. Helffer*
T. Hoffmann-Ostenhof**,***
S. Terracini#

Département de Mathématiques, Univ Paris-Sud and CNRS*
Institut für Theoretische Chemie, Universität Wien**
International Erwin Schrödinger Institute for Mathematical Physics***
Dipartimento di Matematica e Applicazioni, Università di Milano Bicocca#
February 12, 2010

Abstract

In continuation of [20], we analyse the properties of spectral minimal k-partitions of an open set Ω in $\mathbb{R}^3$ which are nodal, i.e. produced by the nodal domains of an eigenfunction of the Dirichlet Laplacian in Ω. We show that such a k-partition is necessarily the nodal one associated with a k-th eigenfunction. Hence we have in this case equality in Courant's nodal theorem.

1 Introduction

Let Ω is an open set[2] in $\mathbb{R}^d$ and let $H(D)$ denote, for any open set D in Ω, the Dirichlet realization of the Laplacian $H(D)$ in D; for a given integer $k \geq 1$, we associate with each k-partition $\mathcal{D} = (D_1, \dots, D_k)$ of Ω (that is with each family of k disjoint open sets $(D_1, \dots, D_k)$ in Ω)) the quantity

$$\Lambda(\mathcal{D}) = \max_j (\lambda(D_j))\,,$$

with $\lambda(D_j)$ denoting the lowest eigenvalue of $H(D_j)$.
Now, let us consider an eigenfunction having exactly k nodal domains of $H(\Omega)$: this produces a k-partition of Ω, which will be called nodal k-partition. Of course the value of Λ for that nodal partition if nothing but the value of the associated eigenvalue λ.

1991 Mathematics Subject Classification 35B05

[2]The precise assumptions of regularity will be given in Sections 2 and 3. We only give in this introduction "rough" statements.

In this paper we are concerned with the extremal values of

$$\mathfrak{L}_k(\Omega) = \inf_{\mathcal{D}} \Lambda(\mathcal{D})$$

and with the associated minimal k-partitions, that is, k-partitions which achieve the infimum.

Our aim is to show, in continuation of [20], that *if a minimal k-partition is a nodal partition, then it necessarily corresponds to the nodal domains of the k-th eigenfunction*. With Courant's nodal theorem is mind, we call these eigenfunctions "Courant-sharp" because they have the maximal number of nodal domains. Hence *Courant sharpness is equivalent to minimality of the corresponding k–partition.*

This result was obtained in dimension two in [20] together with other qualitative results on minimal spectral partitions. In contrast with the two dimensional case, the general structure of k-minimal partitions is only poorly understood in higher dimension. Our Theorem 3.1 summarises the results on the geometry of the boundary of the minimizing partition that can be obtained joining the results [20, 6, 7, 29]. In spite of this lack of information, we shall be able to perform the proof of the result for the 3-dimensional case, exploiting a careful analysis of the nodal sets of eigenfunctions for a class of auxiliary problems.

In Section 2 the main definitions and some 2-dimensional results are presented. The exact statement of the main theorem will be presented in Section 3. In §4, we recall the properties of nodal sets for domains in $\mathbb{R}^3$. The proof is then given in the §5-7 and finally in Section 8 we give two illustrative examples.

Acknowledgements

Many thanks to M. Dauge for discussions and references for the analysis of the regularity of the Dirichlet problem in dihedral like domains. We also thank M. Van den Berg and B. Davies for useful references on L^∞-bounds for the eigenfunctions of the Dirichlet Laplacian, and finally the Schrödinger Institute where part of this work has been done.

2 Definitions, notation and previous results.

We first recall notation, definitions and results extracted essentially from [20].

We consider the Dirichlet Laplacian $H(\Omega)$ on a bounded domain $\Omega \subset \mathbb{R}^d$. Under some weak regularity assumption, $H(\Omega)$ is selfadjoint when viewed as the Friedrichs extension of the quadratic form associated to H with form domain $W_0^{1,2}(\Omega)$ and form core $C_0^\infty(\Omega)$. We are interested in the eigenvalue problem for $H(\Omega)$ and note that $H(\Omega)$ has discrete spectrum $\sigma(H(\Omega))$. We denote by $\{\lambda_k\}_{k\in\mathbb{N}\setminus\{0\}}$ the ordered sequence of eigenvalues, such that the associated eigenfunctions u_k can be chosen to form an orthonormal basis for $L^2(\Omega)$. We shall denote for any open domain D by $\lambda(D)$ the lowest eigenvalue of $H(D)$ with Dirichlet boundary condition

$$\lambda(D) = \lambda_1(H(D))\,. \tag{2.1}$$

We know that u_1 can be chosen to be strictly positive in Ω. We define for any function $u \in C_0^0(\overline{\Omega})$

$$N(u) = \overline{\{x \in \Omega \mid u(x) = 0\}} \tag{2.2}$$

and call the components of $\Omega \setminus N(u)$ the nodal domains of u. The number of nodal domains of such a function will be called $\mu(u)$.

We now introduce the notions of partition and spectral minimal partition.

Definition 2.1
Let $1 \leq k \in \mathbb{N}$. We call a ***partition*** *of Ω (or k-partition if we want to indicate the cardinality of the partition) a family $\mathcal{D} = \{D_i\}_{i=1}^k$ of pairwise disjoint open sets such that*

$$\cup_{i=1}^k D_i \subset \Omega. \tag{2.3}$$

It is called ***strong*** *if*

$$\mathrm{Int}\,(\overline{\cup_{i=1}^k D_i}) \setminus \partial\Omega = \Omega. \tag{2.4}$$

We denote by $\mathfrak{O}_k$ the set of such partitions.

We now introduce spectral minimal partitions:

Definition 2.2
For $1 \leq k \in \mathbb{N}$ and $\mathcal{D} \in \mathfrak{O}_k$ we introduce

$$\Lambda(\mathcal{D}) = \max_i \lambda(D_i)\,, \tag{2.5}$$

and

$$\mathfrak{L}_k(\Omega) = \inf_{\mathcal{D}\in\mathfrak{O}_k} \Lambda(\mathcal{D}) \tag{2.6}$$

We call a k-partition $\mathcal{D} \in \mathfrak{O}_k$ a spectral minimal k-partition if $\mathfrak{L}_k(\Omega) = \Lambda(\mathcal{D})$.

Remark 2.3
If $k = 2$, the minimal value $\mathfrak{L}_2$ is the second eigenvalue and any minimal 2–partition is represented as the nodal partition associated to some second eigenfunction.

To each $\mathcal{D}$ we associate a graph $G(\mathcal{D})$ in the following way. We say $D_i, D_j \in \mathcal{D}$ are **neighbors**, if

$$\mathrm{Int}\,(\overline{D_i \cup D_j}) \setminus \partial\Omega \text{ is connected} \tag{2.7}$$

and denote this by $D_i \sim D_j$. To each $D_i \in \mathcal{D}$ we associate a vertex v_i and to each pair $D_i \sim D_j$ we associate an edge $e_{i,j}$. This defines a graph $G(\mathcal{D})$.

Attached to a partition $\mathcal{D}$ we can associate a closed set $N \in \overline{\Omega}$ defined by

$$N(\mathcal{D}) = \overline{\bigcup_i (\partial D_i \cap \Omega)}\,, \tag{2.8}$$

called the boundary of the partition. In the case of a nodal partition (associated to the nodal domains of an eigenfunction) this is simply the nodal set. In the $2D$ case, the boundary sets of minimal partitions exhibit regularity properties which are close to the properties of the nodal sets. We have introduced in [20] a class of sets called *regular* describing these properties. In particular we also introduced the notion of the *equal angle property*, a natural generalisation of the local properties of zero sets of eigenfunctions near points where the eigenfunction vanishes of higher order, see [20] for details.

The following theorem has been proved by Conti-Terracini-Verzini [8, 9, 10].

Theorem 2.4
We assume that the dimension is two. Then for each $k \in \mathbb{N}^*$, there exists a minimal regular[3] strong k-partition.

This existence theorem was completed in [20] by a regularity result.

Theorem 2.5
If the dimension is two, then any minimal spectral k-partition admits a representative which is regular, connected and strong. Moreover these partitions

[3]Except for isolated points, the boundary of the partition consists of C^∞ arcs.

satisfy the equal angle meeting property. Furthermore if $\mathcal{D} = \{D_i\}_{i=1}^k$ is a spectral minimal k-partition, then $\lambda(D_i) = \lambda(D_j)$ for all i, j.

