An interpolating 4-point C^2 ternary non-stationary subdivision scheme with tension control

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Abstract

In this paper we present a non-stationary 4-point ternary interpolatory subdivision scheme which provides the user with a tension parameter that, when increased within its range of definition, can generate C^2-continuous limit curves showing considerable variations of shape.

As a generalization we additionally propose a locally-controlled C^2-continuous subdivision scheme, which allows a different tension value to be assigned to every edge of the original control polygon.

Key words: Subdivision; Interpolation; Curves; Ternary; Locality; Tension control

1 Introduction

Until a few years ago all the work in the area of univariate subdivision was limited to consider just binary scenarios (Dyn and Levin, 1992; Dyn, 2002; Warren and Weimer, 2002). Recent proposals of ternary subdivisions (Hassan et al., 2002; Hassan and Dodgson, 2003; Jeon et al., 2005; Wang and Qin, 2005; Zheng et al., 2005) have introduced new interesting animals in the subdivision zoo, showing the possibility of treating refinement schemes with arity other than two. A general increasing interest to investigating for higher arities has emerged since Hassan et al. (2002) showed that we can achieve higher smoothness and smaller support for the so-called interpolating 4-point stationary scheme, by going from binary (Dyn et al.,

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1987) to ternary. But does this trend continue for the non-stationary schemes too? In Beccari et al. (2007) it was described an interpolating 4-point C^1 binary non-stationary subdivision scheme with a global tension parameter. Aim of this paper is introducing a tension controlled ternary proposal with higher smoothness and smaller support than its binary counterpart.
An interpolating 4-point ternary subdivision scheme maps a polygon $P^j = \{p_i^j\}_{i\in\mathbb{Z}}$ to a refined polygon $P^{j+1} = \{p_i^{j+1}\}_{i\in\mathbb{Z}}$ by applying the following subdivision rules

$$\begin{aligned} p_{3i}^{j+1} &= p_i^j, \\ p_{3i+1}^{j+1} &= a_0^j\ p_{i-1}^j + a_1^j\ p_i^j + a_2^j\ p_{i+1}^j + a_3^j\ p_{i+2}^j, \\ p_{3i+2}^{j+1} &= a_3^j\ p_{i-1}^j + a_2^j\ p_i^j + a_1^j\ p_{i+1}^j + a_0^j\ p_{i+2}^j, \end{aligned} \tag{1}$$

where the coefficients $\{a_i^j\}_{i=0,1,2,3}$ are chosen to satisfy the relation

$$a_0^j + a_1^j + a_2^j + a_3^j = 1. \tag{2}$$

The subdivision step (1) can be compactly written in a single equation of the form

$$p_i^{j+1} = \sum_{k\in\mathbb{Z}} m_{3k-i}^j\ p_k^j \qquad \forall j \in \mathbb{Z}_+$$

where

$$m^j = \left[\ a_3^j,\ a_0^j,\ 0,\ a_2^j,\ a_1^j,\ 1,\ a_1^j,\ a_2^j,\ 0,\ a_0^j,\ a_3^j\ \right] \tag{3}$$

is the so-called mask at the j-th level of refinement. From (2) it immediately follows that

$$\sum_{k\in\mathbb{Z}} m_{3k}^j = 1, \qquad \sum_{k\in\mathbb{Z}} m_{3k+1}^j = 1, \qquad \sum_{k\in\mathbb{Z}} m_{3k+2}^j = 1.$$

Hassan et al. (2002) introduced a stationary interpolatory 4-point scheme of the kind (1) with $\{a_i^j\}_{i=0,1,2,3}$ given by

$$\begin{aligned} a_0^j &\equiv a_0 = -\frac{1}{18} - \frac{1}{6}\mu \\ a_1^j &\equiv a_1 = \frac{13}{18} + \frac{1}{2}\mu \\ a_2^j &\equiv a_2 = \frac{7}{18} - \frac{1}{2}\mu \\ a_3^j &\equiv a_3 = -\frac{1}{18} + \frac{1}{6}\mu. \end{aligned} \tag{4}$$

In the coefficients set (4) μ is a global parameter independent of the refinement level j which, when chosen in the span $\left]\frac{1}{15}, \frac{1}{9}\right[$, allows us to generate C^2-continuous

limit curves. Unfortunately, when varying the value of μ inside such a tight span of definition, it is very difficult to appreciate some significant alterations of the limiting shape.
In this paper we present a ternary 4-point non-stationary interpolatory scheme providing the user with a tension parameter that, when increased within its range of definition, can generate C^2-continuous limit curves showing considerable variations of shape.
In order to include also the possibility of applying a different tension in correspondence of every edge of the original control polygon, we additionally propose a generalization of such a scheme with local parameters.
More precisely, the paper is structured as follows: in section 2 we briefly define the non-stationary ternary interpolatory 4-point scheme with global tension; then in section 3 we study its convergence and we prove that for every choice of the initial tension parameter in the span $[-2, +\infty[\setminus \{-1\}$ the resulting limit curve is C^2-continuous. Successively, in section 4 we analyze the properties of the basis function to underline the advantages of using the 4-point ternary scheme instead of its binary counterpart (Beccari et al., 2007). Finally, in section 5 we present a generalization of this proposal which makes it possible to set a local tension parameter in correspondence of each edge of the initial polyline.

