Compactness results in conformal deformations of Riemannian metrics on manifolds with boundaries

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Abstract. This paper is devoted to the study of a problem arising from a geometric context, namely the conformal deformation of a Riemannian metric to a scalar flat one having constant mean curvature on the boundary. By means of blow-up analysis techniques and the Positive Mass Theorem, we show that on locally conformally flat manifolds with umbilic boundary all metrics stay in a compact set with respect to the C^2-norm and the total Leray-Schauder degree of all solutions is equal to -1. Then we deduce from this compactness result the existence of at least one solution to our problem.

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1. Introduction

Let (M, g) be an n-dimensional compact smooth Riemannian manifold with boundary. For $n = 2$, the well-known Riemann Mapping Theorem states that an open simply connected proper subset of the plane is conformally diffeomorphic to the disk. In what can be seen as a tentative of generalization of the above problem, J. Escobar [5] asked if (M, g) is conformally equivalent to a manifold that has zero scalar curvature and whose boundary has a constant mean curvature.

Setting $\tilde{g} = u^{\frac{4}{n-2}} g$ conformal metric to g, the above problem is equivalent to find a smooth positive solution u to the following nonlinear boundary value problem on (M, g):

$$
\begin{cases}
-\Delta_g u + \dfrac{n-2}{4(n-1)} R_g u = 0, \quad u > 0, & \text{in } \overset{\circ}{M}, \\
\dfrac{\partial u}{\partial \nu} + \dfrac{n-2}{2} h_g u = c u^{\frac{n}{n-2}}, & \text{on } \partial M,
\end{cases}
\tag{$\mathcal{P}$}
$$

where $\overset{\circ}{M} = M \setminus \partial M$ denotes the interior of M, R_g is the scalar curvature, h_g is the mean curvature of ∂M, ν is the outer normal with respect to g, and c is a constant whose sign is uniquely determined by the conformal structure of M. Solutions of equation $(\mathcal{P})$ correspond, up to some positive constant, to critical points of the following function J defined on $H^1(M) \setminus \{0\}$

$$
J(u) = \frac{\displaystyle\int_M \left(|\nabla_g u|^2 + \frac{n-2}{4(n-1)} R_g u^2 \right) dV_g + \frac{n-2}{2} \int_{\partial M} h_g u^2 \, d\sigma_g}{\left(\displaystyle\int_{\partial M} |u|^{\frac{2(n-1)}{n-2}} \, d\sigma_g \right)^{\frac{n-2}{n-1}}}.
$$

The exponent $\frac{2(n-1)}{n-2}$ is critical for the Sobolev trace embedding $H^1(M) \hookrightarrow L^q(\partial M)$. This embedding being not compact, the functional J does not satisfy the Palais-Smale condition. For this reason standard variational methods cannot be applied to find critical points of J.

The regularity of the H^1-solutions of $(\mathcal{P})$ was established by Cherrier [3], and existence results in many cases were obtained by Escobar, see [5, 7]. Related problems regarding conformal deformations of Riemannian metrics on manifolds with boundaries have been studied in [1, 2, 4, 6, 9, 10, 11, 15, 16, 20, 22]; see also the references therein.

To describe our results concerning problem $(\mathcal{P})$, we need the following notation. We use L_g to denote $\Delta_g - (n-2)/[4(n-1)] R_g$, B_g to denote $\partial/\partial\nu + (n-2)/2\, h_g$. Let H denote the second fundamental form of ∂M in (M, g) with respect to the inner normal; we denote its traceless part part by U:

$$
U(X, Y) = H(X, Y) - h_g g(X, Y).
$$

Definition 1.1. *A point $p \in \partial M$ is called an umbilic point if $U = 0$ at p. The boundary of M is called umbilic if every point of ∂M is umbilic.*

Remark 1.2. *The notion of umbilic point is conformally invariant, namely, if $p \in \partial M$ is an umbilic point with respect to g, it is also an umbilic point with respect to the metric $\tilde{g} = \psi^{\frac{4}{n-2}} g$, for any positive smooth function ψ on M.*

Let $\lambda_1(L)$ denote the first eigenvalue of

$$
\begin{cases}
-L_g \varphi = \lambda \varphi, & \text{in } \overset{\circ}{M}, \\
B_g \varphi = 0, & \text{on } \partial M,
\end{cases}
\tag{E_1}
$$

and $\lambda_1(B)$ denote the first eigenvalue of the problem

$$\begin{cases} L_g u = 0, & \text{in } \overset{\circ}{M}, \\ B_g u = \lambda u, & \text{on } \partial M. \end{cases} \tag{E_2}$$

It is well-known (see [5]) that the signs of $\lambda_1(B)$ and $\lambda_1(L)$ are the same and they are conformal invariants.

Definition 1.3. *We say that a manifold is of positive (respectively negative, zero) type if $\lambda_1(L) > 0$ (respectively < 0 , $= 0$).*

In this paper, we give some existence and compactness results concerning $(\mathcal{P})$. We first describe our results for manifolds of positive type.

Let (M, g) be a manifold of positive type. We consider the following problem

$$\begin{cases} -L_g u = 0, \quad u > 0, & \text{in } \overset{\circ}{M}, \\ B_g u = (n-2) u^{\frac{n}{n-2}}, & \text{on } \partial M. \end{cases} \tag{$\mathcal{P}_+$}$$

Let $\mathcal{M}^+$ denote the set of solutions of $(\mathcal{P}_+)$. Then we have

Theorem 1.4. *For $n \geq 3$, let (M, g) be a smooth compact n-dimensional locally conformally flat Riemannian manifold of positive type with umbilic boundary. Then $\mathcal{M}^+ \neq \emptyset$. Furthermore, if (M, g) is not conformally equivalent to the standard ball, then there exists $C = C(M, g)$ such that for all $u \in \mathcal{M}^+$ we have*

$$\frac{1}{C} \leq u(x) \leq C, \quad \forall\, x \in M; \quad \textit{and} \quad \|u\|_{C^2(M)} \leq C,$$

and the total Leray-Schauder degree of all solutions to $(\mathcal{P}_+)$ is -1.

Let us remark that the existence of solutions to $(\mathcal{P}_+)$ under the condition of Theorem 1.4 was already established by Escobar in [5], among other existence results. He obtained, using the Positive Mass Theorem of Schoen-Yau [24], that the infimum of J is achieved. See also [22] for the existence of a solution to $(\mathcal{P}_+)$ of higher energy and higher Morse index. What is new in Theorem 1.4 is the compactness part. In fact we establish a slightly stronger compactness result. Consider, for $1 < q \leq \frac{n}{n-2}$,

$$\begin{cases} -L_g u = 0, \quad u > 0, & \text{in } \overset{\circ}{M}, \\ B_g u = (n-2) u^{q}, & \text{on } \partial M. \end{cases} \tag{$\mathcal{P}_q^+$}$$

Let $\mathcal{M}_q^+$ denote the set of solutions of $(\mathcal{P}_q^+)$ in $C^2(M)$. We have the following

Theorem 1.5. *For $n \geq 3$, let (M, g) be a smooth compact n-dimensional locally conformally flat Riemannian manifold of positive type with umbilic boundary. We assume that (M, g) is not conformally equivalent to the standard ball. Then there exist*

$\delta_0 = \delta_0(M,g) > 0$ and $C = C(M,g) > 0$ such that for all $u \in \bigcup_{1+\delta_0 \le q \le \frac{n}{n-2}} \mathcal{M}_q^+$ we have

$$\frac{1}{C} \le u(x) \le C, \quad \forall\, x \in M, \quad \textit{and} \quad \|u\|_{C^2(M)} \le C.$$

To prove Theorems 1.4 and 1.5 we establish compactness results for all solutions of $(\mathcal{P}_q^+)$ and then show that the total degree of all solutions to $(\mathcal{P}_+)$ is -1. To do this we perform some fine blow-up analysis of possible behaviour of blowing-up solutions of $(\mathcal{P}_q^+)$ which, together with the Positive Mass Theorem by Schoen and Yau [24] (see also [6]), implies energy independent estimates for all solutions of $(\mathcal{P}_q^+)$.

When (M,g) is a n-dimensional ($n \ge 3$) locally conformally flat manifold without boundary, such compactness results based on blow-up analysis and energy independent estimates were obtained by Schoen [23] for solutions of

$$-L_g u = n(n-2)u^q, \quad u > 0, \quad \text{in } M,$$

where $1+\varepsilon_0 < q < \frac{n+2}{n-2}$. In the same paper [23] he also announced, with indications on the proof, the same results for general manifolds. Along the same approach initiated by Schoen, Z. C. Han and Y. Y. Li [10] obtained similar compactness and existence results for the so-called Yamabe like problem on compact locally conformally flat manifolds with umbilic boundary. Other compactness results on Yamabe type equations on three dimensional Riemannian manifolds were obtained by Y. Y. Li and M. J. Zhu [18].

Now we present similar existence and compactness results for manifolds of negative type. Let (M,g) be a compact n-dimensional Riemannian manifold of negative type. Consider for $1 < q \le \frac{n}{n-2}$

$$\begin{cases} -L_g u = 0, \quad u > 0, \quad \text{in } \overset{\circ}{M}, \\ B_g u = -(n-2)u^q, \qquad \text{on } \partial M. \end{cases} \tag{$\mathcal{P}_q^-$}$$

Let $\mathcal{M}_q^-$ denote the set of solutions of $(\mathcal{P}_q^-)$ in $C^2(M)$ and $\mathcal{M}^- = \mathcal{M}^-_{\frac{n}{n-2}}$. We have the following

Theorem 1.6. *For $n \ge 3$, let (M,g) be a smooth compact n-dimensional Riemannian manifold of negative type with boundary. Then $\mathcal{M}^- \ne \emptyset$. Furthermore, there exist $\delta_0 = \delta_0(M,g)$ and $C = C(M,g) > 0$ such that for all $u \in \bigcup_{1+\delta_0 \le q \le \frac{n}{n-2}} \mathcal{M}_q^-$*

$$\frac{1}{C} \le u(x) \le C, \quad \forall\, x \in M; \quad \|u\|_{C^2(M)} \le C,$$

and the total degree of all solutions of $(\mathcal{P}_q^-)$ is -1.

Let us notice that apriori estimates in the above Theorem are due basically to some non-existence Liouville-type Theorems for the limiting equations.

The remainder of the paper is organized as follows. In section 2 we provide the local blow-up analysis. In section 3 we establish the compactness part in Theorems 1.4 and 1.5.

In section 4 we prove existence part of Theorem 1.4 while section 5 is devoted to the proof of Theorem 1.6. Finally, we collect some technical lemmas and well-known results in the appendix.

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2. Local blow-up analysis

In the following, we give the definitions of isolated and isolated simple blow-up, which were first introduced by R. Schoen, see [23], and adapted to the framework of boundary value problems by Y. Y. Li [15], see also [10].

Definition 2.1. *Let (M,g) be a smooth compact n-dimensional Riemannian manifold with boundary, and let $\bar r > 0$, $\bar c > 0$, $\bar x \in M$, $f \in C^0(\overline{B_{\bar r}(\bar x)})$ be some positive function where $B_{\bar r}(\bar x)$ denotes the geodesic ball in (M,g) of radius $\bar r$ centered at $\bar x$. Suppose that, for some sequences $q_i = \frac{n}{n-2} - \tau_i$, $\tau_i \to 0$, $f_i \to f$ in $C^0(\overline{B_{\bar r}(\bar x)})$, $\{u_i\}_{i\in\mathbb{N}}$ solves*

$$\begin{cases} -L_g u_i = 0, \quad u_i > 0, \quad \text{in } B_{\bar r}(\bar x), \\ B_g u_i = (n-2) f_i^{\tau_i} u_i^{q_i}, \quad \text{on } \partial M \cap B_{\bar r}(\bar x). \end{cases} \tag{2.1$_i$}$$

We say that $\bar x$ is an isolated blow-up point of $\{u_i\}_i$ if there exists a sequence of local maximum points x_i of u_i such that $x_i \to \bar x$ and, for some $C_1 > 0$,

$$\lim_{i\to\infty} u_i(x_i) = +\infty \quad \text{and} \quad u_i(x) \le C_1 d(x,x_i)^{-\frac{1}{q_i-1}}, \quad \forall\, x \in B_{\bar r}(x_i),\ \forall\, i.$$

To describe the behaviour of blowing-up solutions near an isolated blow-up point, we define spherical averages of u_i centered at x_i as follows

$$\bar u_i(r) = \fint_{M\cap\partial B_r(\bar x)} u_i = \frac{1}{\mathrm{Vol}_g(M\cap\partial B_r(\bar x))} \int_{M\cap\partial B_r(\bar x)} u_i.$$

Now we define the notion of isolated simple blow-up point.

