PERTURBATION RESULTS OF CRITICAL ELLIPTIC EQUATIONS OF CAFFARELLI-KOHN-NIRENBERG TYPE

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Abstract. We find for small ε positive solutions to the equation

$$-\text{div}\,(|x|^{-2a}\nabla u)-\frac{\lambda}{|x|^{2(1+a)}}\,u=\Big(1+\varepsilon k(x)\Big)\frac{u^{p-1}}{|x|^{bp}}$$

in $\mathbb{R}^N$, which branch off from the manifold of minimizers in the class of radial functions of the corresponding Caffarelli-Kohn-Nirenberg type inequality. Moreover, our analysis highlights the symmetry-breaking phenomenon in these inequalities, namely the existence of non-radial minimizers.

1. Introduction

We will consider the following elliptic equation in $\mathbb{R}^N$ in dimension $N\geq 3$

$$-\text{div}\,(|x|^{-2a}\nabla u)-\frac{\lambda}{|x|^{2(1+a)}}\,u=K(x)\frac{u^{p-1}}{|x|^{bp}},\quad x\in\mathbb{R}^N \tag{1.1}$$

where

$$\begin{aligned}&-\infty<a<\frac{N-2}{2},\quad -\infty<\lambda<\left(\frac{N-2a-2}{2}\right)^2\\ &p=p(a,b)=\frac{2N}{N-2(1+a-b)}\quad\text{and}\quad a\leq b<a+1.\end{aligned} \tag{1.2}$$

For $\lambda=0$ equation (1.1) is related to a family of inequalities given by Caffarelli, Kohn and Nirenberg [6],

$$\|u\|_{p,b}^2:=\left(\int_{\mathbb{R}^N}|x|^{-bp}|u|^p\,dx\right)^{2/p}\leq\mathcal{C}_{a,b}\int_{\mathbb{R}^N}|x|^{-2a}|\nabla u|^2\,dx\qquad\forall u\in C_0^\infty(\mathbb{R}^N). \tag{1.3}$$

For sharp constants and extremal functions we refer to Catrina and Wang [7].
The natural functional space to study (1.1) is $D_a^{1,2}(\mathbb{R}^N)$ defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|\nabla u\|_a:=\|u\|_*=\left[\int_{\mathbb{R}^N}|x|^{-2a}|\nabla u|^2\,dx\right]^{1/2}.$$

We will mainly deal with the perturbative case $K(x)=1+\varepsilon k(x)$, namely with the problem

$$\begin{cases}-\text{div}\,(|x|^{-2a}\nabla u)-\dfrac{\lambda}{|x|^{2(1+a)}}\,u=\big(1+\varepsilon k(x)\big)\dfrac{u^{p-1}}{|x|^{bp}}\\ u\in D_a^{1,2}(\mathbb{R}^N),\quad u>0\text{ in }\mathbb{R}^N\setminus\{0\}.\end{cases} \tag{$\mathcal{P}_{a,b,\lambda}$}$$

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Concerning the perturbation k we assume

$$k \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N). \tag{1.4}$$

Our approach is based on an abstract perturbative variational method discussed by Ambrosetti and Badiale [2], which splits our procedure in three main steps. First we consider the unperturbed problem, i.e. $\varepsilon = 0$, and find a one dimensional manifold of radial solutions. If this manifold is non-degenerate (see Theorem 1.1 below) a one dimensional reduction of the perturbed variational problem in $D_a^{1,2}(\mathbb{R}^N)$ is possible. Finally we have to find a critical point of a functional defined on the real line.
Solutions of $(\mathcal{P}_{a,b,\lambda})$ are critical points in $\mathcal{D}^{1,2}_{a,\lambda}(\mathbb{R}^N)$ of

$$f_\varepsilon(u) := \frac{1}{2}\int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2\,dx - \frac{\lambda}{2}\int_{\mathbb{R}^N} \frac{u^2}{|x|^{2(1+a)}}\,dx - \frac{1}{p}\int_{\mathbb{R}^N}\big(1+\varepsilon k(x)\big)\frac{u_+^p}{|x|^{bp}}\,dx,$$

where $u_+ := \max\{u, 0\}$. For $\varepsilon = 0$ we show that f_0 has a one dimensional manifold of critical points

$$Z_{a,b,\lambda} := \left\{ z_\mu^{a,b,\lambda} := \mu^{-\frac{N-2-2a}{2}} z_1^{a,b,\lambda}\left(\frac{x}{\mu}\right) \mid \mu > 0 \right\},$$

where $z_1^{a,b,\lambda}$ is explicitly given in (2.5) below. These radial solutions were computed for $\lambda = 0$ in [7], the case $a = b = 0$ and $-\infty < \lambda < (N-2)^2/4$ was done by Terracini [12]. The exact knowledge of the critical manifold enables us to clarify the question of non-degeneracy.

Theorem 1.1. *Suppose a, b, λ, p satisfy (1.2). Then the critical manifold $Z_{a,b,\lambda}$ is non-degenerate, i.e.*

$$T_z Z_{a,b,\lambda} = \ker D^2 f_0(z) \quad \forall\, z \in Z_{a,b,\lambda}, \tag{1.5}$$

if and only if

$$b \neq h_j(a,\lambda) := \frac{N}{2}\left[1 + \frac{4j(N+j-1)}{(N-2-2a)^2 - 4\lambda}\right]^{-1/2} - \frac{N-2-2a}{2} \qquad \forall\, j \in \mathbb{N}\setminus\{0\}. \tag{1.6}$$

Figure 1 ($\lambda = 0$ and $h_j(\cdot, 0)$ for $j = 1 \ldots 5$)

The above theorem is rather unexpected as it is explicit. It improves the non-degeneracy results and answers an open question in [1, Rem. 4.2]. Moreover, it fairly highlights the symmetry breaking phenomenon of the unperturbed problem observed in [7], i.e. the existence of non-radial minimizers of

$$\mathcal{C}_{a,b} := \inf_{u\in D_a^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int |x|^{-2a}|\nabla u|^2}{\left(\int |x|^{-bp}|u|^p\right)^{\frac{2}{p}}} = \inf_{u\in D_a^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\|\nabla u\|_a^2}{\|u\|_{p,b}^2}. \tag{1.7}$$

In fact we improve [7, Thm 1.3], where it is shown that there are an open subset $H \subset \mathbb{R}^2$ containing $\{(a,a)\,|\,a<0\}$, a real number $a_0 \le 0$ and a function $h :]-\infty, a_0] \to \mathbb{R}$ satisfying $h(a_0) = a_0$ and $a < h(a) < a+1$ for all $a < a_0$, such that for every $(a,b) \in H \cup \{(a,b) \in \mathbb{R}^2\,|\,a < a_0,\ a < b < h(a)\}$ the minimizer in (1.7) is non-radial (see figure 2 below). We show that one may choose $a_0 = 0$ and $h = h_1(\cdot, 0)$ and obtain, as a consequence of Theorem 1.1 for $\lambda = 0$,

Corollary 1.2. *Suppose a, b, p satisfy (1.2). If $b < h_1(a,0)$, then $\mathcal{C}_{a,b}$ in (1.7) is attained by a non-radially symmetric function.*

region of non-radial minimizers in [7] region of non-radial minimizers given by $h_1(\cdot,0)$

Figure 2

Concerning step two, the one-dimensional reduction, we follow closely the abstract scheme in [2] and construct a manifold $Z^{\varepsilon}_{a,b,\lambda} = \{z_\mu^{a,b,\lambda} + w(\varepsilon,\mu)\,|\,\mu > 0\}$, such that any critical point of f_ε restricted to $Z^{\varepsilon}_{a,b,\lambda}$ is a solution to $(\mathcal{P}_{a,b,\lambda})$. We emphasize that in contrast to the local approach in [2] we construct a manifold which is globally diffeomorphic to the unperturbed one such that we may estimate the difference $\|w(\varepsilon,\mu)\|$ when $\mu \to \infty$ or $\mu \to 0$ (see also [4, 5]). More precisely we show under assumption (1.8) below that $\|w(\varepsilon,\mu)\|$ vanishes as $\mu \to \infty$ or $\mu \to 0$.

