MINIMAL COEXISTENCE CONFIGURATIONS FOR MULTISPECIES SYSTEMS

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Abstract. We deal with strongly competing multispecies systems of Lotka-Volterra type with homogeneous Neumann boundary conditions in dumbbell-like domains. Under suitable non-degeneracy assumptions, we show that, as the competition rate grows indefinitely, the system reaches a state of coexistence of all the species in spatial segregation. Furthermore, the limit configuration is a local minimizer for the associated free energy.

1. Introduction

In this paper we consider the system of $k \geq 2$ elliptic equations

$$-\Delta u_i + u_i = f_i(u_i) - \varkappa\, u_i \sum_{j \neq i} u_j^2, \qquad \text{in } \Omega, \tag{1}$$

for $i = 1, \dots, k$. It models the steady states of k organisms, each of density u_i, which coexist in a smooth, connected, bounded domain $\Omega \subset \mathbb{R}^N$; their dynamics is ruled out by internal growth f_i's and mutual competition of Lotka-Volterra type with parameter $\varkappa > 0$. Systems of this form have attracted considerable attention both in ecology and social sciences since they furnish a relatively simple model to study the behavior of k populations competing for the same resource Ω. One of the main question is to investigate whether *coexistence* may occur, namely the existence of equilibrium configurations where all the densities u_i are strictly positive on sets of positive measure, or the internal dynamic leads to *extinction*, that is steady states where one or more densities are null. Many results are nowadays available, dealing mainly with $k = 2$ populations. We quote among others [15, 16, 18, 19, 20, 21] where, for logistic internal growth $f_i(u) = u(a_i - u)$, both the situation are proved to be possible depending on the relations between the diffusion rates and the coefficients of intra–specific and of inter–specific competition, see also [11, 12].

A different perspective is proposed in [3, 4, 7, 10, 13, 14], where the authors study the effect of very strong competition, letting the parameter $\varkappa$ growing indefinitely. It is observed (see Section 5) that the presence of large interactions of competitive type

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produces, in the limit configuration as $\varkappa \to \infty$, the *spatial segregation* of the densities, meaning that if $(u_i^\varkappa)_{i=1,\dots,k}$ solves (1), then $u_i^\varkappa$ converges (in a suitable sense) to some u_i which satisfies

$$u_i(x) \cdot u_j(x) = 0 \text{ a.e. in } \Omega, \qquad \text{for all } i \neq j. \tag{2}$$

A number of qualitative properties of the possible coexistence states u_i and their supports is proved in [5, 7, 8], with the aim of describing the way the territory is partitioned by the segregated populations. We refer the interested reader to the above quoted papers for details on the regularity theory so far developed and to [4, 6] for some applications.

A further point of interest is to establish if coexistence of the species is possible in a segregated configuration: do all the species survive when the intra specific competition becomes larger and larger? The answer cannot be positive in general: [17] shows that in any *convex domain* the only stable configurations are those where only one specie is alive. It is worth pointing out that in [5, 7]c the strict positivity of each component in the limiting configuration is guaranteed by simply forcing non-homogeneous Dirichlet boundary conditions

$$u_i = \phi_i \quad \text{on } \partial\Omega, \tag{3}$$

with $\phi_i > 0$ on a set of positive $(N-1)$–measure. Coexistence results for competing systems under more natural homogeneous boundary conditions are obtained in [3] for the Dirichlet case

$$u_i = 0 \quad \text{on } \partial\Omega, \tag{4}$$

with interactions of the form $\varkappa u_i \sum_{j\neq i} u_j$. To avoid the extinction predicted by [17], a special class of non-convex domains close to a union of k disjoint balls is considered. Suitable non-degeneracy assumptions on the f_i's allow the application of a domain perturbation technique envisaged in [9] which strongly relies on the continuity of the eigenvalues of the Laplace operator with respect to the domain. It is well known that such a property does not hold in the case of Neumann boundary conditions, see for instance [2]. Hence, in order to treat Neumann no-flux boundary conditions, a different approach is needed.

This is precisely the aim of the present paper: we deal with system (1) coupled with

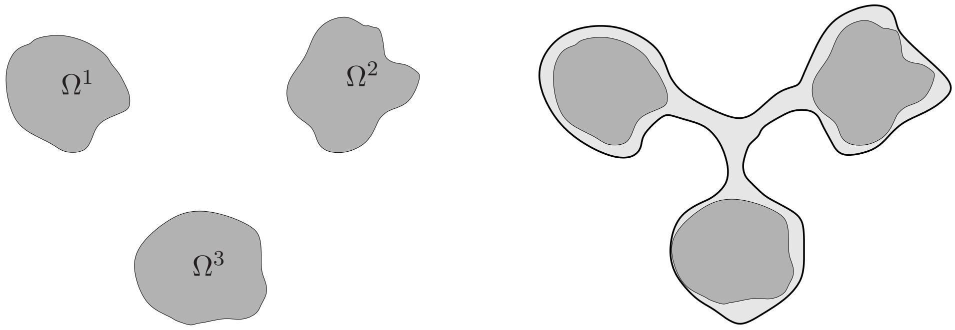
$$\frac{\partial u_i}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{5}$$

in a class of non-convex domains $\Omega = \Omega_\varepsilon$ suitably approximating a given domain Ω_0 composed by k disjoint open sets, see Figure 1.

Due to the variational structure of problem (1), the following *free energy* functional

$$J_\Omega(U) = \sum_{i=1,\dots,k} \left\{ \frac{1}{2} \int_\Omega \big(|\nabla u_i(x)|^2 + |u_i(x)|^2\big) dx - \int_\Omega F_i(u_i(x)) dx \right\}, \tag{6}$$

given by the sum of the internal energies of the k densities u_i, each having internal potential $F_i(x,s) = \int_0^s f_i(x,u) du$, is naturally associated to the system.



(a) example of Ω_0 with $k = 3$ (b) example of Ω_ε approximating Ω_0

FIGURE 1

Our analysis will highlight how the coexistence of all the densities is connected to the following minimization problem: *finding local minimizers of* $J_\Omega(U)$ *in the class of segregated states*

$$\mathcal{U} = \Big\{ U = (u_1, u_2, \dots, u_k) \in \big(H^1(\Omega)\big)^k : \, u_i \geq 0, \; u_i \cdot u_j = 0 \text{ if } i \neq j, \text{ a.e. in } \Omega \Big\}.$$

The problem of the existence of the *global minimum* of $J_\Omega(U)$ in $\mathcal{U}$ was investigated in [5] under the non-homogeneous conditions (3). As we shall see in Theorem 2.1, the global minimizer under homogeneous boundary conditions is in general trivial, namely a k-tuple with all but one component identically null. Hence, the only possibility for finding a *stable* coexistence solution where all the k densities survive, consists in looking for *local* minimizers of J_Ω, see problem $(\boldsymbol{P_\varepsilon})$ below.

Exploiting the variational character of the interaction term in (1) and developing a suitable domain perturbation technique, in this paper we give positive answer to both questions of minimization of J_Ω and occurrence of coexistence states for the system. Our main result can be summarized as follows: under suitable assumptions on f_i's ensuring the existence of a non-degenerate solution to the system on the unperturbed domain Ω_0 (see (10) below), *for all* Ω_ε *close enough to* Ω_0 *and large parameter* $\varkappa$*, there exists* $(u_1^\varkappa, \dots, u_k^\varkappa)$ *solution to* (1) *in* Ω_ε*, whose limit configuration as* $\varkappa \to \infty$ *is a segregated coexistence state* $(u_1, \dots, u_k)$ *with* k *positive components (i.e. each component* $u_i \geq 0$ *and* u_i *is strictly positive on a set of positive measure), characterized as a local minimizer of the free energy* J_{Ω_ε}.

