A new affine stochastic volatility model with Normal variance-mean mixture

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Abstract

We present a new model where the distribution of innovations is a Normal variance-mean mixture. In the model, the mixing process follows an affine Garch model with Gamma innovations, then we obtain a recursive procedure for the characteristic function of the logprices and we evaluate a European call by inverse Fourier Transform. The model admits the Garch model with Gamma innovations and the Variance-Gamma model as special cases.

1 Introduction

Several empirical studies have documented important departures from the assumption of normality of log-returns. Indeed skewness, kurtosis, serial correlation and time-varying volatilities are observed in financial time series. For this reason different models have been investigated in discrete-time and in continuous time.

In continuous time, the Lévy processes seem to be a natural generalization of the Brownian motion. Indeed the Lévy process exihibits right-continuous sample paths with stationary and independent increments. Moreover, the marginal distribution can be easily identified by characteristic function (see Schoutens (2003)). However, the Lévy processes usually represent an incomplete market and therefore we need to choose an equivalent martingale measure. The standard approach is based on Esscher Transform or the Minimal Entropy Martingale Measure (see Hubalek and Sgarra (2006) for a survey and comparison of these measures).

Another way to capture the departure from normality is based on the concept of random time, introduced in finance by Clark (1973). A new process, namely the subordinated process, can be obtained from a primitive stochastic process by using an independent random time change process, referred to as a subordinator (usually an increasing Lévy process). The distribution of this process is closely related to a mixture distribution. In particular, if we consider the time-changed Brownian motion, the distribution at time one is a Normal variance-mean distribution. Some cases considered in the literature are the Variance-Gamma (see Madan and Seneta (1990)), the normal Inverse Gaussian (see Barndorff-Nielsen (1995)) and the hyperbolic and generalized hyperbolic distributions (see Barndorff-Nielsen (1977)).

As far as discrete time models are considered, the main classes are the stochastic volatility models and Garch-like models.

In stochastic volatility models, the distribution of returns is specified indirectly by the structure of the model, indeed there exists a random variable V such that the conditional distribution of log-returns given V is known (usually normal). This kind of assumption is often made in continuous-time where the volatility also follows a diffusion process. The main drawback of this approach is that the stochastic volatility is an undetectable process and this gives rise to an estimation problem.

Garch-like models explicitly model the conditional variance given the past returns observed. For option pricing, the affine Garch models represent a suitable class, since they yield a closed form formula for option prices based on inverse Fourier transform (see Heston and Nandi (2000) for normal innovations, Christoffersen et Al. (2006) for Inverse Gaussian innovations, Bellini and Mercuri (2007) for Gamma innovations and Mercuri (2008) for Tempered Stable innovations).

In this paper, we present a new discrete-time stochastic volatility model where the increments of log-returns follow a conditional Normal variance-mean mixture. The main feature of this model is that the mixing process, following an affine Garch model with Gamma innovations, allows a recursive procedure of characteristic function to be obtained and accordingly we evaluate the European call option by inverse Fourier Transform (see Heston (1993) and Carr-Madan (1999)). We observe that this model encompasses the affine Garch model proposed by Bellini and Mercuri (2007) as special case.

As in other stochastic volatility models we need to identify an equivalent martingale measure. In this paper we use the Conditional Esscher transform proposed by Buhlmann (1996) and widely applied in Garch-like models with non-normal innovation (see Siu et Al. (2004)). The main advantage of this approach is that the conditional distribution of log-returns is still a Normal variance-mean mixture with gamma mixing density.

In Section 2, we review some classical results of Normal variance-mean distribution and we focus on Variance Gamma distribution. In Section 3, we present our model and, following the approach proposed by Heston and Nandi (2000), we obtain a recursive procedure for characteristic function and we achieve the affine Garch model with Gamma innovations and the Variance Gamma model as special cases. In Section 4, we apply the Conditional Esscher transform introduced and we obtain a closed form formula for option prices by inverse Fourier transform (see Heston 1993). Moreover, we show that, under martingale measure, the model can be rewritten with two different specifications of parameters.

In Section 5, the proposed model is calibrated on 738 daily last prices of European options on S&P500 where the moneyness is between 0.975 and 1.025 and the daily quotations span from 12/23/2008 to 02/17/2009. We calibrate our model using Total Relative Pricing Error and Mean Squared Error. In both cases, we obtain a small pricing error. In the Appendix we study the behavior of a European call option when the time interval is no longer unitary; it shrinks. Consequently, under a suitable choice of parameters, we achieve an analytical formula for characteristic function and we reduce significantly the run time.

2 The Normal variance-mean mixture

In this Section, we review the Normal variance-mean mixture and we focus on the Variance-Gamma distribution. The Normal variance-mean mixture with positively continuous mixing density is a random variable defined as:

$$Y \stackrel{d}{=} \mu_0 + \mu V + \sigma \sqrt{V} Z \tag{1}$$

where $\mu_0, \mu \in R$, $\sigma \in [0, +\infty)$, $Z \sim N(0,1)$ and V is a random variable defined on the positive real line with density function $g(V)$ and independent of Z.

The density function is given by:

$$f(y) = \int_0^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2 v}} \exp\left(-\frac{1}{2}\left(\frac{y-(\mu_0+\mu v)}{\sigma\sqrt{v}}\right)^2\right) g(v)\, dv \tag{2}$$

Moreover, if the random variable V admits moment generating function (m.g.f.), the m.g.f. of Y is obtained by:

$$M_Y(c) = \exp(c\mu_0)\, M_V\left(c\mu + c^2\frac{\sigma^2}{2}\right)$$

For instance, the Variance-Gamma distribution (V.G.), introduced in finance by Madan and Seneta (1990), is obtained by choosing a Gamma distribution as mixing density. Let $V \sim \Gamma\left(\frac{1}{v}, \frac{1}{v}\right)$, the m.g.f. is given by:

$$M_Y(c) = \exp(c\mu_0)\left(\frac{1}{1-\left(c\mu v + c^2 v\frac{\sigma^2}{2}\right)}\right)^{1/v} \tag{3}$$

Figure 1 compares the V.G. distribution for different values of parameters (μ, σ, v)

Insert here Figure 1

Table 1 reports mean, variance, skewness and kurtosis:

Insert here Table 1

From Table 1 and Figure 1, we note that the parameter μ controls the skewness, indeed the V.G. distribution is symmetric when $\mu = 0$, positively skewed when $\mu > 0$ and negatively otherwise. Moreover the kurtosis is a decreasing function of v.

3 The model

The aim of this Section is to present our model. We analyze the probabilistic aspects and we provide a recursive procedure for conditional characteristic function.