2 Definition of the scheme

The novel interpolating 4-point ternary subdivision scheme is described by the refinement rules in (1), where the coefficients $\{a_i^j\}_{i=0,1,2,3}$ are given by

$$\begin{aligned} a_0^j &= \frac{1}{60}(-90\gamma^{j+1} - 1) \\ a_1^j &= \frac{1}{60}(90\gamma^{j+1} + 43) \\ a_2^j &= \frac{1}{60}(90\gamma^{j+1} + 17) \\ a_3^j &= \frac{1}{60}(-90\gamma^{j+1} + 1) \end{aligned} \tag{5}$$

with

$$\gamma^{j+1} = -\frac{1}{3(1 - (\beta^{j+1})^2)(1 + \beta^{j+1})} \tag{6}$$

and

$$\beta^{j+1} = \sqrt{2 + \beta^j}, \quad \beta^j \geq -2 \ \ (\beta^j \neq -1) \ \ \forall j \in \mathbb{Z}_+. \tag{7}$$

In this way, given an initial tension $\beta^0 \in [-2, +\infty[\setminus \{-1\}$, the subdivision rules (1), defining points at level $j+1$, are derived by first computing the parameter β^{j+1} through equation (7) and then substituting its value into equation (6) in order to work out the coefficients $\{a_i^j\}_{i=0,1,2,3}$ for the j-th level of refinement.

Remark 1 *Note that, starting from any initial parameter $\beta^0 \geq -2$, we have $2 + \beta^j \geq 0 \ \forall j \in \mathbb{Z}_+$, and so β^{j+1} is always well-defined. The initial value $\beta^0 = -1$ has been discarded in order to avoid the denominator in (6) to vanish. As a result, for each choice of β^0 in $[-2, +\infty[\setminus \{-1\}$, the parameters γ^{j+1} turn out to be well-defined for any j. Such a wide range of definition allows us to get considerable variations of shape in the limit curves (Fig. 1).*

Remark 2 *The discarded value $\beta^0 = -1$ identifies an interesting property of the proposed subdivision scheme: in fact, given a convex control polygon, for any $\beta^0 > -1$ the resulting limit curve lies completely outside of it. In addition, while for $\beta^0 < -1$ it is hard to get an intuition of the final shape from that of the initial polyline, for increasing values of β^0 in the range $]-1, +\infty[$ it is evident that the*

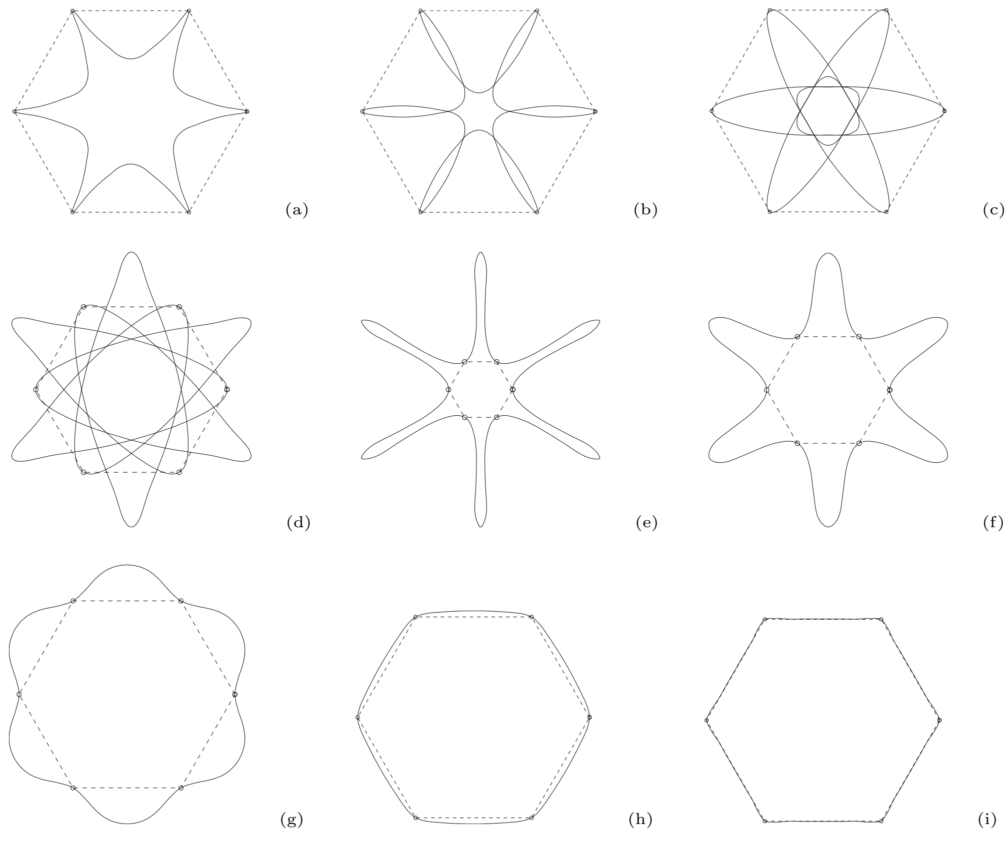

Fig. 1. Interpolation of the vertices of a regular hexagon with increasing values of the global tension parameter β^0: (a) -1.9, (b) -1.5, (c) -1.3, (d) -1.15, (e) -0.9, (f) -0.75, (g) 0, (h) 5, (i) 100.

limit curve progressively tends to shrink to the initial control polygon, so becoming tighter and tighter (Fig. 1).

