From symmetric subdivision masks of Hurwitz type to interpolatory subdivision masks

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Abstract

In this paper we present a general strategy to deduce a family of interpolatory masks from a symmetric Hurwitz non-interpolatory one. This brings back to a polynomial equation involving the symbol of the non-interpolatory scheme we start with. The solution of the polynomial equation here proposed, tailored for symmetric Hurwitz subdivision symbols, leads to an efficient procedure for the computation of the coefficients of the corresponding family of interpolatory masks. Several examples of interpolatory masks associated with classical approximating masks are given.

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1. Introduction

A *subdivision scheme* is an iterative process that produces curves or surfaces from given discrete data by refining these on denser and denser grids.

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In the univariate case, starting with some initial points attached to the integer grid, *i.e.* with $\mathbf{q} = (q_i \,:\, i \in \mathbb{Z})$, one iteratively computes a sequence $\mathbf{q}^n := S_{\mathbf{a}}\mathbf{q}^{n-1} = S_{\mathbf{a}}^n\mathbf{q}^0$ for $n \geq 1$, where $\mathbf{q}^0 \equiv \mathbf{q}$, by repeated application of the rules

$$(S_{\mathbf{a}}\mathbf{q})_i = \sum_{j\in\mathbb{Z}} a_{i-2j}\, q_j, \qquad i \in \mathbb{Z}, \tag{1}$$

which rely on the coefficients a_i, $i \in \mathbb{Z}$. These identify the subdivision operator $S_{\mathbf{a}}$ and the so called *refinement mask* $\mathbf{a} = (a_i \,:\, i \in \mathbb{Z})$, which is an element of $\ell_0(\mathbb{Z})$, *i.e.* of the space of compactly supported sequences of real values. By assigning the values of $S_{\mathbf{a}}^n\mathbf{q}$, $n \in \mathbb{N}_0$, to the denser and denser grids $2^{-n}\mathbb{Z}$, one can then establish a notion of convergence to a continuous limit function by requiring the existence of a uniformly continuous function $f_{\mathbf{q}}$ (depending on the starting sequence $\mathbf{q}$) satisfying

$$\lim_{n\to\infty} \sup_{j\in\mathbb{Z}} \left| (S_{\mathbf{a}}^n\mathbf{q})_j - f_{\mathbf{q}}\left(2^{-n}j\right) \right| = 0 \tag{2}$$

and $f_{\mathbf{q}} \neq 0$ for at least some initial data $\mathbf{q}$ such that $\|\mathbf{q}\|_\infty := \sup_{i\in\mathbb{Z}} |\, q_i| < \infty$. An equivalent description of convergence is to demand the existence of the so called *basic limit function* as the limit of the sequence $S_{\mathbf{a}}^n\boldsymbol{\delta}$ (hereafter, we denote by $\boldsymbol{\delta} = (\delta_{i,0} \,:\, i \in \mathbb{Z})$ the "delta" sequence) that is, the existence of a uniformly continuous function $\phi_{\mathbf{a}}$ such that

$$\lim_{n\to\infty} \sup_{j\in\mathbb{Z}} \left| (S_{\mathbf{a}}^n\boldsymbol{\delta})_j - \phi_{\mathbf{a}}\left(2^{-n}j\right) \right| = 0. \tag{3}$$

In fact, in case of convergence of the subdivision scheme, we have

$$f_{\mathbf{q}} = \sum_{j\in\mathbb{Z}} \phi_{\mathbf{a}}(\cdot - j)\, q_j.$$

Note that the basic limit function is *refinable* with respect to $\mathbf{a}$ since it satisfies the functional equation

$$\phi_{\mathbf{a}} = \sum_{j\in\mathbb{Z}} a_j\; \phi_{\mathbf{a}}\left(2\cdot - j\right)\,. \tag{4}$$

Most of the theory of stationary subdivision, whether convergence takes place, consists of reading off the properties of the basic limit function $\phi_{\mathbf{a}}$ from the mask properties, or equivalently, from the properties of its *symbol*

$$a(z) = \sum_{i\in\mathbb{Z}} a_i\, z^i \qquad z \in \mathbb{C} \setminus \{0\},$$

a Laurent polynomial associated with the mask $\mathbf{a}$. For example, necessary conditions for the subdivision convergence are given in terms of symbol properties as

$$a(1) = 2, \quad a(-1) = 0. \tag{5}$$

For more than an introduction to stationary subdivision, we refer to [2, 7].

A particular class of subdivision schemes are those that refine the sequence $\mathbf{q}$ while keeping the "original data" in the sense that $(S_{\mathbf{a}}\mathbf{q})_{2i} = q_i,\ i \in \mathbb{Z}$. For obvious reasons, such schemes are called *interpolatory* and their refinement mask is of special type since it satisfies

$$a_{2i} = \delta_{i,0}, \quad i \in \mathbb{Z}.$$

Whenever they converge, the associated limit functions are cardinal interpolants to $\mathbf{q}$, *i.e.* $f_{\mathbf{q}}(i) = q_i,\ i \in \mathbb{Z}$ and their basic limit function $\phi_{\mathbf{a}}$ is a cardinal interpolant of the $\boldsymbol{\delta}$ sequence, $\phi_{\mathbf{a}}(i) = \delta_{i,0},\ i \in \mathbb{Z}$. It is easy to show that $a(z)$ is the symbol of an interpolatory scheme if and only if it satisfies

$$a(z) + a(-z) = 2 \qquad z \in \mathbb{C} \setminus \{0\}. \tag{6}$$

Interpolatory subdivision schemes play a crucial role in both geometric modeling and wavelets construction (see [7] and [16], respectively). In fact, on one hand interpolatory methods have the ability to generate curves in a very predictable manner (due to the fact that the produced limit shapes pass through the given control points) which is certainly a desirable feature in curve design. On the other hand, a positive symmetric interpolatory symbol, via its spectral factorization, allows the construction of an orthogonal refinable function, building block of orthonormal wavelets. Despite of their importance and of the recent burgeoning literature in the field of subdivision schemes, very few interpolatory examples are known so far even in the univariate setting. The most celebrated example is the class of Dubuc-Deslauriers (DD) symmetric schemes, first presented in [4].

Interpolatory subdivision is the subject of this paper, where we provide a general strategy to deduce a family of interpolatory masks from a symmetric Hurwitz non-interpolatory one. This brings back to a polynomial equation involving the symbol of the non-interpolatory subdivision scheme we start with. The solution of the polynomial equation here proposed is tailored for symmetric Hurwitz subdivision symbols and leads to an efficient strategy for the computation of the coefficients of the corresponding interpolatory masks. This is particularly true in the case of B-splines and for the masks given in [11].

We remark that a combination of DD-masks, also based on a polynomial equation solution, has been recently considered in [5] by J. De Villiers and K. Hunter. Nevertheless, their approach is less general than ours and no discussion of how to get the solution of the polynomial equation is conducted. We continue by noticing that, in the multivariate case, Jia extended the results in [17] by discussing existence and uniqueness of interpolatory masks induced by Box-splines [14]. Even though his analysis is somehow related to the one here proposed, in his paper no strategy for the derivation of the interpolatory mask coefficients is given. The advantage of our procedure (at present confined to the univariate case) is therefore two-fold: first it allows to generate a whole family of symmetric interpolatory masks and second it provides an efficient method for the construction of their coefficients. In addition, the polynomial formulation seems to be well suited for extensions to the multivariate case which is the subject of our future research.

The paper is organized as follows: in Section 2 the idea of the strategy used to move from a non-interpolatory mask to an interpolatory one is sketched also with the help of some examples. The theoretical foundation of it is then given in Section 3 together with an efficient procedure for the computation of the interpolatory mask coefficients. The closing Section 4 is devoted to the analysis of the B-spline and Gori-Pitolli (GP) cases. In both contexts several examples are also given.

2. Getting the idea

Aim of this section is the description, in a quite heuristic way, of the strategy used to deduce an interpolatory subdivision mask from a non-interpolatory one. It arises from the discussion conducted in [15] for the particular case of the cubic B-spline. Theoretical foundation of the strategy will be given in the next section.