Definition 2.2. *Let $x_i \to \bar x$ be an isolated blow-up point of $\{u_i\}_i$ as in Definition 2.1. We say that $x_i \to \bar x$ is an isolated simple blow-up point of $\{u_i\}$ if, for some positive constants $\tilde r \in (0,\bar r)$ and $C_2 > 1$, the function $\bar w_i(r) := r^{\frac{1}{q_i-1}} \bar u_i(r)$ satisfies, for large i,*

$$\bar w_i'(r) < 0 \quad \text{for } r \text{ satisfying} \quad C_2 u_i^{1-q_i}(x_i) \le r \le \tilde r.$$

Let us introduce the following notation

$$\mathbb{R}^n_+ = \{(x', x^n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \; x^n > 0\}, \quad B^+_\sigma(\bar{x}) = \{x = (x', x^n) \in \mathbb{R}^n_+ : \; |x - \bar{x}| < \sigma\},$$
$$B^+_\sigma = B^+_\sigma(0), \quad \Gamma_1(B^+_\sigma(\bar{x})) = \partial B^+_\sigma(\bar{x}) \cap \partial \mathbb{R}^n_+, \quad \Gamma_2(B_\sigma(\bar{x})) = \partial B_\sigma(\bar{x}) \cap \mathbb{R}^n_+.$$

Let $\{f_i\} \subset C^1(\Gamma_1(B^+_3))$ be a sequence of functions satisfying, for some positive constant C_3,

$$f_i \underset{i\to\infty}{\longrightarrow} f \quad \text{in } C^1(\Gamma_1(B^+_3)), \quad \|f_i\|_{L^\infty(\Gamma_1(B^+_3))} \le C_3 \tag{2.2}$$

where $f \in C^1(\Gamma_1(B^+_3))$ is some positive function. Suppose that $\{v_i\}_i \subset C^2(\overline{B^+_3})$ is a sequence of solutions to

$$\begin{cases} -\Delta v_i = 0, \quad v_i > 0, & \text{in } B^+_3, \\ \dfrac{\partial v_i}{\partial x^n} = -(n-2) f_i^{\tau_i} v_i^{q_i}, & \text{on } \Gamma_1(B^+_3). \end{cases} \tag*{$(2.3)_i$}$$

The following Lemma gives a Harnack inequality.

Lemma 2.3. *Assume* (2.2) *and let* $\{v_i\}_i$ *satisfy* $(2.3)_i$. *Let* $0 < \bar{r} < \frac{1}{8}$, $\bar{x} \in \Gamma_1(\overline{B^+_{1/8}})$ *and suppose that* $x_i \to x$ *is an isolated blow-up point of* $\{v_i\}_i$. *Then, for all* $0 < r < \bar{r}$,

$$\sup_{B^+_{2r}(x_i) \setminus B^+_{\frac{r}{2}}(x_i)} v_i \le C_4 \inf_{B^+_{2r}(x_i) \setminus B^+_{\frac{r}{2}}(x_i)} v_i,$$

where $C_4 > 0$ *is some positive constant independent of* i *and* r.

Proof. Without loss of generality, we assume that $x_i \in \Gamma_1(\overline{B^+_{1/8}})$. For $0 < r < \bar{r}$, let us consider

$$\tilde{v}_i(y) := r^{\frac{1}{q_i - 1}} v_i(ry + x_i).$$

Then $\tilde{v}_i$ satisfies

$$\begin{cases} -\Delta \tilde{v}_i = 0, \quad \tilde{v}_i > 0, & \text{in } A_i, \\ \dfrac{\partial \tilde{v}_i}{\partial y^n} = -(n-2) f_i^{\tau_i}(ry + x_i) \tilde{v}_i^{q_i - 1} \tilde{v}_i, & \text{on } \Gamma_1(A_i), \end{cases}$$

where $A_i = \left\{ y \in \mathbb{R}^n : \; \frac{1}{3} < |y| < 3, \; ry + x_i \in \mathbb{R}^n_+ \right\}$. From Definition 2.1 we know that

$$\tilde{v}_i \le C_1 \quad \text{in } A_i,$$

where C_1 depends neither on r nor on i. In view of (2.2), from Lemma 6.1 (standard Harnack) in the appendix we obtain that for some constant $c > 0$

$$\max_{\widetilde{A}_i} \tilde{v}_i \le c \min_{\widetilde{A}_i} \tilde{v}_i$$

where $\widetilde{A}_i = \left\{ y \in \mathbb{R}^n : \ \frac{1}{2} < |y| < 2, \ ry + x_i \in \mathbb{R}^n_+ \right\}$; the proof of the Lemma is thereby completed. □

Lemma 2.4. *Suppose that* $\{v_i\}_i$ *satisfies* $(2.3)_i$ *and* $\{x_i\}_i \subset \Gamma_1(B_1^+)$ *is a sequence of local maximum points of* $\{v_i\}_i$ *in* $\overline{B_3^+}$ *satisfying*

$$\{v_i(x_i)\} \quad \textit{is bounded,}$$

and, for some constant C_5 *,*

$$|x - x_i|^{\frac{1}{q_i - 1}} v_i(x) \le C_5, \quad \forall\, x \in B_3^+. \tag{2.4}$$

Then

$$\limsup_{i \to \infty} \max_{\overline{B^+_{1/4}(x_i)}} v_i < \infty. \tag{2.5}$$

Proof. By contradiction, suppose that, under the assumptions of the Lemma, (2.5) fails, namely that, along a subsequence, for some $\tilde{x}_i \in \overline{B^+_{1/4}(x_i)}$ we have

$$v_i(\tilde{x}_i) = \max_{\overline{B^+_{1/4}(x_i)}} v_i \underset{i \to \infty}{\longrightarrow} +\infty.$$

It follows from (2.4) that $|\tilde{x}_i - x_i| \to 0$. Let us now consider

$$\xi_i(z) = v_i^{-1}(\tilde{x}_i) v_i(\tilde{x}_i + v_i^{1-q_i}(\tilde{x}_i) z)$$

defined on the set

$$B^{-T_i}_{\frac{1}{8} v_i^{q_i - 1}(\tilde{x}_i)} := \left\{ z \in \mathbb{R}^n : \ |z| < \frac{1}{8} v_i^{q_i - 1}(\tilde{x}_i) \quad \text{and} \quad z^n > -T_i \right\}$$

where $T_i = \tilde{x}_i^n v_i^{q_i - 1}(\tilde{x}_i)$. In view of $(2.3)_i$, ξ_i satisfies

$$\begin{cases} -\Delta \xi_i = 0, \quad \xi_i > 0, \qquad z \in B^{-T_i}_{\frac{1}{8} v_i^{q_i - 1}(\tilde{x}_i)}, \\ \dfrac{\partial \xi_i}{\partial z^n} = -(n-2) f_i^{\tau_i} \xi_i^{q_i}, \quad z \in \partial B^{-T_i}_{\frac{1}{8} v_i^{q_i - 1}(\tilde{x}_i)} \cap \{ z = (z', z^n) \in \mathbb{R}^n : \ z^n = -T_i \}, \end{cases}$$

and

$$\xi_i(z) \le \xi_i(0) = 1, \quad \forall\, z \in B^{-T_i}_{\frac{1}{8} v_i^{q_i - 1}(\tilde{x}_i)}.$$

It follows from (2.4) that

$$|z|^{\frac{1}{q_i - 1}} \xi_i(z) \le C_1, \quad \forall\, z \in B^{-T_i}_{\frac{1}{8} v_i^{q_i - 1}(\tilde{x}_i)}.$$

Since $\{\xi_i\}_i$ is locally bounded, applying L^p-estimates, Schauder estimates, the Harnack inequality, and Lemma 6.1, we have that, up to a subsequence, there exists some positive function ξ such that

$$\lim_{i\to\infty} \|\xi_i - \xi\|_{C^2(\mathbb{R}^n_{-T_i}\cap\overline{B_R})} = 0, \quad \forall\, R > 1,$$

where $\mathbb{R}^n_{-T_i} = \{z = (z', z^n) \in \mathbb{R}^n : \ z^n > -T_i\}$ and, for $T = \lim_{i\to\infty} T_i \in [0, +\infty]$, ξ satisfies

$$\begin{cases} -\Delta\xi = 0, \quad \xi > 0, & \text{in } \mathbb{R}^n_{-T}, \\ \dfrac{\partial\xi}{\partial z^n} = -(n-2)\xi^{\frac{n}{n-2}}, & \text{on } \partial\mathbb{R}^n_{-T}. \end{cases} \tag{2.6}$$

Let us prove that $T < \infty$. Indeed, if we assume by contradiction that $T = +\infty$, we have that ξ is a harmonic bounded function in $\mathbb{R}^n$. The Liouville Theorem yields that ξ is a constant and this is in contradiction with (2.4).

Therefore $T < \infty$. Let us prove that $T = 0$. Since problem (2.6), up to a translation, satisfies the assumptions of the uniqueness Theorem by Li and Zhu [17], we deduce that ξ is of the form

$$\xi(x', x^n) = \left[\frac{\lambda}{(1 + \lambda(x^n - T))^2 + \lambda^2|x' - x'_0|^2}\right]^{\frac{n-2}{2}}$$

for some $\lambda > 0$, $x'_0 \in \mathbb{R}^{n-1}$. Since 0 is a local maximum point for ξ, it follows that $x'_0 = 0$ and $T = 0$. Furthermore the fact that $\xi(0) = 1$ yields $\lambda = 1$. It follows that, for all $R > 1$

$$\min_{\overline{B}^{-T_i}_{Rv_i^{-(q_i-1)}(\tilde{x}_i)}(\tilde{x}_i)} v_i = v_i(\tilde{x}_i) \min_{\overline{B}^{-T_i}_R(0)} \xi_i \underset{i\to\infty}{\longrightarrow} \infty.$$

Since $\{v_i(x_i)\}_i$ is bounded, we have that, for any $R > 1$, $x_i \notin \bar{B}^{-T_i}_{Rv_i^{q_i-1}(\tilde{x}_i)}(\tilde{x}_i)$ for large i, namely

$$R < v_i^{q_i-1}(\tilde{x}_i)|\tilde{x}_i - x_i|.$$

Hence we have that

$$|\tilde{x}_i - x_i|^{\frac{1}{q_i-1}} v_i(\tilde{x}_i) > R^{\frac{1}{q_i-1}}$$

which contradicts (2.4). □

Proposition 2.5. *Let (M, g) be a smooth compact n-dimensional locally conformally flat Riemannian manifold with umbilic boundary, and let $x_i \to \bar{x}$ be an isolated simple blow-up point of $\{u_i\}_i$. Then for any sequences of positive numbers $R_i \to \infty$, $\varepsilon_i \to 0$ there exists a subsequence $\{u_{j_i}\}_i$ (still denoted as $\{u_i\}_i$) such that*

$$r_i := R_i u_i^{1-q_i}(x_i) \underset{i\to\infty}{\longrightarrow} 0, \quad x_i \in \partial M,$$

and

$$\left\|u_i^{-1}(x_i)u_i\big(\exp_{x_i}(yu_i^{1-q_i}(x_i))\big)-\left(\frac{1}{(1+y^n)^2+|y'|^2}\right)^{\frac{n-2}{2}}\right\|_{C^2(B_{3R_i}(0))}$$
$$+\left\|u_i^{-1}(x_i)u_i\big((\exp_{x_i}(yu_i^{1-q_i}(x_i))\big)-\left(\frac{1}{(1+y^n)^2+|y'|^2}\right)^{\frac{n-2}{2}}\right\|_{H^1(B_{3R_i}(0))}<\varepsilon_i.$$

Moreover, for all $2r_i\le d(x,x_i)\le \tilde r/2$,

$$u_i(x)\le C_6u_i^{-1}(x_i)d(x,x_i)^{2-n},$$

where C_6 *is some positive constant independent of* i , *and*

$$u_i(x_i)u_i \underset{i\to\infty}{\longrightarrow} aG(\cdot,\bar x)+b \quad \text{in } C^2_{\rm loc}(\overline{B_{\tilde r}(\bar x)}\setminus\{\bar x\})$$

where $a>0$, b *is some nonnegative function satisfying*

$$\begin{cases} L_gb=0, & \text{in } B_{\tilde r}(\bar x)\setminus\{\bar x\},\\ B_gb=0, & \text{on } B_{\tilde r}(\bar x)\cap\partial M,\end{cases}$$

and $G(\cdot,\bar x)$ *is the Green's function satisfying*

$$\begin{cases} -L_gG(\cdot,\bar x)=0, & \text{in } M\setminus\{\bar x\},\\ B_gG(\cdot,\bar x)=0, & \text{on } \partial M\setminus\{\bar x\}.\end{cases} \tag{2.7}$$

To prove Proposition 2.5 we need some preliminary results. Hence forward we use c , $c_1,\ c_2,\dots$ to denote positive constants which may vary from formula to formula and which may depend only on M , g , n , and $\bar r$.