We will prove the following existence results.

Theorem 1.3. *Suppose a, b, p, λ satisfy (1.2),(1.4) and (1.6) holds. Then problem $(\mathcal{P}_{a,b,\lambda})$ has a solution for all $|\varepsilon|$ sufficiently small if*

$$k(\infty) := \lim_{|x|\to\infty} k(x) \text{ exists and } k(\infty) = k(0) = 0. \tag{1.8}$$

Theorem 1.4. *Assume (1.2),(1.4), (1.6) and*

$$k \in C^2(\mathbb{R}^N),\ |\nabla k| \in L^\infty(\mathbb{R}^N) \text{ and } |D^2 k| \in L^\infty(\mathbb{R}^N). \tag{1.9}$$

Then $(\mathcal{P}_{a,b,\lambda})$ is solvable for all small $|\varepsilon|$ under each of the following conditions

$$\limsup_{|x|\to\infty} k(x) \le k(0) \text{ and } \Delta k(0) > 0, \tag{1.10}$$

$$\liminf_{|x|\to\infty} k(x) \ge k(0) \text{ and } \Delta k(0) < 0. \tag{1.11}$$

Remark 1.5. *Our analysis of the unperturbed problem allows to consider more general perturbation, for instance it is possible to treat equations like*

$$\begin{cases} -\mathrm{div}\,(|x|^{-2a}\nabla u) - \dfrac{\lambda+\varepsilon_1 V(x)}{|x|^{2(1+a)}}\,u = \big(1+\varepsilon_2 k(x)\big)\dfrac{u^{p-1}}{|x|^{bp}} \\ u \in D_a^{1,2}(\mathbb{R}^N), \quad u>0 \text{ in } \mathbb{R}^N\backslash\{0\}. \end{cases}$$

Existence results in this direction are given by Abdellaoui and Peral [1], where the case $a=0$ *and* $b=0$ *and* $\frac{(N-2)^2}{4N}<\lambda<\frac{(N-2)^2}{4}$ *is studied. We generalize some existence results obtained there to arbitrary* a,b *and* λ *satisfying (1.2) and (1.6).*

Problem (1.1), the non-perturbative version of $(\mathcal{P}_{a,b,\lambda})$, was studied by Smets [11] in the case $a=b=0$ and $0<\lambda<(N-2)^2/4$. A variational minimax method combined with a careful analysis and construction of Palais-Smale sequences shows that in dimension $N=4$ equation (1.1) has a positive solution $u\in D_a^{1,2}(\mathbb{R}^N)$ if $K\in C^2$ is positive and satisfies an analogous condition to (1.8), namely $K(0)=\lim_{|x|\to\infty}K(x)$. In our perturbative approach we need not to impose any condition on the space dimension N. Theorem 1.3 gives the perspective to relax the restriction $N=4$ on the space dimension also in the nonperturbative case.

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Preliminaries

Catrina and Wang [7] proved that for $b=a+1$

$$\mathcal{C}_{a,a+1}^{-1}=\mathcal{S}_{a,a+1}=\inf_{D_a^{1,2}(\mathbb{R}^N)\backslash\{0\}}\frac{\int_{\mathbb{R}^N}|x|^{-2a}|\nabla u|^2}{\left(\int_{\mathbb{R}^N}|x|^{-2(1+a)}|u|^2\right)}=\left(\frac{N-2-2a}{2}\right)^2.$$

Hence we obtain for $-\infty<\lambda<\left(\frac{N-2-2a}{2}\right)^2$ a norm, equivalent to $\|\cdot\|_*$, given by

$$\|u\|=\left[\int_{\mathbb{R}^N}|x|^{-2a}|\nabla u|^2\,dx-\lambda\int_{\mathbb{R}^N}\frac{u^2}{|x|^{2(1+a)}}\,dx\right]^{1/2}. \tag{1.12}$$

We denote by $\mathcal{D}_{a,\lambda}^{1,2}(\mathbb{R}^N)$ the Hilbert space equipped with the scalar product induced by $\|\cdot\|$

$$(u,v)=\int_{\mathbb{R}^N}|x|^{-2a}\nabla u\cdot\nabla v\,dx-\lambda\int_{\mathbb{R}^N}\frac{u\,v}{|x|^{2(1+a)}}\,dx.$$

We will mainly work in this space. Moreover, we define by $\mathcal{C}$ the cylinder $\mathbb{R}\times S^{N-1}$. It is is shown in [7, Prop. 2.2] that the transformation

$$u(x)=|x|^{-\frac{N-2-2a}{2}}v\Big(-\ln|x|,\frac{x}{|x|}\Big) \tag{1.13}$$

induces a Hilbert space isomorphism from $\mathcal{D}_{a,\lambda}^{1,2}(\mathbb{R}^N)$ to $H_\lambda^{1,2}(\mathcal{C})$, where the scalar product in $H_\lambda^{1,2}(\mathcal{C})$ is defined by

$$(v_1,v_2)_{H_\lambda^{1,2}(\mathcal{C})}:=\int_{\mathcal{C}}\nabla v_1\cdot\nabla v_2+\left(\left(\frac{N-2-2a}{2}\right)^2-\lambda\right)v_1v_2.$$

Using the canonical identification of the Hilbert space $\mathcal{D}^{1,2}_{a,\lambda}(\mathbb{R}^N)$ with its dual induced by the scalar-product and denoted by $\mathcal{K}$, i.e.

$$\mathcal{K} : \big(\mathcal{D}^{1,2}_{a,\lambda}(\mathbb{R}^N)\big)' \to \mathcal{D}^{1,2}_{a,\lambda}(\mathbb{R}^N),\ (\mathcal{K}(\varphi), u) = \varphi(u) \quad \forall(\varphi, u) \in \big(\mathcal{D}^{1,2}_{a,\lambda}(\mathbb{R}^N)\big)' \times \mathcal{D}^{1,2}_{a,\lambda}(\mathbb{R}^N),$$

we shall consider $f'_\varepsilon(u)$ as an element of $\mathcal{D}^{1,2}_{a,\lambda}(\mathbb{R}^N)$ and $f''_\varepsilon(u)$ as one of $\mathcal{L}(\mathcal{D}^{1,2}_{a,\lambda}(\mathbb{R}^N))$.
If we test $f'_\varepsilon(u)$ with $u_- = \max\{-u, 0\}$ we get

$$\big(f'_\varepsilon(u), u_-\big) = \int_{\mathbb{R}^N} |x|^{-2a} \nabla u \cdot \nabla u_- - \lambda \int_{\mathbb{R}^N} \frac{u u_-}{|x|^{2(1+a)}} - \int_{\mathbb{R}^N} \big(1 + \varepsilon k(x)\big) \frac{u_+^{p-1} u_-}{|x|^{bp}} = -\|u_-\|^2$$

and see that any critical point of f_ε is nonnegative. The maximum principle applied in $\mathbb{R}^N \backslash \{0\}$ shows that any nontrivial critical point is positive in that region. We cannot expect more since the radial solutions to the unperturbed problem ($\varepsilon = 0$) vanish at the origin if $\lambda < 0$ (see (2.5) below). Moreover from standard elliptic regularity theory, solutions to $(\mathcal{P}_{a,b,\lambda})$ are $C^{1,\alpha}(\mathbb{R}^N \setminus \{0\})$, $\alpha > 0$.
The unperturbed functional f_0 is given by

$$f_0(u) := \frac{1}{2} \int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2\, dx - \frac{\lambda}{2} \int_{\mathbb{R}^N} \frac{u^2}{|x|^{2(1+a)}}\, dx - \frac{1}{p} \int_{\mathbb{R}^N} \frac{u_+^p}{|x|^{bp}}\, dx, \quad u \in \mathcal{D}^{1,2}_{a,\lambda}(\mathbb{R}^N)$$

and we may write $f_\varepsilon(u) = f_0(u) + \varepsilon G(u)$, where

$$G(u) := \frac{1}{p} \int_{\mathbb{R}^N} k(x) \frac{u_+^p}{|x|^{bp}}. \tag{1.14}$$

