Uniform Hölder bounds and regularity properties of the limiting profile for highly competing nonlinear systems of Schrödinger equations

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1 Bose–Einstein Condensation

The idea behind Bose–Einstein condensation is that in some physically extreme circumstances, the particles of a quantum gas (photons, atoms) start moving as a single one, in the sense that the governing statistics is essentially obtained by restricting the physical Hilbert space to be the symmetric tensor product of single particle state:

$$\phi(x_1, \ldots, x_N) = \prod_{i=1}^{N} \varphi(x_i) \ .$$

Bose–Einstein condensation was predicted in the 20's, but its first empirical evidence was only obtained in 1995, in experiments performed by groups led by Cornell and Wieman at the University of Colorado at Boulder and by Ketterle at MIT. A rigorous derivation of the model has not been completely settled yet, not even for the plain Bose-Einstein condensation. Lieb, Yngvason, and Seiringer considered a trapped Bose gas consisting of N three-dimensional particles described by the Hamiltonian

$$H_N = \sum_{1 \le j \le N} (\Delta_j + V_{ext}(x_j)) + \sum_{1 \le i < j \le N} V_N(x_i - x_j).$$

where e V_{ext} is an external confining potential and $V_N(x) = N^2 V(Nx)$ is a spherically symmetric repulsive interaction potential.

Letting $N \to \infty$, they showed that the ground state energy $E(N)$ converges to the ground state energy of the Gross-Pitaevskii energy functional, after a suitable normalization,

$$\boxed{\mathcal{E}_{GP}(u) = \int |\nabla u|^2 + V(x)u^2 + 4\pi a_0 u^4 \ .}$$

Lieb and Seiringer also proved that the ground state of the Hamiltonian exhibits complete Bose-Einstein condensation into the minimizer of the Gross-Pitaevskii energy functional. The convergence of the whole dynamics of Bose-Einstein Condensates has been recently derived rigorously in the defocusing case by Adami, Golse and Teta (2007) in one space dimension and by Erdös, Schlein and Yau (2008) in three dimensions. No proof exists in the fcousing case.

E. Lieb, R. Seiringer, J.P. Solovej and J. Yngvason The Mathematics of the Bose Gas and its Condensation, vol. 34, Oberwolfach Seminars Series, Birkhaeuser (2005)

Bose–Einstein condensation has been experimentally observed also in double and triple hyperfine spin states.

Ch. Rüegg et al.: Bose–Einstein condensation of the triple states in the magnetic insulator tlcucl3. *Nature* **423**, 62–65 (2003)

2 Condensation in multiple states

Then the model consists in a system of k Gross-Pitaevski equations:

$$\begin{cases} -\imath\partial_t(\phi_i) = \Delta\phi_i - V_i(x)\phi_i + \mu_i|\phi_i|^2\phi_i - \sum_{j\neq i}\beta_{ij}|\phi_j|^2\phi_i, & i = 1,\dots,k \\ \phi_i \in H^1(\mathbb{R}^N;\mathbb{C}), \qquad N = 1,2,3, \end{cases}$$

The complex valued functions ϕ_i's are the wave functions of the i–th condensate, the functions V_i's represent the trapping potentials, and the positive constants μ_i's and β_{ij}'s are the intraspecies and the interspecies scattering lengths, respectively. The interactions between like particles can be attractive (the focusing case) or repulsive (the defocusing case), while the interactions between the unlike ones are repulsive.

We assume symmetry of the interspecific scattering lenghts, which gives the system a gradient structure. Also, we will deal with repulsive interactions:

$$\beta_{ij} = \beta_{ji} \qquad \text{and} \qquad \beta_{ij} > 0.$$

3 Standing waves

To obtain standing waves we impose

$$\phi_i(t,x) = e^{-\imath\lambda_i t} u_i(x).$$

Now the real functions u_i's solve the elliptic system

$$\begin{cases} -\Delta u_i + [V_i(x) + \lambda_i]\, u_i = \mu_i u_i^3 - \sum_{j\neq i} \beta_{ij} u_j^2 u_i, \qquad i = 1,\dots,k \\ u_i \in H^1(\mathbb{R}^N). \end{cases} \tag{Sys}$$

We think V_i to be trapping potentials or we confine the motion in a bounded domain Ω.

