Moser-Trudinger and Beckner-Onofri's inequalities on the CR sphere

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Abstract. We derive sharp Moser-Trudinger inequalities on the CR sphere. The first type is in the Adams form, for powers of the sublaplacian and for general spectrally defined operators on the space of CR-pluriharmonic functions. We will then obtain the sharp Beckner-Onofri inequality for CR-pluriharmonic functions on the sphere, and, as a consequence, a sharp logarithmic Hardy-Littlewood-Sobolev inequality in the form given by Carlen and Loss.

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0. Introduction

Motivations and history.

The problem of finding optimal Sobolev inequalities continues to be a source of inspiration to many analysts. The literature on the subject is vast and rich. Besides its intrinsic value, the determination of best constants in Sobolev, or Sobolev type, inequalities has almost always revealed or employed deep facts about the geometric structure of the underlying space. More importantly, such constants were often the crucial elements needed to identify extremal geometries, and to solve important problems such as isoperimetric inequalities, eigenvalue comparison theorems, curvature prescription equations, existence of solutions of PDE's, and more.

This kind of research has produced a wealth of conclusive results in the context of Euclidean spaces and Riemannian manifolds. In contrast, very little is known in subRiemannian geometry, even in the simplest cases of the Heisenberg group or the CR sphere; this is especially true with regards to best constants in Sobolev imbeddings and sharp geometric inequalities.

In order to motivate our work, we present three by now classical sharp inequalities on the Euclidean $\mathbb{R}^n, S^n$. First, there is the standard Sobolev imbedding $W^{d/2,2} \hookrightarrow L^{2n/(n-d)}$, $(0 < d < n)$ represented by the optimal inequality

$$\|F\|_q^2 \le C(d,n) \int_X F A_d F \qquad q = \frac{2n}{n-d} \tag{0.1}$$

with $C(n,d) = \omega_n^{-d/n} \Gamma\big(\frac{n-d}{2}\big)/\Gamma\big(\frac{n+d}{2}\big)$, and where ω_n denotes the volume of S^n. For $X = \mathbb{R}^n$ the operator A_d is $\Delta^{d/2}$, and the extremals in (0.1) are dilations and translations of the function $(1+|x|^2)^{-n/q}$. For $X = S^n$ the operator A_d is the spherical picture of $\Delta^{d/2}$, obtained from it via the stereographic projection and conformal invariance. These operators act on the kth order spherical harmonics Y_k of S^n as

$$A_d Y_k = \frac{\Gamma\big(k + \frac{n+d}{2}\big)}{\Gamma\big(k + \frac{n-d}{2}\big)} Y_k. \tag{0.2}$$

When $d = 2$, $A_2 = Y = \Delta_{S^n} + \frac{n(n-2)}{4}$, the conformal Laplacian; for general $d \in (0,n)$ A_d is the intertwining operator of order d for the complementary series representations of $SO(n+1,1)$, and it is an elliptic pseudodifferential operator with the same leading symbol as $(\Delta_{S^n})^{d/2}$. The fundamental solution of A_d is given by the chordal distance function $c_d|\zeta - \eta|^{d-n}$, with $\zeta, \eta \in S^n$, where c_d is the same constant appearing in the fundamental solution (Riesz kernel) for $\Delta^{d/2}$ on $\mathbb{R}^n$. Higher order invariant operators where studied by Branson [Br], and also Graham, Jenne, Mason, Sparling [GJMS].

The extremals for the inequality (0.1) in this case are functions of type $|J_\tau|^{1/q}$ where $|J_\tau|$ denotes the density of the volume change via a conformal transformation τ of S^n.

Both in $\mathbb{R}^n$ and S^n inequality (0.1) is invariant under the action of their conformal group; for example on $\mathbb{R}^n$ in addition to the usual dilation/translation invariance, there is also an invariance under inversion: the action $F \to F\big(x/|x|^2\big)|x|^{-2n/q}$ leaves both sides of (0.1) unchanged. It is this particular aspect that makes this type of operators and inequalities interesting.

For $d = 2$ it was Talenti [Ta] who first derived (0.1) on $\mathbb{R}^n$, followed by Aubin [Au1] on S^n. For general d the inequality is the dual of the sharp Hardy-Littlewood- Sobolev inequality obtained by Lieb [L], a fundamental inequality which concerns the minimization of $\|F * |x|^{-\lambda}\|_q/\|F\|_p$ in the case $p = 2$; as stated, (0.1) appears in [Bec].

Next, there is the limit case $d = n$ of the above imbedding, which gives the so-called exponential class imbedding $W^{n/2,2} \hookrightarrow e^L$, and more generally $W^{n/p,p} \hookrightarrow e^L$, itself a limiting case of $W^{k,p} \hookrightarrow L^{np/(n-kp)}$. In concrete terms the Sobolev imbedding in the critical case $kp = n$ is represented by a Moser-Trudinger-Adams inequality of type

$$\int_X \exp\left[\alpha_{n,p}\left(\frac{|F|}{\|B_{n/p}F\|_p}\right)^{p'}\right] \le c_0 \tag{0.3}$$

where $B_{n/p}$ is a suitable, possibly vector valued, pseudodifferential operator of order n/p, and where the constant $\alpha_{n,p}$ is best, i.e. it cannot be replaced by a higher constant. Here F runs through an appropriate subspace of $W^{n/p,p}$ where $B_{n/p}$ is invertible.

Estimate (0.3) was first studied in the case $p = n$ by Trudinger [Tr] and later by Aubin [Au2], who showed that e^F is locally integrable if $F \in W^{1,n}$. The first sharp result is due to Moser [Mos1] on domains of $\mathbb{R}^n$, later extended to operators of higher order and general p by Adams [Ad]. On the sphere, the first result is by Moser himself [Mos2], in the case $n = p = 2$ and for and general compact Riemannian manifolds of dimension n, and operators of arbitrary order the best optimal inequality is due to Fontana [F]. The operators involved there were either integer powers of the Laplace Beltrami operators, or gradients ot those powers; in particular for (0.3) in the case $p = 2$ we get:

$$\alpha_{n,2} = \frac{n(2\pi)^n}{\omega_{n-1}} = \frac{n!\omega_n}{2}.$$

One of the key ingredients needed in the proof of (0.3) was the precise asymptotic expansion of the fundamental solutions for the operators involved.

Both (0.1) and (0.3) have important applications to isoperimetric problems, curvature prescription equations, and other nonlinear PDE's.

On S^n there is also another form of the exponential class imbedding, namely the so-called Beckner-Onofri inequality

$$\frac{1}{2n!} \fint_{S^n} u A_n u + \fint_{S^n} u - \log \fint_{S^n} e^u \geq 0 \ , \tag{0.4}$$

where $\fint$ denotes the average operator, and where A_n is the intertwining operator defined by (0.2) in the limit case $d = n$, namely with eigenvalues $k(k+1)...(k+n-1)$. Such A_n is sometimes referred to as the Paneitz operator on the sphere, in honor of S. Paneitz who first discovered a fourth order conformally invariant operator on general manifolds. Note that $A_2 = \Delta$. Due to the particular nature of A_n, the functional in (0.4) is invariant under the group action $F \to F \circ \tau + \log |J_\tau|$, where τ is a conformal transformation of S^n and $|J_\tau|$ its associated volume density; this action preserves the exponential integral. This important inequality was first derived by Onofri in dimension 2, but its general $n-$dimensional form was discovered later by Beckner [Bec], via an endpoint differentiation argument based on (0.1) and the sharp Hardy-Littlewood-Sobolev inequality. Later, Chang and Yang [CY] gave an alternative proof of (0.4) by a completely different method, based on an extended and refined version of the original compactness argument used by Onofri.

Estimate (0.4) has relevant applications in spectral geometry and mathematical physics, from comparison theorems for functional determinants to the theory of isospectral surfaces. [Br], [BCY], [CY], [CQ], [O], [OPS].

For the past 15-20 years there has been a growing interest in finding the analogues of the above results in the context of CR geometry. The biggest motivations are certainly the isoperimetric inequality, the isospectral problem, extremals for spectral invariants such as the functional determinant, and several other eigenvalue comparison theorems.

In the CR setting, the first and only known sharp Sobolev embedding estimate of type (0.1) with conformal invariance properties is due to Jerison and Lee [JL1],[JL2], and it holds on the Heisenberg group $\mathbb{H}^n$ and on the CR sphere S^{2n+1} in the case $d = 2$, for the CR invariant Laplacian (which is the standard sublaplacian in the case of $\mathbb{H}^n$). The corresponding version for operators of order $0 < d < Q = 2n+2$, $d \neq 2$, is only conjectured, and involves the intertwining operators $\mathcal{A}_d$ for the complementary series representations of $SU(n+1,1)$. The explicit form of such operators has been known for quite some time, for example by work of Johnson and Wallach [JW], and also Branson, Ólafsson and Ørsted [BOØ], and can be described as follows. Let $\mathcal{H}_{jk}$ be the space of harmonic polynomials of bidegree (j,k) on S^{2n+1}, for $j,k = 0,1,...$; such spaces make up for the standard decomposition of L^2 into $U(n+1)$-invariant and irreducible subspaces. The intertwining operators of order $d < Q$ are characterized (up to a constant) by their action on $Y_{jk} \in \mathcal{H}_{jk}$:

$$\mathcal{A}_d Y_{jk} = \lambda_j(d)\lambda_k(d) Y_{jk}, \qquad \lambda_j(d) = \frac{\Gamma\big(j + \frac{Q+d}{4}\big)}{\Gamma\big(j + \frac{Q-d}{4}\big)}; \tag{0.5}$$

when $d = 2$ this gives the CR invariant sublaplacian. As it turns out these operators have a simple fundamental solution of type $c_d|1 - \zeta \cdot \overline{\eta}|^{\frac{d-Q}{2}}$, where $\zeta, \eta \in S^{2n+1}$, for a suitable constant c_d. The conformally invariant sharp Sobolev inequality that is conjectured to be true is

$$\|F\|_q^2 \leq \frac{1}{\lambda_0(d)^2} \fint_{S^{2n+1}} F \mathcal{A}_d F \qquad q = \frac{2Q}{Q-d} \tag{0.6}$$

with extremals of type $|J_\tau|^{1/p}$, τ a conformal transformation of S^{2n+1}; this is the Jerison-Lee inequality for $d = 2$ but it is an open problem for general d. This conjecture does not seem to appear on any published articles, but it is well known within the group of researchers interested in this type of questions. One of the aspects that makes the CR treatment more difficult, is the lack, to date, of an effective symmetrization technique on the CR sphere or the Heisenberg group, that would allow for example to show the dual version of (0.6), namely the CR Hardy-Littlewood-Sobolev inequality.

Regarding Moser-Trudinger inequalities at the borderline case $d = Q/p$, Cohn and Lu recently made some progress [CoLu1], [CoLu2], deriving the CR analogue of (0.3) with sharp exponential constant in the case of the gradient, $p = Q$, both on $\mathbb{H}^n$ or the CR S^{2n+1} (see also [BMT] for similar results on Carnot groups). In regards to the "correct" CR analogue of Beckner-Onofri's inequality (0.4), the situation is not so obvious. One would certainly start to consider the operator $\mathcal{A}_Q = \lim_{d \to Q} \mathcal{A}_d$, the intertwining or Paneitz operator at the end of the complementary series range; the kernel of this operator is the space of CR-pluriharmonic functions on S^{2n+1}, given by $\mathcal{P} := \bigoplus_j (\mathcal{H}_{j0} \oplus \mathcal{H}_{0j})$. On the basis of (0.4) the natural conjecture would be that for a suitable constant c_n

$$c_n \fint F \mathcal{A}_Q F - \log \fint e^{F - \pi F} \geq 0, \qquad \forall F \in W^{Q/2,2} \tag{0.7}$$

where πF denotes the Cauchy-Szego projection of F on the space $\mathcal{P}$. The Euclidean version (0.4) can be cast in a similar form, with πF being just the average of F. This inequality however is *not* invariant under the conformal action that preserves the exponential integral, i.e. $F \to F \circ \tau + \log |J_\tau|$. On the other hand, the fact that $\mathcal{A}_Q$ has such large kernel $\mathcal{P}$ combined with the invariance of $\mathcal{P}$ under the conformal action (see Prop. 3.2) leads one to think that there should be a CR version of (0.4) which is conformally invariant and whose natural "milieu" is the space of CR-pluriharmonic functions; in this work we show that this is indeed the case.

Main results.

The CR version of Beckner Onofri's inequality proven in this paper is described as follows. Let $\mathcal{A}'_Q$ be the operator acting on CR-pluriharmonic functions as

$$\mathcal{A}'_Q \sum_j (Y_{j0} + Y_{0j}) = \sum_j \lambda_j(Q)(Y_{j0} + Y_{0j}), \qquad \lambda_j(Q) = j(j+1)..(j+n)$$

where $Y_{j0} \in \mathcal{H}_{j0}$, $Y_{0j} \in \mathcal{H}_{0j}$. In Theorem 3.1 we prove that for any real $F \in W^{Q/2,2} \cap \mathcal{P}$ we have

$$\frac{1}{2(n+1)!} \fint_{S^{2n+1}} F \mathcal{A}'_Q F + \fint_{S^{2n+1}} F - \log \fint_{S^{2n+1}} e^F \geq 0. \tag{0.8}$$

The functional in (0.8) is invariant under the conformal action $F \to F \circ \tau + \log |J_\tau|$, where τ is a conformal transformation of S^{2n+1} (i.e. τ is identified with an element of $SU(n+1,1)$), and $|J_\tau|$ its Jacobian determinant. The extremals of (0.8) are precisely the functions $\log |J_\tau|$.

A few remarks are in order. First, the conformal action is an *affine* representation of $SU(n+1,1)$, and the minimal nontrivial closed (real) subspace of L^2 that is invariant under such action is precisely the space of real CR-pluriharmonic functions (Prop. 3.2). This is in contrast with the Euclidean case, for the action induced by $SO(n+1,1)$, since in that case the only invariant closed subspaces of L^2 are the trivial ones. This observation seems to justify (at least partially) that inequality (0.8) could be regarded as the direct CR analogue of (0.4) from the point of view of conformal invariance.

Secondly, the key character in (0.8) is the operator $\mathcal{A}'_Q$, which we call the *conditional intertwinor* of order Q on $\mathcal{P}$. This operator is the CR analogue on $\mathcal{P}$ of the Paneitz, or GJMS, operator A_n on the standard Euclidean sphere, and coincides, up to a multiplicative constant, with the $d-$*derivative* at $d = Q$ of $\mathcal{A}_d$ restricted to $\mathcal{P}$. Moreover, we have

$$\mathcal{A}'_Q F = \prod_{\ell=0}^{n} \left(\tfrac{2}{n}\mathcal{L} + \ell\right) F\,, \qquad F \in \mathcal{P}$$

where $\mathcal{L}$ is the standard sublaplacian on the sphere. To our knowledge such operator is introduced here for the first time.

Finally, if conjecture (0.6) were true then (0.8) would result by the same endpoint differentiation argument used by Beckner to obtain (0.4). The meaning of this is that even though we do not know whether (0.6) holds, we can still consider the functional

$$\frac{1}{\lambda_0(d)^2} \fint G \mathcal{A}_d G \, d\zeta - \left(\fint |G|^q d\zeta \right)^{2/q}, \qquad q = \frac{2Q}{Q-d}$$

and take its $d-$derivative at Q under the restriction $F \in \mathcal{P}$; the result of this operation is the functional in (0.8). This argument will in fact be used to prove the conformal invariance of (0.8) (see Prop. 3.2).

Our proof of (0.8) follows the same general strategy used by Chang-Yang and Onofri. The first step is to show that the functional in (0.8) is bounded below. This is accomplished by a "linearization" procedure from a sharp Adams inequality on the CR sphere derived here for the first time. Indeed a substantial portion of this work is dedicated to inequalities of type

$$\int_{S^{2n+1}} \exp \left[A_d \left(\frac{|F|}{\|B_d F\|_p} \right)^{p'} \right] d\zeta \leq C_0 \tag{0.9}$$

where $0 < d < Q$, $dp = Q$. We will obtain (0.9) for what we call $d-$type operators on Hardy spaces $\mathcal{H}^p$, or $\mathcal{P}^p$ (L^p boundary values of pluriharmonic functions on the ball), and which are essentially finite sums of powers of the sublaplacian, restricted to such spaces, with leading power equal to $d/2$. When $p = 2$, the case of interest for (0.8), we have $A_{Q/2} = \frac{1}{2}(n+1)!\omega_{2n+1}$ and this constant is sharp, i.e. in (0.9) it cannot be replaced by a larger constant, For general p we can only provide upper and lower bounds for the sharp constant. We will also obtain (0.9) on the full $W^{d,p}$ for $B_d = \mathcal{L}^{d/2}$ or $B_d = \mathcal{D}^{d/2}$, where $\mathcal{L}$ is the sublaplacian of the CR sphere, and $\mathcal{D} = \mathcal{L} + \frac{n^2}{4}$ is the conformal sublaplacian. In this case the constants are sharp for any p. In a forthcoming paper we will treat sharp Adams inequalities for more general spectrally defined operators on $W^{d,p}$.

Various instances of inequality (0.9) in different settings and forms can be found in the literature. The most common and basic strategy to prove them goes back to Adams. It reduces the problem to a sharp exponential inequality for the convolution with the fundamental solution of the operator involved. A tipical obstacle comes from the fact that in many cases such fundamental solution is only approximately known, but in [F] Fontana showed that only suitable asymptotics are really needed, and that it is not even necessary to have a convolution operator. Fontana's original work was for a class of differential operators on compact Riemannian manifolds without boundary. In this paper we present a general unified abstract result on exponential integrability in the Adams form, but valid in arbitrary measure spaces, and with rather minimal assumptions. The proof of such result relies on extensions and refinements of the methods used in [F], and its proof will appear in a separate work [FM]. With this approach the key step in the proof of (0.9) consists in proving only a precise asymptotic expansion for the distribution function of the fundamental solution.

Finally, but not less importantly, (0.8) implies the following sharp logarithmic Hardy-Littlewood-Sobolev inequality:

$$(n+1) ⨍ ⨍ \log \frac{1}{|1-\zeta\cdot\overline{\eta}|}\, G(\zeta)G(\eta)d\zeta d\eta \le ⨍ G \log G\, d\zeta \tag{0.10}$$

valid for all $G > 0$ with the right hand side finite, and $⨍ G = 1$. The inequality is conformally invariant under the action $G \to (G\circ\tau)|J_\tau|$, and its extremals are the functions $|J_\tau|$, with τ any conformal transformation. The logarithmic kernel in (0.10) is a fundamental solution of $\mathcal{A}'_Q$ as an operator acting on CR-pluriharmonic functions:

$$(\mathcal{A}'_Q)^{-1}(\zeta,\eta) = -\frac{2}{\Gamma\big(\frac{Q}{2}\big)\omega_{2n+1}}\,\log|1-\zeta\cdot\overline{\eta}|.$$

In the Euclidean context (0.10) was obtained by Carlen and Loss [CL] from the sharp inequality (0.1), cast in its dual form, via endpoint differentiation. In some precise sense (0.10) and (0.8) are dual of one another. Finally we will derive an equivalent version of (0.10) on the Heisenberg group, using the conformal invariance of such inequality.

Ideas for related research

The inequality obtained by Beckner and Onofri turned out to be central in the problem of finding extremal geometries for the functional determinant of certain operators on compact Riemannian manifolds. We expect the same to be true in the case of CR geometry, namely that an explicit computation of functional determinants of conformally invariant operators, at least in low dimensions, would involve the functional in (0.8), and that (0.8) itself would be useful in solving extremal problems.

At the dual end, the third author has shown in [M1] that the logarithmic Hardy-Littlewood-Sobolev inequality on S^n was the analytic expression of an extremal problem for the regularized zeta function of the Paneitz operators. Likewise we expect the same to be true on the CR sphere.

We hope that the results presented in this paper will serve as an incentive to pursue these matters, and in particular to motivate the explicit calculation of functional determinants for low dimensional CR-manifolds.

In memory of Tom Branson.

Tom Branson wrote: *"What I have in mind is to generalize Beckner's sharp, invariant Moser-Trudinger inequality on S^n, which is a fact about conformal geometry, to a fact about CR geometry, and eventually other rank 1 and higher rank geometries"* [Br1]. Chang and Yang gave an alternative, symmetrization-free proof of Beckner's inequality on S^n; it was Branson's idea that we might attempt to *"play the same game"* on the CR sphere. *"This is not just any example; it's the one people will be by far most interested in, because of CR geometry"* [Br1]. The present paper is the result of our efforts to prove that Tom Branson's original intuition was indeed correct: yes, we can play the same game, but on the space of CR-pluriharmonic functions (and with considerably more difficulties).

Tom Branson suddenly passed away in March 2006.

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1. Intertwining operators on the CR sphere.

The Heisenberg group, the complex sphere and the Cayley transform

The Heisenberg group $\mathbb{H}^n$ is $\mathbb{C}^n \times \mathbb{R}$ with elements $u = (z,t)$, $z = (z_1, ..., z_n)$, and with group law

$$(z,t)(z',t') = (z + z', t + t' + 2\mathrm{Im}\, z \cdot \overline{z}')$$

where we set $z \cdot \overline{w} = \sum_1^n z_j \overline{w}_j$, for $w = (w_1, ..., w_n)$. The Lebesegue-Haar measure on $\mathbb{H}^n$ is denoted by $du = dzdt$.

Throughout the paper we will often use the standard notation for the homogeneous dimension of $\mathbb{H}^n$:

$$Q = 2n + 2.$$

The sphere S^{2n+1} is the boundary of the unit ball B of $\mathbb{C}^{n+1}$. In coordinates, $\zeta = (\zeta_1, ..., \zeta_{n+1}) \in S^{2n+1}$ if and only if $\zeta \cdot \overline{\zeta} = \sum_1^{n+1} |\zeta_j|^2 = 1$. The standard Euclidean volume element of S^{2n+1} will be denoted by $d\zeta$.

The Heisenberg group and the sphere are equivalent via the Cayley transform $\mathcal{C} : \mathbb{H}^n \to S^{2n+1} \setminus (0, 0, ..., 0, -1)$ given by

$$\mathcal{C}(z,t) = \Big(\frac{2z}{1+|z|^2+it}, \frac{1-|z|^2-it}{1+|z|^2+it}\Big)$$

and with inverse

$$\mathcal{C}^{-1}(\zeta) = \Big(\frac{\zeta_1}{1+\zeta_{n+1}}, ..., \frac{\zeta_n}{1+\zeta_{n+1}}, Im\frac{1-\zeta_{n+1}}{1+\zeta_{n+1}}\Big).$$

We will use the notation

$$\mathcal{N} = \mathcal{C}(0,0) = (0, 0, ..., 1).$$

The Jacobian determinant (really a volume density) of this transformation is given by

$$|J_{\mathcal{C}}(z,t)| = \frac{2^{2n+1}}{\big((1+|z|^2)^2+t^2\big)^{n+1}}$$

so that

$$\int_{S^{2n+1}} F d\zeta = \int_{\mathbb{H}^n} (F \circ \mathcal{C})|J_{\mathcal{C}}| dz dt$$

The homogeneous norm on $\mathbb{H}^n$ is defined by

$$|(z,t)| = (|z|^4 + t^2)^{1/4}$$

and the distance from $u = (z,t)$ and $v = (z', t')$ is

$$d((z,t),(z',t')) := |v^{-1}u| = \big(|z-z'|^4 + (t - t' - 2\mathrm{Im}\,(z \cdot \overline{z}'))^2\big)^{1/4}$$

On the sphere the distance function is defined as

$$d(\zeta,\eta)^2 := 2|1-\zeta\cdot\overline{\eta}| = \big|\,|\zeta-\eta|^2 - 2i\,\mathrm{Im}\,(\zeta\cdot\overline{\eta})\big| = \big(|\zeta-\eta|^4 + 4\cdot\mathrm{Im}^{\,2}(\zeta\cdot\overline{\eta})\big)^{1/2}$$

and a simple calculation shows that if $u = (z,t), v = (z',t')$, and $\zeta = \mathcal{C}(u)$, $\eta = \mathcal{C}(v)$. then

$$\frac{|1-\zeta\cdot\overline{\eta}|}{2} = |v^{-1}u|^2\big((1+|z|^2)^2+t^2\big)^{-1/2}\big((1+|z'|^2)^2+(t')^2\big)^{-1/2} \qquad (1.1)$$

i.e.

$$d(\zeta,\eta) = d(u,v)\Big(\frac{4}{(1+|z|^2)^2+t^2}\Big)^{1/4}\Big(\frac{4}{(1+|z'|^2)^2+(t')^2}\Big)^{1/4}. \tag{1.2}$$

Sublaplacians on $\mathbb{H}^n$ and S^{2n+1}.

The sublaplacian on $\mathbb{H}^n$ is the second order differential operator

$$\mathcal{L}_0 = -\frac{1}{4}\sum_{j=1}^{n}(X_j^2+Y_j^2)$$

where $X_j = \dfrac{\partial}{\partial x_j} + 2y_j\dfrac{\partial}{\partial t}$, $Y_j = \dfrac{\partial}{\partial y_j} - 2x_j\dfrac{\partial}{\partial t}$, and $\dfrac{\partial}{\partial t}$ denote the basis of the space of left-invariant vector fields on $\mathbb{H}^n$. It's easy to check that

$$\mathcal{L}_0 = -\frac{1}{2}\sum_{j=1}^{n}(Z_j\overline{Z}_j + \overline{Z}_jZ_j)$$

where

$$Z_j = \frac{\partial}{\partial z_j} + i\overline{z}_j\frac{\partial}{\partial t}, \quad \overline{Z}_j = \frac{\partial}{\partial \overline{z}_j} - iz_j\frac{\partial}{\partial t}$$

and with $\dfrac{\partial}{\partial z_j} = \dfrac{1}{2}\Big(\dfrac{\partial}{\partial x_j} - i\dfrac{\partial}{\partial y_j}\Big)$, $\dfrac{\partial}{\partial \overline{z}_j} = \dfrac{1}{2}\Big(\dfrac{\partial}{\partial x_j} + i\dfrac{\partial}{\partial y_j}\Big)$.