Before stating rigorously our assumptions and main results, a further remark is in order. As observed in [5] (see also Theorem 5.1) any $(u_1, \dots, u_k)$ which is a local minimizer of the free energy J_Ω on $\mathcal{U}$, is also a solution of the following system of distributional inequalities:

$$
\begin{cases}
\displaystyle\int_\Omega \big(\nabla u_i(x)\nabla\phi(x) + u_i(x)\phi(x) - f_i(u_i(x))\phi(x)\big)dx \le 0,\\
\displaystyle\int_\Omega (\nabla\widehat{u}_i(x)\nabla\phi(x) + \widehat{u}_i(x)\phi(x) - \widehat{f}(\widehat{u}_i(x))\phi(x))dx \ge 0,
\end{cases}
\tag{7}
$$

$i = 1,\dots,k$, for any non-negative $\phi \in H^1(\Omega)$, where we have denoted $\widehat{u}_i = u_i - \sum_{h\neq i} u_h$ and $\widehat{f}(\widehat{u}_i) = f(u_i) - \sum_{j\neq i} f_j(u_j)$. The link between systems of this form and population dynamics has been pointed out in [3, 5, 7]: as a matter of fact *all* the limiting configurations as $\varkappa \to \infty$ of the solutions to (1) are solutions to (7). In other words, the possibility of coexistence of many species ruled out by strong competition is governed by the system of distributional inequalities (7): its independent study is thus crucial in population dynamics. In this perspective our main result can be reformulated in the following way: *the system of distributional inequalities* (7) *has a solution* $(u_1,\dots,u_k) \in \mathcal{U}$ *with* k *nonnegative and nonzero components.*

2. Assumptions and main results

Description of the domain. We shall work in a class of smooth non-convex domains Ω_ε which generalizes the dumbbell form with many components as in [9]. Let $N \ge 2$ and for $k \in \mathbb{N}$, let

$$\Omega_0 = \Omega^1 \cup \Omega^2 \cup \cdots \cup \Omega^k,$$

where $\Omega^i \subset \mathbb{R}^N$ are open bounded smooth domains with mutually disjoint closures, i.e.

$$\overline{\Omega^i} \cap \overline{\Omega^j} = \emptyset \qquad \text{if } i \neq j. \tag{8}$$

For any $\varepsilon > 0$, let $R_\varepsilon \subset (\mathbb{R}^N \setminus \Omega_0)$ be a bounded measurable set satisfying the properties:

(i) $|R_\varepsilon| \to 0$ as $\varepsilon \to 0$;
(ii) $\Omega_0 \cup R_\varepsilon$ is open and connected;
(iii) $\partial(\Omega_0 \cup R_\varepsilon)$ is smooth.

Here we denote with $|B|$ the Lebesgue measure of any set $B \subset \mathbb{R}^N$. Finally we set

$$\Omega_\varepsilon := \Omega_0 \cup R_\varepsilon.$$

Assumptions on the nonlinearity. For every $i = 1,\dots,k$, let $F_i \in C^2(\mathbb{R})$ with $f_i = F_i'$ satisfying

(F1) $F_i(0) = 0$ and $f_i(0) = 0$;
(F2) there exists $A_i > 0$ such that $\mu_i := F_i(A_i) - \frac{A_i^2}{2} = \max_{t\in[0,+\infty)} \big(F_i(t) - \frac{t^2}{2}\big)$;
(F3) $f_i'(A_i) < 1$.

Noticeable examples of nonlinearities satisfying (F1)–(F3) are logistic type functions of the form $f_i(u) = \lambda u - |u|^{p-1}u$ with $p > 1$ and $\lambda > 1$.

Assumption (F2) implies that $A_i = f_i(A_i)$ and hence the constant function $u \equiv A_i$ is a solution to problem

$$\begin{cases} -\Delta u + u = f_i(u), & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases}$$

in any open smooth domain Ω; moreover $u \equiv A_i$ minimizes the internal energy

$$\int_\Omega \left(\frac{1}{2}|\nabla u_i(x)|^2 + \frac{1}{2}|u_i(x)|^2 - F_i(u_i(x)) \right) dx.$$

Let us denote $w_i = A_i \chi_{\Omega^i}$, $i = 1, \dots, k$, $W = (w_1, w_2, \dots, w_k) \in \left(H^1(\Omega_0)\right)^k$, and set

$$\mu = \sum_{i=1}^k \left\{ \frac{1}{2} \int_{\Omega_0} \left(|\nabla w_i|^2 + |w_i|^2\right) dx - \int_{\Omega_0} F_i(w_i)\, dx \right\} = -\sum_{i=1}^k \mu_i |\Omega^i|. \tag{9}$$

Assumption (F3) implies the following non-degeneracy property: for all i and $u \in H^1(\Omega^i)$

$$\int_{\Omega^i} (|\nabla u|^2 + |u|^2 - f_i'(w_i)u^2) dx \geq \nu \int_{\Omega^i} (|\nabla u|^2 + |u|^2) dx, \tag{10}$$

where $\nu := \min_{i=1,\dots,k}\{1, 1 - f_i'(A_i)\} > 0$.

We are now going to describe the main results of the present paper, starting from the following optimal partition problem.

Problem (P_ε). Find *nontrivial* local minimizers of the functional

$$J_{\Omega_\varepsilon} : \left(H^1(\Omega_\varepsilon)\right)^k \to (-\infty, +\infty],$$
$$J_{\Omega_\varepsilon}(U) = \sum_{i=1,\dots,k} \left\{ \frac{1}{2} \int_{\Omega_\varepsilon} \left(|\nabla u_i(x)|^2 + |u_i(x)|^2\right) dx - \int_{\Omega_\varepsilon} F_i(u_i(x)) dx \right\},$$

among k-tuples $U = (u_1, u_2, \dots, u_k)$ belonging to the class

$$\mathcal{U}_\varepsilon = \left\{ U = (u_1, u_2, \dots, u_k) \in \left(H^1(\Omega_\varepsilon)\right)^k : \ u_i \geq 0, \ u_i \cdot u_j = 0 \text{ if } i \neq j, \text{ a.e. in } \Omega_\varepsilon \right\}.$$

By *nontrivial* we mean that no component u_i of the solution U can be null, i.e. $u_i \not\equiv 0$ for all $i = 1, \dots, k$. As stated in the introduction, we shall prove that in any connected domain and for a wide class of F_i's including logistic-type nonlinearities, any *global minimizer* of the free energy (6) is indeed trivial.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^N$ be a connected open domain and $F_i \in C^2(\mathbb{R})$ satisfy $F_i(0) = 0$ and* (F2)*. Then the infimum*

$$\lambda := \inf_{U \in \mathcal{U}} J_\Omega(U)$$

is achieved by $U_0 = (u_1^0, \dots, u_k^0)$ *with* $u_{i_0}^0 \equiv A_{i_0}$ *and* $u_i^0 \equiv 0$ *for* $i \neq i_0$*, and*

$$\lambda = -\mu_{i_0}|\Omega|,$$

where $\mu_{i_0} = \max_{i\in\{1,\dots,k\}} \mu_i$*. Furthermore, any* k*-tuple achieving* λ *has all but one component identically null.*

In view of the above proposition, there is no hope to find nontrivial solutions to $(\boldsymbol{P_\varepsilon})$ by global minimization. On the contrary, by studying J_{Ω_ε} *near* W, we can find positive answer to the problem. To this aim let us denote by

$$B_\varepsilon^\delta(W) := \Big\{U \in \big(H^1(\Omega_\varepsilon)\big)^k : \|U - W\|_{(H^1(\Omega_0))^k} < \delta\Big\}$$

the set of k-tuples U whose restriction to Ω_0 is close within $\delta > 0$ to W, with respect to the H^1 norm $\|V\|^2_{(H^1(\Omega_0))^k} = \sum_{i=1}^k \|v_i\|^2_{H^1(\Omega_0)}$. Notice that, if $U = (u_1, \dots, u_k) \in B_\varepsilon^\delta(W)$, then each u_i satisfies $\int_{\Omega_i} |u_i - A_i|^2 \le \delta^2$. Hence, if

$$\delta^2 < A_i^2|\Omega_i|, \qquad i = 1, \dots, k, \tag{11}$$

then $u_i \not\equiv 0$. Henceforward, δ will be supposed to satisfy (11), thus ensuring that any $U \in B_\varepsilon^\delta(W)$ is nontrivial.