A natural question is whether a minimal partition is the nodal partition induced by an eigenfunction. We have given in [20] (in the $2D$-case) a simple criterion for a partition to be associated to a nodal set. For this we need some additional definitions. We will say that the graph $G(\mathcal{D})$ is **bipartite** if its vertices can be colored by two colors (two neighbours having different colors). We recall that the graph associated to a collection of nodal domains of an eigenfunction is always bipartite. We have now the following converse theorem [20] :

Theorem 2.6
Assume that the dimension is two and that there is a minimal spectral k-partition (we choose then a strong, regular representative) of Ω such that the associated graph is bipartite. Then this partition is associated to the nodal set of an eigenfunction of $H(\Omega)$ corresponding to an eigenvalue equal to $\mathfrak{L}_k(\Omega)$.

A natural question is now to determine how general is the situation described in the previous theorem. The surprise is that this will only occur in the so called Courant-sharp situation.

Courant's nodal theorem says that the number of nodal domains $\mu(u)$ satisfies $m(u) \leq k$ for each function in the eigenspace of λ_k Then we say, as in [1], that u is **Courant-sharp** if $\mu(u) = k$. For any integer $k \geq 1$, we denote by L_k the smallest eigenvalue for which its eigenspace contains an eigenfunction with k nodal domains. In general we have

$$\lambda_k(\Omega) \leq \mathfrak{L}_k(\Omega) \leq L_k(\Omega) \ . \tag{2.9}$$

The next result of [20] gives the full picture of the equality cases :

Theorem 2.7
Suppose $\Omega \subset \mathbb{R}^2$ is regular. If either $\mathfrak{L}_k = L_k$ or $\lambda_k = \mathfrak{L}_k$, then

$$\lambda_k = \mathfrak{L}_k = L_k \ .$$

In addition, any minimal k-partition admits a representative which is the family of nodal domains of some Courant-sharp eigenfunction u associated with λ_k.

3 The case of dimension 3

We now discuss what can be extended to three dimensions and present our main theorem. In [20] (see also Conti-Terracini-Verzini [8, 9, 10], Bucur-Buttazzo-Henrot [4], Caffarelli-Lin [5, 6] and references therein) the existence of $\mathfrak{L}_k$ together with the existence of some minimal k-partition was shown. In particular, it is shown in [20] that properly normalized eigenfunctions associated with the minimal partition satisfy a certain system of differential inequalities (Theorems 3.4 and 3.8). This fact makes the results of [7] applicable and gives the following result on the structure of the minimal partitions :

Theorem 3.1
Let Ω be an open subset of $\mathbb{R}^d$ with a $\mathcal{C}^2$ boundary. For any k there is a representative[4] for a minimal spectral k-partition which is strong and connected. Its boundary consists of the union of a singular set, having Hausdorff dimension at most $d-2$, and of a collection of analytic codimension 1 manifolds. Furthermore if $\mathcal{D} = \{D_i\}_{i=1}^k$ is a spectral minimal k-partition, then $\lambda(D_i) = \lambda(D_j)$ for all i, j.

Sketch of the Proof.
According to Theorem 3.4 of [20], let $\mathcal{D} = \{D_i\}_{i=1}^k$ be any minimal partition associated with $\mathfrak{L}_k$ and let $(\tilde{\phi}_i)_i$ be any corresponding set of positive eigenfunctions normalized in L^2. Then there are nonnegative coefficients $a_i \geq 0$, not all vanishing, such that the functions $u_i = a_i \tilde{\phi}_i$ satisfy a certain system of differential inequalities, denoted in [20] as (I1) and (I2). From these inequalities, that can be extended through a regular boundary, it is deduced in [7] the validity of the Almgren's monotonicity formula and consequently the fact that the boundary set consists in the union of a singular set, having Hausdorff dimension at most $d-2$, and of a collection of $\mathcal{C}^{1,\alpha}$ manifolds (see also [29] for more details). Using the regularity of the boundary set, one can easily extend Theorem 4.14 in [20] from dimension 2 to any dimension, obtaining positivity of all coefficients a_i and connectedness of the open representative of the minimizing partition. Finally, arguing as in Remark 3.11 in [20] one then conclude that $\lambda(D_i) = \lambda(D_j)$ for all i, j. This last fact also improves the regularity of the regular part subset from $\mathcal{C}^{1,\alpha}$ to $\mathcal{C}^\omega$.

Unfortunately, the information contained in this Theorem are too weak to be used in extending Theorem 2.7 to the higher dimensional case. In contrast, in the proof of the extension of Theorem 2.7, which is a proof by

[4] see [20] for the definition

contradiction, with start with a nodal configuration associated with an eigenfunction. Hence we will exploit the regularity properties of nodal sets which are already proved in the former literature, rather than those of a minimal partition stated in Theorem 3.1. Indeed, our proof relies on the finiteness of the 1-Hausdorff measure of the singular part of the nodal set proved in [18], which is of course stronger than the fact that its Hausdorff dimension is at most one. On the other hand, it requires more stringent regularity of the boundary. The properties of the nodal set will be recalled in Section 4 (Proposition 4.2).

To avoid technical difficulties, we make the following strong but natural assumption.

Assumption 3.2
$\Omega \subset \mathbb{R}^3$ is a bounded domain with $\partial\Omega \in C^\omega$.

This assumption occurs in a related context in [14, 15].

Our main result is the following extension of Theorem 2.7 (Theorem 1.17 in [20]) to dimension 3.

Theorem 3.3
Suppose that Ω satisfies Assumption 3.2. If for some k, $\mathfrak{L}_k = L_k$, then

$$\lambda_k = \mathfrak{L}_k = L_k \,. \tag{3.1}$$

Remark 3.4
Suppose Ω satisfies the previous assumptions. Assume that $\lambda_k(\Omega) < \mathfrak{L}_k(\Omega)$, then $\mathcal{D}$, the spectral minimal k-partition associated with $\mathfrak{L}_k$, is non-nodal, i.e. is not produced by the eigenfunction u_m. This would be non-trivial only if $\mathcal{D}$ consisted of $k' > k$ domains. But this impossible due to Theorem 3.1.

Remark 3.5
As done in [20] for the 2D-case, we observe that Pleijel's sharpened version of Courant's nodal theorem [27] implies that, for any Ω satisfying the assumptions above, there is a $k_0(\Omega)$ such that for $k > k_0(\Omega)$ the minimal spectral partition associated to $\mathfrak{L}_k(\Omega)$ is non-nodal.

Remark 3.6
In addition, if for some k, $\lambda_k = \mathfrak{L}_k$, then (3.1) holds. This fact does not depend on the dimension and is simply based on the variational principle. If φ_i ($i = 1, \dots, k$) is the ground state relative to D_i. There exists indeed

a non trivial combination of the φ_i which is orthogonal to the eigenspace associated with the interval $[0, \lambda_{k-1}]$ for which the energy is λ_k. So by the minimax-principle, it is an eigenfunction.

4 Properties of nodal sets in the case of dimension 3

We consider the eigenvalues and the minimal spectral partitions associated to the Dirichlet problem on Ω. It is more difficult to describe the regularity properties of the nodal sets in three and higher dimensions than for the two dimensional case.

We know that an eigenfunction is analytic (hypoanalyticity of the Laplacian) in Ω and, under Assumption 3.2, it is also standard [22] that an eigenfunction is analytic up to the boundary. In fact we have, see [15], Proposition 4.1, the following more precise result:

Lemma 4.1
Suppose that Ω satisfies Assumption 3.2 and that u is a Dirichlet eigenfunction associated to λ. Then there is an open set $\tilde{\Omega}$ so that $\overline{\Omega} \subset \tilde{\Omega}$ and u extends to a real analytic function $\tilde{u}$ in $\tilde{\Omega}$ satisfying $-\Delta \tilde{u} = \lambda \tilde{u}$ in $\tilde{\Omega}$.

This can be proved in two steps. First one shows that it has an analytic extension. Secondly, one observes that $-\Delta \tilde{u} - \lambda \tilde{u}$ is analytic in a neighborhood of $\bar{\Omega}$ and vanishes in Ω. The result follows by unique continuation.

This result permits us to reduce the analysis of the local properties of nodal sets of eigenfunctions at the boundary to the analysis of the same problem at an interior point.

The next property concerns the Hausdorff measures of the nodal set of an eigenfunction and of the critical points of the nodal set (see [24] for the definition). It is worthwhile noticing that in the C^∞ case the Hausdorff dimension of the singular set can be any number between 0 and 1 as is noted in [18]. This is shown for a smooth divergence type operator. Of course for the analytic case we must have either 0 or 1.