The fundamental solution of $\mathcal{L}_0$ was computed by Folland [Fo1] and

$$\mathcal{L}_0^{-1}(u,v) = C_2\, d(u,v)^{2-Q}, \qquad C_2 = \frac{2^{n-2}\Gamma(\frac{n}{2})^2}{\pi^{n+1}}$$

so that

$$G(u) = \int_{\mathbb{H}^n} C_2|v|^{2-Q}F(v^{-1}u)dv = \int_{\mathbb{H}^n}\mathcal{L}_0^{-1}(u,v)F(v)dv$$

solves $\mathcal{L}_0G = F$.

On the standard sphere, the sublaplacian is defined similarly as

$$\mathcal{L} = -\frac{1}{2}\sum_{j=1}^{n+1}(T_j\overline{T}_j + \overline{T}_jT_j)$$

where

$$T_j = \frac{\partial}{\partial \zeta_j} - \overline{\zeta}_j\mathcal{R}, \qquad \mathcal{R} = \sum_{k=1}^{n+1}\zeta_k\frac{\partial}{\partial \zeta_k}, \tag{1.3}$$

and where the T_j generate the holomorphic tangent space $T_{1,0}S^{2n+1} = T_{1,0}\mathbb{C}^{n+1} \cap \mathbb{C}TS^{2n+1}$. Explicitly

$$\mathcal{L} = -\Delta + \sum_{j,k=1}^{n+1} \zeta_j \overline{\zeta}_k \frac{\partial^2}{\partial \zeta_j \partial \overline{\zeta}_k} + \frac{n}{2}(\mathcal{R} + \overline{\mathcal{R}}) \tag{1.4}$$

with $\Delta = \sum_j \dfrac{\partial^2}{\partial \zeta_j \partial \overline{\zeta}_j}$. The trasversal direction is the real vector field

$$\mathcal{T} = \frac{i}{2}(\mathcal{R} - \overline{\mathcal{R}}) = \frac{i}{2} \sum_{j=1}^{n+1} \left(\zeta_j \frac{\partial}{\partial \zeta_j} - \overline{\zeta}_j \frac{\partial}{\partial \overline{\zeta}_j} \right) \tag{1.5}$$

and $\mathbb{C}TS^{2n+1}$ is generated by the $T_j, \overline{T}_j, \mathcal{T}$.

The *conformal sublaplacian* on the sphere is defined as

$$\mathcal{D} = \mathcal{L} + \frac{n^2}{4}.$$

The fundamental solution of $\mathcal{D}$ has been computed by Geller [Ge] (Thm. 2.1 with $\alpha = 0$ and modulo volume normalization)

$$\mathcal{D}^{-1}(\zeta, \eta) = c_2 \, d(\zeta, \eta)^{2-Q}, \qquad c_2 = \frac{2^{n-1}\Gamma(\frac{n}{2})^2}{\pi^{n+1}} = 2C_2 \tag{1.6}$$

in the sense that for smooth $F : S^{2n+1} \to \mathbb{C}$ the function

$$G(\zeta) = \mathcal{D}^{-1}F(\zeta) = \int_{S^{2n+1}} c_2 d(\zeta, \eta)^{2-Q} F(\eta) d\eta$$

satisfies $\mathcal{D}G = F$.

The peculiarity of $\mathcal{D}$ is its direct relation with $\mathcal{L}_0$ via the Cayley transform:

$$\mathcal{L}_0\Big((2|J_\mathcal{C}|)^{\frac{Q-2}{2Q}} (F \circ \mathcal{C}) \Big) = (2|J_\mathcal{C}|)^{\frac{Q+2}{2Q}} (\mathcal{D}F) \circ \mathcal{C} \tag{1.7}$$

which can be readily established by using the explicit formulas for the fundamental solutions and (1.2). The multiplicative factor 2 in the above formula appears because we use the standard volume elements for $\mathbb{H}^n$ and S^{2n+1} instead of the volume forms associated with the standard contact forms θ_0, and θ of these two spaces. In this case indeed we have that

$$\int_{\mathbb{H}^n} f \, \theta_0 \wedge d\theta_0 ... \wedge d\theta_0 = 2^{2n} n! \int_{\mathbb{H}^n} f dz dt = \int_{S^{2n+1}} F \, \theta \wedge d\theta ... \wedge d\theta = 2^{2n+1} n! \int_{S^{2n+1}} F d\zeta$$

where $f = (F \circ \mathcal{C})(2|J_{\mathcal{C}}|)$ (see Jerison-Lee [JL1)]. This also accounts for the factor 2 in the relation $c_2 = 2C_2$.

Spherical and zonal harmonics on the CR sphere.

The Hilbert space $L^2(S^{2n+1})$, endowed with the inner product

$$(F, G) = \int_{S^{2n+1}} F\overline{G}\, d\zeta$$

can be decomposed as $L^2(S^{2n+1}) = \bigoplus_{j,k\geq 0} \mathcal{H}_{j,k}$, where $\mathcal{H}_{j,k}$ is the space of harmonic polynomials on $\mathbb{C}^{n+1}$ which are homogeneous of degree j, k in the ζ's and $\overline{\zeta}$'s respectively, and restricted to the sphere. The dimension of $\mathcal{H}_{j,k}$ is

$$\dim(\mathcal{H}_{j,k}) = m_{jk} := \frac{(j+n-1)!(k+n-1)!(j+k+n)}{n!(n-1)!j!k!} \tag{1.8}$$

and if $\{Y_{jk}^{\ell}\}$ is an orthonormal basis of $\mathcal{H}_{jk}$ then the zonal harmonics are defined as

$$\Phi_{j,k}(\zeta, \eta) = \sum_{\ell=1}^{m_{jk}} Y_{jk}^{\ell}(\zeta)\overline{Y_{jk}^{\ell}(\eta)}$$

The $\Phi_{j,k}$ are invariant under the transitive action of $U(n)$ and it turns out that

$$\Phi_{jk}(\zeta, \eta) = \Phi_{jk}(\zeta \cdot \overline{\eta}) := \frac{(j+n-1)!(j+k+n)}{\omega_{2n+1} n! j!} (\zeta \cdot \overline{\eta})^{j-k} P_j^{(n-1, j-k)}(2|\zeta \cdot \overline{\eta}|^2 - 1) \tag{1.9}$$

if $k \leq j$, and $\Phi_{jk}(\zeta, \eta) = \Phi_{jk}(\zeta \cdot \overline{\eta}) := \overline{\Phi_{kj}(\zeta \cdot \overline{\eta})}$, if $j \leq k$, where $P_j^{(n,\ell)}$ are the Jacobi polynomials (see [VK], Section 11.3.2).

In particular, since $P_0^{(n-1,j)} \equiv 1$ we have also

$$\Phi_{j0}(\zeta \cdot \overline{\eta}) = \frac{(j+n)!}{j! n! \omega_{2n+1}} (\zeta \cdot \overline{\eta})^j = \frac{\Gamma\big(j + \frac{Q}{2}\big)}{\Gamma(j+1)\Gamma\big(\frac{Q}{2}\big)\omega_{2n+1}} (\zeta \cdot \overline{\eta})^j \tag{1.10}$$

and $\Phi_{0k}(\zeta \cdot \overline{\eta}) = \overline{\Phi_{k0}(\zeta \cdot \overline{\eta})} = \Phi_{k0}(\overline{\zeta} \cdot \eta)$.

If $F \in L^2$ then

$$F(\zeta) = \sum_{j,k\geq 0} \int_{S^{2n+1}} F(\eta) \Phi_{jk}(\zeta \cdot \overline{\eta}) d\eta$$

the series being convergent in L^2.

Hardy spaces and CR-pluriharmonic functions.

In the sequel we will use the following notations

$\mathcal{H} = \bigoplus_{j\geq 0}\mathcal{H}_{j0} = \{L^2 \text{ boundary values of holomorphic functions on the unit ball }\}$

$\overline{\mathcal{H}} = \bigoplus_{j\geq 0}\mathcal{H}_{0j} = \{L^2 \text{ boundary values of antiholomorphic functions on the unit ball }\}$

$\mathcal{P} = \mathcal{H}\oplus\overline{\mathcal{H}} = \{L^2 \text{ CR-pluriharmonic functions }\}$

$\mathbb{R}\mathcal{P} = \{L^2 \text{ real-valued CR-pluriharmonic functions }\}$

$\mathcal{H}_0,\ \overline{\mathcal{H}}_0,\ \mathcal{P}_0,\ \mathbb{R}\mathcal{P}_0 =$ functions in $\mathcal{H}$, $\overline{\mathcal{H}}$, $\mathcal{P}$, $\mathbb{R}\mathcal{P}$ with 0 mean.

The space $\mathcal{H}$ is the classical Hardy space for the boundary of the unit ball of $\mathbb{C}^{n+1}$. The Cauchy-Szego projection $\pi_0 : L^2(S^{2n+1}) \to \mathcal{H}$ is given by the Cauchy-Szego kernel

$$K(\zeta,\eta) = \frac{1}{\omega_{2n+1}(1-\zeta\cdot\overline{\eta})^{n+1}} = \sum_{j\geq 0}\Phi_{j0}(\zeta\cdot\overline{\eta}).$$

The projection operator on $\mathcal{P}$

$$\pi : L^2(S^{2n+1}) \to \mathcal{P}$$

has kernel $2\mathrm{Re}\,K(\zeta,\eta) - \dfrac{1}{\omega_{2n+1}}$. Denote by $\mathcal{P}^\perp$ the orthogonal complement of $\mathcal{P}$, with respect to the standard Hermitian product $\zeta\cdot\overline{\eta}$, i.e.

$$L^2(S^{2n+1}) = \mathcal{P}\oplus\mathcal{P}^\perp.$$

The Hardy spaces for $p>1$ are defined similarly. $\mathcal{H}^p$ will denote the L^p closure of boundary values of holomorphic functions on the unit ball, and likewise for all the other spaces $\mathcal{H}^p_0$, $\mathcal{P}^p$, $\mathcal{P}^p_0, \ldots$ etc. The Cauchy-Szego projection π sends L^p into $\mathcal{H}^p$ boundedly.

Sobolev spaces.

The Sobolev, or Folland-Stein, spaces on $\mathbb{H}^n$ and S^{2n+1} can be defined in terms of the powers of the corresponding conformal sublaplacians. The main references here are for example [ACDB], [ADB], [Fo2]. We summarize the main properties below.

It is well known (see e.g. [St]) that for $Y_{jk}\in\mathcal{H}_{jk}$

$$\mathcal{D}Y_{jk} = \lambda_j\lambda_k Y_{jk}, \qquad \lambda_j = j+\frac{n}{2} \tag{1.11}$$

For $F\in L^2(S^{2n+1})$, we can write $F=\sum_{j,k\geq 0}\sum_{\ell=1}^{m_{jk}} c^\ell_{jk}(F)Y^\ell_{jk}$, and $c^\ell_{jk}(F)=\int FY^\ell_{jk}$; in particular, if $F\in C^\infty(S^{2n+1})$ then (1.11) implies that

$$\sum_{j,k\geq 0}\sum_{\ell=1}^{m_{jk}}(\lambda_j\lambda_k)^d|c^\ell_{jk}(F)|^2 < \infty. \tag{1.12}$$

For $F \in C^\infty(S^{2n+1})$ we then define for any $d \in \mathbb{R}$

$$\mathcal{D}^{d/2}F = \sum_{j,k\geq 0}\sum_{\ell=1}^{m_{jk}} (\lambda_j\lambda_k)^{d/2} c^{\ell}_{jk}(F) Y^{\ell}_{jk} \tag{1.13}$$

so that $\mathcal{D}^{d/2}$ extends naturally to the space of distributions on the sphere. For $d>0$, $p\geq 1$ we let

$$W^{d,p} = \{F \in L^p : \mathcal{D}^{d/2}F \in L^p\}$$

endowed with norm

$$\|F\|_{W^{d,p}} = \|\mathcal{D}^{d/2}F\|_p;$$

the space $W^{d,p}$ is the completion of $C^\infty(S^{2n+1})$ under such norm.

$W^{d,2}$ is the space of F in L^2 so that (1.12) and (1.13) hold, and it's a Hilbert space with inner product and norm

$$(F,G)_{W^{d,2}} = \int_{S^{2n+1}} \mathcal{D}^{d/2}F\, \overline{\mathcal{D}^{d/2}G}, \qquad \|F\|_{W^{d,2}} = (F,F)^{1/2}_{W^{d,2}}.$$

Clearly $\|(I+\mathcal{L})^{d/2}\|_2$ yields an equivalent norm on $W^{d,2}$. Also, if L^2_d denotes the classical Sobolev space on S^{2n+1}, defined as above but using the Laplace-Beltrami Δ rather than $\mathcal{D}$, and with norm $\|F\|_{L^2_d} = \|(I+\Delta)^{d/2}F\|_2$, then

$$L^2_d \hookrightarrow W^{d,2} \hookrightarrow L^2_{d/2}$$

in fact

$$c_1\|F\|_{L^2_{d/2}} \leq \|F\|_{W^{d,2}} \leq c_2\|F\|_{L^2_d}$$

for some $c_1, c_2 > 0$, as one can easily see by comparing the eingenvalues of $\mathcal{D}$ with those of $I+\Delta$ (i.e. $1+(j+k)(j+k+2n)$).

The dual of $W^{d,2}$ is the space of distributions

$$(W^{d,2})' = \{\mathcal{D}^{d/2}F, F \in L^2\}$$

and it coincides with $W^{-d,2}$ defined as the space of distributions T such that $\mathcal{D}^{-d/2}T \in L^2$.

The operators $\mathcal{D}^{d/2}$ and $\mathcal{L}^{d/2}$ are positive and self-adjoint in their domain $W^{d,2}$. The quadratic form $(\mathcal{D}^{d/4}F, \mathcal{D}^{d/4}G)$ allows us to further extend $\mathcal{D}^{d/2}$ and $\mathcal{L}^{d/2}$ to operators defined on $W^{d/2,2}$ (the form domain) valued in $W^{-d/2,2}$. In the sequel we will denote such extensions by $\mathcal{D}^{d/2}$, $\mathcal{L}^{d/2}$, with domain $W^{d/2,2}$.

On the Heisenberg group the Sobolev spaces are defined analogously as the completion of $C^\infty_c(\mathbb{H}^n)$ under the norm $\|(I+\mathcal{L}_0)^{d/2}\|_2$. The resulting space is still denoted by $W^{d,2}$.

The group $SU(n+1,1)$ acts as a group of conformal transformations on S^{2n+1}, and therefore on $\mathbb{H}^n$ by means of the Cayley projection (see [KR1-2]). Recall that a conformal (or contact) transformation, is a diffeomorphism $h:\mathbb{H}^n\to\mathbb{H}^n$ that preserves the contact structure, i.e. if θ_0 is a contact form, then $h^*\theta_0=|J_h|^{2/Q}\theta_0$, where $|J_h|$ is the Jacobian determinant of h. An analogue of the Euclidean Liouville's theorem holds: every C^4 conformal mapping on $\mathbb{H}^n$ comes from the action of an element of $SU(n+1,1)$, and it can be written as composition of

$$\begin{aligned}
\text{left translations}\quad & (z,t)\to(z',t')(z,t)\\
\text{dilations}\quad & (z,t)\to(\delta z,\delta^2 t),\;\;\delta>0\\
\text{rotations}\quad & (z,t)\to(Rz,t),\;\;R\in U(n)\\
\text{inversion}\quad & (z,t)\to\Big(-\frac{z}{|z|^2+it},-\frac{t}{|z|^4+t^2}\Big).
\end{aligned}$$

Let us denote the spaces of conformal transformations of $\mathbb{H}^n$ by $\mathrm{CON}(\mathbb{H}^n)$, and the space of conformal tranformations of S^{2n+1} by $\mathrm{CON}(S^{2n+1}):=\{\tau:\tau=\mathcal{C}\circ h\circ\mathcal{C}^{-1}$ some $h\in\mathrm{CON}(\mathbb{H}^n)\}$. Note that the inversion on $\mathbb{H}^n$ corresponds to the antipodal map $\zeta\to-\zeta$ on S^{2n+1}.

The functions $|J_h|$ with $h\in\mathrm{CON}(\mathbb{H}^n)$, are obtained from $|J_{\mathcal{C}}|$ by left translations and dilations and can be written as (cf [JL2])

$$|J_h(u)|=\frac{C}{\big||z|^2+it+2z\cdot w+\lambda\big|^Q},\qquad C>0,\,w\in\mathbb{C}^n,\,\lambda\in\mathbb{C},\,\mathrm{Re}\,\lambda>|w|^2,\,u=(z,t)\in\mathbb{H}^n.$$

From this formula it's easy to see that the family of functions $|J_\tau|$ with $\tau\in\mathrm{CON}(S^{2n+1})$ can be parametrized as

$$|J_\tau(\zeta)|=\frac{C}{|1-\omega\cdot\zeta|^Q},\quad C>0,\,\omega\in\mathbb{C}^{n+1},\,|\omega|<1,\,\zeta\in S^{2n+1}. \tag{1.14}$$

The following formulas hold:

$$\begin{aligned}
d(h(u),h(v))&=d(u,v)|J_h(u)|^{\frac{1}{2Q}}|J_h(v)|^{\frac{1}{2Q}},\qquad \forall h\in\mathrm{CON}(\mathbb{H}^n)\\
d(\tau(\zeta),\tau(\eta))&=d(\zeta,\eta)|J_\tau(\zeta)|^{\frac{1}{2Q}}|J_\tau(\eta)|^{\frac{1}{2Q}},\qquad \forall \tau\in\mathrm{CON}(S^{2n+1})
\end{aligned}\tag{1.15}$$

These formulas are trivially checked on traslations, rotations, dilations of $\mathbb{H}^n$, and on the inversion of S^{2n+1}; using (1.2) one can cover the remanining cases.

The operators $\mathcal{L}_0$ and $\mathcal{D}$ are intertwining in the sense that for each $f,F\in W^{1,2}$

$$|J_h|^{(Q+2)/(2Q)}(\mathcal{L}_0 f)\circ h=\mathcal{L}_0\big(|J_h|^{(Q-2)/(2Q)}(f\circ h)\big),\;\;\forall h\in\mathrm{CON}(\mathbb{H}^n)$$

$$|J_\tau|^{(Q+2)/(2Q)}(\mathcal{D}F)\circ\tau = \mathcal{D}\big(|J_\tau|^{(Q-2)/(2Q)}(F\circ\tau)\big),\ \ \forall\tau\in \mathrm{CON}(S^{2n+1}) \tag{1.16}$$

To check these formulas it's enough to rewrite them in terms of the inverse operators $\mathcal{L}_0^{-1}$, $\mathcal{D}^{-1}$, and then use the explicit formulas for their kernels and (1.15).

For $0 < d < Q$ the general intertwining operator $\mathcal{A}_d$ of order d is defined by the following property:

$$|J_\tau|^{(Q+d)/(2Q)}(\mathcal{A}_dF)\circ\tau = \mathcal{A}_d\big(|J_\tau|^{(Q-d)/(2Q)}(F\circ\tau)\big),\ \ \forall\tau\in \mathrm{CON}(S^{2n+1}) \tag{1.17}$$

for each $F\in C^\infty(S^{2n+1})$ and hence for each $F\in W^{d/2,2}$ (in fact for each distribution F). In other words, the natural pullback of $\mathcal{A}_d$ by a conformal transformation τ satisfies

$$\tau^*\mathcal{A}_d(\tau^{-1})^* = |J_\tau|^{-(Q+d)/(2Q)}\mathcal{A}_d|J_\tau|^{(Q-d)/(2Q)}$$

where $\tau^*F = F\circ\tau$.

The concept of intertwining operator is more properly understood in the context of representation theory of semisimple Lie groups, in our case $SU(n+1,1)$, see e.g. [Br], [BOØ],[C], [JW]. In particular, for $d\in\mathbb{C}$ the map $u_d:\tau\to\big\{F\to|J_\tau|^{(Q+d)/(2Q)}(F\circ\tau)\big\}$ is a representation of the group $SU(n+1,1)$, modulo identification of the latter with $\mathrm{CON}(S^{2n+1})$; these u_d are known as *principal series representations* of $SU(n+1,1)$, and the ones corresponding to $d\in(-Q,Q)$ are called *complementary series*. The relation (1.17) says that $\mathcal{A}_d$ intertwines the representations u_d and u_{-d}. The present formulation is given in elementary differential-geometric terms, which for our purposes is more than enough (see however [Br], pp 18-19, for a digression on the u_d in more Lie-theoretic language).

It is known, from the above works (see also Appendix A, Prop. A.1), that an operator satisfying (1.17) is diagonal w.r. to the spherical harmonics, and its spectrum is completely determined up to a multiplicative constant by the functions

$$\lambda_j(d) = \frac{\Gamma\big(\frac{Q+d}{4}+j\big)}{\Gamma\big(\frac{Q-d}{4}+j\big)} \sim j^{d/2} \tag{1.18}$$

in the sense that up to a constant the spectrum is precisely $\{\lambda_j(d)\lambda_k(d)\}$. From now on we will choose such constant to be 1, i.e. $\mathcal{A}_d$ will be the operator on $W^{d,2}$ such that

$$\mathcal{A}_dY_{jk} = \lambda_j(d)\lambda_k(d)Y_{jk}, \qquad Y_{jk}\in\mathcal{H}_{jk} \tag{1.19}$$

The form $(\mathcal{A}_d^{1/2}F,\mathcal{A}_d^{1/2}G)$ allows us to extend $\mathcal{A}_d$ to an operator with domain $W^{d/2,2}$ valued in $W^{-d/2,2}$, which we still denote by $\mathcal{A}_d$. The eigenvalues of such operators are still $\lambda_j(d)\lambda_k(d)$, i.e. (1.19) holds, in the sense of forms. Since $\lambda_j(d)>0$ for all $j\geq 0$ then $\mathrm{Ker}\mathcal{A}_d=\{0\}$, and eigenvalue estimate shows easily that $\|\mathcal{A}_{d/2}F\|_2$ or $\|(\mathcal{A}_d)^{1/2}\|_2$ are equivalent to $\|F\|_{W^{d,2}}$, for $0<d<Q$. Observe that in the case $d=2$ we have $\lambda_j(2)=\lambda_j=j+\frac{n}{2}$, and we recover the conformal sublaplacian i.e.

$$\mathcal{A}_2=\mathcal{D}.$$

A fundamental solution of $\mathcal{A}_d$ is given by

$$G_d(\zeta,\eta) := \mathcal{A}_d^{-1}(\zeta,\eta) = \sum_{j,k\geq 0} \frac{\Phi_{jk}(\zeta\cdot\overline{\eta})}{\lambda_j(d)\lambda_k(d)} = c_d\, d(\zeta,\eta)^{d-Q} \tag{1.20}$$

with

$$c_d = \frac{2^{n-\frac{d}{2}}\,\Gamma\big(\frac{Q-d}{4}\big)^2}{\pi^{n+1}\Gamma\big(\frac{d}{2}\big)} \tag{1.21}$$

and where the series converges unconditionally in the sense of distributions, and also in L^2 if $Q/2 < d < Q$. The proof of (1.20) is somehow implicit in the work of Johnson and Wallach [JW], and a similar formula (still quoted from [JW]) appears in [ACDB] (formula (11)), but with different normalizations. The case d an even integer was treated by Graham [Gr], including the expression for the fundamental solution. For the reader's sake in Appendix A we offer a self-contained proof of the spectral characterization of intertwining operators, in the sense of (1.17), and of formula (1.20), using only the explicit knowledge of the zonal harmonics. We note here (but see also Appendix A) that the intertwining property can be checked directly using (1.20) and formulas (1.15), after casting (1.17) in terms of the inverse $\mathcal{A}_d^{-1}$.

We shall be concerned with the intertwining, Paneitz-type operators of order Q. Noticing that

$$\lambda_0(d) = \frac{\Gamma\big(\frac{Q+d}{4}\big)}{\Gamma\big(\frac{Q-d}{4}\big)} \sim \frac{Q-d}{4}\Gamma\big(\frac{Q}{2}\big), \quad d\to Q \tag{1.22}$$

we easily obtain from (1.17) that the operator $\mathcal{A}_Q : W^{d,2}\to\mathcal{P}^\perp$ defined as

$$\mathcal{A}_Q F := \lim_{d\to Q} \mathcal{A}_d F \tag{1.23}$$

the limit being in L^2, satisfies for $F\in W^{Q,2}$

$$|J_\tau|(\mathcal{A}_Q F)\circ\tau = \mathcal{A}_Q(F\circ\tau),\ \ \forall\tau\in \mathrm{CON}(S^{2n+1}) \tag{1.24}$$

or

$$\tau^*\mathcal{A}_Q(\tau^{-1})^* = |J_\tau|^{-1}\mathcal{A}_Q. \tag{1.25}$$

The operator $\mathcal{A}_Q$ can be extended via its quadratic form to an operator, still denoted by $\mathcal{A}_Q$, with domain $W^{Q/2,2}$, kernel $\mathrm{Ker}\mathcal{A}_Q=\mathcal{P}$, valued in $\big(W^{Q/2,2}\big)'=W^{-Q/2,2}$. The identity (1.24) is still valid for $F\in W^{Q/2,2}$ and

$$\mathcal{A}_Q Y_{jk} = \lambda_j(Q)\lambda_k(Q)Y_{jk} = j(j+1)...(j+n)k(k+1)...(k+n)Y_{jk}.$$

Observe that $\|(I+\mathcal{A}_Q)^{1/2}F\|_2$ is equivalent to $\|F\|_{W^{Q/2,2}}$ on the space $W^{Q/2,2}\cap\mathcal{P}^\perp$.