Theorem 2.2. *Assume that (F1)–(F3) hold and let*

$$\lambda_\varepsilon^\delta := \inf_{U\in\mathcal{U}_\varepsilon\cap B_\varepsilon^\delta(W)} J_{\Omega_\varepsilon}(U).$$

Then, there exists $\delta > 0$ *such that, for every* ε *sufficiently small,* $\lambda_\varepsilon^\delta$ *is achieved by a* k*-tuple* $U_\varepsilon = (u_1^\varepsilon, \dots, u_k^\varepsilon)$ *with* $0 \le u_i^\varepsilon \le A_i$ *a.e. in* Ω_ε *and* $u_i^\varepsilon \not\equiv 0$ *for all* $i = 1, \dots, k$*.*

The proof of Theorem 2.2 will be obtained through a careful analysis of the solutions to the original competitive system (1), as the parameter $\varkappa$ of the interspecific competition grows. Our main result reads as follows:

Theorem 2.3. *Assume that (F1)–(F3) hold. Then, there exists* $\delta > 0$ *such that, for every* ε *sufficiently small and* $\varkappa > 0$ *sufficiently large, system* (1) *coupled with* (5) *in* Ω_ε *admits a solution* $U^{\varepsilon,\varkappa} = (u_1^{\varepsilon,\varkappa}, \dots, u_k^{\varepsilon,\varkappa}) \in B_\varepsilon^\delta(W)$ *with the following properties:*

1. $u_i^{\varepsilon,\varkappa} \not\equiv 0$ *for all* $i = 1, \dots, k$*.*
2. $0 \le u_i^{\varepsilon,\varkappa} \le A_i$ *a.e. in* Ω_ε *for all* $i = 1, \dots, k$*.*
3. *There exists* $V^\varepsilon = (v_1^\varepsilon, ..., v_k^\varepsilon) \in \mathcal{U}_\varepsilon \cap B_\varepsilon^\delta(W)$ *such that* $v_i^\varepsilon \not\equiv 0$ *for every* i *and, up to subsequences,* $U^{\varepsilon,\varkappa} \to V^\varepsilon$ *strongly in* $(H^1(\Omega_\varepsilon))^k$ *as* $\varkappa \to +\infty$*. Furthermore,* V^ε *is a local minimizer of* J_{Ω_ε}*, namely* $J_{\Omega_\varepsilon}(V^\varepsilon) = \lambda_\varepsilon^\delta$*.*

The proof of our results relies on the minimization on the whole $B_\varepsilon^\delta(W)$ of an auxiliary functional obtained by penalizing the internal energy J_{Ω_ε} with a positive competition term.

More precisely we shall consider a suitable modification of the functional

$$\sum_{i=1}^{k}\left\{\frac{1}{2}\int_{\Omega_\varepsilon}\left(|\nabla u_i(x)|^2+|u_i(x)|^2\right)dx-\int_{\Omega_\varepsilon}F_i(u_i(x))\,dx\right\}+\varkappa\sum_{\substack{i,j=1\\ i\neq j}}^{k}\int_{\Omega_\varepsilon}u_i(x)^2u_j(x)^2\,dx$$

defined on $(H^1(\Omega_\varepsilon))^k$, see $I_{\varepsilon,\varkappa}$ in (12) below. Due to the variational character of the competition term in (1), by standard Critical Point Theory, any local minimizer of the above function is a (weak) solution to the original system. Section 3 is devoted to the search for a local minimizer of $I_{\varepsilon,\varkappa}$ in $B^\delta_\varepsilon(W)$ and requires the main technical effort of the paper. By developing a domain perturbation argument based on the nondegeneracy condition (10), we shall succeed in proving the existence of a minimizer in small perturbations of the domain Ω_0, for large values of the competition parameter $\varkappa$. In this way, we directly obtain the existence of a positive solution to the competitive system, at any fixed $\varkappa$, see Section 4. In the subsequent Section 5 we perform the asymptotic analysis of these solutions as the competition parameter $\varkappa\to\infty$, showing that the steady states segregate in a nontrivial limit configuration V^ε. The comparison between the minimal energy levels of $I_{\varepsilon,\varkappa}$ and J_{Ω_e} will allow proving that V^ε indeed solves problem $(\boldsymbol{P_\varepsilon})$ on $B^\delta_\varepsilon(W)$. This concludes the proof of Theorem 2.3 and, in turn, that of Theorem 2.2. In the last part of Section 5, we show that any solution to the optimal partition problem $(\boldsymbol{P_\varepsilon})$ satisfies some extremality conditions in the form of inequalities (7). Finally, in the last section we derive some consequences of this fact, and outline further developments of the subject.

3. A variational problem

Aim of this section is to study the minimization of a suitable functional on $(H^1(\Omega_\varepsilon))^k$, which will reveal to be strongly related both to problem $(\boldsymbol{P_\varepsilon})$ and to the original competitive system. The functional is defined as follows:

$$I_{\varepsilon,\varkappa}(U)=\sum_{i=1}^{k}\left\{\frac{1}{2}\int_{\Omega_\varepsilon}\left(|\nabla u_i(x)|^2+|u_i(x)|^2\right)dx-\int_{\Omega_\varepsilon}\widetilde{F}_i(u_i(x))\,dx\right\}+\varkappa\sum_{\substack{i,j=1\\ i\neq j}}^{k}\int_{\Omega_\varepsilon}G_i(u_i(x))G_j(u_j(x))\,dx \tag{12}$$

where

$$\widetilde{F}_i(t)=\begin{cases}0, & \text{if } t\le 0,\\ F_i(t), & \text{if } 0\le t\le A_i,\\ A_it+F_i(A_i)-A_i^2, & \text{if } t\ge A_i,\end{cases}$$

and

$$G_i(t)=\begin{cases}t^2, & \text{if } |t|\le A_i,\\ 2A_i|t|-A_i^2, & \text{if } |t|>A_i.\end{cases}$$

Notice that $I_{\varepsilon,\varkappa} \in C^1\big((H^1(\Omega_\varepsilon))^k, \mathbb{R}\big)$. Aim of this section is to prove

Theorem 3.1. *Assume that (F1)–(F3) hold and let*

$$c_{\varepsilon,\varkappa} := \inf_{U \in \mathcal{U}_\varepsilon \cap B_\varepsilon^\delta(W)} I_{\varepsilon,\varkappa}(U).$$

Then, there exists $\delta > 0$ such that, for every $\varepsilon > 0$ sufficiently small and $\varkappa > 0$ sufficiently large, $c_{\varepsilon,\varkappa}$ is achieved by a k-tuple $U^{\varepsilon,\varkappa} = (u_1^{\varepsilon,\varkappa}, \dots, u_k^{\varepsilon,\varkappa})$ with $0 \le u_i^{\varepsilon,\varkappa} \le A_i$ a.e. in Ω_ε and $u_i^{\varepsilon,\varkappa} \not\equiv 0$ for all $i = 1, \dots, k$.

The first step in this direction consists in proving that the minimum is achieved on the closure of $B_\varepsilon^\delta(W)$, namely the set

$$\overline{B_\varepsilon^\delta(W)} := \Big\{ U \in \big(H^1(\Omega_\varepsilon)\big)^k : \|U - W\|_{(H^1(\Omega_0))^k} \le \delta \Big\}.$$

Lemma 3.2. *For every δ satisfying (11), $\varepsilon \in (0,1)$ and $\varkappa > 0$, the infimum*

$$\Lambda_{\varepsilon,\varkappa} = \inf_{U \in \overline{B_\varepsilon^\delta(W)}} I_{\varepsilon,\varkappa}(U)$$

is achieved by a k-tuple $U^{\varepsilon,\varkappa} = (u_1^{\varepsilon,\varkappa}, \dots, u_k^{\varepsilon,\varkappa})$ where $u_i^{\varepsilon,\varkappa} \not\equiv 0$ and

$$0 \le u_i^{\varepsilon,\varkappa}(x) \le A_i \quad \text{for a.e. } x \in \Omega_\varepsilon. \tag{13}$$