Proposition 4.2
Suppose that Ω satisfies Assumption 3.2 and that u is an eigenfunction of $H(\Omega)$. Then :

- *The zeroset $N(u)$ of an eigenfunction u has finite 2-dimensional Hausdorff measure.*
- *The singular set $\Sigma(u)$, which is defined by*

$$\Sigma(u) = N(u) \cap \{x \in \overline{\Omega} \,:\, |\nabla u(x)| = 0\}\,, \tag{4.1}$$

has finite 1-dimensional Hausdorff measure.

Proof.
This follows either from more general results for the smooth case derived in [21] for the 3D case, see also [18] and [17] for the the higher dimensional case. For the real analytic case we can proceed more directly by investigating the function defined on Ω by :

$$f = |\nabla u|^2 + u^2\,. \tag{4.2}$$

f is real analytic and its zeroset is

$$N(f) = \Sigma(u)\,. \tag{4.3}$$

In order to describe the structure of $N(f)$, let us observe that the real analyticity implies, by a result of S. Lojasievicz [23], that $N(f)$ admits the following stratification :

$$N(f) = \Gamma_0 \cup (\cup_{i=1}^{s} \Gamma_{i1}) \cup (\cup_{j=1}^{r} \Gamma_{j2}) \tag{4.4}$$

with Γ_0 (a finite set) and, for each i, Γ_{i1} an analytic curve such that $\partial\Gamma_{i1} \subset \Gamma_0$, Γ_{j2} an analytic surface such that $\partial\Gamma_{j2} \subset \Gamma_0 \cup (\cup_{i=1}^{s} \Gamma_{i1})$.
Next we want to show that the decomposition of $N(f)$ does not contain a $2D$-component. Because we are in the analytic case, one can use the Cauchy-Kowalewski theorem and get that u is identically 0 near this $2D$-component, hence everywhere by analyticity.

So we have obtained :

Lemma 4.3
Under Assumption 3.2, we have in each relatively compact open set ω in Ω :

$$N(f) \cap \omega = \Gamma_0 \cup (\cup_{i=1}^{s} \Gamma_{i1}) \tag{4.5}$$

with Γ_0 (a finite set) and, for each i, Γ_{i1} an analytic curve such that $\partial\Gamma_{i1} \subset \Gamma_0$.

The same proof applied to u gives

Lemma 4.4
Under Assumption 3.2, we have in each relatively compact open set ω in Ω :

$$N(u) \cap \omega = \Gamma_0 \cup (\cup_{i=1}^{s_1} \Gamma_{i1}) \cup (\cup_{j=1}^{s_2} N_j) \tag{4.6}$$

with Γ_0 (a finite set), for each i, Γ_{i1} an analytic curve such that $\partial\Gamma_{i1} \subset \Gamma_0$, and N_j is a $(2D)$-analytic surface such that $\partial N_j \subset N(f)$.

Remark 4.5
The same proofs can be applied to $\tilde{u}$ and $\tilde{f} = \tilde{u}^2 + |\nabla \tilde{u}|^2$, with the notation of Lemma 4.1. This permits us to replace in the two previous statements ω by $\overline{\Omega}$.

Remark 4.6
Note that the proof of Lojasiewicz implies that the curves in $\Sigma(u)$ have finite length.

We will need the following relation between capacity (defined in the appendix) and Hausdorff measure.

Lemma 4.7
Suppose that $\Omega \subset \mathbb{R}^d$ is a bounded domain and that $\mathcal{E} \subset \Omega$ has finite $(d-2)$-dimensional Hausdorff measure, then $\mathrm{Cap}\,(\mathcal{E}) = 0$.

This is due to [26] (see e.g. Theorem 2.52 in [25]). As a consequence we have :

Proposition 4.8
Under Assumption 3.2 and if u is a real valued eigenfunction of $H(\Omega)$, then $\mathrm{Cap}\,(\Sigma(u)) = 0$.

Proof
We use Lemma 4.1. Hence $\overline{\Omega} \subset \tilde{\Omega}$ for some open domain $\tilde{\Omega}$. Now Lemma 4.7 applies directly. $\square$

We end this section with a property related with the nodal partition associated with an eigenvalue that we will use in the following.

Proposition 4.9
Let u be an eigenfunction of the Dirichlet Laplacian in Ω, and let $N(u)$ its nodal set. Then

- $u \in H_0^1(\Omega \setminus N(u))$
- *u is an eigenfunction of the Laplacian in* $\Omega \setminus \tilde{N}$ *for every* $\tilde{N} \subset N(u)$.

Proof
We only have to prove that $u \in H_0^1(\Omega \setminus N(u))$, the other part of the assertion being obviously true. To this aim, let η be a real smooth function such that $\eta(s) = 0$ for $|s| \leq 1$ and $\eta(s) = s$ for $|s| \geq 2$, and let $u_\epsilon(x) = \epsilon\eta(u(x)/\epsilon)$. As $u \in \mathcal{C}^\infty$ we have that $u_\epsilon \in \mathcal{C}_0^\infty(\Omega \setminus N(u))$. Moreover, as $\epsilon \to 0$, u_ϵ converges to u in the strong H^1 topology. This can be seen as an easy consequence of the Dominated Convergence Theorem, observing that $\int_\Omega 1_{u=0}(x)|\nabla u(x)|^2\, dx$ where $1_{u=0}$ is the characteristic function of $u = 0$.

5 Proof of Theorem 3.3.

5.1 Starting point of the proof

We follow as close as possible the proof given in Section 7 of [20]. We assume by contradiction that :

$$\lambda_k < \mathfrak{L}_k = L_k = \lambda_m \tag{5.1}$$

for some $m > k$.
This implies that there exists an eigenfunction $u = u_m$ with k nodal domains. We also assume **for the moment** that

$$\lambda_{m-1} < \lambda_m < \lambda_{m+1}, \tag{5.2}$$

hence that λ_m is simple. The goal is to show that (5.1) and (5.2) lead to a contradiction.

At the end of the section we will, as in [20], obtain the contradiction without assuming (5.2).

5.2 Abstract properties of the interpolating family $\mathcal{N}(\alpha)$.

The proof of Theorem 2.7 in [20] was based on an explicit construction of a continuously increasing interpolating family between $N(u)$ and $\emptyset$. We can explicitly consider each component of $N(u) \setminus \Sigma(u)$ which was either a closed line or a segment with end points in $\Sigma(u)$. The $(3D)$-construction is more involved and will be given in Section 6.

Our goal in this subsection is to propose to list all the "abstract properties" needed for the proof. We write $u = u_m$ and $N(u) = N(u_m)$. What we need is to construct a continuous increasing family of closed sets $\mathcal{N}(\alpha)_{\alpha\in[0,1]}$ in $\overline{\Omega}$ satisfying four properties.

Property 5.1 [P1]

$$\mathcal{N}(0) = \Sigma(u)\,,\, \mathcal{N}(\alpha) \subset \mathcal{N}(\alpha') \textit{ if } \alpha \le \alpha'\,,\, \mathcal{N}(1) = N(u)\,,$$

Actually, for technical reasons, we will start instead of $\mathcal{N}(0)$, from a suitable neigborhood of $\Sigma(u)$ (see (6.2)) in $N(u)$, noting that the Assumptions (5.1) and (5.2) are still satisfied if Ω is replaced by $\Omega \setminus \mathcal{N}(0)$.
Similarly, we will replace $N(u)$ for the definition of $\mathcal{N}(1)$ by $N(u) \setminus X^+$ where X^+ has capacity 0. The definition of $\mathcal{N}(0)$ and $\mathcal{N}(1)$ will be given in Section 6, respectively in (6.2) and (6.4).

With

$$\Omega(\alpha) = \Omega \setminus \mathcal{N}(\alpha)\,, \tag{5.3}$$

we need the continuity of the eigenvalues with respect to α :

Property 5.2 [P2]
For any ℓ,

$$\alpha \mapsto \lambda_\ell(\Omega(\alpha)) \in C^0([0,1])\,.$$

The continuity of the eigenvalues will ensured by the continuity in capacity of the exhausting family $\Omega(\alpha)$ (see section 7.3) :

Property 5.3 [P3]
$\mathcal{N}(\alpha) \setminus \lim_{\beta\to\alpha\,,\,\beta<\alpha} \mathcal{N}(\beta)$ *has capacity* 0.