In the case d an even integer it is possible to write down a more explicit formula for $\mathcal{A}_d$ as a product of Geller's type operators. In fact, we can recover the operators found by Graham in [Gr]:

Proposition 1.1. *If $d \le Q$ is an even integer, then $\mathcal{A}_d$ is a differential operator and*

$$\mathcal{A}_d = \begin{cases} \prod_{\ell=0}^{\frac{d}{4}-1} \Big(\mathcal{D} - \frac{(2\ell+1)^2}{4} + i(2\ell+1)\mathcal{T}\Big)\Big(\mathcal{D} - \frac{(2\ell+1)^2}{4} - i(2\ell+1)\,\mathcal{T}\Big) & \text{if } \frac{d}{4} \in \mathbb{N} \\ \mathcal{D} \prod_{\ell=0}^{\frac{d-2}{4}} \Big(\mathcal{D} - \ell^2 + 2i\ell\mathcal{T}\Big)\Big(\mathcal{D} - \ell^2 - 2i\ell\,\mathcal{T}\Big) & \text{if } \frac{d-2}{4} \in \mathbb{N}. \end{cases}$$

Proof. We have

$$\lambda_j(d) = \prod_{\ell=0}^{\frac{d}{2}-1} \big(\lambda_j + \ell - \tfrac{d}{4} + \tfrac{1}{2}\big)$$

from which it's easy to check that (recall: $\lambda_j = j + \frac{n}{2}$)

$$\lambda_j(d)\lambda_k(d) = \begin{cases} \prod_{\ell=0}^{\frac{d}{4}-1} \big(\lambda_j^2 - (\ell+\tfrac{1}{2})^2\big)\big((\lambda_k^2 - (\ell+\tfrac{1}{2})^2\big) & \text{if } \frac{d}{4} \in \mathbb{N} \\ \lambda_j\lambda_k \prod_{\ell=0}^{\frac{d-2}{4}} \big(\lambda_j^2 - \ell^2\big)\big(\lambda_k^2 - \ell^2\big) & \text{if } \frac{d-2}{4} \in \mathbb{N}. \end{cases}$$

The proof is completed noticing that $\mathcal{T}Y_{jk} = \frac{i}{2}(j-k)Y_{jk}$, for $Y_{jk} \in \mathcal{H}_{jk}$, and that $\big(\lambda_j^2 - b^2\big)\big(\lambda_k^2 - b^2\big) = \big(\lambda_j\lambda_k - b^2 + b(j-k)\big)\big(\lambda_j\lambda_k - b^2 - b(j-k)\big)$.

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Note in particular that when $d = 4$

$$\mathcal{A}_4 = \Big(\mathcal{L} + \frac{n^2-1}{4}\Big)^2 + \mathcal{T}^2.$$

Also, note that since $\mathcal{T}^2 = -|\mathcal{T}|^2$ then on can isolate the highest order derivatives in the above expression, counting T as an operator of order 2, and obtain

$$\mathcal{A}_d = |2\mathcal{T}|^{d/2} \frac{\Gamma\big(\mathcal{L}|2\mathcal{T}|^{-1} + \frac{2+d}{4}\big)}{\Gamma\big(\mathcal{L}|2\mathcal{T}|^{-1} + \frac{2-d}{4}\big)} + \text{ lower order derivatives.} \tag{1.26}$$

The formula above needs of course to be suitably interpreted, as $\mathcal{T}$ is invertible only on the space $\bigoplus_{j\neq k} \mathcal{H}_{jk}$. For d not an even integer, we speculate that there might still be a way to make sense out of (1.26), as the "leading operator" appearing in that formula, has the same form as the intertwinor on the Heisenberg group (see (1.33)).

Remark. It is possible to show that a fundamental solution for $\mathcal{A}_Q:\mathcal{P}^\perp\to\mathcal{P}^\perp$ is given by

$$\mathcal{A}_Q^{-1}(\zeta,\eta)=\frac{2}{\omega_{2n+1}\Gamma\big(\frac{Q}{2}\big)^2}\ \log^2\frac{d^2(\zeta,\eta)}{2}$$

(up to a CR-pluriharmonic function). This calculation can be effected using the explicit formula for the fundamental solution of $\mathcal{A}_d$, and differentiating twice with respect to d at $d=Q$ (note that the constant c_d has a pole of order two ad $d=Q$).

Conditional intertwinors.

Of particular importance for us, is the existence of another intertwinor of order Q defined on $\mathcal{P}$, which we call the *conditional intertwinor* or *Paneitz operator* on $\mathcal{P}$. This is defined on $C^\infty(S^{2n+1})$ (and hence on the space of distributions) by its action on the spherical harmonics in the following way:

$$\mathcal{A}'_QY_{j0}=\lambda_j(Q)Y_{j0}=j(j+1)...(j+n)Y_{j0},\qquad \mathcal{A}'_QY_{0k}=\lambda_k(Q)Y_{0k}\tag{1.27}$$

and $\mathcal{A}'_QY_{jk}=0$, if $jk>0$. Observe that $\|(I+\mathcal{A}'_Q)^{1/2}F\|_2$ is equivalent to $\|F\|_{W^{Q/2,2}}$ on $W^{Q/2,2}\cap\mathcal{P}$.

We summarize the properties of $\mathcal{A}'_Q$ in the following proposition.

Proposition 1.2. *$\mathcal{A}'_Q$ as defined as in (1.27) is a positive semidefinite, self-adjoint operator, with* $\mathrm{Ker}\mathcal{A}'_Q=\mathcal{P}_0^\perp$. *For each $F\in W^{Q/2,2}\cap\mathcal{P}$ we have*

$$\mathcal{A}'_QF=-\frac{4}{\Gamma\big(\frac{Q}{2}\big)}\frac{\partial}{\partial d}\Big|_{d=Q}(\mathcal{A}_dF)=\lim_{d\to Q}\frac{1}{\lambda_0(d)}\mathcal{A}_dF\tag{1.28}$$

and for every $\tau\in\mathrm{CON}(S^{2n+1})$

$$|J_\tau|(\mathcal{A}'_QF)\circ\tau=\mathcal{A}'_Q(F\circ\tau)+\frac{2}{Q\Gamma\big(\frac{Q}{2}\big)}\mathcal{A}_Q\big(\log|J_\tau|(F\circ\tau)\big).\tag{1.29}$$

Moreover, $\mathcal{A}'_Q$ is a differential operator with

$$\mathcal{A}'_QF=\prod_{\ell=0}^{n}\big(2|\mathcal{T}|+\ell\big)F=\prod_{\ell=0}^{n}\big(\tfrac{2}{n}\mathcal{L}+\ell\big)F,\qquad \forall F\in C^\infty(S^{2n+1})\cap\mathcal{P}\tag{1.30}$$

and it is injective on $\mathcal{P}_0$ with fundamental solution

$$G'_Q(\zeta,\eta):=(\mathcal{A}'_Q)^{-1}(\zeta,\eta)=-\frac{2}{n!\,\omega_{2n+1}}\ \log\frac{d^2(\zeta,\eta)}{2}.\tag{1.31}$$

Note that (1.29) says that the intertwining property in the form (1.24) or (1.25) continues to hold for $\mathcal{A}'_Q$, but modulo distributions that annihilate $\mathcal{P}$ (or modulo functions in $\mathcal{P}^\perp$, if $F \in W^{Q,2}$). Also, $\mathcal{A}'_Q$ is an intertwining operator if seen as an operator from $\mathcal{P}$ to $L^2/\mathcal{P}^\perp$. In particular, the representations intertwined by $\mathcal{A}'_Q$ are the standard shift $\tau \to \{F \to F \circ \tau\}$, on $\mathcal{P}$, and $\tau \to \{\,[F] \to [\,(F \circ \tau)|J_\tau|\,]\,\}$ on $L^2/\mathcal{P}^\perp$.

Proof. The eigenvalues of $\mathcal{A}'_Q$ vanish when $j = 0$ hence $\mathrm{Ker}\mathcal{A}'_Q = \mathcal{P}_0^\perp$. The first identity follows easily from (1.22). To prove (1.29), it's enough to take the d-derivative at Q of (1.17):

$$|J_h|(\mathcal{A}'_Q F) \circ \tau - \frac{2}{Q\Gamma\big(\frac{Q}{2}\big)}|J_\tau| \log |J_\tau|(\mathcal{A}_Q F) \circ \tau = \mathcal{A}'_Q(F \circ \tau) + \frac{2}{Q\Gamma\big(\frac{Q}{2}\big)}\mathcal{A}_Q\big(\log |J_\tau|(F \circ \tau)\big)$$

for each $F \in C^\infty(S^{2n+1})$. We can trivially check (1.30) when F is a spherical harmonic. The last statement (1.31) follows from the formula

$$G'_Q(\zeta, \eta) = 2\mathrm{Re} \sum_{j=1}^{\infty} \sum_{\ell=1}^{m_j} \frac{1}{\lambda_j(Q)} Y^\ell_{j,0} = \frac{2}{\Gamma\big(\frac{Q}{2}\big)\omega_{2n+1}}\, \mathrm{Re} \sum_{j=1}^{\infty} \frac{(\zeta \cdot \overline{\eta})^j}{j}.$$

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Intertwining operators on the Heisenberg group.

For completeness we say a few words for the case of the intertwining operators on $\mathbb{H}^n$.

We already know from (1.7) that there is a direct connection between $\mathcal{A}_2 = \mathcal{D}$ and $\mathcal{L}_0$, via the Cayley transform. To find the analogue situation for $\mathcal{A}_d$ one basically has to find the operator on $\mathbb{H}^n$ with fundamental solution $|u|^{d-Q}$, since this operator is easily checked to be intertwining. This has been done by Cowling [C] and the result can be formulated as follows. Consider the $U(n)-$spherical functions

$$\Phi_{\lambda,k}(z,t) = e^{i\lambda t - |\lambda|\,|z|^2} L_k^{n-1}(|\lambda\,|z|^2), \qquad \lambda \neq 0,\ k \in \mathbb{N}$$

where L_k^{n-1} denote the classical Laguerre polynomial of degree k and order $n-1$. These are the eigenfunctions of the sublaplacian $\mathcal{L}_0$ and of $T = \partial_t$:

$$\mathcal{L}_0 \Phi_{\lambda,k} = |\lambda|(2k+n)\Phi_{\lambda,k}, \qquad T\Phi_{\lambda,k} = i\lambda \Phi_{\lambda,k}.$$

On $\mathbb{H}^n$ there is a notion of "group Fourier transform", which on radial functions (i.e. functions depending only on $|z|$ and t) takes the form

$$\widehat{f}(\lambda, k) = \int_{\mathbb{H}^n} \Phi_{\lambda,k}(z,t) f(z,t)\, dzdt, \qquad f \in L^1(\mathbb{H}^n).$$

With this notation we have

$$\widehat{\mathcal{L}_0 f}(\lambda,k)=|\lambda|(2k+n)\widehat{f}(\lambda,k),\qquad \widehat{Tf}(\lambda,k)=-i\lambda\widehat{f}(\lambda,k).$$

In analogy with the sphere situation, on can show that up to a multiplicative constant there is a unique operator $\mathcal{L}_d$ such that

$$|J_h|^{(Q+d)/(2Q)}(\mathcal{L}_d f)\circ h=\mathcal{L}_d\big(|J_h|^{(Q-d)/(2Q)}(f\circ h)\big),\qquad \forall h\in \mathrm{CON}(\mathbb{H}^n)$$

for $f\in C^\infty(\mathbb{H}^n)$, and such $\mathcal{L}_d$ is characterized by (under our choice of the constant)

$$\widehat{\mathcal{L}_d f}(\lambda,k)=2^{d/2}|\lambda|^{d/2}\,\frac{\Gamma\big(k+\frac{Q+d}{4}\big)}{\Gamma\big(k+\frac{Q-d}{4}\big)}=2^{d/2}|\lambda|^{d/2}\lambda_k(d), \tag{1.32}$$

or, otherwise put,

$$\mathcal{L}_d=|2T|^{d/2}\,\frac{\Gamma\big(\mathcal{L}_0|2T|^{-1}+\frac{2+d}{4}\big)}{\Gamma\big(\mathcal{L}_0|2T|^{-1}+\frac{2-d}{4}\big)}. \tag{1.33}$$

With this particular choice of the multiplicative constant we have

$$\mathcal{L}_2=\mathcal{L}_0,\qquad \mathcal{L}_4=\mathcal{L}_0^2+T^2=\mathcal{L}_0^2-|T|^2$$

$$\mathcal{L}_d\Big((2|J_{\mathcal{C}}|)^{\frac{Q-d}{2Q}}(F\circ\mathcal{C})\Big)=(2|J_{\mathcal{C}}|)^{\frac{Q+d}{2Q}}(\mathcal{A}_dF)\circ\mathcal{C}$$

and a fundamental solution of $\mathcal{L}_d$ is

$$\mathcal{L}_d^{-1}(u,v)=C_d\,|v^{-1}u|^{d-Q},\qquad C_d=\tfrac12\,c_d=\frac{2^{n-\frac{d}{2}-1}\,\Gamma\big(\frac{Q-d}{4}\big)^2}{\pi^{n+1}\Gamma\big(\frac{d}{2}\big)}. \tag{1.34}$$

The proofs of these facts are more or less contained in [C], Thm 8.1, which gives the computation of the group Fourier transform of $|u|^{d-Q}$. Note however, that our proof of the corresponding facts on the sphere (Appendix A) can easily be adapted to this situation.

We remark here that in the case d an even integer the operator $\mathcal{L}_d$ coincides with the operator found by Graham in [Gr].

The intertwinors at level $d=Q$ on $\mathbb{H}^n$ are obtained in the same manner as those for the sphere. There's the operator

$$\mathcal{L}_Q=\lim_{d\to Q}\mathcal{L}_d$$

whose kernel is the space of boundary values of pluriharmonic functions on the Siegel domain (modulo identification of its boundary with $\mathbb{H}^n$). In terms of $\mathcal{A}_Q$ we have

$$\mathcal{L}_Q(F\circ\mathcal{C})=2|J_{\mathcal{C}}|(\mathcal{A}_QF)\circ\mathcal{C}. \tag{1.35}$$

For the conditional intertwinor, we recall that f is the boundary value of a holomorphic resp. antiholomorphic function on the Siegel domain if and only if $\widehat{f}(\lambda,k)=0$ if $k\neq 0$ or $\lambda<0$ resp. $\lambda>0$. So for f a smooth CR-pluriharmonic function on $\mathbb{H}^n$ we can define, in analogy with $\mathcal{A}'_Q$ and via (1.32),

$$\mathcal{L}'_Q f = -\frac{4}{\Gamma\big(\frac{Q}{2}\big)}\frac{\partial}{\partial d}\bigg|_{d=Q}\mathcal{L}_d f = \lim_{d\to Q}\frac{1}{\lambda_0(d)}\,\mathcal{L}_d f = |2T|^{Q/2} f.$$

With this definition we have for a smooth $F\in\mathcal{P}$

$$2|J_{\mathcal{C}}|(\mathcal{A}'_Q F)\circ\mathcal{C} = \mathcal{L}'_Q(F\circ\mathcal{C}) + \frac{2}{Q\Gamma\big(\frac{Q}{2}\big)}\mathcal{L}_Q\big(\log(2|J_{\mathcal{C}}|)(F\circ\mathcal{C})\big)$$

which basically says that the conditional intertwinor on S^{2n+1} is nothing but $|2T|^{Q/2}$ on the $\mathbb{H}^n$-pluriharmonic functions, "lifted" from $\mathbb{H}^n$ to S^{2n+1} via the Cayley map (note that the second term on the right is orthogonal the the pluriharmonics). Also, we have

$$|J_h|(\mathcal{L}'_Q f)\circ h = \mathcal{L}'_Q(f\circ h) + \frac{2}{Q\Gamma\big(\frac{Q}{2}\big)}\mathcal{L}_Q\big(\log|J_h|(f\circ h)\big),\quad h\in \mathrm{CON}(\mathbb{H}^n)$$

analogous to (1.29).

Intertwining operators and change of metric.

The sublaplacian and conformal sublaplacian can be defined intrinsecally on any compact, strictly pseudoconvex CR manifold M, in terms of the contact form θ; see e.g. [JL1], [St]. If $\mathcal{D}_\theta$ is such conformal sublaplacian, then for any positive, smooth W on M there is a simple transformation formula for the conformal sublaplacian under conformal change of contact structure:

$$\mathcal{D}_{W\theta} = W^{-\frac{Q+2}{4}}\mathcal{D}_\theta W^{\frac{Q-2}{4}} \tag{1.36}$$

where, as usual, $Q=2n+2$, and $2n+1$ is the dimension of the manifold. We would like to extend this process to the operators $\mathcal{A}_d$ and $\mathcal{A}'_Q$ on the CR sphere. Unfortunately, there does not exist a general theory of such operators on general CR manifolds, i.e. we do not have available an intrinsic expression of $\mathcal{A}_d$ in terms of the contact structure. However, we are just interested in conformal changes of the standard CR structure of the sphere, and in the eigenvalues of the corresponding operators, so we can easily bypass the above difficulty as follows.

In order to motivate our construction, first observe that the sublaplacian $\mathcal{D}_W$ on $(S^{2n+1},W\theta_0)$ satisfies (1.36) i.e. $\mathcal{D}_W = W^{-\frac{Q+2}{4}}\mathcal{D}\,W^{\frac{Q-2}{4}}$, and that it is a positive self-adjoint operator densely defined on $L^2(S^{2n+1},Wd\zeta)$. By standard facts (which will be recalled below) $\mathcal{D}_W$ has eingevalues $0<\lambda_j(W)\uparrow\infty$, and by the intertwining property

(1.16) (see proof of Prop. 1.3 below) such eigenvalues are invariant under the conformal action that preserves $L^{Q/2}$ norms:

$$\lambda_j(W) = \lambda_j\big((W\circ\tau)|J_\tau|^{2/Q}\big).$$

We can now extend all this to the operators $\mathcal{A}_d$. For $0 < W \in C^\infty(S^{2n+1})$ and $0 < d \le Q$, the L^2 Hermitian products

$$(F,G) = \int_{S^{2n+1}} F\overline{G}d\zeta, \qquad (F,G)_W := \int_{S^{2n+1}} F\overline{G}W^{Q/d}d\zeta$$

are obviously defining equivalent norms on L^2 and the sequilinear forms

$$a_d(F,G) := (\mathcal{A}_d^{1/2}F, \mathcal{A}_d^{1/2}G), \qquad a_d^W(F,G) := (\mathcal{A}_d^{1/2}F, \mathcal{A}_d^{1/2}G)_W$$

are continuous on $W^{d/2,2}$, and define equivalent norms on such space. Likewise, the forms

$$a_Q(F,G) := (\mathcal{A}_Q^{1/2}F, \mathcal{A}_Q^{1/2}G), \qquad a_Q^W(F,G) = (\mathcal{A}_Q^{1/2}F, \mathcal{A}_Q^{1/2}G)_W$$

are continuous on $W^{Q/2,2}$ and $(F,G)+a_Q(F,G)$, $(F,G)_W + a_Q^W(F,g)$ define equivalent norms on $\mathcal{W}^{Q/2,2}\cap\mathcal{P}^\perp$ and the forms

$$a_Q'(F,G) := \big((\mathcal{A}_Q')^{1/2}F, (\mathcal{A}_Q)^{1/2}G\big), \qquad (a_Q')^W(F,G) = \big((\mathcal{A}_Q')^{1/2}F, (\mathcal{A}_Q')^{1/2}G\big)_W$$

are continuous on $W^{Q/2,2}$ and $(F,G)+a_Q'(F,G)$, $(F,G)_W+(a_Q')^W(F,G)$ define equivalent norms on $\mathcal{W}^{Q/2,2}\cap\mathcal{P}$.

By the equivalence of the norms in $L^2(d\zeta)$ and $L^2(Wd\zeta)$ we have that $\mathcal{P}$ and $\mathcal{P}^\perp$ are closed subspaces (in general not orthogonal!) of $L^2(Wd\zeta)$. Therefore, there are orthogonal complements to such spaces, and projections

$$\pi_W : L^2 \to \mathcal{P}, \qquad \pi_W^\perp : L^2 \to \mathcal{P}^\perp.$$

Proposition 1.3. *Let $W \in C^\infty(S^{2n+1})$, with $W > 0$. For $0 < d < Q$ the operator $\mathcal{A}_d(W) := W^{-\frac{Q+d}{2d}}\mathcal{A}_d W^{\frac{Q-d}{2d}}$ is positive, self-adjoint and invertible, and*

$$\big(\mathcal{A}_d^{1/2}(W)F, \mathcal{A}_d^{1/2}(W)G\big)_W = \big(\mathcal{A}_d^{1/2}F, \mathcal{A}_d^{1/2}G\big) \qquad \forall F,G\in W^{d/2,2}. \tag{1.37}$$

There is a sequence $\{\phi_j^W\}$ of eigenfunctions of $\mathcal{A}_d(W)$ which form an orthonormal basis of L^2 w.r. to the product $(F,G)_W$. The corresponding eigenvalues $\{\lambda_j(d,W)\}$ are positive, nondecreasing, and for any $\tau\in \mathrm{CON}(S^{2n+1})$.

$$\lambda_j(d,W) = \lambda_j(d,W_\tau), \qquad W_\tau = (W\circ\tau)|J_\tau|^{d/Q} \tag{1.38}$$

When $d = Q$ the operators

$$\mathcal{A}'_Q(W) := \pi_W W^{-1} \mathcal{A}'_Q, \qquad \mathcal{A}_Q(W) := \pi_W^{\perp} W^{-1} \mathcal{A}_Q \tag{1.39}$$

are positive semidefinite, self-adjoint and invertible on $W^{Q/2,2} \cap \mathcal{P}_0$ and $W^{Q/2,2} \cap \mathcal{P}^{\perp}$ respectively, and in such spaces

$$\big((\mathcal{A}'_Q)^{1/2}(W)F, (\mathcal{A}'_Q)^{1/2}(W)G\big)_W = \big((\mathcal{A}'_Q)^{1/2}F, (\mathcal{A}'_Q)^{1/2}G\big) \tag{1.40}$$

$$\big(\mathcal{A}_Q^{1/2}(W)F, \mathcal{A}_Q^{1/2}(W)G\big)_W = \big(\mathcal{A}_Q^{1/2}F, \mathcal{A}_Q^{1/2}G\big). \tag{1.41}$$

There is a sequence $\{\phi_j^W\}$ of eigenfunctions of $\mathcal{A}'_Q(W)$ which form an orthonormal basis of $\mathcal{P}_0$ w.r. to the product $(F, G)_W$. The corresponding eigenvalues $\{\lambda_j(Q, W)\}$ are positive, nondecreasing, and satisfy (1.38) with $d = Q$. Similarly for $\mathcal{A}_Q(W)$.

Proof. This proposition follows in a more or less straightforward way from the standard spectral theory of forms and operators (e.g. see [Sh], Theorem 7.7). For $0 < d < Q$ identity (1.37) is immediate, and the form $\big(\mathcal{A}_d^{1/2}F, \mathcal{A}_d^{1/2}F\big)$ is equivalent to $\|F\|_{W^{d/2,2}}$. Since $W^{d/2,2}$ is compactly imbedded in L^2 we can find an o.n. basis of eigenfunctions of $A_d(W)$. Identity (1.38) follows easily from the intertwining property (1.17). Indeed, using (1.17) one checks that if λ is an eigenvalue of $\mathcal{A}_d(W_\tau)$ with eigenfunction ϕ then λ is also an eigenvalue of $\mathcal{A}_d(W)$, with eigenfunction $\phi \circ \tau^{-1}$. The proof of the case $d = Q$ is similar, except that the background Hilbert spaces are now $\mathcal{P}$ or $\mathcal{P}^{\perp}$, and that (1.40), (1.41) follow from (1.29) and (1.17) respectively.

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The operators $\mathcal{A}_d(W)$ and $\mathcal{A}'_Q(W)$ are natural generalizations of the corresponding operators $\mathcal{A}_d, \mathcal{A}'_Q$, under conformal change of contact form via W. Indeed, from (1.17), (1.24), (1.29)

$$\tau^* \mathcal{A}_d(W)(\tau^{-1})^* = \mathcal{A}_d(W_\tau) \ \ (0 < d \le Q) \qquad \tau^* \mathcal{A}'_Q(W)(\tau^{-1})^* = \mathcal{A}'_Q(W_\tau).$$

2. Adams inequalities

Adams inequalities in measure spaces.

Let $(M, d\mu)$ be a measure space with $\mu(M) < \infty$. Consider an integral operator

$$Tf(x) = \int_M k(x, y) f(y) d\mu(y)$$

where f is a nonnegative measurable function on M, and $k(x, y)$ a nonnegative measurable function on $M \times M$. Define

$$k^*(t) = \max\big\{ \sup_{x \in M} k^*(x, \cdot)(t), \ \sup_{y \in M} k^*(\cdot, y)(t) \big\}, \qquad t > 0 \tag{2.1}$$

where $k^*(x,\cdot)$ denotes the nonincreasing rearrangement of $k(x,y)$ with respect to the variable y, for fixed x. Note that for convolution operators of type $Tf = K * f$ on a homogeneous space, the function $K^*(t)$ defined by (2.1) coincides with the distribution function of K.

Theorem 2.1. *If*

$$m(k^*,s) := \big|\{t>0:\ k^*(t)>s\}\big| \le As^{-\beta}\big(1+O(\log^{-\gamma} s)\big) \tag{2.2}$$

as $s \to +\infty$, for some $\beta,\gamma > 1$, then there exists a constant $C>0$ s.t.

$$\int_M \exp\left[A^{-1}\left(\frac{|Tf|}{\|f\|_{\beta'}}\right)^{\beta}\right] d\mu \le C \tag{2.3}$$

for each $f \in L^{\beta'}(M)$, with $\dfrac{1}{\beta}+\dfrac{1}{\beta'}=1$.