Lemma 2.6. *Let* $x_i\to 0$ *be an isolated blow-up point of* $\{v_i\}_i$ *with* v_i *solutions of* $(2.3)_i$. *Then, for any* $R_i\to\infty$ *and* $\varepsilon_i\to 0$, *there exists a subsequence of* $\{v_i\}_i$, *still denoted by* $\{v_i\}_i$, *such that*

$$r_i:=R_iv_i^{1-q_i}(x_i)\underset{i\to\infty}{\longrightarrow} 0$$

and

$$\left\|v_i^{-1}(x_i)v_i(xv_i^{1-q_i}(x_i)+x_i)-\left(\frac{1}{(1+x^n)^2+|x'|^2}\right)^{\frac{n-2}{2}}\right\|_{C^2(B^+_{3R_i})}$$
$$+\left\|v_i^{-1}(x_i)v_i(xv_i^{1-q_i}(x_i)+x_i)-\left(\frac{1}{(1+x^n)^2+|x'|^2}\right)^{\frac{n-2}{2}}\right\|_{H^1(B^+_{3R_i})}<\varepsilon_i.$$

Proof. Let us set

$$\tilde v_i(z) = v_i^{-1}(x_i)v_i(v_i^{1-q_i}(x_i)z + x_i), \quad z \in B^{-T_i}_{v_i^{q_i-1}(x_i)},$$

where $T_i = x_i^n v_i^{q_i-1}(x_i)$. It is clear that $\tilde v_i$ satisfies

$$\begin{cases} -\Delta \tilde v_i = 0, & \text{in } B^{-T_i}_{v_i^{q_i-1}(x_i)}, \\ \dfrac{\partial \tilde v_i}{\partial z^n} = -(n-2) f_i^{\tau_i}(v_i^{1-q_i}(x_i)z + x_i)\tilde v_i^{q_i}, & \text{on } \partial B^{-T_i}_{v_i^{q_i-1}(x_i)} \cap \{z \in \mathbb{R}^n : \ z^n = -T_i\}. \end{cases}$$

Let us prove that $\tilde v_i$ is uniformly bounded. By definition of isolated blow-up point, we have that

$$|z|^{\frac{1}{q_i-1}} \tilde v_i(z) \le C_1, \quad \forall\, z \in B^{-T_i}_{v_i^{q_i-1}(x_i)}. \tag{2.8}$$

It follows from (2.8), Lemma 2.3, and the Harnack inequality that $\tilde v_i$ is uniformly bounded in $B^{-T_i}_{v_i^{q_i-1}(x_i)} \cap \bar B_R$ for any $R > 0$. Then, up to a subsequence, setting $T = \lim_{i\to\infty} T_i \in [0, +\infty]$, $\tilde v_i$ converges to some $\tilde v$ in $C^2_{\rm loc}(\mathbb{R}^n_{-T})$ satisfying

$$\begin{cases} -\Delta \tilde v = 0, \quad \tilde v > 0, & \text{in } \mathbb{R}^n_{-T}, \\ \dfrac{\partial \tilde v}{\partial x^n} = -(n-2)\tilde v^{\frac{n}{n-2}}, & \text{on } \partial \mathbb{R}^n_{-T} \quad (\text{if } T < \infty). \end{cases} \tag{2.9}$$

We claim that $T < \infty$. Indeed, if we assume by contradiction that $T = +\infty$, we have that $\tilde v$ is a harmonic bounded function in $\mathbb{R}^n$. By the Liouville Theorem, this implies that $\tilde v$ is a constant and this is in contradiction with (2.8).

Therefore $T < \infty$ and it follows from Li and Zhu uniqueness result [17] that $T = 0$, hence

$$\tilde v(x) = \left(\frac{1}{(1+x^n)^2 + |x'|^2}\right)^{\frac{n-2}{2}}.$$

So, Lemma 2.6 follows. □

Lemma 2.7. *Let $x_i \to 0$ be an isolated simple blow-up point of $\{v_i\}_i$, where v_i are solutions of $(2.3)_i$, and*

$$|x - x_i|^{\frac{1}{q_i-1}} v_i(x) \le C_7, \quad \forall\, x \in B_2^+,$$

for some positive constant C_7 and

$$\bar w_i'(r) < 0, \quad \forall\, r_i \le r \le 2.$$

Then, for each sequence $R_i \to \infty$, there exists $\delta_i > 0$, $\delta_i = O(R_i^{-1+o(1)})$ such that

$$v_i(x) \le C_8 v_i^{-\lambda_i}(x_i)|x - x_i|^{2-n+\delta_i}, \quad \forall\, r_i \le |x - x_i| \le 1,$$

where $r_i = R_i v_i^{1-q_i}(x_i)$, $\lambda_i = (n-2-\delta_i)(q_i-1)-1$, *and* C_8 *is some positive constant independent of* i .

Proof. For any $x \in \{x \in \mathbb{R}^n : r_i < |x - x_i| < 2\}$, using the Harnack inequality we have that

$$|x-x_i|^{\frac{1}{q_i-1}} v_i(x) \le c\bar{v}_i(|x-x_i|)|x-x_i|^{\frac{1}{q_i-1}}.$$

Since the blow-up is isolated simple, we have that the function at the right hand side is decreasing so that we deduce

$$|x-x_i|^{\frac{1}{q_i-1}} v_i(x) \le c\bar{v}_i(r_i) r_i^{\frac{1}{q_i-1}}$$

for some positive constant c . Since

$$\bar{v}_i(r_i) = \frac{1}{|\Gamma_2(B^+_{r_i})|}\int_{\Gamma_2(B^+_{r_i})} v_i$$

from Lemma 2.6 we deduce that for any $r_i < |x - x_i| < 2$

$$|x-x_i|^{\frac{1}{q_i-1}} v_i(x) \le R_i^{\frac{2-n}{2}+o(1)}$$

which yields

$$v_i^{q_i-1}(x) \le c|x-x_i|^{-1} R_i^{\frac{2-n}{2}(q_i-1)+o(1)} = c|x-x_i|^{-1} R_i^{-1+o(1)}. \tag{2.10}$$

Set $T_i = x_i^n v_i^{q_i-1}(x_i)$. From the proof of Lemma 2.6 we know that $\lim_i T_i = 0$. It is not restrictive to suppose that $x_i = (0, 0, \dots, 0, x_i^n)$. Thus we have that

$$|x_i^n| = o(v_i^{1-q_i}(x_i)) = o(r_i).$$

So

$$B_1^+(0)\setminus B_{2r_i}^+(0) \subset \left\{x\in\mathbb{R}^n : \ \frac{3}{2}r_i \le |x-x_i| \le \frac{3}{2}\right\}.$$

Let us apply the Maximum Principle stated in the appendix (Theorem 6.2) with

$$\begin{aligned}
&\Omega = D_i := B_1^+(0)\setminus B_{2r_i}^+(0), \\
&\Sigma = \Gamma_1(D_i) = \partial D_i \cap \partial\mathbb{R}^n_+, \quad && \Gamma = \Gamma_2(D_i) = \partial D_i \cap \mathbb{R}^n_+, \\
&V \equiv 0, && h = (n-2) f_i^{\tau_i} v_i^{q_i-1}, \\
&\psi = v_i, && v = \varphi_i,
\end{aligned}$$

where

$$\varphi_i(x) = M_i(|x|^{-\delta_i} - \varepsilon_i |x|^{-\delta_i-1}x^n) + A v_i^{-\lambda_i}(x_i)(|x|^{2-n+\delta_i} - \varepsilon_i|x|^{1-n+\delta_i}x^n) - \frac{1}{2}v_i(x)$$

with M_i , A , ε_i , $\delta_i = O(R_i^{-1+o(1)})$ to be suitably chosen and $\lambda_i = (n-2-\delta_i)(q_i-1)-1$.

A straightforward calculation gives

$$\begin{aligned}\Delta\varphi_i(x) = M_i|x|^{-\delta_i-2}[-\delta_i(n-2-\delta_i)+O(\varepsilon_i)]\\ +|x|^{-(n-\delta_i)}Av_i^{-\lambda_i}(x_i)[-\delta_i(n-2-\delta_i)+O(\varepsilon_i)], \quad x\in D_i\end{aligned}$$

and, taking into account (2.10), we have

$$\begin{aligned}B\varphi_i =& M_i|x|^{-\delta_i-1}\left(\varepsilon_i - O\left(R_i^{-1+o(1)}\right)\right)\\ &+Av_i^{-\lambda_i}(x_i)|x|^{-n+1+\delta_i}\left(\varepsilon_i - O\left(R_i^{-1+o(1)}\right)\right), \quad \text{on } \Gamma_1(D_i).\end{aligned}$$

Apparently we can find $0<\delta_i = O\left(R_i^{-1+o(1)}\right)$ and $0<\varepsilon_i = O\left(R_i^{-1+o(1)}\right)$, so that

$$\Delta\varphi_i \le 0, \quad \text{in } D_i, \quad \frac{\partial\varphi_i}{\partial x^n}+(n-2)f_i^{\tau_i}v_i^{q_i-1}\le 0, \quad \text{on } \Gamma_1(D_i).$$

Now we check $\varphi_i \ge 0$ on $\Gamma_2(D_i)$. We have that $\Gamma_2(D_i) = \Gamma_{r_i}\cup\Gamma_2(B_1^+)$ where $\Gamma_{r_i} = \{x\in\mathbb{R}^n_+ : |x| = r_i\}$, $\Gamma_2(B_1^+) = \{x\in\mathbb{R}^n_+ : |x|=1\}$. On Γ_{r_i} we have that

$$v_i(x)\le cv_i(x_i)R_i^{2-n} \tag{2.11}$$

for some positive c. Choose A such that

$$Av_i(x_i)R_i^{2-n+\delta_i} - cv_i(x_i)R_i^{2-n}\ge 0.$$

Then by (2.11) and for ε_i small enough we have that $\varphi_i\ge 0$ on Γ_{r_i} and taking $M_i = \max_{\Gamma_2(B_1^+)} v_i$ we obtain $\varphi_i\ge 0$ on $\Gamma_2(B_1^+)$. Then from Theorem 6.2 we derive that $\varphi_i\ge 0$, and hence

$$\begin{aligned}v_i(x)\le& M_i(|x|^{-\delta_i}-\varepsilon_i|x|^{-\delta_i-1}x^n)\\ &+Av_i^{-\lambda_i}(x_i)(|x|^{2-n+\delta_i}-\varepsilon_i|x|^{1-n+\delta_i}x^n) \quad \forall\, x\in D_i.\end{aligned} \tag{2.12}$$

By the Harnack inequality and by the assumption that the blow-up point is isolated simple, we derive

$$M_i\le c\bar v_i(1)\le c\vartheta^{\frac{1}{q_i-1}}\bar v_i(\vartheta) \quad \forall\,\vartheta\in(r_i,1). \tag{2.13}$$

From (2.12) and (2.13) we have that

$$M_i\le c\left\{\vartheta^{\frac{1}{q_i-1}}\left[M_i\vartheta^{-\delta_i}+Av_i^{-\lambda_i}(x_i)\vartheta^{2-n+\delta_i}\right]\right\}$$

which implies

$$M_i\vartheta^{n-2-\delta_i-\frac{1}{q_i-1}}\left(1-c\vartheta^{\frac{1}{q_i-1}-\delta_i}\right)\le cAv_i^{-\lambda_i}(x_i).$$

Choosing ϑ such that $1 - c\vartheta^{\frac{2}{n-2}} > 1/10$, we obtain that

$$M_i \le c v_i^{-\lambda_i}(x_i) \tag{2.14}$$

for some constant $c > 0$. The conclusion of the Lemma follows from (2.12) and (2.14). $\square$

The following Lemma is a consequence of the Pohozaev identity in the appendix (see Theorem 6.3), Lemma 2.6, Lemma 2.7, and standard elliptic arguments..

Lemma 2.8. $\tau_i = O\left(v_i^{-2}(x_i)\right)$. *In particular* $\lim_i v_i^{\tau_i}(x_i) = 1$.