Finally we require that, all along our family, λ_m is a an eigenvalue:

Property 5.4 [P4]
λ_m *is an eigenvalue of $H(\alpha)$ for any $\alpha \in [0,1]$.*

This requirement will be automatically fullfilled, thanks to Proposition 4.9, from the fact that $\mathcal{N}(\alpha)$ is already contained in the nodal set of the selected m-th eigenfunction.

An immediate consequence is the following

Lemma 5.5
Under Assumption [P1], the eigenvalues of $H(\alpha)$ are monotonically increasing for $0 \leq \alpha \leq 1$. Furthermore $\lambda_1(1) = \cdots = \lambda_k(1) = \lambda_m(0)$.

In the 2-dimensional case, the construction was easy because the description of $N(u)$ and $\Sigma(u)$ was explicit. In higher dimensions the situation is more complicated and one cannot hope for such an explicit description of $N(u), \Sigma(u)$ even for the analytic case (see Section 4).
In the construction given below $\mathcal{N}(\alpha) \setminus \lim_{\beta\to\alpha\,,\,\beta<\alpha} \mathcal{N}(\beta)$ will be a union of analytic curves in $N(u) \setminus \Sigma(u)$.

By Lemma 4.7 and a theorem of Gesztesy, Zhao [16] we have

$$\sigma(H(\Omega(0))) = \sigma(H(\Omega))\,. \tag{5.4}$$

where σ denotes the spectrum. Furthermore, thanks to the properties [P2] and [P3], we deduce

$$\lim_{\alpha\to 0} \sigma(H(\Omega(\alpha))) = \sigma(H(\Omega))\,. \tag{5.5}$$

5.3 Continuation of the proof.

We assume that we have constructed an exhausting family satisfying the properties [P1], [P2], [P3] and [P4] and continue to follow the proof of of Theorem 2.7 given in [20]. We are going to treat in full detail only those arguments that differ from the 2-dimensional case and we refer the reader to §7 of [20] for the remaining parts.

Lemma 5.6
There is an $\alpha_1 < 1$ such that λ_m is an eigenvalue of $H(\alpha_1)$ with multiplicity at least 2.

For $\alpha = 0$, λ_m is the m-th eigenvalue and for $\alpha = 1$ it is the k-th eigenvalue with $k < m$. This is then an immediate consequence of properties [P1] and [P2].

We consider at α_1 some normalized real valued eigenfunction of $H(\alpha_1)$ associated with λ_m which is orthogonal to $u = u_m$ and which we call w.

We try to prove that λ_m has multiplicity at least 2 as eigenvalue of $H(\Omega)$ which will be the desired contradiction to (5.2). So we consider for $\beta \in]-\epsilon, \epsilon[$ for sufficiently small $\epsilon > 0$

$$w_\beta = u + \beta w. \tag{5.6}$$

Remember that by assumption $u := u_m$ has k nodal domains.

As in [20] we have the following lemma:

Lemma 5.7
Under Assumptions [P1], [P2], [P3] and [P4], there is an $\epsilon > 0$ such that for $|\beta| \le \epsilon$, w_β has exactly k nodal domains.

Proof
The proof has two parts.
First part : $\mu(w_\beta) \le k$.
Suppose for contradiction w_β has more than k nodal domains. We now consider those nodal domains in all of Ω. Take one of those domains, say D_1 and consider one neighboring domain, say D_2. Then $\partial D_1 \cap \partial D_2 \ne \emptyset$. Hence we can consider some domain $D_1' \subset D_1$ which also neighbors D_2 and introduce $D_2' = \text{Int}\,(\overline{D_2 \cup D_1'})$. Then $\lambda_1(H(D_2')) < \lambda_1(D_2)$. Also the other neighboring domains can be treated the same way and eventually we will obtain a new k-partition with a $\mathfrak{L}_k' < \mathfrak{L}_k$, the desired contradiction.

Here we emphasize that our deformation can be done in a neighborhood of a regular point of $\partial D_1 \cap \partial D_2$.

Second part: $\mu(w_\beta) \ge k$
Next we have to show that $\mu(w_\beta)$ is at least k. To see this we observe that by construction $\Sigma(u) \subset \partial\Omega(\alpha_1)$. Moreover, assuming that we have affected a sign $\pm$ to each component D_i of $\Omega \setminus N(u)$ (in order to have a bipartite associated graph), we observe that our construction of $\Omega(\alpha_1)$ implies that a path joining two D_i's of same sign contained in $\Omega(\alpha_1)$ crosses another D_ℓ of opposite sign.
Now let us choose $x_i \in D_i$ ($i = 1, \dots, k$). It is clear that there exists a $\beta_0 > 0$ such that for $|\beta| \le \beta_0$ we have

$$u(x_i) w_\beta(x_i) > 0\,.$$

Hence, for $i = 1, \dots, k$, there exists a nodal domain $\widehat{D}_i$ of w_β in $\Omega(\alpha_1)$ containing x_i.

It remains to show that x_i can not be connected to x_j (for $j \neq i$) inside $\widehat{D}_i$. Of course, this concerns only two points such that $u(x_i)u(x_j) > 0$.

In the construction of $\Omega(\alpha_1)$, we have opened some windows in the regular $2D$-part of $N(u)$.

Let us consider one window $W_{i\ell}$ contained in ∂D_i. Of course there could be more than one window in ∂D_i and hence the index ℓ. This window connects D_i and a neighboring D_k of opposite sign. We claim that there exists a neighborhood of $W_{i\ell}$ in Ω such that $W_{i\ell}^{nbd} \cap D_j = \emptyset$, $W_{i\ell}^{nbd}$ contains only one window and $x_i \notin W_{i\ell}^{nbd}$. Moreover for β small enough $w_\beta(x)u(x_i) < 0$ for x on $\Sigma_{i\ell k} := \partial W_{i\ell}^{nbd} \cap D_k$ (Here we use that $|\nabla u| \neq 0$ in $W_{i\ell}^{nbd}$).
Now any path in $\Omega(\alpha_1)$ joining x_i and x_j must cross one of the $\Sigma_{i\ell k}$. In particular x_j cannot belong to $\widehat{D}_i$.

We hence have two distinct minimal k-partitions corresponding respectively to u and w_β and it is immediate to see that the associated graphs are the same, hence bipartite.

By construction $w = \frac{1}{\beta}(w_\beta - u)$ and is orthogonal to u. Consequently if we show that the extension $\hat{w}_\beta$ of w_β in Ω by 0 is an eigenfunction of $H(\Omega)$, then $\hat{w} := \frac{1}{\beta}(\hat{w}_\beta - u)$ is an eigenfunction of $H(\Omega)$ which is orthogonal to u and the corresponding eigenvalue λ_m of $H(\Omega)$ cannot be simple and we have a contradiction to the assumption done in the first subsection.

In the $2D$-case, we were applying Theorem 1.14 in [20] to the minimal k-partition created by w_β. Because, we do not have proven this result in the $(3D)$-case, we will come back to a more direct proof related to the fact that we have more information on our partition (and in particular its regularity). The argument is more closed to the approach in [19].

From our construction we know that

$$-\Delta w_\beta = \lambda_m w_\beta \text{ in } \Omega \setminus \mathcal{N}(\alpha_1)\,,$$

Consider the k-partition $\widehat{\mathcal{D}}$ associated with w_β. We know that it is minimal. In particular, for any pair (i,j), $(\widehat{D}_i, \widehat{D}_j)$ is a minimal 2-partition of $\widehat{D}_{ij} := \mathrm{Int}\,(\overline{\widehat{D}_i} \cup \overline{\widehat{D}_j})$.

Let us denote by w_β^i the restriction of w_β to $\widehat{D}_i$ extended by 0 outside $\widehat{D}_i$. From the characterization of the minimal 2-partitions, we obtain that, for any pair (i,j) of neighbors, there exists $\alpha_1(i,j)$ and $\alpha_2(i,j)$ such that

$$\alpha_1(i,j)w_\beta^i + \alpha_2(i,j)w_\beta^j \in H_0^1(\Omega)$$

But looking at this function in the neighborhood of a window between D_i and D_j, we obtain that $\alpha_1(i,j) = \alpha_2(i,j)$.
Hence, we get

$$w_\beta^i + w_\beta^j \in H_0^1(\Omega)\,. \tag{5.7}$$

From this we deduce that the extension of w_β by 0 satisfies $w_\beta \in H_0^1(\Omega)$ and

$$-\Delta w_\beta = \lambda_m w_\beta \text{ in } \Omega \setminus J(\widehat{\mathcal{D}})\,,$$

where $J(\widehat{\mathcal{D}})$ denotes the set of the critical points of w_β in $\Omega \cap \mathcal{N}(\alpha_1)$. It is worthwhile noticing that $J(\widehat{\mathcal{D}})$ has null capacity: indeed it consists either of regular points of the boundary $\partial\Omega(\alpha_1)$, where Proposition 4.2 applies, or of irregular points of $\partial\Omega(\alpha_1)$ which by property [P3] of our construction are included in a set of capacity 0.