The condition $\gamma > 1$ is best possible, in the sense that it is possible to find explicit examples with $\gamma \le 1$ for which (2.3) fails.

The proof of this theorem is measure-theoretic in nature and it is based on the arguments originally used by Adams in [Ad], and later extended by Fontana [F] to a more general setting. A complete proof of the theorem will appear in a forthcoming paper [FM]. In fact, in [FM] a slightly more general result is proven: the conclusion of Theorem 2.1 holds if the integral operator T acts from $L^1(M,d\mu)$ to $L^1(N,d\nu)$, where $(M,d\mu)$ and $(N,d\nu)$ are two (possibly different) measure spaces with finite measure; moreover, it is only required that condition (2.2) holds for $\sup_{x\in N} k^*(x,\cdot)(t)$, rather than $k^*(t)$.

To the authors knowledge almost all the known results concerning Moser-Trudinger inequalities in the Adams form are immediate consequences of the above theorem, where the integral operator T is used to represent f in terms of its derivatives of borderline order. One of the main features of theorem 2.1 is to highlight the fact that exponential integrability in the form (2.3) is a consequence of a single asymptotic estimate on the distribution function of the kernel of T, and that it is stable under suitable perturbations.

The simplest case is on a bounded domain of $\mathbb{R}^n$, with $k(x,y)=|x-y|^{d-n}$, $p=n/d$, in which case $m(k^*,s)=\frac{\omega_{n-1}}{n}\,s^{-p'}$, and one recovers Adams' original results. Other known and new situations are discussed in [FM].

We also remark that in all the known cases where (2.2) holds with an equal sign, the constant A^{-1} in (2.3) turns out to be sharp, i.e. if it is replaced by a larger constant then (2.3) is not true uniformly in f. This fact can be also formalized in the abstract setting of theorem 2.1, under suitable conditions on the kernel $k(x,y)$ [FM].

Adams inequalities for convolution operators on the CR sphere

We would now llike to apply Theorem 2.1 to a wide class of convolution operators on the CR sphere. Let us first introduce some notation:

$$u=(z,t)\in\mathbb{H}^n,\quad \Sigma=\{u\in\mathbb{H}^n: |u|=1\},\quad u^*=(z^*,t^*)=\frac{u}{|u|}\in\Sigma$$

$$\zeta=\mathcal{C}(z,t)\in S^{2n+1},\quad \frac{1-\zeta_{n+1}}{1+\zeta_{n+1}}=|z|^2+it=|u|^2e^{i\theta}, \tag{2.4}$$
$$\mathcal{E}=\mathcal{C}(\Sigma)=\big\{(\zeta_1,...,\zeta_{n+1})\in S^{2n+1}:\ \mathrm{Re}\,\zeta_{n+1}=0\big\}.$$

It's easy to see that a function $g(\zeta,\eta)$ is $U(n+1)-$invariant, i.e. $g(R\zeta,R\eta)=g(\zeta,\eta)$, $\forall R\in U(n+1)$, if and only if $g(\zeta,\eta)=g(\zeta\cdot\overline{\eta})$ for some g defined on the unit disk of $\mathbb{C}$. Furthermore, from (2.4) the function $g(\zeta\cdot\overline{\mathbb{N}})=g(\zeta_{n+1})$ is independent on $\mathrm{Re}\,\zeta_{n+1}$, i.e. it is defined on $\mathcal{E}$, if and only if it is a function of the angle $\theta=\sin^{-1}t^*$.

A measurable function $\phi:\big[-\frac{\pi}{2},\frac{\pi}{2}\big]\to\mathbb{R}$ can be viewed as a function on Σ, via $\phi(\theta)=\phi(\sin^{-1}t^*)$, and we will use the notation

$$\int_\Sigma \phi\,du^*:=\int_\Sigma\phi(\sin^{-1}t^*)du^*=\omega_{2n-1}\int_{-\pi/2}^{\pi/2}\phi(\theta)(\cos\theta)^{n-1}d\theta \tag{2.5}$$

whenever the integrals make sense. The formula on the right in (2.5) is easily checked via polar coordinates. Finally, for $w\in\mathbb{C}$, $|w|<1$ we let

$$\theta=\theta(w)=\arg\frac{1-w}{1+w}\in\Big[-\frac{\pi}{2},\frac{\pi}{2}\Big].$$

The main result of this section is the following:

Theorem 2.2. *Let $0<d<Q$ and $p=\dfrac{Q}{d}$. Define*

$$TF(\zeta)=\int_{S^{2n+1}}G(\zeta,\eta)F(\eta)d\eta,\qquad F\in L^p(S^{2n+1})$$

where

$$G(\zeta,\eta)=g_0\big(\theta(\zeta\cdot\overline{\eta})\big)\,d(\zeta,\eta)^{d-Q}+O\big(d(\zeta,\eta)^{d-Q+\epsilon}\big)=$$
$$=2^{\frac{d-Q}{2}}g_0\big(\theta(\zeta\cdot\overline{\eta})\big)\,|1-\zeta\cdot\overline{\eta}|^{\frac{d-Q}{2}}+O\big(|1-\zeta\cdot\overline{\eta}|^{\frac{d-Q+\epsilon}{2}}\big),\quad \zeta\neq\eta$$

for bounded and measurable $g_0:\big[-\frac{\pi}{2},\frac{\pi}{2}\big]\to\mathbb{R}$, with $\big|O\big(|1-\zeta\cdot\overline{\eta}|^{\frac{d-Q+\epsilon}{2}}\big)\big|\le C|1-\zeta\cdot\overline{\eta}|^{\frac{d-Q+\epsilon}{2}}$, some $\epsilon>0$, and with C independent of ζ,η.

Then, there exists $C_0 > 0$ such that for all $F \in L^p(S^{2n+1})$

$$\int_{S^{2n+1}} \exp\left[A_d\left(\frac{|TF|}{\|F\|_p}\right)^{p'}\right] d\zeta \leq C_0 \tag{2.6}$$

with

$$A_d = \frac{2Q}{\displaystyle\int_\Sigma |g_0|^{p'} du^*} \tag{2.7}$$

for every $F \in L^p(S^n)$, with $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover, if the function $g_0(\theta)$ is Hölderian of order $\sigma \in (0,1]$ then the constant in (2.7) is sharp, in the sense that if it is replaced by a larger constant then there exists a sequence $F_m \in L^p(S^{2n+1})$ such that the exponential integral in (2.6) diverges to $+\infty$ as $m \to \infty$.

In [CoLu1] Cohn and Lu give a similar result in the context of the Heisenberg group, and for kernels of type $G(u) = g(u^*)|u|^{d-Q}$, i.e. without any perturbations. It will be clear from the proof below that a version analogous to Theorem 2.2 holds also on $\mathbb{H}^n$ (thus extending the result in [CoLu1]).

In view of Theorem 2.1 it is enough to find an asymptotic estimate for the the distribution function of G. This is provided by the following:

Lemma 2.3. *Let $G : S^{2n+1} \times S^{2n+1} \setminus \{(\zeta,\zeta), \zeta \in S^{2n+1}\} \to \mathbb{R}$, be measurable and such that*

$$G(\zeta,\eta) = g\big(\theta(\zeta\cdot\overline{\eta})\big)\, |1-\zeta\cdot\overline{\eta}|^{-\alpha} + O\big(|1-\zeta\cdot\overline{\eta}|^{-\alpha+\epsilon}\big), \quad \zeta \neq \eta$$

some bounded and measurable $g : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$, with $\left|O\big(|1-\zeta\cdot\overline{\eta}|^{-\alpha+\epsilon}\big)\right| \leq C|1-\zeta\cdot\overline{\eta}|^{-\alpha+\epsilon}$, some $\epsilon > 0$, and with C independent of ζ, η. Then, for each $\eta \in S^{2n+1}$ and as $s \to +\infty$

$$\big|\{\zeta :\, |G(\zeta,\eta)| > s\}\big| = s^{-Q/2\alpha}\, \frac{2^{Q/2-1}}{Q} \int_\Sigma |g|^{Q/2\alpha} du^* + O\big(s^{-Q/2\alpha-\sigma}\big) \tag{2.8}$$

for a suitable $\sigma > 0$.

Proof. Let

$$|G(\zeta,\eta)| \leq |g(\theta)|\, |1-\zeta\cdot\overline{\eta}|^{-\alpha} + C|1-\zeta\cdot\overline{\eta}|^{-\alpha+\epsilon}.$$

Since the right hand side is rotation invariant we have

$$\lambda(s) := |\{\zeta :\, |G(\zeta,\eta)| > s\}| \leq |\{\zeta : |g(\theta)|\, |1-\zeta_{n+1}|^{-\alpha} + C|1-\zeta_{n+1}|^{-\alpha+\epsilon} > s\}|.$$

Now

$$|g(\theta)||1-\zeta_{n+1}|^{-\alpha} + C|1-\zeta_{n+1}|^{-\alpha+\epsilon} > s \quad \Longrightarrow \quad |1-\zeta_{n+1}| \leq s^{-1/\alpha}\big(|g(\theta)| + C|1-\zeta_{n+1}|^{\epsilon}\big)^{1/\alpha} \leq C\, s^{-1/\alpha}$$

hence

$$\lambda(s) \le |\{\zeta : \, |1-\zeta_{n+1}| \le s^{-1/\alpha}(|g(\theta)| + Cs^{-\epsilon/\alpha})^{1/\alpha}\}|.$$

Let

$$\phi(u^*, s) = s^{-1/\alpha}(|g(\theta)| + Cs^{-\epsilon/\alpha})^{1/\alpha} < 1, \qquad s \ge s_0$$

so that from (1.2) and polar coordinates

$$\begin{aligned}
\lambda(s) &\le \int_{\left\{\frac{2|u|^2}{(1+2|z|^2+|u|^4)^{1/2}} \le \phi(u^*,s)\right\}} \frac{2^{Q-1}}{\left(1+2|z|^2+|u|^4\right)^{n+1}} du \\
&= \int_\Sigma du^* \int_{\left\{\frac{2r^2}{(1+2r^2|z^*|^2+r^4)^{1/2}} \le \phi(u^*,s)\right\}} \frac{2^{Q-1} r^{Q-1}}{\left((1+2r^2|z^*|^2+r^4\right)^{n+1}} dr \\
&\le \int_\Sigma du^* \int_{\left\{2r^2 \le \phi(u^*,s)\left(1+\phi(u^*,s)\right)\right\}} \frac{2^{Q-1} r^{Q-1}}{\left((1+2r^2|z^*|^2+r^4\right)^{n+1}} dr \\
&\le \frac{2^{Q/2-1}}{Q} \int_\Sigma \phi(u^*,s)^{Q/2}\left(1+\phi(u^*,s)\right)^{Q/2} du^*. \\
&\le s^{-Q/2\alpha} \frac{2^{Q/2-1}}{Q} \int_\Sigma |g|^{Q/2\alpha} du^* + O\left(s^{-Q/2\alpha-\sigma}\right)
\end{aligned}$$

the last inequality being a consequence of $|(x+y)^a - y^a| \le Cx^{\min\{1,a\}}$, valid for $a > 0$, $x, y \in [0, K]$ some fixed $K > 0$, and with C depending only on K and a.

To show the reverse inequality, by hypothesis

$$|G(\zeta,\eta)| \ge |g(\theta)|\, |1-\zeta\cdot\overline{\eta}|^{-\alpha} - D|1-\zeta\cdot\overline{\eta}|^{-\alpha+\epsilon}$$

for some $D > 0$, so that

$$\lambda(s) \ge |\{\zeta : |g(\theta)|\, |1-\zeta_{n+1}|^{-\alpha} - D|1-\zeta_{n+1}|^{-\alpha+\epsilon} > s\}|.$$

If $|g(\theta)||1-\zeta_{n+1}|^{-\alpha} - D|1-\zeta_{n+1}|^{-\alpha+\epsilon} > s$, then

$$|1-\zeta_{n+1}| < s^{-1/\alpha}\Big(|g(\theta)| - D|1-\zeta_{n+1}|^{\epsilon}\Big)^{1/\alpha} \le s^{-1/\alpha}|g(\theta)|^{1/\alpha}.$$

Hence

$$\begin{aligned}
&\Big\{\zeta : \, |g(\theta)| > D|g(\theta)|^{\epsilon/\alpha}s^{-\epsilon/\alpha},\, |1-\zeta_{n+1}| < s^{-1/\alpha}\left(|g(\theta)| - D|g(\theta)|^{\epsilon/\alpha}s^{-\epsilon/\alpha}\right)^{1/\alpha}\Big\} \subseteq \\
&\qquad \subseteq \Big\{\zeta : \, |1-\zeta_{n+1}| < s^{-1/\alpha}\Big(|g(\theta)| - D|1-\zeta_{n+1}|^{\epsilon}\Big)^{1/\alpha}\Big\}
\end{aligned}$$

Without loss of generality we can assume that $\epsilon < \alpha$, so that if

$$E_s = \left\{u^* \in \Sigma : \, |g(\theta)| > D|g(\theta)|^{\epsilon/\alpha}s^{-\epsilon/\alpha}\right\} = \left\{u^* : |g(\theta)| > D^{\alpha/(\alpha-\epsilon)}s^{-\epsilon/(\alpha-\epsilon)}\right\}$$

and

$$\phi(u^*,s)=s^{-1/\alpha}\Big(|g(\theta)|-D|g(\theta)|^{\epsilon/\alpha}s^{-\epsilon/\alpha}\Big)^{1/\alpha}\chi_{E_s}(u^*)\le 1,\quad s\ge s_0$$

then

$$\begin{aligned}\lambda(s)\ge|\{\zeta:\,|1-\zeta_{n+1}|<\phi(u^*,s)\}|&=\int_{\Big\{\frac{2|u|^2}{(1+2|z|^2+|u|^4)^{1/2}}\le\phi(u^*,s)\Big\}}\frac{2^{Q-1}}{\big(1+2|z|^2+|u|^4\big)^{n+1}}du\\
&\ge\int_{\{2|u|^2\le\phi(u^*,s)\}}\frac{2^{Q-1}}{\big(1+|u|^2\big)^Q}du=2^{Q-1}\int_{\{2|u|^2\le\phi(u^*,s)\}}\big(1+O(|u|)\big)=\\
&=\frac{2^{Q/2-1}}{Q}\int_\Sigma\Big[\phi(u^*,s)^{Q/2}+O\big(\phi(u^*,s)^{(Q+1)/2}\big)\Big]du^*\\
&=s^{-Q/2\alpha}\frac{2^{Q/2-1}}{Q}\int_{E_s}|g|^{Q/2\alpha}du^*+O(s^{-Q/2\alpha-\sigma'})\\
&=s^{-Q/2\alpha}\frac{2^{Q/2-1}}{Q}\int_\Sigma|g|^{Q/2\alpha}du^*+O(s^{-Q/2\alpha-\sigma}),\end{aligned}$$

for suitable $\sigma,\sigma'>0$.

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Theorem 2.2 is a straight consequence of Theorem 2.1 and Lemma 2.3. The statement about sharpness, on the other hand, follows from the same general philosophy originally used by Adams and later by Fontana, Cohn-Lu and many others. In our case it is possible to check that the sequence F_m in the statement of Theorem 2.2., can be chosen as

$$F_m(\eta)=\begin{cases}|G(\mathcal{N},\eta)|^{d/(Q-d)}\,\mathrm{sgn}\big(G(\mathcal{N},\eta)\big) & \text{if } |G(\mathcal{N},\eta)|\le m\ ,\ d(\mathcal{N},\eta)\ge 2m^{-2/(Q-d)}\\ 0 & \text{otherwise.}\end{cases}\tag{2.9}$$

The calculations are similar in spirit as those in [CoLu1], with some added complications given that we are now working on the sphere rather than $\mathbb{H}^n$. The smoothness hypothesis on g can be also relaxed a little to a Dini-type condition such as that of [CoLu1]. More detailed work on this will appear in [FM], where the proof of the sharpness statement appears as an immediate application of general "abstract" theorems on measure spaces.

Adams inequalities for operators of $d-$type on Hardy spaces

For a given $d\in(0,Q)$, we say that a densely defined and self-adjoint operator P_d on $\mathcal{H}$ is of *$d-$type* if

$$P_dY_{j0}=\mu_{j0}Y_{j0},\qquad\forall Y_{j0}\in\mathcal{H}_{j0}\tag{2.10}$$

for a given sequence $\{\mu_{j0}\}$ such that for $j\to\infty$

$$0\le\mu_{00}\le\mu_{10}\le\mu_{20}\le\ldots\qquad\mu_{j0}=j^{d/2}+a_1j^{d/2-\epsilon_1}+\ldots+a_mj^{d/2-\epsilon_m}+O(j^{d/2-\epsilon_{m+1}})\tag{2.11}$$

for some $0 < \epsilon_1 < \epsilon_2 < ... < \epsilon_{m+1}$ with $\frac{Q-d}{2} < \epsilon_{m+1}$. From this condition it follows that $\mathrm{Ker}(P_d)$ is finite dimensional, and that P_d is a continuous operator from $W^{d,2} \cap \mathcal{H}$ to $\mathcal{H}$. More generally, one defines operators of $d-$type on $\mathcal{H}^p$ as densely defined operators satisfying (2.10) and (2.11). Note that by (2.11) the operator P_d can be written on $C^\infty \cap \mathcal{H}^p$ as a finite sum of powers of the sublaplacian, up to a smoothing operator. P_d is a continuous operator from $W^{d,p} \cap \mathcal{H}^p$ to $\mathcal{H}^p$ and invertible if restricted to $\mathrm{Ker}(P_d)^\perp$ with

$$\mathrm{Ker}(P_d)^\perp := \Big\{F \in \mathcal{H}^p : \int_{S^{2n+1}} F\phi_k = 0,\, k = 1, ...m\Big\}$$

and where $\phi_1, ..., \phi_m$ denote a basis of $\mathrm{Ker}(P_d)$, the null space of P_d. Operators of $d-$type on $\overline{\mathcal{H}}^p$ and $\mathcal{P}^p$ are defined similary, and the spectrum of such operators is denoted by $\{\mu_{0j}\}$ and $\{\mu_{j0},\, \mu_{0j}\}$ respectively, where the μ's satisfy a condition of type (2.11).

From the previous section it is clear that in order to obtain Adams inequalities for operators of this sort, all we need to do is to have an estimate on their fundamental solutions. Here's the result we need:

Proposition 2.4. *If P_d is an operator of $d-$type on $\mathcal{H}^p$ then a fundamental solution of P_d is defined by the formula*

$$P_d^{-1}(\zeta,\eta) := \lim_{R\to 1^-} \sum_{\mu_{j0}\neq 0} \frac{\Phi_{j0}(\zeta\cdot\overline{\eta})}{\mu_{j0}}\, R^j$$

in the sense of distributions and pointwise for $\zeta \neq \eta$ and . Moreover, the following expansion holds:

$$P_d^{-1}(\zeta,\eta) = \frac{\Gamma\big(\frac{Q-d}{2}\big)}{\omega_{2n+1}\, n!}\, (1-\zeta\cdot\overline{\eta})^{\frac{d-Q}{2}} + O\big(|1-\zeta\cdot\overline{\eta}|^{\frac{d-Q}{2}+\epsilon}\big)$$

for a suitable $\epsilon > 0$. Likewise, if P_d is an operator of $d-$type on $\mathcal{P}^p$ then a fundamental solution of P_d is defined by the formula

$$P_d^{-1}(\zeta,\eta) := \lim_{R\to 1^-} \Bigg\{ \sum_{\mu_{j0}\neq 0} \frac{\Phi_{j0}(\zeta\cdot\overline{\eta})}{\mu_{j0}}\, R^j + \sum_{\mu_{0j}\neq 0} \frac{\Phi_{0j}(\zeta\cdot\overline{\eta})}{\mu_{0j}}\, R^j \Bigg\}$$

in the sense of distributions and pointwise for $\zeta \neq \eta$. Moreover,

$$\begin{aligned} P_d^{-1}(\zeta,\eta) &= \frac{2\Gamma\big(\frac{Q-d}{2}\big)}{\omega_{2n+1}\, n!}\, \mathrm{Re}\,(1-\zeta\cdot\overline{\eta})^{\frac{d-Q}{2}} + O\big(|1-\zeta\cdot\overline{\eta}|^{\frac{d-Q}{2}+\epsilon}\big) \\ &= g_d(\theta)\, d(\zeta,\eta)^{d-Q} + O\big(d(\zeta,\eta)^{d-Q+\epsilon}\big) = \\ &= 2^{\frac{d-Q}{2}} g_d(\theta)\, |1-\zeta\cdot\overline{\eta}|^{\frac{d-Q}{2}} + O\big(|1-\zeta\cdot\overline{\eta}|^{\frac{d-Q}{2}+\epsilon}\big) \end{aligned}$$

for a suitable $\epsilon > 0$, with

$$g_d(\theta) = \frac{2^{\frac{Q-d}{2}+1}\, \Gamma\big(\frac{Q-d}{2}\big)}{\omega_{2n+1}\, n!}\, \cos\big(\tfrac{Q-d}{2}\,\theta\big). \tag{2.12}$$

Proof. Suppose that $\mu_{j0} > 0$ for $j \geq j_0$, so that for $0 < r < 1$

$$\sum_{\mu_{j0}\neq 0} \frac{\Phi_{j0}}{\mu_{j0}} R^j = \frac{1}{n!\omega_{2n+1}} \sum_{j\geq j_0} \frac{\Gamma\big(j+\frac{Q}{2}\big)}{\Gamma(j+1)} \frac{(R\,\zeta\cdot\overline{\eta})^j}{\mu_{j0}}$$

The given hypothesis on the μ_{j0}'s implies (in fact it is equivalent to)

$$\frac{1}{\mu_{j0}} \frac{\Gamma\big(j+\frac{Q}{2}\big)}{\Gamma(j+1)} = j^{\frac{Q-d}{2}-1} + \alpha_1 j^{\frac{Q-d}{2}-1-\epsilon_1} + \ldots \alpha_m j^{\frac{Q-d}{2}-1-\epsilon_m} + \alpha j^{-1} + O(j^{-\epsilon'}), \qquad j \to \infty.$$

for possibly a new set of ϵ's such that $0 < \epsilon_1 < \epsilon_2 < \ldots < \epsilon_m$, with $\frac{Q-d}{2} - \epsilon_m > 0$, and for some $\epsilon' > 1$. Here we use that

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = z^{\alpha-\beta}\big(1 + c_1 z^{-1} + c_2 z^{-2} + \ldots + c_N z^{-N} + O(z^{-N})\big)$$

any $N > 0$, for $|z| \to \infty$. Using the above expansions we can then write

$$\frac{1}{\mu_{j0}} \frac{\Gamma\big(j+\frac{Q}{2}\big)}{\Gamma(j+1)} = \frac{\Gamma\big(j+\frac{Q-d}{2}\big)}{\Gamma(j+1)} + \sum_{p=1}^{m} \alpha_p \frac{\Gamma\big(j+\frac{Q-d}{2}-\epsilon_p\big)}{\Gamma(j+1)} + \alpha j^{-1} + O(j^{-\epsilon'})$$

and the same expansion, with possibily different α's and ϵ's, holds for the μ_{0j}. Using the binomial expansion we get

$$\sum_{j\geq j_0} \frac{\Gamma\big(j+\frac{Q}{2}\big)}{\Gamma(j+1)} \frac{(R\,\zeta\cdot\overline{\eta})^j}{\mu_{j0}} = \Gamma\big(\tfrac{Q-d}{2}\big)\,(1-R\,\zeta\cdot\overline{\eta})^{\frac{d-Q}{2}} + \sum_{p=1}^{m} \alpha_p\, \Gamma\big(\tfrac{Q-d-\epsilon_p}{2}\big)(1-R\,\zeta\cdot\overline{\eta})^{\frac{d+\epsilon_p-Q}{2}}$$
$$- \alpha \log(1-R\,\zeta\cdot\overline{\eta}) + O(1).$$

This identity implies the statements in the case P_d defined on $\mathcal{H}^p$. The case P_d defined on $\mathcal{P}^p$ follows immediately.

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Theorem 2.5. *Let P_d be an operator of d-type on $\mathcal{P}^p$, then there is $C_0 > 0$ such that for any $F \in W^{d,p} \cap \mathcal{P}^p \cap \mathrm{Ker}(P_d)^\perp$ and with $p = \frac{Q}{d},\ \frac{1}{p} + \frac{1}{p'} = 1$ we have*

$$\int_{S^{2n+1}} \exp\left[A_d\left(\frac{|F|}{\|P_d F\|_p}\right)^{p'}\right] d\zeta \leq C_0 \tag{2.13}$$

with

$$A_d = \frac{2Q}{\displaystyle\int_\Sigma |g_d|^{p'}\, du^*} \tag{2.14}$$

and $g_d(\theta)$ as in (2.12). In the special case $d = Q/2$ (i.e. $p = p' = 2$)

$$A_{Q/2} = \frac{\omega_{2n+1}(n+1)!}{2} = (n+1)\pi^{n+1} \tag{2.15}$$

and this constant is sharp, i.e. it cannot be replaced by a larger constant in (2.13).

If P_d is of $d-$type on $\mathcal{H}^p$, then for any $F \in W^{d,p} \cap \mathcal{H}^p \cap \mathrm{Ker}(P_d)^\perp$ both (2.13) and (2.14) hold with $g_d = \dfrac{2^{\frac{Q-d}{2}}\Gamma\big(\frac{Q-d}{2}\big)}{n!\omega_{2n+1}}$. In the special case $d = Q/2$ we have $A_{Q/2} = \omega_{2n+1}(n+1)! = 2(n+1)\pi^{n+1}$ and this constant is sharp.