Given a filtered complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with $t = 0, 1, \ldots, T$, the stock price process has an exponential form:

$$S_t = S_{t-1} \exp(Y_t)$$

the log-returns are defined as:

$$\begin{cases} Y_t = r + \lambda V_t + \sigma \sqrt{V_t} Z_t \\ V_t | F_{t-1} \sim \Gamma(a h_t, 1) \\ Z_t \sim N(0,1) \\ h_t = \alpha_0 + \alpha_1 V_{t-1} + \beta_1 h_{t-1} \end{cases} \tag{4}$$

where V_t follows an affine Garch model with Gamma innovations. Moreover the conditional distribution is a Normal variance-mean mixture where the mixing density is a Gamma distribution. The first four conditional moments of log-returns are given by:

$$\begin{cases} E(Y_t \mid F_{t-1}) = r + a\lambda h_t \\ E\left(Y_t^2 \middle| F_{t-1}\right) = r^2 + \left(\sigma^2 + 2r\lambda + \lambda^2\right) a h_t + \lambda^2 a^2 h_t^2 \\ E\left(Y_t^3 \middle| F_{t-1}\right) = r^3 + a\left[3r\lambda(r+\lambda) + \left[3\lambda\sigma^2 + \lambda^3(a h_{t-1} + 2)\right](a h_{t-1} + 1)\right] h_{t-1} \\ E\left(Y_t^4 \middle| F_{t-1}\right) = r^4 + \left[4r^3\lambda + \left[\left(6r^2\lambda^2 + 6r^2\sigma^2 + 12r\lambda\sigma^2\right) + \right.\right. \\ \qquad \left.\left. + \left[\left(4r\lambda^3 + 6\lambda^2\sigma^2\right) + \left(\lambda^4 + 3\sigma^4\right)(a h_t + 3)\right](a h_t + 2)\right](a h_t + 1)\right] a h_t \end{cases} \tag{5}$$

The conditional variance is given by

$$Var_{t-1} = \left(\sigma^2 + \lambda^2\right) a h_t \tag{6}$$

therefore by choosing:

$$a = \frac{1}{(\sigma^2 + \lambda^2)}$$

the h_t process can be interpreted as variance dynamics.

In order to derive the conditional distribution of $\log(S_T)$, given the information at time t, we follow the approach proposed in Heston and Nandi (2000). We define the conditional m.g.f. as:

$$\varphi_t(c) := E\left[\exp\left(c\log\left(S_T\right)\right)|F_t\right] \tag{7}$$

and we claim that $\varphi_t(c)$ has following form:

$$\varphi_t(c) = S_t^c \exp\left(A(t:T,c) + B(t:T,c)\, h_{t+1}\right) \tag{8}$$

We assume the equation (8) holds at time $t+1$ and, by the iteration law of conditional expected value, we get the conditional m.g.f at time t:

$$\begin{aligned}
\varphi_t(c) &= E\left[E\left[\varphi_{t+1}(c)\,|F_{t+1}\right]|F_t\right] = \\
&= E\left[\exp\left(c\log\left(S_{t+1}\right) + A(t+1:T,c) + B(t+1:T,c)\, h_{t+2}\right)|F_t\right] = \\
&= S_t^c \exp\left(cr + A(t+1:T,c) + \alpha_0 B(t+1:T,c) + \beta_1 B(t+1:T,c)\, h_{t+1}\right) * \\
&* E\left[\exp\left[\left(c\lambda + \alpha_1 B(t+1:T,c)\right) V_{t+1} + c\sigma\sqrt{V_{t+1}} Z_{t+1}\right]|F_t\right].
\end{aligned}$$

by applying the m.g.f. of Normal variance-mean mixture with Gamma mixing density, we get:

$$\begin{aligned}
\varphi_t(c) &= S_t^c \exp\left[cr + A(t+1:T,c) + \alpha_0 B(t+1:T,c) + \beta_1 B(t+1:T,c)\, h_{t+1}\right] * \\
&* \left[1 - \left(c\lambda + \alpha_1 B(t+1:T,c) + \frac{c^2\sigma^2}{2}\right)\right]^{-ah_{t+1}}
\end{aligned}$$

where $A(t:T,c)$ and $B(t:T,c)$ follow the system below:

$$\begin{cases} A(t:T,c) = cr + A(t+1:T,c) + \alpha_0 B(t+1:T,c) \\ B(t:T,c) = \left[\beta_1 B(t+1:T,c) - a\log\left[1 - \left(c\lambda + \alpha_1 B(t+1:T,c) + \frac{c^2\sigma^2}{2}\right)\right]\right] \end{cases} \tag{9}$$

with terminal conditions:

$$\begin{cases} A(T:T,c) = 0 \\ B(T:T,c) = 0 \end{cases}$$

Since $\varphi_t(c)$ is the conditional m.g.f., $\varphi_t(ic)$ is the characteristic function of logarithm of spot price. Therefore, the conditional distribution is achieved by inverse Fourier transform.

Figure 2 shows the conditional distribution of log-returns for different values of parameters.

Insert here Figure 2

As shown in Figure 2, we see that, as the V. G. distribution, the λ parameter influences the asymmetry. Since we have a right-tailed distribution when $\lambda > 0$, a left-tailed distribution when $\lambda < 0$ and a symmetric distribution when $\lambda = 0$. Moreover the kurtosis is a decreasing function of the a parameter.

As special cases, we obtain the affine Garch Gamma model when $\sigma = 0$ and the discrete time version of V.G. model, when $\alpha_0 = 0, \alpha_1 = 0, \beta_1 = 1$. Indeed, in the previous case the recursive system is:

$$\begin{cases} A(t:T,c) = cr + A(t+1:T,c) \\ B\left(t:T,c\right) = \left[B\left(t+1:T,c\right) - a\log\left[1-\left(c\lambda+\frac{c^2\sigma^2}{2}\right)\right]\right] \end{cases}$$

Given the terminal condition, we obtain the explicit solution for coefficients:

$$\begin{cases} A(t:T,c) = c\left(T-t\right)r \\ B\left(t:T,c\right) = -2a(T-t)\log\left[1-\left(c\lambda+\frac{c^2\sigma^2}{2}\right)\right] \end{cases}$$

and the conditional m.g.f. is given by

$$\varphi_t\left(c\right) = S_t^c \exp\left(c\left(T-t\right)r\right)\left[1-\left(c\lambda+\frac{c^2\sigma^2}{2}\right)\right]^{-2a\bar{h}(T-t)} \tag{10}$$

Remark 1 *A similar model can be constructed by using the Inverse Gaussian Garch model instead of the Gamma Garch model. In this case, the model is a generalization of the CHJ model (see Christoffersen et Al. (2004)) and the Normal Inverse Gaussian model (See Barndorff-Nielsen (1995)). Alternatively we can use the Tempered Stable Garch model (See Mercuri (2008)).*

4 Option pricing formula

In this Section we analyze the option pricing issue. Our model represents an incomplete market. Indeed we have an infinity of states of nature and so it is not possible to replicate each pay-off using only two assets. Therefore we have the classic problem of choosing an equivalent martingale measure.

In our model, given the simplicity of the conditional moment generating function, a natural choice seems to be the Conditional Esscher transform proposed by Buhlmann et Al. (1996) and applied in Garch framework (see Siu et Al. (2004)).

Following the same notation of Siu et Al. (2004), the conditional m.g.f. of log-returns can be written as:

$$\begin{aligned} M_{Y_k|F_{k-1}}\left(c\right) &= E\left[\exp\left(cY_k\right)|\,F_{k-1}\right] = \\ &= \exp\left(cr\right)\left(\frac{1}{1-\left(c\lambda+c^2\frac{\sigma^2}{2}\right)}\right)^{ah_k} \end{aligned}$$

The Esscher equation and the θ Esscher parameter are given by:

$$\exp(r) = \frac{\exp((\theta+1)r)\left(\frac{1}{1-\left((\theta+1)\lambda+(\theta+1)^2\frac{\sigma^2}{2}\right)}\right)^{ah_k}}{\exp(\theta r)\left(\frac{1}{1-\left(\theta\lambda+\theta^2\frac{\sigma^2}{2}\right)}\right)^{ah_k}} \tag{11}$$

and θ^* admits an explicit solution

$$\theta^* = -\left(\frac{\lambda}{\sigma^2}+\frac{1}{2}\right)$$

The conditional moment generating function, under martingale measure, is obtained as:

$$\begin{aligned} M^Q_{Y_k|F_{k-1}}(c) &= \frac{M_{Y_k|F_{k-1}}(c+\theta^*)}{M_{Y_k|F_{k-1}}(\theta^*)} = \\ &= \exp(cr)\left(\frac{1}{1-\left(c\frac{\sigma^2}{\frac{1}{4}\sigma^2-2}+c^2\frac{\sigma^2}{2-\frac{1}{4}\sigma^2}\right)}\right)^{ah_k} \end{aligned} \tag{12}$$

The innovations are distributed as a Normal variance-mean mixture with Gamma mixing density.

