COEXISTENCE AND SEGREGATION FOR STRONGLY COMPETING SPECIES IN SPECIAL DOMAINS.

MONICA CONTI, VERONICA FELLI

ABSTRACT. We deal with strongly competing multispecies systems of Lotka-Vol–terra type with homogeneous Dirichlet boundary conditions. For a class of non-convex domains composed by balls connected with thin corridors, we show the occurrence of pattern formation (coexistence and spatial segregation of all the species), as the competition grows indefinitely. As a result we prove the existence and uniqueness of solutions for a remarkable system of differential inequalities involved in segregation phenomena and optimal partition problems.

1. INTRODUCTION

In this paper we consider the system of $k \geq 2$ elliptic equations

$$-\Delta u_i = f_i(x, u_i) - \kappa u_i \sum_{j \neq i} u_j \qquad \text{in } \Omega, \tag{1}$$

for $i = 1, \ldots, k$, where $\Omega \subset \mathbb{R}^N$ is a smooth, connected, bounded domain. Systems of this form model the steady states of k organisms which coexist in the area Ω. The function u_i represents the population density of the i-th species (hence only $u_i \geq 0$ are considered) and f_i describes the internal dynamic of u_i. The coupling between different equations is the classical Lotka-Volterra interaction term: the positive constant κ prescribes the competitive character of the relationship between u_i and u_j and its largeness measure the strength of the competition.

Systems of this form have attracted considerable attention both in ecology and social science since they furnish a relatively simple model to study phenomena of extinction, coexistence and segregation states of populations. Several theoretical studies have been carried out in this direction (see e.g. [13, 15, 17, 20, 21, 24, 25]), mainly in the case of two competing species and for the logistic nonlinearities $f_i(u) = u(a_i - u)$. In those papers it is shown that both coexistence and exclusion may

Date: Revised version of November 16, 2007.

Research partially supported the Italian MIUR Research Projects *Metodi Variazionali ed Equazioni Differenziali Nonlineari*,

occur, depending on the relations between the diffusion rates, the coefficients of intra–specific and of inter–specific competitions.

In this paper we face the multispecies Lotka-Volterra system (1) in the different perspective investigated by [5] (see also [9, 12]) and study the possibility of coexistence governed by very strong competition. As we shall discuss in detail in Section 3, the presence of large interactions of competitive type produces the spatial segregation of the densities in the limit configuration as $\kappa \to \infty$, namely if $(u_i^\kappa)_{i=1,\dots,k}$ solves (1), then, for all $i = 1, \dots, k$, u_i^κ converges to some u_i in $H^1(\Omega)$ which satisfies

$$u_i(x) \cdot u_j(x) = 0 \text{ a.e. in } \Omega, \qquad \text{for all } i \neq j, \tag{2}$$

so that $\{u_i > 0\} \cap \{u_j > 0\} = \emptyset$. Furthermore, in the limit, the densities satisfy a system of differential inequalities of the form

$$\begin{cases} -\Delta u_i \leq f_i(x, u_i), & \text{in } \Omega, \\ -\Delta \widehat{u}_i \geq \widehat{f}_i(x, \widehat{u}_i), & \text{in } \Omega, \end{cases} \tag{3}$$

where $\widehat{u}_i := u_i - \sum_{j \neq i} u_j$ and $\widehat{f}_i(x, \widehat{u}_i) := f_i(x, u_i) - \sum_{j \neq i} f_j(x, u_j)$, in the sense of definition 1.1. The link between the differential inequalities (3) and population dynamics is reinforced by considering another class of segregation states between species, governed by a minimization principle rather than strong competition–diffusion. In [3] (see also [2, 4]), the following energy functional

$$J(U) = \sum_{i=1,\dots,k} \left\{ \int_\Omega \left(\frac{1}{2} |\nabla u_i(x)|^2 - F_i(x, u_i(x)) \right) dx \right\},$$

given by the sum of the internal energies of k positive densities u_i having internal potentials $F_i(x, s) = \int_0^s f_i(x, u) du$, was considered. The problem of finding the *minimum* of $J(U)$ in the class of k-tuples $U = (u_1, \dots, u_n)$ satisfying $u_j \cdot u_i = 0$ a.e. on Ω for $i \neq j$ was investigated in [3], where it is proved that any non-trivial minimizer U (if it exists) satisfies the differential inequalities (3).

This further motivates the study of the solutions of (2–3) as a natural step in the understanding of segregation phenomena occurring in population dynamics. Remarkably enough, (3) coupled with (2) can be naturally interpreted as a *free boundary problem*: the unknown free boundary set is given by

$$\mathcal{F} = \bigcup_{i=1}^k \partial\{x \in \Omega : u_i(x) > 0\},$$

which represents the collection of the boundaries of the disjoint supports of the densities. The properties of $\mathcal{F}$ are ruled out by the validity of global differential inequalities in the form (3) and provide information about how the segregation occurs, in particular about the way the territory is partitioned by the segregated populations. In this direction, in [1, 2, 3, 5, 7] a number of qualitative properties both of u_i and the free boundary set $\mathcal{F}$ is exhibited. We refer the interested reader to [6] for a brief review of the regularity theory for $\mathcal{F}$ so far developed, and to the above quoted papers for proofs and details.

Another question of particular interest is the existence of a *strictly positive* solution to (2–3), that is a solution of the differential inequalities (3) with each component $u_i \geq 0$ and u_i positive on a set of positive measure. As a matter of fact, since all the asymptotic states of the Lotka-Volterra system have to satisfy (2–3), the existence of such a solution is necessary to ensure that *all* the species survive under strong competition. It has to be stressed that in [3, 5], the strict positivity is guaranteed by assuming positive boundary values for each component, in the form $u_i = \phi_i$ on $\partial\Omega$ with $\phi_i > 0$ on a set of positive $(N-1)$–measure.

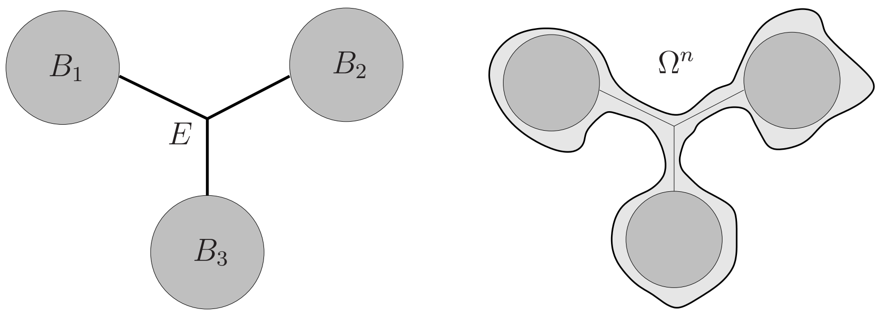
Hence a major problem consists in proving the existence of a positive solution under natural boundary conditions, such as Dirichlet or Neumann homogeneous boundary conditions. This is precisely the problem we face in this paper: we consider (2–3) with the Dirichlet condition

$$u_i = 0 \qquad \text{on } \partial\Omega, \tag{4}$$

and we look for a strictly positive solution $U = (u_1, \dots, u_k)$. The interesting case of Neumann condition will be treated elsewhere, see the concluding remarks.

This is an interesting and mathematically challenging problem: we cannot expect in general to avoid extinction of one or more strongly competing species. For instance, if Ω is convex, it is shown in [19] that two competing species can not coexist under strong competition. Dually, the main variational procedure leading to solutions of (3), that is the minimization of the internal energy J, may fail under Dirichlet homogeneous conditions, since it in general provides k-tuple of the form $(0, \dots, u_i, \dots 0)$ with all but one component identically null, see [26] for a related result.

Therefore, some mechanism of different nature must occur in order to ensure coexistence of the species. In line with [14] and [24] dealing with two populations in planar domains of dumbbell shape, in the present paper we show that the geometry of Ω can play a crucial role in this. As in [8], we consider a class of non convex domains Ω^n, $n \in \mathbb{N}$, essentially composed by k balls connected by thin corridors, as depicted in Figure 1 (see Section 1.1 for the precise definition). Under the main assumption that the Dirichlet problems on each ball admit a nondegenerate local minimizer, we are able to prove existence and uniqueness of positive solutions to



(a) the set $\Omega^0 = B_1 \cup B_2 \cup B_3$ and segments E joining the balls

(b) sets Ω obtained by small perturbation of Ω^0.

Figure 1

the free boundary problem (2–3), where each component is close to such a local minimizer (Theorem 1).

The second result we obtain (Theorem 2), concerns the multispecies Lotka-Volterra system endowed with the Dirichlet null condition. Under the same topological and nondegeneration assumptions, we first prove the existence of a positive solution, provided that the competition parameter κ is large enough. This is by itself an interesting result in the framework of multispecies systems. In fact, in spite of the rich literature dealing with the case $k = 2$ of two populations, the case of $k \geq 3$ species is much harder. We quote for instance [15, 22, 23, 20] for three-species competing systems with cross-diffusion and [10, 11] for the Lotka-Volterra model, where various sufficient conditions for coexistence are provided, depending on the values of the parameters involved in the equations.

Next we perform the asymptotic analysis as κ grows to infinity, and we prove that this solution converges to the unique segregation state found in Theorem 1. The biological implication of this result is now clear: all the species survive under strong competition in a segregating configuration. Furthermore, as we shall see, they divide the domain in such a way that the i-th species does not invade the native territory B_j of the other populations, phenomenon that we call *non invading property*.

Both results come from the study of a multispecies system that can be seen as a generalized Lotka-Volterra model with presence of spatial barriers localized in the balls. We shall introduce it in (6), after some rigorous definitions and precise statements of our results.

1.1. **Assumptions and main results.** Let $\Omega^0 := \bigcup_{i=1}^k B_i$ be a finite union of open balls $B_i \subset \mathbb{R}^N$ such that $\overline{B_i}$ are disjoint, $i = 1, ..., k$. Following [8], we consider a sequence of domains $\{\Omega^n\}_{n\in\mathbb{N}}$ approximating Ω^0 in the following sense: there exists a compact zero measure set $E \subset \mathbb{R}^N$ such that

(i) for any compact set $K \subset \Omega^0$, $\Omega^n \supset K$ provided n is large;

(ii) for any open set $U \supset E \cup \overline{\Omega^0}$, $\Omega^n \subset U$ provided n is large,

see Figure 1. Let us fix a bounded smooth domain Ω strictly containing $\overline{\Omega^0} \cup \overline{\Omega^n}$ for all $n \in \mathbb{N}$. Notice that if $\tilde{\Omega} \subset \Omega$ and $u \in H_0^1(\tilde{\Omega})$, it is possible to extend u to an element of $H_0^1(\Omega)$ by defining it to be zero outside of Ω. Thus in all the paper we shall think of all our functions as being in $H_0^1(\Omega)$. We will make the following set of assumptions (for every $i = 1, \dots, k$):

(F1) $f_i(x,s) : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, it is odd and C^1 in the variable s, uniformly in x;

(F2) $|f_i'(x,s)| = O(|s|^{q-1})$ for large $|s|$, uniformly in x for some $q < \frac{N+2}{N-2}$ ($q < \infty$ if $N = 2$),

where $f_i'(x,s) = \partial_s f_i(x,s)$. Furthermore, for any $i = 1, \dots, k$, we assume that the problem

$$\begin{cases} -\Delta u = f_i(x,u), & \text{in } B_i, \\ u = 0, & \text{on } \partial B_i, \end{cases} \tag{5}$$

admits a positive solution $u_i^0 \in H_0^1(B_i) \cap L^\infty(B_i)$ which is nondegenerate in the following sense:

(ND) there exists $\varepsilon > 0$ such that

$$\int_{B_i} (|\nabla w|^2 - f_i'(x,u_i^0)w^2)dx \geq \varepsilon \int_{B_i} |\nabla w|^2 dx,$$

for every $w \in H_0^1(B_i)$ and for every i.

