On the behavior of solutions to Schrödinger equations near an isolated singularity of the electromagnetic potential

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joint work with **Alberto Ferrero** and **Susanna Terracini**

Schrödinger operators with singular potentials

In quantum mechanics, the hamiltonian of a non-relativistic charged particle in an **electromagnetic** field has the form

$$(-i\nabla + \boldsymbol{\mathcal{A}})^2 + V$$

$\boldsymbol{\mathcal{A}} : \mathbb{R}^N \to \mathbb{R}^N$ magnetic potential associated to the magnetic field $B = \operatorname{curl} \boldsymbol{\mathcal{A}}$.

$V : \mathbb{R}^N \to \mathbb{R}$ electric potential.

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For $\boldsymbol{N \geqslant 2}$, we consider **singular homogeneous electromagnetic** potentials which make the operator *invariant* by scaling

$$\boldsymbol{\mathcal{A}}(x) = \frac{\mathbf{A}\left(\frac{x}{|x|}\right)}{|x|} \qquad \mathbf{A} \in C^1(\mathbb{S}^{N-1}, \mathbb{R}^N)$$

and

$$V(x) = -\frac{a\left(\frac{x}{|x|}\right)}{|x|^2} \qquad a \in L^\infty(\mathbb{S}^{N-1}, \mathbb{R})$$

A prototype in dimension 2

Aharonov-Bohm magnetic potentials are associated to thin solenoids: if the radius of the solenoid tends to zero while the flux through it remains constant, then the particle is subject to a δ-type magnetic field, which is called Aharonov-Bohm field. An associated vector potential in $\mathbb{R}^2$ is

$$\boldsymbol{\mathcal{A}}(x_1, x_2) = \boldsymbol{\alpha}\bigg(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2} \bigg), \quad (x_1, x_2) \in \mathbb{R}^2,$$

with $\boldsymbol{\alpha} =$ circulation of $\boldsymbol{\mathcal{A}}$ around the solenoid.

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Aharonov-Bohm vector potentials are

- singular at 0,
- homogeneous of degree -1
- transversal, i.e. $\mathbf{A}(\theta) \cdot \theta = 0$ for all $\theta \in \mathbb{S}^{N-1}$ **(TC)**

Singular homogeneous electric potentials

which scale as the laplacian arise in nonrelativistic molecular physics, see **[J. M. Lévy-Leblond, Phys. Rev. (1967)]**. The potential describing the interaction between an electric charge and the dipole moment $\mathbf{D} \in \mathbb{R}^N$ of a molecule has the form

$$V(x) = -\frac{\lambda\,(x \cdot \mathbf{d})}{|x|^3} \quad \text{in} \quad \mathbb{R}^N,$$

where $\lambda \propto$ magnitude of the dipole moment $\mathbf{D}$

$\mathbf{d} = \mathbf{D}/|\mathbf{D}| =$ orientation of $\mathbf{D}$.

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Schrödinger operators with dipole-type potentials $\frac{a(x/|x|)}{|x|^2}$ *are studied in*

[Terracini, Adv. Diff. Equations (1996)]

[F.-Marchini-Terracini, Discr. Contin. Dyn. Syst. (2008)]

[F.-Marchini-Terracini, Indiana Univ. Math. J., to appear]

Problem:

describe the asymptotic behavior at the singularity of solutions to equations associated to Schrödinger operators with singular electromagnetic potentials of type

$$\mathcal{L}_{\mathbf{A},a} := \left(-i\,\nabla + \frac{\mathbf{A}\left(\frac{x}{|x|}\right)}{|x|}\right)^2 - \frac{a\left(\frac{x}{|x|}\right)}{|x|^2}$$

in a domain $\Omega \subset \mathbb{R}^N$ *containing either the origin or a neighborhood of* ∞.