The model can be rewritten by choosing following conditions:

$$\lambda_Q = \frac{\sigma^2}{\frac{1}{4}\sigma^2-2} \tag{13}$$

$$\frac{(\sigma_Q)^2}{2} = \frac{\sigma^2}{2-\frac{1}{4}\sigma^2} \tag{14}$$

and then

$$\lambda_Q = -\frac{(\sigma_Q)^2}{2} \tag{15}$$

In order to ensure the positivity of σ_Q, we need

$$2-\frac{1}{4}\sigma^2 > 0$$

accordingly we have a restriction of σ under real measure, moreover from the condition (15) and (5) , the conditional distribution is again a Normal variance-mean mixture with Gamma mixing density shifted from the right to the left if $\lambda > 0$.

Under the martingale measure Q, the log-returns dynamics are shown in the following

$$\begin{cases} S_t = S_{t-1}\exp\left(Y_t^Q\right) \\ Y_t^Q = r - \frac{\sigma_Q^2}{2}V_t + \sigma_Q\sqrt{V_t}Z_t \\ V_t|\,F_{t-1} \sim \Gamma(ah_t, 1) \\ h_t = \alpha_0 + \alpha_1 V_{t-1} + \beta_1 h_{t-1} \end{cases} \tag{16}$$

To identify the model, we need five parameters a, α_0, α_1, β_1 and σ^Q of which the first four are the same for real measure and then they can be estimated from the underlying asset. Alternatively, defining the process h_t^Q as:

$$h_t^Q := Var_{t-1}^Q\left(Y_t\right) = a\left[\sigma_Q^2 + \lambda_Q^2\right] h_t$$

and letting

$$\begin{cases} \alpha_0^Q = a\left[\sigma_Q^2 + \lambda_Q^2\right]\alpha_0, & \alpha_1^Q = a\left[\sigma_Q^2 + \lambda_Q^2\right]\alpha_1 \\ \beta_1^Q = \beta_1 & a^Q = \frac{1}{\left[\sigma_Q^2 + \lambda_Q^2\right]} \end{cases}$$

the model becomes:

$$\begin{cases} S_t = S_{t-1}\exp\left(Y_t^Q\right) \\ Y_t^Q = r - \frac{\sigma_Q^2}{2}V_t^Q + \sigma^Q\sqrt{V_t^Q}Z_t \\ V_t^Q\Big| F_{t-1} \sim \Gamma\left(a^Q h_t^Q, 1\right) \\ h_t^Q = \alpha_0^Q + \alpha_1^Q V_{t-1} + \beta_1^Q h_{t-1}^Q \end{cases} \tag{17}$$

In this way, the model is identified by four parameters $\left(\sigma_Q, \alpha_0^Q, \alpha_1^Q, \beta_1^Q\right)$ but only β_1^Q remains the same under real measure.

In order to check the accuracy of the procedure for both systems, we compare the option prices obtained by the inverse Fourier Transform (FT.) with the Monte Carlo simulation (MC.) in Tables 2, 3 and 4:

Insert here Tables 2,3,4

Figure 3 compares the density of log-price under real measure with corresponding density under equivalent martingale measure obtained by Conditional Esscher transform:

Insert here Figure 3

5 Calibration

The aim of this Section is to investigate the ability of the model to describe market option prices. The classic approach, based on minimization between prices predicted by model and market option data, presents several difficulties. Indeed, the option pricing formula is not a linear function of parameters, it is possible to arrive a local minimum and the error surface is not smooth. A range of objective functions have been proposed in financial literature, we perform all calibrations using two objective functions, the root of Relative Mean Squared Error $\left(\sqrt{RMSE}\right)$ defined as:

$$\sqrt{RMSE} = \sqrt{\sum_i \left(\frac{C_i^{theo} - C_i^{mkt}}{C_i^{mkt}} \right)^2}$$

and the root of Mean Squared Error ($\sqrt{MSE}$) as:

$$\sqrt{MSE} = \min \sqrt{\sum_i \omega_i \left(C_i^{theo} - C_i^{mkt} \right)^2}$$

where the weights ω_i are usually chosen in order to assign more relevance to at-the money options, whose bid-ask spreads are typically smaller. Possible choices consist in choosing the reciprocal of the bid-ask spreads or even the Black-Scholes Vega. In calibration exercises, we consider only option prices in which the moneyness is between 0.975 and 1.025; in this way, we can omit the weights.

In order to limit the number of parameters, we consider only the system (17) and we estimate the parameters $\left(\sigma^Q, \alpha_0^Q, \alpha_1^Q, \beta\right)$ from data. We consider a dataset composed of 738 daily last prices of European options on S&P500 where the daily quotations range from 12/23/2008 to 02/17/2009. We perform all minimization by the Newton-Raphson algorithm and Tables 5 and 6 report the results for each quotation day and for both calibrations:

Insert here Tables 5, and 6.

In both cases, a small pricing error results but the procedure based on $\sqrt{MSE}$ seems to give more stable parameters over a period of time.

It is also well known that the error pricing surface, usually, is not a convex function and can be irregular with respect to parameters. For these reasons, in Figures 4 and 5, we conduct an analysis of the sections of the error pricing surface at 29st quotation day.

Insert here Figures 4 and 5

In general, the surfaces appear to be sufficiently smooth and, although the estimates obtained can reach a local minimum, the Newton-Raphson algorithm seems to return acceptable values of parameters.

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6 Appendix

In this Appendix we investigate the behavior of a European call option when the length of time interval Δ is no longer unitary but shrinks. For this reason, we analyze the behavior of conditional m.g.f of the log-price and we provide two closed form formula for m.g.f under suitable choices of parameters. Under martingale measure, the price dynamics are defined as:

$$\begin{cases} S_t = S_{t-\Delta}\exp\left(Y_t\right) \\ Y_t = r\Delta - \dfrac{\sigma_q^2}{2}V_t + \sigma_q\sqrt{V_t}Z_t \\ V_t|\, F_{t-\Delta} \sim \Gamma\left(ah_t, 1\right) \\ h_t = \alpha_0 + \alpha_1 V_{t-\Delta} + \beta_1 h_{t-\Delta} \end{cases} \tag{18}$$