2. The unperturbed problem

Critical points of the unperturbed functional f_0 solve the equation

$$\begin{cases} -\mathrm{div}\,(|x|^{-2a} \nabla u) - \dfrac{\lambda}{|x|^{2(1+a)}}\, u = \dfrac{1}{|x|^{bp}}\, u^{p-1} \\ u \in \mathcal{D}^{1,2}_{a,\lambda}(\mathbb{R}^N), \quad u > 0 \text{ in } \mathbb{R}^N \backslash \{0\}. \end{cases} \tag{2.1}$$

To find all radially symmetric solutions u of (2.1), i.e. $u(x) = u(r)$, where $r = |x|$, we follow [7] and note that if u is radial, then equation (2.1) can be written as

$$-\frac{u''}{r^{2a}} - \frac{N - 2a - 1}{r^{2a+1}}\, u' - \frac{\lambda}{r^{2(a+1)}}\, u = \frac{1}{r^{bp}}\, u^{p-1}. \tag{2.2}$$

Making now the change of variable

$$u(r) = r^{-\frac{N-2-2a}{2}} \varphi(\ln r), \tag{2.3}$$

we come to the equation

$$-\varphi'' + \left[\left(\frac{N - 2 - 2a}{2} \right)^2 - \lambda \right] \varphi - \varphi^{p-1} = 0. \tag{2.4}$$

All positive solutions of (2.4) in $H^{1,2}(\mathbb{R})$ are the translates of

$$\varphi_1(t) = \left[\frac{N(N-2-2a)\sqrt{(N-2-2a)^2 - 4\lambda}}{4(N - 2(1+a-b))} \right]^{\frac{N-2(1+a-b)}{4(1+a-b)}} \cdot$$

$$\cdot \left(\cosh \frac{(1+a-b)\sqrt{(N-2-2a)^2 - 4\lambda}}{N - 2(1+a-b)}\, t \right)^{-\frac{N-2(1+a-b)}{2(1+a-b)}},$$

namely $\varphi_\mu(t) = \varphi_1(t - \ln \mu)$ for some $\mu > 0$ (see [7]). Consequently all radial solutions of (2.1) are dilations of

$$z_1^{a,b,\lambda}(x) = \left[\frac{N(N-2-2a)\sqrt{(N-2-2a)^2-4\lambda}}{N-2(1+a-b)}\right]^{\frac{N-2(1+a-b)}{4(1+a-b)}} \cdot$$

$$\cdot \left[|x|^{\left(1-\frac{\sqrt{(N-2-2a)^2-4\lambda}}{N-2-2a}\right)\frac{(N-2-2a)(1+a-b)}{N-2(1+a-b)}}\left[1+|x|^{\frac{2(1+a-b)\sqrt{(N-2-2a)^2-4\lambda}}{N-2(1+a-b)}}\right]\right]^{-\frac{N-2(1+a-b)}{2(1+a-b)}} \tag{2.5}$$

and given by

$$z_\mu^{a,b,\lambda}(x) = \mu^{-\frac{N-2-2a}{2}} z_1^{a,b,\lambda}\Big(\frac{x}{\mu}\Big), \quad \mu > 0.$$

Using the change of coordinates in (2.3), respectively (1.13), and the exponential decay of $z_\mu^{a,b,\lambda}$ in these coordinates it is easy to see that the map $\mu \mapsto z_\mu^{a,b,\lambda}$ is at least twice continuously differentiable from $(0, \infty)$ to $\mathcal{D}^{1,2}_{a,\lambda}(\mathbb{R}^N)$ and we obtain

Lemma 2.1. *Suppose a, b, λ, p satisfy (1.2). Then the unperturbed functional f_0 has a one dimensional C^2-manifold of critical points $Z_{a,b,\lambda}$ given by $\{z_\mu^{a,b,\lambda} \mid \mu > 0\}$. Moreover, $Z_{a,b,\lambda}$ is exactly the set of all radially symmetric, positive solutions of (2.1) in $\mathcal{D}^{1,2}_{a,\lambda}(\mathbb{R}^N)$.*

In order to apply the abstract perturbation method we need to show that the manifold $Z_{a,b,\lambda}$ satisfy a non-degeneracy condition. This is the content of Theorem 1.1.

Proof of Theorem 1.1. The inclusion $T_{z_\mu^{a,b,\lambda}} Z_{a,b,\lambda} \subseteq \ker D^2 f_0(z_\mu^{a,b,\lambda})$ always holds and is a consequence of the fact that $Z_{a,b,\lambda}$ is a manifold of critical points of f_0. Consequently, we have only to show that $\ker D^2 f_0(z_\mu^{a,b,\lambda})$ is one dimensional. Fix $u \in \ker D^2 f_0(z_\mu^{a,b,\lambda})$. The function u is a solution of the linearized problem

$$-\mathrm{div}\,(|x|^{-2a}\nabla u) - \frac{\lambda}{|x|^{2(a+1)}} u = \frac{p-1}{|x|^{bp}} (z_\mu^{a,b,\lambda})^{p-2} u. \tag{2.6}$$

We expand u in spherical harmonics

$$u(r\vartheta) = \sum_{i=0}^{\infty} \vec{v}_i(r)\vec{Y}_i(\vartheta), \quad r \in \mathbb{R}^+, \quad \vartheta \in \mathbb{S}^{N-1},$$

where $\vec{v}_i(r) = \int_{\mathbb{S}^{N-1}} u(r\vartheta)\vec{Y}_i(\vartheta)\,d\vartheta$ and $\vec{Y}_i$ denotes the orthogonal i-th spherical harmonic jet satisfying for all $i \in \mathbb{N}_0$

$$-\Delta_{\mathbb{S}^{N-1}} \vec{Y}_i = i(N+i-2)\vec{Y}_i. \tag{2.7}$$

Since u solves (2.6) the functions $\vec{v}_i$ satisfy for all $i \geq 0$

$$-\frac{\vec{v}_i''}{r^{2a}} \vec{Y}_i - \frac{N-1-2a}{r^{2a+1}} \vec{v}_i' \vec{Y}_i - \frac{\vec{v}_i}{r^{2(a+1)}} \Delta_\vartheta \vec{Y}_i - \frac{\lambda}{r^{2(a+1)}} \vec{v}_i \vec{Y}_i = \frac{p-1}{r^{bp}} (z_\mu^{a,b,\lambda})^{p-2} \vec{v}_i \vec{Y}_i$$

and hence, in view of (2.7),

$$-\frac{\vec{v}_i''}{r^{2a}} - \frac{N-1-2a}{r^{2a+1}} \vec{v}_i' + \frac{i(N+i-2)}{r^{2(a+1)}} \vec{v}_i - \frac{\lambda}{r^{2(a+1)}} \vec{v}_i = \frac{p-1}{r^{bp}} (z_\mu^{a,b,\lambda})^{p-2} \vec{v}_i. \tag{2.8}$$