Note that this implies that the linearized problem at u_i^0

$$-\Delta v - f_i'(x,u_i^0)v = 0, \quad v \in H_0^1(B_i),$$

has only the trivial solution. This is precisely the assumption used in [8]. Condition **(ND)** is stronger and essentially means that u_i^0 is a local nondegenerate minimizer of the energy on B_i (see also [3]). As a model for f_i we can consider logistic type nonlinearities $f_i(x,s) = \lambda(s - |s|^{p-1}s)$, $p > 1$. It is well known that if $\lambda > \lambda_1(B_i)$ (being $\lambda_1(B_i)$ the first eigenvalue of $-\Delta$ in B_i with homogeneous Dirichlet boundary conditions) the elliptic problem (5) has a unique positive solution which is a nondegenerate global minimum for the energy.

Since u_i^0 is an isolated solution to (5), the parameter $\delta > 0$ appearing in all the paper will be assumed small enough to ensure that, for all i:

if $u_i \in H_0^1(B_i)$ *is a solution to* (5) *such that* $\|u_i - u_i^0\|_{H_0^1(B_i)} \le \delta$, *then* $u_i \equiv u_i^0$.

We shall also denote as U^0 the k-tuple $(u_1^0, \dots, u_k^0)$ and as $U = (u_1, \dots, u_k)$ generic k-tuples in $\big(H_0^1(\Omega^n)\big)^k$. Let us clarify the meaning of solution to differential inequalities (3) in the following definition.

Definition 1.1. *A solution to* (3) *is an k-tuple $U = (u_1, \dots, u_k)$ such that, for every $i = 1, \dots, k$ and $\phi \in H_0^1(\Omega)$, $\phi \ge 0$ a.e. in Ω, there holds*

$$\int_\Omega \nabla u_i(x) \cdot \nabla \phi(x)\, dx \le \int_\Omega f_i(x, u_i(x))\phi(x)\, dx$$

and

$$\int_\Omega \nabla \widehat{u}_i(x) \cdot \nabla \phi(x)\, dx \ge \int_\Omega \widehat{f}_i(x, \widehat{u}_i(x))\phi(x)\, dx.$$

The following theorem ensures the existence of a unique segregated solution to (3) in the perturbed domain Ω^n which is H^1-close to U^0.

Theorem 1. *Let us define*

$$\mathcal{S}(\Omega^n) = \left\{ \begin{array}{l} (u_1, \dots, u_k) \in \big(H_0^1(\Omega^n)\big)^k \ : \ u_i \ge 0, \ u_i \cdot u_j = 0, \ \text{if } i \ne j \\ -\Delta u_i \le f_i(x, u_i), \ -\Delta \widehat{u}_i \ge \widehat{f}_i(x, \widehat{u}_i), \ \text{in } \Omega^n, \ i = 1, \dots, k \end{array} \right\}.$$

Then, there exists $\delta > 0$ such that, for any n sufficiently large, the class $\mathcal{S}(\Omega^n)$ contains an element $U = (u_1, \dots, u_k) \in \big(H_0^1(\Omega^n)\big)^k$ such that $\|u_i - u_i^0\|_{H_0^1(\Omega^n)} < \delta$ and $u_i \equiv 0$ in B_j for all $j \ne i$. Moreover, U is the unique element of $S(\Omega^n)$ such that $\|U - U^0\|_{(H_0^1(\Omega^n))^k} < \delta$, where U^0.

As we already announced, the proof of Theorem 1 relies on a careful analysis of the following auxiliary system:

$$\text{(6)} \qquad \begin{cases} -\Delta u_i = f_i(x, u_i) - \kappa u_i \displaystyle\sum_{j \ne i} u_j - \kappa u_i \sum_{j \ne i} u_j^0 - \kappa u_i^0 \sum_{j \ne i} u_j, & \text{in } \Omega^n, \\ u_i = 0, & \text{on } \partial\Omega^n, \end{cases}$$

for $i = 1, \dots, k$.

This system can be seen as a modification of the Lotka-Volterra model, through linear terms which are localized in the single balls: this feature will be crucial in order to obtain solutions with the *non-invading* property. Notice that, due to the presence of the barriers, systems (6) lack of the maximum principle, so that we cannot ensure the positivity of its solutions nor even, by now, the competitive character of the model. As we shall see, this will cause some technical difficulties.

Nonetheless, by careful energy estimates and eigenvalue theory, the system at fixed κ will be shown to be suitably nondegenerate on Ω^n, if n is large enough. This will allow the application of a degree technique introduced by Dancer [8] to control domain perturbation in the case of certain nonlinear equations. As a result we will prove the existence of a solution U^κ of the system, which is close to U^0. The major feature of this approach is that the whole procedure turns out to be *uniform* with respect to κ. This uniformity will allow to perform successfully the asymptotic analysis of the solutions to the auxiliary system as the competition parameter goes to infinity. The final result can be collected in the following form.

Theorem 2. *There exists $\delta > 0$ such that, for any κ and n sufficiently large, both the Lotka-Volterra system* (1) *with Dirichlet boundary conditions* (4) *on Ω^n and the modified model* (6)*, admit a solution $U^\kappa = (u_1^\kappa, \dots, u_k^\kappa) \in \left(H_0^1(\Omega^n)\right)^k$ such that $\|U^\kappa - U^0\|_{(H_0^1(\Omega^n))^k} < \delta$, and, in the case of* (1)*, U^κ is strictly positive. Furthermore, as $\kappa \to \infty$, $u_i^\kappa \to u_i$ strongly in $H^1(\Omega^n)$, where the k-tuple $U = (u_1, \dots, u_k)$ is the unique element in $\mathcal{S}(\Omega^n)$ close to U^0.*

1.2. **Plan of the paper.** In Section 2 we establish some preliminary facts that shall be used throughout the paper, in particular we discuss the nondegeneracy of the problems in Ω^0. Section 3 is devoted to perform the asymptotic analysis of the solutions to the auxiliary system as $\kappa \to \infty$. In Section 4 we prove the uniqueness of the solution to (3) close to U^0, as stated in the uniqueness part of Theorem 1. Section 5 is devoted to the proof of the existence of a solution U close to U^0 for system (6), when the domain is close enough to Ω^0 and the competition is large. We conclude section 5 by presenting the proofs of Theorems 1 and 2 and giving some final remarks. A final appendix collects some technical proofs and lemmas used throughout the paper.

2. Preliminary results

In this section we further modify the Lotka-Volterra system in order to ensure sign conditions and boundedness of its solutions. Furthermore we derive by condition **(ND)** the main nondegeneracy properties holding in the unperturbed set Ω^0.

Let us consider the following system:

$$
\begin{cases}
-\Delta u_i = f_i(x, [u_i + u_i^0]^+ - u_i^0) - \kappa [u_i + u_i^0]^+ \sum_{j \neq i} [u_j + u_j^0]^+, & \text{in } \tilde{\Omega}, \\
u_i = 0, & \text{on } \partial\tilde{\Omega},
\end{cases}
\tag{7}
$$

for $i = 1, \dots, k$, where $\Omega^0 \subset \tilde{\Omega} \subseteq \Omega$.

Here and throughout the paper the symbol $[t(x)]^+$ will denote the positive part of t, namely $[t(x)]^+ = \max\{t(x), 0\}$. The motivation for this choice is contained in the following lemma

Lemma 2.1. *Let $U = (u_1, \dots, u_k)$ be a solution of system* (7)*. Then for all $i = 1, \dots, k$, $u_i \geq -u_i^0$ in $\tilde{\Omega}$. In particular they solve* (6) *and satisfy $u_i(x) \geq 0$ for $x \in B_j$ when $j \neq i$.*

Proof. Let $v_i = u_i + u_i^0$ with its equation

$$-\Delta v_i = f_i(x, [u_i + u_i^0]^+ - u_i^0) + f_i(x, u_i^0) - \kappa [v_i]^+ \sum_{j \neq i} [v_j]^+.$$

Then it suffices to test the equation by $-[u_i + u_i^0]^-$, recalling that f_i is odd. ∎

Remark 2.1. Notice that the original system (1) can be recovered in this model by the formal identification $u_i^0 \equiv 0$. In particular by Lemma 2.1 it turns out that the solutions obtained with $f_i(x, [u_i]^+)$ instead of $f_i(x, u_i)$ satisfy $u_i \geq 0$ for all i, and thus are nonnegative solutions for the Lotka-Volterra system (1).

2.1. **Differential inequalities.** Let $(u_1, \dots, u_k)$ be a solution to (7). Since by Lemma 2.1 each u_i satisfies $u_i + u_i^0 \geq 0$, the coupling term has negative sign and we immediately have

$$-\Delta u_i \leq f_i(x, u_i). \tag{8}$$

Furthermore, by a straightforward calculation we obtain an opposite differential inequality for $\widehat{u}_i$:

$$-\Delta \widehat{u}_i \geq f_i(x, u_i) - \sum_{j \neq i} f_j(x, u_j) = \widehat{f}_i(x, \widehat{u}_i). \tag{9}$$

It turns out that the solutions of (7) satisfy the differential inequalities (3).

2.2. **Uniform L^∞ bounds.** We now suitably modify f_i in order to ensure that the solutions of the new system are bounded in L^∞. This is based on the following result due to Dancer [8]: *for n sufficiently large, the problem*

$$\begin{cases} -\Delta u = f_i(x, u), & \text{in } \Omega^n, \\ u = 0, & \text{on } \partial\Omega^n, \end{cases}$$

admits a positive solution $\phi_i^n \in H_0^1(\Omega^n)$ which is close to $\sum u_i^0$ in some $L^r(\Omega)$ $(r > 1)$.