As shown in Section 2, the conditional m.g.f. is log-linear w.r.t. variance and the time-dependent coefficients are obtained by solving the system below:

$$\begin{cases} A(t:T,c) = cr\Delta + A(t+\Delta:T,c) + \alpha_0 B\left(t+\Delta:T,c\right) \\ B\left(t:T,c\right) = \left[\beta_1 B\left(t+\Delta:T,c\right) - a\log\left[1-\left(\alpha_1 B\left(t+\Delta:T,c\right) + \left(c^2-c\right)\frac{\sigma_q^2}{2}\right)\right]\right] \end{cases} \tag{19}$$

with final conditions

$$\begin{cases} A(T:T,c)=0 \\ B(t:T,c)=0 \end{cases}$$

To analyze the behavior of system (19), firstly we consider the incremental ratios of coefficients $A(t:T,c), B(t:T,c)$

$$\frac{A(t+\Delta:T,c)-A(t:T,c)}{\Delta} = \frac{-cr\Delta-\alpha_0'\Delta B(t+\Delta:T,c)}{\Delta}$$

$$\frac{B(t+\Delta:T,c)-B(t:T,c)}{\Delta} = \frac{1-\beta_1}{\Delta}B(t+\Delta:T,c) + \frac{a\log\left[1-\left(\alpha_1 B(t+\Delta:T,c)+\left(c^2-c\right)\frac{\sigma_v^2}{2}\right)\right]}{\Delta}$$

Unfortunately when the time interval Δ shrinks, the result is not unique but it depends on the behavior of parameters, indeed by defining:

$$\begin{cases} \alpha_0 := \alpha_0'\Delta, \quad \alpha_1 := \alpha_1'\Delta \\ \sigma_v^2 := \sigma^2\Delta, \quad \beta_1 := 1-\beta_1^{'}\Delta \\ a := a \end{cases} \tag{20}$$

and as limit, we get the following differential equations:

$$\begin{cases} \dfrac{\partial B(t:T,c)}{\partial t} = CB(t:T,c) - D \\ \dfrac{\partial A(t:T,c)}{\partial t} = -cr - \alpha_0' B(t:T,c) \end{cases} \tag{21}$$

where

$$D := a\left(c^2-c\right)\frac{\sigma^2}{2}$$
$$C := \left(\beta_1' - a\alpha_1'\right)$$

given final conditions, we get an analytical solutions of coefficients:

$$\begin{cases} B(t:T,c) = D\frac{[1-\exp(-C(T-t))]}{C} \\ A(t:T,c) = cr(T-t) + \alpha_0'\dfrac{D}{C}(T-t) - \dfrac{\alpha_0'}{C}B(t:T,c) \end{cases} \tag{22}$$

Alternatively, we obtain another limit price by using

$$\begin{cases} \alpha_0 := \alpha_0'\Delta, \quad \alpha_1 := \alpha_1'\Delta^2 + \alpha_1''\Delta \\ \sigma_v^2 := \sigma^2\Delta^2, \quad \beta_1 := 1-\beta_1^{'}\Delta - a'\alpha_1'' \\ a := \dfrac{a'}{\Delta} \end{cases} \tag{23}$$

following the same approach, we obtain the system below:

$$
\begin{cases}
\dfrac{\partial B\left(t:T,c\right)}{\partial t}=-D+CB\left(t:T,c\right)-E\left[B\left(t:T,c\right)\right]^{2}\\
\dfrac{\partial A\left(t:T,c\right)}{\partial t}=-cr-\alpha_{0}^{\prime}B\left(t:T,c\right)\\
D:=a^{\prime}\left(c^{2}-c\right)\dfrac{\sigma^{2}}{2}\\
C:=\left(\beta_{1}^{\prime}-a^{\prime}\alpha_{1}^{\prime}\right)\\
E:=\left(\alpha_{1}^{\prime\prime}\right)^{2}\dfrac{a^{\prime}}{2}
\end{cases}
\tag{24}
$$

with final conditions

$$
\begin{cases}
B\left(T:T,c\right)=0\\
A\left(T:T,c\right)=0
\end{cases}
$$

The system (24) is more general. Indeed, as expected, we recover the system (21) choosing $\alpha_1^{\prime\prime}=0$, moreover a similar system is achieved by Heston[1] (1993). The system (24) allows an explicit solution given by:

$$
\begin{cases}
d_{1}=\sqrt{C^{2}-4DE}\\
f_{\pm}=\frac{C\pm d_{1}}{2E}\\
g:=\frac{f_{-}}{f_{+}}\\
B\left(t:T,c\right)=f_{-}\frac{1-\exp\left(-d_{1}\left(T-t\right)\right)}{1-g\exp\left(-d_{1}\left(T-t\right)\right)}\\
A\left(t:T,c\right)=cr\left(T-t\right)+\alpha_{0}^{\prime}\left\{f_{-}*T-\frac{1}{E}\log\left(\frac{1-g\exp\left(-d_{1}\left(T-t\right)\right)}{1-g}\right)\right\}
\end{cases}
\tag{25}
$$

In order to check our result, Tables 7, 8 and 9 provide a comparison between the recursive procedure analyzed in discrete time and option prices obtained by system (22), in Tables 10, 11 and 12 we consider the system (25). We report the run time in number of seconds per option and we measure the error between the two procedures by Mean Relative Absolute error (MRAE) defined as:

$$
MRAE=\frac{1}{N}\sum\nolimits_{i=1}^{N}\left|\frac{Call_{i}^{\Delta}-Call_{i}^{\lim}}{0.5\left(Call_{i}^{\Delta}+Call_{i}^{\lim}\right)}\right|
$$

Insert here Tables 7, 8 and 9

Insert here Tables 10, 11 and 12

[1] We obtain exactly Heston's formula modifying the dynamics of log-returns as follow:

$$
Y_{t}=r\Delta+\lambda_{0}h_{t}+\lambda_{1}V_{t}+\sigma_{v}\sqrt{V_{t}}Z_{t}
$$

and posing:

$$
\begin{cases}
\alpha_{0}:=\alpha_{0}^{\prime}\Delta,\quad \alpha_{1}:=\alpha_{1}^{\prime}\Delta^{2}+\alpha_{1}^{\prime\prime}\Delta\\
\sigma_{v}^{2}:=\sigma^{2}\Delta^{2},\quad \beta_{1}:=1-\beta_{1}^{\prime}\Delta-a^{\prime}\alpha_{1}^{\prime\prime}\\
a:=\dfrac{a^{\prime}}{\Delta},\quad \lambda_{1}:=-\lambda_{0}^{\prime}\Delta-\dfrac{\mu}{2}\Delta^{2}\\
\lambda_{0}=\lambda_{0}^{\prime}\Delta
\end{cases}
$$

Tab.1. Moments of the Variance-Gamma distribution

	$VG(\mu,\sigma,v)$
mean	$\mu_0 + \mu v$
variance	$\sigma^2 + v\mu^2$
skewness	$\frac{\mu v\left(3\sigma^2 + 2v\mu^2\right)}{\left(\sigma^2 + v\mu^2\right)^{3/2}}$
kurtosis	$3\left(1 + 2v - \frac{v\sigma^4}{\left(\sigma^2 + v\mu^2\right)^2}\right)$

Tab.2. Comparison between the Monte Carlo and semi-analytical formula

T= 15, N = 10000

K	MC	FT	K	MC	FT
0.9	0.1356	0.1336	0.9	0.1365	0.1336
0.95	0.1040	0.1013	0.95	0.1045	0.1013
0.975	0.0903	0.0874	0.975	0.0907	0.0874
1	0.0782	0.0750	1	0.0786	0.0750
1.025	0.0674	0.0642	1.025	0.0679	0.0642
1.05	0.0580	0.0549	1.05	0.0587	0.0549
1.1	0.0430	0.0401	1.1	0.0437	0.0401

The Table shows a comparison between Monte Carlo simulation and semi-analytical formula for European call options. The left side reports option prices obtained by choosing λ_Q = -0.0050, $\sigma_Q = 0.1001$, $a = 3$, $\alpha_0 = 0.05$, $\alpha_1 = 0.12$, $\beta_1 = 0.08$, $h_0 = 0.15$, or equivalently λ_Q = -0.0050, $\sigma_Q = 0.1001$, $a_Q = 99.6256$, $\alpha_0^Q = 0.0015$, $\alpha_1^Q = 0.0036$, $\beta_1^Q = 0.0800$ $h_0^Q = 0.0045$ on the right-hand side.