Making in (2.8) the transformation (2.3) we obtain the equations

$$-\vec{\varphi}_i'' - \beta \cosh^{-2}\big(\gamma(t - \ln \mu)\big)\vec{\varphi}_i = \left(\lambda - \left(\frac{N-2-2a}{2}\right)^2 - i(N+i-2)\right)\vec{\varphi}_i, \quad i \in \mathbb{N}_0,$$

where

$$\beta = \frac{N(N+2(1+a-b))((N-2-2a)^2-4\lambda)}{4(N-2(1+a-b))^2} \text{ and } \gamma = \frac{(1+a-b)\sqrt{(N-2-2a)^2-4\lambda}}{N-2(1+a-b)},$$

which is equivalent, through the change of variable $\zeta(s) = \varphi(s+\ln\mu)$, to

$$-\vec{\zeta}_i'' - \beta\cosh^{-2}(\gamma s)\vec{\zeta}_i = \left(\lambda - \left(\frac{N-2-2a}{2}\right)^2 - i(N+i-2)\right)\vec{\zeta}_i, \quad i \in \mathbb{N}_0. \tag{2.9}$$

It is known (see [8],[10, p. 74]) that the negative part of the spectrum of the problem

$$-\zeta'' - \beta\cosh^{-2}(\gamma s)\zeta = \nu\zeta$$

is discrete, consists of simple eigenvalues and is given by

$$\nu_j = -\frac{\gamma^2}{4}\left(-(1+2j)+\sqrt{1+4\beta\gamma^{-2}}\right)^2, \quad j \in \mathbb{N}_0, \quad 0 \le j < \frac{1}{2}\left(-1+\sqrt{1+4\beta\gamma^{-2}}\right).$$

Thus we have for all $i \ge 0$ that zero is the only solution to (2.9) if and only if

$$A_i(a,\lambda) \ne B_j(a,b,\lambda) \text{ for all } 0 \le j < \frac{N}{2(1+a-b)}, \tag{2.10}$$

where

$$A_i(a,\lambda) = \lambda - \left(\frac{N-2-2a}{2}\right)^2 - i(N+i-2)$$

and

$$B_j(a,b,\lambda) = -\frac{((N-2-2a)^2-4\lambda)(1+a-b)^2}{4(N-2(1+a-b))^2}\left[-2j + \frac{N}{1+a-b}\right]^2.$$

Note that $A_0(a,\lambda) = B_1(a,b,\lambda)$, $A_i(a,\lambda) \ge A_{i+1}(a,\lambda)$ and $B_j(a,b,\lambda) \le B_{j+1}(a,b,\lambda)$, which is shown in figure 3 below.

Figure 3

Hence (2.10) is satisfied for $i \ge 1$ if and only if $B_0(a,b,\lambda) \ne A_i(a,b,\lambda)$, which is equivalent to $b \ne h_i(a,\lambda)$. On the other hand for $i = 0$ equation (2.9) has a one dimensional space of nonzero solutions. Hence, $\ker D^2 f_0(z_\mu^{a,b,\lambda})$ is one dimensional if and only if $b \ne h_i(a,\lambda)$ for any $i \ge 1$, which proves the claim. □

Proof of Corollary 1.2. We define I on $D_a^{1,2}(\mathbb{R}^N)\backslash\{0\}$ by the right hand side of (1.7), i.e.

$$I(u) := \frac{\|\nabla u\|_a^2}{\|u\|_{p,b}^2}.$$

I is twice continuously differentiable and

$$(I'(u),\varphi)=\frac{2}{\|u\|_{p,b}^2}\Big(\int_{\mathbb{R}^N}|x|^{-2a}\nabla u\nabla\varphi-\frac{\|\nabla u\|_a^2}{\|u\|_{p,b}^p}\int_{\mathbb{R}^N}|x|^{-bp}|u|^{p-2}u\varphi\Big).$$

Moreover, for positive critical points u of I a short computation leads to

$$\begin{aligned}(I''(u)\varphi_1,\varphi_2)=&\frac{2}{\|u\|_{p,b}^2}\Big(\int_{\mathbb{R}^N}|x|^{-2a}\nabla\varphi_1\nabla\varphi_2-\frac{\|\nabla u\|_a^2}{\|u\|_{p,b}^p}(p-1)\int_{\mathbb{R}^N}|x|^{-bp}u^{p-2}\varphi_1\varphi_2\Big)\\&+(p-2)\frac{2\|\nabla u\|_a^2}{\|u\|_{p,b}^{2p+2}}\Big(\int_{\mathbb{R}^N}|x|^{-bp}u^{p-1}\varphi_1\Big)\Big(\int_{\mathbb{R}^N}|x|^{-bp}u^{p-1}\varphi_2\Big).\end{aligned}$$

Obviously I is constant on $Z_{a,b,0}$ and we obtain for $z_1:=z_1^{a,b,0}$ and all $\varphi_1,\varphi_2\in\mathcal{D}^{1,2}_{a,\lambda}(\mathbb{R}^N)$

$$\begin{aligned}(I'(z_1),\varphi_1)&=\frac{2}{\|z_1\|_{p,b}^2}(f_0'(z_1),\varphi_1)=0,\\(I''(z_1)\varphi_1,\varphi_2)&=\frac{2}{\|u\|_{p,b}^2}(f_0''(z_1)\varphi_1,\varphi_2)\\&\quad+(p-2)\frac{2}{\|z_1\|_{p,b}^{p+2}}\Big(\int_{\mathbb{R}^N}|x|^{-bp}z_1^{p-1}\varphi_1\Big)\Big(\int_{\mathbb{R}^N}|x|^{-bp}z_1^{p-1}\varphi_2\Big).\end{aligned}\tag{2.11}$$

From the proof of Theorem 1.1 we know that for $b<h_1(a,0)$ there exist functions $\hat\varphi\in D^{1,2}_a(\mathbb{R}^N)$ of the form $\hat\varphi(x)=\bar\varphi(|x|)Y_1(x/|x|)$, where Y_1 denotes one of the first spherical harmonics, such that $(f_0''(z_1)\hat\varphi,\hat\varphi)<0$. By (2.11) we get $(I''(z_1)\hat\varphi,\hat\varphi)<0$ because the integral $\int|x|^{-bp}z_1^{p-1}\hat\varphi=0$. Consequently $\mathcal{C}_{a,b}$ is strictly smaller than $I(z_1)=I(z_\mu^{a,b,0})$. Since all positive radial solutions of (2.1) are given by $z_\mu^{a,b,0}$ (see Lemma 2.1) and the infimum in (1.7) is attained (see [7, Thm 1.2]) the minimizer must be non-radial. □

As a particular case of Theorem 1.1 we can state

Corollary 2.2. (i) *If $0<a<\frac{N-2}{2}$ and $0\le\lambda<\left(\frac{N-2-2a}{2}\right)^2$ then $Z_{a,b,\lambda}$ is non-degenerate for any b between a and $a+1$.*
(ii) *If $a=0$ and $0\le\lambda<\left(\frac{N-2-2a}{2}\right)^2$, then $Z_{0,b,\lambda}$ is degenerate if and only if $b=\lambda=0$.*

Remark 2.3. *If $a=b=\lambda=0$, equation (2.1) is invariant not only by dilations but also by translations. The manifold of critical points is in this case $N+1$-dimensional and given by the translations and dilations of $z_1^{0,0,0}$. Hence the one dimensional manifold $Z_{0,0,0}$ is degenerate. However, the full $N+1$-dimensional critical manifold is non-degenerate in the case $a=b=\lambda=0$ (see [3]).*

3. The finite dimensional reduction

We follow the perturbative method developed in [2] and show that a finite dimensional reduction of our problem is possible whenever the critical manifold is non-degenerated. For simplicity of notation we write z_μ instead of $z_\mu^{a,b,\lambda}$ and Z instead of $Z_{a,b,\lambda}$ if there is no possibility of confusion.