4 Ground states

In the defocusing case, i.e. ($\mu_i < 0$) we associate with the system the energy:

$$\mathcal{E}_\beta(u_1,\dots,u_k) = \sum_{i=1}^{k}\left[\int_{\mathbb{R}^N}\frac{1}{2}|\nabla u_i|^2 + \frac{1}{2}V_i(x)u^2 - \frac{\mu_i}{4}\int_{\mathbb{R}^N} u_i^4\,dx\right] + \frac{\beta}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{k}\int_{\mathbb{R}^N} u_i^2u_j^2\,dx$$

It is a coercive functional. Ground states are minimizers of the energy under the mass constraint:

$$\int_{\mathbb{R}^N} u_i^2 = 1 \qquad i = 1,\dots,k.$$

We define

$$c_\beta = \min\{\mathcal{E}_\beta(u_1,\dots,u_k)\ :\ \int_{\mathbb{R}^N} u_i^2 = 1 \qquad i = 1,\dots,k\}\ .$$

The functional associated with the focusing case ($\mu_i > 0$) is:

$$J_\beta(u_1,\dots,u_k) = \sum_{i=1}^{k}\left[\int_{\mathbb{R}^N}\frac{1}{2}|\nabla u_i|^2 + \frac{1}{2}V_i(x)u^2 - \frac{\mu_i}{4}\int_{\mathbb{R}^N} u_i^4\,dx\right] + \frac{\beta}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{k}\int_{\mathbb{R}^N} u_i^2u_j^2\,dx$$

in this case, solutions having minimal solutions are saddle points or, equivalently, minimizers of the functional on the Nehari's manifold. Again, we define

$$c_\beta = \min_{(u_1,\dots,u_k)}\ \max_{r_i>0} J_\beta(r_1u_1,\dots,r_ku_k)\ .$$

5 Segregation for ground states

We are interested in the limit $\beta \to \infty$. The limiting functionals are obviously

$$\mathcal{E}_\infty(u_1, \cdots, u_k) = \begin{cases} \mathcal{E}_1(u_1, \cdots, u_k), & \text{if } u_i(x)u_j(x) \equiv 0 \ , \forall i \neq j \\ +\infty & \text{otherwise.} \end{cases}$$

Ground states, in the defocusing case are minimizers of

$$c_\infty = \min_{\substack{\int_\Omega u_i^2 = 1 \ , i,j=1,\dots,k \\ u_i(x)u_j(x) \equiv 0 \text{ for } i \neq j}} \mathcal{E}_\beta(u_1, \dots, u_k) \ . \tag{LVP}$$

and in the focusing case

$$c_\infty = \min_{\substack{(u_1,\dots,u_k) \\ u_i(x)u_j(x) \equiv 0 \text{ for } i \neq j}} \max_{r_i > 0} J_\beta(r_1 u_1, \dots, r_k u_k) \ . \tag{LVP}$$

It is not difficult to prove that,

Theorem 1 *As $\beta \to +\infty$, there holds*

- $c_\beta \nearrow c_\infty$;
- *there is strong H^1–convergence of the minimizers to a minimizer of (LVP).*

For a fixed k, as the interspecific competition goes to infinity, the wave amplitudes u_i's segregate, that is, their supports tend to be disjoint

This phenomenon, called phase separation, has been studied, in the case of $\mu_i > 0$ (focusing), starting from

M. Conti, S. Terracini, G. Verzini, An optimal partition problem related to non linear eigenvalues, *J. Funct. Anal.* **198** (2003), no. 1, 160-196;

M. Conti, S. Terracini, G. Verzini, On a class of optimal partition problems related to the Fučík spectrum and to the monoticity formulae, *Calc. Var.* **22** (2005), 45-72.

and, in the case $\mu_i < 0$, for least energy solutions in bounded domains:

S.M. Chang, C.S. Lin, T.C. Lin, and W.W. Lin: Segregated nodal domains of two-dimensional multispecies Bose-Einstein condensates. *Phys. D* **196**, 341–361 (2004)

M. Squassina, S. Zuccher,: Numerical computations for the spatial segregation limit of some 2D competition-diffusion systems, Adv.Math. Sci. Appl. 18 (2008), 83104.

6 Optimal partition problems

In the limit, we find both a limiting profile, a minimizer of (LVP), and a partition of the original domain, which is optimal with respect to the sum of the ground state energies of the subdomains. Indeed, let us define

$$\varphi_i(\omega) = \min_{\substack{u\in H_0^1(\omega)\\ \int u^2=1}} \left[\int_\omega \frac{1}{2}|\nabla u|^2 + \frac{1}{2}V_i(x)u^2 - \frac{\mu_i}{4}\int_\omega u^4\,dx \right]$$

or, in the focusing case,

$$\varphi_i(\omega) = \min_{u\in H_0^1(\omega)} \max_{r>0} \left[\int_\omega \frac{r^2}{2}|\nabla u|^2 + \frac{r^2}{2}V_i(x)u^2 - \frac{r^4\mu_i}{4}\int_\omega u^4\,dx \right]$$

Given such a ground state energy as a function of the domain, the unknown is the union of the interfaces: in other words we are lead to solve the optimal partition problem:

$$\inf_{(\omega_i)\in\mathcal{P}} \sum_{i=1}^{k} \varphi_i(\omega_i) \qquad \text{(OPP)}$$

where

$$\mathcal{P} := \left\{ (\omega_1,\ldots,\omega_k) : \bigcup_{i=1}^{k} \overline{\omega}_i = \overline{\Omega},\ \omega_i\cap\omega_j = \emptyset \text{ if } i\neq j \right\}.$$

As the real unknown is the nodal set determined by the optimal partition. Hence, we can regard the phase segregation as a free boundary problem.