End of the proof
The general case does not introduce additional difficulties in comparison, with that of the $2D$-case (case (b) in the proof of Theorem 1.17 in [20]).

6 Effective construction of the interpolating family

The remaining point is to construct an explicit family satisfying the abstract properties. Note that in a close context a construction was proposed in [3] but this does[5] not seem to be directly applicable.

We start by observing that $N(u)\setminus\Sigma(u)$ has a nice differentiable structure. Moreover, according to Lemmas 4.3 and 4.4 together with Remark 4.5,

$$N(u) = \Sigma(u) \cup (\cup_i N_i) \tag{6.1}$$

[5]The first author thanks D. Bucur for useful discussions.

where the family of the N_i is finite, each N_i is a regular "open" connected submanifold of dimension 2 in Ω whose boundary points are points of $\Sigma(u)$.

Now we use the function $f = |\nabla u|^2$ as a measure of the distance from $\Sigma(u)$ in $N(u)$. Indeed, the family $\mathcal{U}^\delta = f^{-1}([0,\delta[)$ is a fundamental system of neighborhoods of $\Sigma(u)$. We select those indices for which $\partial\mathcal{U}^\delta$ is a regular submanifold of $N(u) \setminus \Sigma(u)$ and we call $\mathcal{J} \subset \mathbb{R}$ the corresponding set. We can apply Sard's Lemma to the restriction of f to each N_i to prove that transversality holds for almost every $\delta > 0$ (here transversality means that $\nabla_{|N_i} f$ is transverse to $\partial\mathcal{U}^\delta$):

Proposition 6.1
There is a full measure set $\mathcal{J}$ such that, for every $\delta \in \mathcal{J}$, $\partial\mathcal{U}^\delta$ is a smooth submanifold of $N(u) \setminus \Sigma(u)$.

In particular, for any $\bar{\delta} > 0$, there exists $\delta \in \mathcal{J}$ such that $\mathrm{dist}\,(\partial U^\delta, \Sigma(u)) < \bar{\delta}$.

We denote

$$N_i^\delta = N_i \setminus \mathcal{U}^\delta(\Sigma(u)) \qquad , \qquad N^\delta = \bigcup_i N_i^\delta \,,$$

and

$$\Sigma^\delta = \partial\mathcal{U}^\delta(\Sigma(u)) \,.$$

Notice that, for each $\delta \in \mathcal{J}$, N^δ is a 2-dimensional compact manifold whose boundary Σ^δ is a smooth 1-dimensional submanifold.

We are going to deal with the steepest ascending flow Φ^t associated with a small perturbation of f. We remark that, because of the transverse intersection, N^δ is positively invariant under this flow: i.e., $\Phi^t(N^\delta) \subset N^\delta$, for every positive t.

Next, using again standard transversality theory (Sard's lemma), for each fixed $\delta \in \mathcal{J}$ we can take a smooth perturbation $\varphi : N^\delta \to \mathbb{R}$ of f restricted to N^δ which is a Morse function, whose associated flow enjoys all the standard nondegeneracy properties (Morse–Smale), that is:

- φ has a finite number of critical points of Morse index $(0, 1, 2)$ (corresponding to local minima, saddle point or maxima),
- the stable and unstable manifolds of critical points intersect transversally along heteroclinic lines joining two of them.

A basic reference on gradient flow of Morse-Smale type is [28]. Moreover, for $\delta \in \mathcal{J}$, we can assume that the normal derivative of φ is not zero on Σ^δ. Moreover, we can extend φ smoothly to the whole of $N(u)$, in such a way that it vanishes in a small neighborhood of $\Sigma(u)$.

Let us consider the gradient flow Φ^t associated with $\nabla\varphi$ on $N(u)$. By construction, $\nabla\varphi$ agrees with $\nabla_{|N_i} f$ at the boundary of N^δ: then, as already pointed out, since φ increases along the flow lines, N^δ is positively strictly invariant and Σ^δ is its entrance set with respect to the flow Φ^t ($t > 0$). Now, consider the (finite) set of critical points $K = \{x_\ell \in N^\delta : \nabla\varphi(x_\ell) = 0\}$, each with its Morse index $m(x_\ell) \in \{0, 1, 2\}$ and a pair of stable (unstable) manifolds $W^s(x_\ell)$ (resp. $W^u(x_\ell)$).

Among all critical points, we select the local maxima $K_{\text{Max}} = \{x \in K : m(x) = 2\}$ and the local minima $K_{\text{Min}} = \{x \in K : m(x) = 0\}$. Now, removing all the local minima K_{Min}, the flow Φ^t has the global attractor

$$X^+ = \bigcup_{m(x_\ell) \geq 1} \overline{W^u(x_\ell)} \cup K_{\text{Max}}$$

which is a union of compact manifolds having at most dimension 1.

We can provide a uniform estimate for the time of absorption of X^+.

Proposition 6.2
For every $\nu > 0$ small enough there exists $T > 0$ such that $\forall x \in N^\delta$ with $d(x, K_{Min}) > \nu$, $d(\Phi^t(x), X^+) < \nu$ for every $t > T$.

Proof. Indeed, assume the proposition was false. Then, for some $\bar\nu$ ($\bar\nu$ must be taken small enough so that the flow exits the balls of radius $\bar\nu$ around local minimizers), there would be a sequence (x_n, t_n) such that $d(x_n, K_{\text{Min}}) > \bar\nu$, $t_n \to +\infty$ and $d(\Phi^{t_n}(x_n), X^+) \geq \bar\nu$. Hence we have $d(\Phi^t(x_n), K_{\text{Min}}) > \bar\nu$ for every $t > 0$. Up to a subsequence, we can assume $\Phi^{t_n}(x_n) \to y \notin X^+$, and hence, $d(\Phi^t(y), K_{\text{Min}}) > \bar\nu$, for every $t \in \mathbb{R}$. Consequently, the α–limit of y –i.e. the limit as $t \to -\infty$ of $\Phi^t(y)$– of y can not be a local minimum, thus the orbit of y is an heteroclinic connection between two critical points with non vanishing Morse index. As such it lies entirely on X^+, while we have $y \notin X^+$, a contradiction.

We now describe what is our initial $\Omega(0)$. In the construction of the interpolating family, instead removing $\Sigma(u)$, we will remove a full neighborhood of $\Sigma(u)$ together with a suitable neighborhood of the local minimizers of φ:

indeed, we define

$$\mathcal{N}(0) = \overline{\mathcal{U}^\delta \cup \bigcup_{m(x_\ell)=0} N(u) \cap B(x_\ell, \nu)}\,. \tag{6.2}$$

and

$$\Omega(0) := \Omega \setminus \mathcal{N}(0)\,. \tag{6.3}$$

Let us denote the boundary of $\mathcal{N}(0)$ by :

$$\Sigma^+ = \partial\mathcal{U}^\delta \cup \bigcup_{m(x_\ell)=0} N(u) \cap \partial B(x_\ell, \nu)\ .$$

Because of our transversality and invariance assumptions, the following holds true.

Lemma 6.3
Σ^+ is a smooth, compact 1-dimensional manifold, transverse to the flow. It cuts $N(u)$ into two parts: $\mathcal{N}(0)$ and its complement. Moreover $\mathcal{N}(0)$ is invariant by the flow for negative times.

It is clear from the previous discussion that the parameters δ and ν can be taken small enough so that the capacity of $\Omega \setminus \Omega(0)$ is as small as we wish. Next we define our interpolating family simply as the flow evolution of the starting set.

$$\begin{aligned} &\mathcal{N}(\alpha) = \Omega \setminus \Omega(\alpha) = \Phi^{\alpha/(1-\alpha)}(\mathcal{N}(0))\ , \qquad \forall \alpha \in [0,1) \\ &\mathcal{N}(1) = N(u) \setminus X^+\ . \end{aligned} \tag{6.4}$$

In order to show that it fits the requirements for an interpolation family, we need the following proposition.