Lemma 3.1. *Suppose a,b,λ,p satisfy (1.2) and v is a measurable function such that the integral $\int|v|^{\frac{p}{p-2}}|x|^{-bp}$ is finite. Then the operator $J_v:D^{1,2}_{a,\lambda}(\mathbb{R}^N)\to D^{1,2}_{a,\lambda}(\mathbb{R}^N)$, defined by*

$$J_v(u):=\mathcal{K}\Big(\int_{\mathbb{R}^N}|x|^{-pb}vu\,\cdot\Big),\tag{3.1}$$

is compact.

Proof. Fix a sequence $(u_n)_{n\in\mathbb{N}}$ converging weakly to zero in $D^{1,2}_{a,\lambda}(\mathbb{R}^N)$. To prove the assertion it is sufficient to show that up to a subsequence $J_v(u_n) \to 0$ as $n \to \infty$. Using the Hilbert space isomorphism given in (1.13) we see that the corresponding sequence $(v_n)_{n\in\mathbb{N}}$ converges weakly to zero in $H^{1,2}_\lambda(\mathcal{C})$. Since $(v_n)_{n\in\mathbb{N}}$ converges strongly in $L^2(\Omega)$ for all bounded domains Ω in $\mathcal{C}$, we may extract a subsequence that converges to zero pointwise almost everywhere. Going back to $\mathcal{D}^{1,2}_{a,\lambda}(\mathbb{R}^N)$ we may assume that this also holds for $(u_n)_{n\in\mathbb{N}}$. By Hölder's inequality and (1.3)

$$\begin{aligned}\|J_v(u_n)\| &\le \sup_{\|h\|_{D^{1,2}_{a,\lambda}(\mathbb{R}^N)}\le 1} \int_{\mathbb{R}^N} |x|^{-pb}|v||u_n||h| \\ &\le \sup_{\|h\|_{D^{1,2}_{a,\lambda}}\le 1} \Big(\int_{\mathbb{R}^N} |x|^{-pb}|h|^p\Big)^{1/p}\Big(\int_{\mathbb{R}^N} |x|^{-pb}|v|^{\frac{p}{p-1}}|u_n|^{\frac{p}{p-1}}\Big)^{(p-1)/p} \\ &\le C\Big(\int_{\mathbb{R}^N} |x|^{-pb}|v|^{\frac{p}{p-1}}|u_n|^{\frac{p}{p-1}}\Big)^{(p-1)/p}.\end{aligned}$$

To show that the latter integral converges to zero we use Vitali's convergence theorem given for instance in [9, 13.38]. Obviously the functions $|\cdot|^{-pb}|v|^{\frac{p}{p-1}}|u_n|^{\frac{p}{p-1}}$ converge pointwise almost everywhere to zero. For any measurable $\Omega \subset \mathbb{R}^N$ we may estimate using Hölder's inequality

$$\begin{aligned}\int_\Omega |x|^{-pb}|v|^{\frac{p}{p-1}}|u_n|^{\frac{p}{p-1}} &\le \Big(\int_\Omega |x|^{-pb}|v|^{\frac{p}{p-2}}\Big)^{(p-2)/(p-1)}\Big(\int_\Omega |x|^{-pb}|u_n|^p\Big)^{1/(p-1)} \\ &\le C\Big(\int_\Omega |x|^{-pb}|v|^{\frac{p}{p-2}}\Big)^{(p-2)/(p-1)}\end{aligned}$$

for some positive constant C. Taking Ω a set of small measure or the complement of a large ball and the use of Vitali's convergence theorem prove the assertion. □

Lemma 3.1 immediately leads to

Corollary 3.2. *For all $z \in Z$ the operator $f_0''(z) : D^{1,2}_{a,\lambda}(\mathbb{R}^N) \to D^{1,2}_{a,\lambda}(\mathbb{R}^N)$ may be written as $f_0''(z) = id - J_{|z|^{p-2}}$ and is consequently a self-adjoint Fredholm operator of index zero.*

Define for $\mu > 0$ the map $U_\mu : D^{1,2}_{a,\lambda}(\mathbb{R}^N) \to D^{1,2}_{a,\lambda}(\mathbb{R}^N)$ by

$$U_\mu(u) := \mu^{-\frac{N-2-2a}{2}} u\Big(\frac{x}{\mu}\Big).$$

It is easy to check that U_μ conserves the norms $\|\cdot\|$ and $\|\cdot\|_{p,b}$, thus for every $\mu > 0$

$$(U_\mu)^{-1} = (U_\mu)^t = U_{\mu^{-1}} \text{ and } f_0 = f_0 \circ U_\mu \tag{3.2}$$

where $(U_\mu)^t$ denotes the adjoint of U_μ. Twice differentiating the identity $f_0 = f_0 \circ U_\mu$ yields for all $h_1, h_2, v \in D^{1,2}_{a,\lambda}(\mathbb{R}^N)$

$$(f_0''(v)h_1, h_2) = (f_0''(U_\mu(v))U_\mu(h_1), U_\mu(h_2)),$$

that is

$$f_0''(v) = (U_\mu)^{-1} \circ f_0''(U_\mu(v)) \circ U_\mu \quad \forall v \in D^{1,2}_{a,\lambda}(\mathbb{R}^N). \tag{3.3}$$

Differentiating (3.2) we see that $U(\mu,z):=U_\mu(z)$ maps $(0,\infty)\times Z$ into Z, hence

$$\frac{\partial U}{\partial z}(\mu,z)=U_\mu:\ T_zZ\to T_{U_\mu(z)}Z \text{ and } U_\mu:\ (T_zZ)^\perp\to(T_{U_\mu(z)}Z)^\perp. \tag{3.4}$$

If the manifold Z is non-degenerated the self-adjoint Fredholm operator $f_0''(z_1)$ maps the space $D^{1,2}_{a,\lambda}(\mathbb{R}^N)$ into $T_{z_1}Z^\perp$ and $f_0''(z_1)\in\mathcal{L}(T_{z_1}Z^\perp)$ is invertible. Consequently, using (3.3) and (3.4), we obtain in this case

$$\|(f_0''(z_1))^{-1}\|_{\mathcal{L}(T_{z_1}Z^\perp)}=\|(f_0''(z))^{-1}\|_{\mathcal{L}(T_zZ^\perp)}\quad\forall z\in Z. \tag{3.5}$$