Questions:

- definition of a class of the admissible partitions;
- regularity of the free boundary;
- regularity of the limiting configuration;
- qualitative properties of the subdomains and the free boundaries.

A remark on connectedness:

Obviously, the number of connected domains of segregation is at least the number of different phases surviving in the limit. For the minimal solutions, the limiting states have connected supports Indeed, in the case $\mu_i > 0$, it is easy to prove that the supports of the limiting segregated states solve an optimal partition problem, where the total cost is additive (and strictly positive) with respect to the disjoint union; this penalizes non connected supports. On the other hand, when $\mu_i < 0$, it results from numerical evidence.

Optimal partitions problems for functions of the eigenvalues:

Bucur-Buttazzo-Henrot

Buttazzo-Timofte

Conti-T-Verzini

Caffarelli-Lin

Helffer-Hoffmann-Ostenhof-T

Bourdin-Bucur-Oudet.

7 Some extremality conditions for the limiting problem

Introduce the notation:

$$f_i(x,u) = -\lambda_i u_i + \mu_i u_i^3$$

$$\widehat{u}_i = u_i - \sum_{j\neq i} u_j, \qquad \widehat{f}(x,\widehat{u}_i) = f_i(x,u_i) - \sum_{j\neq i} f_j(x,u_j).$$

Let us define the class $\mathcal{S}$ as

$$\mathcal{S} = \left\{ U = (u_1,\ldots,u_k) \in (H_0^1(\Omega))^k : \begin{array}{l} u_i \geq 0,\ u_i \cdot u_j = 0 \text{ if } i \neq j, \text{ in } \Omega \\ -\Delta u_i \leq f_i(x,u_i(x)), \\ -\Delta \widehat{u}_i \geq \widehat{f}_i(x,\widehat{u}_i(x)) \end{array} \right\}.$$

In fact, we have

Theorem 2 *Let $\bar{U} = (\bar{u}_1,\ldots,\bar{u}_k)$ a minimizer of the minimization problem (LVP) then*

$$(\bar{u}_1,\ldots,\bar{u}_k) \in \mathcal{S}.$$

8 Regularity in the class $\mathcal{S}$

Thus the study of $\mathcal{S}$ provides the information on the segregated states induced by strong competition. In particular:

- the elements in the class $\mathcal{S}$ are Lipschitz continuous;
- their nodal set has some regularity properties.

Conti M., Terracini S., Verzini G., *A variational problem for the spatial segregation of reaction–diffusion systems*, Indiana Univ. Math. J. 54 (2005), no. 3, 779–815.

Kelei Wang and Zhitao Zhang,*Some New Results in Competing Systems with Many Species*, preprint 2009

Further regularity for energy minimizing configurations:

L. Caffarelli, F.-H- Lin, Singularly perturbed elliptic systems and multi-valued harmonic functions with free boundaries, *J. Amer. Math. Soc.* **21** (2008), 847-862;

L. A. Caffarelli, A. L. Karakhanyan, and F. Lin,*The geometry of solutions to a segregation problem for non-divergence systems*, Journal of Fixed Point Theory and Applications 5 (1009), 319-351

- the free boundary, up to a set of Hausdorff dimension at most, $N-2$ consists of a collection of $\mathcal{C}^{1,\alpha}$ $N-1$–dimensional manifolds.

9 Sign changing solutions and the case of two competing densities

In the focusing case, let us assume $\lambda_i = \mu_i = 1$. Then, the limiting profiles are associated with the changing–sign solutions of the scalar equation

$$-\Delta w + w = w^3. \qquad \text{(SE)}$$

Indeed, the system of inequalities in the definition of the class $\mathcal{S}$ read

$$-\Delta(u_i - u_j) \geq -(u_i - u_j) + (u_i - u_j)^3 \,, \qquad i, j = 1, 2 \,, i \neq j$$

and hence, as u_1 and u_2 have disjoint supports:

$$u_1 = w^+ \qquad u_2 = w^- \,.$$

Of course, as its nodal partition is optimal with respect to the (OOP), this solution minimizes the energy among all sign–changing solutions and has exactly two nodal regions.

This particular type of sign changing solutions was first discovered in

Castro A., Cossio J. and Neuberger J. M., *A sign-changing solution for a superlinear Dirichlet problem.* Rocky Mountain J. Math. 27 (1997), no. 4, 1041–1053

Convergence of the solution to the system and relation with the sign–changing solution is in

Conti M., Terracini S., Verzini G., *Nehari method for PDE's and competing species systems*,Ann. Inst. H. Poincaré Anal. Non Linéaire 19 (2002), no. 6, 871–888

10 Segregation for bound states

We see that there are two intertwined directions of investigation:

- seek solutions of (Sys), for large β, emanating from a given sign-changing solution of (SE). This analysis involves aspects of the behaviour of minimax under Γ–convergence and continuation of solutions.