3 Convergence analysis

Goal of this section is showing that, given an initial polygon P^0, the subdivision scheme we have presented in section 2 allows us to define an increasingly dense collection of polygons P^j that converges to a C^2-continuous limit curve for any choice of the initial tension parameter β^0 in the span $[-2,+\infty[\setminus\{-1\}$. In order to prove this, we analyze the smoothness properties of our scheme by exploiting the well-known results by Dyn and Levin (1995), which relate the convergence of a non-stationary scheme to its asymptotically equivalent counterpart. Hence we start by observing that the parameter β^j converges to 2 as $j \to \infty$, i.e., in practice the non-stationary subdivision scheme we have proposed converges to the stationary one defined by coefficients in (4) with $\mu = \frac{1}{10}$, which is known to be C^2 (Hassan et al., 2002). To this aim we recall the following property of monotonic sequences.

Proposition 3 *A monotonic and bounded sequence is always convergent. In particular, given a monotonic sequence $\{\beta^j\}_{j\in\mathbb{N}}$,*

- *if $\{\beta^j\}_{j\in\mathbb{N}}$ is non decreasing and upper bounded, then it converges to the upper bound of the values it assumes;*
- *if $\{\beta^j\}_{j\in\mathbb{N}}$ is non increasing and lower bounded, then it converges to the lower bound of the values it assumes.*

Lemma 4 *For the sequence defined by*

$$\begin{cases} \beta^0 & \in & [-2,+\infty[\setminus\{-1\} \\ \beta^{j+1} & = & \sqrt{2+\beta^j} \end{cases} \tag{8}$$

it holds:

- *if $\beta^0 = 2$, then $\beta^j = 2\ \forall j > 0$ and the sequence $\{\beta^j\}_{j\in\mathbb{N}}$ is stationary;*
- *if $\beta^0 \in [-2,2[\setminus\{-1\}$, then $\beta^j \in [0,2[\ \forall j > 0$ and the sequence $\{\beta^j\}_{j\in\mathbb{N}}$ is strictly increasing;*
- *if $\beta^0 \in]2,+\infty[$, then $\beta^j \in]2,+\infty[\ \forall j > 0$ and the sequence $\{\beta^j\}_{j\in\mathbb{N}}$ is strictly decreasing.*

PROOF. Note that, if $\beta^0 = 2$, then $\beta^j = 2\ \forall j > 0$. Moreover, for every $\beta^0 \in [-2,+\infty[\setminus\{-1\}$, it always holds $\beta^j \in [0,+\infty[\ \forall j > 0$. Thus, for any $j > 0$, $\{\beta^j\}_{j\in\mathbb{N}}$ is strictly increasing if and only if $\beta^j \in [0,2[\ \forall j > 0$ (that is, if and only if $\beta^0 \in [-2, 2[\setminus\{-1\})$. Analogously, for any $j > 0$, $\{\beta^j\}_{j\in\mathbb{N}}$ is strictly decreasing if and only if $\beta^j > 2\ \forall j > 0$ (that is, if and only if $\beta^0 > 2$). $\square$

Proposition 5 *Given the initial parameter* $\beta^0 \in [-2, +\infty[\setminus\{-1\}$, *the recurrence relation in (7) satisfies the property*

$$\lim_{j\to+\infty} \beta^j = 2.$$

PROOF. Since the sequence in (8) is

- monotonic non decreasing for $\beta^0 \in\ [-2, 2]\setminus\{-1\}$,
- monotonic non increasing for $\beta^0 \in\ [2, +\infty[$,

by Proposition 3 we can conclude that in both cases $\{\beta^j\}_{j\in\mathbb{N}}$ is convergent and it converges to 2. $\square$

Proposition 6 *The non-stationary subdivision scheme defined by coefficients in (5) is asymptotically equivalent to the stationary scheme having coefficients in (4) with* $\mu = \frac{1}{10}$. *Moreover it generates* C^2*-continuous limit curves.*

PROOF. In order to prove that the proposed non-stationary scheme converges to a C^2-continuous limit curve, we compute its second divided difference mask and we show that the associated limit curves are C^0-continuous. Since the mask of the scheme has the expression

$$m^j = \frac{1}{60}\Big[-90\gamma^{j+1}+1,\ -90\gamma^{j+1}-1,\ 0,\ 90\gamma^{j+1}+17,\ 90\gamma^{j+1}+43,\ 60,$$
$$90\gamma^{j+1}+43,\ 90\gamma^{j+1}+17,\ 0,\ -90\gamma^{j+1}-1,\ -90\gamma^{j+1}+1\Big],$$

its related first divided differences are

$$d^j_{(1)} = \frac{1}{60}\Big[-90\gamma^{j+1}+1,\ -2,\ 90\gamma^{j+1}+1,\ 18,\ 24,$$
$$18,\ 90\gamma^{j+1}+1,\ -2,\ -90\gamma^{j+1}+1\Big].$$