Lemma 2.9. *Under the same assumptions of Lemma 2.7, we have that for some positive constant* $C_9 > 0$

$$v_i(x_i)v_i(x) \le C_9|x - x_i|^{2-n}, \quad \forall x \in B_3^+, \tag{2.15}$$

and

$$v_i(x_i)v_i \underset{i\to\infty}{\longrightarrow} a|x|^{2-n} + b \quad \textit{in } C^2_{\mathrm{loc}}(B_1^+ \setminus \{0\})$$

where a *is a positive constant and* $b \ge 0$ *satisfies*

$$\begin{cases} -\Delta b = 0, & \textit{in } B_1^+, \\ \dfrac{\partial b}{\partial \nu} = 0, & \textit{on } \Gamma_1(B_1^+). \end{cases}$$

Proof. The inequality in Lemma 2.9 for $|x - x_i| < r_i$ follows immediately from Lemma 2.6 and Lemma 2.8. Let $e \in \mathbb{R}^n$, $e \in \Gamma_2(B_1^+)$, and set

$$\tilde{v}_i(x) = v_i^{-1}(x_i + e)v_i(x).$$

Then $\tilde{v}_i$ satisfies

$$\begin{cases} -\Delta \tilde{v}_i = 0, \quad \tilde{v}_i > 0, & \text{in } B_2^+, \\ \dfrac{\partial \tilde{v}_i}{\partial x^n} = -(n-2)f_i^{\tau_i}v_i^{q_i-1}(x_i+e)\tilde{v}_i^{q_i}, & \text{on } \Gamma_1(B_2^+). \end{cases}$$

Using Lemma 2.3 and some standard elliptic estimates, it follows, after taking a subsequence, that $\tilde{v}_i$ converges in $C^2_{\mathrm{loc}}(\overline{B_2^+ \setminus \{0\}})$ to some positive function $\tilde{v} \in C^2_{\mathrm{loc}}(\overline{B_2^+ \setminus \{0\}})$ satisfying

$$\begin{cases} -\Delta \tilde{v} = 0, & \text{in } B_2^+ \setminus \{0\}, \\ \dfrac{\partial \tilde{v}}{\partial x^n} = 0, & \text{on } \Gamma_1(B_2^+) \setminus \{0\}, \end{cases} \tag{2.16}$$

where we have used Lemma 2.7 to derive the second equation in (2.16). By Schwartz reflection, we obtain a function (still denoted by $\tilde{v}$) in B_2 satisfying

$$\Delta \tilde{v} = 0, \quad \text{in } B_2 \setminus \{0\}.$$

So by Böcher's Theorem, see e.g. [13], it follows that $\tilde v(x) = a_1|x|^{2-n} + b_1$, where $a_1 \ge 0$, $\Delta b_1 = 0$, and $\frac{\partial b_1}{\partial x^n} = 0$ on $\Gamma_1(B_2^+)$. Furthermore $\tilde v$ has to be singular at $x = 0$. Indeed it follows from Lemma 2.3 and some standard elliptic estimates that for $0 < r < 2$,

$$\lim_{i\to\infty} v_i^{-1}(x_i+e) r^{\frac{1}{q_i-1}} \bar v_i(r) = r^{\frac{n-2}{2}} \xi(r)$$

where

$$\xi(r) = \fint_{\Gamma_2(B_r^+)} \tilde v.$$

Therefore, it follows from the definition of isolated simple blow-up point that $r^{\frac{n-2}{2}}\bar\xi(r)$ is decreasing , which is impossible if ξ is regular at the origin. It follows that $a_1 > 0$.

We first establish the inequality in Lemma 2.9 for $|x - x_i| = 1$. Namely, we prove that

$$v_i(x_i+e)v_i(x_i) \le c, \tag{2.17}$$

for some constant $c > 0$. Suppose that (2.17) is not true, then along some subsequence we have

$$\lim_{i\to\infty} v_i(x_i+e)v_i(x_i) = \infty.$$

Multiply $(2.3)_i$ by $v_i^{-1}(x_i+e)$ and integrate by parts over B_1^+ to obtain

$$0 = \int_{B_1^+} (-\Delta v_i) v_i^{-1}(x_i+e) = v_i^{-1}(x_i+e)\int_{\partial B_1^+} \frac{\partial v_i}{\partial \nu}.$$

Hence from the boundary condition in $(2.3)_i$ we have that

$$0 = (n-2)v_i^{-1}(x_i+e)\int_{\Gamma_1(B_1^+)} f_i^{\tau_i} v_i^{q_i} + v_i^{-1}(x_i+e)\int_{\Gamma_2(B_1^+)} \frac{\partial v_i}{\partial \nu}.$$

Then we have

$$\begin{aligned}
&\lim_{i\to\infty}\left((n-2)v_i^{-1}(x_i+e)\int_{\Gamma_1(B_1^+)} f_i^{\tau_i} v_i^{q_i}\right) = -\lim_{i\to\infty}\left(\int_{\Gamma_2(B_1^+)} \frac{\partial \tilde v_i}{\partial \nu}\right)\\
&\quad = -\int_{\Gamma_2(B_1^+)} \frac{\partial \tilde v}{\partial \nu} = (n-2)a_1\int_{\Gamma_2(B_1^+)} |x|^{1-n} + \int_{\Gamma_2(B_1^+)} \frac{\partial b_1}{\partial \nu}\\
&\quad = (n-2)a_1|\Gamma_2(B_1^+)| + \int_{B_1^+}(-\Delta b_1) = (n-2)a_1|S_+^{n-1}| > 0.
\end{aligned} \tag{2.18}$$

On the other hand, in view of Lemma 2.6, Lemma 2.7, and (2.17), it is easy to check that

$$(n-2)v_i^{-1}(x_i+e)\int_{\Gamma_1(B_1^+)} f_i^{\tau_i} v_i^{q_i} = o(1)v_i^{-1}(x_i)v_i^{-1}(x_i+e) \underset{i\to\infty}{\longrightarrow} 0$$

which is in contradiction with (2.18).

So we have established the inequality for $|x-x_i|=1$. To establish the inequality for $r_i \le |x-x_i| \le 3$, it is sufficient to scale the problem to reduce it to the case $|x-x_i|=1$.

It follows from the above that $w_i = v_i(x_i)v_i \to w$ in $C^2_{\mathrm{loc}}(B_1^+ \setminus \{0\})$ where $w(x) = aG(\bar{x}, x) + b$, for some positive constant a and a function $b \ge 0$ satisfying

$$\begin{cases} \Delta b = 0, & \text{in } B_1, \\ \dfrac{\partial b}{\partial \nu} = 0, & \text{on } \Gamma_1(B_1^+). \quad \square \end{cases}$$

Proof of Proposition 2.5. Since M is locally conformally flat and the boundary of M is umbilic, we can find a diffeomorphism $\varphi: B_2^+ \to B_{\bar{r}}(\bar{x})$ and $f \in C^2(\overline{B_2^+})$ some positive function such that $\varphi(0) = \bar{x}$ and $\varphi^* g = f^{\frac{4}{n-2}} g_0$, where g_0 is the flat metric in B_2^+ . Let $v_i = f u_i \circ \varphi$. It follows from the conformal invariance of L_g and B_g that v_i satisfies equation $(2.3)_i$. So the proof of Proposition 2.5 can be easily deduced from Lemma 2.6 and Lemma 2.9. $\square$

Proposition 2.10. *Let (M,g) be a smooth compact n-dimensional locally conformally flat Riemannian manifold with umbilic boundary and $x_i \to \bar{x}$ be an isolated blow-up point of $\{u_i\}_i$, where u_i are solutions of $(2.1)_i$. Then it is necessarily an isolated simple blow-up point.*

Due to the conformal invariance of L_g and B_g , the proof of Proposition 2.10 is reduced to the proof of the following

Proposition 2.11. *Let $x_i \to 0$ be an isolated blow-up point of $\{v_i\}_i$, where v_i are solutions of $(2.3)_i$. Then it is an isolated simple blow-up point.*

Proof. It follows from Lemma 2.6 that

$$\bar{w}_i'(r) < 0 \quad \text{for every} \quad C_2 v_i^{1-q_i}(x_i) \le r \le r_i. \tag{2.19}$$

Suppose that the blow-up is not simple; then there exist some sequences of positive numbers $\tilde{r}_i \to 0$, $\tilde{c}_i \to \infty$, satisfying $\tilde{c}_i v_i^{1-q_i}(x_i) \le \tilde{r}_i$ such that after passing to a subsequence

$$\bar{w}_i'(\tilde{r}_i) \ge 0. \tag{2.20}$$

It follows from (2.19) and (2.20) that $\tilde{r}_i \ge r_i$ and $\bar{w}_i$ has at least one critical point in the interval $[r_i, \tilde{r}_i]$. Let μ_i be the smallest critical point of $\bar{w}_i$ in this interval. It is clear that

$$\tilde{r}_i \ge \mu_i \ge r_i \quad \text{and} \quad \lim_{i \to \infty} \mu_i = 0.$$

Consider now

$$\xi_i(x) = \mu_i^{\frac{1}{q_i - 1}} v_i(\mu_i x + x_i).$$

Set $T_i = x_i^n/\mu_i$ and $T = \lim_i T_i$. Then we have that ξ_i satisfies the following

$$\begin{cases} -\Delta \xi_i = 0, \quad \xi_i > 0, & \text{in } B^{-T_i}_{1/\mu_i}, \\ -\dfrac{\partial \xi_i}{\partial x^n} = (n-2) f_i^{\tau_i} \xi_i^{q_i}, & \text{on } \partial B^{-T_i}_{1/\mu_i} \cap \{x^n = -T_i\}, \\ |x|^{\frac{1}{q_i-1}} \xi_i(x) \le C_{10}, & \text{in } B^{-T_i}_{1/\mu_i}, \\ \lim\limits_{i\to\infty} \xi_i(0) = \infty \quad \text{and } 0 \text{ is a local maximum point of } \xi_i, \\ r^{\frac{1}{q_i-1}} \bar{\xi}_i(r) \qquad \text{has negative derivative in} \quad C_{10}\xi_i(0)^{1-q_i} < r < 1, \\ \dfrac{d}{dr}\left(r^{\frac{1}{q_i-1}} \bar{\xi}_i(r) \right)\Big|_{r=1} = 0. \end{cases} \tag{2.21}$$

It is easy, arguing as we did before (e.g. see the proof of Lemma 2.9), to see that $\{\xi_i\}_i$ is locally bounded and then converges to some function ξ satisfying

$$\begin{cases} -\Delta \xi = 0, \quad \xi > 0, & \text{in } \mathbb{R}^n_{-T}, \\ -\dfrac{\partial \xi}{\partial x^n} = (n-2)\xi^{\frac{n}{n-2}}, & \text{on } \partial \mathbb{R}^n_{-T}. \end{cases}$$

By the Liouville Theorem and the uniqueness result by Li and Zhu [17] of the appendix we deduce that $T = 0$. Since 0 is an isolated simple blow-up point, by Lemma 2.9 we have that

$$\xi_i(0)\xi_i(x) \underset{i\to\infty}{\longrightarrow} a|x|^{2-n} + b = h(x) \quad \text{in} \quad C^2_{\mathrm{loc}}(B_1^+ \setminus \{0\}) \tag{2.22}$$

where $a > 0$ and b is some harmonic function satisfying

$$\begin{cases} -\Delta b = 0, & \text{in } \mathbb{R}^n_+, \\ \dfrac{\partial b}{\partial x^n} = 0, & \text{on } \partial \mathbb{R}^n_+ \setminus \{0\}. \end{cases}$$

By the Maximum Principle we see that $b \ge 0$. Now, reflecting b to be defined on all $\mathbb{R}^n$ and denoting the resulting function by $\tilde{b}$, we deduce from the Liouville Theorem that $\tilde{b}$ is a constant and so b is a constant. Using the last equality in (2.21) and (2.22), we deduce easily that $a = b$. Hence $h(x) = a(|x|^{2-n} + 1)$. Therefore by Corollary 6.4 in the appendix we have that

$$\lim_{r\to 0} \int_{\Gamma_2(B_r^+)} B(r, x, h, \nabla h) < 0 \tag{2.23}$$

where B is given by

$$B(x, r, h, \nabla h) = \frac{n-2}{2}\frac{\partial h}{\partial \nu} h + \frac{1}{2} r \left(\frac{\partial h}{\partial \nu}\right)^2 - \frac{1}{2} r |\nabla_{\tan} h|^2 \tag{2.24}$$

where $\nabla_{\tan} h$ is the tangent component of ∇h. From another part, using Lemma 6.3 in the appendix, Lemma 2.6, and Lemma 2.9, we deduce

$$\int_{\Gamma_2(B_r^+)} B(r, x, \xi_i, \nabla \xi_i) \geq O\left(v_i^{-2}(x_i)\right) \tau_i + O\left(v_i^{-(q_i+1)}(x_i)\right).$$