Tab.3. Comparison between the Monte Carlo and semi-analytical formula

T= 30, N = 10000

K	MC	FT
0.9	0.1651	0.1632
0.95	0.1366	0.1350
0.975	0.1240	0.1224
1	0.1123	0.1108
1.025	0.1015	0.1002
1.05	0.0916	0.0905
1.1	0.0745	0.0736

K	MC	FT
0.9	0.1631	0.1632
0.95	0.1351	0.1350
0.975	0.1225	0.1224
1	0.1110	0.1108
1.025	0.1004	0.1002
1.05	0.0907	0.0905
1.1	0.0740	0.0736

The Table shows a comparison between Monte Carlo simulation and semi-analytical formula for European call options. The left side reports option prices obtained by choosing λ_Q = -0.0050, $\sigma_Q = 0.1001$, $a = 3$, $\alpha_0 = 0.05$, $\alpha_1 = 0.12$, $\beta_1 = 0.08$, $h_0 = 0.15$, or equivalently λ_Q = -0.0050, $\sigma_Q = 0.1001$, $a_Q = 99.6256$, $\alpha_0^Q = 0.0015$, $\alpha_1^Q = 0.0036$, $\beta_1^Q = 0.0800$ $h_0^Q = 0.0045$ on the right-hand side.

Tab.4. Comparison between the Monte Carlo and semi-analytical formula

T= 45, N = 10000

K	MC	FT
0.9	0.1847	0.1856
0.95	0.1584	0.1593
0.975	0.1464	0.1473
1	0.1352	0.1361
1.025	0.1247	0.1257
1.05	0.1150	0.1160
1.1	0.0978	0.0987

K	MC	FT
0.9	0.1869	0.1856
0.95	0.1607	0.1593
0.975	0.1487	0.1473
1	0.1375	0.1361
1.025	0.1271	0.1257
1.05	0.1174	0.1160
1.1	0.1000	0.0987

The Table shows a comparison between the Monte Carlo simulation and semi-analytical formula for European call options. The left side reports option prices obtained by choosing λ_Q = -0.0050, $\sigma_Q = 0.1001$, $a = 3$, $\alpha_0 = 0.05$, $\alpha_1 = 0.12$, $\beta_1 = 0.08$, $h_0 = 0.15$, or equivalently λ_Q = -0.0050, $\sigma_Q = 0.1001$, $a_Q = 99.6256$, $\alpha_0^Q = 0.0015$, $\alpha_1^Q = 0.0036$, $\beta_1^Q = 0.0800$ $h_0^Q = 0.0045$ on the right-hand side.

Tab.5. Parameters from $\sqrt{MSE}$

t	σ_Q	α_0^Q	α_1^Q	β	$\frac{1}{N}\sqrt{MSE}$
1	$2.08*10^{-2}$	$2.06*10^{-4}$	$3.55*10^{-5}$	0.48	0.94
2	$2.07*10^{-2}$	$2.07*10^{-4}$	$2.73*10^{-5}$	0.44	0.99
3	$2.20*10^{-2}$	$2.20*10^{-4}$	$1.92*10^{-5}$	0.43	0.99
4	$1.96*10^{-2}$	$1.95*10^{-4}$	$2.93*10^{-5}$	0.46	1.16
5	$2.29*10^{-2}$	$2.21*10^{-4}$	$1.79*10^{-6}$	0.40	1.23
6	$1.99*10^{-2}$	$1.77*10^{-4}$	$1.88*10^{-6}$	0.38	1.43
7	$1.82*10^{-2}$	$1.71*10^{-4}$	$1.05*10^{-5}$	0.39	2.23
8	$1.81*10^{-2}$	$1.70*10^{-4}$	$1.31*10^{-4}$	0.40	2.21
9	$4.27*10^{-2}$	$5.53*10^{-5}$	$2.58*10^{-5}$	0.26	1.10
10	$4.18*10^{-2}$	$3.32*10^{-5}$	$2.32*10^{-5}$	0.26	0.57
11	$2.13*10^{-2}$	$2.11*10^{-4}$	$7.66*10^{-5}$	0.19	2.08
12	$2.17*10^{-2}$	$2.35*10^{-4}$	$8.22*10^{-5}$	0.19	2.06
13	$1.61*10^{-2}$	$2.17*10^{-4}$	$6.93*10^{-5}$	0.25	1.74
14	$1.90*10^{-2}$	$3.05*10^{-4}$	$7.64*10^{-5}$	0.20	2.47
15	$1.61*10^{-2}$	$2.16*10^{-4}$	$6.61*10^{-5}$	0.23	2.12
16	$1.56*10^{-2}$	$2.02*10^{-4}$	$6.47*10^{-5}$	0.24	0.73
17	$1.60*10^{-2}$	$2.11*10^{-4}$	$6.94*10^{-5}$	0.24	0.71
18	$2.22*10^{-2}$	$3.80*10^{-4}$	$7.64*10^{-5}$	0.16	0.63
19	$1.82*10^{-2}$	$2.71*10^{-4}$	$5.15*10^{-5}$	0.15	0.85
20	$1.97*10^{-2}$	$3.17*10^{-4}$	$5.96*10^{-5}$	0.15	0.81
21	$1.78*10^{-2}$	$2.59*10^{-4}$	$6.16*10^{-5}$	0.17	0.68
22	$1.83*10^{-2}$	$2.72*10^{-4}$	$6.87*10^{-5}$	0.18	0.65
23	$1.67*10^{-2}$	$2.25*10^{-4}$	$5.81*10^{-5}$	0.18	0.49
24	$1.39*10^{-2}$	$1.61*10^{-4}$	$3.40*10^{-5}$	0.20	0.70
25	$1.71*10^{-2}$	$2.44*10^{-4}$	$6.21*10^{-5}$	0.23	1.35
26	$1.82*10^{-2}$	$2.79*10^{-4}$	$7.41*10^{-5}$	0.24	1.67
27	$1.37*10^{-2}$	$1.49*10^{-4}$	$6.47*10^{-5}$	0.34	1.71
28	$1.32*10^{-2}$	$1.29*10^{-4}$	$5.63*10^{-5}$	0.28	0.93
29	$1.34*10^{-2}$	$1.25*10^{-4}$	$6.42*10^{-5}$	0.29	0.55
30	$1.44*10^{-2}$	$1.61*10^{-4}$	$4.81*10^{-5}$	0.27	0.36
31	$1.54*10^{-2}$	$1.72*10^{-4}$	$2.62*10^{-5}$	0.27	0.56
32	$1.53*10^{-2}$	$1.93*10^{-4}$	$7.98*10^{-9}$	0.36	0.39
33	$1.86*10^{-2}$	$3.00*10^{-4}$	$1.32*10^{-5}$	0.39	1.67
34	$1.64*10^{-2}$	$2.29*10^{-4}$	$1.50*10^{-5}$	0.39	1.25
35	$1.50*10^{-2}$	$1.67*10^{-4}$	$1.52*10^{-9}$	0.33	0.65
36	$1.37*10^{-2}$	$1.57*10^{-4}$	$3.17*10^{-5}$	0.45	1.74
37	$1.89*10^{-2}$	$3.18*10^{-5}$	$2.79*10^{-4}$	0.16	1.71