Hence the second divided difference mask turns out to be

$$d^j_{(2)} = \frac{1}{20}\Big[-90\gamma^{j+1}+1,\ 90\gamma^{j+1}-3,\ 90\gamma^{j+1}+3,\ -180\gamma^{j+1}+18,$$
$$90\gamma^{j+1}+3,\ 90\gamma^{j+1}-3,\ -90\gamma^{j+1}+1\Big].$$

In this way, by applying Proposition 5, it follows that

$$d^\infty_{(2)} = \lim_{j\to+\infty} d^j_{(2)} = \frac{1}{60}\left[-7,\ 1,\ 19,\ 34,\ 19,\ 1,\ -7\,\right].$$

This is the mask of the second divided differences of the stationary scheme having coefficients in (4) with $\mu = \frac{1}{10}$. Thus, since such a stationary refinement is C^2 for values of the parameter μ in $\left]\frac{1}{15}, \frac{1}{9}\right[$, the scheme associated with $d^{\infty}_{(2)}$ will be C^0. Now, if

$$\sum_{j=0}^{+\infty} \| d^j_{(2)} - d^{\infty}_{(2)} \|_{\infty} < +\infty, \tag{9}$$

the two difference schemes are asymptotically equivalent, and then we can conclude that the scheme associated with $d^j_{(2)}$ is C^0 too (Dyn and Levin, 1995). Since

$$\| d^j_{(2)} - d^{\infty}_{(2)} \|_{\infty} = \frac{1}{20} \max\left\{ 2\left| -90\gamma^{j+1} + \frac{10}{3}\right|, \left| 90\gamma^{j+1} - \frac{10}{3}\right| \right\} = \frac{1}{3}\left| -27\gamma^{j+1} + 1 \right|,$$

verifying condition (9) reduces to prove the convergence of the series

$$\sum_{j=0}^{+\infty} | -27\gamma^{j+1} + 1|, \tag{10}$$

which clearly depends on the parameter γ^{j+1}. Now, as γ^{j+1} is expressed in terms of the tension parameter β^{j+1} through relation (6), we will study the behavior of (10) as β^{j+1} varies in the interval $[0, +\infty[$. In particular, since

$$-27\gamma^{j+1} + 1 = 0 \iff \gamma^{j+1} = \frac{1}{27} \iff \beta^{j+1} = 2,$$

$$-27\gamma^{j+1} + 1 > 0 \iff \gamma^{j+1} < \frac{1}{27} \iff 0 \le \beta^{j+1} < 1 \ \cup \ \beta^{j+1} > 2,$$

and

$$-27\gamma^{j+1} + 1 < 0 \iff \gamma^{j+1} > \frac{1}{27} \iff 1 < \beta^{j+1} < 2,$$

we should study the convergence of (10) separating the analysis into the following three cases:

1. $\beta^{j+1} = 2$ (i.e. $\beta^0 = 2$)
2. $0 \le \beta^{j+1} < 1 \ \cup \ \beta^{j+1} > 2$ (i.e. $-2 \le \beta^0 < 1, \ \beta^0 \neq -1 \ \cup \ \beta^0 > 2$)
3. $1 < \beta^{j+1} < 2$ (i.e. $1 < \beta^0 < 2$).

1. Case 1: $\quad \| d^j_{(2)} - d^{\infty}_{(2)} \|_{\infty} = 0.$

   Convergence of (10) trivially follows.

2. Case 2: $\quad \| d^j_{(2)} - d^{\infty}_{(2)} \|_{\infty} = \frac{1}{3}(-27\gamma^{j+1} + 1).$

We have to prove that

$$\sum_{j=0}^{+\infty}(-27\gamma^{j+1}+1)=\sum_{j=0}^{+\infty}\frac{9}{(1-(\beta^{j+1})^2)(1+\beta^{j+1})}+1<+\infty.$$

To this aim we exploit the ratio test. Since $\frac{9}{(1-(\beta^{j+1})^2)(1+\beta^{j+1})}+1>0$, then

$$\frac{\frac{9}{(1-(\beta^{j+2})^2)(1+\beta^{j+2})}+1}{\frac{9}{(1-(\beta^{j+1})^2)(1+\beta^{j+1})}+1}<1 \iff \frac{(1+\beta^{j+1})^2}{(1-\beta^{j+2})(1+\beta^{j+2})^2}<\frac{1}{(1-\beta^{j+1})}. \tag{11}$$

At this point,

2.1. If $0\le\beta^{j+1}<1$, from equation (11) we get

$$\frac{(1-(\beta^{j+1})^2)(1+\beta^{j+1})}{(1-(\beta^{j+2})^2)(1+\beta^{j+2})}<1.$$