Let $S_{\mathbf{a}^k}$ be the subdivision operator associated with the subdivision mask

$$\mathbf{a}^k = (\cdots, 0, 0, a_0, a_1, \cdots, a_k, 0, 0, \cdots)$$

and with the subdivision symbol $a_k(z) = \sum_{j=0}^{k} a_j z^j$.

Let $\mathcal{H}_k$ be the matrix of order k associated with the polynomial $a_k(z)$

$$\mathcal{H}_k = \begin{pmatrix} a_1 & a_3 & a_5 & a_7 & \cdots & 0 \\ a_0 & a_2 & a_4 & a_6 & \cdots & 0 \\ 0 & a_1 & a_3 & a_5 & \cdots & 0 \\ 0 & a_0 & a_2 & a_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_k \end{pmatrix}, \tag{7}$$

namely

$$\mathcal{H}_k = (h_{i,j}^{(k)}), \quad h_{i,j}^{(k)} = a_{2j-i},\ 1 \le i, j \le k \tag{8}$$

and denote by $\mathcal{A}_k$ the leading principal submatrix of order $k-1$. Whenever $\mathcal{H}_k$ is non singular, $\mathcal{H}_k$ and $\mathcal{A}_k$ are related by

$$\det \mathcal{H}_k = a_k \det \mathcal{A}_k$$

and, moreover, by

$$\mathcal{H}_k^{-1} = \left(\begin{array}{c|c} \mathcal{A}_k^{-1} & \mathbf{0} \\ \hline * & a_k^{-1} \end{array} \right).$$

It is not difficult to see that the application of the subdivision operator $S_{\mathbf{a}^k}$ defined as in (1),

$$\mathbf{v} = S_{\mathbf{a}^k}\mathbf{u}, \qquad v_{k+i} = \sum_{j\in\mathbb{Z}} a_{k+i-2j}\, u_j, \qquad i \in \mathbb{Z},$$

exploiting a Matlab-like notation can be locally written in terms of the matrix $\mathcal{H}_k$ as

$$\mathbf{v}[2k-1\colon\ -1\colon k]^T = \mathcal{H}_k\ \mathbf{u}[k-1\colon\ -1\colon 0]^T. \tag{9}$$

Now, if $\mathcal{H}_k$ is an invertible matrix so is $\mathcal{A}_k$, and from (9) we can write

$$\mathbf{u}[k-1\colon -1\colon 0]^T = \mathcal{H}_k^{-1}\ \mathbf{v}[2k-1\colon -1\colon k]^T = \mathcal{H}_k^{-1}\mathcal{H}_k\ \mathbf{u}[k-1\colon -1\colon 0]^T. \tag{10}$$

For $1 \le i \le k-1$, let us introduce the polynomials

$$p_k^i(z) := \sum_{\ell=1}^{k-1} (\mathcal{A}_k^{-1})_{i,\ell}\ z^{\ell-1} = \sum_{\ell=1}^{k-1} p_{i,\ell}\ z^{\ell-1}, \quad (p_{i,\ell} = 0 \text{ if } \ell < 1 \text{ or } \ell > k-1), \tag{11}$$

whose coefficients are determined by the entries of $\mathcal{A}_k$. Moreover, define the bi-infinite triangular Toeplitz matrix $\mathcal{T}^i = (t^i_{k,s})$ associated with $p^i_k(z)$ by setting $t^i_{k,s} = p_{i,s-k}$, $s, k \in \mathbb{Z}$.
Then, we can derive $k-1$ interpolatory masks from a non-interpolatory one by taking the following subdivision rules for $i = 1, \cdots, k-1$

$$\boldsymbol{v}^{(new)} = \mathcal{T}^i \cdot S_{\mathbf{a}^k}\mathbf{u}, \quad 1 \leq i \leq k-1.$$

From (10) it follows that these masks are indeed interpolatory. In addition, it can be easily shown that the symbol associated with the novel subdivision operator $\mathcal{T}^i \cdot S_{\mathbf{a}^k}$ is a suitable shift of $a_k(z)\, p^i_k(z)$. For a reason that will be clear soon, the shift we will consider is given by the factor z^{2i-1}. The shifted symbol will be denoted by

$$m^i_k(z) := \frac{a_k(z)\, p^i_k(z)}{z^{2i-1}}\,. \tag{12}$$

Though the theoretical analysis of the strategy sketched above will be given in the next section, to better understand it we continue by discussing a few examples.

2.1. Examples

We consider B-spline subdivision schemes of order k, $k \geq 2$, having symbol $a_k(z) = \dfrac{(1+z)^k}{2^{k-1}}$, $k \geq 2$. Since for $k = 2$ the matrix $\mathcal{A}_2$ is $\mathcal{A}_2 = (1)$, the process doesn't change the subdivision mask which is in fact interpolatory already. In case $k = 3$ we get the symbol of quadratic B-splines $a_3(z) = \frac{(1+z)^3}{4}$, defining the matrix

$$\mathcal{A}_3 = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix},$$

which is invertible. Using the coefficients of the first row of its inverse, $(\frac{3}{2}, -\frac{1}{2})$, we define the interpolatory symbol

$$m^1_3(z) = \frac{(1+z)^3}{4}\left(\frac{3}{2} - \frac{1}{2}z\right) z^{-1}$$

having mask

$$\mathbf{m}^1_3 = \frac{1}{8}\left(\cdots,\ 0, 3,\ 8,\ 6,\ 0,\ -1,\ 0, \cdots\right), \tag{13}$$

while using the coefficients of the second row of its inverse, $(-\frac{1}{2}, \frac{3}{2})$, we define the interpolatory symbol

$$m_3^2(z) = \frac{(1+z)^3}{4}\left(-\frac{1}{2} + \frac{3}{2}z\right) z^{-3}$$

whose associated mask is

$$\mathbf{m}_3^2 = \frac{1}{8}\left(\cdots,\ 0,\ -1,\ 0,\ 6,\ 8,\ 3,\ 0, \cdots\right). \tag{14}$$

Figure 1 shows the results obtained when applying 10 steps of the stationary subdivision schemes based on the masks (13) and (14), respectively.

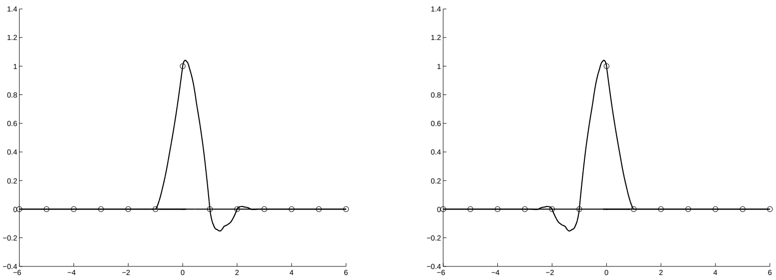

Fig. 1. *Plot of* $S^{10}_{\mathbf{m}_3^1}\boldsymbol{\delta}$ *(left) and of* $S^{10}_{\mathbf{m}_3^2}\boldsymbol{\delta}$ *(right).*

For $k = 4$ we deal with the symbol of cubic B-splines, $a_4(z) = \frac{(1+z)^4}{8}$, defining the matrix

$$\mathcal{A}_4 = \frac{1}{8}\begin{pmatrix} 4 & 4 & 0 \\ 1 & 6 & 1 \\ 0 & 4 & 4 \end{pmatrix},$$

with inverse given by

$$\mathcal{A}_4^{-1} = \frac{1}{2}\begin{pmatrix} 5 & -4 & 1 \\ -1 & 4 & -1 \\ 1 & -4 & 5 \end{pmatrix}.$$