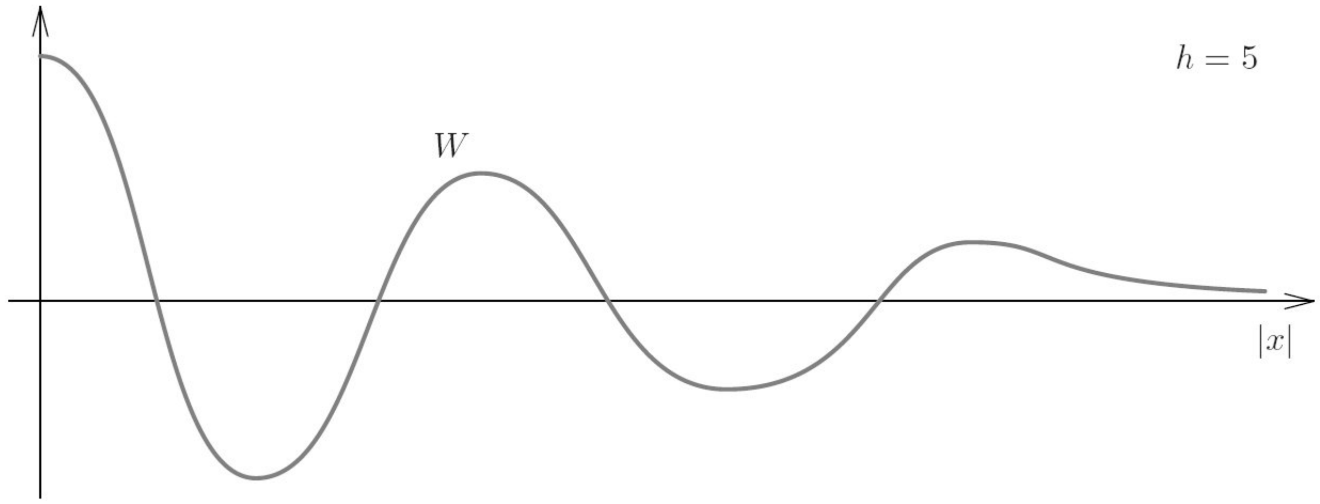
or

- find excited states for (Sys) and then prove their convergence to some limiting profiles (possibly in connections with (SE). This involves compctness and regularity issues in and also matching conditions at the interfaces:

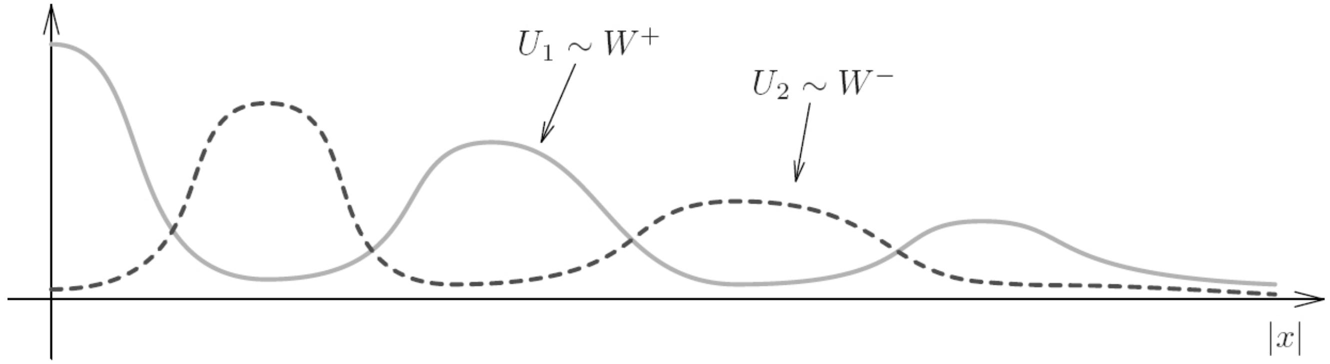
  - do families of bounded solutions converge in spaces of Hölder continuous (or Lipschitz) functions?
  - what are the properties of their limiting profiles?
  - of the nodal sets?
  - are there natural matching conditions at the interface? For example,is it true that for any limit (u_∞, v_∞) of solutions to thes system it happens that $w = u_\infty - v_\infty$ solves the associated single equation?

11 Phase separation in the radially symmetric case

According to Nehari, (see also Bartsch and Willem), for any $h \in \mathbb{N}$ equation (SE) possesses radial solutions with exactly $h-1$ changes of sign, that is h nodal components ("bumps"), with a variational characterization. Wei and Weth have shown that, in the case of $k = 2$ components, there are



solutions (u_1, u_2) such that the difference $u_1 - u_2$, for large values of β, approaches such a nodal solution:



➡ Can we construct similar solutions for systems of $k \geq 3$ components?