Now, as $\beta^{j+2}=\sqrt{2+\beta^{j+1}}$, thus $1-(\beta^{j+2})^2=-(1+\beta^{j+1})$, from which it follows

$$\frac{(\beta^{j+1})^2-1}{1+\beta^{j+2}}<1$$

and, due to the fact that $1+\beta^{j+2}>0$, we further obtain

$$(\beta^{j+1})^2-1<1+\beta^{j+2}.$$

Again, as $(\beta^{j+1})^2=2+\beta^j$, then $\beta^j<\beta^{j+2}$. Therefore, since whenever $0\le\beta^{j+1}<1$ the sequence $\{\beta^j\}_{j\in\mathbb{N}}$ is strictly increasing, this last statement is trivially verified and hence the ratio test allows us to prove the convergence of (10).

2.2. If $\beta^{j+1}>2$, from equation (11) we get

$$\frac{(1-(\beta^{j+1})^2)(1+\beta^{j+1})}{(1-(\beta^{j+2})^2)(1+\beta^{j+2})}>1.$$

Now, as $\beta^{j+2}=\sqrt{2+\beta^{j+1}}$, thus $1-(\beta^{j+2})^2=-(1+\beta^{j+1})$, from which it follows

$$\frac{(\beta^{j+1})^2-1}{1+\beta^{j+2}}>1$$

and, due to the fact that $1+\beta^{j+2}>0$, we further obtain

$$(\beta^{j+1})^2-1>1+\beta^{j+2}.$$

Again, as $(\beta^{j+1})^2=2+\beta^j$, then $\beta^j>\beta^{j+2}$. Therefore, since whenever $\beta^{j+1}>2$ the sequence $\{\beta^j\}_{j\in\mathbb{N}}$ is strictly decreasing, this last statement is trivially verified and hence the ratio test allows us to prove the convergence of (10).

3. Case 3: $\| d_{(2)}^j - d_{(2)}^\infty \|_\infty = \frac{1}{3}(27\gamma^{j+1} - 1)$.

   Thus we have to prove that

$$\sum_{j=0}^{+\infty}(27\gamma^{j+1} - 1) = \sum_{j=0}^{+\infty} \frac{9}{((\beta^{j+1})^2 - 1)(\beta^{j+1} + 1)} - 1 < +\infty.$$

   To this aim we exploit the ratio test. Since $\frac{9}{((\beta^{j+1})^2-1)(\beta^{j+1}+1)} - 1 > 0$, then

$$\frac{\frac{9}{((\beta^{j+2})^2-1)(\beta^{j+2}+1)} - 1}{\frac{9}{((\beta^{j+1})^2-1)(\beta^{j+1}+1)} - 1} < 1 \iff \frac{1}{(\beta^{j+2} - 1)(\beta^{j+2} + 1)^2} < \frac{1}{(\beta^{j+1} - 1)(\beta^{j+1} + 1)^2}.$$

   At this point, from condition $\beta^{j+1} \in]1, 2[$ it follows $\beta^{j+1} - 1 > 0$. Therefore

$$\frac{((\beta^{j+1})^2 - 1)(\beta^{j+1} + 1)}{((\beta^{j+2})^2 - 1)(\beta^{j+2} + 1)} < 1.$$

   Now, as $\beta^{j+2} = \sqrt{2 + \beta^{j+1}}$, thus $(\beta^{j+2})^2 - 1 = \beta^{j+1} + 1$, from which it turns out

$$\frac{(\beta^{j+1})^2 - 1}{\beta^{j+2} + 1} < 1$$

   and consequently

$$(\beta^{j+1})^2 - 1 < \beta^{j+2} + 1.$$

   Again, due to the fact that $(\beta^{j+1})^2 = 2 + \beta^j$, we get $\beta^j < \beta^{j+2}$. Therefore, since whenever $1 < \beta^{j+1} < 2$ the sequence $\{\beta^j\}_{j\in\mathbb{N}}$ is strictly increasing, this last statement is trivially verified and hence the ratio test allows us to prove the convergence of (10).

In this way, by unifying the three cases above, we can conclude that (9) is verified for any choice of the initial tension parameter $\beta^0 \in [-2, +\infty[\setminus\{-1\}$. Hence the non-stationary subdivision scheme defined by coefficients in (5) is asymptotically equivalent to the stationary scheme having coefficients in (4) with $\mu = \frac{1}{10}$. $\square$

4 Basis function

The basis function of a subdivision scheme is the limit function for the data

$$p_i^0 = \begin{cases} 1, & \text{if } i = 0, \\ 0, & \text{if } i \neq 0. \end{cases} \tag{12}$$

By Proposition 6 it follows that the basis function defined by the scheme introduced in section 2 belongs to $C^2(\mathbb{R})$. We show now that it is symmetric about the Y-axis and it possesses a compact support over the interval $[-\frac{5}{2}, \frac{5}{2}]$ (Fig.2).