Proposition 6.4
For every $T > 0$,

$$\begin{aligned} &\bigcup_{0\le t\le T} \Phi^t\left(\mathcal{N}(0)\right) = \Phi^T\left(\mathcal{N}(0)\right) \\ &\Phi^T\left(\mathcal{N}(0)\right) \setminus \bigcup_{0\le t<T} \Phi^t\left(\mathcal{N}(0)\right) = \Phi^T\left(\Sigma^+\right) \\ &\bigcup_{t>0} \Phi^t\left(\mathcal{N}(0)\right) = N(u) \setminus X^+\ , \end{aligned} \tag{6.5}$$

This is non-trivial only if $\mathcal{D}$ would consist of $k' > k$ domains. But this impossible due to Theorem 3.1.

Proof.
Going back to our Lemma 6.3, as the flow turns on, we can see the boundary $\Phi^t(\Sigma^+)$ moving towards the interior of $N(u) \setminus \mathcal{N}(0)$. Of course the moving boundary will keep the property of transversality with respect to the flow. Hence the first two assertions are straightforward consequences of the definition. The third point follows directly from Proposition 6.4.

As a consequence of the above proposition, the family is continuous in capacity:

Lemma 6.5
For every $\alpha \in [0,1]$, there holds:

$$\lim_{\substack{\beta\to\alpha\\ \beta\in[0,1]}} \mathrm{Cap}\,(\overline{\Omega(\beta)\setminus\Omega(\alpha)}) = 0\,. \tag{6.6}$$

Moreover, for every $\varepsilon > 0$, the parameters can be chosen so that

$$\mathrm{Cap}\,(\overline{\Omega\setminus\Omega(0)}) < \varepsilon. \tag{6.7}$$

Proof.
When $\alpha \in [0,1[$ this is a consequence of the fact that

$$\lim_{\substack{\beta\to\alpha\\ \beta\in[0,1]}} \overline{\Omega(\beta)\setminus\Omega(\alpha)} = \Phi^{\alpha/(1-\alpha)}(\Sigma^+)$$

and the last set has null capacity. The continuity at $\alpha = 1$ follows again from Proposition 6.4. To prove the last assertion, just consider that, for δ and ν sufficiently small, $\mathcal{N}(0)$ can be included in an arbitrarily small neighborhood of the singular set $\Sigma(u)$ together with a finite number of arbitrarily small balls.

Joining the last lemma together with the results of next Section 7.3, we can finally conclude that

Proposition 6.6
The exhausting family defined in (6.4) satisfies [P1], [P2] and [P3] for suitable values of the parameters δ and ν.

7 Continuity of eigenvalues

7.1 Main result.

This section is devoted to the proof of the continuity of eigenvalues for families of domains which are continuous with respect to capacity. This result is probably known but, since we could not find it in the literature, we prefer to give an explicit proof. We refer to §6 in the book [2] for a systematic exposition of the continuity properties of eigenvalues with respect to variations of the domains, in connection to other types of domain approximations and with Mosco and γ–convergence.

Theorem 7.1
Let Ω be a bounded domain in $\mathbb{R}^3$. Let Ω_n a sequence of open subsets of Ω, converging to an open $\widehat{\Omega}$ in capacity, in the sense that:

$$\lim_{n\to+\infty} \operatorname{Cap}(\widehat{\Omega}\triangle\Omega_n) = 0\,. \tag{7.1}$$

Then, for any $j \in \mathbb{N}^*$,

$$\lim_{n\to+\infty} \lambda_j(\Omega_n) = \lambda_j(\widehat{\Omega})\,. \tag{7.2}$$

7.2 Around L^∞-boundedness of the eigenfunctions.

To prove our theorem, we make use of an L^∞ bound on normalized eigenfunctions. To our purposes, the bound may depend on the eigenvalue but should be uniform with respect to families of domains which are continuous in capacity. The L^∞ bound for the eigenfunctions is[6] a result of Davies [12] (Lemma 3.1 together with the remarks at the end of the paper) or [13] (Example 2.1.8 on page 62-63). More precisely, if Ω is any bounded subset of $\mathbb{R}^d$, then the heat kernel $K_0(t,x,y)$ of $\exp -tH(\Omega)$ satisfies the pointwise bound

$$0 \le K_0(t,x,y) \le (4\pi t)^{-\frac{d}{2}} e^{-|x-y|^2/4t}\,,$$

This implies for a suitable choice of t that an j'th normalized eigenfunction Φ_j (associated with the eigenvalue λ_j) of $H(\Omega)$ satisfies

$$\|\Phi_j\|_\infty \le e^{1/8\pi} \lambda_j^{d/4}.$$

In our application j will be fixed. The dependence on the open set Ω is only through λ_j and will be easy to control by monotonicity.

[6]We thank M. Van den Berg for mentioning to us these references.

Remark 7.2
One can also think of using Theorem 1.1 [11] (case (ii)) showing that for open sets $\Omega \subset \mathbb{R}^3$ with vertices (as we will construct later) then there exists $\epsilon > 0$ such that an eigenfunction u belongs to $W^{1,3+\epsilon}(\Omega)$. The statement implies that u is bounded but the control of the uniformity with respect to Ω seems more difficult.

7.3 Proof of Theorem 7.1

In what follows we use the characterization of the spaces $H_0^1(\Omega_n)$ (and similarly for $\widehat{\Omega}$) as:

$$H_0^1(\Omega_n) := \overline{C_0^\infty(\Omega_n)}^{H^1(\Omega_n)}$$

Step 1. We first prove upper semi–continuity, i.e. that

$$\limsup_{n\to+\infty} \lambda_j(\Omega_n) \leq \lambda_j(\widehat{\Omega}).$$

For this, we only need to find a j-dimensional subspace $\mathcal{E}_{j,n}$ in $H_0^1(\Omega_n)$ and ϵ_n such that $\lim_{n\to+\infty} \epsilon_n = 0$ and

$$Q_n(\Phi) \leq (\lambda_j(\widehat{\Omega}) + \epsilon_n)||\Phi||^2\,, \tag{7.3}$$

for all $\Phi \in \mathcal{E}_{j,n}$. Here $Q_n(\Phi)$ is the Dirichlet form :

$$\Phi \mapsto Q_n(\Phi) = \int_{\Omega_n} |\nabla\Phi(x)|^2\, dx\,. \tag{7.4}$$

Let us construct $\mathcal{E}_{j,n}$. Our assumption (7.1) gives (see Proposition A.1) the existence of maps $\eta_n \in H_0^1(\Omega)$ such that, $0 \leq \eta_n \leq 1$, $\eta_n = 0$ in $\widehat{\Omega}\triangle\Omega_n$, $\eta_n = 1$ in a compact set K_n of Ω_n and such that

$$\lim_{n\to+\infty} \int (|\nabla\eta_n|^2 + (1-\eta_n)^2)\, dx = 0\,.$$

Let $\widehat{\mathcal{E}_j}$ some[7] spectral space attached to $\lambda_1(\widehat{\Omega}), \ldots, \lambda_j(\widehat{\Omega})$. For any $\Phi \in \widehat{\mathcal{E}_j}$, we have

$$Q_\infty(\Phi) \leq \lambda_j||\Phi||^2\,. \tag{7.5}$$

We now define $\mathcal{E}_{j,n}$ by

$$\mathcal{E}_{j,n} = \eta_n\widehat{\mathcal{E}_j}\,.$$

[7] In the case that λ_j is not simple, we make a choice!

We write $\Phi = \sum_{\ell=1}^{j} a_\ell \varphi_\ell$. Then by the regularity of the eigenfunctions of $-\Delta^D$ in $\widehat{\Omega}$, we obtain the existence of C_j such that

$$||\Phi||_\infty \leq C_j ||\Phi||_2 \,, \; \forall \Phi \in \mathcal{E}_j \,. \tag{7.6}$$

We now compute $Q_n(\eta_n \Phi)$. By testing the equation for Φ with $\eta_n^2 \Phi$ we find, using the L^∞–estimate above,

$$\begin{array}{ll} Q_n(\eta_n\Phi) & = \int_\Omega |\nabla(\eta_n\Phi)|^2\,dx \\ & = \int_\Omega \eta_n^2 |\nabla\Phi|^2\,dx + 2\int_\Omega \eta_n \nabla\eta_n \nabla\Phi\,dx + \int_\Omega |\nabla\eta_n|^2\,dx \\ & = \int_\Omega \eta_n^2 \left(\sum_\ell^j \lambda_\ell a_\ell \varphi_\ell\right)\left(\sum_\ell^j a_\ell\varphi_\ell\right)\,dx + \int_\Omega |\nabla\eta_n|^2 |\Phi|^2\,dx \\ & \leq (\lambda_j + C_j(\lambda_j ||(1-\eta^2)||_2 + ||\nabla\eta_n||_2))\,||\Phi||^2\,. \end{array} \tag{7.7}$$

We now observe that, there exists a sequence γ_n tending to 0 such that, for $\Phi \in \mathcal{E}_j$

$$(1-\gamma_n)||\Phi||^2 \leq ||\eta_n\Phi||^2 \leq ||\Phi||^2\,. \tag{7.8}$$

This achieves the proof of the first step.