Proof. We first observe that $\frac{1}{2}t^2 - \widetilde{F}_i(t) \ge \frac{1}{2}A_i^2 - F_i(A_i)$ for all $t \in \mathbb{R}$, hence, being the coupling term nonnegative, for all $U = (u_1, \dots, u_k) \in \big(H^1(\Omega_\varepsilon)\big)^k$

$$I_{\varepsilon,\varkappa}(U) \ge \sum_{i=1}^k \int_{\Omega_\varepsilon} \bigg(\frac{1}{2}|u_i|^2 - \widetilde{F}_i(u_i)\bigg) dx \ge \sum_{i=1}^k \bigg(\frac{A_i^2}{2} - F_i(A_i)\bigg) |\Omega_\varepsilon|,$$

and hence $\Lambda_{\varepsilon,\varkappa} > -\infty$. Let $\big\{U_n = (u_1^n, \dots, u_k^n)\big\}_{n\in\mathbb{N}}$ be a minimizing sequence, i.e. $U_n \in \overline{B_\varepsilon^\delta(W)}$ and $\lim_{n\to+\infty} I_{\varepsilon,\varkappa}(U_n) = \Lambda_{\varepsilon,\varkappa}$. We notice that, by definition of $\widetilde{F}_i$ and the fact that $w_i \ge 0$ a.e., we can choose U_n such that $u_i^n \ge 0$ a.e. in Ω_ε for all $i = 1, \dots, k$ (otherwise we take $((u_1^n)^+, \dots, (u_k^n)^+)$ with $(u_i^n)^+ := \max\{u_i^n, 0\}$ as a new minimizing sequence). Letting $V_n = (v_1^n, \dots, v_k^n)$ with $v_i^n = \min\{u_i^n, A_i\}$, it is easy to verify that $V_n \in \overline{B_\varepsilon^\delta(W)}$ and $I_{\varepsilon,\varkappa}(V_n) \le I_{\varepsilon,\varkappa}(U_n)$. Then also $\big\{V_n\big\}_{n\in\mathbb{N}}$ is a minimizing sequence.

Since $\big\{V_n\big\}_{n\in\mathbb{N}}$ is a minimizing sequence and it is uniformly bounded , it is easy to realize that $\big\{V_n\big\}_{n\in\mathbb{N}}$ is bounded in $\big(H^1(\Omega_\varepsilon)\big)^k$, hence there exists a subsequence, still denoted as $\big\{V_n\big\}_{n\in\mathbb{N}}$, which converges to some $V = (v_1, \dots, v_k) \in \big(H^1(\Omega_\varepsilon)\big)^k$ weakly in $\big(H^1(\Omega_\varepsilon)\big)^k$, strongly in $\big(L^2(\Omega_\varepsilon)\big)^k$ and a.e. in Ω_ε. A.e. convergence implies that $0 \le v_i \le A_i$ a.e. in Ω_ε, while weakly lower semi-continuity implies that $V \in \overline{B_\varepsilon^\delta(W)}$. From $0 \le v_i \le A_i$ and the Dominated Convergence Theorem, it follows that

$$\lim_{n\to+\infty}\int_{\Omega_\varepsilon}\widetilde{F}_i(v_i^n(x))\,dx=\int_{\Omega_\varepsilon}\widetilde{F}_i(v_i(x))\,dx,$$
$$\lim_{n\to+\infty}\int_{\Omega_\varepsilon}G_i(v_i^n(x))G_j(v_j^n(x))\,dx=\int_{\Omega_\varepsilon}G_i(v_i(x))G_j(v_j(x))\,dx,$$

for every $i,j=1,\dots,k$, which, together with lower semi-continuity, yields

$$\Lambda_{\varepsilon,\varkappa}\le I_{\varepsilon,\varkappa}(V)\le\liminf_{n\to+\infty}I_{\varepsilon,\varkappa}(V_n)=\lim_{n\to+\infty}I_{\varepsilon,\varkappa}(V_n)=\Lambda_{\varepsilon,\varkappa},$$

thus proving that V attains $\Lambda_{\varepsilon,\varkappa}$.

Finally, if $v_i\equiv 0$ in Ω_i then $\|v_i-A_i\|^2_{H^1(\Omega^i)}=\int_{\Omega^i}A_i^2dx\le\delta^2$, in contradiction with the choice of δ as in (11). □

A major effort is now needed to show that the minimum provided by Lemma 3.2 indeed belongs to the open set $B^\varepsilon_\delta(W)$. The crucial ingredient in this direction consists in providing suitable estimates of the minimal level $\Lambda_{\varepsilon,\varkappa}$, which require the following technical lemma.

Lemma 3.3. *For every $\eta>0$ there exists $\delta_\eta>0$ such that if $U=(u_1,\dots,u_k)\in\overline{B^{\delta_\eta}_\varepsilon(W)}$ and $|u_i(x)|\le A_i$ for a.e. $x\in\Omega_0$ and for all $i=1,\dots,k$, then*

$$\sum_{i=1}^k\int_{\Omega_0}\left[F_i(u_i)-F_i(w_i)-f_i(w_i)(u_i-w_i)-\frac12 f_i'(w_i)(u_i-w_i)^2\right]dx\le\eta\|U-W\|^2_{(H^1(\Omega_0))^k}. \tag{14}$$

Proof. We have

$$\begin{aligned}
&\int_{\Omega_0}\left[F_i(u_i)-F_i(w_i)-F_i'(w_i)(u_i-w_i)-\frac12F_i''(w_i)(u_i-w_i)^2\right]dx\\
&=\int_{\Omega_0}\left[\int_0^1\left[\left(\frac{d}{dt}F_i(t\,u_i+(1-t)w_i)\right)-F_i'(w_i)(u_i-w_i)-t\,F_i''(w_i)(u_i-w_i)^2\right]dt\right]dx\\
&=\int_{\Omega_0}\left[\int_0^1\left[F_i'(t\,u_i+(1-t)w_i)-F_i'(w_i)-t\,F_i''(w_i)(u_i-w_i)\right](u_i-w_i)\,dt\right]dx\\
&=\int_{\Omega_0}\left[\int_0^1\left(\int_0^1\left(\frac{d}{ds}F_i'\big(s(t\,u_i+(1-t)w_i)+(1-s)w_i\big)\right)ds\right.\right.\\
&\qquad\qquad\left.\left.-\,t\,F_i''(w_i)(u_i-w_i)\right)(u_i-w_i)\,dt\right]dx.
\end{aligned}$$

Hence, by Hölder's inequality,

$$\begin{aligned}
&\int_{\Omega_0}\Big[F_i(u_i)-F_i(w_i)-F_i'(w_i)(u_i-w_i)-\frac{1}{2}F_i''(w_i)(u_i-w_i)^2\Big]\,dx\\
&\quad\le\int_{\Omega_0}\bigg[\int_0^1\bigg(\int_0^1\big(F_i''\big(st(u_i-w_i)+w_i\big)-F_i''(w_i)\big)t(u_i-w_i)^2ds\bigg)\,dt\bigg]\,dx\\
&\quad\le\|u_i-w_i\|^2_{L^p(\Omega_0)}\iint_{(0,1)\times(0,1)}t\big\|F_i''\big(st(u_i-w_i)+w_i\big)-F_i''(w_i)\big\|_{L^{\frac{p}{p-2}}(\Omega_0)}\,ds\,dt,
\end{aligned}$$

where $p=2^*$ for $N\ge 3$ and $p\in(2,+\infty)$ for $N=2$. The conclusion follows now from Sobolev's embedding and the continuity of the operator

$$\begin{aligned}
F_i'':\big\{v\in H^1(\Omega_0):\ |v(x)|\le 3A_i\big\}&\to L^{\frac{p}{p-2}}(\Omega_0),\\
v&\mapsto F_i''(v),
\end{aligned}$$

which can be easily proved using the Dominated Convergence Theorem. □

Remark 3.4. *According to Lemma 3.3, besides (11) in the sequel we assume*

$$0<\delta\le\delta_0$$

with δ_0 small enough in such a way that inequality (14) *with $\eta=\min\{\frac{\nu}{4},\frac{1}{8}\}$ holds for all functions $U=(u_1,\dots,u_k)\in\overline{B_\varepsilon^{\delta_0}(W)}$ satisfying $|u_i(x)|\le A_i$ a.e. in Ω_0. We also require that $\delta_0\le A_i^2/4$ and finally that condition* (17) *in Lemma 3.6 is satisfied.*

By exploiting the separation of the Ω^i's as in (8), for every $i=1,\dots,k$, we can construct test functions $\varphi^i\in H^1(\mathbb{R}^N)$ satisfying

$$0\le\varphi^i(x)\le A_i\quad\text{a.e. in }\mathbb{R}^N,\tag{15}$$

$\varphi_i(x)=0$ for all $x\in\Omega_0\setminus\Omega_i$, $\varphi_i(x)=A_i$ if $x\in\Omega_i$, and $\varphi_i\cdot\varphi_j=0$ a.e. in $\mathbb{R}^N$ if $i\ne j$. This allows us to provide an estimate from above of the value $\Lambda_{\varepsilon,\varkappa}$ in terms of the total free-energy of W.