Let n_0 large such that $\Omega^n \subset \Omega^{n_0}$ for all $n \geq n_0$ and denote $\phi_i^{n_0}$ simply ϕ_i. Let us define

$$\tilde{f}_i(x,s) = \begin{cases} f_i(x,s), & \text{if } s \leq \phi_i(x), \\ f_i(x,\phi_i(x)), & \text{if } s > \phi_i(x). \end{cases}$$

Lemma 2.2. *For $n \geq n_0$, let $u_i \in H_0^1(\Omega^n)$ such that $-\Delta u_i \leq \tilde{f}_i(x,u_i)$ in Ω^n. Then $u_i \leq \phi_i$ a.e. in Ω.*

Proof. Summing up the differential inequalities for ϕ_i and u_i it holds

$$\begin{cases} -\Delta(\phi_i - u_i) \geq f_i(x,\phi_i) - \tilde{f}_i(x,u_i), & \text{in } \Omega^n, \\ \quad \phi_i - u_i \geq 0, & \text{on } \partial\Omega^n. \end{cases}$$

Set $\omega = \{x \in \Omega^n : \phi_i < u_i\}$: note that ω is strictly contained in Ω^n by the boundary conditions. Hence by testing the first inequality with $-(\phi_i - u_i)^-$ we obtain

$$\int_\omega |\nabla(\phi_i - u_i)^-|^2 \, dx \leq -\int_\omega \big(f_i(x,\phi_i) - \tilde{f}_i(x,u_i)\big)(\phi_i - u_i)^- \, dx = 0,$$

which implies $(\phi_i - u_i)^- = 0$ and so $\phi_i \geq u_i$. ■

As a consequence, any solution $(u_1, \dots, u_k)$ either to (7) or to (3), with $\tilde{f}_i$ instead of f_i in Ω^n, satisfies

$$-u_i^0 \leq u_i \leq \phi_i, \quad i = 1, \dots, k. \tag{10}$$

In particular, any solution with u_i close to u_i^0 will be a true solution of the original problem with f_i. Moreover, by virtue of a classical strong maximum principle and Harnack's inequality, any solution $(u_1, \dots, u_k)$ to (7) satisfies either $u_i \equiv \phi_i$, or $u_i \equiv -u_i^0$, or $-u_i^0 < u_i < \phi_i$.

Notations. Throughout all the paper we shall work with $\tilde{f}_i$ instead of f_i, denoting $\tilde{f}_i$ simply by f_i. $\mathrm{supp}(u_i)$ will denote the set $\{u_i > 0\}$.

2.3. **Nondegeneracy in Ω^0.** Let us now consider (7) in Ω^0: then we immediately realize that $U^0 = (u_1^0, \dots, u_k^0)$ is a solution of the problem. Furthermore, it comes from **(ND)** that U^0 is an isolated solution, uniformly in κ.

Theorem 2.1. *There exists $\bar{\kappa} > 0$ and $\delta > 0$ such that if U^κ is a solution of (7) in Ω^0 such that $\|U^\kappa - U^0\| < \delta$ in $\big(H_0^1(\Omega^0)\big)^k$, then $U^\kappa \equiv U^0$ for all $\kappa \geq \bar{\kappa}$.*

An analogous result holds for the solutions to (2)-(3), thanks to the following sign condition prescribed by the validity of (9)

Lemma 2.3. *Let $(u_1,\dots,u_k)$ be solution of $-\Delta\widehat{u}_i \geq \widehat{f}(x,\widehat{u}_i)$ in B_i for some i. Assume that $u_i \cdot u_j = 0$ if $i \neq j$ and that $\|u_j - u_j^0\|_{H_0^1(\Omega)} \leq \delta$ for all $j = 1,\dots,k$. Then, if δ is small enough, $\widehat{u}_i \geq 0$.*

Theorem 2.2. *Let $(u_1,\dots,u_k) \in \mathcal{S}(\Omega^0)$ such that $\|u_i - u_i^0\|_{H_0^1(\Omega^0)} \leq \delta$ for all $i = 1,\dots,k$. Then, if δ is small enough, $u_i \equiv u_i^0$ for all $i = 1,\dots,k$.*

All these results are crucial in what follows, but since the proofs are somewhat technical, we postpone them in the Appendix.

3. Asymptotic analysis as $\kappa \to \infty$

This section is devoted to establish the link between the population systems and the original set of differential inequalities (3). To this aim, throughout the whole section let $\delta > 0$ and assume that *there exists $(u_1^\kappa,\dots,u_k^\kappa)$ solution to (7) such that $\|u_i^\kappa - u_i^0\|_{H^1(\Omega)} \leq \delta$ for all large κ.* Our main result is

Theorem 3.1. *Let $\tilde{\Omega}$ be a connected domain such that $\Omega^0 \subset \tilde{\Omega} \subseteq \Omega$. For each κ let $U^\kappa = (u_1^\kappa,\dots,u_k^\kappa)$ be a solution of (7) in $\tilde{\Omega}$ such that $\|U^\kappa - U^0\|_{(H_0^1(\Omega))^k} < \delta$. Then, if δ is small enough, there exists $U \in \big(H_0^1(\tilde{\Omega})\big)^k$ such that, for all $i = 1,\dots,k$:*

(i) *up to subsequences, $u_i^\kappa \to u_i$ strongly in H^1 as $\kappa \to \infty$,*
(ii) *$u_i > 0$ in B_i,*
(iii) *if $i \neq j$ then $u_i = 0$ a.e. in B_j,*
(iv) *if $i \neq j$ then $u_i \cdot u_j = 0$ a.e. in Ω,*
(v) *$(u_1,\dots,u_k)$ satisfy the differential inequalities (3).*

The proof of this fact is obtained through the next Lemmas 3.1, 3.2 and 3.3.

Lemma 3.1. *Under the assumptions of Theorem 3.1, if δ is small enough there exists $U \in \big(H_0^1(\tilde{\Omega})\big)^k$ such that, for all $i = 1,\dots,k$:*

(i) *up to subsequences, $u_i^\kappa \rightharpoonup u_i$ weakly in $H^1(\Omega)$ as $\kappa \to \infty$,*
(ii) *$u_i \geq 0$ in $\Omega \setminus B_i$,*
(iii) *if $i \neq j$ then $u_i = 0$ a.e. in B_j,*
(iv) *if $i \neq j$ then $u_i \cdot u_j = 0$ a.e. in Ω,*
(v) *$(u_1,\dots,u_k)$ satisfy the differential inequalities (3).*

Proof. Since U^κ is bounded in $(H^1(\Omega))^k$ by assumption, we immediately obtain the existence of a weak limit U such that, up to subsequences, $u_i^\kappa \rightharpoonup u_i$ in $H^1(\Omega)$. Since each u_i^κ is positive on B_j when $j \neq i$ by Lemma 2.1, property *(ii)* comes from almost everywhere pointwise convergence. Furthermore, the differential inequalities

(8) and (9) for u_i^κ pass to the weak limit, so (v) is already proved. Let us discuss properties (iii) and (iv). By testing (6) times $u_i^\kappa + u_i^0$ we obtain

$$\kappa \int_\Omega (u_i^\kappa + u_i^0)^2 \sum_{j\neq i}(u_j^\kappa + u_j^0) \quad \text{is bounded uniformly in } \kappa,$$

hence, since $u_j^\kappa + u_j^0 \geq 0$ for all j,

$$\int_\Omega (u_i^\kappa + u_i^0)^2 \sum_{j\neq i}(u_j^\kappa + u_j^0) \to 0, \qquad \text{as } \kappa \to \infty.$$

Passing to the limit for $U^\kappa \rightharpoonup U$ we obtain, for all $i \neq j$, $i, j = 1, ..., k$

$$u_i(x)\cdot u_j(x) + u_i^0(x)\cdot u_j(x) + u_i(x)\cdot u_j^0(x) = 0, \qquad \forall x \in \Omega. \tag{11}$$

Let $x \in \tilde{\Omega} \setminus \cup B_i$: then (11) ensures $u_i(x)\cdot u_j(x) = 0$ for all $i \neq j$.

Claim. If $x \in B_i$ then $u_j(x) = 0$ for all $j \neq i$.

Let $x \in B_i$ for some fixed i. If $u_i(x) = 0$ then (11) becomes $u_i^0(x)\cdot u_j(x) = 0$ and hence $u_j(x) = 0$, for all $j \neq i$. If $u_i(x) > -u_i^0(x)$ we have $u_j(x)\big(u_i(x) + u_i^0(x)\big) = 0$ implying again $u_j(x) = 0$ for all $j \neq i$. Finally, let $u_i(x) = -u_i^0(x)$. Since $u_j \cdot u_h = 0$ in B_i for all $j \neq h$, $j, h \neq i$, then there exists at most one index different from i (say j) where $u_j(x) > 0$. Set ω_j the connected component of $\{u_j > 0\}$ which is contained in the set $\{y \in B_i : u_i(y) = -u_i^0(y)\}$, and such that $x \in \omega_j$. Then by (9), since $u_h = 0$ in ω_j for all $h \neq i, j$, we have

$$-\Delta(u_i - u_j) \geq f_i(\cdot, u_i) - f_j(\cdot, u_j) \qquad \text{in } \omega_j.$$

Adding $-\Delta u_i^0 = f_i(\cdot, u_i^0)$ we get

$$-\Delta(u_i - u_j + u_i^0) \geq f_i(\cdot, u_i) - f_j(\cdot, u_j) + f_i(\cdot, u_i^0), \qquad \text{in } \omega_j.$$

Test this equation times $-[u_i - u_j + u_i^0]^-$. Note that $[u_i - u_j + u_i^0]^- \equiv u_j|_{\omega_j}$, providing

$$\int_{\omega_j} |\nabla u_j|^2 \leq \int_{\omega_j} f_j(x, u_j)u_j \leq \|f_j(\cdot, u_j)\|_{L^{N/2}(\omega_j)}\|u_j\|^2_{L^{2^*}(\omega_j)}.$$

By **(F1)**, **(F2)**, since $\|u_j\|_{H^1(\omega_j)} \leq \delta$, we have $\|f_j(\cdot, u_j)\|_{L^{N/2}(\omega_j)} \leq C\,\delta$ which implies $u_j \equiv 0$ in ω_j if δ is small enough, giving rise to a contradiction. This proves the claim. ■

Lemma 3.2. *Under the assumptions of Theorem 3.1, if δ is small enough, then $u_i > 0$ in the whole of B_i. In particular $B_i \subset \operatorname{supp}(u_i)$.*

Proof. By Theorem 3.1, we already know that $u_i \geq 0$ in $\Omega \setminus B_i$. Furthermore, $u_i \cdot u_j = 0$ for $i \neq j$ in Ω and $(u_1, \dots, u_k)$ satisfies (9). Therefore Lemma 2.3 yields $\widehat{u}_i = u_i - \sum_{j \neq i} u_j \geq 0$ in B_i. Since by (ii) of Theorem 3.1 $u_j \geq 0$ in B_i, then $u_i \geq 0$ in B_i. Hence $u_i \geq 0$ in Ω and it is not identically null by its closeness to u_i^0; the strict positivity now comes from the Harnack inequality. ∎

Lemma 3.3. *Under the assumption of Theorem 3.1, the convergence $u_i^\kappa \to u_i$ is strong in $H_0^1(\Omega)$ (up to subsequences), where $U = (u_1, \dots, u_k)$ is as in Lemma 3.1.*