Remark. The proof below does not seem to yield the sharp constants for $p \neq 2$ (see (2.19)).

Proof. Estimate (2.13), for P_d acting either on $\mathcal{P}^p$ or $\mathcal{H}^p$, follows at once from Theorem 2.1, Lemma 2.3 and Prop. 2.3. As for the case $d = Q/2$, the computation of $A_{Q/2}$ is based on (2.5) and the formula

$$\int_0^{\pi/2} \cos^2\big(\tfrac{n+1}{2}\theta\big)(\cos\theta)^{n-1}d\theta = \frac{1}{2}\int_0^{\pi/2}(\cos\theta)^{n-1}d\theta = \frac{\sqrt{\pi}\,\Gamma\big(\frac{n}{2}\big)}{4\,\Gamma\big(\frac{n+1}{2}\big)}$$

together with the duplication formula $\Gamma(n) = (2\pi)^{-\frac{1}{2}}2^{n-\frac{1}{2}}\Gamma\big(\frac{n}{2}\big)\Gamma\big(\frac{n+1}{2}\big)$.

To find an upper bound for the best constant in (2.13), we consider for $0 < R < 1$

$$f_R(\zeta) = \mathrm{Re}\sum_{k\geq k_0}\frac{\Gamma\big(k+\frac{d}{2}\big)}{\Gamma\big(\frac{d}{2}\big)\Gamma(k+1)}R^k\zeta_{n+1}^k = \mathrm{Re}\,(1-R\zeta_{n+1})^{-\frac{d}{2}} + O(1)$$

where $\mu_{k0}\mu_{0k} > 0$ for $k \geq k_0$.

From the proof of Prop. 2.3 we know that

$$\begin{aligned} n!\omega_{2n+1}P_d^{-1}(\zeta,\eta) &= 2\Gamma\big(\tfrac{Q-d}{2}\big)\mathrm{Re}\,(1-\zeta\cdot\overline{\eta})^{\frac{d-Q}{2}} + \sum_{i=1}^m a_i\Gamma\big(\tfrac{Q-d-\epsilon_i}{2}\big)(1-\zeta\cdot\overline{\eta})^{\frac{d+\epsilon_i-Q}{2}} \\ &+ \sum_{j=1}^{\ell} b_j\Gamma\big(\tfrac{Q-d-\sigma_j}{2}\big)(1-\overline{\zeta}\cdot\eta)^{\frac{d+\sigma_j-Q}{2}} - a\log(1-\zeta\cdot\overline{\eta}) - b\log(1-\overline{\zeta}\cdot\eta) + O(1), \end{aligned}$$

for some constants $a_i, b_j, a, b, \epsilon_i > 0, \sigma_j > 0$. If $\widetilde{P}_d$ is the operator with fundamental solution

$$\frac{\Gamma\big(\frac{Q-d}{2}\big)}{n!\omega_{2n+1}}(1-\zeta\cdot\overline{\eta})^{\frac{d-Q}{2}} = \frac{1}{n!\omega_{2n+1}}\sum_0^\infty\frac{\Gamma\big(k+\frac{Q-d}{2}\big)}{\Gamma\big(k+1\big)}(\zeta\cdot\overline{\eta})^k.$$

then from (1.10)

$$\widetilde{P}_d^{-1} f_R(\zeta) = \frac{1}{2}\sum_{k=1}^{\infty} \frac{\Gamma\big(k+\frac{d}{2}\big)\Gamma\big(k+\frac{Q-d}{2}\big)}{\Gamma\big(\frac{d}{2}\big)\Gamma(k+1)\Gamma\big(k+\frac{Q}{2}\big)}\, R^k \zeta_{n+1}^k + O(1) = \frac{1}{2\Gamma\big(\frac{d}{2}\big)} \log\frac{1}{1-R\zeta_{n+1}} + O(1)$$

and the corresponding formula is valid for the conjugate operator. Likewise, for an operator $\widehat{P}_d$ with fundamental solution of type $(1-\zeta\cdot\overline{\eta})^{\frac{d'-Q}{2}},\ d<d'<Q$ or of type $\log(1-\zeta\cdot\overline{\eta})$, and their conjugate operators, we obtain $\widehat{P}_d^{-1} f_R(\zeta) = O(1)$. Hence we obtain

$$P_d^{-1} f_R(\zeta) = \frac{1}{\Gamma\big(\frac{d}{2}\big)} \log\frac{1}{|1-R\zeta_{n+1}|} + O(1)$$

Let $F_R(\zeta) = f_R(\zeta)\Gamma\big(\frac{d}{2}\big)\big(\log\frac{1}{1-R}\big)^{-1}$ and $\Omega_R = \{\zeta\in S^{2n+1} : |1-\zeta_{n+1}| < 1-R\}$. Then $F_R \in \mathcal{P}^p \cap \mathrm{Ker}(P_d)^{\perp}$ and

$$P_d^{-1} F_R(\zeta) = 1+\psi_R(\zeta), \qquad |\psi_R(\zeta)| \le C\big(\log\frac{1}{1-R}\big)^{-1} = o(1),\ \zeta\in\Omega_R,\ R\to 1 \quad (2.16)$$

(use that, on Ω_R, $\Big|\log\frac{1-R}{|1-R\zeta_{n+1}|}\Big| \le \log(1+|\zeta_{n+1}|) \le \log 2$).

Now if (2.13) holds with constant B, then

$$|\Omega_R| \exp\left[B\left(\frac{1-o(1)}{\|F_R\|_p}\right)^{p'}\right] \le C_0$$

It's easy to check that $|\Omega_R| \sim c(1-R)^{n+1}$, as $R\to 1$ (for example as in the proof of Lemma 2.3), with $\alpha=1, s=(1-R)^{-1}$, which also gives $c = 2^n|\Sigma|Q^{-1})$. From this we obtain

$$B \le \frac{Q}{2} \lim_{R\to 1} \log\frac{1}{1-R}\ \|F_R\|_p^{p'} \qquad (2.17)$$

provided that the limit exists, which is what we are going to prove now.

We have

$$\|F_R\|_p^{p'} \sim \left(\frac{\Gamma\big(\frac{d}{2}\big)}{\log\frac{1}{1-R}}\right)^{p'} \left(\int_{S^{2n+1}} \big|\,\mathrm{Re}\,(1-R\zeta_{n+1})^{-\frac{d}{2}}\big|^p d\zeta\right)^{p'-1}$$

so it's enough to evaluate

$$\lim_{R\to 1}\left(\log\frac{1}{1-R}\right)^{-1} \int_{S^{2n+1}} \big|\,\mathrm{Re}\,(1-R\zeta_{n+1})^{-\frac{d}{2}}\big|^{\frac{Q}{d}} d\zeta. \qquad (2.18)$$

To this end let us recall that $1-\zeta_{n+1} = (1+\zeta_{n+1})|u|^2 e^{i\theta}$, i.e. $\zeta_{n+1} = \dfrac{1-|u|^2e^{i\theta}}{1+|u|^2e^{i\theta}}$. Observe that in (2.18) we only need to integrate over the region $|u|<1$. Letting $\epsilon = \frac{1-R}{1+R}$, and $\varphi_\epsilon(u) = \arg(\epsilon+|u|^2e^{i\theta})$ we have, for $|u|<1$,

$$\begin{aligned}\mathrm{Re}\,(1-R\zeta_{n+1})^{-d/2} &\sim 2^{-d/2}\mathrm{Re}\Big[(\epsilon+|u|^2e^{i\theta})^{-d/2}(1+|u|^2e^{i\theta})^{d/2}\Big]\\ &= 2^{-d/2}(\epsilon^2+2\epsilon|u|^2\cos\theta+|u|^4)^{-d/4}\big[\cos\big(\tfrac{d}{2}\varphi_\epsilon\big)+O(|u|^2)\big].\end{aligned}$$

Using the Cayley transform and polar coordinates

$$\begin{aligned}
&\Big(\log\frac{1}{1-R}\Big)^{-1}\int_{S^{2n+1}}\big|\operatorname{Re}(1-R\zeta_{n+1})^{-\frac{d}{2}}\big|^{\frac{Q}{d}}d\zeta \\
&\sim \frac{2^{-Q/2}}{\log\frac{1}{\epsilon}}\int_{|u|<1}\frac{2^{2n+1}\big[|\cos\big(\frac{d}{2}\varphi_\epsilon\big)|^{Q/d}+O(|u|^2)\big]}{(\epsilon^2+2\epsilon|u|^2\cos\theta+|u|^4)^{Q/4}(1+2|u|^2|z^*|^2+|u|^4)^{n+1}}du \\
&\sim \frac{2^n}{\log\frac{1}{\epsilon}}\int_0^1\int_\Sigma\frac{r^{Q-1}\big|\cos\big(\frac{d}{2}\,\varphi_\epsilon\big)\big|^{\frac{Q}{d}}}{(\epsilon^2+r^4+2r^2\epsilon\cos\theta)^{Q/4}}dr du^* \\
&= \frac{2^{n-1}}{\log\frac{1}{\epsilon}}\int_0^{1/\epsilon}\int_\Sigma\big|\cos\big(\tfrac{d}{2}\,\tan^{-1}\tfrac{r\sin\theta}{1+r\cos\theta}\big)\big|^{\frac{Q}{d}}\Big(\frac{r^2}{1+r^2+2r\cos\theta}\Big)^{Q/4}\frac{dr}{r}du^* \\
&\sim 2^{n-1}\int_\Sigma\big|\cos\big(\tfrac{d}{2}\theta\big)\big|^{\frac{Q}{d}}du^*
\end{aligned}$$

so finally we obtain

$$B\le\frac{Q}{2}\big(\Gamma\big(\tfrac{d}{2}\big)\big)^{\frac{Q}{Q-d}}\Big[2^{Q/2-2}\int_\Sigma\big|\cos\big(\tfrac{d}{2}\theta\big)\big|^{\frac{Q}{d}}du^*\Big]^{\frac{d}{Q-d}}.$$

If B_{sharp} is the sharp constant in (2.13) then we have the bounds

$$\frac{(2Q)^{1/p'}}{\|g_d\|_{p'}}\le B_{\mathrm{sharp}}^{1/p'}\le(2Q)^{1/p'}\,\frac{n!\omega_{2n+1}}{8}\,\|g_{Q-d}\|_p \tag{2.19}$$

where g_d is defined as in (2.12) and where $\|g_d\|_p$ is the norm in the space $L^p(\Sigma)$. The right hand side in (2.19) is strictly bigger than the left hand side, unless $p=2$, i.e. $d=Q/2$, in which case they are equal. This can be seen via Hölder's inequality

$$\|g_{Q-d}\|_p\|g_d\|_{p'}\ge\int_\Sigma g_{Q-d}(\theta)g_d(\theta)du^*=\frac{8}{n!\omega_{2n+1}} \tag{2.20}$$

where the last identity follows from

$$\int_0^{\pi/2}\cos(a\theta)(\cos\theta)^{n-1}d\theta=\frac{2^{-n}\pi\Gamma(n)}{\Gamma\big(\frac{n+1+a}{2}\big)\Gamma\big(\frac{n+1-a}{2}\big)}, \tag{2.21}$$

which can easily be verified by induction. Equalities in (2.19) hold if and only if equality holds on the left in (2.20), which happens if and only if g_d is a multiple of g_{Q-d}, or $d=Q/2$. Moreover in this case we obtain the sharp constant given in (2.15).

The argument for the case P_d acting on $\mathcal{H}^p$ is similar. This time we consider

$$f_R(\zeta)=\sum_{k\ge k_0}\frac{\Gamma\big(k+\frac{d}{2}\big)}{\Gamma\big(\frac{d}{2}\big)\Gamma(k+1)}R^k\zeta_{n+1}^k=\operatorname{Re}(1-R\zeta_{n+1})^{-\frac{d}{2}}+O(1)$$

where $\mu_{k0} > 0$ for $k \geq k_0$, which obviously belongs to $\mathcal{H} \cap \operatorname{Ker}(P_d)^\perp$. The argument proceeds as before. We have now

$$n!\omega_{2n+1} P_d^{-1}(\zeta,\eta) = \Gamma\big(\tfrac{Q-d}{2}\big)\mathrm{Re}\,(1-\zeta\cdot\overline{\eta})^{\frac{d-Q}{2}} + (\text{ more regular kernels })$$

and

$$P_d^{-1} f_R(\zeta) = \frac{1}{\Gamma\big(\frac{d}{2}\big)} \log \frac{1}{1-R\zeta_{n+1}} + O(1)$$

so if we let $F_R(\zeta) = f_R(\zeta)\Gamma\big(\frac{d}{2}\big)\big(\log\frac{1}{1-R}\big)^{-1}$, then (2.16), (2.17) still holds and

$$\|F_R\|_p^{p'} \sim \left(\frac{\Gamma\big(\frac{d}{2}\big)}{\log\frac{1}{1-R}}\right)^{p'} \left(\int_{S^{2n+1}} |1-R\zeta_{n+1}|^{-\frac{Q}{2}} d\zeta\right)^{p'-1}$$

and one can proceed as before (with easier calculations) with g_d constant, given in the statement of the theorem. Details are left to the reader.

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Adams inequalities for powers of sublaplacians and related operators

In this section we obtain sharp Adams inequalities for $\mathcal{L}^{d/2}$, and more generally for powers of operators of type $L_{a,b} := a\mathcal{L}\pi + b\mathcal{L}\pi^\perp$, where $\pi^\perp := I - \pi$ on L^p. We will need these inequalities later on (see the proof of Prop. 3.4). Again, the first step is to have estimates on the fundamental solutions of these operators; for sake of exposition we postpone the proofs of these estimates at the end of this section.

The starting point is an explicit formula for the fundamental solution of the powers of the $\mathbb{H}^n$ sublaplacian:

$$\mathcal{L}_0^{-d/2}(u,0) = \tfrac{1}{2}\, G_d(\theta)|u|^{d-Q} \tag{2.22}$$

$$G_d(\theta) = \frac{2^{n+1}\Gamma\big(\frac{Q-d}{2}\big)}{\pi^{n+1}\Gamma\big(\frac{d}{2}\big)}\,\mathrm{Re}\left\{e^{i\frac{Q-d}{2}\theta}\int_0^\infty \Big(\frac{s}{1-e^{-2s}}\Big)^{\frac{d}{2}-1} \frac{e^{-ns}}{(e^{2i\theta}+e^{-2s})^{\frac{Q-d}{2}}}\,ds\right\} \tag{2.23}$$

which was derived first by [BDR] in case d an even integer, and later by [CT] for any $d < Q$ using the heat kernel approach.

Proposition 2.6. *The fundamental solution of $\mathcal{D}^{d/2}$, for $0 < d < Q$, satisfies*

$$\begin{aligned}\mathcal{D}^{-d/2}(\zeta,\eta) &= G_d(\theta) d(\zeta,\eta)^{d-Q} + O\big(d(\zeta,\eta)^{d-Q+\epsilon}\big)\\ &= 2^{\frac{d-Q}{2}} G_d(\theta)|1-\zeta\cdot\overline{\eta}|^{\frac{d-Q}{2}} + O\big(|1-\zeta\cdot\overline{\eta}|^{\frac{d-Q+\epsilon}{2}}\big)\end{aligned} \tag{2.24}$$

with $G_d(\theta)$ as in (2.23), a bounded and positive function. Moreover,

$$|\mathcal{D}^{-Q/2}(\zeta,\eta)| \leq C\big(1 + \big|\log|1-\zeta\cdot\overline{\eta}|\,\big|\big), \qquad |\mathcal{D}^{-d/2}(\zeta,\eta)| \leq C \quad \textit{if } d > Q. \tag{2.25}$$

Corollary 2.7. *The fundamental solution of $\mathcal{L}^{d/2}$ $(0 < d < Q)$ satisfies*

$$\mathcal{L}^{-d/2}(\zeta,\eta) = G_d(\theta)d(\zeta,\eta)^{d-Q} + O\big(d(\zeta,\eta)^{d-Q+\epsilon}\big) \tag{2.26}$$

with $G_d(\theta)$ as in (2.23).

Corollary 2.8. *Let $0 < d < Q$ and*

$$L_{a,b} := a\mathcal{L}\pi + b\mathcal{L}\pi^\perp, \qquad a,b>0.$$

Then $L_{a,b}^{d/2}$ is continuous on $W^{d,p}$ and invertible on the subspace of functions with zero mean. Its fundamental solution satisfies

$$L_{a,b}^{-d/2}(\zeta,\eta) = 2^{\frac{d-Q}{2}}\left[\frac{g_d(\theta)}{(an/2)^{d/2}} + \frac{g_d^\perp(\theta)}{b^{d/2}}\right]|1-\zeta\cdot\overline{\eta}|^{\frac{d-Q}{2}} + O\big(|1-\zeta\cdot\overline{\eta}|^{\frac{d-Q}{2}+\epsilon}\big) \tag{2.27}$$

$$g_d^\perp(\theta) = G_d(\theta) - \frac{g_d(\theta)}{(n/2)^{d/2}} \tag{2.28}$$

for a suitable $\epsilon > 0$, and with $g_d(\theta)$ as in (2.12), and $G_d(\theta)$ as in (2.23).

The following result yields more information on the function $G_d(\theta)$, and it will be useful in the explicit computation of sharp Adams constants for the case $p = 2$.

Proposition 2.9. *$G_d(\theta)$ has the following trigonometric expansion*

$$G_d(\theta) = \sum_{k=0}^{\infty}\frac{g_{k,d}(\theta)}{\lambda_k^{d/2}} \tag{2.29}$$

where

$$g_{k,d}(\theta) = \frac{2^{\frac{Q-d}{2}+1}}{\omega_{2n+1}n!}\sum_{\ell=0}^{k}\frac{(-1)^\ell\Gamma(k-\ell+d/2-1)\Gamma(\ell+n-d/2+1)}{\Gamma(d/2-1)\Gamma(k-\ell+1)\Gamma(\ell+1)}\,\cos\big[\big(2\ell+\tfrac{Q-d}{2}\big)\theta\big] \tag{2.30}$$

if $d \neq 2$, with the series converging in the sense of distributions, and

$$g_{k,2}(\theta) = \frac{(-1)^k 2^{n+1}}{\omega_{2n+1}n!}\cdot\frac{\Gamma(k+n)}{\Gamma(k+1)}.$$

Moreover,

$$\int_\Sigma g_{k,d}g_{j,Q-d}du^* = \frac{4\,\Gamma(k+n)}{\pi^{n+1}\Gamma(n)\Gamma(k+1)}\,\delta_{j,k}. \tag{2.31}$$

In particular note that $g_d(\theta)(n/2)^{-d/2}$ is the first term in the expansion (2.29), and this justifies the notation $g_d^\perp$ in (2.28) (Cf. also (2.20)). Formula (2.29) appeared in [BDR], for the case d an even integer, while the orthogonality relation (2.31) seems to be new.

The following is now an immediate consequence of the above results combined with Theorem 2.2, (for the sharpness statement see (2.9) and the comment thereafter):

Theorem 2.10. *Let $L_{a,b} = a\mathcal{L}\pi + b\mathcal{L}\pi^\perp$ $(a, b > 0)$. Then there is $C_0 > 0$ so that for any $F \in W^{d,p}$ with zero mean and with $p = \frac{Q}{d}$, $\frac{1}{p} + \frac{1}{p'} = 1$*

$$\int_{S^{2n+1}} \exp\left[A_d(a,b)\left(\frac{|F|}{\|L_{a,b}^{d/2}F\|_p}\right)^{p'}\right] d\zeta \le C_0 \tag{2.32}$$

with

$$A_d(a,b) = \frac{2Q}{\displaystyle\int_\Sigma \left|\frac{g_d(\theta)}{(an/2)^{d/2}} + \frac{g_d^\perp(\theta)}{b^{d/2}}\right|^{p'} du^*} \tag{2.33}$$

and the constant $A_d(a,b)$ is sharp. If $d = \frac{Q}{2}$, or $p = p' = 2$

$$A_{Q/2}(a,b) = \frac{\omega_{2n+1}(n+1)!}{2\left[\left(\frac{2}{an}\right)^{n+1} + \frac{1}{b^{n+1}}\displaystyle\sum_{k=1}^{\infty}\binom{k+n-1}{n-1}\left(k+\frac{n}{2}\right)^{-n-1}\right]}. \tag{2.34}$$

Corollary 2.11. *There is $C_0 > 0$ so that for any $F \in W^{d,p}$ with zero mean and with $p = \frac{Q}{d}$, $\frac{1}{p} + \frac{1}{p'} = 1$*

$$\int_{S^{2n+1}} \exp\left[A_d\left(\frac{|F|}{\|\mathcal{L}^{d/2}F\|_p}\right)^{p'}\right] d\zeta \le C_0$$

with

$$A_d = \frac{2Q}{\displaystyle\int_\Sigma |G_d(\theta)|^{p'}\, du^*} \tag{2.35}$$

and the constant A_d is sharp. If $d = \frac{Q}{2}$, or $p = p' = 2$

$$A_{Q/2} = \frac{(n+1)(n-1)!\,\pi^{n+1}}{\displaystyle\sum_{k=0}^{\infty}\frac{(k+n-1)!}{k!\left(k+\frac{n}{2}\right)^{n+1}}}. \tag{2.36}$$

In particular,

$$A_{Q/2} = \begin{cases} 4 & \text{if } n = 1 \\ 18\pi & \text{if } n = 2 \\ \dfrac{192\,\pi^2}{12-\pi^2} & \text{if } n = 3. \end{cases}$$

Remarks.

1. The constant in (2.36) can be computed in principle for any given n, by using partial fractions and the values of the Hurwitz zeta function $\sum_0^\infty (k+a)^{-s}$, when $a = n/2$ and s is even.

2. By means of Prop. 2.6 Corollary 2.11 above holds also for $\mathcal{D}^{d/2}$ with the same constant as in (2.35) (and for all functions in $W^{d,p}$).

Proof of Proposition 2.6. Start with formula (1.7), which for convenience we rewrite as

$$\mathcal{L}_0\Big(J^{\frac{Q-2}{4}}(F\circ\mathcal{C})\Big) = J^{\frac{Q+2}{4}}(\mathcal{D}F)\circ\mathcal{C} \tag{2.37}$$

with

$$J = (2|J_{\mathcal{C}}|)^{\frac{2}{Q}} = \frac{4}{1+2|z|^2+|u|^4} = 4 + O(|u|^\epsilon) \tag{2.38}$$

for any $0 < \epsilon \le 2$.

By (1.2), if $\zeta = \mathcal{C}(u)$, $\eta = \mathcal{C}(v)$

$$d(\zeta,\eta) = d(u,v)J(u)^{\frac14}J(v)^{\frac14}. \tag{2.39}$$

Let's first assume that $d = 2N$ is an even integer. If $N = 1$ then use (1.6). If $N > 1$, from (1.6)

$$\mathcal{D}^{-N}(\zeta,\eta) = c_2^N \int_{(S^{2n+1})^{N-1}} \prod_{i=1}^N d(\zeta^{i-1},\zeta^i)^{2-Q} d\zeta^1 ... d\zeta^{N-1} \tag{2.40}$$

with $\zeta^0 = \zeta$, $\zeta^N = \eta$. It's easy to see that this quantity is bounded in (ζ,η) over the region $d(\zeta,\eta) \ge 1$, (write S^{2n+1} as a union of the regions $\{(\zeta^1,...,\zeta^{N-1}) \in (S^{2n+1})^{N-1} : d(\zeta^{j-1},\zeta^j) = \max_{1\le i\le N} d(\zeta^{i-1},\zeta^i)\}$, $j = 1,...,N-1$; in that region $d(\zeta^{j-1},\zeta^j) \ge N^{-1}$), so it's enough to consider the case $d(\zeta,\eta) \le 1$.

From (2.39) and (2.38) with suitably small ϵ, and with $\zeta^i = \mathcal{C}(u^i)$

$$
\begin{aligned}
\mathcal{D}^{-N}(\zeta,\eta) &= c_2^N J(u_0)^{\frac{2-Q}{4}} J(u_N)^{\frac{2-Q}{4}} \int_{(\mathbb{H}^n)^{N-1}} \prod_{i=1}^{N} d(u^{i-1},u^i)^{2-Q} \prod_{i=1}^{N-1} J(u^i)^{\frac{2-Q}{2}} |J_{\mathcal{C}}(u^i)| du^1 ... du^{N-1} \\
&= 2^{-(N-1)} c_2^N J(u^0)^{\frac{2-Q}{4}} J(u^N)^{\frac{2-Q}{4}} \int_{(\mathbb{H}^n)^{N-1}} \prod_{i=1}^{N} d(u^{i-1},u^i)^{2-Q} \prod_{i=1}^{N-1} J(u^i) du^1 ... du^{N-1} \\
&= 2^{N-1} c_2^N J(u^0)^{\frac{2-Q}{4}} J(u^N)^{\frac{2-Q}{4}} \int_{(\mathbb{H}^n)^{N-1}} \prod_{i=1}^{N} d(u^{i-1},u^i)^{2-Q} \Big(1 + O(|u^1|^\epsilon + ... |u^{N-1}|^\epsilon)\Big) du^1 ... du^{N-1} \\
&= 2^{2N-1} J(u^0)^{\frac{2-Q}{4}} J(u^N)^{\frac{2-Q}{4}} \mathcal{L}_0^{-N}(u^0,u^N) + \\
&\quad + 2^{N-1} c_2^N J(u^0)^{\frac{2-Q}{4}} J(u^N)^{\frac{2-Q}{4}} \sum_{k=1}^{N-1} \int_{(\mathbb{H}^n)^{N-1}} \prod_{i=1}^{N} |(u^i)^{-1} u^{i-1}|^{2-Q} |u^k|^\epsilon du^1 ... du^{N-1}
\end{aligned}
\tag{2.41}
$$

By translation and homogeneity each term in the sum is $O(|(u^N)^{-1}u^0|^{2N+\epsilon-Q})$, where ϵ very small, and so from (2.22) we obtain

$$
\begin{aligned}
\mathcal{D}^{-N}(\zeta,\eta) &= 4^{N-1} J(u)^{\frac{2-Q}{4}} J(v)^{\frac{2-Q}{4}} \Big[G_{2N}(\theta) d(u,v)^{2N-Q} + O\big(d(u,v)^{2N+\epsilon-Q}\big)\Big] \\
&= 4^{N-1} J(u)^{\frac{1-N}{2}} J(v)^{\frac{1-N}{2}} \Big[G_{2N}(\theta) d(\zeta,\eta)^{2N-Q} + O\big(d(u,v)^{2N+\epsilon-Q}\big)\Big]
\end{aligned}
$$

By rotation invariance of $\mathcal{D}$, we can assume that $\eta = \mathcal{N}$, i.e. $v = 0$, and the condition $d(\zeta,\eta) \le 1$ implies (using (2.39)) that $|u| \le 3^{-1/2}$. Under these condition the above formula immediately gives (2.24).