In a joint paper with G. Verzini, we extend this result to the case of an arbitrary number of components , proving the existence of solutions to (SE) with the property that, for β large, each component u_i is near the sum of some non–consecutive bumps of $|W|$.

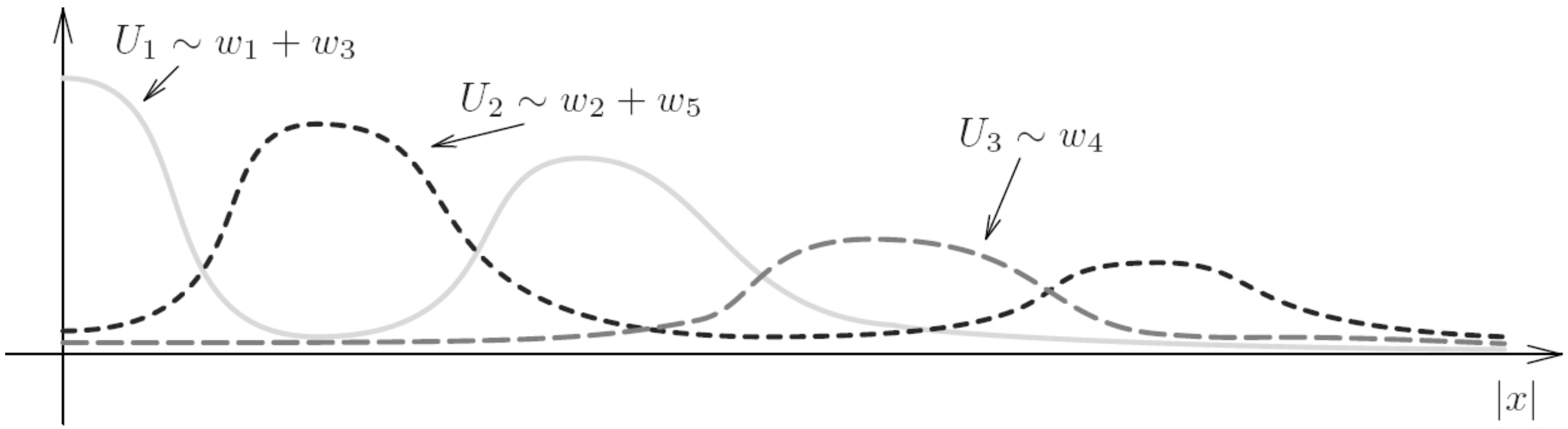


Figure 1: The corresponding solution of a three component (Sys), according to T.–Verzini

Furthermore, we can prescribe the correspondence between such bumps of $|W|$ and the index i of the component u_i. This, compared with the case $k = 2$, provides a much richer structure of the solution set for (Sys). This goal will be achieved by a suitable construction inspired by to extended Nehari's method.

S. Terracini and G. Verzini, *Multipulse phases in k–mixtures of Bose–Einstein condensates*, to appear on Arch. Rational Mech. and Anal.

12 Bounds in Hölder spaces and Lipschitz regularity of the limiting profile

For the sake of simplicity we take only two densities and we assume $V_i(x) \equiv 0$, $\lambda_i = \mu_i = \pm 1$ and $\beta_{ij} = \beta$, for every i and j, and $N = 2, 3$, even though our method works also in more general cases and for any number of components.

Let $\Omega \subset \mathbb{R}^N$ be a regular bounded domain. For every fixed $\beta > 0$, let us consider the system:

$$\begin{cases} -\Delta u_\beta + \lambda_\beta u_\beta = \mu_1 u_\beta^3 - \beta u_\beta v_\beta^2 & \text{in } \Omega \\ -\Delta v_\beta + \mu_\beta v_\beta = \mu_2 v_\beta^3 - \beta u_\beta^2 v_\beta & \text{in } \Omega \\ u_\beta,\ v_\beta \in H_0^1(\Omega), \quad u_\beta, v_\beta \geq 0 & \text{in } \Omega \end{cases} \tag{Sys}$$

where $\lambda_\beta, \mu_\beta \in \mathbb{R}$ is a bounded sequence, $\mu_1, \mu_2 \in \mathbb{R}$ are fixed constants.

First we prove uniform Hölder bounds:

Theorem 3 (Noris,Tavares,T,Verzini) *Let u_β, v_β be solutions of (Sys) uniformly bounded in $L^\infty(\Omega)$. Then for every $\alpha \in (0,1)$ there exists $C > 0$ (independent of β) such that, for all $\beta > 0$*

$$\max_{x,y\in\overline{\Omega}} \frac{|u_\beta(x) - u_\beta(y)|}{|x-y|^\alpha}, \ \max_{x,y\in\overline{\Omega}} \frac{|v_\beta(x) - v_\beta(y)|}{|x-y|^\alpha} \leq C.$$

In addition we have:

Theorem 4 (NTTV) *Let u_β, v_β be solutions of (Sys) uniformly bounded in $L^\infty(\Omega)$. Then there exist limits $(u,v) \in C^{0,\alpha}$, $\forall\ \alpha \in (0,1)$, such that up to a subsequence there holds*