Proof. In order to prove the strong convergence of u_i^κ to u_i in $H_0^1(\Omega)$, let us consider the functions $\hat{u}_i = u_i - \sum_{j \neq i} u_j$, which satisfy the inequality (9) in $\tilde{\Omega}$. Since from Lemma 3.2 $u_i \geq 0$, testing (9) with u_i we obtain

$$\int_\Omega u_i f_i(x, u_i) \leq \int_\Omega |\nabla u_i|^2. \tag{12}$$

Testing

$$-\Delta u_i^\kappa \leq f_i(x, u_i^\kappa), \quad \text{in } \tilde{\Omega},$$

with $u_i^\kappa + u_i^0$ (which is positive in view of Lemma 2.1) we have

$$\int_\Omega \nabla u_i^\kappa \cdot \nabla u_i^0 + \int_\Omega |\nabla u_i^\kappa|^2 \leq \int_\Omega u_i^0 f_i(x, u_i^\kappa) + \int_\Omega u_i^\kappa f_i(x, u_i^\kappa). \tag{13}$$

The uniform L^∞-bound provided in Section 2.2 and the Dominated Convergence Theorem allows to pass to the limit in (13) thus obtaining

$$\int_{B_i} \nabla u_i^0 \cdot \nabla u_i + \limsup_{\kappa \to \infty} \int_\Omega |\nabla u_i^\kappa|^2 \leq \int_{B_i} u_i^0 f_i(\cdot, u_i) + \int_\Omega u_i f_i(\cdot, u_i).$$

Since by Theorem 3.1, u_i solves $-\Delta u_i = f_i(x, u_i)$ in B_i, testing with u_i^0 we have $\int_{B_i} \nabla u_i \cdot \nabla u_i^0 = \int_{B_i} u_i^0 f_i(x, u_i)$, thus implying

$$\limsup_{\kappa \to \infty} \int_\Omega |\nabla u_i^\kappa|^2 \leq \int_\Omega u_i f_i(\cdot, u_i). \tag{14}$$

Now (12), (14), and the lower semi-continuity of the norms yield

$$\lim_{\kappa \to \infty} \int_\Omega |\nabla u_i^\kappa|^2 = \int_\Omega |\nabla u_i|^2.$$

The strong convergence follows easily from weak convergence and convergence of norms. ∎

Remark 3.1. Notice that the above analysis can be performed also for the Lotka-Volterra system (1), with some differences. In particular, following the proof of Lemma 3.1, the segregation property (iv) immediately follows by (11) which reduces in this case to $u_i \cdot u_j = 0$. On the contrary we cannot prove, at the moment, the noninvading property (iii).

4. Uniqueness of the asymptotic limit.

As in the previous section, let us here assume that the system (7) does have a solution on Ω^n for all κ large. Our goal now consists in proving that the class $\mathcal{S}(\Omega^n)$ contains one single element which is close to U^0; it is worth noticing that U^0 does not belong to $\mathcal{S}(\Omega^n)$, since the differential inequalities involving the hat operation cannot hold outside Ω^0.

Theorem 4.1. *For δ sufficiently small and n sufficiently large, the class $\mathcal{S}(\Omega^n)$ has at most a unique element U such that $\|U - U^0\|_{((H_0^1(\Omega))^k} < \delta$.*

Proof. By Theorem 3.1, let $U^n \in \mathcal{S}(\Omega^n)$ be the asymptotic limit of the solutions to (7), so that U^n enjoys the noninvading property. Now assume by contradiction the existence of $V^n \in \mathcal{S}(\Omega^n)$, such that $U^n \neq V^n$.

Claim 1. Letting $n \to \infty$, both $U^n \to U^0$ and $V^n \to U^0$ weakly in $H_0^1(\Omega)$ (hence strongly in $L^p(\Omega)$ for all $1 \leq p < 2^*$).

It suffices to prove the claim for U^n. Since U^n is bounded in $(H^1(\Omega))^k$, there exists $U \in (H^1(\Omega))^k$ such that $u_i^n \rightharpoonup u_i$ weakly in $H^1(\Omega)$, strongly in all $L^p(\Omega)$ with subcritical p. We are going to prove that $U \in \mathcal{S}(\Omega^0)$ so that $U \equiv U^0$ in light of Lemma 2.2. To this aim notice that the differential inequalities characterizing $\mathcal{S}(\Omega^0)$ are satisfied by u_i^n for all n, hence they pass to the weak limit. It remains to prove that $u_i \in H_0^1(\Omega^0)$. To this aim notice that, for all open sets $\mathcal{V}$ containing $\overline{\Omega}^0 \cup E$, we have that

$$\operatorname{supp} u_i^n \subset \Omega^n \subset \mathcal{V},$$

provided that n is sufficiently large. Hence

$$\operatorname{supp} u_i \subset \mathcal{V} \quad \text{for all open sets } \mathcal{V} \text{ containing } \overline{\Omega}^0 \cup E,$$

which implies that $u_i = 0$ a.e. in $\Omega \setminus \left(\overline{\Omega}^0 \cup E\right)$. Since $\partial\Omega \cup E$ has measure zero, $u_i = 0$ on $\Omega \setminus \overline{\Omega}^0$, and the smoothness of $\partial\Omega^0$ ensures that $u_i \in H_0^1(\Omega^0)$ (see [18]).

Let us now start the argument that will lead to a contradiction. By setting $\omega_i^n = \{u_i^n > 0\}$, we have $\widehat{u}_i^n = u_i^n$ in ω_i^n and the following hold:

$$\begin{aligned} -\Delta u_i^n &= f_i(x, u_i^n), && \text{in } \omega_i^n, \\ -\Delta v_i^n &\leq f_i(x, v_i^n), && \text{in } \omega_i^n. \end{aligned}$$

If we now consider

$$w_i^n = \frac{v_i^n - u_i^n}{\|V^n - U^n\|_{L^2(\Omega)}},$$

we have

$$\begin{cases} -\Delta w_i^n \le a_i^n(x) w_i^n, & \text{in } \omega_i^n, \\ -\Delta \widehat{w}_i^n \ge b_i^n(x) \widehat{w}_i^n, & \text{in } \omega_i^n, \end{cases} \tag{15}$$

where $a_i^n(x) = \frac{f_i(x,v_i^n) - f_i(x,u_i^n)}{v_i^n - u_i^n}$ and $b_i^n(x) = \frac{\widehat{f}_i(x,\widehat{v}_i^n) - f_i(x,u_i^n)}{\widehat{v}_i^n - u_i^n}$. Notice that $a_i^n(x) \in L^\infty$ independently of n in light of the a priori estimates in Remark 10 and Lemma 2.2 and since $f_i'(\cdot, 0)$ is bounded. We assert that this is true for the second quotient too. To see this, remember that $v_i^n \cdot v_j^n = 0$ in Ω and notice that $b_i^n(x) = \frac{f_i(x,v_i^n(x)) - f_i(x,u_i^n(x))}{v_i^n(x) - u_i^n(x)}$ for $x \in \Omega$ such that $v_i^n(x) > 0$. On the other side, if $v_j(x) > 0$ for some $j \neq i$, then $b_i^n(x) = \frac{f_i(x,u_i^n(x)) + f_j(x,v_j^n(x))}{u_i^n(x) + v_j^n(x)}$. Hence the same argument used to estimate $a_i^n(x)$ provides an L^∞ control for b_i^n, uniformly in n. As a consequence, by testing the differential inequalities in (15) with $[w_i^n]^+$ and $-[\widehat{w}_i^n]^-$ respectively, we easily obtain that w_i^n is bounded in $H^1(\Omega)$. Since this is true for all $i = 1, \dots, k$, there exists $W = (w_1, \dots, w_k) \in (H^1(\Omega))^k$ such that $w_i^n \rightharpoonup w_i$ weakly in $H^1(\Omega)$, strongly in L^2 so that $w_i \neq 0$ for some i.

Claim 2. $w_i \in H_0^1(B_i)$ *and* $-\Delta w_i \le f_i'(x, u_i^0) w_i$ *in* B_i.

Reasoning as in Claim 1, we can easily prove that $w_i \in H_0^1(B_i)$.

Let $\phi \ge 0$ such that $\phi \in C_0^\infty(B_i)$. Since by Theorem 3.2 we know that $B_i \subset \omega_i^n$, we can test the first inequality in (15) with ϕ, providing

$$\int_{B_i} \nabla w_i^n \nabla \phi - a_i^n(x) w_i^n \phi \le 0.$$

By the strong convergence of U^n and V^n to U^0 in $L^p(\Omega)$ for all $1 \le p < 2^*$ and by the continuity of the Nemytskij operator $f_i' : L^{N(q-1)/2}(\Omega) \to L^{N/2}(\Omega)$, see (25), it is easy to realize that $a_i^n \to f_i'(\cdot, u_i^0)$ in $L^{N/2}(\Omega)$ as $n \to \infty$. Hence we can pass to the limit and we find

$$\int_{B_i} \nabla w_i \nabla \phi - f_i'(x, u_i^0) w_i \phi \le 0. \tag{16}$$

By exploiting the same argument starting by the second inequality in (15), we can prove the opposite inequality, namely $-\Delta w_i \ge f_i'(x, u_i^0) w_i$ in B_i. To this aim, we first notice that, setting $A_n^{i,j} := \{x \in B_i : v_j^n(x) > 0\}$, there holds $\lim_{n\to+\infty} \mu(A_n^j) = 0$ if

$j \neq i$. Indeed, since $A_n^{i,j} \subset \{x \in \Omega \setminus B_j : v_i^n(x) = 0\}$ for every $j \neq i$, we have that

$$o(1) = \int_{B_i} |u_i^0 - v_i^n|^2 \, dx \geq \int_{\{x \in B_i : v_i^n(x) = 0\}} |u_i^0|^2 \, dx \geq \int_{A_n^{i,j}} |u_i^0|^2 \, dx$$

as $n \to +\infty$. Due to the absolute continuity of the Lebesgue measure μ with respect to the measure $A \mapsto \int_A |u_i^0|^2 \, dx$ in B_i, we deduce that

$$\lim_{n \to +\infty} \mu(A_n^{i,j}) = 0. \tag{17}$$

In B_i we can write b_i^n as

$$\begin{aligned} b_i^n(x) = & \frac{f_i(x, v_i^n(x)) - f_i(x, u_i^n(x))}{v_i^n(x) - u_i^n(x)} \chi_{B_i \cap \operatorname{supp} v_i^n}(x) \\ & + \sum_{j \neq i} \frac{f_i(x, u_i^n(x)) + f_j(x, v_j^n(x))}{u_i^n(x) + v_j^n(x)} \chi_{A_n^{i,j}}(x). \end{aligned}$$

From $\lim_{n \to +\infty} \mu(A_n^{i,j}) = 0$, the a priori estimates in Remark 10 and Lemma 2.2, and since $f_i'(\cdot, 0)$ is bounded, we deduce that $b_i^n(x) \to f_i'(x, u_i^0(x))$ for a.e. $x \in B_i$. From the uniform L^∞-boundedness of b_i^n and the Dominated Convergence Theorem, we conclude that $b_i^n \to f_i'(\cdot, u_i^0)$ in $L^{N/2}(\Omega)$ as $n \to \infty$. Hence testing the second inequality in (15) with $\phi \in C_0^\infty(B_i)$, $\phi \geq 0$, and passing to the limit as $n \to +\infty$ we obtain that

$$\int_{B_i} \nabla \widehat{w}_i \nabla \phi - f_i'(x, u_i^0) \widehat{w}_i \phi \geq 0. \tag{18}$$