Multiplying by $\xi_i^2(0)$ we derive that

$$\lim_{r \to 0} \int_{\Gamma_2(B_r^+)} B(r, x, h, \nabla h) \geq 0,$$

which is in contradiction with (2.23). Therefore our Proposition is proved. □

3. Compactness results for manifolds of positive type

We point out that if q stays strictly below the critical exponent $\frac{n}{n-2}$ and strictly above 1, the compactness of solutions of $(\mathcal{P}_q)$ is much easier matter since it follows directly from the nonexistence of positive solutions to the global equation which one arrives at after a rather standard blow-up argument. Namely we prove

Theorem 3.1. *Let (M, g) be a smooth compact n-dimensional Riemannian manifold with boundary. Then for any $\delta_1 > 0$ there exists a constant $C = C(M, g, \delta_1) > 0$ such that for all $u \in \bigcup_{1+\delta_1 \leq q \leq \frac{n}{n-2} - \delta_1} \mathcal{M}_q^+$ we have*

$$\frac{1}{C} \leq u(x) \leq C, \quad \forall\, x \in M; \quad \|u\|_{C^2(M)} \leq C.$$

Proof. Suppose that the Theorem were false. Then, in view of the Harnack inequality (see Lemma 6.1 in the appendix) and standard elliptic estimates, we would find sequences $\{q_i\}_i$ and $\{u_i\}_i \subset \mathcal{M}_{q_i}$ satisfying

$$\lim_{i \to \infty} q_i = q \in \left]1, \frac{n}{n-2}\right[\quad \text{and} \quad \lim_{i \to \infty} \max_M u_i = \infty.$$

Let p_i be the maximum point of u_i; it follows from the Maximum Principle that $p_i \in \partial M$. Let x be a geodesic normal coordinate system in a neighbourhood of p_i given by $\exp_{p_i}^{-1}$. We write $u_i(x)$ for $u_i(\exp_{p_i}(x))$. We rescale x by $y = \lambda_i x$ with $\lambda_i = u_i^{q_i-1}(p_i) \to \infty$ and define

$$\hat{v}_i(y) = \lambda_i^{-\frac{1}{q_i-1}} u_i(\lambda_i^{-1} y).$$

Clearly $\hat{v}_i(0) = 1$ and $0 \leq \hat{v}_i \leq 1$. Let $\delta > 0$ be some small positive number independent of i. We write $g(x) = g_{ab}(x)\, dx^a dx^b$ for $x \in \exp_{p_i}^{-1}(B_\delta(p_i) \cap M)$. Define

$$g^{(i)}(y) = g_{ab}(\lambda_i^{-1} y)\, dy^a dy^b.$$

Then $\hat{v}_i$ satisfies

$$\begin{cases} -L_{g^{(i)}}\hat{v}_i = 0, \quad \hat{v}_i > 0, & \text{in } \lambda_i \exp^{-1}(B_\delta(p_i) \cap M), \\ B_{g^{(i)}}\hat{v}_i = (n-2)\hat{v}_i^{q_i}, & \text{on } \lambda_i \exp^{-1}(B_\delta(p_i) \cap \partial M). \end{cases}$$

Applying L^p-estimates and Schauder estimates, we know that, after passing to a subsequence and a possible rotation of coordinates , $\hat{v}_i$ converges to a limit $\hat{v}$ in C^2-norm on any compact subset of $\{y \in \mathbb{R}^n : \ y^n \geq 0\}$, where

$$\begin{cases} -\Delta \hat{v} = 0, & \text{in } \mathbb{R}^n_+, \\ -\dfrac{\partial \hat{v}}{\partial y^n} = (n-2)\hat{v}^q, & \text{on } \partial\mathbb{R}^n_+, \\ \hat{v}(0) = 1, \quad \hat{v} \geq 0. \end{cases} \tag{3.1}$$

It follows from the Liouville Theorem by Hu [12] that (3.1) has no solution. This is a contradiction, thus we have established Theorem 3.1. $\square$

The compactness of solutions of $(\mathcal{P}_q)$ is much more difficult to establish when allowing q to be close to $\frac{n}{n-2}$, since the corresponding global equation does have solutions. On the other hand, due to the Liouville Theorem and Liouville-type Theorem by Li-Zhu [17] on the half-space $\mathbb{R}^n_+$, we have the following Proposition similar to Lemma 3.1 of [25] and Proposition 1.1 of [10].

Proposition 3.2. *Let (M, g) be a smooth compact n-dimensional Riemannian manifold with boundary. For any $R \geq 1$, $0 < \varepsilon < 1$, there exist positive constants $\delta_0 = \delta_0(M, g, R, \varepsilon)$, $c_0 = c_0(M, g, R, \varepsilon)$, and $c_1 = c_1(M, g, R, \varepsilon)$ such that for all u in*

$$\bigcup_{\frac{n}{n-2} - \delta_0 \leq q \leq \frac{n}{n-2}} \mathcal{M}_q^+$$

with $\max_M u \geq c_0$, there exists $\mathcal{S} = \{p_1, \dots, p_N\} \subset \partial M$ with $N \geq 1$ such that

(i) *each p_i is a local maximum point of u in M and*

$$\overline{B_{\bar{r}_i}(p_i)} \cap \overline{B_{\bar{r}_j}(p_j)} = \emptyset, \quad \text{for } i \neq j,$$

where $\bar{r}_i = Ru^{1-q}(p_i)$ and $\overline{B_{\bar{r}_i}(p_i)}$ denotes the geodesic ball in (M, g) of radius $\bar{r}_i$ and centered at p_i ;

(ii)
$$\left\| u^{-1}(p_i)u(\exp_{p_i}(yu^{1-q}(p_i))) - \left(\frac{1}{(1+x^n)^2 + |x'|^2}\right)^{\frac{n-2}{2}} \right\|_{C^2(B^M_{2R}(0))} < \varepsilon$$

where
$$B^M_{2R}(0) = \{y \in T_{p_i}M : \ |y| \leq 2R, \ u^{1-q}(p_i)y \in \exp^{-1}_{p_i}(B_\delta(p_i))\},$$

$y = (y', y^n) \in \mathbb{R}^n$;

(iii) $d^{\frac{1}{q-1}}(p_j, p_i)u(p_j) \geq c_0$, *for* $j > i$, *while* $d(p, \mathcal{S})^{\frac{1}{q-1}}u(p) \leq c_1$, $\forall\, p \in M$, *where* $d(\cdot, \cdot)$ *denotes the distance function in metric* g .

The proof of Proposition 3.2 will follow from the following Lemma.

Lemma 3.3. *Let* (M, g) *be a smooth compact* n*-dimensional Riemannian manifold. Given* $R \leq 1$ *and* $\varepsilon < 1$, *there exist positive constants* $\delta_0 = \delta_0(M, g, R, \varepsilon)$ *and* $C_0 = C_0(M, g, R, \varepsilon)$ *such that, for any compact* $K \subset M$ *and any* $u \in \bigcup_{\frac{n}{n-2}-\delta_0 \leq q \leq \frac{n}{n-2}} \mathcal{M}_q$ *with* $\max_{p \in \overline{M \setminus K}} d^{\frac{1}{q-1}}(p, K)u(p) \leq C_0$, *we have that there exists* $p_0 \in M \setminus K$ *which is a local maximum point of* u *in* M *such that* $p_0 \in \partial M$ *and*

$$\left\| u^{-1}(p_0)u(\exp_{p_0}(yu^{1-q}(p_0)) - \left(\frac{1}{(1+x^n)^2 + |x'|^2}\right)^{\frac{n-2}{2}} \right\|_{C^2(B^M_{2R}(0))} < \varepsilon$$

where $B^M_{2R}(0)$ *is as in Proposition 3.2,* $d(p, K)$ *denotes the distance of* p *to* K , *with* $d(p, K) = 1$ *if* $K = \emptyset$.

Proof. Suppose the contrary, then there exist compacta $K_i \subset M$, $\frac{n}{n-2} - \frac{1}{i} \leq q_i \leq \frac{n}{n-2}$, and solutions u_i of $(\mathcal{P}_{q_i})$ such that

$$\max_{p \in \overline{M \setminus K_i}} d^{\frac{1}{q_i-1}}(p, K_i)u_i(p) \geq i.$$

It is easy to deduce from the Hopf Lemma that $u_i > 0$ in M . Let $\hat{p}_i \in \overline{M \setminus K_i}$ be such that

$$d^{\frac{1}{q_i-1}}(\hat{p}_i, K_i)u_i(\hat{p}_i) = \max_{p \in \overline{M \setminus K_i}} d^{\frac{1}{q_i-1}}(p, K_i)u_i(p).$$

Let x be a geodesic normal coordinate system in a neighbourhood of $\hat{p}_i$ given by $\exp^{-1}_{\hat{p}_i}$. We write $u_i(x)$ for $u_i(\exp_{\hat{p}_i}(x))$ and denote $\lambda_i = u_i^{q_i-1}(p_i)$. We rescale x by $y = \lambda_i x$ and define $\hat{v}_i(y) = \lambda_i^{-\frac{1}{q_i-1}}u_i(\lambda_i y)$. By standard blow-up arguments and the Liouville Theorem, one can prove that $d(\hat{p}_i, \partial M) \to 0$. Fix some small positive constant $\delta > 0$ independent of i such that $\partial M \cap B_\delta(\hat{p}_i) \neq \emptyset$. We may assume without loss of generality, by taking δ smaller, that $\exp^{-1}_{\hat{p}_i}(\partial M) \cap B_\delta(0)$ has only one connected component, and may arrange to let the closest point on $\exp^{-1}_{\hat{p}_i}(\partial M) \cap B_\delta(0)$ to 0 to be at $(0, \ldots, 0, -t_i)$ and

$$\exp^{-1}_{\hat{p}_i}(\partial M) \cap B_\delta(0) = \partial \mathbb{R}^n_+ \cap B^M_\delta(0)$$

is a graph over $(x^1, \ldots, x^{n-1})$ with horizontal tangent plane at $(0, \ldots, -t_i)$ and uniformly bounded second derivatives. In $\exp^{-1}_{\hat{p}_i}(B_\delta(\hat{p}_i))$ we write $g(x) = g_{ab}(x)\, dx^a dx^b$. Define

$$g^{(i)}(y) = g_{ab}(\lambda_i^{-1} y)\, dy^a dy^b.$$

Then $\hat v_i$ satisfies

$$\begin{cases} -L_{g^{(i)}}\hat v_i = 0, \quad \hat v_i > 0,\\ B_{g^{(i)}}\hat v_i = (n-2)\hat v_i^{q_i}. \end{cases}$$

Note that $\lambda_i d(\hat p_i, K_i)\to\infty$ and, for $|y|\le \frac14\lambda_i d(\hat p_i,K_i)$ with $x=\lambda_i^{-1}y\in \exp_{\hat p_i}^{-1}(B_\delta(\hat p_i))$, we have

$$d(x,K_i)\ge \frac12 d(\hat p_i,K_i),$$

and therefore

$$\left(\frac12 d(\hat p_i,K_i)\right)^{\frac{1}{q_i-1}} u_i(x)\le d(x,K_i)^{\frac{1}{q_i-1}}u_i(x)\le d(\hat p_i,K_i)^{\frac{1}{q_i-1}}u_i(\hat p_i)$$

which implies , for all $|y|\le\frac14\lambda_i d(\hat p_i,K_i)$ with $\lambda_i^{-1}y\in\exp_{\hat p_i}^{-1}(B_\delta(\hat p_i))$, that

$$\hat v_i(y)\le 2^{\frac{1}{q_i-1}}.$$

Standard elliptic theories imply that there exists a subsequence, still denoted by $\hat v_i$, such that, for $T=\lim_i \lambda_i d(\hat p_i,\partial M)\in[0,+\infty]$, $\hat v_i$ converges to a limit $\hat v$ in C^2-norm on any compact set of $\{y=(y^1,\dots,y^n)\in\mathbb{R}^n:\ y^n\ge -T\}$, where $\hat v>0$ satisfies

$$\begin{cases} -\Delta\hat v = 0, & \text{in } \{y^n>-T\},\\ -\dfrac{\partial\hat v}{\partial y^n} = (n-2)\hat v^{\frac{n}{n-2}}, & \text{on } \{y^n=-T\}, \quad \text{if } T<+\infty. \end{cases}$$

It follows from the Liouville Theorem that $T<+\infty$, and, from the Liouville-type Theorem of Li-Zhu [17], that

$$\hat v(x',x^n)=\left(\frac{1}{(1+(x^n-T))^2+|x'-x_0'|^2}\right)^{\frac{n-2}{2}}.$$

Set $\hat y=(\hat y',-T)$. It follows from the explicit form of $\hat v_i$ that there exist $y_i\to\hat y$ which are local maximum points of $\hat v_i$ such that $\hat v_i(y_i)\to\lambda^{\frac{n-2}{2}}=\max\hat v$.