Therefore, we can define three interpolatory symbols

$$\begin{aligned} m_4^1(z) &= \tfrac{(1+z)^4}{8}\left(\tfrac{5}{2} - 2z + \tfrac{1}{2}z^2\right) z^{-1}, \\ m_4^2(z) &= \tfrac{(1+z)^4}{8}\left(-\tfrac{1}{2} + 2z - \tfrac{1}{2}z^2\right) z^{-3}, \\ m_4^3(z) &= \tfrac{(1+z)^4}{8}\left(\tfrac{1}{2} - 2z + \tfrac{5}{2}z^2\right) z^{-5}, \end{aligned}$$

with corresponding masks

$$\begin{aligned}
\mathbf{m}_4^1 &= \tfrac{1}{16}\,(\,\cdots,0,\ 5,\ 16,\ 15,\ 0,\ -5,\ 0,\ 1,\ 0,\ \cdots\,)\,,\\
\mathbf{m}_4^2 &= \tfrac{1}{16}\,(\,\cdots,0,\ -1,\ 0,\ 9,\ 16,\ 9,\ 0,\ -1,\ 0,\ \cdots\,)\,,\\
\mathbf{m}_4^3 &= \tfrac{1}{16}\,(\,\cdots,0,\ 1,\ 0,\ -5,\ 0,\ 15,\ 16,\ 5,\ 0,\ \cdots\,)\,.
\end{aligned} \tag{15}$$

Note that $\mathbf{m}_4^2$ is the mask of the celebrated Dubuc-Deslauriers 4-point scheme [4, 9]. Figure 2 shows the results obtained when applying 10 steps of the stationary subdivision schemes based on the masks (15).

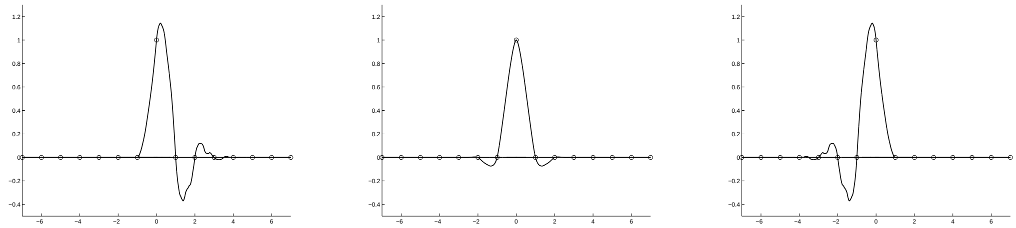

Fig. 2. *Plot of* $S^{10}_{\mathbf{m}_4^1}\boldsymbol{\delta}$ *(left),* $S^{10}_{\mathbf{m}_4^2}\boldsymbol{\delta}$ *(center),* $S^{10}_{\mathbf{m}_4^3}\boldsymbol{\delta}$ *(right).*

3. Hurwitz symmetric masks

To set the theoretical foundation of the strategy sketched in the previous section, we restrict the analysis to the case of a symmetric mask whose symbol is a Hurwitz polynomial of degree $k \in \mathbb{N}_+$, *i.e.* a polynomial of degree k with all zeros in the left-half plane. Thus, we continue by considering $a_k(z)$ a symmetric Hurwitz polynomial symbol of degree k defined by

$$a_k(z) = a_0 + a_1 z + \ldots + a_k z^k,$$

subjected to the constraints

$$a_j = a_{k-j}, \quad 0 \le j \le k\,. \tag{16}$$

Remark 1. *We remark that the refinable limit function of a subdivision scheme based on a Hurwitz symmetric mask is symmetric and totally positive which is known to be a very important property, for example in CAGD applications [10]. The stipulation on the symbol has two consequences. First* *it makes possible to prove the invertibility of $\mathcal{H}_k$ by characterizing the polynomials $p_k^i(z)$ in (11) as the unique solution of a certain polynomial equation. Secondly, it leads to very effective numerical methods for solving this equation. While the Hurwitz property is essential for the second result, the first one could be obtained under very relaxed conditions, requiring merely that $a_k(z)$ is a symmetric polynomial such that $a_k(z)$ and $a_k(-z)$ are relatively prime. In this way the applicability of our approach can be extended to deal with a wider class of symmetric masks.*

3.1. Existence and characterization of the polynomials $p_k^i(z)$

Aim of this subsection is two-fold. First we consider the existence and the characterization of the polynomials $p_k^i(z)$ mentioned in (11), then we discuss a technique for their construction.

Theorem 2. *Let $a_k(z)$ be a symmetric Hurwitz polynomial with $\mathcal{H}_k$ its associated matrix of order k. The polynomial $p_k^i(z)$ with coefficients given by the entries of the i-th row of $\mathcal{H}_k^{-1}$, $1 \le i \le k-1$, is the unique polynomial of degree less than k such that*

$$a_k(z)p_k^i(z) - a_k(-z)p_k^i(-z) = 2\, z^{2i-1}, \qquad 1 \le i \le k-1. \tag{17}$$

Proof. Let us introduce the matrix $\mathcal{R}_k \in \mathbb{R}^{2k\times 2k}$ defined by

$$\mathcal{R}_k = \left(\begin{array}{c|c} J_k \mathcal{H}_k^T J_k & 0 \\ \hline 0 & \mathcal{H}_k^T D_k \end{array}\right),$$

where J_k denotes the $k \times k$ reversion matrix having unit antidiagonal entries, *i.e.*, $J_k = (\delta_{i,n-j+1})$ where δ is the Kronecker symbol, and, moreover,

$$D_k = \mathrm{diag}[-1, (-1)^2, \ldots, (-1)^{k-1}, (-1)^k].$$

The proof of the Theorem follows by relating the matrix $\mathcal{R}_k$ with the resultant matrix generated by the polynomials $a_k(z)$ and $a_k(-z)$.
Let $P_k \in \mathbb{R}^{2k\times 2k}$, $P_k = (\delta_{i,\sigma(j)})$ be the permutation matrix associated with the "perfect shuffle" permutation given by

$$\sigma \;:\; \{1, \ldots, 2k\} \to \{1, \ldots, 2k\}, \quad \sigma(j) = \begin{cases} (j+1)/2, & \text{if } j \text{ is odd;} \\ j/2 + k, & \text{if } j \text{ is even.} \end{cases}$$

There follows that

$$P_k \mathcal{R}_k = P_k \left(\frac{J_k \mathcal{H}_k^T J_k}{0} \right) + P_k \left(\begin{array}{c} 0 \\ \mathcal{H}_k^T D_k \end{array} \right).$$

In addition, let $G_k \in \mathbb{R}^{2k \times 2k}$ be the matrix defined by

$$G_k = \left(\begin{array}{c|c} I_k & -D_k \\ \hline D_k & I_k \end{array} \right).$$

By performing one step of block Gaussian elimination we find that

$$G_k = \left(\begin{array}{c|c} I_k & 0 \\ \hline D_k & I_k \end{array} \right) \left(\begin{array}{c|c} I_k & -D_k \\ \hline 0 & 2I_k \end{array} \right),$$

which yields the following block characterization of the inverse of G_k

$$G_k^{-1} = \frac{1}{2} \left(\begin{array}{c|c} I_k & D_k \\ \hline -D_k & I_k \end{array} \right).$$