From (17) and L^2-convergence of w_j^n to w_j, for all $j \neq i$ there holds

$$\int_{B_i} |w_j|^2 \, dx = \lim_{n \to +\infty} \int_{B_i} |w_j^n|^2 \, dx = \lim_{n \to +\infty} \int_{A_n^{i,j}} |w_j^n|^2 \, dx = 0.$$

Therefore $w_j = 0$ a.e. in B_i for every $j \neq i$, and

$$\widehat{w}_i = w_i \quad \text{in } B_i. \tag{19}$$

From (16), (18), and (19), we conclude that w_i is a nontrivial solution to the linearized equation $-\Delta w_i = f_i'(x, u_i^0) w_i$ in B_i with boundary condition $w_i = 0$ on ∂B_i. This provides a contradiction with the nondegeneracy assumption **(ND)**. ■

Remark 4.1. We note that the weak $H^1(\Omega)$-convergence stated in Claim 1 of the above proof, is actually strong. Indeed from Theorem 3.1(v), we have that

$$\|u_i^n\|_{H^1(\Omega)}^2 = \int_{\operatorname{supp} u_i^n} |\nabla u_i^n|^2 \, dx = \int_\Omega \chi_{\{\operatorname{supp} u_i^n\}}(x) f_i(x, u_i^n(x)) u_i^n(x) \, dx.$$

By the choice of Ω^n, Theorem 3.1(iii), and pointwise convergence of u_i^n to u_i^0, it follows that $\chi_{\{\operatorname{supp} u_i^n\}} \to \chi_{B_i}$ a.e. in Ω. Hence the uniform L^∞-bound provided in Section 2.2 and the Dominated Convergence Theorem allows to pass to the limit in the right hand side, thus obtaining

$$\lim_{n\to+\infty} \|u_i^n\|^2_{H^1(\Omega)} = \int_{B_i} f_i(x, u_i^0(x))u_i^0(x)\, dx = \|u_i^0\|^2_{H^1(\Omega)}.$$

Strong $H^1(\Omega)$-convergence follows now from weak convergence and convergence of norms.

5. Coexistence in the Lotka-Volterra models.

This section is devoted to prove the existence of solutions to the auxiliary system when the domain is sufficiently close to Ω^0 and the interspecific competition is sufficiently strong. Precisely, we shall prove

Theorem 5.1. *For any κ and n sufficiently large, the system with barriers* (6) *admits a solution $U^\kappa = (u_1^\kappa, \dots, u_k^\kappa) \in \big(H_0^1(\Omega^n)\big)^k$ which is close to $U^0 = (u_1^0, \dots, u_k^0)$ in $\big(H_0^1(\Omega^n)\big)^k$.*

In light of Remark 2.1, the above theorem immediately provides

Corollary 5.1. *For any κ and n sufficiently large, the Lotka-Volterra system* (1) *admits a solution $U^\kappa = (u_1^\kappa, \dots, u_k^\kappa) \in \big(H_0^1(\Omega^n)\big)^k$ which is close to $U^0 = (u_1^0, \dots, u_k^0)$ in $\big(H_0^1(\Omega^n)\big)^k$ and satisfies $u_i \geq 0$ for all i.*

The proof of Theorem 5.1 is obtained by using a standard topological degree technique (see e.g. [16]) and it is based on the ideas introduced in [8] in order to control the perturbation of the domain. As a first step, we introduce suitable operators which allow to reformulate the existence of solutions to (7) as a fixed point problem. For all integers $n = 0, 1, \dots,$ we define

$$A^{n,\kappa}:\ \big(H_0^1(\Omega)\big)^k \to \big(H_0^1(\Omega)\big)^k, \quad A^{n,\kappa} := L^n \circ F^{n,\kappa} \circ i^n,$$

where $i^n : \big(H_0^1(\Omega)\big)^k \to \big(H^1(\Omega^n)\big)^k$ is the restriction $i^n(u_1, \dots, u_k) = (u_1|_{\Omega^n}, \dots, u_k|_{\Omega^n})$,

$$F^{n,\kappa}:\ \big(H^1(\Omega^n)\big)^k \to \big(H^{-1}(\Omega^n)\big)^k,$$
$$F^{n,\kappa}(U) = f_i(\cdot, [u_i + u_i^0]^+ - u_i^0) - \kappa[u_i + u_i^0]^+ \sum_{j\neq i} [u_j + u_j^0]^+,$$

and

$$L^n:\ \big(H^{-1}(\Omega^n)\big)^k \to \big(H_0^1(\Omega^n)\big)^k \hookrightarrow \big(H_0^1(\Omega)\big)^k$$

is defined as: $L^n(h_1,\dots,h_k)=(u_1,\dots,u_k)$ if and only if $-\Delta u_i=h_i$ in Ω^n, $u_i=0$ on $\partial\Omega^n$, for all $i=1,\dots,k$.

With the above notation, it turns out that the solutions of (7) in Ω^n are in 1-1 correspondence with the fixed points of $A^{n,\kappa}$. We are going to prove the existence of fixed points of $A^{n,\kappa}$ by showing that the Leray-Schauder degree of the map $\mathrm{Id}-A^{n,\kappa}$ in a small ball centered at U^0 is different from 0. We recall that the Leray-Schauder degree is well defined for operators which differ from the identity for a compact map. To this aim, we notice that it is not restrictive to assume that $A^{n,\kappa}$ is compact from $\big(H^1_0(\Omega)\big)^k$ into itself. Indeed, if $N<6$, the growth of the nonlinearity $\tilde q=\max\{2,q\}$ is subcritical, i.e. $\tilde q<\frac{N+2}{N-2}$ and compactness is guaranteed by the Sobolev-Rellich embedding Theorem. Otherwise, for $N\ge 6$, using the L^∞ bounds proved in Section 2.2, compactness can be recovered by truncating the coupling term, thus obtaining a subcritical nonlinearity without affecting the proofs.

The following lemma allows to compute the topological degree of the unperturbed problem. In the sequel, we will use the notation $A'(U)$ to denote the Fréchét derivative at $U\in X$ of any differentiable map A from a Banach space X to the Banach space Y.

Lemma 5.1. *Let $\varepsilon>0$ as in assumption* ***(ND)***. *There exists $\bar\kappa$ such that for all $\kappa>\bar\kappa$, the eigenvalues of $\mathrm{Id}-\big(A^{0,\kappa}\big)'(U^0)$ in $\big(H^1_0(\Omega^0)\big)^k$ are greater than ε. In particular, the kernel of $\mathrm{Id}-\big(A^{0,\kappa}\big)'(U^0)$ is trivial.*

Proof. Let us preliminarly notice that, by Lemma 6.2 in the appendix, the map

$$F^{0,\kappa}:\ \big(H^1(\Omega^0)\big)^k\to\big(H^{-1}(\Omega^0)\big)^k$$

is Fréchet differentiable at U^0 and

$$\big(F^{0,\kappa}\big)'(U^0)[V]=\mathrm{Jac}\,G^\kappa(U^0)V,\quad \text{for all } V\in\big(H^1(\Omega^0)\big)^k, \tag{20}$$

where

$$G^\kappa:\ \mathbb{R}^k\to\mathbb{R}^k,\quad G^\kappa(U)=f_i(\cdot,u_i)-\kappa u_i\sum_{j\neq i}u_j-\kappa u_i^0\sum_{j\neq i}u_j-\kappa u_i\sum_{j\neq i}u_j^0,$$

and $\mathrm{Jac}\,G^\kappa(U^0)$ denotes the Jacobian matrix of G^κ at U^0.

Let us set $\mathcal{L}_\kappa:=\mathrm{Id}-\big(A^{0,\kappa}\big)'(U^0)$ and write $(H^1_0(\Omega^0))^k$ as the direct sum

$$(H^1_0(\Omega^0))^k=\bigoplus_{i=1}^k\mathcal{H}_i,$$

where

$$\mathcal{H}_i=H^1_0(B_i)\times H^1_0(B_{i+1(\mathrm{mod}\ k)})\times H^1_0(B_{i+2(\mathrm{mod}\ k)})\times\cdots\times H^1_0(B_{i+k-1(\mathrm{mod}\ k)}).$$

Spaces $\mathcal{H}_i$ are mutually orthogonal and $\mathcal{L}_\kappa|_{\mathcal{H}_i}:\ \mathcal{H}_i\to\mathcal{H}_i$, so that it is enough to prove that 0 is not an eigenvalue of $\mathcal{L}_\kappa|_{\mathcal{H}_i}$ for all $i=1,\dots,k$.

If λ is an eigenvalue of $\mathcal{L}_\kappa$ in $\mathcal{H}_1$, then there exists $V=(v_1,\dots,v_k)\in\mathcal{H}_1$ such that $(v_1,\dots,v_k)\neq(0,\dots,0)$ and

$$-(1-\lambda)\Delta V=\operatorname{Jac}G^\kappa(U^0)V,$$

i.e.

$$-(1-\lambda)\Delta v_i=\Big(f_i'(\cdot,u_i^0)-2\kappa\sum_{j\neq i}u_j^0\Big)v_i-2\kappa u_i^0\sum_{j\neq i}v_j,\quad \text{in } \Omega^0, \tag{21}$$

for all $i=1,\dots,k$. Since $(v_1,\dots,v_k)\neq(0,\dots,0)$, there exists ℓ such that $v_\ell\not\equiv 0$. Equation (21) for $i=\ell$ in B_ℓ reads as

$$-(1-\lambda)\Delta v_\ell=f_\ell'(\cdot,u_\ell^0)v_\ell,\quad \text{in } B_\ell,$$

hence $\lambda\geq\varepsilon$ in view of assumption **(ND)**.