Lemma 3.5. *For every $\varepsilon\in(0,1)$, there exists τ_ε such that $\tau_\varepsilon\to 0$ as $\varepsilon\to 0$ and*

$$\Lambda_{\varepsilon,\varkappa}\le\mu+\tau_\varepsilon$$

for all $\varkappa>0$, with μ given by (9)*.*

Proof. Let $\varphi_\varepsilon^i\in H^1(\Omega_\varepsilon)$ be the restriction of φ_i to Ω_ε. Notice that $(\varphi_\varepsilon^1,\varphi_\varepsilon^2,\dots,\varphi_\varepsilon^k)\in\overline{B_\varepsilon^\delta(W)}$ and that $\varphi_\varepsilon^i\cdot\varphi_\varepsilon^j\equiv 0$ if $i\ne j$. Hence we have

$$\begin{aligned}
\Lambda_{\varepsilon,\varkappa}&\le I_{\varepsilon,\varkappa}(\varphi_\varepsilon^1,\varphi_\varepsilon^2,\dots,\varphi_\varepsilon^k)\\
&=\mu+\sum_{i=1}^k\bigg\{\frac{1}{2}\int_{R_\varepsilon}\big(|\nabla\varphi_\varepsilon^i(x)|^2+|\varphi_\varepsilon^i(x)|^2\big)\,dx-\int_{R_\varepsilon}F_i(\varphi_\varepsilon^i)\,dx\bigg\}\\
&=\mu+\tau_\varepsilon,
\end{aligned}$$

where

$$\tau_\varepsilon = \sum_{i=1}^{k} \left\{ \frac{1}{2} \int_{R_\varepsilon} \left(|\nabla \varphi^i(x)|^2 + |\varphi^i(x)|^2 \right) dx - \int_{R_\varepsilon} F_i(\varphi^i)\, dx \right\}.$$

Since $|R_\varepsilon| \to 0$ as $\varepsilon \to 0$, then $\tau_\varepsilon \to 0$, proving the stated estimate. □

Lemma 3.6. *For every $\varepsilon \in (0,1)$, there exists σ_ε such that $\sigma_\varepsilon \to 0$ as $\varepsilon \to 0$ and*

$$\| U^{\varepsilon,\varkappa} - W \|^2_{(H^1(\Omega_0))^k} \le \sigma_\varepsilon$$

for every $\varkappa > \max\limits_{i \neq j} \frac{2 f_i'(0)}{A_j^2}$.

Proof. From (13), we can write $\Lambda_{\varepsilon,\varkappa} = I^1_{\varepsilon,\varkappa} + I^2_{\varepsilon,\varkappa}$ where

$$I^1_{\varepsilon,\varkappa} = \sum_{i=1}^{k} \left\{ \frac{1}{2} \int_{\Omega_0} \left(|\nabla u_i^{\varepsilon,\varkappa}|^2 + |u_i^{\varepsilon,\varkappa}|^2 \right) dx - \int_{\Omega_0} F_i(u_i^{\varepsilon,\varkappa})\, dx + \varkappa \sum_{j \neq i} \int_{\Omega_0} (u_i^{\varepsilon,\varkappa})^2 (u_j^{\varepsilon,\varkappa})^2\, dx \right\}$$

$$I^2_{\varepsilon,\varkappa} = \sum_{i=1}^{k} \left\{ \frac{1}{2} \int_{R_\varepsilon} \left(|\nabla u_i^{\varepsilon,\varkappa}|^2 + |u_i^{\varepsilon,\varkappa}|^2 \right) dx - \int_{R_\varepsilon} F_i(u_i^{\varepsilon,\varkappa})\, dx + \varkappa \sum_{j \neq i} \int_{R_\varepsilon} (u_i^{\varepsilon,\varkappa})^2 (u_j^{\varepsilon,\varkappa})^2\, dx \right\}.$$

Since by assumption $-\Delta w_i + w_i = f_i(w_i)$ in Ω_0, we can write each term in $I^1_{\varepsilon,\varkappa}$ as follows

$$\begin{aligned}
&\frac{1}{2} \int_{\Omega_0} \left(|\nabla u_i^{\varepsilon,\varkappa}|^2 + |u_i^{\varepsilon,\varkappa}|^2 \right) dx - \int_{\Omega_0} F_i(u_i^{\varepsilon,\varkappa})\, dx + \varkappa \sum_{j \neq i} \int_{\Omega_0} (u_i^{\varepsilon,\varkappa})^2 (u_j^{\varepsilon,\varkappa})^2\, dx \\
&\quad = \frac{1}{2} \int_{\Omega_0} \left(|\nabla w_i|^2 + |w_i|^2 \right) dx - \int_{\Omega_0} F_i(w_i)\, dx \\
&\qquad + \frac{1}{2} \int_{\Omega_0} \left(|\nabla (u_i^{\varepsilon,\varkappa} - w_i)|^2 + |(u_i^{\varepsilon,\varkappa} - w_i)|^2 \right) dx - \int_{\Omega_0} \left(F_i(u_i^{\varepsilon,\varkappa}) - F_i(w_i) \right) dx \\
&\qquad + \int_{\Omega_0} \left(\nabla w_i \cdot \nabla (u_i^{\varepsilon,\varkappa} - w_i) + w_i (u_i^{\varepsilon,\varkappa} - w_i) \right) dx + \varkappa \sum_{j \neq i} \int_{\Omega_0} (u_i^{\varepsilon,\varkappa})^2 (u_j^{\varepsilon,\varkappa})^2\, dx \\
&\quad = -\mu_i |\Omega^i| + \alpha^1_{\varepsilon,\varkappa,i} + \alpha^2_{\varepsilon,\varkappa,i} \\
&\qquad - \int_{\Omega_0} \left(F_i(u_i^{\varepsilon,\varkappa}) - F_i(w_i) - f_i(w_i)(u_i^{\varepsilon,\varkappa} - w_i) - \frac{1}{2} f_i'(w_i)(u_i^{\varepsilon,\varkappa} - w_i)^2 \right) dx.
\end{aligned}$$

where

$$\alpha^1_{\varepsilon,\varkappa,i} = \frac{1}{2} \| u_i^{\varepsilon,\varkappa} - w_i \|^2_{H^1(\Omega^i)} - \frac{1}{2} \int_{\Omega^i} f_i'(A_i)(u_i^{\varepsilon,\varkappa} - w_i)^2 + \varkappa \sum_{j \neq i} \int_{\Omega^i} (u_i^{\varepsilon,\varkappa})^2 (u_j^{\varepsilon,\varkappa})^2\, dx$$

and

$$\alpha^2_{\varepsilon,\varkappa,i} = \frac{1}{2} \sum_{j \neq i} \int_{\Omega^j} \left(|\nabla u_i^{\varepsilon,\varkappa}|^2 + |u_i^{\varepsilon,\varkappa}|^2 - \left[f_i'(0) - 2\varkappa \sum_{h \neq i} (u_h^{\varepsilon,\varkappa})^2 \right] |u_i^{\varepsilon,\varkappa}|^2 \right) dx.$$

From (10) it follows that

$$\alpha^1_{\varepsilon,\varkappa,i} \geq \frac{\nu}{2}\|u_i^{\varepsilon,\varkappa} - w_i\|^2_{H^1(\Omega^i)}. \tag{16}$$