Hence, we obtain that

$$P_k \mathcal{R}_k G_k = P_k \left(\frac{J_k \mathcal{H}_k^T J_k}{0} \right) \left(\begin{array}{c|c} I_k & -D_k \end{array} \right) + P_k \left(\frac{0}{\mathcal{H}_k^T D_k} \right) \left(\begin{array}{c|c} D_k & I_k \end{array} \right),$$

which implies

$$P_k \mathcal{R}_k G_k = \left(\begin{array}{c|c} P_k \left(\frac{J_k \mathcal{H}_k^T J_k}{\mathcal{H}_k^T} \right) & P_k \left(\frac{(-1)^k J_k \mathcal{H}_k^T D_k J_k}{\mathcal{H}_k^T D_k} \right) \end{array} \right)$$

since $-D_k = (-1)^k J_k D_k J_k$. Now it is worth noting that the two block columns of $P_k \mathcal{R}_k G_k$ have basically the same structure. For the first component we find that

$$\Gamma_+ = (\gamma_{i,j}^+) = P_k \left(\frac{J_k \mathcal{H}_k^T J_k}{\mathcal{H}_k^T} \right)$$

where from (8) we get

$$\gamma_{i,j}^+ = \begin{cases} a_{k+j-i} \text{ if } i \text{ odd}, \\ a_{i-j} \text{ if } i \text{ even}; \end{cases}$$

and, therefore, from (16) we obtain

$$\gamma_{i,j}^+ = a_{k+j-i}, \quad 1 \leq i \leq 2k, \ 1 \leq j \leq k.$$

Similarly, we deduce that

$$\Gamma_- = (\gamma^-_{i,j}) = P_k \left(\frac{(-1)^k J_k \mathcal{H}_k^T D_k J_k}{\mathcal{H}_k^T D_k} \right),$$

where from (8) it follows

$$\gamma^-_{i,j} = \begin{cases} a_{k+j-i}(-1)^{j-1} & \text{if } i \quad \text{odd}, \\ a_{i-j}(-1)^j & \text{if } i \text{ even}; \end{cases}$$

and, therefore, again by using (16) we find that

$$\gamma^-_{i,j} = a_{k+j-i}(-1)^{j-i}, \quad 1 \le i \le 2k, \; 1 \le j \le k.$$

In this way we may conclude that

$$P_k \mathcal{R}_k G_k = \mathcal{S}_k(a_k^+, a_k^-),$$

where $\mathcal{S}_k(a_k^+, a_k^-) \in \mathbb{R}^{2k \times 2k}$ is the resultant matrix of order $2k$ associated with the polynomials $a_k^+ := a_k(z)$ and $a_k^- := a_k(-z)$. From the Hurwitz property we get that $a_k(z)$ and $a_k(-z)$ are relatively prime and, therefore, $\mathcal{S}_k(a_k^+, a_k^-)$ is nonsingular. This implies that $\mathcal{R}_k$, $\mathcal{H}_k$ and $\mathcal{A}_k$ are nonsingular matrices. In particular, the i-th row $\boldsymbol{x}_i^T$ of $\mathcal{H}_k^{-1}$ satisfies

$$\mathcal{R}_k \left(\frac{\mathbf{0}}{D_k \boldsymbol{x}_i} \right) = \left(\frac{\mathbf{0}}{\boldsymbol{e}_i} \right)$$

which yields

$$P_k \mathcal{R}_k G_k G_k^{-1} \left(\frac{\mathbf{0}}{D_k \boldsymbol{x}_i} \right) = P_k \left(\frac{\mathbf{0}}{\boldsymbol{e}_i} \right),$$

and, therefore,

$$\frac{1}{2} \mathcal{S}_k(a_k^+, a_k^-) \left(\frac{\boldsymbol{x}_i}{D_k \boldsymbol{x}_i} \right) = \boldsymbol{e}_{2i}.$$

By setting

$$\boldsymbol{x}_i = [x_0^{(i)}, \ldots, x_{k-1}^{(i)}]^T, \quad p_k^i(z) = x_0^{(i)} + x_1^{(i)} z + \ldots + x_{k-1}^{(i)} z^{k-1},$$

we conclude that $p_k^i(z)$ is the unique polynomial of degree less than k such that

$$a_k(z) p_k^i(z) - a_k(-z) p_k^i(-z) = 2\, z^{2i-1}, \quad 1 \le i \le k-1.$$

$\square$

By multiplying (17) by z we find that

$$a_k(z)(zp_k^i(z)) + a_k(-z)(-zp_k^i(-z)) = 2\,z^{2i}, \quad 1 \leq i \leq k-1. \tag{18}$$

Effective computational methods for solving such kind of equations had already appeared in the literature. In fact, polynomial equations of the more general form

$$a(z)p(-z) + a(-z)p(z) = b(z), \tag{19}$$

with

$$\deg(a(z)), \deg(p(z)) \leq k, \ \deg(b(z)) \leq 2k, \quad b(z) = b(-z), \tag{20}$$

play a key role in several different contexts, including control theory. A solution method based on polynomial manipulations and related to the Routh-Hurwitz theory has been described in [13]. In the following we pursue a different approach which is more suited for the applications we have in mind (see subsections 4.1 and 4.2).

Let us assume that all the polynomials in (19) are suitably represented by using the Bernstein type polynomial basis $(1-z)^k$, $(1-z)^{k-1}(1+z), \ldots, (1-z)(1+z)^{k-1}$, $(1+z)^k$ of the vector space of real polynomials of degree less than or equal to k. That is,

$$a(z) = \sum_{j=0}^{k} \hat{a}_j (1-z)^{k-j}(1+z)^j, \qquad p(z) = \sum_{j=0}^{k} \hat{p}_j (1-z)^{k-j}(1+z)^j.$$

Moreover let us assume that $b(z)$ is also suitably represented by using the polynomial basis $(1-z)^{2k}$, $(1-z)^{2k-1}(1+z), \ldots, (1-z)(1+z)^{2k-1}$, $(1+z)^{2k}$ of the vector space of real polynomials of degree less than or equal to $2k$, namely

$$b(z) = \sum_{j=0}^{2k} \hat{b}_j (1-z)^{2k-j}(1+z)^j.$$

Since $b(z) = b(-z)$ we obtain that

$$b(z) = \sum_{j=0}^{2k} \hat{b}_j (1-z)^{2k-j}(1+z)^j = \sum_{j=0}^{2k} \hat{b}_j (1+z)^{2k-j}(1-z)^j = b(-z)$$

and, hence,

$$\hat{b}_j = \hat{b}_{2k-j} \quad 0 \le j \le 2k.$$

Observe that

$$a(z) = (1-z)^k \hat{a}\left(\frac{1+z}{1-z}\right), \qquad a(-z) = (1+z)^k \hat{a}\left(\frac{1-z}{1+z}\right),$$

where

$$\hat{a}(w) := \sum_{j=0}^{k} \hat{a}_j w^j, \qquad w := \frac{1+z}{1-z}.$$

Similarly we can write

$$p(z) = (1-z)^k \hat{p}\left(\frac{1+z}{1-z}\right), \quad p(-z) = (1+z)^k \hat{p}\left(\frac{1-z}{1+z}\right), \quad \hat{p}(w) := \sum_{j=0}^{k} \hat{p}_j w^j,$$

and

$$b(z) = (1-z)^k (1+z)^k \left(\hat{b}\left(\frac{1+z}{1-z}\right) + \hat{b}\left(\frac{1-z}{1+z}\right)\right), \quad \hat{b}(w) := \frac{\hat{b}_k}{2} + \sum_{j=1}^{k} \hat{b}_{k+j} w^j.$$

In this way the relation (19) can be restated as

$$\hat{a}\left(\frac{1+z}{1-z}\right)\hat{p}\left(\frac{1-z}{1+z}\right) + \hat{a}\left(\frac{1-z}{1+z}\right)\hat{p}\left(\frac{1+z}{1-z}\right) = \hat{b}\left(\frac{1+z}{1-z}\right) + \hat{b}\left(\frac{1-z}{1+z}\right), \tag{21}$$

or, equivalently, as

$$\hat{a}(w)\hat{p}\left(\frac{1}{w}\right) + \hat{a}\left(\frac{1}{w}\right)\hat{p}(w) = \hat{b}(w) + \hat{b}\left(\frac{1}{w}\right). \tag{22}$$

Note that the Moebius transformation $z \to w = \dfrac{1+z}{1-z}$ maps the left half-plane into the open unit disc so that $\hat{a}(w)$ has all its zeros of modulus less than 1. The equation (22) reduces to a Toeplitz-plus Hankel linear system. Since $\hat{a}(w)$ is stable it follows that the coefficient matrix is invertible and, hence, $\hat{p}(w)$ is uniquely determined from the coefficients of $\hat{a}(w)$ and of $\hat{b}(w)$. In [1] superfast methods are devised for solving the linear system at the cost of $O(k \log^2 k)$ arithmetic operations. In many cases of interest for the applications under consideration, the polynomial $\hat{a}(w)$ has a very simple form so that the computation can be dramatically simplified (compare with subsections 4.1 and 4.2). For the polynomial $b(z) = z^{2i}$, $1 \leq i \leq k-1$ in (18), the next proposition provides explicit and easily computable expressions for the coefficients.