Step 2. Now we prove that, $j \in \mathbb{N}^*$,

$$\liminf_{n\to+\infty} \lambda_j(\Omega_n) \geq \lambda_j(\widehat{\Omega}).$$

First of all, by selecting a subsequence such that $\mathrm{cap}(\Omega_n \triangle \Omega) < 1/2^n$ and by replacing Ω_n with $\widehat{\Omega} \cup \bigcup_{k\geq n} \Omega_k$ we can reduce to the case of decreasing sequences.

Let us consider, for a given j, a converging sequence of normalized eigenfunctions $\varphi_{j,n}$ in $H_0^1(\Omega_n)$ attached to $\lambda_j(\Omega_n) =:= \lambda_{j,n}$. We denote its limit by $\lambda_{j,\infty}$.
We now observe that, there exists a constant C such that

$$||\varphi_{j,n}||_{L^\infty(\Omega)} + ||\varphi_{j,n}||_{H_0^1(\Omega)} \leq C\,. \tag{7.9}$$

Extracting possibly a subsequence, we can assume that $\varphi_{j,n}$ weakly converges in $H_0^1(\Omega)$ and (by compactness) strongly in $L^2(\Omega)$ to some v_j in the unit sphere of L^2. We also deduce a uniform bound on the L^∞ norm of the $\varphi_{j,n}$'s and, of course, of their limit v_j.

Let η_ϵ be as in Proposition A.1 be vanishing on $\widehat{\Omega}\triangle\Omega_n = \Omega_n \setminus \widehat{\Omega}$ for each n sufficiently large. Then $\eta_\epsilon \varphi_{j,n}$ and $\eta_\epsilon v_j \in H_0^1(\Omega_n)$; we also remark that

$\eta_\epsilon \varphi_{j,n}$ converges weakly in $H^1(\widehat{\Omega})$ and strongly in L^2 to $\eta_\epsilon v_j$. Hence testing the equation

$$-\Delta \varphi_{j,n} = \lambda_{j,n} \varphi_{j,n} \text{ in } \Omega_n \,, \tag{7.10}$$

with $\varphi_{j,n} - \eta_\epsilon v_j$ and passing to the limit first with respect to n and then with respect to ϵ, we infer the convergence of the norms and hence the strong convergence of $\varphi_{j,n}$ to v_j. Therefore $v_j \in H_0^1(\widehat{\Omega})$. In addition, we have

$$-\Delta v_j = \lambda_{j,\infty} v_j \text{ in } \widehat{\Omega} \,, \tag{7.11}$$

in the sense of distributions. Hence, as $v_j \not\equiv 0$, $\lambda_{j,\infty}$ is an eigenvalue of the Dirichlet Laplacian in $\widehat{\Omega}$.

In this way we have proved that the sequence of eigenvalues if the approximating domains do converge to an eigenvalue of the limiting domain. With a simple inductive argument it is now quite easy to finish the proof. Indeed, it is clear from Step 1 that the sequence of first eigenvalues of the approximating domains converges to the first eigenvalue of $\widehat{\Omega}$. Let us assume that continuity has been proved up to the j-th eigenvalue. If this last eigenvalue is simple, then the sequence of the $(j + 1)$-th eigenvalues must converge to some eigenvalue which, by Step 1, can be only the $(j + 1)$-th eigenvalue of the limiting domain.

To control the case of multiple j-th eigenvalue it is enough to consider the full family of the first j orthonormal converging eigenfunctions and to select a sequence of $(j + 1)$-th eigenfunctions orthogonal to this family. Again, passing to the limit, the upper semicontinuity proved in Step 1 allows to conclude that

$$\lim_{n\to+\infty} \lambda_{j+1}(\Omega_n) = \lambda_{j+1}(\widehat{\Omega}) \,. \tag{7.12}$$

8 Some Examples.

In this last section we consider two explicit examples of nodal, respectively non-nodal minimal partitions.

8.1 Cylindrical domains.

As first example of application, we can consider a cylinder

$$\Omega = \omega \times]0, \ell[\,, \tag{8.1}$$

where ω is a bounded domain with analytic boundary in $\mathbb{R}^2$ or a suitable polygon like a rectangle, a half disk or another domain which can be extended analytically. We want to investigate whether $\lambda_3(\Omega)$ has a Courant-sharp eigenfunction or not.

First consider the eigenvalues associated to $\Omega = \Omega(\ell)$. Let $\gamma_1 < \gamma_2 \leq \gamma_3 \leq$ be the increasing eigenvalues of the 2-dimensional Dirichlet problem $-\Delta$ on ω. Then $\lambda_1(\ell) = \lambda_1(\Omega(\ell)) = \gamma_1 + \frac{\pi^2}{\ell^2}$ and the spectrum of $H(\Omega(\ell))$ is given by

$$\{\gamma_i + \frac{k^2\pi^2}{\ell^2}\}_{i,k}. \tag{8.2}$$

Proposition 8.1
Under assumption 8.1, if

$$\ell^2 \geq 8\pi^2(\gamma_2 - \gamma_1)^{-1}\,, \tag{8.3}$$

then any minimal 3*-partition is nodal and the nodal partition is given by*

$$\omega \times \Big( ]0\,, \frac{\ell}{3}[\, \cup \,]\frac{\ell}{3}\,, \frac{2\ell}{3}[\, \cup \,]\frac{2\ell}{3}\,, \ell[\Big). \tag{8.4}$$

If

$$3\pi^2(\gamma_3 - \gamma_1)^{-1} < \ell^2 < 8\pi^2(\gamma_2 - \gamma_1)^{-1}\,, \tag{8.5}$$

no minimal 3*-partition can be nodal.*

Proof. (8.2) implies that

$$\lambda_3(\ell) \in \Big\{ \gamma_1 + \frac{4\pi^2}{\ell^2}, \, \gamma_1 + \frac{9\pi^2}{\ell^2}, \, \gamma_2 + \frac{\pi^2}{\ell^2}, \, \gamma_3 + \frac{\pi^2}{\ell^2} \Big\}. \tag{8.6}$$

Courant's nodal theorem implies $\gamma_2 + \frac{4\pi^2}{\ell^2} > \lambda_3$ because the associated eigenfunction has 4 nodal domains. We know for sure that if $\lambda_3 = \gamma_1 + \frac{9\pi^2}{\ell^2}$ there is a nodal 3-partition. This can happen only if $\gamma_1 + \frac{9\pi^2}{\ell^2} < \gamma_2 + \frac{\pi^2}{\ell^2}$ and this leads to (8.3).

If $\lambda_3(\ell) = \gamma_1 + \frac{4\pi^2}{\ell^2}$ then we must have $\lambda_3 \leq \gamma_3 + \frac{\pi^2}{\ell^2}$ and if $\lambda_3 = \gamma_2 + \frac{\pi^2}{\ell^2}$ then $\lambda_3 \leq \gamma_1 + \frac{9\pi^2}{\ell^2}$. Those inequalities yield (8.5) and hence by remark 3.4 the result. This ends the proof.

Finally we note that if we know that γ_3 is associated to an eigenfunction on ω with 3 nodal domains that then $\ell^2 < 3\pi^2/(\gamma_3 - \gamma_1)$ implies also that there is a nodal 3-partition.