Lemma 3.3. *Suppose a,b,p,λ satisfy (1.2) and (1.4) holds. Then there exists a constant $C_1=C_1(\|k\|_\infty,a,b,\lambda)>0$ such that for any $\mu>0$ and for any $w\in\mathcal{D}^{1,2}_{a,\lambda}(\mathbb{R}^N)$*

$$|G(z_\mu+w)|\le C_1\big(\||k|^{1/p}z_\mu\|^p_{p,b}+\|w\|^p\big) \tag{3.6}$$

$$\|G'(z_\mu+w)\|\le C_1\big(\||k|^{1/p}z_\mu\|^{p-1}_{p,b}+\|w\|^{p-1}\big) \tag{3.7}$$

$$\|G''(z_\mu+w)\|\le C_1\big(\||k|^{1/p}z_\mu\|^{p-2}_{p,b}+\|w\|^{p-2}\big). \tag{3.8}$$

Moreover, if $\lim_{|x|\to\infty}k(x)=:k(\infty)=0=k(0)$ then

$$\||k|^{1/p}z_\mu\|_{p,b}\to 0 \text{ as } \mu\to\infty \text{ or } \mu\to 0. \tag{3.9}$$

Proof. (3.6)-(3.8) are consequences of (1.3) and Hölder's inequality. We will only show (3.8) as (3.6)-(3.7) follow analogously. By Hölder's inequality and (1.3)

$$\begin{aligned}\|G''(z_\mu+w)\|&\le(p-1)\sup_{\|h_1\|,\|h_2\|\le1}\int_{\mathbb{R}^N}\frac{|k(x)|}{|x|^{bp}}|z_\mu+w|^{p-2}|h_1||h_2|\\&\le(p-1)\||k|^{1/p}\|^2_\infty\sup_{\|h_1\|,\|h_2\|\le1}\||k|^{1/p}(z_\mu+w)\|^{p-2}_{p,b}\|h_1\|_{p,b}\|h_2\|_{p,b}\\&\le c(\|k\|_\infty,a,b,\lambda)\,\||k|^{1/p}(z_\mu+w)\|^{p-2}_{p,b}.\end{aligned}$$

Using the triangle inequality and again (1.3) we obtain (3.8).
Under the additional assumption $k(0)=k(\infty)=0$ estimate (3.9) follows by the dominated convergence theorem and

$$\int_{\mathbb{R}^N}\frac{|k(x)|}{|x|^{bp}}z_\mu^p=\int_{\mathbb{R}^N}\frac{|k(\mu x)|}{|x|^{bp}}z_1^p.$$

□

Lemma 3.4. *Suppose a,b,p,λ satisfy (1.2) and (1.4) and (1.5) hold. Then there exist constants $\varepsilon_0,C>0$ and a smooth function*

$$w=w(\mu,\varepsilon):\quad(0,+\infty)\times(-\varepsilon_0,\varepsilon_0)\ \longrightarrow\ \mathcal{D}^{1,2}_{a,\lambda}(\mathbb{R}^N)$$

such that for any $\mu>0$ and $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$

$$w(\mu,\varepsilon)\ \text{ is orthogonal to }\ T_{z_\mu}Z \tag{3.10}$$

$$f'_\varepsilon\big(z_\mu+w(\mu,\varepsilon)\big)\in T_{z_\mu}Z \tag{3.11}$$

$$\|w(\mu,\varepsilon)\|\le C\,|\varepsilon|. \tag{3.12}$$

Moreover, if (1.8) holds then

$$\|w(\mu,\varepsilon)\|\to 0 \text{ as } \mu\to 0 \text{ or } \mu\to\infty. \tag{3.13}$$

Proof. Define $H : (0,\infty) \times D^{1,2}_{a,\lambda}(\mathbb{R}^N) \times \mathbb{R} \times \mathbb{R} \to D^{1,2}_{a,\lambda}(\mathbb{R}^N) \times \mathbb{R}$

$$H(\mu, w, \alpha, \varepsilon) := (f'_\varepsilon(z_\mu + w) - \alpha\dot{\xi}_\mu, (w, \dot{\xi}_\mu)),$$

where $\dot{\xi}_\mu$ denotes the normalized tangent vector $\frac{d}{d\mu} z_\mu$. If $H(\mu, w, \alpha, \varepsilon) = (0,0)$ then w satisfies (3.10)-(3.11) and $H(\mu, w, \alpha, \varepsilon) = (0,0)$ if and only if $(w, \alpha) = F_{\mu,\varepsilon}(w, \alpha)$, where

$$F_{\mu,\varepsilon}(w, \alpha) := -\Big(\frac{\partial H}{\partial(w,\alpha)}(\mu, 0, 0, 0)\Big)^{-1} H(\mu, w, \alpha, \varepsilon) + (w, \alpha).$$

We prove that $F_{\mu,\varepsilon}(w, \alpha)$ is a contraction in some ball $B_\rho(0)$, where we may choose the radius $\rho = \rho(\varepsilon) > 0$ independent of $z \in Z$. To this end we observe

$$\Big(\Big(\frac{\partial H}{\partial(w,\alpha)}(\mu, 0, 0, 0)\Big)(w, \beta), (f''_0(z_\mu)w - \beta\dot{\xi}_\mu, (w, \dot{\xi}_\mu))\Big) = \|f''_0(z_\mu)w\|^2 + \beta^2 + |(w, \dot{\xi}_\mu)|^2, \tag{3.14}$$

where

$$\Big(\frac{\partial H}{\partial(w,\alpha)}(\mu, 0, 0, 0)\Big)(w, \beta) = (f''_0(z_\mu)w - \beta\dot{\xi}_\mu, (w, \dot{\xi}_\mu)).$$

From Corollary 3.2 and (3.14) we infer that $\big(\frac{\partial H}{\partial(w,\alpha)}(\mu, 0, 0, 0)\big)$ is an injective Fredholm operator of index zero, hence invertible and by (3.5) and (3.14) we obtain

$$\Big\|\Big(\frac{\partial H}{\partial(w,\alpha)}(\mu, 0, 0, 0)\Big)^{-1}\Big\| \le \max\big(1, \|(f''_0(z_\mu))^{-1}\|\big) = \max\big(1, \|(f''_0(z_1))^{-1}\|\big) =: C_*. \tag{3.15}$$

Suppose $(w, \alpha) \in B_\rho(0)$. We use (3.3) and (3.15) to see

$$\begin{aligned}
\|F_{\mu,\varepsilon}(w, \alpha)\| &\le C_*\Big\|\Big(H(\mu, w, \alpha, \varepsilon) - \Big(\frac{\partial H}{\partial(w,\alpha)}(\mu, 0, 0, 0)\Big)(w, \alpha)\Big)\Big\| \\
&\le C_*\|f'_\varepsilon(z_\mu + w) - f''_0(z_\mu)w\| \\
&\le C_*\int_0^1 \|f''_0(z_\mu + tw) - f''_0(z_\mu)\| \text{ dt } \|w\| + C_*|\varepsilon|\|G'(z_\mu + w)\| \\
&\le C_*\int_0^1 \|f''_0(z_1 + tU_{\mu^{-1}}(w)) - f''_0(z_1)\| \text{ dt } \|w\| + C_*|\varepsilon|\|G'(z_\mu + w)\| \\
&\le C_*\rho \sup_{\|w\|\le\rho} \|f''_0(z_1 + w) - f''_0(z_1)\| + C_*|\varepsilon| \sup_{\|w\|\le\rho} \|G'(z_\mu + w)\|.
\end{aligned} \tag{3.16}$$