Define $p_i=\exp_{\hat p_i}(\lambda_i^{-1}y_i)$, then $p_i\in M\setminus K_i$ is a local maximum point of u_i, and if we repeat the scaling with p_i replacing $\hat p_i$, we still obtain a new limit v. Due to our choice, $v(0)=1$ is a local maximum, so $T=0$ and

$$\left\| u_i^{-1}(p_i)u_i(\exp_{p_i}(yu_i^{1-q_i}(p_i)) - \left(\frac{1}{(1+x^n)^2+|x'|^2}\right)^{\frac{n-2}{2}}\right\|_{C^2(B_{2R}^M(0))} < \varepsilon$$

which leads to a contradiction. $\square$

Proof of Proposition 3.2. First we apply Lemma 3.3 by taking $K = \emptyset$ and $d(p, K) \equiv 1$ to obtain $p_1 \in \partial M$ which is a maximum point of u and (i) of Lemma 3.3 holds. If

$$\max_{p \in M \setminus K_1} d^{\frac{1}{q-1}}(p, K_1) u(p) \le C_0,$$

where $K_1 = \overline{B_{\bar r_1}(p_1)}$, we stop. Otherwise we apply again Lemma 3.3 to obtain $p_2 \in \partial M$. It is clear that we have $\overline{B_{\bar r_1}(p_1)} \cap \overline{B_{\bar r_2}(p_2)} = \emptyset$ by taking ε small from the beginning. We continue the process. Since there exists $a(n) > 0$ such that $\int_{B_{\bar r_i}(p_i)} u_i^{q_i+1} \ge a(n)$, our process will stop after a finite number of steps. Thus we obtain $\mathcal{S} = \{p_1, \dots, p_N\} \subset \partial M$ as in (ii) and

$$d^{\frac{1}{q-1}}(p, \mathcal{S}) u(p) \le C_0,$$

for any $p \in M \setminus \mathcal{S}$. Clearly, we have that item (iii) holds. □

Though Proposition 3.2 states that u is very well approximated in strong norms by standard bubbles in disjoint balls $B_{\bar r_1}(p_1), \dots, B_{\bar r_N}(p_N)$, it is far from the compactness result we wish to prove. Interactions between all these bubbles have to be analyzed to rule out the possibility of blowing-ups.

The next Proposition rules out possible accumulations of these bubbles, and this implies that only isolated blow-up points may occur to a blowing-up sequence of solutions.

Proposition 3.4. *Let (M, g) be a smooth compact n-dimensional locally conformally flat Riemannian manifold with umbilic boundary. For suitably large R and small $\varepsilon > 0$, there exist $\delta_1 = \delta_1(M, g, R, \varepsilon)$ and $d = d(M, g, R, \varepsilon)$ such that for all u in*

$$\bigcup_{\frac{n}{n-2} - \delta_1 \le q \le \frac{n}{n-2}} \mathcal{M}_q^+$$

with $\max_M u \ge C_0$, we have

$$\min\{d(p_i, p_j) : \ i \ne j, \ 1 \le i, j \le N\} \ge d$$

where $C_0, p_1, \dots, p_N$ are given by Proposition 3.2.

Proof. By contradiction, suppose that the conclusion does not hold, then there exist sequences $\frac{n}{n-2} - \frac{1}{i} \le q_i \le \frac{n}{n-2}$, $u_i \in \mathcal{M}_{q_i}$ such that $\min\{d(p_{i,a}, p_{i,b}), \ 1 \le a, b \le N\} \to 0$ as $i \to +\infty$ where $p_{i,1}, \dots, p_{i,N}$ are the points given by Proposition 3.2. Notice that when we apply Proposition 3.2 to determine these points, we fix some large constant R, and then some small constant $\varepsilon > 0$ (which may depend on R), and in all the arguments i will be large (which may depend on R and ε). Let

$$d_i = d(p_{i,1}, p_{i,2}) = \min_{a \ne b} d(p_{i,a}, p_{i,b})$$

and

$$p_0 = \lim_{i \to +\infty} p_{i,1} = \lim_{i \to +\infty} p_{i,2} \in \partial M.$$

Since M is locally conformally flat with umbilic boundary, one can find a diffeomorphism

$$\Phi: \quad B_2^+ \longrightarrow B_\delta(p_0), \quad \Phi(0) = p_0 \tag{3.2}$$

with $\Phi^\star g = f^{\frac{4}{n-2}} g_0$ where g_0 is the flat metric in B_2^+ and $f \in C^2(\overline{B_2^+})$ is some positive function. It follows from the conformal invariance of L_g and B_g that, for $v_i = f u_i \circ \Phi$,

$$\begin{cases} -\Delta v_i = 0, \quad v_i > 0, \qquad \text{in } B_2^+, \\ \dfrac{\partial v_i}{\partial x^n} = -(n-2) f^{\tau_i} v_i^{q_i}, \quad \text{on } \Gamma_1(B_2^+). \end{cases} \tag{3.3}$$

We can assume without loss of generality that $x_{i,a} = \Phi^{-1}(p_{i,a})$ are local maxima of v_i, so it is easy to see that

$$v_i(x_{i,a}) \longrightarrow +\infty, \tag{3.4}$$

$$d\Big(x, \bigcup_a \{x_{i,a}\}\Big)^{\frac{1}{q_i-1}} v_i(x) \le c_1, \quad \forall\, x \in B_1^+, \tag{3.5}$$

$$0 < \sigma_i := |x_{i,1} - x_{i,2}| \longrightarrow 0,$$

$$\sigma_i^{\frac{1}{q_i-1}} v_i(x_i, x) \ge \frac{R^{\frac{n-2}{2}}}{c_2} \quad \text{for } a = 1, 2, \tag{3.6}$$

where $c_1, c_2 > 0$ are some constants independent of i, ε, R. Without loss of generality, we assume that $x_{i,1} = (0, \dots, x_{i,1}^n)$. Consider

$$w_i(y) = \sigma_i^{\frac{1}{q_i-1}} v_i(x_{i,1} + \sigma_i y)$$

and set, for $x_{i,a} \in \overline{B_1^+}$, $y_{i,a} = \frac{x_{i,a} - x_{i,1}}{\sigma_i}$ and $T_i = \frac{1}{\sigma_i} x_{i,a}^n$. Clearly, w_i satisfies

$$\begin{cases} -\Delta w_i(y) = 0, \quad w_i > 0, \qquad \text{in } \left\{ |y| < \dfrac{1}{\sigma_i},\ y^n > -T_i \right\}, \\ \dfrac{\partial w_i}{\partial y^n} = -(n-2) f^{\tau_i}(x_{i,1} + \sigma_i y) w_i^{q_i}, \quad \text{on } \left\{ |y| < \dfrac{1}{\sigma_i},\ y^n = -T_i \right\}. \end{cases} \tag{3.7}$$

It follows that

$$|y_{i,a} - y_{i,b}| \ge 1, \quad \forall\, a \ne b, \quad y_{i,1} = 0, \quad |y_{i,2}| = 1. \tag{3.8}$$

After passing to a subsequence, we have

$$\bar{y} = \lim_{i \to +\infty} y_{i,2}, \quad |\bar{y}| = 1.$$

It follows easily from (3.4), (3.5), and (3.6) that

$$\begin{cases} w_i(0) \ge c_0' \quad w_i(y_{i,2}) \ge c_0', \\ \text{each } y_{i,a} \text{ is a local maximum point of } w_i, \\ \min\limits_a |y - y_{i,a}|^{\frac{1}{q_i-1}} w_i(y) \le c_1, \\ |y| \le \dfrac{1}{2\sigma_i}, \quad y^n \ge -T_i \end{cases}$$

where $c_0' > 0$ is independent of i. At this point we need the following Lemma which is a direct consequence of Lemma 2.4.

Lemma 3.5. *If along some subsequence both $\{y_{i,a_i}\}$ and $w_i(y_{i,a_i})$ remain bounded, then along the same subsequence*

$$\limsup_{i\to+\infty} \max_{B_{1/4}^{-T_i}(y_{i,a_i})} w_i < \infty,$$

where $B_{1/4}^{-T_i}(y_{i,a_i}) = \{y : \ |y - y_{i,a_i}| < 1/4, \ y^n > -T_i\}$.

Due to Proposition 2.11 and Lemma 3.5, all the points y_{i,a_i} are either regular points of w_i or isolated simple blow-up points. We deduce, using Lemma 2.9, Lemma 3.5, (3.7), and (3.8) that

$$w_i(0) \longrightarrow +\infty, \quad w_i(y_{i,2}) \longrightarrow +\infty.$$

It follows that $\{0\}$, $\{y_{i,2} \to \bar{y}\}$ are both isolated simple blow-up points. Let $\tilde{w}_i = w_i(0)w_i$. It follows from Lemma 2.9 that there exists $\widetilde{\mathcal{S}}_1$ such that $\{0, \bar{y}\} \subset \widetilde{\mathcal{S}}_1 \subset \mathcal{S}$,

$$\min\{|x-y| : \ x, y \in \widetilde{\mathcal{S}}_1, \ x \neq y\} \geq 1,$$

and

$$w_i(0)w_i \underset{i\to\infty}{\longrightarrow} h \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^n_{-T} \setminus \widetilde{\mathcal{S}}_1)$$

where h satisfies

$$\begin{cases} \Delta h = 0, & \text{in } \mathbb{R}^n_{-T} \setminus \widetilde{\mathcal{S}}_1, \\ \dfrac{\partial h}{\partial y^n} = 0, & \text{on } \partial\mathbb{R}^n_{-T} \setminus \widetilde{\mathcal{S}}_1. \end{cases}$$

Making an even extension of h across the hyperplane $\{y^n = -T\}$, we obtain $\tilde{h}$ satisfying $\Delta\tilde{h} = 0$ on $\mathbb{R}^n \setminus \widetilde{\mathcal{S}}_1$. Using Böcher's Theorem, the fact that $\{0, \bar{y}\} \subset \widetilde{\mathcal{S}}_1$, and the Maximum Principle, we obtain some nonnegative function $b(y)$ and some positive constants $a_1, a_2 > 0$ such that

$$\begin{cases} b(y) \geq 0, & y \in \mathbb{R}^n \setminus \{\widetilde{\mathcal{S}}_1 \setminus \{0, \bar{y}\}\}, \\ \Delta b(y) = 0, & y \in \mathbb{R}^n \setminus \{\widetilde{\mathcal{S}}_1 \setminus \{0, \bar{y}\}\}, \\ \dfrac{\partial b}{\partial \nu} = 0, & \text{on } \partial\mathbb{R}^n_+ \setminus \{\widetilde{\mathcal{S}}_1 \setminus \{0, \bar{y}\}\}, \end{cases}$$

and $h(y) = a_1|x|^{2-n} + a_2|x-\bar{y}|^{2-n} + b$, $y \in \mathbb{R}^n \setminus \widetilde{\mathcal{S}}_1$. Therefore there exists $A > 0$ such that

$$h(y) = a_1|y|^{2-n} + A + O(|y|)$$

for y close to zero. Using Lemma 6.3 and Corollary 6.4 in the appendix, we obtain a contradiction as in Proposition 2.11. The proof of our Proposition is thereby complete. □

Proof of Theorem 1.5. Let f_1 be an eigenfunction of problem (E_1) associated to $\lambda_1(L)$. Taking if necessary $|f_1|$, we can assume $f_1 \geq 0$. By the Maximum Principle

$f_1 > 0$ in $\overset{\circ}{M}$ and by the Hopf Maximum Principle $f_1 > 0$ on ∂M. Thus $f_1 > 0$ in M. Consider the metric $g_1 = f_1^{\frac{4}{n-2}} g$. Then $R_{g_1} > 0$ and $h_{g_1} \equiv 0$. We will work with g_1 instead of g. For simplicity of notation, we still denote it as g. Then we can assume $R_g > 0$ and $h_g \equiv 0$ without loss of generality, so that $B_g = \partial/\partial\nu$.