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- **Linear perturbation of $\mathcal{L}_{\mathbf{A},a}$:**

$$\boxed{\mathcal{L}_{\mathbf{A},a} u = h(x)\, u}$$

with $h \in L^\infty_{\rm loc}(\Omega \setminus \{0\})$ negligible with respect to $\frac{1}{|x|^2}$ near the singularity

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- **Semilinear equations of type**

$$\mathcal{L}_{\mathbf{A},a} u = f(x, u(x))$$

with f having at most critical growth.

References

Regularity properties of solutions to Schrödinger equations with less singular magnetic and electric potentials:

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[Chabrowski-Szulkin, Topol. Methods Nonlinear Anal., (2005)]: *boundedness and decay at* ∞ *of solutions for* $N \geqslant 3$, $L^2_{\rm loc}$ *magnetic potentials, and electric potentials with* $L^{N/2}$ *negative part.*

References

Behavior of solutions to Schrödinger equations with singular inverse square electric potentials for $\mathbf{A} = 0$ (i.e. no magnetic vector potential):

[F.-Schneider, Adv. Nonl. Studies (2003)] : *Hölder continuity results for degenerate elliptic equations with singular weights; include asymptotics of solutions near the pole for potentials* $\frac{\lambda}{|x|^2}$.

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[Pinchover, Ann. IHP Anal. Nonlinaire (1994)]: *existence of the limit at the singularity of any quotient of two positive solutions in some linear and semilinear cases.*

Remark

Comparison and maximum principles play a crucial role both in **[F.-Marchini-Terracini (2008)]** and **[Pinchover (1994)]**.

In the presence of a singular magnetic potential, comparison methods are no more available, preventing us from a direct extension of the aforementioned results.

We overcome this difficulty by a Almgren type monotonicity formula and blow-up methods.

The angular operator

We aim to describe the rate and the shape of the singularity of solutions, by relating them to the eigenvalues and the eigenfunctions of a Schrödinger operator on the sphere $\mathbb{S}^{N-1}$ corresponding to the angular part of $\mathcal{L}_{\mathbf{A},a}$:

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For $a \in L^\infty(\mathbb{S}^{N-1}, \mathbb{R})$ and $\mathbf{A} \in C^1(\mathbb{S}^{N-1}, \mathbb{R}^N)$, the operator $L_{\mathbf{A},a}$ on $\mathbb{S}^{N-1}$ admits a diverging sequence of real eigenvalues

$$\mu_1(\mathbf{A},a) \leqslant \mu_2(\mathbf{A},a) \leqslant \cdots \leqslant \mu_k(\mathbf{A},a) \leqslant \cdots.$$

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Positivity of the quadratic form associated to $\mathcal{L}_{\mathbf{A},a}$ is ensured by

$$\boxed{\mu_1(\mathbf{A}, a) > -\left(\frac{N-2}{2}\right)^2 \qquad \textbf{(PD)}}$$

The functional setting

$\mathcal{D}^{1,2}_*(\mathbb{R}^N, \mathbb{C}) :=$ completion of $C^\infty_{\rm c}(\mathbb{R}^N \setminus \{0\}, \mathbb{C})$ with respect to

$$\|u\|_{\mathcal{D}^{1,2}_*(\mathbb{R}^N,\mathbb{C})} := \left(\int_{\mathbb{R}^N} \left(\left|\nabla u(x)\right|^2 + \frac{|u(x)|^2}{|x|^2} \right) dx \right)^{1/2}.$$

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Under assumptions **(TC)** and **(PD)**, $\mathcal{D}^{1,2}_*(\mathbb{R}^N,\mathbb{C}) = \mathcal{D}^{1,2}_{\mathbf{A},a}(\mathbb{R}^N)$, where $\mathcal{D}^{1,2}_{\mathbf{A},a}(\mathbb{R}^N)$ is the completion of $C^\infty_c(\mathbb{R}^N\setminus\{0\},\mathbb{C})$ with respect to

$$\|u\|_{\mathcal{D}^{1,2}_{\mathbf{A},a}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N}\left[\left|\left(\nabla+i\,\frac{\mathbf{A}\big(x/|x|\big)}{|x|}\right)u(x)\right|^2-\frac{a\big(x/|x|\big)}{|x|^2}|u(x)|^2\right]dx\right)^{1/2}$$

Moreover the norms $\|\cdot\|_{\mathcal{D}^{1,2}_*(\mathbb{R}^N,\mathbb{C})}$ and $\|\cdot\|_{\mathcal{D}^{1,2}_{\mathbf{A},a}(\mathbb{R}^N)}$ are equivalent.