Proposition 3. *We have*

$$z^{2i} = \frac{1}{2^{2k}} \sum_{s=0}^{2k} \rho_s^{(i)} \, (1+z)^s (1-z)^{2k-s}, \quad 1 \leq i \leq k-1,$$

where

$$\rho_s^{(i)} = \sum_{j=2i-2k+s}^{2i} (-1)^j \binom{2i}{j} \binom{2k-2i}{s-j}, \quad 0 \leq s \leq 2k. \tag{23}$$

Proof. From

$$z^{2i} = z^{2i} \cdot (1)^{2k-2i} = \left(\frac{1+z}{2} - \frac{1-z}{2} \right)^{2i} \left(\frac{1+z}{2} + \frac{1-z}{2} \right)^{2k-2i}$$

it follows

$$z^{2i} = \left[\sum_{j=0}^{2i} \binom{2i}{j} (-1)^j \left(\frac{1+z}{2} \right)^j \left(\frac{1-z}{2} \right)^{2i-j} \right] \cdot \\ \cdot \left[\sum_{m=0}^{2k-2i} \binom{2k-2i}{m} \left(\frac{1+z}{2} \right)^m \left(\frac{1-z}{2} \right)^{2k-2i-m} \right]$$

which gives

$$z^{2i} = \frac{1}{2^{2k}} \sum_{s=0}^{2k} \rho_s^{(i)} \, (1+z)^s (1-z)^{2k-s},$$

where

$$\rho_s^{(i)} = \sum_{j=2i-2k+s}^{2i} (-1)^j \binom{2i}{j} \binom{2k-2i}{s-j}, \quad 0 \leq s \leq 2k.$$

□

Observe that the symmetry property $\rho_s^{(i)} = \rho_{2k-s}^{(i)}$ can easily be checked by performing the substitution $2i - j = \ell$ in (23). Indeed, we have

$$\rho_s^{(i)} = \sum_{j=2i-2k+s}^{2i} (-1)^j \binom{2i}{j}\binom{2k-2i}{s-j} = \sum_{j=\max\{2i-2k+s,0\}}^{\min\{2i,s\}} (-1)^j \binom{2i}{j}\binom{2k-2i}{s-j},$$

$2i - \max\{2i-s, 0\} = \min\{2i, s\}, \quad 2i - \min\{2i, 2k-s\} = \max\{2i-2k+s, 0\}$

and, hence, by using $\binom{k}{h} = \binom{k}{k-h}$,

$$\rho_{2k-s}^{(i)} = \sum_{j=\max\{2i-s,0\}}^{\min\{2i,2k-s\}} (-1)^j \binom{2i}{j}\binom{2k-2i}{2k-s-j} = \sum_{\ell=\max\{2i-2k+s,0\}}^{\min\{2i,s\}} (-1)^\ell \binom{2i}{\ell}\binom{2k-2i}{\ell-s}.$$

A recursive scheme for the computation of all the coefficients $\rho_s^{(i)}$, $1 \le i \le k-1$, $0 \le s \le 2k$ using $O(k^2)$ arithmetic operations was devised in [12].

3.2. Properties of the Laurent polynomials $m_k^i(z)$

We continue by investigating the properties of the family of masks whose symbol has been denoted by $m_k^i(z)$. As stated in the next two propositions, these masks are interpolatory and show "related" symmetry.

Proposition 4. *Given a symmetric Hurwitz-type degree-k polynomial $a_k(z)$ such that $a_k(1) = 2$, $a_k(-1) = 0$, the Laurent polynomials*

$$m_k^i(z) := \frac{a_k(z)p_k^i(z)}{z^{2i-1}}, \quad 1 \le i \le k-1, \tag{24}$$

are interpolatory symbols and satisfy

$$m_k^i(1) = 2, \quad m_k^i(-1) = 0, \quad 1 \le i \le k-1.$$

Proof. From the fundamental relation

$$a_k(z)p_k^i(z) - a_k(-z)p_k^i(-z) = 2\, z^{2i-1}, \quad 1 \le i \le k-1, \tag{25}$$

it follows that $p_k^i(1) = 1$ and, moreover, we can recast the equation as in (18) in the form

$$\frac{z a_k(z)p_k^i(z)}{z^{2i}} + \frac{(-z a_k(-z)p_k^i(-z))}{z^{2i}} = 2, \tag{26}$$

meaning that the schemes $m_k^i(z) := \dfrac{a_k(z)p_k^i(z)}{z^{2i-1}}$, $1 \le i \le k-1$ are interpolatory since they satisfy $m_k^i(z) + m_k^i(-z) = 2$. Last, the fact that $a_k(-1) = 0$ implies that $m_k^i(-1) = 0$ so that $m_k^i(1) = 2$ and the proof is complete. $\square$

Remark 5. *It is useful to note that any affine combination of the schemes* $m_k^i(z)$, $1 \le i \le k-1$ *is still interpolatory and that specific affine combinations of these masks can be used to get* symmetric *interpolatory masks.*

We now continue by analyzing the property of polynomial generation for the interpolatory mask $m_k^i(z)$. To this aim we recall the following result already given in [2] and more recently investigated in [6].

Lemma 6 (Polynomial Generation)**.** *For a non-singular subdivision scheme* $S_{\mathbf{c}}$ *with symbol* $c(z)$*, the condition*

$$c(z) \text{ is divisible by } (1+z)^{d_G+1} \tag{27}$$

is equivalent to the property that for any polynomial p *of degree* $d \le d_G$ *there exists some initial data* $\mathbf{q}^0$ *such that* $S_{\mathbf{c}}^\infty \mathbf{q}^0 = p$*. Moreover,* $\mathbf{q}^0$ *is sampled from a polynomial of the same degree and with the same leading coefficient.*

From the previous Lemma it trivially follows

Corollary 7. *If the symmetric Hurwitz-type subdivision mask* $a_k(z)$ *generates polynomials of degree* d_G*, then the associated family of interpolatory masks* $m_k^i(z)$ *has as well the ability to generate degree-*d_G *polynomials.*

Remark 8. *From the above corollary we can actually deduce that the subdivision schemes with symbols* $m_k^i(z)$*, due to their interpolatory nature, reproduce polynomials of degree* d_G *which means that, assumed* $\mathbf{q}^0 = \{p(i),\ i \in \mathbb{Z}\}$*, where* p *is a degree-*d_G *polynomial, then* $S_{\mathbf{m}_k^i}^\infty \mathbf{q}^0 = p$*. It also means that the interpolatory subdivision scheme has approximation order* d_G+1 *even though the non-interpolatory scheme we started with has a lower approximation order, which is certainly an important fact.*

Now, we will introduce a strategy to "symmetrize" the interpolatory subdivision masks $m_k^i(z)$, $1 \le i \le k-1$, which consists in taking specific convex combinations of them. In general, we can prove the following result.

Proposition 9. *Let* $m_k^i(z),\ 1 \le i \le k-1$ *be the interpolatory subdivision masks defined in (24). The symbols*

$$\begin{cases} s_k^{n-\ell,n+\ell+1}(z) := \frac{1}{2}\left(m_k^{n-\ell}(z) + \ m_k^{n+\ell+1}(z)\right), & \ell = 0,\ldots,n-1 \\ & \textit{for } k = 2n+1, \\ s_k^{n+1-\ell,n+\ell+1}(z) := \frac{1}{2}\left(m_k^{n+1-\ell}(z) + \ m_k^{n+\ell+1}(z)\right), & \ell = 1,\ldots,n \\ & \textit{for } k = 2n+2, \end{cases} \tag{28}$$

are interpolatory and symmetric.