Analogously we get for $(w_1, \alpha_1), (w_2, \alpha_2) \in B_\rho(0)$

$$\begin{aligned}
\frac{\|F_{\mu,\varepsilon}(w_1, \alpha_1) - F_{\mu,\varepsilon}(w_2, \alpha_2)\|}{C_*\|w_1 - w_2\|} &\le \frac{\|f'_\varepsilon(z_\mu + w_1) - f'_\varepsilon(z_\mu + w_2) - f''_0(z_\mu)(w_1 - w_2)\|}{\|w_1 - w_2\|} \\
&\le \int_0^1 \|f''_\varepsilon(z_\mu + w_2 + t(w_1 - w_2)) - f''_0(z_\mu)\|\, dt \\
&\le \int_0^1 \|f''_0(z_\mu + w_2 + t(w_1 - w_2)) - f''_0(z_\mu)\|\, dt \\
&\quad + |\varepsilon| \int_0^1 \|G''(z_\mu + w_2 + t(w_1 - w_2))\|\, dt \\
&\le \sup_{\|w\|\le 3\rho} \|f''_0(z_1 + w) - f''_0(z_1)\| + |\varepsilon| \sup_{\|w\|\le 3\rho} \|G''(z_\mu + w)\|.
\end{aligned}$$

We may choose $\rho_0 > 0$ such that

$$C_* \sup_{\|w\|\le 3\rho_0} \|f_0''(z_1+w) - f_0''(z_1)\| < \frac{1}{2}$$

and $\varepsilon_0 > 0$ such that

$$2\varepsilon_0 < \Big(\sup_{z\in Z, \|w\|\le 3\rho_0} \|G''(z+w)\|\Big)^{-1} C_*^{-1} \text{ and } 3\varepsilon_0 < \Big(\sup_{z\in Z, \|w\|\le \rho_0} \|G'(z+w)\|\Big)^{-1} C_*^{-1}\rho_0.$$

With these choices and the above estimates it is easy to see that for every $z_\mu \in Z$ and $|\varepsilon| < \varepsilon_0$ the map $F_{\mu,\varepsilon}$ maps $B_{\rho_0}(0)$ in itself and is a contraction there. Thus $F_{\mu,\varepsilon}$ has a unique fix-point $(w(\mu,\varepsilon), \alpha(\mu,\varepsilon))$ in $B_{\rho_0}(0)$ and it is a consequence of the implicit function theorem that w and α are continuously differentiable.
From (3.16) we also infer that $F_{z,\varepsilon}$ maps $B_\rho(0)$ into $B_\rho(0)$, whenever $\rho \le \rho_0$ and

$$\rho > 2|\varepsilon|\big(\sup_{\|w\|\le\rho} \|G'(z+w)\|\big)C_*.$$

Consequently due to the uniqueness of the fix-point we have

$$\|(w(z,\varepsilon), \alpha(z,\varepsilon))\| \le 3|\varepsilon|\big(\sup_{\|w\|\le\rho_0} \|G'(z+w)\|\big)C_*,$$

which gives (3.12). Let us now prove (3.13). Set

$$\rho_\mu := \min\left\{4\varepsilon_0 C_* C_1 \| |k|^{1/p} z_\mu\|_{p,b}^{p-1}, \rho_0, \Big(\frac{1}{8\varepsilon_0 C_1 C_*}\Big)^{\frac{1}{p-2}}\right\}$$

where C_1 is given in Lemma 3.3. In view of (3.7) we have that for any $|\varepsilon| < \varepsilon_0$ and $\mu > 0$

$$2|\varepsilon| C_* \sup_{\|w\|\le\rho_\mu} \|G'(z_\mu+w)\| \le 2|\varepsilon| C_* C_1 \| |k|^{1/p} z_\mu\|_{p,b}^{p-1} + 2|\varepsilon| C_* C_1 \rho_\mu^{p-2}\rho_\mu.$$

Since $\rho_\mu^{p-2} \le \frac{1}{8\varepsilon_0 C_1 C_*}$ we have,

$$2|\varepsilon| C_* \sup_{\|w\|\le\rho_\mu} \|G'(z_\mu+w) < 2|\varepsilon| C_* C_1 \| |k|^{1/p} z_\mu\|_{p,b}^{p-1} + \frac{1}{2}\rho_\mu \le \rho_\mu,$$

so that, by the above argument, we can conclude that $F_{\mu,\varepsilon}$ maps $B_{\rho_\mu}(0)$ into $B_{\rho_\mu}(0)$. Consequently due to the uniqueness of the fix-point we have

$$\|w(\mu,\varepsilon)\| \le \rho_\mu.$$

Since by (3.9) we have that $\rho_\mu \to 0$ for $\mu \to 0$ and for $\mu \to +\infty$, we get (3.13). □

Under the assumptions of Lemma 3.4 we may define for $|\varepsilon| < \varepsilon_0$

$$Z^\varepsilon_{a,b,\lambda} := \big\{u \in \mathcal{D}^{1,2}_{a,\lambda}(\mathbb{R}^N) \,|\, u = z_\mu^{a,b,\lambda} + w(\mu,\varepsilon),\ \mu \in (0,\infty)\big\}. \tag{3.17}$$

Note that Z^ε is a one dimensional manifold.

Lemma 3.5. *Under the assumptions of Lemma 3.4 we may choose $\varepsilon_0 > 0$ such that for every $|\varepsilon| < \varepsilon_0$ the manifold Z^ε is a natural constraint for f_ε, i.e. every critical point of $f_\varepsilon|_{Z^\varepsilon}$ is a critical point of f_ε.*

Proof. Fix $u \in Z^\varepsilon$ such that $f_\varepsilon|'_{Z^\varepsilon}(u) = 0$. In the following we use a dot for the derivation with respect to μ. Since $(\dot z_\mu, w(\mu,\varepsilon)) = 0$ for all $\mu > 0$ we obtain

$$(\ddot z_\mu, w(\mu,\varepsilon)) + (\dot z_\mu, \dot w(\mu,\varepsilon)) = 0. \tag{3.18}$$

Moreover differentiating the identity $z_\mu = U_\sigma z_{\mu/\sigma}$ with respect to μ we obtain

$$\dot z_\sigma = \frac{1}{\sigma} U_\sigma \dot z_1 \text{ and } \ddot z_\sigma = \frac{1}{\sigma^2} U_\sigma \ddot z_1. \tag{3.19}$$

From (3.11) we get that $f'_\varepsilon(u) = c_1 \dot z_\mu$ for some $\mu > 0$. By (3.18) and (3.19)

$$\begin{aligned} 0 &= (f'_\varepsilon(u), \dot z_\mu + \dot w(\mu,\varepsilon)) = c_1(\dot z_\mu, \dot z_\mu + \dot w(\mu,\varepsilon)) \\ &= c_1 \mu^{-2}\big(\|\dot z_1\|^2 - (\ddot z_1, U_{\mu^{-1}} w(\mu,\varepsilon))\big) = c_1\mu^{-2}\big(\|\dot z_1\|^2 - \|\ddot z_1\| O(1)\varepsilon\big). \end{aligned}$$

Finally we see that for small $\varepsilon > 0$ the number c_1 must be zero and the assertion follows. □

In view of the above result we end up facing a finite dimensional problem as it is enough to find critical points of the functional $\Phi_\varepsilon : (0,\infty) \to \mathbb{R}$ given by $f_\varepsilon|_{Z^\varepsilon}$.

4. Study of Φ_ε

In this section we will assume that the critical manifold is non-degenerate, i.e. (1.5), such that the functional Φ_ε is defined. To find critical points of $\Phi_\varepsilon = f_\varepsilon|_{Z^\varepsilon}$ it is convenient to introduce the functional Γ given below.