(i) $u_\beta \to u, v_\beta \to v$ *in* $C^{0,\alpha}(\Omega) \cap H^1(\Omega)$, $\forall\ \alpha \in (0,\alpha)$;

(ii) $u \cdot v \equiv 0$ *in* Ω *and* $\int_\Omega \beta u_\beta^2 v_\beta^2 \to 0$ *as* $\beta \to +\infty$;

(iii) the limiting functions u, v satisfy the following equations:

$$\begin{cases} -\Delta u + \lambda u = \mu_1 u^3 & in\ \{u > 0\}, \\ -\Delta v + \mu v = \mu_2 v^3 & in\ \{v > 0\}, \end{cases}$$

where $\lambda := \lim_{\beta\to+\infty} \lambda_\beta$, $\mu := \lim_{\beta\to+\infty} \mu_\beta$.

We can improve the regularity result for the limiting profile.

Theorem 5 (NTTV) *The limiting profile (u,v) is Lipschitz continuous in the interior of Ω.*

B. Noris, H. Tavares, Terracini S. and G. Verzini, *Uniform Hölder bounds for nonlinear Schrödinger systems with strong competition*, CPAM, to appear

This result generalizes:

J.C. Wei and T. Weth Asymptotic behavior of solutions of planar systems with strong competition. *Nonlinearity* **21**, 305–317 (2008)

13 Ideas of the proof

The proof of the uniform Hölder estimates goes by contradiction.

Blow up
+
Liouville type theorem
(based on a perturbed Alt-Caffarelli-Friedman
monotonicity formula)
+
Almgren's frequency formula
(gradient structure of the system)

Some similar argument were developed in:

M. Conti, S. Terracini, G. Verzini, Adv. Math. (2005)

Caffarelli, L. A.; Roquejoffre, J.-M. Arch. Ration. Mech. Anal. 183 (2007)

As no extremality condition nor differential inequalities (class $\mathcal{S}$) are known for the excited states, we need a new tool that allows to reach the contradiction.

14 More on the nodal set of the limiting profile

It works in any space dimension (but $N = 2, 3$if we want 4 to be subcritical) for general systems of k equations.

Theorem 6 (Tavares,T) *Let (u, v) the limiting profile as before and $\mathcal{N}$ its nodal set:*

$$\mathcal{N} = \{x \in \Omega \ : \ u(x) = v(x) = 0\} \ .$$

Then,

$$\boxed{\mathcal{N} = \mathcal{N}_s \cup \mathcal{N}_r}$$

where

- $\mathcal{N}_s$ *has Hausdorff dimension at most* $N - 2$;
- $\mathcal{N}_r$ *is a union of* $\mathcal{C}^{1,\alpha}$ *codimension 1 surfaces.*
- *for every* $x \in \mathcal{N}_r$ *the nodal set locally separates the domain in two parts. Rename* u *and* v *the components supported in the two subdomain. Then,*

$$\boxed{|\nabla u(x)| = |\nabla v(x)|}$$

15 Equations for the limiting profiles

We are dealing with locally lipschitz solutions of

$$-\Delta u_i = f_i(u_i) - \mu_i \qquad \text{in } \mathcal{D}'(\Omega), \quad i = 1, \dots, h.$$

Where

- $f_i : \mathbb{R}^+ \to \mathbb{R}$ are some C^1 functions such that $f_i(s) = O(s)$ when $s \to 0$;
- $\mu_i \in \mathcal{M}(\Omega) = (C_0(\Omega))'$ are some nonnegative (finite) regular Borel measures, each concentrated on the nodal set $\Gamma_U = \{x \in \Omega : \ U(x) = 0\}$,
- Define for every $x_0 \in \Omega$ and $r \in (0, d(x_0, \partial\Omega))$ the energy

$$E(r) = E(x_0, U, r) = \frac{1}{r^{N-2}} \int_{B_r(x_0)} \left(|\nabla U|^2 - F(U) \cdot U \right)$$

Then $E(x_0, U, \cdot)$ is an absolutely continuous function on r and there holds

$$\frac{d}{dr} E(x_0, U, r) = \frac{2}{r^{N-2}} \int_{\partial B_r(x_0)} (\partial_\nu U)^2 \, d\sigma + R(x_0, U, r),$$

The remainder

$$R(x_0,U,r)=\frac{1}{r^{N-1}}\int_{B_r(x_0)}\left((N-2)F(U)\cdot U-2N\sum_i f_i(u_i)\right)+\\ +\frac{1}{r^{N-2}}\int_{\partial B_r(x_0)}\left(2\sum_i f_i(u_i)-F(U)\cdot U\right).$$

acts a perturbation of the main term.

When

$$\frac{d}{dr}E(x_0,U,r)=\frac{2}{r^{N-2}}\int_{\partial B_r(x_0)}(\partial_\nu U)^2\,d\sigma+R(x_0,U,r),$$

holds, we say that u satisfies the Pohožaev identity.