The Table reports the daily parameters obtained by minimizing the root of Mean Squared Error. Daily quotations range from 12/23/2008 to 02/17/2009

Tab.6. Parameters from $\sqrt{RMSE}$

t	σ_Q	α_0^Q	α_1^Q	β	$\frac{1}{N}\sqrt{RMSE}$
1	8.77*10-2	3.69*10-4	9.41*10-4	9.14*10-2	1.40*10-2
2	9.02*10-2	3.55*10-4	9.76*10-4	8.48*10-2	1.54*10-2
3	9.14*10-2	3.55*10-4	1.00*10-4	8.38*10-2	1.55*10-2
4	9.30*10-2	3.41*10-4	1.10*10-4	7.59*10-2	1.84*10-2
5	7.91*10-2	3.19*10-4	6.87*10-4	8.27*10-2	1.95*10-2
6	6.48*10-2	2.45*10-4	3.26*10-4	8.52*10-2	2.49*10-2
7	4.79*10-2	2.28*10-4	4.33*10-4	1.23*10-2	4.42*10-2
8	6.21*10-2	2.35*10-4	8.93*10-4	8.85*10-3	4.27*10-2
9	2.82*10-2	3.43*10-5	1.63*10-4	8.28*10-3	1.53*10-2
10	2.71*10-2	2.40*10-4	1.61*10-4	1.61*10-5	8.24*10-3
11	2.22*10-2	2.94*10-4	1.88*10-4	5.16*10-5	3.01*10-3
12	2.45*10-2	1.94*10-4	2.26*10-4	5.54*10-4	2.70*10-2
13	2.12*10-2	2.32*10-4	2.67*10-4	4.41*10-2	2.33*10-2
14	2.30*10-2	1.84*10-6	1.30*10-5	4.39*10-2	3.07*10-2
15	4.45*10-3	9.11*10-7	6.16*10-6	0.35	0.18
16	1.46*10-3	2.18*10-7	1.39*10-6	0.35	1.90*10-2
17	1.94*10-3	1.38*10-7	2.45*10-6	0.35	1.02*10-2
18	2.02*10-3	4.00*10-6	2.62*10-6	0.34	1.83*10-2
19	2.28*10-3	5.11*10-6	3.35*10-6	0.35	1.60*10-2
20	2.70*10-3	7.29*10-6	4.68*10-6	0.34	1.35*10-2
21	4.43*10-3	1.96*10-5	1.24*10-5	0.33	1.61*10-2
22	4.33*10-3	1.63*10-5	1.18*10-5	0.32	1.36*10-2
23	4.77*10-3	2.27*10-5	1.35*10-5	0.30	2.09*10-2
24	8.17*10-3	6.69*10-5	3.84*10-5	0.27	2.37*10-2
25	5.24*10-3	1.08*10-5	1.88*10-5	0.30	2.66*10-2
26	5.24*10-3	3.51*10-6	1.93*10-5	0.29	2.20*10-2
27	4.70*10-3	3.14*10-6	1.54*10-5	0.29	1.91*10-2
28	5.38*10-3	5.21*10-6	2.02*10-5	0.29	1.10*10-2
29	4.63*10-3	6.58*10-6	1.43*10-5	0.32	1.25*10-2
30	3.87*10-3	4.81*10-6	9.96*10-6	0.32	1.90*10-2
31	1.64*10-3	8.83*10-6	1.81*10-6	0.32	1.03*10-2
32	2.13*10-3	1.47*10-6	3.04*10-6	0.32	2.45*10-2
33	2.22*10-3	4.93*10-6	3.29*10-6	0.32	2.79*10-2
34	2.16*10-3	4.68*10-6	3.09*10-6	0.32	3.86*10-2
35	3.30*10-3	1.09*10-5	7.18*10-6	0.31	2.39*10-2
36	3.59*10-3	1,27*10-5	8.29*10-6	0.31	2.40*10-2
37	3.30*10-3	5,56*10-5	3.20*10-5	0.32	2.38*10-2

The Table reports the daily parameters obtained by minimizing the root of Relative Mean Squared Error. Daily quotations range from 12/23/2008 to 02/12/2009.

Tab.7. Comparison between discrete time and option pricing limit formula

T=10

Δ	K = 0.925	K = 0.95	K = 0.975	K = 1	K = 1.025	K = 1.05	K = 1.075	sec./(numb)	MRAE
1	$8.0917*10^{-2}$	$6.1089*10^{-2}$	$4.4060*10^{-2}$	$3.0264*10^{-2}$	$1.9776*10^{-2}$	$1.2305*10^{-2}$	$7.3088*10^{-3}$	0.0429	0.0212
0.1	$8.1122*10^{-2}$	$6.1471*10^{-2}$	$4.4591*10^{-2}$	$3.0858*10^{-2}$	$2.0323*10^{-2}$	$1.2722*10^{-2}$	$7.5678*10^{-3}$	0.1524	0.0021
0.01	$8.1142*10^{-2}$	$6.1508*10^{-2}$	$4.4644*10^{-2}$	$3.0917*10^{-2}$	$2.0377*10^{-2}$	$1.2764*10^{-2}$	$7.5933*10^{-3}$	1.6344	$2.08*10^{-4}$
Limit price	$8.1144*10^{-2}$	$6.1513*10^{-2}$	$4.4649*10^{-2}$	$3.0923*10^{-2}$	$2.0383*10^{-2}$	$1.2768*10^{-2}$	$7.5962*10^{-3}$	0.0254	-

The Table shows a comparison between European call options in discrete time and option pricing limit formula using the parameters defined by (20) $\alpha_0' = 0.72$, $\alpha_1' = 0.11$, $\sigma^2 = 0.0154$, $\beta_1' = 0.27$, $a = 1$.

Tab.8. Comparison between discrete time and option pricing limit formula

T=20

Δ	K = 0.925	K = 0.95	K = 0.975	K = 1	K = 1.025	K = 1.05	K = 1.075	sec./(numb)	MRAE
1	$9.4467*10^{-2}$	$7.7554*10^{-2}$	$6.2624*10^{-2}$	$4.9732*10^{-2}$	$3.8842*10^{-2}$	$2.9847*10^{-2}$	$2.2577*10^{-2}$	0.0531	0.0056
0.1	$9.4617*10^{-2}$	$7.7761*10^{-2}$	$6.2870*10^{-2}$	$4.9993*10^{-2}$	$3.9095*10^{-2}$	$3.0070*10^{-2}$	$2.2752*10^{-2}$	0.2677	$5.67*10^{-4}$
0.01	$9.4633*10^{-2}$	$7.7782*10^{-2}$	$6.2895*10^{-2}$	$50019*10^{-2}$	$3.9121*10^{-2}$	$3.0093*10^{-2}$	$2.2770*10^{-2}$	3.9927	$5.68*10^{-5}$
Limit price	$9.4635*10^{-2}$	$7.7785*10^{-2}$	$6.2898*10^{-2}$	$5.0022*10^{-2}$	$3.9124*10^{-2}$	$3.0095*10^{-2}$	$2.2772*10^{-2}$	0.0236	-

The Table shows a comparison between European call option in discrete time and the limit option pricing formula using parameters defined by (20) $\alpha_0' = 0.72$, $\alpha_1' = 0.11$, $\sigma^2 = 0.0154$, $\beta_1' = 0.27$, $a = 1$.