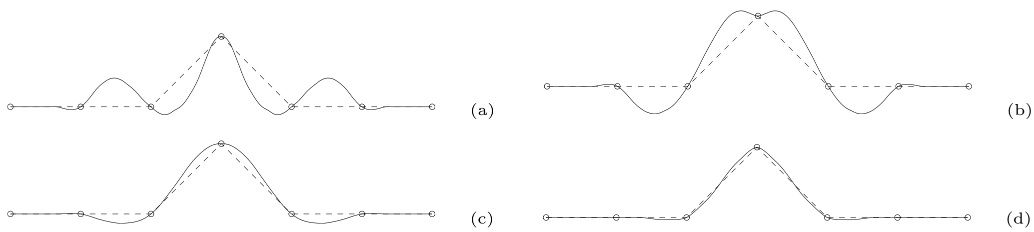

Fig. 2. Basis functions for increasing values of the global tension parameter β^0: (a) -1.5, (b) -0.5, (c) 0.5, (d) 10.

Proposition 7 *The basis function F defined by the scheme introduced in section 2 is symmetric about the Y-axis.*

PROOF. Let us define the set $D_n := \{\frac{i}{3^n} \mid i \in \mathbb{Z}\}$ such that the restriction of the basis function F to D_n satisfies $F\left(\frac{i}{3^n}\right) = p_i^n$ for all $i \in \mathbb{Z}$ and prove the thesis by induction on n.
First of all we observe that $F(i) = p_i^0 = p_{-i}^0 = F(-i)\ \forall i \in \mathbb{Z}$, and thus $F\left(\frac{i}{3^n}\right) = F\left(-\frac{i}{3^n}\right)\ \forall i \in \mathbb{Z},\ n = 0$.
Now, assuming $F\left(\frac{i}{3^n}\right) = F\left(-\frac{i}{3^n}\right)\ \forall i \in \mathbb{Z}$, it follows that $p_i^n = F\left(\frac{i}{3^n}\right) = F\left(-\frac{i}{3^n}\right) = p_{-i}^n\ \forall i \in \mathbb{Z}$, and consequently

$$F\left(\frac{3i}{3^{n+1}}\right) = p_{3i}^{n+1} = p_{-3i}^{n+1} = F\left(-\frac{3i}{3^{n+1}}\right),$$

$$\begin{aligned} F\left(\frac{3i+1}{3^{n+1}}\right) &= p_{3i+1}^{n+1} = a_0^n\ p_{i-1}^n + a_1^n\ p_i^n + a_2^n\ p_{i+1}^n + a_3^n\ p_{i+2}^n \\ &= a_3^n\ p_{-i-2}^n + a_2^n\ p_{-i-1}^n + a_1^n\ p_{-i}^n + a_0^n\ p_{-i+1}^n = p_{-3i-1}^{n+1} = F\left(-\frac{3i+1}{3^{n+1}}\right), \end{aligned}$$

$$\begin{aligned} F\left(\frac{3i+2}{3^{n+1}}\right) &= p_{3i+2}^{n+1} = a_3^n\ p_{i-1}^n + a_2^n\ p_i^n + a_1^n\ p_{i+1}^n + a_0^n\ p_{i+2}^n \\ &= a_0^n\ p_{-i-2}^n + a_1^n\ p_{-i-1}^n + a_2^n\ p_{-i}^n + a_3^n\ p_{-i+1}^n = p_{-3i-2}^{n+1} = F\left(-\frac{3i+2}{3^{n+1}}\right). \end{aligned}$$

Hence $F\left(\frac{i}{3^n}\right) = F\left(-\frac{i}{3^n}\right)\ \forall i \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$.
As a consequence, from the continuity of F it holds $F(x) = F(-x)$ for all $x \in \mathbb{R}$, which completes the proof. $\square$

Proposition 8 *The basis function F defined by the scheme introduced in section 2 has support width $s = 5$, i.e. it vanishes outside the interval $[-\frac{5}{2}, \frac{5}{2}]$.*

PROOF. Since the basis function F is the limit function of the scheme for the data in (12), its support width s can be determined by computing how far the effect of the non-zero vertex p_0^0 will propagate along by. As the mask m^j is an 11-long sequence, by centering it on that vertex, the distance to the last of its non-zero coefficients is equal to 5 on each side and, after each refinement, it is reduced by the factor $\frac{1}{3}$. Therefore, at the first subdivision step, the influence of the non-zero vertex p_0^0 extends a distance $\frac{5}{3}$ on each side; during the second step that last non-zero coefficient itself causes a further effect of $\frac{5}{3^2}$, and successive iterations push it out by $\frac{5}{3^3}, \cdots$. Hence, after N subdivisions, the furthest non-zero vertex will be at $5\left(\frac{1}{3}+\frac{1}{3^2}+...+\frac{1}{3^N}\right)=\frac{5}{3}\sum_{j=0}^{N-1}\frac{1}{3^j}$. Since, being $|\frac{1}{3}|<1$, the geometric sequence can be summed to give the extended distance on each side, we can conclude that, in the limit, the total influence of the original non-zero vertex will propagate along by $s=2\,\frac{5}{3}\sum_{j=0}^{+\infty}\frac{1}{3^j}=\frac{10}{3}\frac{1}{1-\frac{1}{3}}=5$. $\square$

5 Local tensions

The uniform subdivision scheme described in section 2 allows the user to choose an initial tension value β^0 which is updated at each refinement step through relation (7). The parameter β^0 acts like a global tension, i.e. its choice affects the shape of the whole limit curve (Fig.1).