16 Ideas of the proof

We follow the approach by Caffarelli and Lin.

Consider a ball $B_r(x_0) \subset \Omega$. and define the Almgren quotient:

$$N(x_0, r) := \frac{r \int_{B_r(x_0)} |\nabla u|^2 dx}{\int_{\partial B_r(x_0)} u^2 d\sigma(x)}.$$

Define

$$\mathcal{N}_s = \{x \in \mathcal{N} \ : \ \lim_{r \to 0} N(x, r) > 1\} \qquad \mathcal{N}_r = \{x \in \mathcal{N} \ : \ \lim_{r \to 0} N(x, r) = 1\}$$

- First we wish to apply Federer's reduction principle:
  - Almgren monotonicity formula at nodal points
  - bounds in Hölder spaces $\Longrightarrow$ convergence of blow-up sequences
  - classification of conic solutions satisfying a Pohozaev–type identity
- Next we analyze the non singular part of the free boundary:
  - flatness of the boundary (Reifenberg condition is satisfied for any constant of flatness)
  - clean–up lemma
  - nondegeneracy of the associated harmonic boundary measure up to a nullset.
- Reflection principle:
  - Pohozaev identity $\Longrightarrow$ equality of the gradients on the two sides.

17 Blow up

We assume by contradiction that the Hölder norm of (u_β, v_β) is not uniformly bounded, for a certain $\alpha \in (0,1)$.

$$L_\beta := \frac{|u_\beta(x_\beta) - u_\beta(y_\beta)|}{|x_\beta - y_\beta|^\alpha} = \max\{\|u_\beta\|_{C^{0,\alpha}}, \|v_\beta\|_{C^{0,\alpha}}\} \longrightarrow +\infty, \qquad \text{as } \beta \to +\infty.$$

We zoom in at x_β and normalize:

$$\bar{u}_\beta(x) = \frac{1}{L_\beta r_\beta^\alpha} u_\beta(x_\beta + r_\beta x)$$
$$\bar{v}_\beta(x) = \frac{1}{L_\beta r_\beta^\alpha} v_\beta(x_\beta + r_\beta x), \qquad x \in \Omega_\beta = \frac{\Omega - x_\beta}{r_\beta}$$

- Notice that $L_\beta \to +\infty$, whereas $r_\beta \to 0$, hence the behavior of the rescaled functions is not known.
- The rescaled functions are uniformly bounded in α–Hölder norm and

$$\max\left\{[\bar{u}_\beta]_{C^{0,\alpha}(\overline{\Omega}_\beta)}, [\bar{v}_\beta]_{C^{0,\alpha}(\overline{\Omega}_\beta)}\right\} = 1,$$

achieved by u_β.

In addition the rescaled functions satisfy the following system in Ω_β:

$$
\begin{cases}
-\Delta \bar{u}_\beta + \lambda_\beta r_\beta^2 \bar{u}_\beta & = \mu_1 M_\beta \bar{u}_\beta^3 - \beta M_\beta \bar{u}_\beta \bar{v}_\beta^2 \\
-\Delta \bar{v}_\beta + \mu_\beta r_\beta^2 \bar{v}_\beta & = \mu_2 M_\beta \bar{v}_\beta^3 - \beta M_\beta \bar{u}_\beta^2 \bar{v}_\beta \\
\bar{u}_\beta,\ \bar{v}_\beta \in H_0^1(\Omega_\beta),
\end{cases}
$$

where

$$
M_\beta := L_\beta^2 r_\beta^{2\alpha+2}.
$$

While $M_\beta \to 0$, the actual behaviour of the solutions depends on the character of the sequence $\beta M_\beta := \beta L_\beta^2 r_\beta^{2\alpha+2}$.

If

$$
\liminf_{\beta \to +\infty} \beta M_\beta < +\infty,
$$

then the blow–up limits satisfy a differential system. Otherwise, if

$$
\liminf_{\beta \to +\infty} \beta M_\beta = +\infty,
$$

we can say that the two components segregate, each being harmonic on its support.

18 A first Liouville–type result

The following result is a consequence of the well-known monotonicity formula by Alt, Caffarelli, Friedman for harmonic functions with disjoint supports.

Theorem 7 *Let* $u_i \in H^1_{\text{loc}}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ $i = 1, \ldots, k$ *be nonnegative functions such that* $u_i \cdot u_j \equiv 0$ *when* $i \neq j$*. Assume moreover that*

$$-\Delta u_i \leq 0, \quad \text{in } \mathbb{R}^N \ , \forall i = 1, \ldots, k$$

and

$$u_i(x_0) = 0 \ , \quad \forall i = 1, \ldots, k.$$

Assume moreover that for some $\alpha \in (0,1)$ *there holds*

$$\sup_{x,y \in \mathbb{R}^N} \frac{|u_i(x) - u_i(y)|}{|x-y|^\alpha} < \infty \ , \quad \forall i = 1, \ldots, k.$$

Then al u_i *but possibly one vanish identically.*

♠ What happens to the last component?