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$$N \geqslant 3$$

Hardy's inequality

$$\left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2}\, dx \leqslant \int_{\mathbb{R}^N} |\nabla u|^2\, dx$$

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$$\Downarrow$$

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If $\boxed{N = 2}$, **(TC)** holds (i.e. $\mathbf{A}(\theta) \cdot \theta = 0$), and

$$\boxed{\Phi_{\mathbf{A}} := \frac{1}{2\pi} \int_0^{2\pi} \alpha(t)\, dt \notin \mathbb{Z} \qquad \textbf{(ND)}}$$

where $\alpha(t) := \mathbf{A}(\cos t, \sin t) \cdot (-\sin t, \cos t)$

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$\Downarrow$ **[Laptev-Weidl (1999)]**

$C^\infty_c(\mathbb{R}^N \setminus \{0\}, \mathbb{C})$ functions satisfy the following Hardy inequality:

$$\underbrace{\Big(\min_{k\in\mathbb{Z}} |k - \Phi_{\mathbf{A}}|\Big)^2}_{\textit{optimal}} \int_{\mathbb{R}^2} \frac{|u(x)|^2}{|x|^2}\,dx \leqslant \int_{\mathbb{R}^2} \left|\nabla u(x) + i\,\frac{\mathbf{A}\big(x/|x|\big)}{|x|}\,u(x)\right|^2 dx$$

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$\mathcal{D}^{1,2}_*(\mathbb{R}^N,\mathbb{C}) = \mathcal{D}^{1,2}_{\mathbf{A}}(\mathbb{R}^2) =$ completion w.r.t. $\left\|\nabla u+i\,\frac{\mathbf{A}\big(x/|x|\big)}{|x|}\,u\right\|_{L^2(\mathbb{R}^2,\mathbb{C})}$

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In an open bounded domain $\Omega \subset \mathbb{R}^N$, $0 \in \Omega$, let

$$H^1_*(\Omega,\mathbb{C}) = \text{ completion of } \left\{ \begin{array}{l} u \in H^1(\Omega,\mathbb{C}) \cap C^\infty(\Omega,\mathbb{C}) : \\ u \text{ vanishes in a neighborhood of } 0 \end{array} \right\} \text{ w.r.t.}$$

$$\|u\|_{H^1_*(\Omega,\mathbb{C})} = \left(\|\nabla u\|^2_{L^2(\Omega,\mathbb{C}^N)} + \|u\|^2_{L^2(\Omega,\mathbb{C})} + \left\| \frac{u}{|x|} \right\|^2_{L^2(\Omega,\mathbb{C})} \right)^{1/2}.$$

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- If $N \geqslant 3$, $H^1_*(\Omega) = H^1(\Omega, \mathbb{C})$ and their norms are equivalent.
- If $N = 2$, $H^1_*(\Omega)$ is strictly smaller than $H^1(\Omega, \mathbb{C})$.

The Almgren frequency function

Studying regularity of area-minimizing surfaces of codimension $\geqslant 1$, in 1979 Almgren introduced the *frequency function*

$$N(r) = \frac{r \int_{B_r} |\nabla u|^2 \, dx}{\int_{\partial B_r} u^2}$$

and observed that, if u is harmonic, then $\boxed{N \nearrow \text{ in } r}$.

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"*frequency*": if u is a harmonic function in $\mathbb{R}^2$ homogeneous of degree k $\big(u_k(r,\theta) = a_k r^k \sin(k\theta)\big)$, then $N(r) = k$.