Proof. Since $m_k^i(z) = \dfrac{a_k(z)p_k^i(z)}{z^{2i-1}},\ 1 \le i \le k-1$ with $a_k(z)$ a symmetric symbol, we work with the polynomials $\frac{p_k^i(z)}{z^{2i-1}}, 1 \le i \le k-1$ only. We start by using the fact that, since $\mathcal{A}_k$ is a centrosymmetric matrix, so is its inverse $(b_{i,j}) = \mathcal{B} := \mathcal{A}_k^{-1}$. This means that

$$b_{i,j} = b_{k-i+1,k-j+1}, \quad i,j = 1,\ldots,k-1. \tag{29}$$

If k is odd, the first expression in (28) is

$$\frac{1}{2}\frac{a_k(z)}{z^{2(n-\ell)-1}}\left(p_k^{n-\ell}(z) + z^{-4\ell-2}p_k^{n+\ell+1}(z)\right).$$

Since

$$p_k^{n-\ell}(z) = \sum_{j=1}^{2n} b_{n-\ell,j}\ z^j, \qquad z^{-4\ell-2}p_k^{n+\ell+1}(z) = \sum_{j=-4\ell-1}^{2n-4\ell-2} b_{n+\ell+1,j+4\ell+2}\ z^j,$$

the polynomial $c(z) := p_k^{n-\ell}(z) + z^{-4\ell-2}p_k^{n+\ell+1}(z)$ can be written as

$$\sum_{j=-4\ell-1}^{2n} c_j z^j \ \text{ where } \ c_j := \begin{cases} b_{n+\ell+1,j+4\ell+2}, & j = -4\ell-1,\cdots,0; \\ b_{n-\ell,j} + b_{n+\ell+1,j+4\ell+2}, & j = 1,\cdots,2n-4\ell-2; \\ b_{n-\ell,j}, & j = 2n-4\ell-1\cdots,2n. \end{cases}$$

The symmetry request for the first (and for the last) $4\ell+2$ elements of $c(z)$, *i.e.* the request that

$$b_{n+\ell+1,j+4\ell+2} = b_{n-\ell,2n-j-4\ell-1}, \quad j = -4\ell-1,\cdots,0,$$

follows from (29), from which also follows symmetry on the remaining coefficients of $c(z)$, *i.e.*

$$c_j = c_{2n-4\ell-1-j} \quad j = 1, \cdots, 2n - 4\ell - 2.$$

In fact,

$$c_j = b_{n-\ell,j} + b_{n+\ell+1,j+4\ell+2} = b_{n+\ell+1,2n-j+1} + b_{n-\ell,2n-4\ell-j-1} = c_{2n-4\ell-1-j}.$$

The case k even can be treated in a similar way. □

Remark 10. *We observe that any average of masks in (28) is also a symmetric interpolatory mask resulting in what we can call an "higher order" average. Examples of higher order averages will be given in the next section.*

The symmetrization of the interpolatory masks via average of two of them, may increase by one the order of polynomial generation. In fact, as a consequence of [3, Corollary 1] we have

Corollary 11. *Let the symbol $m_k^i(z)$ contain the odd degree factor $(1+z)^{2d-1}$. Then the symmetric masks defined in (28) contain also the even degree factor $(1+z)^{2d}$.*

4. Classical non-interpolatory Hurwitz symmetric masks: B-spline and GP masks

This section is devoted to the solution of the polynomial equation (18) in case $a_k(z)$ is either the symbol of a B-spline or of a GP function of order k. Even though the latter class of functions includes B-splines, for the sake of clarity we keep the two examples separated.

4.1. The B-spline case

In case we deal with a B-spline of order k whose symbol is the degree-k polynomial $a_k(z) = \dfrac{(1+z)^k}{2^{k-1}}$, equation (18) reads as

$$(1+z)^k(zp_k^i(z)) + (1-z)^k(-zp_k^i(-z)) = 2^k z^{2i}, \quad 1 \le i \le k-1, \tag{30}$$

which gives us a simple way to compute the coefficients of the polynomial $p_k^i(z)$. For the sake of simplicity we refer to the polynomial $-zp_k^i(-z)$ as to $p(z)$. Since $a(z) = (1+z)^k$ we find that $\hat{a}(w) = w^k$ and, therefore, the solution $\hat{p}(w)$ of (22) can immediately be reconstructed from the coefficients of $\hat{b}(w)$ given in Proposition 3. Indeed, we obtain that

$$w^k \sum_{j=0}^{k} \hat{p}_j w^{-j} + \frac{1}{w^k} \sum_{j=0}^{k} \hat{p}_j w^j = \rho_0^{(i)} w^{-k} + \ldots + \rho_{k-1}^{(i)} w^{-1} + \rho_k^{(i)} + \rho_{k+1}^{(i)} w + \ldots + \rho_{2k}^{(i)} w^k \tag{31}$$

which gives

$$\hat{p}_j = 2^{-k} \rho_j^{(i)}, \ 0 \le j \le k-1, \quad \hat{p}_k = 2^{-k-1} \rho_k^{(i)}. \tag{32}$$

By using the formulae stated in Proposition 3 we get an explicit representation of the coefficients of the solution polynomial $p(z)$ expressed in the Bernstein-like basis.

For the sake of illustration, we describe some examples of interpolatory subdivision masks derived through the explained strategy. In particular, we consider the case $k = 5$ *i.e.* the quartic B-spline symbol $a_5(z) = \dfrac{(1+z)^5}{16}$. The following four different polynomials are found

$$\begin{aligned}
p_5^1(z) \ &= \tfrac{1}{8} \left(35 - 47z + 25z^2 - 5z^3\right), \\
p_5^2(z) \ &= \tfrac{1}{8} \left(-5 + 25z - 15z^2 + 3z^3\right), \\
p_5^3(z) \ &= \tfrac{1}{8} \left(3 - 15z + 25z^2 - 5z^3\right), \\
p_5^4(z) \ &= \tfrac{1}{8} \left(-5 + 25z - 47z^2 + 35z^3\right),
\end{aligned}$$

giving rise to the respective interpolatory masks

$$\begin{aligned}
\mathbf{m}_5^1 \ &= \tfrac{1}{128} \left(\cdots, \ 0, \ 35, \ 128, \ 140, \ 0, \ -70, \ 0, \ 28, \ 0, \ -5, \ 0, \ \cdots \right), \\
\mathbf{m}_5^2 \ &= \tfrac{1}{128} \left(\cdots, \ 0, \ -5, \ 0, \ 60, \ 128, \ 90, \ 0, \ -20, \ 0, \ 3, \ 0, \ \cdots \right), \\
\mathbf{m}_5^3 \ &= \tfrac{1}{128} \left(\cdots, \ 0, \ 3, \ 0, \ -20, \ 0, \ 90, \ 128, \ 60, \ 0, \ -5, \ 0, \ \cdots \right), \\
\mathbf{m}_5^4 \ &= \tfrac{1}{128} \left(\cdots, \ 0, \ -5, \ 0, \ 28, \ 0, \ -70, \ 0, \ 140, \ 128, \ 35, \ 0, \ \cdots \right).
\end{aligned} \tag{33}$$

Figure 3 shows the results obtained when applying 10 steps of the stationary subdivision schemes based on the masks (33).