This can be seen as an extension of the following form of the usual Liouville Theorem:

Remark 1 *Let u be a harmonic function in $\mathbb{R}^N$ such that for some $\alpha \in (0,1)$ there holds*

$$\sup_{x,y\in\mathbb{R}^N} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < \infty.$$

Then u is constant.

Remark 2 *This false for $\alpha = 1$: just take $u(x) = x_1$. Analogously, it is possible to see that also system* (S) *below admits non trivial solutions which are globally bounded in Lipschitz norm; these are the main reasons for which our strategy, as it is, can not apply to prove uniform Lipschitz estimates.*

In order to face the case

$$\liminf_{\beta\to+\infty} \beta M_\beta < +\infty,$$

we need a result similar to Proposition 7, for functions u, v which do not have disjoint supports, but are positive solutions in $H^1_{\text{loc}}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ of the system

$$\begin{cases} -\Delta u = -uv^2 & \text{in } \mathbb{R}^N \\ -\Delta v = -u^2 v & \text{in } \mathbb{R}^N. \end{cases} \tag{S}$$

19 A second Liouville–type result

As a consequence of a perturbed Alt-Caffarelli-Friedman monotonicity formula we have a second Liouville–type result:

Theorem 8 (Conti,T,Verzini) *Let $k \geq 2$ and let $U = (u_1, \dots, u_k)$ be a solution of*

$$\begin{cases} -\Delta u_i(x) = -u_i(x) \sum_{j \neq i} \beta_{ij} u_j^2(x) & x \in \mathbb{R}^N \\ u_i(x) \geq 0 & x \in \mathbb{R}^N \end{cases}$$

for every i. Let $\alpha \in (0,1)$ such that

$$\max_{i=1,\dots,k} \sup_{x \in \mathbb{R}^N} \frac{|u_i(x)|}{1 + |x|^\alpha} < \infty.$$

Then all components (but possibly one) vanish.

♠ Here the non vanishing component is necessarily constant.

20 The Almgren's frequency formula

Let u be a function in $\Omega \subset \mathbb{R}^N$ and consider a ball $B_r(x_0) \subset \Omega$. We define the following quantities

$$E(x_0, r) = \frac{1}{r^{N-2}} \int_{B_r(x_0)} |\nabla u|^2 dx,$$
$$H(x_0, r) = \frac{1}{r^{N-1}} \int_{\partial B_r(x_0)} u^2 d\sigma(x),$$
$$N(x_0, r) = \frac{E(x_0, r)}{H(x_0, r)}.$$

If u is harmonic in Ω then

(i) for every $x_0 \in \Omega$, $\frac{d}{dr} N(x_0, r) \geq 0$;

(ii) $\frac{d}{dr} \log H(x_0, r) = \frac{2}{r} N(x_0, r)$;

(iii) If $N(x_0, r) \equiv \gamma$ then $u(x) = r^\gamma g(\theta)$ (polar coordinates around x_0).

21 A refined Liouville–type Theorem

The Almgren's frequency formula allows us to strenghten Theorem 7 into the following non existence result

Theorem 9 (Noris,Tavares,T,Verzini) *Let* $u_i \in H^1_{\mathrm{loc}}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ $i = 1, \ldots, k$ *be nonnegative functions such that* $u_i \cdot u_j \equiv 0$ *when* $i \neq j$*. Assume moreover that*

$$-\Delta u_i = 0, \quad \text{where } u_i > 0 \, , \quad \forall i = 1, \ldots, k$$

and

$$u_i(x_0) = 0 \, , \quad \forall i = 1, \ldots, k.$$

Assume moreover that for some $\alpha \in (0, 1)$ *there holds*

$$\sup_{x,y \in \mathbb{R}^N} \frac{|u_i(x) - u_i(y)|}{|x - y|^\alpha} < \infty \, , \quad \forall i = 1, \ldots, k.$$

Assume moreover that $\sum_i u_i$ *satisfy* $(i) - (iii)$ *at each* x_0 *common zero of the* u_i*'s, then all component but one vanish identically and the last is constant.*

A key point is to classify the homogeneous solutions of the system.

Theorem 10 (Helffer, Hoffmann-Ostenhof, T) *Let* $u : \mathbb{R}^3 \to \mathbb{R}$ *be continuous and such that*

$$u(rx) = r^\alpha u(x),$$

and

$$-\Delta u = 0 \quad \text{where } u \neq 0\,.$$

Assume moreover that u *has at least three nodal regions on the sphere. Then* $\alpha \geq 3/2$.

Theorem 11 (Tavares,T) *Let* u *be as before and assume moreover that* u *satisfy the Pohožaev identity. Then*

➡ *either* $\alpha \geq \frac{3}{2}$, *or* $\alpha = 1$ *and* u *is linear.*