Now assume $d = 2\alpha$, with $\alpha \in (0,1)$. By a well known formula ([RS], p. 317) for $0 < \alpha < 1$

$$
\mathcal{D}^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{-\alpha} (\mathcal{D}+\lambda)^{-1} d\lambda.
$$

From (2.37) we obtain

$$
\Big((\mathcal{D}+\lambda)^{-1} F\Big) \circ \mathcal{C} = \Big(J^{-\frac{Q+2}{4}} \mathcal{L}_0 J^{\frac{Q-2}{4}} + \lambda\Big)^{-1} (F \circ \mathcal{C}) = J^{\frac{2-Q}{4}} (\mathcal{L}_0 + \lambda J)^{-1} J^{\frac{Q+2}{4}} (F \circ \mathcal{C})
$$

that is, if $\zeta = \mathcal{C}(u)$, $\eta = \mathcal{C}(v)$ and using (2.38)

$$
(\mathcal{D}+\lambda)^{-1}(\zeta,\eta) = 2J(u)^{\frac{2-Q}{4}} J(v)^{\frac{2-Q}{4}} \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{-\alpha} (\mathcal{L}_0 + \lambda J)^{-1}(u,v) d\lambda
$$

and also

$$
(\mathcal{L}_0 + \lambda J)^{-1}(u,v) = \int_0^\infty e^{-\tau(\mathcal{L}_0 + \lambda J)}(u,v) d\tau
$$

Now we would like to invoke the Feynman-Kac formula, in order to write $e^{-\tau(\mathcal{L}_0+\lambda J)}(u,v)$ explicitly, and so a small preamble is necessary. First, the heat kernel $P(\tau,u,v)$ for the sublaplacian $\mathcal{L}_0$ is a symmetric, positive C^∞ function [G], that defines a strongly continuous, positivity preserving contraction semigroup on $C_0(\mathbb{H}^n)$. The corresponding Brownian motion with initial distribution δ_u=Dirac's measure at $u\in\mathbb{H}^n$ is defined by a measure μ_u on the Skorohod space $\mathcal{P}$ of sample paths B (right continuous functions with left limits on $[0,\infty)$, valued in the compactification of $\mathbb{H}^n$, and staying at the point at infinity after hitting it), endowed with the σ-algebra $\mathcal{F}_0$ generated by the coordinate functions on $\mathcal{P}$. The conditional or pinnned measure $\mu^0_{u,v,t}$ on $\mathcal{P}$ is the probability measure defined by the transition function

$$P_{v,\tau}(\sigma,u,w)=\frac{P(\tau-\sigma,v,w)P(\sigma,w,u)}{P(t,u,v)} \tag{2.42}$$

on the space $(\mathcal{P},\mathcal{F}_{0,\tau})$, where $\mathcal{F}_{0,\tau}$ is generated by the coordinate functions in $\mathcal{P}$ up to time τ. Such measure identifies the paths that with probabiliy 1 start at u and end at v at time τ. The nonhomogeneous pinned measure, or conditional Wiener measure, is then defined as

$$\mu_{u,v,\tau}=P(\tau,u,v)\mu^0_{u,v,\tau},$$

ad the Feynman-Kac formula (or one version of it) states that

$$e^{-\tau(\mathcal{L}_0+\lambda J)}(u,v)=\int_{\mathcal{P}}\exp\Big(-\lambda\int_0^\tau J(B_\sigma)d\sigma\Big)\,d\mu_{u,v,\tau}(B)$$

with continuity in (u,v) (see DvC]...).

With the aid of this formula we easily conclude that

$$\begin{aligned}\mathcal{D}^{-\alpha}(\zeta,\eta)&=2J(u)^{\frac{2-Q}{4}}J(v)^{\frac{2-Q}{4}}\frac{\sin\pi\alpha}{\pi}\Gamma(1-\alpha)\int_0^\infty\int_{\mathcal{P}}\Big(\int_0^\tau J(B_\sigma)d\sigma\Big)^{\alpha-1}d\mu_{u,v,\tau}(B)\,d\tau\\&=2J(u)^{\frac{2-Q}{4}}J(v)^{\frac{2-Q}{4}}\frac{1}{\Gamma(\alpha)}\int_0^\infty\tau^{\alpha-1}\int_{\mathcal{P}}\Big(\int_0^1 J(B_{\sigma\tau})d\sigma\Big)^{\alpha-1}d\mu_{u,v,\tau}(B)\,d\tau\end{aligned}$$

But

$$\Big(\int_0^1 J(B_{\sigma\tau})d\sigma\Big)^{\alpha-1}=4^{\alpha-1}+O\Big(\int_0^1\big(|B_{\sigma\tau}|^{2(1-\alpha)}+|B_{\sigma\tau}|^{4(1-\alpha)}\big)d\sigma\Big),$$

by Jensen's inequality and the fact that $J\le 4$, so that

$$\begin{aligned}\mathcal{D}^{-\alpha}(\zeta,\eta)=2J(u)^{\frac{2-Q}{4}}J(v)^{\frac{2-Q}{4}}\Big[&4^{\alpha-1}\mathcal{L}_0^{-\alpha}(u,v)+\\&+O\Big(\int_0^\infty\tau^{\alpha-1}\int_{\mathcal{P}}\int_0^1\big(|B_{\sigma\tau}|^{2(1-\alpha)}+|B_{\sigma\tau}|^{4(1-\alpha)}\big)d\sigma d\mu_{u,v,\tau}(B)\,d\tau\Big)\Big].\end{aligned}$$

By rotation invariance of $\mathcal{D}$ we can assume $v=0$, from which we can handle the first term of the above identity as we did for (2.41):

$$2\cdot 4^{\alpha-1}J(u)^{\frac{2-Q}{4}}J(v)^{\frac{2-Q}{4}}\mathcal{L}_0^{-\alpha}(u,v)=4^{\alpha-1}J(u)^{\frac{2-Q}{4}}J(v)^{\frac{2-Q}{4}}G_{2\alpha}(\theta)d(u,v)^{2\alpha-Q}$$
$$=4^{\alpha-1}J(u)^{\frac{1-\alpha}{2}}J(v)^{\frac{1-\alpha}{2}}G_{2\alpha}(\theta)d(\zeta,\eta)^{2\alpha-Q}=G_{2\alpha}(\theta)d(\zeta,\eta)^{2\alpha-Q}+O\big((d(\zeta,\eta)^{2\alpha+\epsilon-Q}\big)$$

To estimate the error term note first the formula

$$\int_{\mathcal{P}}\int_0^1 f(B_{\tau\sigma})d\sigma d\mu_{u,v,\tau}(B)=\int_{\mathbb{H}^n}\int_0^1 P(\tau\sigma,u,w)P(\tau-\tau\sigma,w,v)f(w)d\sigma dw$$

which is a consequence of (2.42), and valid for nonnegative measurable f. From (4.7) of [BGG] we have the estimate

$$P(\tau,u,0)\leq C\tau^{-Q/2}e^{-c|u|^2/\tau}$$

for some constants $C,c>0$ and therefore, for $\gamma=2(1-\alpha)$ or $\gamma=4(1-\alpha)$

$$\int_0^\infty \tau^{\alpha-1}\int_{\mathcal{P}}\int_0^1 |B_{\sigma\tau}|^\gamma d\sigma d\mu_{u,v,\tau}(B)\,d\tau\leq$$
$$\leq C\int_0^1\big(\sigma(1-\sigma)\big)^{-Q/2}d\sigma\int_0^\infty\tau^{\alpha-1-Q}\int_{\mathbb{H}^n}e^{-c|w^{-1}u|^2/(\tau\sigma)-c|w|^2/(\tau-\tau\sigma)}|w|^\gamma dwd\tau.$$

We estimate the (σ,τ) integral by splitting the σ-interval in half.

$$\int_0^{1/2}d\sigma\int_0^\infty\big(\sigma(1-\sigma)\big)^{-Q/2}\tau^{\alpha-1-Q}e^{-c|w^{-1}u|^2/(\tau\sigma)-c|w|^2/(\tau-\tau\sigma)}d\tau\leq$$
$$\leq C\int_0^{1/2}d\sigma\int_0^\infty\sigma^{-Q/2}\tau^{\alpha-1-Q}e^{-c|w^{-1}u|^2/(\tau\sigma)-2c|w|^2/\tau}d\tau$$
$$\leq\int_1^\infty d\sigma\int_0^\infty\sigma^{Q/2-2}\tau^{Q-\alpha-1}e^{-c\tau\sigma c|w^{-1}u|^2-2c\tau|w|^2}d\tau$$

Using the estimate $\int_1^\infty\sigma^\lambda e^{-\sigma A}d\sigma\leq CA^{-\lambda-1}e^{-A/2}$ (for $\lambda,A>0$) we get that the above integral is bounded above by

$$|w^{-1}u|^{2-Q}\int_0^\infty\tau^{Q/2-\alpha}e^{-c\tau|w^{-1}u|^2/2-2c\tau|w|^2}d\tau=C|w^{-1}u|^{2-Q}\Big(|w^{-1}u|^2+|w|^2\Big)^{-Q/2+\alpha-1}.$$

Likewise,

$$\int_{1/2}^1 d\sigma\int_0^\infty\big(\sigma(1-\sigma)\big)^{-Q/2}\tau^{\alpha-1-Q}e^{-c|w^{-1}u|^2/(\tau\sigma)-c|w|^2/(\tau-\tau\sigma)}d\tau\leq$$
$$\leq|w|^{2-Q}\int_0^\infty\tau^{Q/2-\alpha}e^{-2c\tau|w^{-1}u|^2-c\tau|w|^2/2}d\tau=C|w|^{2-Q}\Big(|w^{-1}u|^2+|w|^2\Big)^{-Q/2+\alpha-1}.$$

Combining these two estimates together

$$\int_0^\infty \tau^{\alpha-1}\int_{\mathcal{P}}\int_0^1 |B_{\sigma\tau}|^\gamma d\sigma d\mu_{u,v,\tau}(B)\,d\tau \le$$
$$\le C\int_{\mathbb{H}^n}\big(|w^{-1}u|^{2-Q}+|w|^{2-Q}\big)\big(|w^{-1}u|^2+|w|^2\big)^{-Q/2+\alpha-1}|w|^\gamma dw \le C|u|^{2\alpha-Q+\gamma}$$

where the last inequality comes from the homogeneity in u of the last integral, and the fact that the integrand is in L^1, for $\gamma = 2(1-\alpha)$ or $\gamma = 4(1-\alpha)$ and $u \ne 0$, (which makes the whole thing continuous on the Heisenberg sphere $|u| = 1$). Note that $2\alpha+\gamma < 4 \le Q$.

The general case $d/2 = N+\alpha,\ N \in \mathbb{N},\ \alpha \in (0,1)$ can be handled by writing $\mathcal{D}^{-d/2} = \mathcal{D}^{-N}\mathcal{D}^{-\alpha}$ and combining the asymptotic expansions so far obtained.

Notice that the positivity and boundedness of G_d is simply due to the homogeneity of the Heisenberg heat kernel $P(\tau,\lambda u,0) = \lambda^{-Q}P(\tau\lambda^{-2},u,0)$. With this formula we have

$$\tfrac12\, G_d(\theta) = \frac{1}{\Gamma\big(\frac d2\big)}\int_0^\infty \tau^{d/2-1}P(\tau,u^*,0)d\tau$$

which is positive and bounded, as the heat kernel is smooth and positive away from the origin.

Finally, if $d \ge Q$, from (2.40) (which is valid for any N) we need only assume $d(\zeta,\eta) \le 1$, and also $\eta = \mathcal{N}$, $v = 0$, and $|u| \le 3^{-1/2}$. Moreover

$$\mathcal{D}^{-d/2}(\zeta,\eta) = \mathcal{D}^{-\frac{d-Q+1}{2}}\mathcal{D}^{-\frac{Q-1}{2}}(\zeta,\eta) \le C\int_{S^{2n+1}} d(\zeta,\xi)^{d+1-2Q}d(\xi,\eta)^{-1}d\xi =$$
$$= CJ(u)^{\frac{d+1-2Q}{4}}J(v)^{-\frac14}\int_{\mathbb{H}^n} d(u,w)^{d+1-2Q}d(w,v)^{-2}J(w)^{d/4}dw \le C\int_{\mathbb{H}^n}\frac{|w^{-1}u|^{d+1-2Q}|w|^{-1}}{1+|w|^d}\,dw$$

and by splitting the integral over the two regions $\{w:\ |w^{-1}u| \le 5|u|\}$ and $\{w:\ |w^{-1}u| \ge 5|u|\}$ one can easily see that the integral is bounded in u if $d > Q$ or $O(\log|u|^{-1})$ if $d > Q$.

///

Proof of Corollary 2.7. Recall that $\mathcal{D} = \mathcal{L} + \frac{n^2}{4} = \mathcal{L} + \lambda_0^2$. For any $N \in \mathbb{N}$

$$(\lambda_j\lambda_k - \lambda_0^2)^{-d/2} = (\lambda_j\lambda_k)^{-d/2}\sum_{p=0}^N \frac{\Gamma(p+\frac d2)}{\Gamma(p+1)\Gamma\big(\frac d2\big)}\left(\frac{\lambda_0^2}{\lambda_j\lambda_k}\right)^p + O\big((\lambda_j\lambda_k)^{-N-1}\big)$$

for $(j,k) \ne (0,0)$. Now it's easy to see that for any fixed $\alpha > Q/2$ and any j,k

$$\|\Phi_{jk}\|_\infty = (\lambda_j\lambda_k)^{\alpha/2}\|\mathcal{D}^{-\alpha/2}\Phi_{jk}\|_\infty \le C(\lambda_j\lambda_k)^{\alpha/2}\|\Phi_{jk}\|_2 =$$
$$= Cm_{jk}(\lambda_j\lambda_k)^{\alpha/2} \le C(\lambda_j+\lambda_k)(\lambda_j\lambda_k)^{n-1+\alpha/2}$$

with C independent of j,k. This means that for N large enough the series $\sum_{j,k}(\lambda_j\lambda_k)^{-N-1}\Phi_{jk}$ converges absolutely and uniformly on $S^{2n+1}\times S^{2n+1}$, and that the asymptotic expansion of $\mathcal{L}^{-d/2}$ is determined by that of $\mathcal{D}^{-d/2}+a_1\mathcal{D}^{-d/2-1}+...+a_N\mathcal{D}^{-d/2-N}$, some constants a_j, which gives (2.26) by the previous proposition.

///

Proof of Corollary 2.8. It's enough to observe that the operator $\left(\frac{2}{n}\mathcal{L}\right)^{d/2}\pi$ satisfies the hypothesis of Prop. 2.3, as an operator on $\mathcal{P}$. The rest is a consequence of Prop. 2.3 and Corollary 2.7.

///

Proof of Proposition 2.9. The expansion (2.29) follows easily as in [BDR], but using formula (2.23) and writing $(1-e^{-2s})^{d/2-1}$ and $(e^{2i\theta}+e^{-2s})^{-(Q-d)/2}$ as binomial series. To show (2.31) we proceed by brute calculations, leaving some details for the reader to check.

Assume for now that d is not an even integer. From (2.21) it is straightforward to check that for given $\ell'\in\mathbb{N}$

$$\begin{aligned}&\int_\Sigma g_{k,d}\cos\big[\big(2\ell'+\tfrac{d}{2}\big)\theta\big]du^*=2\omega_{2n-1}\int_0^{\pi/2}g_{k,d}(\theta)\cos\big[\big(2\ell'+\tfrac{d}{2}\big)\theta\big](\cos\theta)^{n-1}d\theta=\\&=\frac{2^{2-d/2}}{\Gamma\big(\frac{d}{2}-1\big)}\sum_{\ell=0}^{k}\frac{(-1)^\ell\,\Gamma\big(k-\ell+\frac{d}{2}-1\big)\Gamma\big(\ell+n+1-\frac{d}{2}\big)}{\Gamma(k-\ell+1)\Gamma(\ell+1)\Gamma\big(\ell-\ell'+n+1-\frac{d}{2}\big)\Gamma\big(\ell'-\ell+\frac{d}{2}\big)}.\end{aligned}\tag{2.43}$$

The sum in (2.43) can be evaluated explicitly as follows. Write the right hand side of (2.43) as

$$\begin{aligned}&\frac{2^{2-d/2}(-1)^{\ell'+k+1}}{\Gamma\big(\frac{d}{2}-1\big)\Gamma(k+1)}\cdot\frac{\Gamma\big(n+1-\frac{d}{2}\big)\Gamma\big(-\ell'-\frac{d}{2}+1\big)}{\Gamma\big(-k-\frac{d}{2}+2\big)\Gamma\big(-\ell'+n+1-\frac{d}{2}\big)}\times\\&\quad\times\,{}_3F_2\Big(-k,n-\frac{d}{2}+1,-\ell'-\frac{d}{2}+1;-k-\frac{d}{2}+2,-\ell'+n+1-\frac{d}{2}\Big)\end{aligned}\tag{2.44}$$

where ${}_3F_2(-k,a,b;c,d)$ denotes the classical terminating Saalschützian (balanced) hypergeometric series (evaluated at 1), i.e. with $d=1+a+b-c-k$. Such sum can be explicitly evaluated using Saalschütz's formula (see e.g. [EMOT] 2.1.5 (30), 4.4 (3))

$${}_3F_2(-k,a,b;c,d)=\frac{(c-a)_k(c-b)_k}{(c)_k(c-a-b)_k},\qquad(\alpha)_k=\alpha(\alpha+1)...(\alpha+k-1).$$

With this formula we obtain that (2.44) vanishes if $\ell' < k$, whereas if $\ell' = k$ then (2.43) equals

$$\frac{2^{2-d/2}(-1)^k\,\Gamma(k+n)}{\Gamma(n)\,\Gamma\big(k+\frac{d}{2}\big)}.$$

If now $j \leq k$, write $g_{j,Q-d}$ using (2.30), but as a sum over ℓ' and integrate against $g_{k,d}$: if $j < k$ each term is 0, if $j = k$ then the only term that survives is the one corresponding to $\ell' = k$ which yields precisely (2.31).

The case d an even integer follows now by continuity.

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3. Beckner-Onofri's inequalities

The goal of this section is to establish the sharp Beckner-Onofri inequality for real CR-pluriharmonic functions on the sphere:

Theorem 3.1. *For any $F \in W^{Q/2,2} \cap \mathbb{R}\mathcal{P}$ we have the inequality*

$$\frac{1}{2(n+1)!}\fint F\mathcal{A}'_Q F\,d\zeta + \fint F\,d\zeta - \log\fint e^F\,d\zeta \geq 0. \tag{3.1}$$

The inequality is invariant under the conformal group of S^{2n+1}, in the sense that the functional on the left hand side is invariant under the action $F \to F_\tau = F\circ\tau + \log|J_\tau|$, for $\tau \in \mathrm{CON}(S^{2n+1})$. Equality in (3.1) holds if and only if $F = \log|J_\tau|$, for some $\tau \in \mathrm{CON}(S^{2n+1})$.

There is a corresponding version of (3.1) for general complex-valued CR-pluriharmonic functions F:

$$\frac{1}{2(n+1)!}\fint \overline{F}\mathcal{A}'_Q F\,d\zeta + \fint \mathrm{Re}\,F\,d\zeta - \log\fint e^{\mathrm{Re}\,F}\,d\zeta \geq 0.$$

but it is a trivial consequence of the real-valued case.

As we mentioned in the introduction, the proof of this theorem is based on the original compactness argument given by Onofri in dimension 2, and later perfected and extended to any dimensions by Chang-Yang, to provide an alternative proof of Beckner's result.

Define once and for all

$$\mathcal{J}[F] = \frac{1}{2(n+1)!}\fint F\mathcal{A}'_Q F\,d\zeta + \fint F\,d\zeta - \log\fint e^F\,d\zeta,$$

for any $F \in W^{Q/2,2} \cap \mathbb{R}\mathcal{P}$.

We divide the proof in three main steps:

I. Conformal invariance of $\mathcal{J}$

II. Existence of a minimum for $\mathcal{J}$

III. Characterization of the minimum.

Step I: Conformal invariance of $\mathcal{J}$.

Proposition 3.2. *The conformal action $F \to F_\tau = F \circ \tau + \log |J_\tau|$ preserves $\mathbb{R}\mathcal{P}$ and $W^{Q/2,2} \cap \mathbb{R}\mathcal{P}$. Moreover, such spaces are the minimal closed subspaces of $L^2(S^{2n+1})$, $W^{Q/2,2}$ respectively, which are invariant under the conformal action. Finally, $\mathcal{J}[F_\tau] = \mathcal{J}[F]$, for all $F \in W^{Q/2,2} \cap \mathbb{R}\mathcal{P}$.*

Proof. Since the conformal transformations are restrictions of biholomorphic mappings on the ball, then $F \circ \tau \in \mathbb{R}\mathcal{P}$ if $F \in \mathbb{R}\mathcal{P}$, and likewise for the invariance of $W^{Q/2,2} \cap \mathbb{R}\mathcal{P}$. For τ conformal, using (1.14) we see that that $\log |J_\tau| \in \mathbb{R}\mathcal{P}$. Any subspace M of L^2 invariant under the action must contain the orbit of the function 0, i.e. all functions of type $\log |J_\tau|$; thus (still from (1.14)) every function of type $C - Q \operatorname{Re} \log(1 - \zeta \cdot \omega)$ must be in M, for any given $\omega \in \mathbb{C}^{n+1}$, $|\omega| < 1$. If M is also closed, then it contains all ω-partial derivatives of such functions, evaluated at $\omega = 0$, and therefore M contains every real pluriharmonic polynomial and hence all of $\mathbb{R}\mathcal{P}$.

Next consider the functional

$$\mathcal{J}_d[G] = \frac{1}{\lambda_0(d)^2} \fint G \mathcal{A}_d G \, d\zeta - \left(\fint |G|^{1/\theta} d\zeta \right)^{2\theta}$$

with $\theta = \dfrac{Q-d}{2Q}$. This functional is invariant under the action $G \to G_{\tau,\theta} = (G \circ \tau)|J_\tau|^\theta$; this follows from (1.17). One easily checks that as $\theta \to 0$ (i.e. $d \to Q$)

$$\mathcal{J}_d[1 + \theta F] = \frac{\theta^2}{\lambda_0(d)^2} \fint F \mathcal{A}_d F \, d\zeta + 2\theta \fint F \, d\zeta - 2\theta \log \fint e^F \, d\zeta + O(\theta^2)$$

so that if $F \in W^{Q/2,2} \cap \mathbb{R}\mathcal{P}$, using (1.28) we obtain

$$\left. \frac{d}{d\theta} \right|_{\theta=0} \mathcal{J}_d[1 + \theta F] = 2\mathcal{J}[F]$$

On the other hand $G_{\tau,\theta} = (1 + \theta F)_\tau = 1 + \theta F_\tau + O(\theta^2)$ so that $\mathcal{J}_d[(1 + \theta F)_\tau] = \mathcal{J}_d[1 + \theta F_\tau] + O(\theta^2)$ and by differentiation this implies $\mathcal{J}[F] = \mathcal{J}[F_\tau]$ if $F \in W^{Q/2,2} \cap \mathbb{R}\mathcal{P}$.

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Note. In the Euclidean S^n the minimal subspace of L^2 invariant under the conformal action is the whole L^2. Indeed, in that case, the $\log|J_\tau|$ are of type $C - n\log|1-\omega\cdot\zeta|$, with $\omega \in \mathbb{R}^{n+1}$, $|\omega|<1$. An argument similar to the one used in the above proof shows that the orbit of the function 0 is dense in L^2.

We remark that the proof above is an adaptation of Beckner's argument in [Bec]. Another possible proof of Prop. 3.2 can be given directly as in [CY], without appealing to the intertwining property of $\mathcal{A}_d$, but working directly with $\mathcal{A}'_Q$. We chose Beckner's argument since it shows how the putative sharp, conformally invariant Sobolev inequality $\mathcal{J}_d[G]\geq 0$ i.e.

$$⨍ G\mathcal{A}_d G\,d\zeta \geq \left[\frac{\Gamma\big(\frac{Q+d}{4}\big)}{\Gamma\big(\frac{Q-d}{4}\big)}\right]^2 \|G\|_q^2, \qquad q=\frac{2Q}{Q-d} \tag{3.2}$$

would imply Beckner-Onofri's inequality (3.1), for the Hardy space. Inequality (3.2), or its dual "Hardy-Littlewood-Sobolev" form, is only known for $d=2$, and due to Jerison and Lee [JL1,2].

Step II: Existence of a minimum for $\mathcal{J}$.