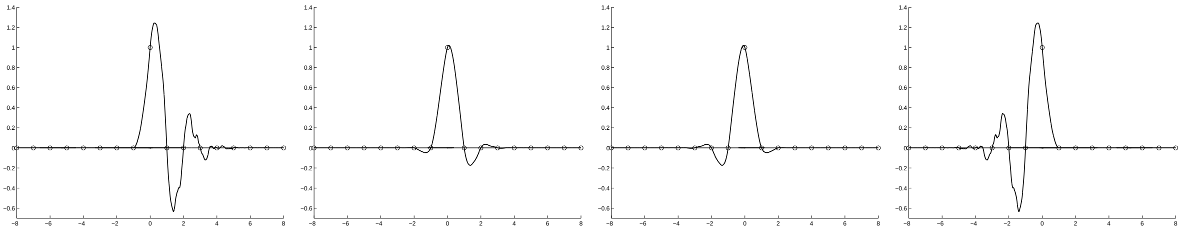

Fig. 3. *From left to right: plot of* $S^{10}_{\mathbf{m}_5^i}\boldsymbol{\delta},\ i=1,\cdots,4.$

Next, according to Proposition 9 we consider the symmetrized schemes

$$s_5^{1,4}(z) = \frac{1}{2}\left(m_5^1(z) + m_5^4(z)\right), \quad s_5^{2,3}(z) = \frac{1}{2}\left(m_5^2(z) + m_5^3(z)\right),$$

and

$$s_5^{1,2,3,4}(z) := \frac{1}{2}\left(s_5^{1,4}(z) + s_5^{2,3}(z)\right) = \frac{1}{4}\left(m_5^1(z) + m_5^2(z) + m_5^3(z) + m_5^4(z)\right).$$

The elements of the corresponding masks are:

$$\mathbf{s}_5^{1,4} = \frac{1}{256}(-5,\ 0,\ 28,\ 0,\ -70,\ 0,\ 175,\ 256,\ 175,\ 0,\ -70,\ 0,\ 28,\ 0,\ -5),$$

$$\mathbf{s}_5^{2,3} = \frac{1}{256}(3,\ 0,\ -25,\ 0,\ 150,\ 256,\ 150,\ 0,\ -25,\ 0,\ 3),$$

and $\mathbf{s}_5^{1,2,3,4} := \frac{1}{2}\left(\mathbf{s}_5^{1,4} + \mathbf{s}_5^{2,3}\right)$ that is

$$\mathbf{s}_5^{1,2,3,4} = \frac{1}{512}(-5,\ 0,\ 31,\ 0,\ -95,\ 0,\ 325,\ 512,\ 325,\ 0,\ -95,\ 0,\ 31,\ 0,\ -5).$$

Note that $\mathbf{s}_5^{2,3}$ is the mask of the celebrated Dubuc-Deslauriers 6-point scheme [4].

Figure 4 shows the results obtained when applying 10 steps of the stationary subdivision schemes based on the above given symmetric interpolatory masks.

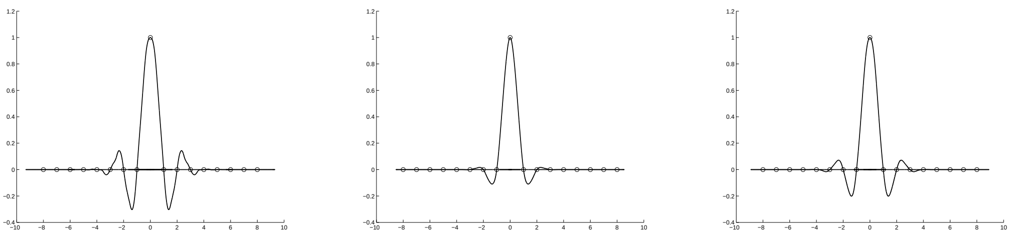

Fig. 4. *From left to right: plot of* $S^{10}_{\mathbf{s}_5^{1,4}}\boldsymbol{\delta}$, $S^{10}_{\mathbf{s}_5^{2,3}}\boldsymbol{\delta}$ *and* $S^{10}_{\mathbf{s}_5^{1,2,3,4}}\boldsymbol{\delta}$.

4.2. The GP case

Another class of masks whose associated symbol is a symmetric Hurwitz polynomial is the class of masks introduced in [11], hereinafter referred to as GP-masks. These masks, also characterized by positiveness of the coefficients, for fixed (k,ℓ), $k>2$ and $\ell>0$, are defined as

$$g_i^{k,\ell} = \frac{1}{2^{k-1+\ell}}\left(\binom{k}{i} + 4(2^\ell-1)\binom{k-2}{i-1}\right), \quad i=0,\cdots,k, \tag{34}$$

and are associated with the Hurwitz symmetric degree-k polynomials

$$g_{k,\ell}(z) = \frac{(1+z)^{k-2}}{2^{k-2}}\frac{\left(z^2+(2^{\ell+2}-2)z+1\right)}{2^{\ell+1}}. \tag{35}$$

Note that $g_{k,\ell}(z)$ is a convex combination of B-spline symbols of order k and $k-2$ with a convex combination parameter depending on ℓ, *i.e.*

$$g_{k,\ell}(z) = \frac{1}{2^\ell}\,a_k(z) + \left(1-\frac{1}{2^\ell}\right) z\,a_{k-2}(z), \quad 0 \le \frac{1}{2^\ell} \le 1,$$

and that

$$g_{k,0}(z) = a_k(z), \quad g_{k,\infty}(z) = z a_{k-2}(z)$$

are B-spline symbols. Even so, the refinable functions associated with convergent GP symbols $g_{k,\ell}(z)$ *are not B-splines.*

In view of the discussion had in Section 4.1, it is convenient to write the quadratic polynomial in (35), $z^2+(2^{\ell+2}-2)z+1$, as

$$(1-2^\ell)(1-z)^2 + 2^\ell(1+z)^2,$$

and derive the polynomial $\hat{a}(w) = \hat{a}_{k-2}w^{k-2} + \hat{a}_k w^k$ in (22) explicitly as

$$\hat{a}(w) = (1-2^\ell)w^{k-2} + 2^\ell w^k.$$

The above expression of $\hat{a}(w)$ allows us to provide an efficient strategy for the computation of $\hat{p}(w) = \sum_{j=0}^k \hat{p}_j w^j$ which reduces to the solution of a 3×3 linear system. In fact, a direct comparison of the polynomial coefficients in the left hand side of (22), *i.e.*,

$$\left(\hat{a}_{k-2}w^{k-2} + \hat{a}_k w^k\right)\left(\hat{p}_0 + \cdots + \hat{p}_k w^{-k}\right) + \left(\hat{a}_{k-2}w^{2-k} + \hat{a}_k w^{-k}\right)\left(\hat{p}_0 + \cdots + \hat{p}_k w^k\right),$$

with those in the right hand side of (22), *i.e.*,

$$2^{\ell-k-1}(\rho_0^{(i)}w^{-k} + \ldots + \rho_{k-1}^{(i)}w^{-1} + \rho_k^{(i)} + \rho_{k+1}^{(i)}w + \ldots + \rho_{2k}^{(i)}w^k),$$

leads to the following relations for the coefficients $\hat{p}_j,\ j = 0, \cdots, k-3$,

$$\begin{aligned}
&\hat{a}_k\hat{p}_0 = 2^{\ell-k-1}\rho_0^{(i)},\\
&\hat{a}_k\hat{p}_1 = 2^{\ell-k-1}\rho_1^{(i)},\\
&\hat{a}_k\hat{p}_j + \hat{a}_{k-2}\hat{p}_{j-2} = 2^{\ell-k-1}\rho_j^{(i)}, \quad j = 2, \cdots k-3,
\end{aligned} \tag{36}$$

and the linear system

$$\begin{cases}
\hat{a}_{k-2}\hat{p}_k + \hat{a}_k\hat{p}_{k-2} & = 2^{\ell-k-1}\left(\rho_{k+2}^{(i)} - \hat{a}_{k-2}\hat{p}_{k-4}\right)\\
(\hat{a}_k + \hat{a}_{k-2})\,\hat{p}_{k-1} & = 2^{\ell-k-1}\left(\rho_{k+1}^{(i)} - \hat{a}_{k-2}\hat{p}_{k-3}\right)\\
\hat{a}_{k-2}\hat{p}_{k-2} + \hat{a}_k\hat{p}_k & = 2^{\ell-k-2}\rho_k^{(i)},
\end{cases}$$

to be solved for getting the remaining coefficients

$$\begin{aligned}
\hat{p}_{k-2} &= \frac{2^{\ell-k-2}(\hat{a}_{k-2}\rho_k^{(i)} - 2\hat{a}_k\rho_{k+2}^{(i)} + 2\hat{a}_k\hat{a}_{k-2}\hat{p}_{k-4})}{(\hat{a}_{k-2})^2 - (\hat{a}_k)^2},\\
\hat{p}_{k-1} &= \frac{2^{\ell-k-1}(\rho_{k+1}^{(i)} - \hat{a}_{k-2}\hat{p}_{k-3})}{\hat{a}_{k-2} + \hat{a}_k},\\
\hat{p}_k &= \frac{2^{\ell-k-2}(-\hat{a}_k\rho_k^{(i)} + 2\hat{a}_{k-2}\rho_{k+2}^{(i)} - 2(\hat{a}_{k-2})^2\hat{p}_{k-4})}{(\hat{a}_{k-2})^2 - (\hat{a}_k)^2}.
\end{aligned}$$