Tab.9. Comparison between discrete time and option pricing limit formula

T=30

Δ	K = 0.925	K = 0.95	K = 0.975	K = 1	K = 1.025	K = 1.05	K = 1.075	sec./(numb)	MRAE
1	$1.0653*10^{-1}$	$9.0882*10^{-2}$	$7.6821*10^{-2}$	$6.4343*10^{-2}$	$5.3410*10^{-2}$	$4.3948*10^{-2}$	$3.5857*10^{-2}$	0.0687	0.0028
0.1	$1.0664*10^{-1}$	$9.1027*10^{-2}$	$7.6985*10^{-2}$	$6.4515*10^{-2}$	$5.3578*10^{-2}$	$4.4103*10^{-2}$	$3.5989*10^{-2}$	0.4404	$2.87*10^{-4}$
0.01	$1.0666*10^{-1}$	$9.1042*10^{-2}$	$7.7001*10^{-2}$	$6.4532*10^{-2}$	$5.3595*10^{-2}$	$4.4118*10^{-2}$	$3.6003*10^{-2}$	7.9983	$2.87*10^{-5}$
Limit price	$1.0666*10^{-1}$	$9.1043*10^{-2}$	$7.7003*10^{-2}$	$6.4534*10^{-2}$	$5.3597*10^{-2}$	$4.412*10^{-2}$	$3.6004*10^{-2}$	0.0261	-

The Table shows a comparison between European call option in discrete time and the limit option pricing formula using parameters defined by (20) $\alpha_0' = 0.72$, $\alpha_1' = 0.11$, $\sigma^2 = 0.0154$, $\beta_1' = 0.27$, $a' = 1$.

Tab.10. Comparison between discrete time and option pricing limit formula

T=10

Δ	K = 0.925	K = 0.95	K = 0.975	K = 1	K = 1.025	K = 1.05	K = 1.075	sec./(numb)	MRAE
1	$8.0989*10^{-2}$	$6.0400*10^{-2}$	$4.2544*10^{-2}$	$2.8364*10^{-2}$	$1.8205*10^{-2}$	$1.1494*10^{-2}$	$7.2405*10^{-3}$	0.0467	0.0334
0.1	$8.1143*10^{-2}$	$6.0952*10^{-2}$	$4.3542*10^{-2}$	$2.9580*10^{-2}$	$1.9237*10^{-2}$	$1.2116*10^{-2}$	$7.4808*10^{-3}$	0.1651	0.0032
0.01	$8.1159*10^{-2}$	$6.1003*10^{-2}$	$4.3635*10^{-2}$	$2.9691*10^{-2}$	$1.9334*10^{-2}$	$1.2177*10^{-2}$	$7.5060*10^{-3}$	1.7874	$3.21*10^{-4}$
Limit price	$8.1161*10^{-2}$	$6.1009*10^{-2}$	$4.3645*10^{-2}$	$2.9703*10^{-2}$	$1.9344*10^{-2}$	$1.2184*10^{-2}$	$7.5088*10^{-3}$	0.0293	-

The Table shows a comparison between European call option in discrete time and the limit option pricing formula using parameters defined by (23) $\alpha_0' = 0.72$, $\alpha_1' = 0.11$, $\alpha_1' = 1$, $\sigma^2 = 0.0154$, $\beta_1' = 0.27$, $a' = 1$.

Tab.11. Comparison between discrete time and option pricing limit formula

T=20

Δ	K = 0.925	K = 0.95	K = 0.975	K = 1	K = 1.025	K = 1.05	K = 1.075	sec./(numb)	MRAE
1	$9.3381*10^{-2}$	$7.5811*10^{-2}$	$6.0397*10^{-2}$	$4.7306*10^{-2}$	$3.6550*10^{-2}$	$2.7959*10^{-2}$	$2.1254*10^{-2}$	0.0560	0.0206
0.1	$9.3846*10^{-2}$	$7.6552*10^{-2}$	$6.1346*10^{-2}$	$4.8343*10^{-2}$	$3.7527*10^{-2}$	$2.8761*10^{-2}$	$2.1818*10^{-2}$	0.3054	0.0019
0.01	$9.3890*10^{-2}$	$7.6620*10^{-2}$	$6.1433*10^{-2}$	$4.8438*10^{-2}$	$3.7617*10^{-2}$	$2.8836*10^{-2}$	$2.1871*10^{-2}$	4.5406	$1.90*10^{-4}$
Limit price	$9.3895*10^{-2}$	$7.6628*10^{-2}$	$6.1442*10^{-2}$	$4.8448*10^{-2}$	$3.7626*10^{-2}$	$2.8844*10^{-2}$	$2.1877*10^{-2}$	0.0287	-

The Table shows a comparison between European call option in discrete time and the limit option pricing formula using parameters defined by (23) $\alpha_0' = 0.72$, $\alpha_1' = 0.11$, $\alpha_1' = 1$, $\sigma^2 = 0.0154$, $\beta_1' = 0.27$, $a' = 1$.

Tab.12. Comparison between discrete time and option pricing limit formula

T=30

Δ	K = 0.925	K = 0.95	K = 0.975	K = 1	K = 1.025	K = 1.05	K = 1.075	sec./(numb)	MRAE
1	$1.0493*10^{-1}$	$8.8822*10^{-2}$	$7.4442*10^{-2}$	$6.1834*10^{-2}$	$5.0966*10^{-2}$	$4.1743*10^{-2}$	$3.4028*10^{-2}$	0.0681	0.0156
0.1	$1.0553*10^{-1}$	$8.9601*10^{-2}$	$7.5350*10^{-2}$	$6.2794*10^{-2}$	$5.1899*10^{-2}$	$4.2579*10^{-2}$	$3.4715*10^{-2}$	0.4283	0.0014
0.01	$1.0559*10^{-1}$	$8.9672*10^{-2}$	$7.5432*10^{-2}$	$6.2880*10^{-2}$	$5.1983*10^{-2}$	$4.2655*10^{-2}$	$3.4778*10^{-2}$	7.8447	$1.41*10^{-4}$
Limit price	$1.0559*10^{-1}$	$8.9680*10^{-2}$	$7.5441*10^{-2}$	$6.2890*10^{-2}$	$5.1993*10^{-2}$	$4.2664*10^{-2}$	$3.4785*10^{-2}$	0.0274	-

The Table shows a comparison between European call option in discrete time and the limit option pricing formula using parameters defined by (23) $\alpha_0' = 0.72$, $\alpha_1' = 0.11$, $\alpha_1' = 1$, $\sigma^2 = 0.0154$, $\beta_1' = 0.27$, $a' = 1$.