From now one we will denote the average of $F\in L^1(S^{2n+1})$ by

$$\widetilde{F} = ⨍ F = \frac{1}{\omega_{2n+1}}\int_{S^{2n+1}} F.$$

Proposition 3.3 (Provisional Beckner-Onofri's inequalities). *For $F\in W^{Q/2,2}\cap\mathbb{R}\mathcal{P}$ we have*

$$\frac{1}{2(n+1)!}⨍ F\mathcal{A}'_Q F\,d\zeta + ⨍ F - \log ⨍ e^F\,d\zeta + C \geq 0 \tag{3.3}$$

for some $C\geq 0$. If $\lambda>0$ then for all $F\in W^{Q/2,2}$ and with $L_\lambda = \frac{2}{n}\mathcal{L}\pi + \lambda^{2/Q}\mathcal{L}\pi^\perp$

$$A_n(\lambda)⨍ FL_\lambda^{Q/2}F\,d\zeta + ⨍ F - \log ⨍ e^F\,d\zeta + C_\lambda \geq 0 \tag{3.4}$$

for some $C_\lambda>0$ and with

$$A_n(\lambda) = \frac{1}{2(n+1)!}\left[1+\frac{1}{\lambda}\sum_{k=1}^\infty \frac{(k+n-1)!}{(n-1)!k!\big(k+\frac{n}{2}\big)^{n+1}}\right] \tag{3.5}$$

Proof. This is a standard argument based on the Adams inequalities (2.13) and (2.32) for the operators $(\mathcal{A}'_Q)^{1/2}$ and $L_\lambda^{Q/4} = (\frac{2}{n}\mathcal{L})^{Q/4}\pi + \sqrt{\lambda}\,\mathcal{L}^{Q/4}\pi^\perp$. If an inequality of type

$$\int_{S^{2n+1}} \exp\left(B\,\frac{|F-\widetilde{F}|^2}{\|TF\|_2^2}\right)d\zeta \leq C_0$$

holds for one of the above operators T and for either $W^{Q/2,2}\cap\mathbb{R}\mathcal{P}$ or $F\in W^{Q/2,2}$ and with zero mean, then letting $\mu = B^{1/2}(F-\widetilde{F})$, $\nu=\frac{1}{2}B^{-1/2}\|TF\|_2^2$ and expanding $(\mu-\nu)^2\ge 0$ we get

$$\frac{1}{4B}\,\|TF\|_2^2 - \log\int e^{F-\widetilde{F}}d\zeta + \log C_0 \ge 0$$

which implies (3.3) and (3.4).

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Remark. We note that (3.3) is valid with $P_{Q/2}^2$ in place of $\mathcal{A}'_Q$, where $P_{Q/2}$ is any operator as in Prop. 2.3, with $d=Q/2$ and with kernel $\mathcal{H}_{00}$ (i.e. the constants).

From (3.3) we now know that $\mathcal{J}$ is a functional that is bounded below on $W^{Q/2,2}\cap\mathbb{R}\mathcal{P}$. The goal now is to show that the minimizing sequence is actually bounded on such space. The first key step is the following Aubin's type inequality, used in the Euclidean setting first by Onofri and Aubin and then by Chang-Yang:

Proposition 3.4 (Aubin's type inequality). *For given $\sigma>\frac{1}{2}$, there exist constants $C_1(\sigma)$, $C_2(\sigma)$ such that for any $W^{Q/2,2}\cap\mathbb{R}\mathcal{P}$ with $\int_{S^{2n+1}}\zeta_j\,e^F d\zeta=0$ for $j=1,2...,n+1$, the following estimate holds*

$$\frac{\sigma}{2(n+1)!}\fint F\mathcal{A}'_Q F\,d\zeta + \fint F\,d\zeta - \log\fint e^F d\zeta + C_1(\sigma)\|\mathcal{L}^{\frac{Q-1}{4}}F\|_2^2 + C_2(\sigma)\ge 0 \qquad (3.6)$$

The proof below is an adaptation of the one in [CY], Lemma 4.6. We present it here because in our case there is an added difficulty, namely that the localization argument (multiplication by cutoff functions) inherent in the proof does not preserve the class $\mathcal{P}$.

Proof. Assume for the moment that $F\in W^{Q/2,2}$, and WLOG assume that $\int_{S^{2n+1}}e^F=\omega_{2n+1}$. Cover S^{2n+1} with $2(2n+2)=2Q$ congruent spherical caps, by considering a cube inscribed inside the sphere, with side $L=2/\sqrt{2n+2}$. By rotation we can assume that if

$$\Omega^1_{\delta_1}=\{x\in S^{2n+1}:\ \delta_1\le x_{2n+2}\le 1\},\qquad \delta_1<\frac{1}{\sqrt{2n+2}}$$

then

$$\int_{\Omega^1_{\delta_1}} e^F \ge \frac{\omega_{2n+1}}{2Q} \qquad (3.7)$$

It's not hard to show that using the hypothesis $\int_{S^{2n+1}}x_{2n+2}e^F=0$, if

$$\Omega^2_{\delta_2}=\{x\in S^{2n+1}:\ -1\le x_{2n+2}\le -\delta_2\},\qquad \delta_2<\frac{\delta_1}{4Q}$$

then

$$\int_{\Omega^2_{\delta_2}} e^F \geq \delta_2 \omega_{2n+1}. \tag{3.8}$$

Let ϕ_1, ϕ_2 be cutoff functions such that $0 \leq \phi_j \leq 1$ and

$$\phi_j = \begin{cases} 1 & \text{on } \Omega^j_{\delta_j} \\ 0 & \text{on } S^{2n+1} \setminus \Omega^j_{\delta_j/2} \end{cases}$$

Consider the operator $L_\lambda = \frac{2}{n}\mathcal{L}\pi + \lambda^{2/Q}\mathcal{L}\pi^\perp$, so that from (3.4), (3.7) we obtain

$$\begin{aligned} \frac{\omega_{2n+1}}{2Q} &\leq \int_{\Omega^1_{\delta_1}} e^F \leq e^{\widetilde{F}} \int_{\Omega^1_{\delta_1}} e^{(F-\widetilde{F})\phi_1} \leq e^{\widetilde{F}} \omega_{2n+1} \fint e^{(F-\widetilde{F})\phi_1} \\ &\leq \omega_{2n+1} e^{\widetilde{F}} e^{C_\lambda} \exp\left[A_n(\lambda) \fint (F-\widetilde{F})\phi_1 L_\lambda^{Q/2}(F-\widetilde{F})\phi_1 + \fint (F-\widetilde{F})\phi_1\right] \end{aligned} \tag{3.9}$$

with $A_n(\lambda)$ as in (3.5), and likewise, using (3.4) and (3.8)

$$\delta_2\omega_{2n+1} \leq \omega_{2n+1} e^{\widetilde{F}} e^{C_\lambda} \exp\left[A_n(\lambda) \fint (F-\widetilde{F})\phi_2 L_\lambda^{Q/2}(F-\widetilde{F})\phi_2 + \fint (F-\widetilde{F})\phi_2\right]. \tag{3.10}$$

Now we claim that, for k an even integer and $\epsilon > 0$

$$\begin{aligned} \Bigg| \int_{S^{2n+1}} (F-\widetilde{F})\phi_j L_\lambda^k (F-\widetilde{F})\phi_j - \left(\tfrac{2}{n}\right)^k \int_{S^{2n+1}} \phi_j^2 \left(\pi \mathcal{L}^{k/2} F\right)^2 - \lambda^{2k/Q} \int_{S^{2n+1}} \phi_j^2 \left(\pi^\perp \mathcal{L}^{k/2} F\right)^2 \Bigg| \\ \leq \epsilon \int_{S^{2n+1}} \left(L_\lambda^{k/2} F\right)^2 + C(\lambda,\epsilon) \int_{S^{2n+1}} F \mathcal{L}^{k-1} F, \end{aligned} \tag{3.11}$$

whereas if k is odd then

$$\begin{aligned} \Bigg| \int_{S^{2n+1}} (F-\widetilde{F})\phi_j L_\lambda^k (F-\widetilde{F})\phi_j - \left(\tfrac{2}{n}\right)^k \int_{S^{2n+1}} \phi_j^2 \left|\nabla_H \pi \mathcal{L}^{\frac{k-1}{2}} F\right|^2 - \\ - \lambda^{2k/Q} \int_{S^{2n+1}} \phi_j^2 \left|\nabla_H \pi^\perp \mathcal{L}^{\frac{k-1}{2}} F\right|^2 \Bigg| \leq \epsilon \int_{S^{2n+1}} \left(L_\lambda^{k/2} F\right)^2 + C(\lambda,\epsilon) \int_{S^{2n+1}} F \mathcal{L}^{k-1} F. \end{aligned} \tag{3.12}$$

Here ∇_H denotes the so-called horizontal gradient defined on complex valued functions as

$$\nabla_H F = \sum_{j=0}^{n+1} (\overline{T}_j \overline{F}\, T_j + T_j \overline{F}\, \overline{T}_j)$$

the T_j being the generators of $T_{1,0}(S^{2n+1})$ defined in (1.3). Such gradient satisfies the identities

$$\nabla_H G \cdot \overline{\nabla_H F} = \frac{1}{2} \sum_{j=1}^{n+1} (\overline{T_j G} T_j F + T_j \overline{G}\, \overline{T}_j F)$$

$$\int_{S^{2n+1}} \overline{G}\mathcal{L}F = \int_{S^{2n+1}} \nabla_H G \cdot \overline{\nabla_H F}$$

Note that $\int_{S^{2n+1}} |\nabla_H L_\lambda^{\frac{k-1}{2}} F|^2 = \int_{S^{2n+1}} (L_\lambda^{\frac{k}{2}} F)^2$. The proof of these estimates is given in the appendix, but the gist of it is that one can commute ϕ_j with either the projection or L_λ^k, gaining one derivative of F. If n is odd, using (3.11) (with $k = n+1$) we get for $j = 0, 1$

$$\int_{S^{2n+1}} (F - \widetilde{F})\phi_j L_\lambda^{Q/2}(F - \widetilde{F})\phi_j \leq \int_{\Omega^j_{\delta_j/2}} \Big[\big(\tfrac{2}{n}\big)^k \big(\pi \mathcal{L}^{k/2} F\big)^2 + \lambda^{2k/Q}\big(\pi^\perp \mathcal{L}^{k/2} F\big)^2\Big]$$
$$+ \epsilon \int_{S^{2n+1}} (L_\lambda^{\frac{Q}{4}} F)^2 + C(\lambda, \epsilon)\|\mathcal{L}^{\frac{Q-1}{4}} F\|_2^2.$$

Using these last inequalities in (3.9), (3.10) multipliying the resulting estimates out, and taking square roots we get

$$\sqrt{\frac{\delta_1}{2Q}} \leq e^{\widetilde{F}} \exp\Big[\Big(\tfrac{1}{2}A_n(\lambda) + \epsilon\Big) \fint F L_\lambda^{Q/2} F + C_1(\lambda,\epsilon)\|\mathcal{L}^{\frac{Q-1}{4}} F\|_2^2 + C_2(\lambda)\Big].$$

or

$$\Big(\tfrac{1}{2}A_n(\lambda) + \epsilon\Big) \fint F L_\lambda F + \fint F + C_1(\lambda,\epsilon)\|\mathcal{L}^{\frac{Q-1}{4}} F\|_2^2 + C_2(\lambda) \geq 0$$

for some constants $C_1(\lambda,\epsilon), C_2(\lambda)$. The case n even is the same, just use (3.12) rather than (3.11).

Now, for given $\sigma > \frac{1}{2}$ we can certainly find λ, ϵ so that $\frac{1}{2}A_n(\lambda) + \epsilon = \dfrac{\sigma}{2(n+1)!}$, and specializing to $F \in W^{Q/2,2} \cap \mathcal{P}$ we obtain

$$\frac{\sigma}{2(n+1)!} \fint F\big(\tfrac{2}{n}\mathcal{L}\big)^{Q/2} F\, d\zeta + \fint F\, d\zeta + C_1(\sigma)\|\mathcal{L}^{\frac{Q-1}{4}} F\|_2^2 + C_2(\sigma) \geq 0.$$

Since on $\mathcal{P}$ we have $\big(\frac{2}{n}\mathcal{L}\big)^{Q/2} \leq \mathcal{A}'_Q$ we also obtain (3.6), under the condition $\fint e^F = 1$ (for the unconstrained case just replace F in the above inequality by $F - \log \fint e^F$).

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We would like to make an important remark at this point. The very nature of the center of mass hypothesis in the above lemma makes it almost impossible to avoid the use of cutoff functions, in order to proceed with the localization argument; the authors were unable to conceive a different argument working exclusively inside the class $\mathcal{P}$. This justifies our choice of the operator L_λ, which allows us to temporarily exit the space $\mathcal{P}$. Our choice is not the only one. For example, in the same spirit as in [CY] one could try to use the operator $\frac{2}{n}\mathcal{L}$, i.e. L_λ with $\lambda^{4/Q}=\frac{2}{n}$. This operator satisfies $\int F\big(\frac{2}{n}\mathcal{L}\big)^{Q/2}F \le \int F\mathcal{A}'_Q F$ for F in the Hardy space, however to make the argument work the Adams constant $\widetilde{A}_{Q/2}$ corresponding to $\big(\frac{2}{n}\mathcal{L}\big)^{Q/2}$, should satisfy $2\widetilde{A}_{Q/2} > A_{Q/2}$ with $A_{Q/2}$ as in (2.15). Using (2.34) we obtain

$$\frac{A_{Q/2}}{\widetilde{A}_{Q/2}} = \frac{\big(\frac{n}{2}\big)^{n+1}}{(n-1)!}\sum_{k=0}^{\infty}\frac{(k+n-1)!}{k!\,\big(k+\frac{n}{2}\big)^{n+1}}$$

which is less than 2 only for $n=1,2$ (in which cases one can indeed use $\frac{2}{n}\mathcal{L}$ to prove (3.6)), and seems to have exponential growth in n.

The proof of the existence of the minimum for $\mathcal{J}$ can now proceed in more or less the same way as in [CY]. Let

$$\mathcal{S}_0 = \Big\{F \in W^{Q/2,2}\cap \mathbb{R}\mathcal{P} : \;⨍ e^F\,d\zeta = 1,\; ⨍ \zeta e^F\,d\zeta = 0\Big\}$$

and let us prove that a minimum of $\mathcal{J}$ exists in $\mathcal{S}_0$. First note that for any $F\in W^{Q/2,2}$ there exists $\tau \in \mathrm{CON}(S^{2n+1})$ such that $⨍ \zeta e^{F_\tau} = 0$. The proof of this is the same as the corresponding statement in the Euclidean case (see e.g. [O],[CY1]). Next observe that minimizing $\mathcal{J}$ over $W^{Q/2,2}\cap\mathbb{R}\mathcal{P}$ is equivalent to minimizing $\mathcal{J}$ over $\mathcal{S}_0$, byt the above fact and conformal invariance of $\mathcal{J}$.

Pick a minimizing sequence $F_k\in\mathcal{S}_0$, with $\mathcal{J}[F_k]\to \inf J$. Let's first prove that

$$⨍ F_k\mathcal{A}'_Q F_k \le C_2 + C_1\|\mathcal{L}^{\frac{Q-1}{4}}F_k\|_2^2. \tag{3.13}$$

From (3.6), for a fixed $\frac12<\sigma<1$,

$$\mathcal{J}[F_k] + C_1(\sigma)\|\mathcal{L}^{\frac{Q-1}{4}}F_k\|_2^2 + C_2(\sigma) \ge \frac{1-\sigma}{2(n+1)!}⨍ F_k\mathcal{A}'_Q F_k$$

and since F_k is minimizing we obtain (3.13). Now let's prove that F_k can be chosen so that

$$\|\mathcal{L}^{\frac{Q-1}{4}}F_k\|_2 \le C. \tag{3.14}$$

For this we use the Ekeland principle (see e.g. [DeF], Thm 4.4.) to ensure that $\mathcal{J}'[F_k]\to 0$ in $W^{-Q/2,2}\cap\mathbb{R}\mathcal{P}$, where $\mathcal{J}'$ denotes the Gateaux derivative of $\mathcal{J}$. Thus, $<\mathcal{J}'[F_k],\phi>=\int H_k\phi$ with

$$H_k := \mathcal{A}'_Q F_k - (n+1)!\,\pi(e^{F_k}-1) \to 0 \quad \text{in } W^{-Q/2,2}\cap\mathbb{R}\mathcal{P}$$

i.e.

$$F_k - \widetilde{F}_k = (\mathcal{A}'_Q)^{-1}H_k + (n+1)!(\mathcal{A}'_Q)^{-1}\pi(e^{F_k}-1) \tag{3.15}$$

If $0 < 2\alpha < Q$, such as $\alpha = \frac{Q-1}{2}$, the operator $\mathcal{A}'_Q \mathcal{L}^{-\alpha/2}\pi$, with eigenvalues $\left(\frac{n}{2}k\right)^{-\alpha/2}\lambda_Q(k)$, is of the type described by (2.10), (2.11), with $d = Q - \alpha$, hence by Proposition 2.4 we have

$$|\mathcal{L}^{\alpha/2}(\mathcal{A}'_Q)^{-1}\pi(\zeta,\eta)| \leq C|1-\zeta\cdot\overline{\eta}|^{-\alpha/2}.$$

So

$$\int_{S^{2n+1}} \left|\mathcal{L}^{\alpha/2}(\mathcal{A}'_Q)^{-1}\,\pi(e^{F_k}-1)\right|^2 d\zeta \leq C\int_{S^{2n+1}}\left(\int_{S^{2n+1}}|1-\zeta\cdot\overline{\eta}|^{-\alpha/2}|e^{F_k(\eta)}-1|d\eta\right)^2 d\zeta$$
$$\leq C\bigg(\int_{S^{2n+1}}\int_{S^{2n+1}}|e^{F_k(\eta)}-1|d\eta d\zeta\bigg)\int_{S^{2n+1}}\int_{S^{2n+1}}|1-\zeta\cdot\overline{\eta}|^{-\alpha}|e^{F_k(\eta)}-1|d\eta d\zeta \leq C$$

(here we used that $⨍ e^{F_k} = 1$ and that $\int |1-\zeta\cdot\overline{\eta}|^{-\alpha} = C_\alpha$ for any $\eta \in S^{2n+1}$, since $2\alpha < Q$). On the other hand, looking at the eigenvalues of $\mathcal{L}^{\alpha/2}(\mathcal{A}'_Q)^{-1}$

$$\int_{S^{2n+1}} \left|\mathcal{L}^{\alpha/2}(\mathcal{A}'_Q)^{-1}H_k\right|^2 d\zeta \leq C\|H_k\|^2_{W^{\alpha-Q,2}} \leq C\|H_k\|^2_{W^{-Q/2,2}} \leq C$$

since $\|H_k\|_{W^{-Q/2,2}} \to 0$. All this with (3.15), $2\alpha = Q-1$, and $\mathcal{L}^{\alpha/2}(F_k - \widetilde{F}_k) = \mathcal{L}^{\alpha/2}F_k$, proves (3.14).

Finally, by Jensen's inequality $\widetilde{F} \leq 0$ and since $\mathcal{J}[F_k] \to \inf \mathcal{J}$ then

$$|\widetilde{F}_k| = -⨍ F_k = -\mathcal{J}[F_k] + \frac{1}{2(n+1)!}⨍ F_k\mathcal{A}'_Q F_k \leq C + \frac{1}{2(n+1)!}⨍ F_k\mathcal{A}'_Q F_k \leq C'$$

by (3.13) and (3.14). From this we deduce

$$⨍ |F_k|^2 = ⨍ |F_k - \widetilde{F}_k|^2 + |\widetilde{F}_k|^2 \leq C_1\|\mathcal{L}^{Q/4}F_k\|_2^2 + C_2 \leq C$$

and therefore the minimizing sequence is bounded in $W^{Q/2,2}$. Now the standard argument goes like this: find a subsequence F_{k_i} converging in L^2 and pointwise a.e. to an F_0, and weakly in $W^{Q/2,2}$. Clearly $F_0 \in \mathbb{R}\mathcal{P}$, and from the Adams inequality as $i \to \infty$, perhaps along another subsequence,

$$1 = ⨍ e^{F_{k_i}} \to ⨍ e^{F_0} \qquad 0 = ⨍ \zeta_j e^{F_{k_i}} \to ⨍ \zeta_j e^{F_0}, \;\; j = 1,2...,n+1$$

(this is because $e^{F_{k_i}}$ is bounded in L^2, hence up to a subsequence it is weakly convergent, and its weak limit coincides with e^{F_0} a.e.). Since $⨍ F_k \to ⨍ F_0$ and $\mathcal{J}[F_k]$ converges, then also $⨍ F_k\mathcal{A}'_Q F_k$ converges, and by standard results its limit is $\geq ⨍ F_0\mathcal{A}'_Q F_0$, but it cannot be greater, since the sequence is minimizing for $\mathcal{J}$. This shows that $\mathcal{J}[F_k] \to \mathcal{J}[F_0] = \inf \mathcal{J}$.

Step III: Characterization of the minimum.

As in [CY] the problem of describing the minimum will be related to the first nonzero eigenvalue of a conformally invariant operator in the conformal class of the standard contact form, specifically the operator $\mathcal{A}'_Q(W)$ introduced in Prop. 1.3. According to Prop. 1.3, if $W \in C^\infty(S^{2n+1})$ then $\mathcal{A}'_Q(W)$ acting on $W^{Q/2,2} \cap \mathbb{R}\mathcal{P}_0$, with inner product $(F,G)_W = \int FGW$, have positive eigenvalues $0 < \lambda_1(Q,W) \le \lambda_2(Q,W) \le ...$ (each counted with its multiplicity), and

$$\lambda_1(Q,W) = \inf\left\{ \frac{(\phi, \mathcal{A}'_Q\phi)}{(\phi,\phi)_W}, \quad \phi \in W^{Q/2,2} \cap \mathbb{R}\mathcal{P}, \int_{S^{2n+1}} \phi W d\zeta = 0 \right\} \tag{3.16}$$

Note that $(\phi, \mathcal{A}'_Q\phi) = (\phi, \mathcal{A}'_Q(W)\phi)_W$.

Proposition 3.5. *Suppose that $\mathcal{F}_0 \in \mathcal{S}_0$ is a minimum for $\mathcal{J}$, then $F_0 \in C^\infty(S^{2n+1})$ and $\lambda_1(e^{F_0}) \ge (n+1)!$.*

Proof. The function F_0 must satisfy

$$\frac{d}{dt}\bigg|_{t=0} \mathcal{J}[F_0 + t\phi] = ⨍ \phi\left(\frac{1}{2(n+1)!}\mathcal{A}'_Q F_0 + (e^{F_0} - 1)\right) = 0 \qquad \forall \phi \in W^{Q/2,2} \cap \mathbb{R}\mathcal{P}$$

i.e. $\frac{1}{2(n+1)!}\mathcal{A}'_Q F_0 + \pi(e^{F_0} - 1) = 0$, from which and from (1.31) we easily deduce that $F_0 \in C^\infty(S^{2n+1})$. On the other hand F_0 must also satisfy

$$\frac{d^2}{dt^2}\bigg|_{t=0} \mathcal{J}[F_0 + t\phi] = \frac{1}{(n+1)!} ⨍ \phi \mathcal{A}'_Q \phi + \left(⨍ \phi e^{F_0}\right)^2 - ⨍ \phi^2 e^{F_0} \ge 0$$

and from (3.16) we have $\lambda_1(e^{F_0}) \ge (n+1)!$.

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The next result is a Hersch's type "isoperimetric" inequality for the first Q reciprocal eigenvalues. In the Euclidean case the inequality appeared first in [H] and it was later extended in [CY].

Notice that in our notation, when $W \equiv 1$ on S^{2n+1} we have

$$\lambda_k(Q,1) = \lambda_1(Q) = (n+1)!, \quad k = 1, 2, ..., 2n+2$$

since the bottom eigenvalue for $\mathcal{A}'_Q$ is $(n+1)!$ counted with multiplicity $m_{01} + m_{10} = 2n+2$ (see (1.8)), its eigenspace being generated by the coordinate functions $\zeta_1, ..., \zeta_{n+1}$ and $\overline{\zeta}_1, ..., \overline{\zeta}_{n+1}$.

Proposition 3.6. *For $W \in C^\infty(S^{2n+1})$, $W > 0$ and $⨍ W = 1$ we have*

$$\sum_{j=1}^{2n+2} \frac{1}{\lambda_j(Q,W)} \geq \sum_{j=1}^{2n+2} \frac{1}{\lambda_j(Q,1)} = \frac{2n+2}{\lambda_1(Q)} = \frac{2}{n!} \tag{3.17}$$

In particular,

$$\lambda_1(Q,W) \leq \lambda_1(Q,1) = (n+1)! \tag{3.18}$$

and equality holds in (3.17) or (3.18) if and only if $W = |J_\tau|$ for some $\tau \in \mathrm{CON}(S^{2n+1})$.