If λ is an eigenvalue of $\mathcal{L}_\kappa$ in $\mathcal{H}_i$ for $i\neq 1$, then there exists $V=(v_1,\dots,v_k)\in\mathcal{H}_i$, $V\neq(0,\dots,0)$, which solves (21). Let ℓ be such that $v_\ell\not\equiv 0$, then equation (21) in $B_{i+\ell-1}$ reads as

$$-(1-\lambda)\Delta v_\ell=\big(f_\ell'(\cdot,0)-2\kappa u_{i+\ell-1}^0\big)v_\ell,\quad v_\ell\in H_0^1(B_{i+\ell-1}).$$

Testing the above equation with v_ℓ we find

$$\begin{aligned}(1-\lambda)\int_{B_{i+\ell-1}}|\nabla v_\ell|^2\,dx&=\int_{B_{i+\ell-1}}\big(f_\ell'(\cdot,0)-2\kappa u_{i+\ell-1}^0\big)v_\ell^2\,dx\\&\leq\int_{B_{i+\ell-1}}\big(f_\ell'(\cdot,0)-2\kappa u_{i+\ell-1}^0\big)^+v_\ell^2\,dx\\&\leq S^{-1}\bigg(\int_{B_{i+\ell-1}}|\nabla v_\ell|^2\,dx\bigg)\|\big(f_\ell'(\cdot,0)-2\kappa u_{i+\ell-1}^0\big)^+\|_{L^{N/2}(B_{i+\ell-1})}\end{aligned}$$

where S is the best constant in the Sobolev embedding. Therefore

$$\lambda\geq 1-S^{-1}\|\big(f_\ell'(\cdot,0)-2\kappa u_{i+\ell-1}^0\big)^+\|_{L^{N/2}(B_{i+\ell-1})}. \tag{22}$$

By the Dominated Convergence Theorem, $\|\big(f_\ell'(\cdot,0)-\kappa u_{i+\ell-1}^0\big)^+\|_{L^{N/2}(B_{i+\ell-1})}\to 0$ as $\kappa\to+\infty$ for any ℓ and i, hence we can find $\bar\kappa$ such that for all $\kappa\geq\bar\kappa$, for all i and ℓ

$$\|\big(f_\ell'(\cdot,0)-2\kappa u_{i+\ell-1}^0\big)^+\|_{L^{N/2}(B_{i+\ell-1})}<S(1-\varepsilon).$$

With this choice of $\bar\kappa$, from (22) it follows that if λ is an eigenvalue of $\mathcal{L}_\kappa$ in $\mathcal{H}_i$ for $i\neq 1$, then $\lambda\geq\varepsilon$; in particular $\lambda\neq 0$. The proof is thereby complete. ∎

Lemma 5.2. *There exist $\bar\kappa$ and $\bar n$ such that, for all $n \geq \bar n$ and $\kappa \geq \bar\kappa$, there holds*

$$U \neq tA^{0,\kappa}(U) + (1-t)A^{n,\kappa}(U)$$

for all $t \in [0,1]$ and $U \in \big(H^1_0(\Omega)\big)^k$ such that $\|U - U^0\|_{(H^1_0(\Omega))^k} = \delta$.

Proof. Arguing by contradiction, we assume there exist sequences $n_j \to \infty$ and $\kappa_j \to \infty$, $t_j \in [0,1]$, and $U^j \in \big(H^1_0(\Omega)\big)^k$ such that $\|U^j - U^0\|_{(H^1_0(\Omega))^k} = \delta$ and

$$U^j = t_j A^{0,\kappa_j}(U^j) + (1-t_j)A^{n_j,\kappa_j}(U^j). \tag{23}$$

Since A^{n_j,κ_j} takes values in $\big(H^1_0(\Omega^{n_j})\big)^k$, we have that $U^j \in \big(H^1_0(\Omega^{n_j})\big)^k$. Taking the laplacian of both sides in (23), we obtain that U^j solve

$$\begin{cases} -\Delta U^j = F^{\kappa_j}(U^j), \\ U^j \in \big(H^1_0(\Omega^{n_j})\big)^k. \end{cases}$$

Since $\{U^j\}_j$ is bounded in $\big(H^1_0(\Omega)\big)^k$, up to a subsequence, U^j converges weakly in $\big(H^1_0(\Omega)\big)^k$ to some $U = (u_1, \dots, u_k) \in \big(H^1_0(\Omega)\big)^k$. By Theorem 3.1, we know that $u_i \cdot u_j = 0$ for $i \neq j$, $u_i \geq 0$ in Ω and that the k-tuple $(u_1, \dots, u_k)$ solves the differential inequality (9), i.e.

$$-\Delta \widehat{u}_i \geq \widehat{f}_i(x, \widehat{u}_i), \quad \text{in } \Omega.$$

As a matter of fact, arguing as in Theorem 4.1 (see the proof of the Claim 1), it is possible to prove that that $u_i \in H^1_0(\Omega^0)$, hence $U \in \mathcal{S}(\Omega^0)$. Furthermore, since the convergence of U^j to U is actually strong in $\big(H^1_0(\Omega)\big)^k$ by Lemma 3.3, we have $\sum_i \|u_i - u_i^0\|^2_{H^1_0(B_i)} = \|U - U^0\|^2_{(H^1_0(\Omega))^k} = \delta^2 > 0$. This implies the existence of i such that $u_i \not\equiv u_i^0$, in contradiction with Theorem 2.2. ∎

We now have all the ingredients to conclude the proof of Theorem 5.1.

Proof of Theorem 5.1. In view of Theorem 2.1, for all $\kappa \geq \bar\kappa$ we can compute the Leray-Schauder degree

$$\deg\big(\operatorname{Id} - A^{0,\kappa}, B_{(H^1_0(\Omega))^k}(U^0, \delta), 0\big).$$

We learn by Lemma 5.1 that it turns out to be equal to $+1$. In light of Lemma 5.2, for $n \geq \bar n$ and $\kappa \geq \bar\kappa$, it makes sense to compute the Leray-Schauder degree

$$I = \deg\big(\operatorname{Id} - A^{n,\kappa}, B_{(H^1_0(\Omega))^k}(U^0, \delta), 0\big).$$

By the homotopy invariance property, we have that

$$\deg\big(\operatorname{Id} - A^{n,\kappa}, B_{(H^1_0(\Omega))^k}(U^0, \delta), 0\big) = \deg\big(\operatorname{Id} - A^{0,\kappa}, B_{(H^1_0(\Omega))^k}(U^0, \delta), 0\big)$$

and hence $I = +1$. As a consequence $A^{n,\kappa}$ has a fixed point in $B_{(H_0^1(\Omega))^k}(U^0, \delta)$ which provides a solution $U = (u_1, ..., u_k)$ to (7) in Ω^n, which is close to U^0. To conclude the proof, it only remains to show that U is a solution to (6) with the original nonlinearity f_i, and this simply follows from (10). ■

Collecting all the results so far obtained, we can finally prove or main theorems.

Proof of Theorem 1. For a fixed sufficiently large n, let us consider the sequence U^κ of solutions to (6) as in Theorem 5.1. As $\kappa \to \infty$, thanks to Theorem 3.1 we know that U^κ converges strongly to some $U = (u_1, \dots, u_n)$ in $(H_0^1(\Omega^n))^k$, such that U is H^1-close to U^0, $u_i \geq 0$ for all i, $u_i \cdot u_j = 0$ if $i \neq j$, U has the non invading property and satisfies the differential inequalities (3). Hence U belongs to $\mathcal{S}(\Omega^n)$. The uniqueness is ensured by Theorem 4.1. ■

Proof of Theorem 2. The existence of a solution U^κ close to U^0 for the two systems is proved in Theorem 5.1 and the subsequent corollary. The asymptotic analysis as $\kappa \to \infty$ has been carried out for (6) in Section 3 and all the results directly come from Theorem 3.1. For the Lotka-Volterra model (1) we still have that U^k converges to an element of $\mathcal{S}(\Omega^n)$ by Remark 3.1. ■

5.1. **Concluding remarks.** In this paper we have restricted our discussion to homogeneous Dirichlet boundary conditions. It has to be stressed that the technique here employed cannot be used to treat the Neumann no-flux boundary conditions

$$\frac{\partial u_i}{\partial \nu} = 0, \qquad \text{on } \partial\Omega^n,$$

two major obstacles being the difficulty in constructing suitable extension operators and the lack of continuity of the eigenvalues of the Laplacian under Neumann boundary conditions with respect to the perturbation of the domain. This will be object of forthcoming studies.

On the other side, our results can be promptly extended to a great variety of competitive models, not necessarily of Lotka-Volterra type, since they essentially depend only on the validity of the differential inequalities (9).

Acknowledgements. The authors wish to express their gratitude to Prof. Susanna Terracini for interesting comments and discussions. They are also indebted with the unknown referees for their helpful remarks.

6. APPENDIX

In this appendix, we collect some lemmas used throughout the paper and the proofs of our most technical results. The following simple lemma is needed to prove Theorem 2.1, i.e. to prove isolation of U^0.

Lemma 6.1. *For all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $i = 1, \dots, k$ and $u \in H_0^1(\Omega^0)$, $\|u - u_i^0\|_{H_0^1(\Omega^0)} \le \delta$ implies*

$$\left| \int_{\Omega^0} \big(f_i(x, u_i) - f_i(x, u_i^0) - f_i'(x, u_i^0)(u_i - u_i^0)\big)(u_i - u_i^0)\, dx \right| \le \varepsilon \|u_i - u_i^0\|^2_{H_0^1(\Omega^0)}. \tag{24}$$

Proof. Call I the integral in (24). We can estimate as follows:

$$\begin{aligned} I &= \left| \int_{\Omega^0} \left[\int_0^1 \big(f_i'(x, tu_i + (1-t)u_i^0) - f_i'(x, u_i^0)\big)(u_i - u_i^0)^2\, dt \right] dx \right| \\ &\le \|u_i - u_i^0\|^2_{L^{2^*}(\Omega^0)} \int_0^1 \|f_i'(\cdot, tu_i + (1-t)u_i^0) - f_i'(\cdot, u_i^0)\|_{L^{N/2}(\Omega^0)} dt \\ &\le S^{-1} \|u_i - u_i^0\|^2_{H_0^1(\Omega^0)} \int_0^1 \|f_i'(\cdot, tu_i + (1-t)u_i^0) - f_i'(\cdot, u_i^0)\|_{L^{N/2}(\Omega^0)} dt \end{aligned}$$

where S is the best constant of the Sobolev embedding $H_0^1 \hookrightarrow L^{2^*}$. By continuity of the Nemytskij operator $f_i' : H_0^1(\Omega^0) \to L^{N/2}(\Omega^0)$, $u \mapsto f_i'(\cdot, u(\cdot))$, there exists $\delta > 0$ such that

$$\|w - u_0^i\|_{H_0^1(\Omega^0)} \le \delta \Rightarrow \|f_i'(\cdot, w) - f_i'(\cdot, u_i^0)\|_{L^{N/2}(\Omega^0)} \le S\varepsilon. \tag{25}$$

This provides the proof. ■

Let us now prove Theorem 2.1, which has played a crucial role in the degree argument developed in Section 5, as it ensures the isolation of the solution to the unperturbed problem.