On the other hand, from Hölder's and Sobolev's inequalities it follows that

$$\alpha^2_{\varepsilon,\varkappa,i} \geq \frac{1}{2}\sum_{j\neq i}\int_{\Omega^j}\bigg(\bigg(1 - \bigg\|\Big[f_i'(0) - 2\varkappa\sum_{h\neq i}(u_h^{\varepsilon,\varkappa})^2\Big]^+\bigg\|_{L^{\frac{p}{p-2}}(\Omega^j)} S^{-1}_{p,j}\bigg)|\nabla u_i^{\varepsilon,\varkappa}|^2 + |u_i^{\varepsilon,\varkappa}|^2\bigg)\,dx.$$

where $p = 2^*$ for $N \geq 3$ and $p \in (2, +\infty)$ for $N = 2$, and $S_{p,j}$ is the best constant in the Sobolev embedding $H^1(\Omega^j) \hookrightarrow L^p(\Omega^j)$. Let us denote

$$A^\delta_{\varkappa,j} = \{x \in \Omega^j : |u_j^{\varepsilon,\varkappa} - A_j|^2 > \delta\}.$$

Hence

$$\delta^2 \geq \int_{\Omega^j}|u_j^{\varepsilon,\varkappa} - A_j|^2\,dx \geq \delta|A^\delta_{\varkappa,j}|$$

and then $|A^\delta_{\varkappa,j}| < \delta$. In particular, if δ is such that

$$\delta^{\frac{p-2}{p}}|(f_i'(0))^+| < \frac{S_{p,j}}{2}, \tag{17}$$

there holds

$$\bigg\|\Big[f_i'(0) - 2\varkappa\sum_{h\neq i}(u_h^{\varepsilon,\varkappa})^2\Big]^+\bigg\|_{L^{\frac{p}{p-2}}(A^\delta_{\varkappa,j})} < \frac{S_{p,j}}{2}. \tag{18}$$

In $\Omega^j \setminus A^\delta_{\varkappa,j}$, there holds $u_j^{\varepsilon,\varkappa} > A_j - \sqrt{\delta} > \frac{A_j}{2}$ for δ small as in Remark 3.4. Then, if $\varkappa > 2f_i'(0)/A_j^2$,

$$f_i'(0) - 2\varkappa\sum_{h\neq i}(u_h^{\varepsilon,\varkappa})^2 < 0 \quad \text{in } \Omega^j \setminus A^\delta_{\varkappa,j}. \tag{19}$$

Collecting (18) and (19), we deduce that, for $\varkappa > \frac{2f_i'(0)}{A_j^2}$,

$$\bigg\|\Big[f_i'(0) - 2\varkappa\sum_{h\neq i}(u_h^{\varepsilon,\varkappa})^2\Big]^+\bigg\|_{L^{\frac{p}{p-2}}(\Omega^j)} S^{-1}_{p,j} < \frac{1}{2},$$

and therefore

$$\alpha^2_{\varepsilon,\varkappa,i} \geq \frac{1}{4}\sum_{j\neq i}\|u_i^{\varepsilon,\varkappa} - w_i\|^2_{H^1(\Omega^j)}. \tag{20}$$

From (16) and (20), we obtain that

$$\alpha^1_{\varepsilon,\varkappa,i} + \alpha^2_{\varepsilon,\varkappa,i} \geq \min\left\{\frac{\nu}{2}, \frac{1}{4}\right\}\|u_i^{\varepsilon,\varkappa} - w_i\|^2_{H^1(\Omega_0)}.$$

By Lemma 3.3 and Remark 3.4 we have that

$$\sum_{i=1}^{k}\int_{\Omega_0}\left[F_i(u_i^{\varepsilon,\varkappa})-F_i(w_i)-f_i(w_i)(u_i^{\varepsilon,\varkappa}-w_i)-\frac{1}{2}f_i'(w_i)(u_i^{\varepsilon,\varkappa}-w_i)^2\right]dx$$
$$\le \min\left\{\frac{\nu}{4},\frac{1}{8}\right\}\|U^{\varepsilon,\varkappa}-W\|^2_{(H^1(\Omega_0))^k}.$$

Hence

$$I^1_{\varepsilon,\varkappa}\ge \mu+\min\left\{\frac{\nu}{4},\frac{1}{8}\right\}\|U_\varepsilon-W\|^2_{(H^1(\Omega_0))^k}. \tag{21}$$

On the other hand, $I^2_{\varepsilon,\varkappa}$ can be promptly estimated by

$$I^2_{\varepsilon,\varkappa}\ge -|R_\varepsilon|\sum_{i=1}^{k}\mu_i \tag{22}$$

with μ_i as in (F2). Combining inequalities (21) and (22), it follows that

$$\Lambda_{\varepsilon,\varkappa}\ge \mu+\eta\|U_\varepsilon-W\|^2_{(H^1(\Omega_0))^k}-|R_\varepsilon|\sum_{i=1}^{k}\mu_i, \tag{23}$$

where $\eta=\min\{\frac{\nu}{4},\frac{1}{8}\}>0$. From Lemma 3.5 and (23), we infer that $\|U_\varepsilon-W\|^2_{(H^1(\Omega_0))^k}\le\sigma_\varepsilon$ with $\sigma_\varepsilon=\frac{1}{\eta}(\tau_\varepsilon+|R_\varepsilon|\sum_{i=1}^{k}\mu_i)$, concluding the proof. □

Proof of Theorem 3.1. In order to conclude the proof of the theorem, it is sufficient to consider $U^{\varepsilon,\varkappa}$ provided by Lemma 3.2. If $\varkappa$ is large enough, we can apply Lemma 3.6 to infer that $U^{\varepsilon,\varkappa}\in B^\delta_\varepsilon(W)$, provided that ε is sufficiently small. Hence $U^{\varepsilon,\varkappa}$ attains $c_{\varepsilon,\varkappa}=\Lambda_{\varepsilon,\varkappa}$ and it is a local minimizer of $I_{\varepsilon,\varkappa}$ on the open set $B^\delta_\varepsilon(W)$, with all the required properties. □

4. Competitive systems

In this section we prove the existence of solutions to the competitive system

$$\begin{cases}-\Delta u_i+u_i=f_i(u_i)-2\varkappa u_i\sum_{j\ne i}u_j^2, & \text{in } \Omega_\varepsilon,\\ \dfrac{\partial u_i}{\partial\nu}=0, & \text{on } \partial\Omega_\varepsilon,\end{cases} \tag{24}$$

for $i=1,\dots,k$.

Theorem 4.1. *There exists $\delta>0$ such that for $\varepsilon>0$ sufficiently small and $\varkappa>0$ sufficiently large, system (24) admits a solution $U^{\varepsilon,\varkappa}=(u_1^{\varepsilon,\varkappa},\dots,u_k^{\varepsilon,\varkappa})\in B^\delta_\varepsilon(W)$ such that, for all $i=1,\dots,k$, $u_i^{\varepsilon,\varkappa}\not\equiv 0$ and*

$$0\le u_i^{\varepsilon,\varkappa}\le A_i \quad \text{a.e. in } \Omega_\varepsilon. \tag{25}$$

Proof. By standard Critical Point Theory, see e.g. [1], the critical points of $I_{\varepsilon,\varkappa}$ on $(H^1(\Omega_\varepsilon))^k$ give rise to weak (and by regularity classical) solutions to

$$\begin{cases} -\Delta u_i + u_i = \widetilde{f}_i(u_i) - \varkappa g_i(u_i) \sum_{j\neq i} G_j(u_j), & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial u_i}{\partial \nu} = 0, & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{26}$$

where

$$\widetilde{f}_i(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ f_i(t), & \text{if } 0 \leq t \leq A_i, \\ A_i, & \text{if } t \geq A_i, \end{cases}$$

and

$$g_i(t) = \begin{cases} 2t, & \text{if } |t| \leq A_i, \\ 2A_i \operatorname{sgn}(t), & \text{if } |t| > A_i. \end{cases}$$

Notice that a solution to (26) satisfying (25) is also a solution of (24). Now the proof of the theorem immediately follows by considering $U^{\varepsilon,\varkappa} \in B^\delta_\varepsilon(W)$ as in Theorem 3.1; since it is a local minimizer of $I_{\varepsilon,\varkappa}$, it is a free critical point of $I_{\varepsilon,\varkappa}$ and hence solves (26). By the validity of (13) we finally deduce that $U^{\varepsilon,\varkappa}$ is actually a solution to (24), thus completing the proof.

□

5. The optimal partition problem

In this section we deal with problem $(\boldsymbol{P_\varepsilon})$, namely we look for local minimizers of the free energy on segregated states. The localization of the problem is essentially motivated by the fact that any global minimizer of the free energy in a connected domain is trivial, as stated in Proposition 2.1, the proof of which is given below.