Proof. The proof of this uses the variational characterization of the sums of reciprocals (see [CY], or [Ban], (3.7))

$$\sum_{j=1}^{2n+2} \frac{1}{\lambda_j(Q,W)} = \max \sum_{j=1}^{2n+2} \frac{(\phi_j,\phi_j)_W}{\left(\phi_j, \mathcal{A}'_Q(W)\phi_j\right)_W} = \max \sum_{j=1}^{2n+2} \frac{(\phi_j,\phi_j)_W}{(\phi_j, \mathcal{A}'_Q\phi_j)} \tag{3.19}$$

the maximum being over those $\phi_j \in W^{Q/2,2} \cap \mathbb{R}\mathcal{P}$ such that $⨍ \phi_j W = ⨍ \phi_j \overline{\mathcal{A}'_Q \phi_k} = 0$, for $j,k = 1,...,2n+2$, $j \neq k$. It's easy to see that the maximum is attained at $\phi_1,...,\phi_{2n+2}$ if and only if each ϕ_j is an eigenfunction of $\lambda_j(W)$. By conformal invariance of the eigenvalues, we can assume that $⨍ \zeta_j W = 0$, $j = 1,...,n+1$. Hence, choosing $\zeta_j, \overline{\zeta}_j$ as ϕ_j in (3.19), and since

$$(\zeta_j, \mathcal{A}'_Q \zeta_j) = \lambda_1(Q) \int_{S^{2n+1}} |\zeta_j|^2 d\zeta = \frac{\omega_{2n+1}}{n+1} \lambda_1(Q)$$

we obtain

$$\sum_{j=1}^{2n+2} \frac{1}{\lambda_j(Q,W)} \geq \frac{n+1}{\lambda_1(Q)\omega_{2n+1}} \sum_{j=1}^{n+1} \int_{S^{2n+1}} (|\zeta_j|^2 + |\overline{\zeta}_j|^2) W(\zeta) d\zeta = \frac{2(n+1)}{\lambda_1(Q)}$$

which is (3.17). Equality in (3.17) implies that each ζ_j, $\overline{\zeta}_j$ is an eigenfunction of $\mathcal{A}'_Q(W)$ with eigenvalue $\lambda_1(Q)$, which implies $\left(\phi, \mathcal{A}'_Q(W)\zeta_1\right)_W = \lambda_1(Q)(\phi,\zeta_1)_W$ for all $\phi \in C^\infty(S^{2n+1})$, but this means $(\phi,\zeta_1) = (\phi,\zeta_1)_W$ for all ϕ and this implies $W \equiv 1$ on S^{2n+1}. So if W has vanishing center of mass then equality holds if and only if $W \equiv 1$, so if we start from any W by conformal invariance we have equality in (3.17) if and only if W is in the conformal orbit of the constant function 1, i.e. $W = |J_\tau|$, some τ.

Estimate (3.18) follows from the monotonicity of the eigenvalues, and equality in (3.18) implies equality in (3.17).

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To finish up the proof of Theorem 3.1, if F_0 is a minimum for $\mathcal{J}$ then $F_0 \in C^\infty(S^{2n+1})$ and by the previous propositions $\lambda_1(e^{F_0}) = \lambda_1(Q) = (n+1)!$, which is true if and only if $e^{F_0} = |J_\tau|$ for some $\tau \in \mathrm{CON}(S^{2n+1})$, and this concludes the proof.

4. The logarithmic Hardy-Littlewood-Sobolev inequalities

In this final section we use the Beckner-Onofri inequality (3.1) to give a proof of (0.10), i.e. the CR version of the inequality due to Carlen and Loss in the Euclidean setting [CL]. The procedure is at this point fairly standard, see for example [Bec] and [Ok]. The proof below is essentially the one in [Ok].

Theorem 4.1 (Log HLS inequality). *For any measurable $G : S^{2n+1} \to \mathbb{R}$, with $G \geq 0$ and $⨍ G = 1$ we have*

$$(n+1) ⨍ ⨍ \log \frac{1}{|1-\zeta\cdot\overline{\eta}|} G(\zeta)G(\eta)d\zeta d\eta \leq ⨍ G \log G \, d\zeta \tag{4.1}$$

with equality if and only if $G = |J_\tau|$, some $\tau \in \mathrm{CON}(S^{2n+1})$.

In view of (1.31) the inequality (4.1) can be restated as

$$\frac{(n+1)!}{2} ⨍ (G-1)(\mathcal{A}'_Q)^{-1}(G-1) \leq ⨍ G \log G \tag{4.2}$$

Just like the Euclidean case it is possible to state an equivalent result on the Heisenberg group, via the Cayley transform:

Corollary 4.2 (Log HLS inequality on $\mathbb{H}^n$). *For any measurable $g : \mathbb{H}^n \to \mathbb{R}$ with $g \geq 0$, $\int_{\mathbb{H}^n} g = \omega_{2n+1}$ and $\int_{\mathbb{H}^n} g \log(1+|u|^2) < \infty$ we have*

$$(n+1) ⨍_{\mathbb{H}^n} ⨍_{\mathbb{H}^n} \log \frac{2}{|v^{-1}u|^2} g(u)g(v)dudv \leq ⨍_{\mathbb{H}^n} g \log g + \log 2 \tag{4.3}$$

where $⨍_{\mathbb{H}^n} = \frac{1}{\omega_{2n+1}} \int_{\mathbb{H}^n}$. Equality in (4.3) occurs if and only if $g = (|J_{\mathcal{C}}| \circ h)|J_h|$ for some $h \in \mathrm{CON}(\mathbb{H}^n)$.

Proof of Theorem 4.1. Let $G \in L^2$, with $⨍ G = 1$ and let $F = (n+1)!(\mathcal{A}'_Q)^{-1} G \in \mathbb{R}\mathcal{P}_0$, so that $\pi_0 G = \frac{1}{(n+1)!}\mathcal{A}'_Q F$, where $\pi_0 = \pi - ⨍$ is the projection on $\mathbb{R}\mathcal{P}_0$. Using Beckner-Onofri's inequality

$$\frac{(n+1)!}{2}\fint G(\mathcal{A}'_Q)^{-1}G = \frac{1}{2}\fint GF = \fint GF - \frac{1}{2(n+1)!}\fint F\mathcal{A}'_Q F \le \fint GF - \log\fint e^F. \quad (4.4)$$

Now use Jensen's inequality to deduce

$$\log\fint e^F = \log\fint e^{F-\log G}G \ge \fint (F-\log G)G \quad (4.5)$$

and so we obtain (4.2). Moreover, equality in (4.2) implies equality in (4.4) and (4.5), i.e. (by Theorem 3.1) $F = \log|J_\tau|$ some $\tau \in \mathrm{CON}(S^{2n+1})$, and $F - \log G$ =constant, or $G = C|J_\tau|$; since G has mean 1, then we finally have $G = |J_\tau|$ for some τ.

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Proof of Corollary 4.2. First observe that if $g : \mathbb{H}^n \to \mathbb{R}$ and $G : S^{2n+1} \to \mathbb{R}$ are related by $g = (G\circ\mathcal{C})|J_\mathcal{C}|$ then $\fint G = \fint_{\mathbb{H}^n} g = 1$ (with the above convention on the average on $\mathbb{H}^n$). Moreover, since $|1-\zeta\cdot\overline{\eta}| = 2^{-\frac{n}{n+1}}|J_\mathcal{C}(u)|^{\frac{1}{Q}}|J_\mathcal{C}(v)|^{\frac{1}{Q}}|v^{-1}u|^2$ (if $\mathcal{C}(u)=\zeta$ $\mathcal{C}(v)=\eta$) then

$$\begin{aligned}
&(n+1)\fint\fint \log\frac{1}{|1-\zeta\cdot\overline{\eta}|}\,G(\zeta)G(\eta)d\zeta d\eta - \fint G\log G \\
&= (n+1)\fint_{\mathbb{H}^n}\fint_{\mathbb{H}^n}\log\Big(2^{\frac{n}{n+1}}|v^{-1}u|^{-2}|J_\mathcal{C}(u)|^{-\frac{1}{Q}}|J_\mathcal{C}(v)|^{-\frac{1}{Q}}\Big)g(u)g(v)dudv- \\
&\qquad\qquad - \fint_{\mathbb{H}^n} g\log g + \fint_{\mathbb{H}^n} g\log|J_\mathcal{C}| \\
&= (n+1)\fint_{\mathbb{H}^n}\fint_{\mathbb{H}^n}\log\frac{2}{|v^{-1}u|^2}g(u)g(v)dudv - \fint_{\mathbb{H}^n} g\log g - \log 2.
\end{aligned}$$

This identity easily implies the statement. The given integral condition on g is to guarantee that $\int g\log g$ is finite if and only if $\int G\log G$ is finite, where g and G are related as above.

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Note that with the same argument as in the proof of the Corollary above one can see that the log HLS functional (on S^{2n+1} or $\mathbb{H}^n$) is invariant under the conformal action.

5. Appendix

A. Intertwining operators on S^{2n+1}

In this appendix we give an explicit calculation of the spectrum of the intertwining operators $\mathcal{A}_d$, as defined by (1.17); a consequence of this calculation will be formula (1.20) up to a constant, and a further calculation will yield the explicit constant given in (1.21). The proof below is inspired by the method used by Johnson and Wallach [JW], but it is rather self-contained and uses no apparatus from representation theory, other than the knowledge of the zonal harmonics Φ_{jk}. We believe that our calculation is actually sligthly simpler than that in [JW], at least in our context. In [Br] and [BOØ] there is another derivation of the spectrum of intertwining operators, again via the theory of spherical principal series representations of semisimple Lie groups ($SU(n+1,1)$ in our case), and the results there are quite general.

Proposition A.1. *Suppose that an operator $\mathcal{A}_d$ $(0<d<Q)$ is intertwining, i.e.*

$$|J_\tau|^{\frac{Q+d}{2Q}}(\mathcal{A}_d F)\circ\tau=\mathcal{A}_d\big(|J_\tau|^{\frac{Q-d}{2Q}}(F\circ\tau)\big),\quad \forall\tau\in \mathrm{CON}(S^{2n+1}) \tag{5.1}$$

for each $F\in C^\infty(S^{2n+1})$. Then $\mathcal{A}_d$ is diagonal with respect to the spherical harmonics, and for every $Y_{jk}\in\mathcal{H}_{jk}$

$$\mathcal{A}_d Y_{jk}=c\lambda_j(d)\lambda_k(d)Y_{jk}$$

for some constant $c\in\mathbb{R}$. In particular, the operator $\mathcal{A}_d$ with eigenvalues $\lambda_j(d)\lambda_k(d)$ is intertwining, and has fundamental solution

$$\mathcal{A}_d^{-1}(\zeta,\eta)=c_d d(\zeta,\eta)^{d-Q},\qquad c_d=\frac{2^{n-\frac{d}{2}}\,\Gamma\big(\frac{Q-d}{4}\big)^2}{\pi^{n+1}\Gamma\big(\frac{d}{2}\big)}.$$

Proof. The fact that $\mathcal{A}_d$ is diagonal follows from Schur's lemma, and the irreducibility of the spaces $\mathcal{H}_{jk}$. Suppose that $\mathcal{A}_d\Phi_{jk}=\lambda_{j,k}\Phi_{jk}$ with $\lambda_{j,k}=\lambda_{k,j}\in\mathbb{R}$ recall that

$$\Phi_{j,k}(\zeta,\eta)=\Phi_{j,k}(\zeta\cdot\overline{\eta}):=\frac{(k+n-1)!(j+k+n)}{\omega_{2n+1}n!k!}(\overline{\zeta}\cdot\eta)^{k-j}P_j^{(n-1,k-j)}(2|\zeta\cdot\overline{\eta}|^2-1)$$

if $j\le k$, and $\Phi_{jk}(\zeta,\eta)=\Phi_{jk}(\zeta\cdot\overline{\eta}):=\overline{\Phi_{kj}(\zeta\cdot\overline{\eta})}$, if $k\le j$. From now on we choose $\eta=\mathfrak{N}$ and denote

$$\Psi_{j,k}(\zeta\cdot\overline{\mathfrak{N}})=\Psi_{j,k}(z)=\overline{z}^{k-j}P_j^{(n-1,k-j)}(2|z|^2-1),\qquad z=\zeta\cdot\overline{\mathfrak{N}}=\zeta_{n+1}$$

so that still $\mathcal{A}_d\Psi_{j,k}=\lambda_{j,k}\Psi_{j,k}$.

Consider the family of dilations of $\mathbb{H}^n$, which on the sphere take the form

$$\tau_\lambda(\zeta) = \tau_\lambda(\zeta', \zeta_{n+1}) = \left(\frac{2\lambda\zeta'}{1+\zeta_{n+1}+\lambda^2(1-\zeta_{n+1})}, \frac{1+\zeta_{n+1}-\lambda^2(1-\zeta_{n+1})}{1+\zeta_{n+1}+\lambda^2(1-\zeta_{n+1})} \right)$$

The Jacobian of the map is given by (with $z = \zeta_{n+1}$)

$$|J_{\tau_\lambda}| = \left| \frac{2\lambda}{1+z+\lambda^2(1-z)} \right|^Q$$

and

$$\frac{d}{d\lambda}\bigg|_{\lambda=1} |J_{\tau_\lambda}|^{a/Q} = \frac{a}{2}(z+\overline{z}).$$

Also, $\frac{d}{d\lambda}\Big|_{\lambda=1} (\tau_\lambda\zeta \cdot \overline{\mathfrak{N}}) = z^2 - 1$ so that

$$\begin{aligned} \frac{d}{d\lambda}\bigg|_{\lambda=1} |J_{\tau_\lambda}|^{a/Q}\big(\Psi_{jk}\circ\tau_\lambda\big) &= \frac{a}{2}(z+\overline{z})\overline{z}^{\,k-j} P_j^{(n-1,k-j)}(2|z|^2-1)+ \\ &+ (k-j)(-1+\overline{z}^{\,2})\overline{z}^{\,k-j-1} P_j^{(n-1,k-j)}(2|z|^2-1)+ \\ &+ 2(z+\overline{z})(|z|^2-1)\overline{z}^{\,k-j}\frac{d}{dx}P_j^{(n-1,k-j)}(2|z|^2-1). \end{aligned} \tag{5.2}$$

The above quantity is a polynomial in $z, \overline{z}$, with highest order monomials that are multiples of $z^j\overline{z}^{\,k+1}$ and $z^{j+1}\overline{z}^{\,k}$. The projection of (5.2) on $\mathcal{H}_{j+1,k} \bigoplus \mathcal{H}_{j,k+1}$ gives, for fixed $0 \leq j < k$

$$\begin{aligned} &\frac{d}{d\lambda}\bigg|_{\lambda=1} |J_{\tau_\lambda}|^{a/Q}\big(\Psi_{j,k}\circ\tau_\lambda\big)\Big|_{\mathcal{H}_{j+1,k}\oplus\mathcal{H}_{j,k+1}} = \\ &= A\overline{z}^{\,k-j-1}P_{j+1}^{(n-1,k-j-1)}(2|z|^2-1) + B\overline{z}^{\,k-j+1}P_j^{(n-1,k-j+1)}(2|z|^2-1) \end{aligned} \tag{5.3}$$

and for $j = k$

$$\begin{aligned} &\frac{d}{d\lambda}\bigg|_{\lambda=1} |J_{\tau_\lambda}|^{a/Q}\big(\Psi_{j,k}\circ\tau_\lambda\big)\Big|_{\mathcal{H}_{j+1,j}\oplus\mathcal{H}_{j,j+1}} = \\ &\qquad = AzP_j^{(n-1,1)}(2|z|^2-1) + B\overline{z}\,P_j^{(n-1,1)}(2|z|^2-1) \end{aligned} \tag{5.4}$$

and the goal is to determine A and B. In order to do this we consider the case z real and z imaginary, and compare the coefficients of the highest order powers in (5.2) and (5.3); the formula we need here is that for a Jacobi polynomial of order j the coefficient of x^j is

$$\frac{1}{j!}\frac{d^j}{dx^j}P_j^{(\alpha,\beta)}(x) = \frac{1}{2^j j!}\frac{\Gamma(2j+\alpha+\beta+1)}{\Gamma(j+\alpha+\beta+1)}.$$

For z real, a comparison of the coefficients of z^{k+j+1} from $(A.3)$ and $(5.3), (5.4)$ gives

$$\frac{\Gamma(k+j+n)}{j!\Gamma(k+n)}(a+k+j) = A\,\frac{\Gamma(k+j+n+1)}{(j+1)!\Gamma(k+n)} + B\,\frac{\Gamma(k+j+n+1)}{j!\Gamma(k+n+1)}$$

or

$$a+k+j = A\,\frac{k+j+n}{j+1} + B\,\frac{k+j+n}{k+n}. \tag{5.5}$$

On the other hand, if z is purely imaginary the same comparison yields

$$(-i)^{k-j+1}(k-j)\,\frac{\Gamma(k+j+n)}{j!\Gamma(k+n)} = A(-i)^{k-j-1}\frac{\Gamma(k+j+n+1)}{(j+1)!\Gamma(k+n)} + B(-i)^{k-j+1}\,\frac{\Gamma(k+j+n+1)}{j!\Gamma(k+n+1)}$$

or

$$k-j = -A\,\frac{k+j+n}{j+1} + B\,\frac{k+j+n}{k+n}. \tag{5.6}$$

Solving (5.5) and (5.6) in A and B

$$A = \Big(\frac{a}{2}+j\Big)\frac{j+1}{k+j+n}\,, \qquad B = \Big(\frac{a}{2}+k\Big)\frac{k+n}{k+j+n}$$

which means, for $0 \le j \le k$,

$$\begin{aligned} &\frac{d}{d\lambda}\Big|_{\lambda=1} |J_{\tau_\lambda}|^{a/Q}\big(\Psi_{jk}\circ\tau_\lambda\big)\Big|_{\mathcal{H}_{j+1,k}\oplus\mathcal{H}_{j,k+1}} = \\ &\quad = \Big(\frac{a}{2}+j\Big)\frac{j+1}{k+j+n}\Psi_{j+1,k} + \Big(\frac{a}{2}+k\Big)\frac{k+n}{k+j+n}\Psi_{j,k+1}. \end{aligned} \tag{5.7}$$

Differentiating in λ the intertwining relation (5.1) applied to $\Psi_{j,k}$ i.e.

$$\lambda_{j,k}|J_{\tau_\lambda}|^{\frac{Q+d}{2Q}}(\Psi_{j,k}\circ\tau_\lambda) = \mathcal{A}_d\big(|J_{\tau_\lambda}|^{\frac{Q-d)}{2Q}}(\Psi_{j,k}\circ\tau_\lambda)\big)$$

(it's easy to see that differentiation in λ commutes with $\mathcal{A}_d$) and using (5.7)

$$\begin{aligned} &\lambda_{j,k}\Big(\frac{Q+d}{4}+j\Big)\frac{j+1}{k+j+n}\Psi_{j+1,k} + \lambda_{j,k}\Big(\frac{Q+d}{4}+k\Big)\frac{k+n}{k+j+n}\Psi_{j,k+1} = \\ &= \lambda_{j+1,k}\Big(\frac{Q-d}{4}+j\Big)\frac{j+1}{k+j+n}\Psi_{j+1,k} + \lambda_{j,k+1}\Big(\frac{Q-d}{4}+k\Big)\frac{k+n}{k+j+n}\Psi_{j,k+1} \end{aligned}$$

which implies

$$\lambda_{j+1,k} = \lambda_{j,k}\,\frac{\frac{Q+d}{4}+j}{\frac{Q-d}{4}+j}, \qquad \lambda_{j,k+1} = \lambda_{j,k}\,\frac{\frac{Q+d}{4}+k}{\frac{Q-d}{4}+k} \qquad k \ge j \ge 0$$

and therefore

$$\lambda_{j,k}=\lambda_{0,k}\,\frac{\Gamma\big(\frac{Q+d}{4}+j\big)}{\Gamma\big(\frac{Q-d}{4}+j\big)}=\lambda_{0,0}\frac{\Gamma\big(\frac{Q+d}{4}+j\big)}{\Gamma\big(\frac{Q-d}{4}+j\big)}\frac{\Gamma\big(\frac{Q+d}{4}+k\big)}{\Gamma\big(\frac{Q-d}{4}+k\big)}.$$

The proof of the last statement follows from the fact that the convolution operator $\mathcal{B}_d$ with kernel $d(\zeta,\eta)^{d-Q}$ is intertwining, but with d replaced by $-d$:

$$\mathcal{B}_d\big(|J_\tau|^{\frac{Q+d}{2Q}}(G\circ\tau)\big)=|J_\tau|^{\frac{Q-d}{2Q}}(\mathcal{B}_dG)\circ\tau$$

which can be checked directly on the dilations, translations (and trivially rotations and the inversion), using formulas (1.15).

From this and the previous calculations (which are valid also for $-Q<d<0$) we deduce (note: $\lambda_j(-d)=\lambda_j(d)^{-1}$)

$$\int_{S^{2n+1}}d(\zeta,\eta)^{d-Q}Y_{jk}d\eta=\frac{c}{\lambda_j(d)\lambda_k(d)}Y_{jk}.$$

Now set $j=k=0$, and by an elementary computation

$$\int_{S^{2n+1}}d(\zeta,\eta)^{d-Q}d\eta=2^{\frac{d-Q}{2}}\int_{S^{2n+1}}|1-\zeta\cdot\overline{\eta}|^{\frac{d-Q}{2}}d\eta=2^{\frac{d-Q}{2}}\omega_{2n+1}\frac{\Gamma\big(\frac{Q}{2}\big)\Gamma\big(\frac{d}{2}\big)}{\Gamma\big(\frac{Q+d}{4}\big)^2}$$

so that

$$c=\lambda_0(d)^2\omega_{2n+1}\frac{\Gamma\big(\frac{Q}{2}\big)\Gamma\big(\frac{d}{2}\big)}{2^{\frac{Q-d}{2}}\Gamma\big(\frac{Q+d}{4}\big)^2}=\omega_{2n+1}\frac{\Gamma\big(\frac{Q}{2}\big)\Gamma\big(\frac{d}{2}\big)}{2^{\frac{Q-d}{2}}\Gamma\big(\frac{Q-d}{4}\big)^2}=\frac{1}{c_d}.$$

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B. Proof of (3.11)

Let F have zero mean, and assume k even. We have $L_\lambda^k=\big(\frac{2}{n}\big)^k\pi\mathcal{L}^k+\lambda^{2k/Q}\pi^\perp\mathcal{L}^k$, and (for $\phi\in C^\infty$)

$$\int_{S^{2n+1}}\phi FL_\lambda^k\phi F=\big(\tfrac{2}{n}\big)^k\int_{S^{2n+1}}\big[\pi\mathcal{L}^{k/2}(\phi F)\big]^2+\lambda^{2k/Q}\int_{S^{2n+1}}\big[\pi^\perp\mathcal{L}^{k/2}(\phi F)\big]^2 \tag{5.8}$$

so let us first consider the first term. Using the definition of $\mathcal{L}$ we can write

$$\mathcal{L}^{k/2}(\phi F)=\phi\mathcal{L}^{k/2}F+\sum_I\phi_IT_IF$$

where the sum is finite, over a suitable set of multiindeces $I=\{i_1,...,i_\ell\}$, $\ell<k$, and where $T_I=T'_{i_1}...T'_{i_\ell}$, the T'_j being either T_j or $\overline{T}_j$, and ϕ_I a smooth function. Apply π to this formula and square it; the leading term is $(\pi\phi\mathcal{L}^{k/2}F)^2$, and the remainder terms are estimated using the following inequalities:

i) $\|\pi G\|_2 \leq \|G\|_2$

ii) $\|T_I F\|_2 \leq C\|\mathcal{L}^{\frac{k-1}{2}} F\|_2$, if I has length $< k$

iii) $\|\pi\mathcal{L}^{k/2} F T_I F\|_1 \leq \epsilon\|\pi\mathcal{L}^{k/2}F\|_2^2 + C(\epsilon)\|\mathcal{L}^{\frac{k-1}{2}} F\|_2^2$

For ii) see for example [ADB], for an o.n. base of $T_{1,0}$ rather than the T_j. Observe that ii) is also valid for I empty, i.e. for $\|F\|_2$, since F has zero mean.

Thus we are reduced to estimate the last two terms of the identity

$$\int \left[\pi(\phi\mathcal{L}^{k/2}F)\right]^2 = \int \phi^2(\pi\mathcal{L}^{k/2}F)^2 + \int \Big([\pi,\phi]\mathcal{L}^{k/2}F\Big)^2 + 2\int \Big([\pi,\phi]\mathcal{L}^{k/2}F\Big)\phi\pi\mathcal{L}^{k/2}F$$

where $[\pi,\phi] = \pi\phi - \phi\pi$. In order to do this we just have to justify that if T_j is as in (1.3) then the operator $T = T_j[\pi,\phi]$ (and hence $[\pi,\phi]T_j$) is bounded on L^2. This is a consequence of the famous $T1-$theorem by David-Journe, in the context of spaces of homogeneous type (such as the CR sphere); see for example [DJS]. Indeed one can write down explicitly the kernel of such operator, using the Cauchy-Szego kernel, and check that it is a Calderon-Zygmund kernel, with $T1 = T^*1 = 0$.

This given, we can easily estimate the second and third term with $\epsilon\|\pi\mathcal{L}^{k/2}F\|_2^2 + C(\epsilon)\|\mathcal{L}^{\frac{k-1}{2}}F\|_2^2$. This takes care of the first term on the right-hand side of (5.8); to deal with the second term in (5.8), we argue exactly in the same manner. This shows (3.11) in case k even.

For k odd, the proof of (3.12) is completely similar, except one has to start from $\int \pi\mathcal{L}^{\frac{k-1}{2}}(F\phi)\pi\mathcal{L}^{\frac{k+1}{2}}(F\phi)$. Using the same product rule as above and the commutator estimate, the leading term is given by

$$\int \phi^2\pi\mathcal{L}^{\frac{k-1}{2}}F\,\pi\mathcal{L}^{\frac{k+1}{2}}F = \int \phi^2\left|\nabla_H\pi\mathcal{L}^{\frac{k-1}{2}}F\right|^2 + \int \pi\mathcal{L}^{\frac{k-1}{2}}(F\phi)\nabla_H\phi^2\nabla_H\pi\mathcal{L}^{\frac{k-1}{2}}F$$

and it's easy to see that the second term is bounded above by

$$\epsilon\int \left|\nabla_H\pi\mathcal{L}^{\frac{k-1}{2}}F\right|^2 + C(\epsilon)\|\mathcal{L}^{\frac{k-1}{2}}F\|_2^2 = \epsilon\|\mathcal{L}^{k/2}\pi F\|_2^2 + C(\epsilon)\|\mathcal{L}^{\frac{k-1}{2}}F\|_2^2.$$

The remainder terms are estimated similarly.

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