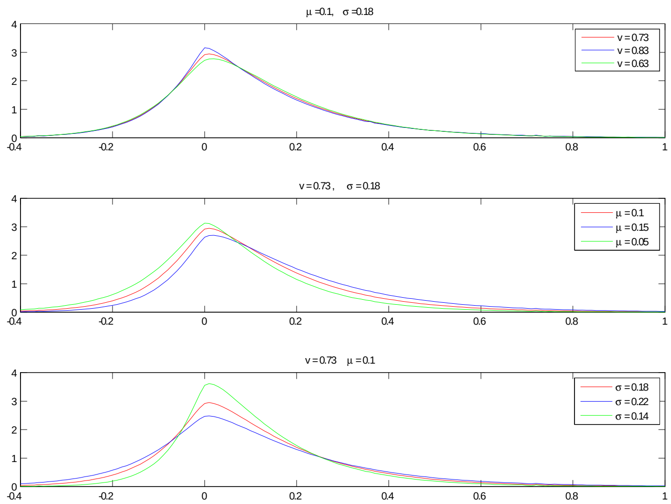

Fig.1. Behavior of Variance-Gamma distristribution as function of v (on the top) μ (in the middle) and σ (at the bottom)

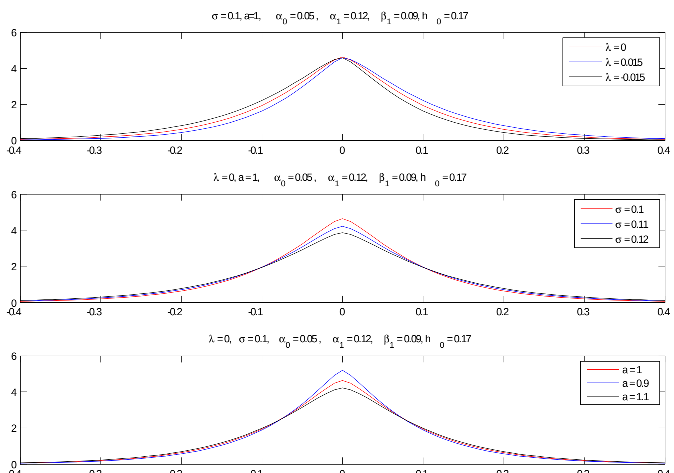

Fig.2. Behavior of the conditional distristribution of $\log(S_{30})$ given the information at time 0, as function of λ (on the top) σ (in the middle) and a (at the bottom)

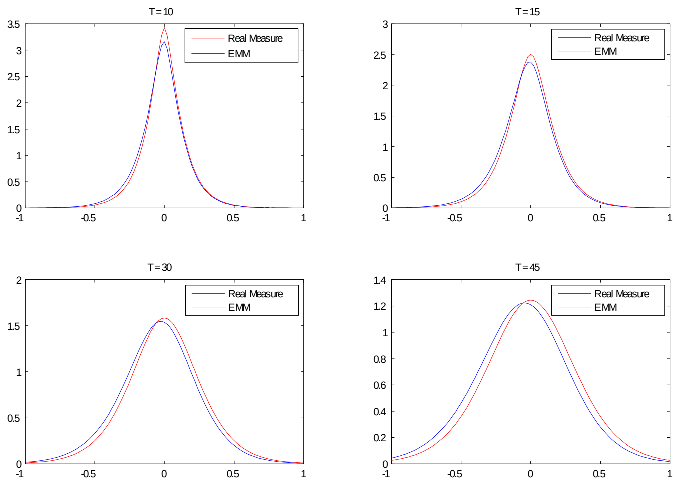

Fig.3. Each panel compares the distribution under real measure with that obtained by means of the conditional Esscher transform. The first panel reports the conditional distribution (10-day horizon), the second panel the 15-days horizon, the third panel the 30-days horizon and the last panel shows the 60-day horizon distribution. The distribution is constructed by inverse Fourier transform. $\lambda = 0$, $\sigma = 0.1$, $\lambda_Q =$ -0.005, $\sigma_Q = 0.1001$, $a = 3$, $\alpha_0 = 0.05$, $\alpha_1 = 0.12$, $\beta_1 = 0.08$, $h_0 = 0.15$

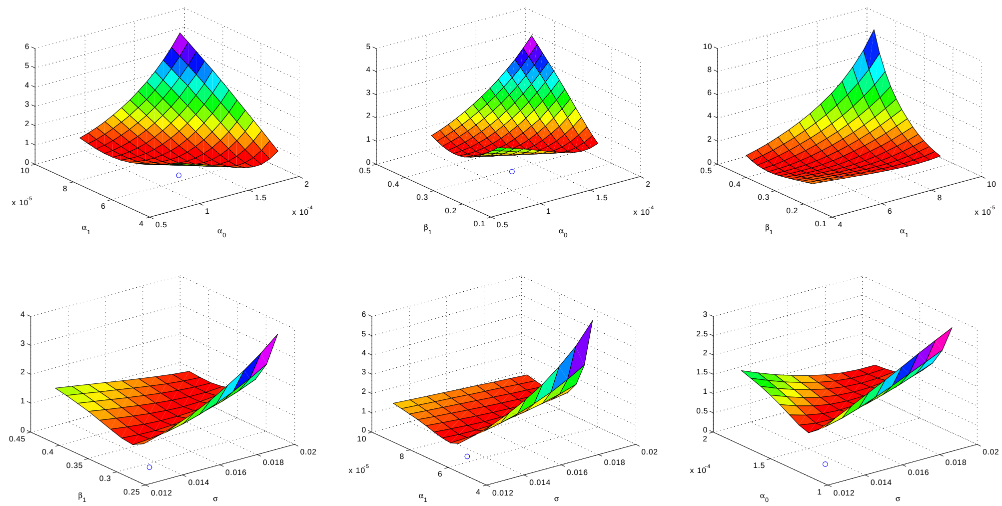

Fig.4. Sections of the root of Mean Squared Error at 29st quotation day. $\sigma_Q = 134 * 10^{-2}$, $\alpha_0^Q = 1.25 * 10^{-4}$, $\alpha_1^Q = 6.42 * 10^{-5}$, $\beta = 0.29$.

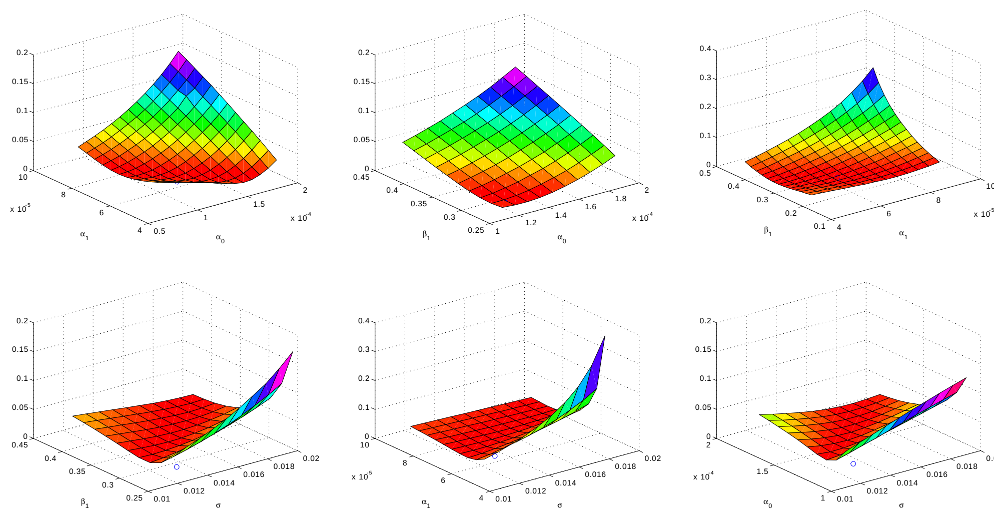

Fig.5. Sections of the root of Relative Mean Squared Error at 29st quotation day. $\sigma_Q = 4.63 * 10^{-3}$, $\alpha_0^Q = 6.58 * 10^{-6}$, $\alpha_1^Q = 1.43 * 10^{-5}$, $\beta = 0.32$.