Lemma 4.1. *Suppose a, b, p, λ satisfy (1.2) and (1.4) holds. Then*

$$\Phi_\varepsilon(\mu) = f_0(z_1) - \varepsilon\Gamma(\mu) + o(\varepsilon), \tag{4.1}$$

where $\Gamma(\mu) = G(z_\mu)$. In particular, there is $C > 0$, independent of μ and ε, such that

$$|\Phi_\varepsilon(\mu) - (f_0(z_1) - \varepsilon\Gamma(\mu))| \le C\big(\|w(\varepsilon,\mu)\|^2 + (1+|\varepsilon|)\|w(\varepsilon,\mu)\|^p + |\varepsilon|\|w(\varepsilon,\mu)\|\big). \tag{4.2}$$

Consequently, if there exist $0 < \mu_1 < \mu_2 < \mu_3 < \infty$ such that

$$\Gamma(\mu_2) > \max(\Gamma(\mu_1), \Gamma(\mu_3)) \text{ or } \Gamma(\mu_2) < \min(\Gamma(\mu_1), \Gamma(\mu_3)) \tag{4.3}$$

then Φ_ε will have a critical point, if $\varepsilon > 0$ is sufficiently small.

Proof. Note that for all $\mu > 0$ we have $f_0(z_\mu) = f_0(z_1)$,

$$\|z_\mu\|^2 = \int_{\mathbb{R}^N} \frac{z_\mu^p}{|x|^{bp}} \text{ and } \big(z_\mu, w(\varepsilon,\mu)\big) = \int_{\mathbb{R}^N} \frac{z_\mu^{p-1} w(\varepsilon,\mu)}{|x|^{bp}}. \tag{4.4}$$

From (4.4) we infer

$$\Phi_\varepsilon(\mu) = \frac12 \int_{\mathbb{R}^N} \frac{z_\mu^p}{|x|^{bp}} + \frac12 \|w(\varepsilon,\mu)\|^2 + \int_{\mathbb{R}^N} \frac{z_\mu^{p-1} w(\varepsilon,\mu)}{|x|^{bp}} - \frac1p \int_{\mathbb{R}^N} \frac{(1+\varepsilon k)\big(z_\mu + w(\varepsilon,\mu)\big)^p}{|x|^{bp}}$$

and

$$f_0(z_1) = f_0(z_\mu) = \frac12\|z_\mu\|^2 - \frac1p \int_{\mathbb{R}^N} \frac{z_\mu^p}{|x|^{bp}} = \left(\frac12 - \frac1p\right)\int_{\mathbb{R}^N} \frac{z_\mu^p}{|x|^{bp}}.$$

Hence

$$\Phi_\varepsilon(\mu) = f_0(z_1) - \varepsilon\Gamma(\mu) + \frac12\|w(\varepsilon,\mu)\|^2 - \frac1p H_\varepsilon(\mu), \tag{4.5}$$

where

$$H_\varepsilon(\mu) = \int_{\mathbb{R}^N} \frac{\big(z_\mu + w(\varepsilon,\mu)\big)^p - z_\mu^p - p\, z_\mu^{p-1} w(\varepsilon,\mu) + \varepsilon k\big((z_\mu + w(\varepsilon,\mu))^p - z_\mu^p\big)}{|x|^{bp}}.$$

Using the inequality

$$(z+w)^{s-1} - z^{s-1} - (p-1)z^{s-2}w \le \begin{cases} C(z^{s-3}w^2 + w^{s-1}) & \text{if } s \ge 3 \\ C\, w^{s-1} & \text{if } 2 < s < 3, \end{cases}$$

where $C = C(s) > 0$, with $s = p + 1$ and Hölder's inequality we have for some $c_2, c_3 > 0$

$$
\begin{aligned}
|H_\varepsilon(\mu)| &\le \int_{\mathbb{R}^N} \frac{\left|\big(z_\mu + w(\varepsilon,\mu)\big)^p - z_\mu^p - p\, z_\mu^{p-1} w(\varepsilon,\mu)\right|}{|x|^{bp}} + |\varepsilon| \int_{\mathbb{R}^N} \frac{|k| \left((z_\mu + w(\varepsilon,\mu))^p - z_\mu^p\right)}{|x|^{bp}} \\
&\le c_2 \left[\int_{\mathbb{R}^N} \frac{z_\mu^{p-2} w^2(\varepsilon,\mu)}{|x|^{bp}} + \int_{\mathbb{R}^N} \frac{|w(\varepsilon,\mu)|^p}{|x|^{bp}} + |\varepsilon| \int_{\mathbb{R}^N} \frac{z_\mu^{p-1}|w(\varepsilon,\mu)|}{|x|^{bp}} + |\varepsilon| \int_{\mathbb{R}^N} \frac{|w(\varepsilon,\mu)|^p}{|x|^{bp}} \right] \\
&\le c_3 \left[\|w(\varepsilon,\mu)\|^2 + (1 + |\varepsilon|)\|w(\varepsilon,\mu)\|^p + |\varepsilon| \|w(\varepsilon,\mu)\| \right]
\end{aligned}
$$

and the claim follows. □

Although it is convenient to study only the reduced functional Γ instead of Φ_ε, it may lead in some cases to a loss of information, i.e. Γ may be constant even if k is a non-constant function. This is due to the fact that the critical manifold consists of radially symmetric functions. Thus Γ is constant for every k that has constant mean-value over spheres, i.e.

$$\frac{1}{r^{N-1}} \int_{\partial B_r(0)} k(x)\, dS(x) \equiv \text{const} \qquad \forall r > 0.$$

In this case we have to study the functional $\Phi_\varepsilon(\mu)$ directly.

Proof of Theorem 1.3. By (1.8), (3.9), (3.13) and (4.2)

$$\lim_{\mu \to 0^+} \Phi_\varepsilon(\mu) = \lim_{\mu \to +\infty} \Phi_\varepsilon(\mu) = f_0(z_1).$$

Hence, either the functional $\Phi_\varepsilon \equiv f_0(z_1)$, and we obtain infinitely many critical points, or $\Phi_\varepsilon \not\equiv f_0(z_1)$ and Φ_ε has at least a global maximum or minimum. In any case Φ_ε has a critical point that provides a solution of $(\mathcal{P}_{a,b,\lambda})$. □

The next lemma shows that it is possible (and convenient) to extend the C^2- functional Γ by continuity to $\mu = 0$. The proof of this fact is analogous to the one in [3, Lem. 3.4] and we omit it here.

Lemma 4.2. *Under the assumptions of Lemma 4.1*

$$\Gamma(0) := \lim_{\mu \to 0} \Gamma(\mu) = k(0) \frac{1}{p} \|z_1\|_{p,b}^p \quad \textit{and} \tag{4.6}$$

$$\frac{1}{p} \liminf_{|x| \to \infty} k(x) \|z_1\|_{p,b}^p \le \liminf_{\mu \to \infty} \Gamma(\mu) \le \limsup_{\mu \to \infty} \Gamma(\mu) \le \frac{1}{p} \limsup_{|x| \to \infty} k(x) \|z_1\|_{p,b}^p. \tag{4.7}$$

If, moreover, (1.9) holds we obtain

$$\Gamma'(0) = 0 \ \textit{and}\ \Gamma''(0) = \frac{\Delta k(0)}{Np} \int |x|^2 \frac{z_1(x)^p}{|x|^{bp}}. \tag{4.8}$$

Proof of Theorem 1.4. To see that assumptions (1.10) and (1.11) give rise to a critical point we use the functional Γ. Condition (1.10) and Lemma 4.2 imply that Γ has a global maximum strictly bigger than $\Gamma(0)$ and $\limsup_{\mu \to \infty} \Gamma(\mu)$. Consequently Φ_ε has a critical point in view of Lemma 4.1. The same reasoning yields a critical point under condition (1.11). □

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