Proof of Proposition 2.1. By a direct computation, for any $U = (u_1, \dots, u_k) \in \mathcal{U}$

$$\begin{aligned} J_\Omega(U) &\geq \sum_{i=1}^k \int_\Omega \left[\frac{|u_i|^2}{2} - F_i(u_i)\right] dx \geq \sum_{i=1}^k \left(\frac{|A_i|^2}{2} - F_i(A_i)\right) |\{x \in \Omega : u_i(x) > 0\}| \\ &\geq -\mu_{i_0} \sum_{i=1}^k |\{x \in \Omega : u_i(x) > 0\}| \geq -\mu_{i_0}|\Omega| = J_\Omega(U_0). \end{aligned} \tag{27}$$

On the other hand for any nontrivial k-uple $U = (U_1, \dots, U_k) \in \mathcal{U}$ there exists j such that $|\nabla u_j| \not\equiv 0$ and hence the inequality in the first line of (27) is strict. Therefore $J_\Omega(U) > J_\Omega(U_0)$ and U cannot be a global minimizer. □

A nontrivial solution to the local minimization problem will be provided by a limit configuration of solutions to the competitive system. To this aim we shall perform the asymptotic analysis of the solutions to (24) found in Theorem 4.1 as $\varkappa \to +\infty$.

PROOF OF THEOREM 2.2 AND 2.3. Let $U^{\varepsilon,\varkappa} = (u_1^{\varepsilon,\varkappa}, ..., u_k^{\varepsilon,\varkappa})$ be the solution of system (24) obtained in Theorem 4.1 by minimizing $I_{\varepsilon,\varkappa}$ on $B_\delta^\varepsilon(W)$, hence $I_{\varepsilon,\varkappa}(U^{\varepsilon,\varkappa}) = c_{\varepsilon,\varkappa}$ as in Theorem 3.1. In particular

$$c_{\varepsilon,\varkappa} \geq \frac{1}{2}\|U^{\varepsilon,\varkappa}\|^2_{(H^1(\Omega_\varepsilon))^k} - |\Omega_\varepsilon| \sum_i \max_{t\in[0,A_i]} |F_i(t)|. \tag{28}$$

For every $U \in \mathcal{U}_\varepsilon \cap B_\varepsilon^\delta(W)$, define $\widetilde{U}$ by setting $\tilde{u}_i(x) = \min\{u_i(x), A_i\}$. Then the following inequalities hold

$$J_{\Omega_\varepsilon}(U) \geq J_{\Omega_\varepsilon}(\widetilde{U}) = I_{\varepsilon,\varkappa}(U) \geq c_{\varepsilon,\varkappa},$$

implying

$$\lambda_\varepsilon^\delta \geq c_{\varepsilon,\varkappa}. \tag{29}$$

From (28) and (29) we obtain that

$$\|U^{\varepsilon,\varkappa}\|^2_{(H^1(\Omega_\varepsilon))^k} \leq 2c_{\varepsilon,\varkappa} + 2|\Omega_\varepsilon| \sum_i \max_{t\in[0,A_i]} |F_i(t)| \leq 2\lambda_\varepsilon^\delta + 2|\Omega_\varepsilon| \sum_i \max_{t\in[0,A_i]} |F_i(t)|.$$

Hence $u_i^{\varepsilon,\varkappa}$ is bounded in $H^1(\Omega_\varepsilon)$ uniformly with respect to $\varkappa$, then there exists a weak limit v_i^ε such that, up to subsequences, $u_i^{\varepsilon,\varkappa} \rightharpoonup v_i^\varepsilon$ in $H^1(\Omega_\varepsilon)$ as $\varkappa \to +\infty$. Also, by lower semicontinuity of the norm, we learn that $V^\varepsilon \in B_\delta^\varepsilon(W)$, hence, by (11), $v_i^\varepsilon \not\equiv 0$ for all i. Let us now multiply the equation of $u_i^{\varepsilon,\varkappa}$ times $u_i^{\varepsilon,\varkappa}$ on account of the boundary conditions: then

$$\varkappa \int_\Omega (u_i^{\varepsilon,\varkappa})^2 \sum_{j\neq i} (u_j^{\varepsilon,\varkappa})^2 \quad \text{is bounded uniformly in } \varkappa,$$

hence

$$\int_\Omega (u_i^{\varepsilon,\varkappa})^2 \sum_{j\neq i} (u_j^{\varepsilon,\varkappa})^2 \to 0, \qquad \text{as } \varkappa \to \infty.$$

By the pointwise convergence $u_i^{\varepsilon,\varkappa}(x) \to v_i^\varepsilon(x)$ a.e. $x \in \Omega_\varepsilon$, we infer that $v_i^\varepsilon(x) \geq 0$ and $v_i^\varepsilon(x) \cdot v_j^\varepsilon(x) = 0$ for almost every x, hence $V^\varepsilon \in \mathcal{U}_\varepsilon$.

Also, by the positivity of the interaction term, we know that $c_{\varepsilon,\varkappa} \leq c_{\varepsilon,\varkappa'}$ when $\varkappa \leq \varkappa'$: hence the sequence of critical levels $c_{\varepsilon,\varkappa}$ converges to some $\lambda \leq \lambda_\varepsilon^\delta$ as $\varkappa \to +\infty$. Since by the Dominated Convergence Theorem (recall that $0 \leq u_i^{\varepsilon,\varkappa} \leq A_i$)

$$\int_{\Omega_\varepsilon} F_i(u_i^{\varepsilon,\varkappa})\, dx = \int_{\Omega_\varepsilon} \widetilde{F}_i(u_i^{\varepsilon,\varkappa})\, dx \to \int_{\Omega_\varepsilon} F_i(v_i^\varepsilon)\, dx, \qquad \varkappa \to \infty,$$

the following chain of inequalities holds:

$$
\begin{aligned}
\lambda_\varepsilon^\delta &\geq \lim_{\varkappa\to\infty} c_{\varepsilon,\varkappa} = \lim_{\varkappa\to\infty} I_{\varepsilon,\varkappa}(U^{\varepsilon,\varkappa}) \\
&= \limsup_{\varkappa\to\infty} \left[\sum_{i=1}^k \left\{ \frac{1}{2}\|u_i^{\varepsilon,\varkappa}\|^2_{H^1(\Omega_\varepsilon)} - \int_{\Omega_\varepsilon} \widetilde{F}_i(u_i^{\varepsilon,\varkappa})\,dx \right\} + \varkappa \sum_{\substack{i,j=1\\ i\neq j}}^k \int_\Omega (u_i^{\varepsilon,\varkappa})^2 (u_j^{\varepsilon,\varkappa})^2 \right] \\
&\geq \limsup_{\varkappa\to\infty} \sum_{i=1}^k \left\{ \frac{1}{2}\|u_i^{\varepsilon,\varkappa}\|^2_{H^1(\Omega_\varepsilon)} - \int_{\Omega_\varepsilon} \widetilde{F}_i(u_i^{\varepsilon,\varkappa})\,dx \right\} \\
&\geq \liminf_{\varkappa\to\infty} \sum_{i=1}^k \left\{ \frac{1}{2}\|u_i^{\varepsilon,\varkappa}\|^2_{H^1(\Omega_\varepsilon)} - \int_{\Omega_\varepsilon} \widetilde{F}_i(u_i^{\varepsilon,\varkappa})\,dx \right\} \\
&\geq \sum_{i=1}^k \left\{ \frac{1}{2}\|v_i^{\varepsilon}\|^2_{H^1(\Omega_\varepsilon)} - \int_{\Omega_\varepsilon} F_i(v_i^{\varepsilon})\,dx \right\} = J_{\Omega_\varepsilon}(V^\varepsilon) \geq \lambda_\varepsilon^\delta.
\end{aligned}
$$

Therefore all the above inequalities are indeed equalities. In particular $J_{\Omega_\varepsilon}(V^\varepsilon) = \lambda_\varepsilon^\delta$, meaning that V^ε solves $(\boldsymbol{P_\varepsilon})$ on $B_\delta^\varepsilon(W)$, giving the proof of Theorem 2.2.