Proof of Theorem 2.1. Assume that there exists a sequence U^κ of solutions to (6) such that $u_i^\kappa > -u_i^0$ for all i and $U^\kappa \to U^0$ as $\kappa \to \infty$. Let us set $V^\kappa = U^\kappa - U^0$ and by subtracting the respective differential equations we obtain, for all $i = 1, ..., k$:

$$-\Delta v_i^\kappa = f_i(x, u_i^\kappa) - f_i(x, u_i^0) - \kappa v_i^\kappa \sum_{j \ne i} v_j^\kappa - 2\kappa u_i^\kappa \sum_{j \ne i} u_j^0 - 2\kappa u_i^0 \sum_{j \ne i} u_j^\kappa, \qquad \text{in } \Omega^0.$$

Let us add and subtract the term $f_i'(x,u_i^0)v_i^\kappa$; then multiply by v_i^κ and integrate on B_h for a fixed h. We have

$$\int_{B_h} \big[|\nabla v_i^\kappa|^2 - f_i'(x,u_i^0)|v_i^\kappa|^2 - \big(f_i(x,u_i^\kappa) - f_i(x,u_i^0) - f_i'(x,u_i^0)v_i^\kappa\big)v_i^\kappa \\ +2\kappa u_i^0(\sum_{j\neq i} v_j^\kappa)v_i^\kappa + 2\kappa(\sum_{j\neq i} u_j^0)|v_i^\kappa|^2 + \kappa(\sum_{j\neq i} v_j^\kappa)|v_i^\kappa|^2\big] = 0. \tag{26}$$

In particular, since $v_j|_{B_h} = u_j$ if $j \neq h$ while $v_h|_{B_h} = u_h - u_h^0$, by choosing $h \neq i$ we have

$$\int_{B_h} |\nabla v_i^\kappa|^2 - \int_{B_h} \big(f_i(x,u_i^\kappa) - f_i(x,0) - f_i'(x,0)v_i^\kappa\big)v_i^\kappa \\ = \int_{B_h} \big(f_i'(x,0) - \kappa u_h^0 - \kappa \sum_{j\neq i} u_j^\kappa\big)|v_i^\kappa|^2.$$

Let $0 < \varepsilon < 1$ be given: if κ is large enough, in light of Lemma 6.1 and since $u_i(x) > 0$ for $x \in B_j$ when $j \neq i$, we know that

$$(1-\varepsilon)\int_{B_h} |\nabla v_i^\kappa|^2 \leq \int_{B_h} \big[f_i'(x,0) - \kappa(u_h^\kappa + u_h^0)\big]^+ |v_i^\kappa|^2 \\ \leq \|[f_i'(x,0) - \kappa(u_h^\kappa + u_h^0)]^+\|_{L^{N/2}(B_h)} \|v_i^\kappa\|^2_{L^{2^*}(B_h)}.$$

Claim. The $L^{N/2}$ norm of $[f_i'(\cdot,0) - \kappa(u_h^\kappa + u_h^0)]^+$ can be made arbitrarily small by letting $\kappa \to \infty$.

Let us first note that $u_h^\kappa + u_h^0 > 0$ by assumption, hence

$$\|[f_i'(\cdot,0) - \kappa(u_h^\kappa + u_h^0)]^+\|_{L^{N/2}(\omega)} < \|[f_i'(\cdot,0)]^+\|_{L^{N/2}(\omega)} \leq \Big(\sup_{x\in\Omega^0}[f_i'(x,0)]^+\Big)\mu(\omega)^{2/N},$$

for any measurable $\omega \subset \Omega^0$. Secondly, since $\|u_h^\kappa - u_h^0\|_{H_0^1(\Omega^0)} \leq \delta$, then by the Sobolev embedding

$$\delta^2 \geq \int_{\Omega^0} |\nabla(u_h^\kappa - u_h^0)|^2 \geq S\Big(\int_{A_\delta} |u_h^\kappa - u_h^0|^{2^*}\Big)^{\frac{2}{2^*}} \geq \delta\, \mu(A_\delta^\kappa)^{2/2^*}$$

where

$$A_\delta^\kappa = \{x \in B_h : |u_h^\kappa(x) - u_h^0(x)|^2 > \delta\}.$$

Choose δ small (independent of κ) enough so that

$$\Big(\sup_{x\in\Omega^0}[f_i'(x,0)]^+\Big)^{N/2} \cdot \mu(A_\delta^\kappa) \leq \frac{1}{4}\big(S(1-\varepsilon)\big)^{N/2}.$$

Let us now fix $r > 0$ such that

$$\Big(\sup_{x\in\Omega^0}[f_i'(x,0)]^+\Big)^{N/2}\cdot\mu(B_h\setminus B_h(r))\le\frac{1}{4}\big(S(1-\varepsilon)\big)^{N/2},$$

where $B_h(r)$ denotes the ball of radius r and with the same center as B_h. We note that there exists $m > 0$ such that $u_h^0(x) \ge m$ for all $x \in B_h(r)$. Also, for $0 < \sqrt{\delta} < m/2$, we have $u_h^\kappa + u_h^0 > m/2$ in $B_h(r)\setminus A_\delta^\kappa$. With this choice we finally have $\bar\kappa$ such that, for all $\kappa \ge \bar\kappa$, it holds $[f_i'(x,0) - \kappa(u_h^\kappa + u_h^0)]^+(x) = 0$ for any x in $B_h(r)\setminus A_\delta^\kappa$. Summing up, the above argument provides

$$\begin{aligned}\|[f_i'(\cdot,0)-\kappa(u_h^\kappa+u_h^0)]^+\|^{N/2}_{L^{N/2}(B_h)} &\le \|[f_i'(\cdot,0)-\kappa(u_h^\kappa+u_h^0)]^+\|^{N/2}_{L^{N/2}(B_h(r)\setminus A_\delta^\kappa)}\\ &\quad+\|[f_i'(\cdot,0)-\kappa(u_h^\kappa+u_h^0)]^+\|^{N/2}_{L^{N/2}((B_h\setminus B_h(r))\cup A_\delta^\kappa)}\\ &\le\frac{1}{2}\big(S(1-\varepsilon)\big)^{N/2},\end{aligned}$$

for κ large enough, and proves the Claim. As a consequence, if κ is large enough, we obtain $v_i^\kappa|_{B_h} \equiv 0$ for all $h \ne i$. By considering this information in (26) for the choice $h = i$ we get

$$\int_{B_i}\big(|\nabla v_i^\kappa|^2 - f_i'(x,u_i^0)|v_i^\kappa|^2\big) = \int_{B_i}\big(f_i(x,u_i^\kappa) - f_i(x,u_i^0) - f_i'(x,u_i^0)v_i^\kappa\big)v_i^\kappa,$$

for κ large enough. In light of assumption **(ND)** the left hand side is always bigger than $\varepsilon\|v_i^\kappa\|^2$ for some positive ε. On the other side Lemma 6.1 ensures that the right hand side is lessen by $\varepsilon/2\|v_i^\kappa\|^2$ if $\|v_i^\kappa\|$ is suitably small. Hence we reach a contradiction for κ large enough, unless $v_i^\kappa \equiv 0$ for all $i = 1,...,k$, that means $U^\kappa \equiv U^0$. ∎

Proof of Lemma 2.3. Let i be fixed and consider the differential inequality for $\widehat{u}_i$,

$$-\Delta\widehat{u}_i \ge \widehat{f}_i(x,\widehat{u}_i), \qquad \text{in } B_i.$$

Let us test the above inequality with $-\widehat{u}_i^-$, and denote $\omega_i := \{\widehat{u}_i^- > 0\}$. This provides

$$\int_{\omega_i}|\nabla(\widehat{u}_i^-)|^2 \le -\int_{\omega_i}\frac{\widehat{f}_i(x,\widehat{u}_i^-)}{\widehat{u}_i^-}(u_i^-)^2 \le M|\mu(\omega_i)|^{2/N}S^{-1}\int_{\omega_i}|\nabla(\widehat{u}_i^-)^2|, \tag{27}$$

where $M := \|\widehat{f}_i(x,\widehat{u}_i^-)/\widehat{u}_i^-\|_{L^\infty}$ is finite by the a priori L^∞ estimate for u_i as in (10) and taking into account that $f_j'(0)$ is finite for all j. Now, since $\|\widehat{u}_i - u_i^0\|_{H_0^1(B_i)} \le k\delta$, we have

$$k^2\delta^2 \ge \int_{B_i}|\nabla(\widehat{u}_i - u_i^0)|^2 \ge \text{const}\Big(\int_{B_i}|\widehat{u}_i - u_i^0|^{2^*}\Big)^{2/2^*} \ge \text{const}\Big(\int_{\omega_i}|u_i^0|^{2^*}\Big)^{2/2^*}.$$

By absolute continuity of Lebesgue integral, we can choose δ sufficiently small to ensure that $\mu(\omega_i) < (S/2M)^{N/2}$. Hence by (27) we find

$$\int_{\omega_i} |\nabla(\widehat{u_i}^-)|^2 \le \frac{1}{2}\int_{\omega_i} |\nabla(\widehat{u_i}^-)^2|,$$

which provides $\widehat{u_i}^- \equiv 0$. ∎

Proof of Theorem 2.2. By Lemma 2.3 we know that $\widehat{u}_i \ge 0$ in B_i for all i. Since $u_j \ge 0$ for all j and the supports are disjoint, $\sum_{j\ne i} u_j = (\widehat{u}_i)^- = 0$, implying $\widehat{u}_i \equiv u_i$. Hence by coupling the differential inequalities for u_i and $\widehat{u}_i$ we obtain that u_i is a solution to

$$-\Delta u_i = f_i(x, u_i), \qquad \text{in } B_i,$$

with null boundary conditions. Hence by assumption **(ND)** we obtain $u_i \equiv u_i^0$. ∎

The following lemma establishes the Fréchet differentiability of the map $F^{0,\kappa}$ defined in Section 5.

Lemma 6.2. *For any $r \in \big[\frac{2N\tilde{q}}{N+2}, \frac{2N}{N-2}\big]$, the Nemytskij operator*

$$F^{0,\kappa}: \ \big(L^r(\Omega^0)\big)^k \to \big(L^{\frac{r}{\tilde{q}}}(\Omega^0)\big)^k,$$
$$F^{0,\kappa}(U) = f_i(\cdot, [u_i + u_i^0]^+ - u_i^0) - \kappa[u_i + u_i^0]^+ \sum_{j\ne i}[u_j + u_j^0]^+,$$

is Fréchet differentiable at U^0 and

$$\big(F^{0,\kappa}\big)'(U^0)[V] = \bigg(\Big(f_i'(\cdot, u_i^0) - 2\kappa\sum_{j\ne i} u_j^0\Big)v_i - 2\kappa u_i^0 \sum_{j\ne i} v_j\bigg)_{i=1,\dots,k}.$$

Proof. We mean to prove that for all $i = 1, \dots, k$

$$\bigg\| f_i(\cdot, [u_i + u_i^0]^+ - u_i^0) - f_i(\cdot, u_i^0) - f_i'(\cdot, u_i^0)(u_i - u_i^0)$$
$$- \kappa\Big[[u_i + u_i^0]^+ \sum_{j\ne i}[u_j + u_j^0]^+ - 2\Big(\sum_{j\ne i} u_j^0\Big)(u_i - u_i^0) - 2u_i^0\sum_{j\ne i}(u_j - u_j^0)\Big]\bigg\|_{(L^{r/\tilde{q}}(\Omega^0))^k}$$
$$= o\big(\|U - U^0\|_{(L^r(\Omega^0))^k}\big) \quad \text{as } \|U - U^0\|_{(L^r(\Omega^0))^k} \to 0.$$

We have that

$$\frac{\Big\| f_i(\cdot, [u_i + u_i^0]^+ - u_i^0) - f_i(\cdot, u_i^0) - f_i'(\cdot, u_i^0)(u_i - u_i^0)\Big\|^{r/\tilde{q}}_{(L^{r/\tilde{q}}(\Omega^0))^k}}{\|u_i - u_i^0\|^{r/\tilde{q}}_{(L^r(\Omega^0))^k}} \le I_1 + I_2,$$

where

$$I_1 = \frac{\int_{\{u_i+u_i^0>0\}} \left|f_i(\cdot,u_i) - f_i(\cdot,u_i^0) - f_i'(\cdot,u_i^0)(u_i-u_i^0)\right|^{r/\tilde q}}{\left(\int_{\Omega^0} \left|u_i-u_i^0\right|^r\right)^{1/\tilde q}},$$

$$I_2 = \frac{\int_{\{u_i+u_i^0<0\}} \left|2f_i(\cdot,u_i^0) + f_i'(\cdot,u_i^0)(u_i-u_i^0)\right|^{r/\tilde q}}{\left(\int_{\{u_i+u_i^0<0\}} \left|u_i-u_i^0\right|^r\right)^{1/\tilde q}}.$$