Finally, by setting

$$\hat{a}_{k-2} = 1 - 2^\ell, \quad \hat{a}_k = 2^\ell, \quad \gamma = -\frac{\hat{a}_{k-2}}{\hat{a}_k} = 1 - 2^{-\ell}$$

we arrive at the following explicit representation of the solution of the linear system (36)

$$\hat{p}_j = \frac{1}{2^{k+1}} \sum_{s=0}^{\lfloor j/2 \rfloor} \gamma^s \rho_{j-2s}^{(i)}, \quad 0 \le j \le k-3. \tag{37}$$

We conclude the GP analysis with some examples of interpolatory subdivision masks derived through the explained strategy from a GP symbol. We consider the GP symbol $g_{4,2}(z) = \dfrac{(1+z)^2}{2^2}\dfrac{(z^2+14z+1)}{2^3}$ with associated refinable function displayed in the next picture (Figure 5).

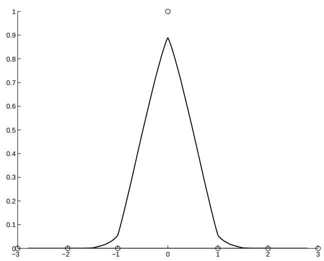

Fig. 5. *Plot of the limit function for $g_{4,2}(z)$.*

Using (37) we can construct three different polynomials

$$\begin{aligned} p^1_{4,2}(z) &= \tfrac{1}{14}\,(29 - 16z + z^2)\,, \\ p^2_{4,2}(z) &= \tfrac{1}{14}\,(-1 + 16z - z^2)\,, \\ p^3_{4,2}(z) &= \tfrac{1}{14}\,(1 - 16z + 29z^2)\,, \end{aligned}$$

and the corresponding interpolatory masks

$$\begin{aligned} \mathbf{m}^1_{4,2} &= \tfrac{1}{448}\,(\,\cdots,\ 0,\ 29,\ 448,\ 615,\ 0,\ -197,\ 0,\ 1,\ 0,\ \cdots\,)\,, \\ \mathbf{m}^2_{4,2} &= \tfrac{1}{448}\,(\,\cdots,\ 0,\ -1,\ 0,\ 225,\ 448,\ 225,\ 0,\ -1,\ 0,\ \cdots\,)\,, \qquad (38) \\ \mathbf{m}^3_{4,2} &= \tfrac{1}{448}\,(\,\cdots,\ 0,\ 1,\ 0,\ -197,\ 0,\ 615,\ 448,\ 29,\ 0,\ \cdots\,)\,. \end{aligned}$$

Figure 6 shows the results obtained when applying 10 steps of the stationary subdivision schemes based on the masks (38).

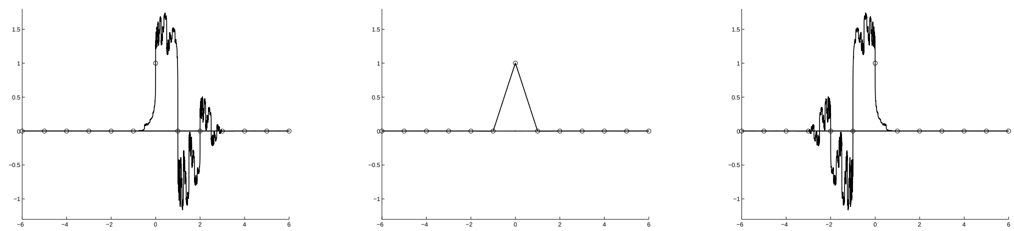

Fig. 6. *From left to right: plot of* $S^{10}_{\mathbf{m}^1_{4,2}}\boldsymbol{\delta}$, $S^{10}_{\mathbf{m}^2_{4,2}}\boldsymbol{\delta}$ *and* $S^{10}_{\mathbf{m}^3_{4,2}}\boldsymbol{\delta}$.

As it can be easily observed from Figure 6, the subdivision schemes $\mathbf{m}^1_{4,2}$ and $\mathbf{m}^3_{4,2}$ do not seem to be convergent. Differently, $\mathbf{m}^2_{4,2}$ is a special member of the family of C^1 interpolatory 4-point schemes presented in [8], corresponding to a mask of the form $(-w,\ 0,\ \frac{1}{2}+w,\ 1,\ \frac{1}{2}+w,\ 0,\ -w)$, with parameter $w=\frac{1}{448}$.
Next, according to Proposition 9, we construct the symmetrized interpolatory symbols

$$s^{1,3}_{4,2}(z)=\frac{1}{2}\left(m^1_{4,2}(z)+m^3_{4,2}(z)\right),$$

$$s^{1,2,3}_{4,2}(z):=\frac{1}{2}\left(s^{1,3}_{4,2}(z)+m^2_{4,2}(z)\right)=\frac{1}{4}\left(m^1_{4,2}(z)+2m^2_{4,2}(z)+m^3_{4,2}(z)\right).$$

The elements of the corresponding masks are:

$$\mathbf{s}^{1,3}_{4,2}=\frac{1}{896}(1,\ 0,\ -197,\ 0,\ 644,\ 896,\ 644,\ 0,\ -197,\ 0,\ 1),$$

$$\mathbf{s}^{1,2,3}_{4,2}=\frac{1}{1792}(1,\ 0,\ -199,\ 0,\ 1094,\ 1792,\ 1094,\ 0,\ -199,\ 0,\ 1),$$

and the results obtained when applying 10 steps of the stationary subdivision schemes based on these masks are shown in Figure 7.

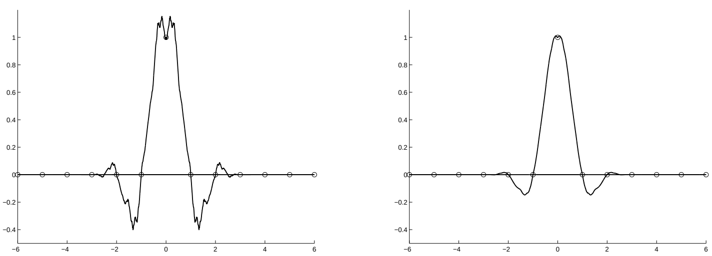

Fig. 7. *Plot of* $S^{10}_{\mathbf{s}^{1,3}_{4,2}}\boldsymbol{\delta}$ *(left) and of* $S^{10}_{\mathbf{s}^{1,2,3}_{4,2}}\boldsymbol{\delta}$ *(right).*

5. Conclusions and future work

This paper describes a general strategy to construct a family of interpolatory masks from a symmetric Hurwitz non-interpolatory one. A way to symmetrize the so obtained masks is also proposed together with an efficient

technique for the computation of the interpolatory mask coefficients. Even if some specific examples of interpolatory subdivision schemes deduced from approximating subdivision schemes were already proposed in the literature (see, e.g., [15]), to our knowledge this is the first time the theoretical foundation of a general strategy is given with the help of linear algebra together with an efficient algorithm for the computation of its coefficients.
In our understanding this is a first step in the analysis of a similar strategy suited to the bivariate setting or, more generally, to the multivariate one. In addition, it is our intention to generalize the proposed idea in the direction of taking affine combinations of non-interpolatory masks ending with an interpolatory mask with specific, and possibly enhanced, properties.

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