Moreover $\lim_{\varkappa\to+\infty} \|U^{\varepsilon,\varkappa}\|_{(H^1(\Omega_\varepsilon))^k} = \|V^\varepsilon\|_{(H^1(\Omega_\varepsilon))^k}$ which, together with weak convergence, implies that the convergence $U^{\varepsilon,\varkappa} \to V^\varepsilon$ is actually strong in $(H^1(\Omega_\varepsilon))^k$. We also deduce that

$$
\lim_{\varkappa\to\infty} \varkappa \int_\Omega (u_i^{\varepsilon,\varkappa})^2 \sum_{j\neq i} (u_j^{\varepsilon,\varkappa})^2 = 0.
$$

The proof of the Theorem 2.3 is thereby complete. □

5.1. **Extremality conditions.** Once the existence of a solution for the optimal partition problem $(\boldsymbol{P_\varepsilon})$ is known, we can appeal to [5] to derive some interesting properties of U_ε. In particular, since U^ε is a local minimizer of the free energy J_{Ω_ε}, we can prove that its components are solution of a remarkable system of distributional inequalities.

Theorem 5.1. *Let $U^\varepsilon \in B_\varepsilon^\delta(W)$ be a solution to problem $(\boldsymbol{P_\varepsilon})$. Then U^ε is a solution of the $2k$ distributional inequalities* (7)*, namely, for every i and every $\phi \in H^1(\Omega_\varepsilon)$ such that $\phi \geq 0$ a.e. in Ω_ε, there holds*

$$
\begin{cases}
\displaystyle\int_{\Omega_\varepsilon} \big(\nabla u_i^\varepsilon \nabla\phi + u_i^\varepsilon\phi - f_i(u_i^\varepsilon)\phi\big)\,dx \leq 0, \\
\displaystyle\int_{\Omega_\varepsilon} \big(\nabla \widehat{u}_i^\varepsilon \nabla\phi + \widehat{u}_i^\varepsilon\phi - \widehat{f}_i(\widehat{u}_i^\varepsilon)\phi\big)\,dx \geq 0,
\end{cases}
$$

where $\widehat{u}_i = u_i - \sum_{h\neq i} u_h$ and $\widehat{f}(\widehat{u}_i) = f_i(u_i) - \sum_{j\neq i} f_j(u_j)$.

The proof can be obtained as in [5, Theorem 5.1], the only difference being that here we are dealing with local (and not global) minima of the free energy. For the reader's convenience, we sketch the main steps.

PROOF. We argue by contradiction, hence, to prove the first inequality, we assume that there exists one index j and $\phi \in H^1(\Omega_\varepsilon)$ such that $\phi \geq 0$ and

$$\int_{\Omega_\varepsilon} \left(\nabla u_j^\varepsilon \nabla \phi + u_j^\varepsilon \phi - f_j(u_j^\varepsilon)\phi\right) > 0. \tag{30}$$

For $t \in (0,1)$ we consider $V = (v_1, \dots, v_k)$ defined as

$$v_i = \begin{cases} u_i^\varepsilon & \text{if } i \neq j \\ (u_i^\varepsilon - t\phi)^+ & \text{if } i = j. \end{cases}$$

We notice that $V \in \mathcal{U}_\varepsilon$. Moreover, since that map $z \mapsto [z]^+$ is continuous from $H^1(\Omega_\varepsilon)$ to $H^1(\Omega_\varepsilon)$ and $U^\varepsilon \in B_\delta^\varepsilon(W)$, we learn that $V \in B_\delta^\varepsilon(W)$ for all t small enough. In light of (30) it is immediate to check that $J_{\Omega_\varepsilon}(V) < J_{\Omega_\varepsilon}(U^\varepsilon) = \min\{J_{\Omega_\varepsilon}(U),\, U \in B_\delta^\varepsilon(W) \cap \mathcal{U}_\varepsilon\}$ for t small enough, a contradiction. Let now j and $\phi \in H^1(\Omega_\varepsilon)$, $\phi \geq 0$, such that

$$\int_\Omega \left(\nabla \widehat{u}_j^\varepsilon \nabla \phi + u_i^\varepsilon \phi - \widehat{f}(\widehat{u}_j^\varepsilon)\phi\right) < 0.$$

Again, we show that the value of the functional can be lessen by replacing U with an appropriate new function V close to W. This is defined as $V = (v_1, \dots, v_k)$ with

$$v_i = \begin{cases} (\widehat{u}_j + t\phi)^+, & \text{if } i = j \\ (\widehat{u}_j + t\phi)^- \chi_{\{u_i > 0\}}, & \text{if } i \neq j. \end{cases}$$

Simple computations lead to

$$J_{\Omega_\varepsilon}(V) - J_{\Omega_\varepsilon}(U^\varepsilon) = t \int_{\Omega_\varepsilon} \left(\nabla \widehat{u}_j^\varepsilon \nabla \phi + \widehat{u}_i^\varepsilon \phi - \widehat{f}_j(\widehat{u}_j^\varepsilon)\phi\right) + o(t),$$

which leads to a contradiction if t is small enough. □

6. Conclusions and final remarks

As a final step of our study, we have proved the existence of an element $(u_1, \dots, u_k)$ with k non-trivial components in the functional class

$$\mathcal{S}(\Omega) = \left\{ \begin{array}{c} (u_1, \cdots, u_k) \in (H^1(\Omega))^k : \ u_i \geq 0,\, u_i \not\equiv 0,\, u_i \cdot u_j = 0 \text{ if } i \neq j, \\ \int_\Omega \left(\nabla u_i \nabla \phi + u_i \phi - f_i(u_i)\phi\right) \leq 0 \text{ and } \int_\Omega \left(\nabla \widehat{u}_i \nabla \phi + \widehat{u}_i \phi - \widehat{f}(\widehat{u}_i)\phi\right) \geq 0 \\ \text{for every } i = 1, \dots, k \text{ and } \phi \in H^1(\Omega) \text{ such that } \phi \geq 0 \text{ a.e. in } \Omega \end{array} \right\}$$

when $\Omega = \Omega_\varepsilon$ with small ε.

In particular, by choosing test functions ϕ with compact support in Ω_ε, we learn that any element of $\mathcal{S}(\Omega_\varepsilon)$ is a solution (in distributional sense) of the following $2k$ differential inequalities:

$$\begin{cases} -\Delta u_i \leq f_i(u_i), & \text{in } \Omega_\varepsilon, \\ -\Delta \widehat{u}_i \geq \widehat{f}(\widehat{u}_i), & \text{in } \Omega_\varepsilon. \end{cases} \tag{31}$$

By appealing to the interior regularity theory developed in [5, Section 8], we know that any u_i is locally Lipschitz continuous and, in particular, the set $\omega_i = \{x \in \Omega_\varepsilon : \ u_i(x) > 0\}$ is an open (nonempty) set. Hence by (7) we obtain that $u_i\big|_{\omega_i}$ is solution of

$$-\Delta u_i + u_i = f_i(u_i), \qquad \text{in } \omega_i,$$

subject to the boundary condition

$$\frac{\partial u_i}{\partial \nu} = 0, \qquad \text{on } \partial\Omega_\varepsilon \cap \omega_i.$$

This suggest that the validity of (7) not only implies the differential inequalities (31) in Ω_ε, but it also contains boundary conditions on $\partial\Omega_\varepsilon$ in some Neumann form, the major difficulty being to give functional sense to "$\frac{\partial u_i}{\partial \nu}$" on the whole of $\partial\Omega_\varepsilon$. A rigorous analysis of this point requires the development of a regularity theory for the class $\mathcal{S}(\Omega)$ up to the boundary, that will be object of future studies.

As a final remark, we point out that the strategy carried out in the present paper can be easily extended to the case in which the number k of the species is less than the number of the connected components of the unperturbed domain Ω_0. Indeed, in such a case we can still write Ω_0 as $\Omega^1 \cup \Omega^2 \cup \cdots \cup \Omega^k$ with $\Omega^i \subset \mathbb{R}^N$ not necessarily connected and repeat, step by step, our arguments which do not actually require connectedness of the sets Ω^j. In this way we can produce solutions both to $(\boldsymbol{P_\varepsilon})$ and to (1) near any function $W = (w_1, \dots, w_k)$ with w_i supported in more than one connected components of Ω_0 for some i.

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