Mimicking the proof of Lemma 6.1, we can easily prove that $I_1 \to 0$ as $u_i \to u_i^0$ in $L^r(\Omega)$. Denoting by $\omega_i := \{x \in \Omega : u_i(x) + u_i^0(x) < 0\}$, we observe that

$$\int_\Omega |u_i - u_i^0|^r \geq \int_{\omega_i} |u_i^0|^r,$$

hence $|\omega_i| \to 0$ as $u_i \to u_i^0$ in $L^r(\Omega)$. From assumptions **(F1)**,**(F2)** we have that $f_i(x,s) \leq \text{const}\,(|s| + |s|^{\tilde q})$, hence

$$\frac{\left(\int_{\omega_i}\left|f_i(\cdot,u_i^0)\right|^{\frac{r}{\tilde q}}\right)^{\frac{\tilde q}{r}}}{\left(\int_{\omega_i}\left|u_i-u_i^0\right|^r\right)^{1/r}} \leq \frac{\left(\int_{\omega_i}\left|u_i^0\right|^{\frac{r}{\tilde q}}\right)^{\frac{\tilde q}{r}}}{\left(\int_{\omega_i}\left|u_i^0\right|^r\right)^{\frac{1}{r}}} + \frac{\left(\int_{\omega_i}\left|u_i^0\right|^{r}\right)^{\frac{\tilde q}{r}}}{\left(\int_{\omega_i}\left|u_i^0\right|^r\right)^{\frac{1}{r}}} \leq |\omega_i|^{\frac{\tilde q-1}{r}} + \left(\int_{\omega_i}\left|u_i^0\right|^r\right)^{\frac{\tilde q-1}{r}} = o(1), \tag{28}$$

as $u_i \to u_i^0$ in $L^r(\Omega)$. Moreover

$$\frac{\left(\int_{\omega_i}\left|f_i'(\cdot,u_i^0)(u_i-u_i^0)\right|^{\frac{r}{\tilde q}}\right)^{\frac{\tilde q}{r}}}{\left(\int_{\omega_i}\left|u_i-u_i^0\right|^r\right)^{1/r}} \leq \frac{\|u-u_i^0\|_{L^r(\omega_i)}\|f_i'(\cdot,u_i^0)\|_{L^{\frac{r}{\tilde q-1}}(\omega_i)}}{\|u-u_i^0\|_{L^r(\omega_i)}} = o(1), \tag{29}$$

as $u_i \to u_i^0$ in $L^r(\Omega)$. From (28) and (29) it follows that $I_2 = o(1)$ as $u_i \to u_i^0$ in $L^r(\Omega)$. Hence

$$\left\|f_i(\cdot,[u_i+u_i^0]^+ - u_i^0) - f_i(\cdot,u_i^0) - f_i'(\cdot,u_i^0)(u_i-u_i^0)\right\|_{(L^{r/\tilde q}(\Omega^0))^k} = o\big(\|u_i-u_i^0\|_{(L^{r/\tilde q}(\Omega^0))^k}\big)$$

as $u_i \to u_i^0$ in $L^r(\Omega)$. On the other hand

$$\begin{aligned}
&\Big\|[u_i+u_i^0]^+\sum_{j\neq i}[u_j+u_j^0]^+-2\Big(\sum_{j\neq i}u_j^0\Big)(u_i-u_i^0)-2u_i^0\sum_{j\neq i}(u_j-u_j^0)\Big\|_{(L^{r/\tilde q}(\Omega^0))^k}\\
&\leq \sum_{j\neq i}\Big(\int_{\Omega^0\setminus(\omega_i\cup\omega_j)}|u_i-u_i^0|^{\frac{r}{\tilde q}}|u_j-u_j^0|^{\frac{r}{\tilde q}}+\int_{\omega_i\cup\omega_j}|2u_j^0(u_i-u_i^0)+2u_i^0(u_j-u_j^0)|^{\frac{r}{\tilde q}}\Big)^{\frac{\tilde q}{r}}\\
&\leq \text{const}\sum_{j\neq i}\Big(\|u_i-u_i^0\|^{\frac{r}{\tilde q}}_{L^r(\Omega)}\|u_j-u_j^0\|^{\frac{r}{\tilde q}}_{L^{r/(\tilde q-1)}(\Omega)}\\
&\qquad+\|u_i-u_i^0\|^{\frac{r}{\tilde q}}_{L^r(\Omega)}\|u_j^0\|^{\frac{r}{\tilde q}}_{L^{r/(\tilde q-1)}(\omega_i\cup\omega_j)}+\|u_j-u_j^0\|^{\frac{r}{\tilde q}}_{L^r(\Omega)}\|u_i^0\|^{\frac{r}{\tilde q}}_{L^{r/(\tilde q-1)}(\omega_i\cup\omega_j)}\Big)^{\frac{\tilde q}{r}}\\
&= o(1) \quad \text{as } \big\|U-U^0\big\|_{(L^r(\Omega^0))^k}\to 0.
\end{aligned}$$

The proof is thereby complete. ■

Since we are actually working with the truncation $\tilde f_i$ instead of f_i and $\tilde f_i$ is not C^1 with respect to the second variable, it is worth noticing that this does not create any problem when linearizing the operator at U^0 and the linearization of the truncated operator is still given by (20). Being the proof very similar to the proof of Lemma 6.2, we omit it.

References

[1] M. Conti, S. Terracini, G. Verzini, *Nehari's Problem and Competing Species Systems*, Ann. Inst. H. Poincaré, AN 19, 6 (2002) 871–888.

[2] M. Conti, S. Terracini, G. Verzini, *An optimal partition problem related to nonlinear eigenvalues*, Journal of Funct. Anal. 198, 1 (2003) 160-196.

[3] M. Conti, S. Terracini, G. Verzini, *A variational problem for the spatial segregation of reaction–diffusion systems*, Indiana Univ. Math. J. 54, (2005), 3, 779-815.

[4] M. Conti, S. Terracini, G. Verzini, *On a class of optimal partition problems related to the Fučík spectrum and to the monotonicity formulae*, Calculus of Variations 22, (2005), 1, 45-72.

[5] M. Conti, S. Terracini, G. Verzini, *Asymptotic estimates for the spatial segregation of competitive systems*, Advances in Mathematics 195, (2005), 2, 524-560.

[6] M. Conti, S. Terracini, G. Verzini, *A regularity theory for optimal partition problems*, SPT 2004 – Symmetry and perturbation theory, 91–98, World Sci. Publ., Hackensack, NJ, 2005.

[7] M. Conti, S. Terracini, G. Verzini, *Uniqueness and Least Energy Property for Solutions to Strongly Competing Systems* Interfaces and Free Boundaries

[8] E.N. Dancer, *The effect of domain shape on the number of positive solutions of certain nonlinear equations* J. Differential Equations 74 (1988), 120–156.

[9] E.N. Dancer, Y.H. Du, *Competing species equations with diffusion, large interactions, and jumping nonlinearities*, J. Differential Equations 114 (1994), 434–475.
[10] E.N. Dancer, Y.H. Du, *Positive solutions for a three-species competition system with diffusion. I. General existence results*, Nonlinear Anal., 24 (1995) 3, 337–357.
[11] E.N. Dancer, Y.H. Du, *Positive solutions for a three-species competition system with diffusion. II. The case of equal birth rates*, Nonlinear Anal., 24 (1995) 3, 359–373.
[12] E.N. Dancer, Z.M. Guo, *Uniqueness and stability for solutions of competing species equations with large interactions*, Comm. Appl. Nonlinear Anal. 1 (1994), 19–45.
[13] E.N. Dancer, D. Hilhorst, M. Mimura, L.A. Peletier, *Spatial segregation limit of a competition–diffusion system*, European J. Appl. Math. 10 (1999), 97–115.
[14] S.I. Ei, Q. Fang, M. Mimura, *Effect of domain shape on coexistence problems in a competition-diffusion system*, J. Math. Biol., 29 (1991), 219–237.
[15] J.C. Eilbeck, J.E. Furter, J. López-Gómez, *Coexistence in the competition model with diffusion*, J. Diff. Eq. 107 (1994) 1, 96–139.
[16] I. Fonseca, W. Gangbo, *Degree theory in analysis and applications*, Oxford Lecture Series in Mathematics and its Applications, 2. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.
[17] C. Gui, Y. Lou, *Uniqueness and nonuniqueness of coexistence states in the Lotka-Volterra competition model*, Comm. Pure Appl. Math., 47 (1994) 12, 1571–1594.
[18] D. Gilbarg, N. Trudinger "Elliptic Partial Differential Equations of Second Order", Springer, Berlin, 1977.
[19] K. Kishimoto, H.F. Weinberger, *The spatial homogeneity of stable equilibria of some reaction–diffusion system on convex domains*, J. Diff. Eq. 48 (1985), 15–21.
[20] P. Korman, A. Leung *On the existence and uniqueness of positive steady states in Lotka–Volterra ecological models with diffusion*, Appl. Anal. 26 (1987), 145–160.
[21] A.C. Lazer, P.J. McKenna, *On steady state solutions of a system of reaction–diffusion equations from biology*, Nonlinear Anal. TMA 6 (1982), 523–530.
[22] Y. Lou, S. Martínez, W-M. Ni, *On* 3×3 *Lotka-Volterra competition systems with cross-diffusion*, Discrete Contin. Dynam. Systems, 6 (2000) 1, 175–190.
[23] Y. Lou, W-M. Ni, *Diffusion, self-diffusion and cross-diffusion*, J. Differential Equations, 131 (1996) 1, 79–131.
[24] H. Matano, M. Mimura, *Pattern formation in competition-diffusion systems in nonconvex domains*, Publ. Res. Inst. Math. Sci., 19 (1983) 3, 1049–1079.
[25] N. Shigesada, K. Kawasaki, E. Teramoto, *The effects of interference competition on stability, structure and invasion of a multispecies system*, J. Math. Biol., 21 (1984) 2, 97–113.
[26] G. Sweers, *A sign-changing global minimizer on a convex domain*, Progress in partial differential equations: elliptic and parabolic problems (Pont-à-Mousson, 1991), 251–258, Pitman Res. Notes Math. Ser., 266, Longman Sci. Tech., Harlow, 1992.

Monica Conti
Dipartimento di Matematica "F.Brioschi"
Politecnico di Milano
Via Bonardi 9, I-20133 Milano, Italy
monica.conti@mate.polimi.it,

Veronica Felli
Dipartimento di Matematica e Applicazioni
Università degli Studi di Milano-Bicocca
Via Cozzi 53, I-20125 Milano, Italy
veronica.felli@unimib.it.