In view of L^p-estimates, Schauder estimates, and Lemma 6.1, we only need to establish the L^∞-bound of u. Arguing by contradiction, suppose there exist sequences $q_i = \frac{n}{n-2} - \tau_i$, $\tau_i \geq 0$, $\tau_i \to 0$, and $u_i \in \mathcal{M}_{q_i}$ such that

$$\max_M u_i \underset{i\to\infty}{\longrightarrow} \infty.$$

It follows from Proposition 2.10, Theorem 3.1, and Proposition 3.4 that, after passing to a subsequence, $\{u_i\}_i$ has N $(1 \leq N < \infty)$ isolated simple blow-up points denoted by $\{p^1, \dots, p^N\}$. Let $\{p_i^1, \dots, p_i^N\}$ denote the local maximum points as in Definition 2.1. It follows from Proposition 2.5 that

$$u_i(p_i^1)u_i \underset{i\to\infty}{\longrightarrow} h \quad \text{in } C^2_{\text{loc}}(M \setminus \{p^1, \dots, p^N\}).$$

Using Proposition 2.5 and subtracting to the function h the contribution of all the poles $\{p^1, \dots, p^N\} \subset \partial M$, we obtain

$$u_i(p_i^1)u_i \underset{i\to\infty}{\longrightarrow} \sum_{\ell=1}^N a_\ell G(\cdot, p^\ell) + \tilde{b} \quad \text{in } C^2_{\text{loc}}(M \setminus \{p^1, \dots, p^N\})$$

where $a_\ell > 0$, $G(\cdot, p^\ell)$ is as in (2.7), and $\tilde{b}$ satisfies

$$\begin{cases} L_g \tilde{b} = 0, & \text{in } M, \\ B_g \tilde{b} = 0, & \text{on } \partial M. \end{cases}$$

Since $\lambda_1(L) > 0$ we deduce that $\tilde{b} = 0$ and $G(\cdot, p^\ell) > 0$ (recall that we have chosen g such that $R_g > 0$ and $h_g \equiv 0$). Since M is compact and locally conformally flat with umbilic boundary, for every p^ℓ there exist $\rho > 0$ uniform and $g_2 = f_2^{\frac{4}{n-2}} g$, for $f_2 \in C^2(\overline{B_{2\rho}(p^\ell)})$, such that g_2 is Euclidean in a neighbourhood of p^ℓ and $h_g = 0$ on $\partial M \cap B_\rho(p^\ell)$. It is standard to see that the Green's function $\widehat{G}(x, p^\ell)$ of g_2 has the following expansion near p^ℓ in geodesic normal coordinates

$$\widehat{G}(x, p^\ell) = |x|^{2-n} + A + O(|x|).$$

It follows then from the Positive Mass Theorem by Schoen and Yau [24] as it was extended to locally conformally flat manifolds with umbilic boundary by Escobar [6] that $A \geq 0$ with equality if and only if (M, g) is conformally equivalent to the standard ball. Let v_i be as in (3.3). Recall that $\Phi(p^1) = 0$, so we can deduce that $x_i \to 0$ is an isolated simple blow-up point of $\{v_i\}_i$ and

$$v_i(x_i)v_i \underset{i\to\infty}{\longrightarrow} \tilde{h} \quad \text{in } C^2_{\text{loc}}(\overline{B_1^+} \setminus \{0\})$$

where $\tilde{h}(x) = |x|^{2-n} + \widetilde{A} + O(|x|)$ for some $\widetilde{A} > 0$. Applying Lemma 6.3 and Corollary 6.4 of the appendix, we reach as usual a contradiction. The Theorem is then proved. $\square$

4. Existence results for manifolds of positive type

In this section we prove the existence part of Theorem 1.4, using the compactness results of the previous section and the Leray-Schauder degree theory.

We assume $R_g > 0$ and $h_g \equiv 0$ without loss of generality (see the beginning of the proof of Theorem 1.5) so that $B_g = \partial/\partial\nu$. For $1 \leq q \leq \frac{n}{n-2}$, consider the problem

$$\begin{cases} L_g u = 0, & \text{in } \overset{\circ}{M}, \\ \dfrac{\partial u}{\partial \nu} = v, & \text{on } \partial M, \end{cases} \tag{$\mathcal{P}_v$}$$

which defines an operator

$$\begin{aligned} T: \quad C^{2,\alpha}(M)^+ &\longrightarrow C^{2,\alpha}(M) \\ v &\longmapsto Tv = u \end{aligned}$$

where $C^{2,\alpha}(M)^+ := \{u \in C^{2,\alpha}(M): \ u > 0 \ \text{ in } \ M\}$, $0 < \alpha < 1$ and Tv is the unique solution of problem $(\mathcal{P}_v)$. Set

$$E(v) := \int_M (-L_g v)v + \int_{\partial M} (B_g v)v = \int_M |\nabla_g v|^2 + \frac{n-2}{4(n-1)} \int_M R_g v^2$$

and consider the problem

$$\begin{cases} -L_g v = 0, \quad v > 0, & \text{in } \overset{\circ}{M}, \\ B_g v = (n-2)E(v)v^q, & \text{on } \partial M. \end{cases} \tag{4.1}$$

We have the following Lemma

Lemma 4.1. *There exists some positive constant $C = C(M,g)$ such that, for all $1 \leq q \leq \frac{n}{n-2}$ and v satisfying* (4.1)*, we have*

$$\frac{1}{C} < v < C, \quad \text{in } M. \tag{4.2}$$

Proof. First of all, notice that, in view of the Harnack inequality and Lemma 6.1, it is enough to prove the upper bound. Multiplying (4.1) by v and integrating by parts, we obtain

$$(n-2)E(v)\int_{\partial M} v^{q+1} = \int_M |\nabla_g v|^2 + \frac{n-2}{4(n-1)} \int_M R_g v^2 \tag{4.3}$$

which yields $E(v) > 0$. It is easy to check that $u = E(v)^{\frac{1}{q-1}} v > 0$ satisfies

$$\begin{cases} -L_g u = 0, \quad u > 0, & \text{in } \overset{\circ}{M}, \\ B_g u = (n-2)u^q, & \text{on } \partial M. \end{cases}$$

It follows from Theorem 3.1 and Proposition 3.4 that there exists $\delta_0 > 0$ such that for $1 + \delta_0 \le q \le \frac{n}{n-2}$

$$\frac{1}{c_1} \le E(v)^{\frac{1}{q-1}} v \le c_1 \tag{4.4}$$

for some positive constant c_1. From (4.3) we know that $(n-2)E(v) \int_{\partial M} v^{q+1} = E(v)$, so that

$$\int_{\partial M} v^{q+1} = \frac{1}{n-2}. \tag{4.5}$$

Next (4.4) and (4.5) yield

$$\frac{1}{c_2} \le E(v) \le c_2 \tag{4.6}$$

for some positive c_2. Then (4.4) and (4.6) give (4.2) for $1 + \delta_0 \le q \le \frac{n}{n-2}$. For $1 \le q \le 1 + \delta_0$ we apply Lemma 6.5 to obtain $E(v) \le c_3$ for a positive constant c_3 and then standard elliptic estimates to obtain the upper bound for v. □

For $0 < \alpha < 1$, $1 \le q \le \frac{n}{n-2}$, we define a map

$$\begin{aligned} F_q : \quad C^{2,\alpha}(M)^+ &\longrightarrow C^{2,\alpha}(M) \\ v &\longmapsto F_q v = v - T(E(v)v^q). \end{aligned}$$

For $\Lambda > 1$, let

$$D_\Lambda = \left\{ v \in C^{2,\alpha}(M),\ \|v\|_{C^{2,\alpha}(M)} < \Lambda,\ \min_M v > \frac{1}{\Lambda} \right\}. \tag{4.7}$$

Let us notice that F_q is a Fredholm operator and $0 \notin F_q(\partial D_\Lambda)$ thanks to Lemma 4.1. Consequently, by the homotopy invariance of the Leray-Schauder degree (see [21] for a comprehensive introduction to Leray-Schauder degree and its properties), we have

$$\deg(F_q, D_\Lambda, 0) = \deg(F_1, D_\Lambda, 0), \quad \forall\, 1 \le q \le \frac{n}{n-2}.$$

It is easy to see that $F_1(v) = 0$ if and only if $E(v) = \lambda_1(B)$ and $v = \sqrt{\lambda_1(B)} f_2$, where f_2 is an eigenfunction of (E_2) associated to $\lambda_1(B)$. Let $\bar{v} = \sqrt{\lambda_1(B)} f_2$.

Lemma 4.2. *$F_1'(\bar{v})$ is invertible with exactly one simple negative eigenvalue. Therefore* $\deg(F_1, D_\Lambda, 0) = -1$.

Proof. This can be proved by quite standard arguments, one can follow, up to minor modifications, the derivation of similar results in [10, pp. 528-529]. We omit the proof. □

For $s \in [0,1]$, let us consider the homotopy

$$\begin{aligned} G_s: \quad C^{2,\alpha}(M)^+ &\longrightarrow C^{2,\alpha}(M) \\ v &\longmapsto G_s(v) = v - T_{\frac{n}{n-2}}\left([(n-2)s + (1-s)E(v)]v^{\frac{n}{n-2}}\right). \end{aligned}$$

Arguing as in Lemma 4.1, one easily deduces

Lemma 4.3. *There exists $\overline{\Lambda} > 2$ depending only on (M,g) such that*

$$G_s(u) \neq 0 \quad \forall\, \Lambda \geq \overline{\Lambda}, \quad \forall\, 0 \leq s \leq 1, \quad \forall\, u \in \partial D_\Lambda.$$

Proof of Theorem 1.4 completed. Using Lemma 4.3 and the homotopy invariance of the Leray-Schauder degree, we have for all $\Lambda \geq \overline{\Lambda}$,

$$\deg(G_1, D_\Lambda, 0) = \deg(G_0, D_\Lambda, 0).$$

Observing that

$$\begin{aligned} G_1(u) &= u - T_{\frac{n}{n-2}}\left((n-2)u^{\frac{n}{n-2}}\right), \\ G_0(u) &= F_{\frac{n}{n-2}}(u) \end{aligned}$$

and using Lemma 4.2, we have that for Λ sufficiently large

$$\deg(G_1, D_\Lambda, 0) = -1,$$

which, in particular, implies that $\mathcal{M} \cap D_\Lambda \neq \emptyset$. We have thus completed the proof of the existence part of Theorem 1.4. $\square$

5. Compactness and existence results for manifolds of negative type

In this section we establish Theorem 1.6. Let f_2 be a positive eigenfunction of (E_2) corresponding to $\lambda_1(B)$ and set $g_2 = f_2^{\frac{4}{n-2}} g$. It follows that $R_{g_2} \equiv 0$ and $h_{g_2} < 0$. We will work throughout this section with g_2 instead of g and we still denote it by g .

We first prove compactness part in Theorem 1.6. Due to the Harnack inequality, Lemma 6.1, elliptic estimates, and Schauder estimates, we need only to establish the L^∞-bound. We use a contradiction argument. Suppose the contrary, that there exist sequences $\{q_i\}_i$, $\{u_i\}_i \in \mathcal{M}^-_{q_i}$ satisfying

$$q_i \underset{i\to\infty}{\longrightarrow} q_0 \in \left]1, \frac{n}{n-2}\right] \quad \text{and} \quad \lim_{i\to\infty} \max_M u_i = +\infty.$$

Let $x_i \in \partial M$ such that $u_i(x_i) = \max_M u_i \to +\infty$. Let $y^1, \ldots, y^n$ be the geodesic normal coordinates given by some exponential map, with $\partial/\partial y^n = -\nu$ at x_i . Consider

$$\tilde{u}_i(z) = u_i^{-1}(x_i) u_i \left(\exp_{x_i}(u_i^{1-q_i}(x_i) z)\right).$$

Reasoning as in Theorem 3.1, we obtain that $\tilde{u}_i$ converges in C^2_{loc}-norm to some $\tilde{u}$ satisfying

$$\begin{cases} -\Delta\tilde{u} = 0, \quad \tilde{u} > 0, \quad \text{in } \mathbb{R}^n_+, \\ \dfrac{\partial\tilde{u}}{\partial z^n} = (n-2)u^{q_0}, \qquad \text{on } \partial\mathbb{R}^n_+, \end{cases} \tag{4.8}$$

with $\tilde{u}(0) = 1$, $0 < \tilde{u} \le 1$ on $\mathbb{R}^n_+$. Using the Liouville-type Theorem of Lou-Zhu [19], we obtain that (4.8) has no solution satisfying $\tilde{u}(0) = 1$ and $0 < \tilde{u} \le 1$.