8.2 The cuboid.

We can also consider a cuboid, i.e. $\Omega =]0,a[\times]0,b[\times]0,c[$, where a, b, c are chosen such that the eigenvalues $\lambda_{mnk} = \pi^2(\frac{m^2}{a^2} + \frac{n^2}{b^2} + \frac{k^2}{c^2})$ are simple. Then we can show the following:

Proposition 8.2
If $\min(m, n, k) \geq 2$ *then* λ_{mnk} *is not Courant sharp. This means that the spectral minimal partition associated to* $\mathfrak{L}_{mnk}$ *is non-nodal.*

To see this we just have to show that $\lambda_{2,2,2} > \lambda_8(\Omega)$. It suffices following a similar argument of [20] to show that $\lambda_{3,1,1}, \lambda_{1,3,1}, \lambda_{1,1,3} > \lambda_{2,2,2}$ leads to a contradiction. This means that $\lambda_{2,2,2}$ is not Courant sharp. We have

$$\pi^2(\frac{9}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}) > 4\pi^2(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}) := \lambda_{2,2,2}$$
$$\pi^2(\frac{1}{a^2} + \frac{9}{b^2} + \frac{1}{c^2}) > \lambda_{2,2,2},\ \pi^2(\frac{1}{a^2} + \frac{1}{b^2} + \frac{9}{c^2}) > \lambda_{2,2,2}.$$

Adding up these three inequalities, we obtain

$$11(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}) > 12(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}),$$

a contradiction. Here we use that, if a Courant sharp eigenfunction has some connected subfamily of nodal domains, then the corresponding restriction is associated with a Courant sharp eigenfunction of this subdomain. This can be found in [1].

Finally we can deduce from results in [20] the following:
If $k = 1$, $\min\{m, n\} \geq 3$ or if $k = 1$ and $m = 2$, $n \geq 4$, then also the corresponding eigenfunctions are not Courant sharp.
Of course the indices k, m, n above can be permuted.

A Capacity

Denote by $C(\mathcal{E})$ the capacity of a set $\mathcal{E}$. Following [2]

$$\mathrm{Cap}(\mathcal{E}) = \inf_{u \in \mathcal{U}_{\mathcal{E}}} \left\{ \int_{\mathbb{R}^d} (|\nabla u|^2 + |u|^2) dx \right\} \tag{A.1}$$

where

$$\mathcal{U}_{\mathcal{E}} = \{u \in W^{1,2}(\mathbb{R}^d) \bigm| u \geq 1 \text{ a.e in a neighborhood of } \mathcal{E}\}. \tag{A.2}$$

A local version of capacity, the capacity of a subset $\mathcal{E}$ of D is defined as follows:

$$\mathrm{Cap}(\mathcal{E}, D) = \inf\Big\{ \int_D |\nabla u|^2 dx \mid u \in W_0^{1,2}(D) \cap U_{\mathcal{E}} \Big\}. \tag{A.3}$$

Proposition A.1
Let Σ be compactly contained set in an open set Ω and having finite 1-*dimensional Hausdorff measure. Then, for any $\epsilon > 0$ there exists $\eta \in C_0^\infty(\Omega)$ such that $\eta = 0$ on Σ, $\eta = 1$ in the complement of a neighborhood $\mathcal{U}(\Sigma)$ of Σ, and*

$$\int_\Omega |\nabla \eta|^2 \, dx < \epsilon \, .$$

This is a standard result. The conclusion is simply a reformulation of the property that Σ has zero relative capacity with respect to Ω.

References

[1] A. Ancona, B. Helffer, and T. Hoffmann-Ostenhof. Nodal domain theorems à la Courant. Documenta Mathematica 9, p. 283-299 (2004).

[2] D. Bucur and G. Buttazzo, Variational Methods in Shape Optimization Problems. Birkhäuser 2005.

[3] D. Bucur, G. Buttazzo, and I. Figuereido, On the attainable eigenvalues of the Laplace operator. Siam J. Math. Anal 30 (3), p. 527-536 (1999).

[4] D. Bucur, G. Buttazzo, and A. Henrot, Existence results for some optimal partition problems. Adv. Math. Sci. Appl. 8, p. 571-579 (1998).

[5] L.A. Caffarelli and F.H. Lin, An optimal partition problem for eigenvalues. Journal of scientific Computing 31 (1/2). DOI: 10.1007/s10915-006-9114.8 (2007)

[6] L.A. Caffarelli and F.H. Lin, Singularly perturbed elliptic systems and multi-valued harmonic functions with free boundaries. J. Amer. Math. Soc. 21 (3), p. 847-862 (2008).

[7] L. A. Caffarelli, A. L. Karakhanyan, and F-H Lin, The geometry of solutions to a segregation problem for nondivergence systems J. Fixed Point Theory and Applications 5, p. 319-351 (2009).

[8] M. Conti, S. Terracini, and G. Verzini, An optimal partition problem related to nonlinear eigenvalues. J. Funct. Anal. 198, p. 160-196 (2003).

[9] M. Conti, S. Terracini, and G. Verzini, A variational problem for the spatial segregation of reaction-diffusion systems. Indiana Univ. Math. J. 54 (3), p. 779-815 (2005).

[10] M. Conti, S. Terracini, and G. Verzini. On a class of optimal partition problems related to the Fučik spectrum and to the monotonicity formula. Calc. Var. 22, p. 45-72 (2005).

[11] M. Dauge, Neumann and mixed problems on curvilinear polyhedra. Integral Equations Operator Theory 15, p. 227-261 (1992).

[12] E.B. Davies, Properties of the Green's functions of some Schrödinger operators. J. London Math. Soc. (2) 7, p. 483-491 (1973).

[13] E.B. Davies, *Heat kernels and Spectral Theory,* Cambridge University Press 1989,

[14] H. Donnelly and C. Fefferman, Nodal sets of eigenfunctions on Riemannian manifolds. Inv. Math. 93, p. 161-183 (1988).

[15] H. Donnelly and C. Fefferman, Nodal sets of eigenfunctions, Riemannian manifolds with boundary, Analysis etc, research papers published in honor of Jürgen Moser's 60th birthday. Edited by Paul H. Rabinowitz and Eduard Zehnder. p. 251-262. Academic Press, Boston, MA, (1990).

[16] F. Gesztesy and Z. Zhao, Domain perturbations, Brownian motion, and ground states of Dirichlet Schrödinger operators. Math. Z. 215, p. 143-150 (1994).

[17] Q. Han, R. Hardt, and F.H. Lin, Geometric measure of singular set of elliptic equations. Comm. Pure. Appl. Math. 51, p. 1425-1443, 1998. errata, *ibid.* 52, 271, 1999.

[18] R. Hardt, M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof and N. Nadirashvili. Critical sets of solutions to elliptic equations. J. Differential Geom. 51, p. 359-373 (1999).

[19] B. Helffer and T. Hoffmann-Ostenhof, Converse spectral problems for nodal domains. Moscow Mathematical Journal 7, p. 67-84 (2007).

[20] B. Helffer, T. Hoffmann-Ostenhof, and S. Terracini, Nodal domains and spectral minimal partitions. Annales de l'Institut Henri Poincaré : section Analyse nonlinéaire 26, p. 101-138 (2009).

[21] M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, and N. Nadirashvili, Critical sets of smooth solutions to elliptic equations in dimension 3. Indiana Univ. Math J. 45, p. 15-37 (1996).

[22] J.L. Lions and E. Magenes, *Problèmes aux limites non-homogènes II.* Dunod.

[23] S. Lojasiewicz, Sur le problème de la division. Studia Mathematica 18, p. 87-136 (1959).

[24] P. Mattila, *Geometry of sets and measure on Euclidean spaces.* Cambridge University Press, 1995.

[25] J. M. Michael and W.P. Ziemer, The Wiener criterion and quasilinear uniformly elliptic equations. Ann. Inst. Henri Poincaré, Analyse non linéaire 4, p. 453-486 (1987).

[26] Norman G. Meyers, A theory of capacities for potentials of functions in Lebesgue classes. Math. Scand. 26, p. 255-292 (1971).

[27] A. Pleijel, Remarks on Courant's nodal theorem. Comm. Pure. Appl. Math. 9, p. 543-550 (1956).

[28] S. Smale, Morse inequalities for a dynamical system, Bull. Amer. Math. Soc. 66, p. 43-49 (1960).

[29] H. Tavares and S. Terracini, Regularity of the nodal set of segregated critical configurations with a weak riflection property. preprint (2010).

B. Helffer: Département de Mathématiques, Bat. 425, Université Paris-Sud, 91 405 Orsay Cedex, France.

email: Bernard.Helffer@math.u-psud.fr

T. Hoffmann-Ostenhof: Institut für Theoretische Chemie, Universität Wien, Währinger Strasse 17, A-1090 Wien, Austria and International Erwin Schrödinger Institute for Mathematical Physics, Boltzmanngasse 9, A-1090 Wien, Austria.

email: thoffman@esi.ac.at

S. Terracini: Università di Milano Bicocca, Via Cozzi, 53 29125 Milano (Italy).

email: Susanna.Terracini@unimib.it