In this section a generalization of that scheme is presented, which enables to set a different parameter for each edge of the starting control polygon (Fig. 4). This means that, assigning an initial tension value β_i^0 to every segment $\overline{p_i^0 p_{i+1}^0}$, after j iterations a tension β_i^j will be associated with $\overline{p_i^j p_{i+1}^j}$. Moreover, since after each refinement, two new points are inserted between two old ones, it is possible to establish an ideal correspondence between every edge of the coarse control polygon and the three new ones of the refined polyline. Let β_i^j be the tension parameter associated with the segment $\overline{p_i^j p_{i+1}^j}$. Such an edge is split into the three new ones $\overline{p_{3i}^{j+1} p_{3i+1}^{j+1}}$, $\overline{p_{3i+1}^{j+1} p_{3i+2}^{j+1}}$, $\overline{p_{3i+2}^{j+1} p_{3(i+1)}^{j+1}}$. Thus, according to (7), we will make them inherit respectively the tension values $\beta_{3i}^{j+1}=\beta_{3i+1}^{j+1}=\beta_{3i+2}^{j+1}=\sqrt{2+\beta_i^j}$. To match up with this pattern, the subdivision rules in (1) will be consequently modified in such a way the coefficients a_k^j $(k=0,1,2,3)$ would include a subscript i underlining their dependence on the tension value β_i^j. Hence they will be described by the following relations

$$
\begin{aligned}
p_{3i}^{j+1} &= p_i^j \\
p_{3i+1}^{j+1} &= a_{0,i}^j\; p_{i-1}^j + a_{1,i}^j\; p_i^j + a_{2,i}^j\; p_{i+1}^j + a_{3,i}^j\; p_{i+2}^j \\
p_{3i+2}^{j+1} &= a_{3,i}^j\; p_{i-1}^j + a_{2,i}^j\; p_i^j + a_{1,i}^j\; p_{i+1}^j + a_{0,i}^j\; p_{i+2}^j
\end{aligned}
\tag{13}
$$

where

$$
\begin{aligned}
a_{0,i}^j &= \frac{1}{60}(-90\gamma_i^{j+1}-1) \\
a_{1,i}^j &= \frac{1}{60}(90\gamma_i^{j+1}+43)
\end{aligned}
\tag{14}
$$

$$a^j_{2,i} = \frac{1}{60}(90\gamma_i^{j+1} + 17)$$
$$a^j_{3,i} = \frac{1}{60}(-90\gamma_i^{j+1} + 1)$$

and

$$\gamma_i^{j+1} = -\frac{1}{3(1-(\beta_i^{j+1})^2)(1+\beta_i^{j+1})}.$$

Remark 9 *Note that property (2) still holds for the definition of the coefficients in (14).*

The mask of the local scheme at the j-th refinement level thus becomes

$$m_i^j = \left[\ a^j_{3,i},\ a^j_{0,i},\ 0,\ a^j_{2,i},\ a^j_{1,i},\ 1,\ a^j_{1,i},\ a^j_{2,i},\ 0,\ a^j_{0,i},\ a^j_{3,i}\ \right]. \tag{15}$$

From equation (15) it is evident that the tension value β_i^0 assigned to each edge $\overline{p_i^0 p_{i+1}^0}$ influences the limit shape only in the restricted region confined between its two endpoints (Fig. 3).

Proposition 10 *The non-stationary subdivision scheme defined by (14)-(15) is asymptotically equivalent to the stationary scheme defined by (3)-(4) with $\mu = \frac{1}{10}$. Moreover it generates C^2-continuous limit curves.*

PROOF. In order to prove that the non-stationary subdivision scheme with mask (14)-(15) converges to C^2-continuous limit curves, we compute the associated first divided difference mask

$$\begin{aligned} d^j_{i,(1)} = \frac{1}{60}\Big[& -90\gamma_i^{j+1}+1,\ 90\gamma_{i-1}^{j+1}-90\gamma_i^{j+1}-2,\ 90\gamma_{i-1}^{j+1}+1,\ 18,\ 24,\ 18, \\ & 90\gamma_i^{j+1}+1,\ 90\gamma_i^{j+1}-90\gamma_{i-1}^{j+1}-2,\ -90\gamma_{i-1}^{j+1}+1 \Big] \end{aligned} \tag{16}$$

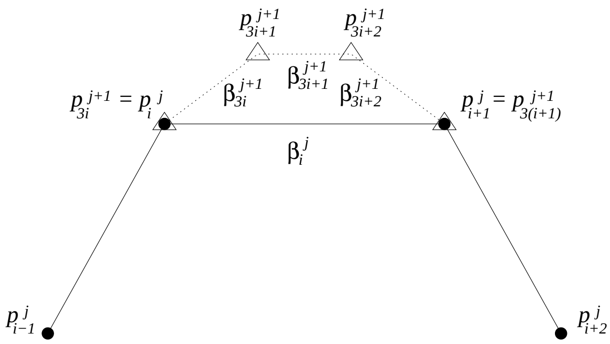

Fig. 3. Interpolation of the vertices of a given control polygon by using the ternary subdivision scheme with local tensions.