We prove now the existence part of Theorem 1.6. Let

$$E(u,v) = \int_M \nabla_g u \cdot \nabla_g v + \frac{n-2}{2}\int_{\partial M} h_g uv$$

and $E(u) = E(u,u)$. Let us observe that one can choose f_2 such that $E(f_2) = -1$. Consider for, $1 \le q \le \frac{n}{n-2}$,

$$\begin{cases} \Delta_g v = 0, \quad v > 0, \quad \text{in } \overset{\circ}{M}, \\ B_g v = E(v)v^q, \qquad \text{on } \partial M. \end{cases} \tag{4.9}$$

Arguing as in Lemma 4.1 and using Lemma 6.6 one can prove

Lemma 5.1. *There exists some constant $C = C(M,g) > 0$ such that for $1 \le q \le \frac{n}{n-2}$ and v satisfying (4.9) we have*

$$\frac{1}{C} < v < C.$$

Let $\lambda_1(B) < \lambda_2(B) < \dots$ denote all the eigenvalues of (E_2). Pick some constant $A \in (-\lambda_2(B), -\lambda_1(B))$. For $0 < \alpha < 1$ and $1 \le q \le \frac{n}{n-2}$, we define

$$\widetilde{T}_A : \quad C^{2,\alpha}(M)^+ \longrightarrow C^{2,\alpha}(M),$$

which associates to $v \in C^{2,\alpha}(M)^+$ the unique solution of

$$\begin{cases} L_g u = 0, & \text{in } \overset{\circ}{M}, \\ (B_g + A)u = v, & \text{on } \partial M \end{cases}$$

and $F_q(v) = v - \widetilde{T}_A(E(v)v^q + Av)$. For $\Lambda > 1$, let $D_\Lambda \subset C^{2,\alpha}(M)^+$ be given as in (4.7). It follows from Lemma 5.1 that $0 \notin F_q(\partial D_\Lambda)$, for all $1 \le q \le \frac{n}{n-2}$. Consequently,

$$\deg(F_q, D_\Lambda, 0) = \deg(F_1, D_\Lambda, 0), \quad \forall\, 1 \le q \le \frac{n}{n-2}.$$

Arguing as we did in Lemma 4.2, we obtain

Lemma 5.2. *Suppose $\lambda_1(B) < 0$ and $R_g \equiv 0$. Then*

$$\deg(F_q, D_\Lambda, 0) = -1, \quad \forall\, 1 \le q \le \frac{n}{n-2}.$$

Now we define for $1 \leq q \leq \frac{n}{n-2}$, $\widetilde{T}_q$ as follows

$$\begin{aligned} \widetilde{T}_q : \quad C^{2,\alpha}(M)^+ &\longrightarrow C^{2,\alpha}(M) \\ v &\longmapsto \widetilde{T}_q v = u \end{aligned}$$

where u is the unique solution of

$$\begin{cases} \Delta_g u = 0, & \text{in } \overset{\circ}{M}, \\ (B_g + A)u = -(n-2)v^q + Av & \text{on } \partial M. \end{cases}$$

Since 0 is not an eigenvalue of B_g+A , $\widetilde{T}_q$ is well defined. It follows from Schauder theory, see e.g. [8], that $\widetilde{T}_q$ is compact. It follows from the compactness part of Theorem 1.6 that there exists $\Lambda >> 1$ depending only on (M, g) such that

$$\left\{ u \in C^{2,\alpha}(M) : \ \left(\mathrm{Id} - \widetilde{T}_{\frac{n}{n-2}}\right) u = 0 \right\} \subset D_\Lambda \quad \text{for every } \Lambda > \bar{\Lambda}.$$

Lemma 5.3. *Suppose that $\lambda_1(B) < 0$ and $R_g \equiv 0$. Then for Λ large enough, we have*

$$\deg \left(\mathrm{Id} - T_{\frac{n}{n-2}}, D_\Lambda, 0 \right) = \deg(F_1, D_\Lambda, 0) = -1.$$

Proof. It follows from the homotopy invariance of the Leray-Schauder degree from one part and Lemma 5.2 from another part. The proof being standard, we omit it. ☐

Proof of Theorem 1.6 completed. The existence part follows from Lemma 5.3 and standard degree theory. Thereby the proof of Theorem 1.6 is established. ☐

Appendix

In this appendix, we present some results used in our arguments. First of all we state a Harnack inequality for second-order elliptic equations with Neumann boundary condition. For the proof one can see [10, Lemma A.1].

Lemma 6.1. *Let L be the operator*

$$Lu = \partial_i(a_{ij}(x)\partial_j u + b_i(x)u) + c_i(x)\partial_i u + d(x)u$$

and assume that for some constant $\Lambda > 1$ the coefficients satisfy

$$\Lambda^{-1}|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \qquad \forall x \in B_3^+ \subset \mathbb{R}^n, \qquad \xi \in \mathbb{R}^n, \tag{6.1}$$

$$|b_i(x)| + |c_i(x)| + |d(x)| \leq \Lambda, \qquad \forall\, x \in B_3^+. \tag{6.2}$$

If $|h(x)| \leq \Lambda$ *for any* $x \in B_3^+$ *and* $u \in C^2(B_3^+) \cap C^1(\overline{B_3^+})$ *satisfies*

$$\begin{cases} -Lu = 0, \quad u > 0, & \text{in } B_3^+, \\ a_{nj}(x)\partial_j u = h(x)u, & \text{on } \Gamma_1(B_3^+), \end{cases}$$

then there exists $C = C(n, \Lambda) > 1$ *such that*

$$\max_{B_1^+} u \leq C \min_{B_1^+} u.$$

In the proofs of our results, we also used the following Maximum Principle.

Theorem 6.2. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *and let* $\partial\Omega = \Gamma \cup \Sigma$, $V \in L^\infty(\Omega)$, *and* $h \in L^\infty(\Sigma)$. *Suppose* $\psi \in C^2(\Omega) \cap C^1(\bar{\Omega})$, $\psi > 0$ *in* $\bar{\Omega}$ *satisfies*

$$\begin{cases} \Delta\psi + V\psi \leq 0, & \text{in } \Omega, \\ \dfrac{\partial\psi}{\partial\nu} \geq h\psi, & \text{on } \Sigma, \end{cases}$$

and $v \in C^2(\Omega) \cap C^1(\bar{\Omega})$ *satisfies*

$$\begin{cases} \Delta v + Vv \leq 0, & \text{in } \Omega, \\ \dfrac{\partial v}{\partial\nu} \geq hv, & \text{on } \Sigma, \\ v \geq 0, & \text{on } \Gamma. \end{cases}$$

Then $v \geq 0$ *in* $\bar{\Omega}$.

We now derive a Pohozaev-type identity for our problem; its proof is quite standard (see [14]).

Lemma 6.3. *Let* v *be a* C^2 *-solution of*

$$\begin{cases} -\Delta v = 0, & \text{in } B_r^+, \\ \dfrac{\partial v}{\partial\nu} = c(n)hv^q, & \text{on } \Gamma_1(B_r^+) = \partial B_r^+ \cap \partial\mathbb{R}^n_+, \end{cases} \tag{6.3}$$

where $1 \leq q \leq \frac{n}{n-2}$ *and* $c(n)$ *is constant depending on* n . *Then*

$$c(n)\left(\frac{n-1}{q-1} - \frac{n-2}{2}\right)\int_{\Gamma_1(B_r^+)} hv^{q+1}\,d\sigma + \frac{c(n)}{q+1}\int_{\Gamma_1(B_r^+)}\sum_{i=1}^{n-1} v^{q+1}\frac{\partial h}{\partial x_i}x_i\,d\sigma$$
$$-\frac{c(n)r}{q+1}\int_{\partial\Gamma_1(B_r^+)} v^{q+1}h\,d\sigma' = \int_{\Gamma_2(B_r^+)} B(x, r, v, \nabla v)\,d\sigma$$

where $\Gamma_2(B_r^+) = \partial B_r^+ \cap \mathbb{R}_+^n$ *and*

$$B(x,r,v,\nabla v) = \frac{n-2}{2}\frac{\partial v}{\partial \nu}v + \frac{1}{2}r\left(\frac{\partial v}{\partial \nu}\right)^2 - \frac{1}{2}r|\nabla_{\tan} v|^2$$

where $\nabla_{\tan} v$ *denotes the component of the gradient* ∇v *which is tangent to* $\Gamma_2(B_r^+)$ *.*

An easy consequence of the previous Lemma is the following

Corollary 6.4. *Let* $v(x) = a|x|^{2-n} + b + O(|x|)$ *for* x *close to* 0 *, with* $a > 0$ *and* $b > 0$ *. There holds*

$$\lim_{r\to 0^+} \int_{\Gamma_2(B_r^+)} B(x,v,\nabla v) < 0.$$

In the proof of Lemma 4.1 we used the following result

Lemma 6.5. *Let* (M,g) *be a smooth compact Riemannian manifold of positive type (namely* $\lambda_1(B) > 0$ *). Let* $\varepsilon_0 > 0$ *,* $1 \le q \le \frac{n}{n-2} - \varepsilon_0$ *. Suppose that* u *satisfies*

$$\begin{cases} -L_g u = 0, \quad u > 0, \quad \text{in } M,\\ \dfrac{\partial u}{\partial \nu} = \mu u^q, \qquad\qquad \text{on } \partial M,\\ \displaystyle\int_{\partial M} u^{q+1} = 1. \end{cases} \tag{6.4}$$

Then

$$0 < \mu = \int_M |\nabla_g u|^2 + \frac{n-2}{4(n-1)} R_g u^2 \le C(M,g,\varepsilon_0).$$

Proof. For $1+\varepsilon_0 \le q \le \frac{n}{n-2} - \varepsilon_0$, it follows from Theorem 3.1 that $C^{-1} \le \mu^{\frac{1}{q-1}} u \le C$, which, together with $\int_{\partial M} u^{q+1} = 1$, gives the claimed estimate. So we have to only establish the estimate for $1 \le q \le 1+\varepsilon_0$. We give a proof for $1 \le q \le \frac{n}{n-2} - \varepsilon_0$. We can choose f_1 such that $E(f_1) = 1$ and recall that f_1 satisfies

$$\begin{cases} -L_g f_1 = 0, \quad f_1 > 0, \quad \text{in } M,\\ \dfrac{\partial f_1}{\partial \nu} = \lambda_1(B) f_1, \qquad \text{on } \partial M. \end{cases}$$

Multiply equation (6.4) by f_1 and integrate by parts to obtain

$$\mu \int_{\partial M} u^q f_1 = \lambda_1(B) \int_{\partial M} f_1 u \tag{6.5}$$

which implies $\mu > 0$. Note that, for $q = 1$, $\mu = \lambda_1(B)$. In the following we assume $1 < q < \frac{n}{n-2} - \varepsilon_0$. Since $1/c \le f_1 \le c$ for some positive c , from (6.5) and the Hölder inequality, we deduce that

$$\mu \|u\|_{L^q(\partial M)}^{q-1} \le c. \tag{6.6}$$

From well-known interpolation inequalities, we deduce

$$\|u\|_{L^{q+1}(\partial M)} \leq \|u\|^{\vartheta}_{L^q(\partial M)} \|u\|^{1-\vartheta}_{L^{2(n-1)/(n-2)}(\partial M)}$$

where

$$\vartheta = \frac{q}{q+1} \cdot \frac{n - nq - 2q}{2(n-1) - nq + 2q}.$$

It is easy to check that $0 < \vartheta < 1$, $\vartheta^{-1} \leq c$, and $(1-\vartheta)^{-1} \leq c$.

Testing (6.4) by u, we easily find that

$$\mu = \int_M \left(|\nabla_g u|^2 + \frac{n-2}{4(n-1)} R_g u^2 \right).$$

Therefore, from the Sobolev embedding Theorems, we deduce

$$1 = \|u\|_{L^{q+1}(\partial M)} \leq c \|u\|^{\vartheta}_{L^q(\partial M)} \mu^{\frac{1-\vartheta}{2}} = c \left(\mu \|u\|^{\frac{2\vartheta}{1-\vartheta}}_{L^q(\partial M)} \right)^{\frac{1-\vartheta}{2}}. \tag{6.7}$$

Combining (6.6) and (6.7), we have that

$$\mu^{1 - \frac{(1-\vartheta)(q-1)}{2\vartheta}} \leq c. \tag{6.8}$$

For $1 \leq q \leq \frac{n}{n-2} - \varepsilon_0$, we have that

$$1 - \frac{(1-\vartheta)(q-1)}{2\vartheta} \geq \delta(\varepsilon_0) > 0. \tag{6.9}$$

The thesis follows from (6.8) and (6.9). □

The analogue for the negative case is

Lemma 6.6. *Let (M,g) be a smooth compact Riemannian manifold with $\lambda_1(B) < 0$ and $h_g \equiv 0$. Let $\varepsilon_0 > 0$ and $1 \leq q < \infty$. Suppose that u satisfies* (6.4)*. Then*

$$0 < -\mu = -\int_M \left(|\nabla_g u|^2 + \frac{n-2}{4(n-1)} R_g u^2 \right) \leq -\frac{n-2}{4(n-1)} \int_M R_g u^2 \leq C(M,g).$$

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