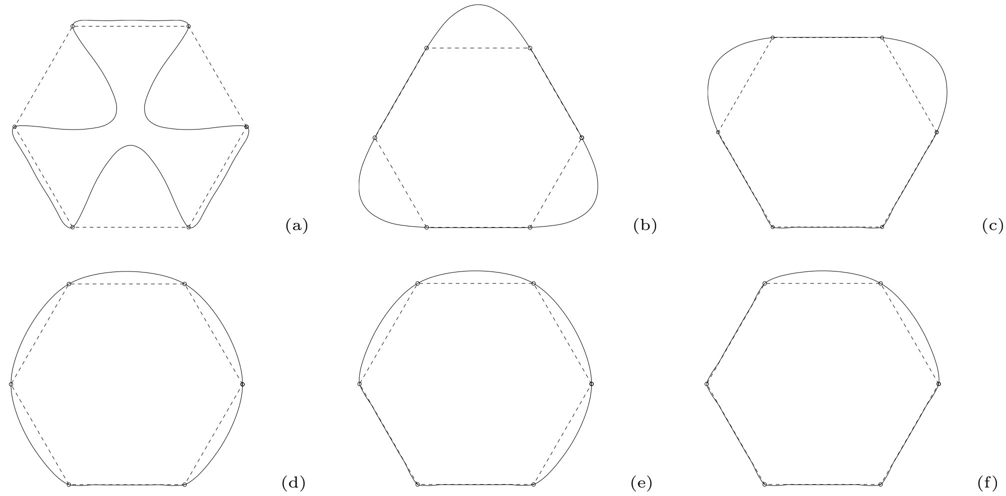

Fig. 4. Interpolation of the vertices of a regular hexagon by using the following local tensions: (a) $[-1.5,\ 5,\ -1.5,\ 5,\ -1.5,\ 5]$, (b) $[100,\ -0.2,\ 100,\ -0.2,\ 100,\ -0.2]$, (c) $[0,\ 100,\ 0,\ 100,\ 100,\ 100]$, (d) $[2,\ 2,\ 2,\ 2,\ 100,\ 2]$, (e) $[2,\ 2,\ 2,\ 100,\ 100,\ 2]$, (f) $[2,\ 2,\ 100,\ 100,\ 100,\ 100]$.

and we derive from it the second divided difference mask

$$
\begin{aligned}
d^j_{i,(2)} = \frac{1}{20}\Big[& -90\gamma_i^{j+1}+1,\ 180\gamma_{i-1}^{j+1}-90\gamma_i^{j+1}-3,\ 180\gamma_{i-1}^{j+1}-90\gamma_{i-2}^{j+1}+3,\\
& -90\gamma_{i-2}^{j+1}-90\gamma_i^{j+1}+18,\ 180\gamma_{i-1}^{j+1}-90\gamma_i^{j+1}+3,\\
& 180\gamma_{i-1}^{j+1}-90\gamma_{i-2}^{j+1}-3,\ -90\gamma_{i-2}^{j+1}+1\Big].
\end{aligned}
\tag{17}
$$

Then we only need to show that the scheme defined by (17) is C^0-continuous. From Proposition 5 it can be easily seen that $\lim_{j\to+\infty} d^j_{i,(2)} = d^\infty_{(2)}$, namely $d^j_{i,(2)}$ converges to the second divided difference mask of the C^2-continuous stationary scheme (3)-(4) with $\mu = \frac{1}{10}$. To conclude the proof it is therefore sufficient to check whether the two subdivision schemes are asymptotically equivalent, i.e. they satisfy

$$
\sum_{j=0}^{+\infty} \parallel d^j_{i,(2)} - d^\infty_{(2)} \parallel_\infty < +\infty.
$$

This last statement can be easily verified, since, as it was already shown in Proposition 6, $\sum_{j=0}^{+\infty} \mid -27\gamma_i^{j+1}+1 \mid < +\infty,\ \forall i,j \in \mathbb{Z}$. $\square$

Remark 11 *By setting all initial tensions equal to the same value β^0, we get the uniform tension controlled interpolating 4-point non-stationary scheme defined in section 2.*

6 Conclusions

In this paper we have introduced a novel non-stationary ternary 4-point interpolatory subdivision scheme which provides the user with a tension parameter that, when increased within its range of definition, can generate C^2-continuous limit curves showing considerable variations of shape. Such a scheme repairs the drawbacks of its stationary analogue (Hassan et al., 2002), which does not give the possibility to appreciate significant shape modifications whenever C^2-continuity is enforced. Moreover, if we compare it with its non-stationary binary counterpart (Beccari et al., 2007), we can see that it possesses higher smoothness (it is C^2 instead of C^1) while having smaller support (whose length is 5 instead of 6).
In order to include also the capability of adjusting the limit shape only in restricted regions, we have generalized our proposal to a C^2-continuous locally-controlled interpolatory 4-point ternary scheme.

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