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The Almgren monotonicity formula was used in

- **[Garofalo-Lin, Indiana Univ. Math. J. (1986)]**: generalization to variable coefficient elliptic operators in divergence form (unique continuation)
- **[Athanasopoulos-Caffarelli-Salsa, Amer. J. Math. (2008)]**: regularity of the free boundary in obstacle problems.
- **[Caffarelli-Lin, J. AMS (2008)]** regularity of free boundary of the limit components of singularly perturbed elliptic systems.

The Almgren type frequency function

In an open bounded $\Omega \ni 0$, let u be a $H^1_*(\Omega)$-weak solution to $\mathcal{L}_{\mathbf{A},a} u = h(x)\, u$, with h satisfying

$$h \in L^\infty_{\rm loc}(\Omega \setminus \{0\}, \mathbb{C}), \quad |h(x)| = O(|x|^{-2+\varepsilon}) \text{ as } |x| \to 0 \qquad (\boldsymbol{H_0})$$

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For small $r > 0$ define

$$D(r) = \frac{1}{r^{N-2}} \int_{B_r} \left[\left| \nabla u + i \frac{\mathbf{A}(\frac{x}{|x|})}{|x|} u \right|^2 - \frac{a(\frac{x}{|x|})}{|x|^2} |u|^2 - (\Re h)|u|^2 \right] dx,$$

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If **(PD)** holds and $u \not\equiv 0$,
$\Rightarrow H(r) > 0$ for small $r > 0$ $\rightsquigarrow$

Almgren type frequency function

$$\mathcal{N}(r) = \mathcal{N}_{u,h}(r) = \frac{D(r)}{H(r)}$$

is well defined in a suitably small interval $(0, \bar{r})$.

The Almgren type frequency function

$\mathcal{N} \in W^{1,1}_{\mathrm{loc}}(0,\overline{r})$ and, in a distributional sense and for a.e. $r \in (0,\overline{r})$,

$$\mathcal{N}'(r) = \frac{2r\left[\left(\int_{\partial B_r}\left|\frac{\partial u}{\partial \nu}\right|^2 dS\right)\left(\int_{\partial B_r}|u|^2 dS\right) - \left(\int_{\partial B_r}\Re\left(u\frac{\partial \overline{u}}{\partial \nu}\right) dS\right)^2\right]}{\left(\int_{\partial B_r}|u|^2 dS\right)^2} + \frac{\alpha_h(r)}{\int_{\partial B_r}|u|^2 dS}$$

where $\alpha_h(r) = 2\Big[\int_{B_r}\Re(h(x)\,\overline{u(x)}\,(x\cdot\nabla u(x)))\,dx$

$$+\frac{N-2}{2}\int_{B_r}(\Re h(x))|u(x)|^2\,dx - \frac{r}{2}\int_{\partial B_r}(\Re h(x))|u(x)|^2\,dS\Big]$$

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$$\mathcal{N}'(r) = \underbrace{\frac{2r\left[\left(\int_{\partial B_r}\left|\frac{\partial u}{\partial \nu}\right|^2 dS\right)\left(\int_{\partial B_r}|u|^2 dS\right) - \left(\int_{\partial B_r}\Re\left(u\frac{\partial \overline{u}}{\partial \nu}\right) dS\right)^2\right]}{\left(\int_{\partial B_r}|u|^2 dS\right)^2}}_{\substack{\textit{Schwarz's inequality} \\ \geq \\ 0}} + \frac{\alpha_h(r)}{\int_{\partial B_r}|u|^2 dS}$$

where $\alpha_h(r) = 2\left[\int_{B_r}\Re\left(h(x)\,\overline{u(x)}\,(x\cdot\nabla u(x))\right)dx + \frac{N-2}{2}\int_{B_r}(\Re h(x))|u(x)|^2\,dx - \frac{r}{2}\int_{\partial B_r}(\Re h(x))|u(x)|^2\,dS\right]$

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$$(\boldsymbol{H_0}) \Longrightarrow \left|\frac{\alpha_h(r)}{\int_{\partial B_r}|u|^2 dS}\right| \leqslant \text{const}\, r^{-1+\varepsilon}$$

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$\Longrightarrow$ the limit $\gamma := \lim_{r\to 0^+}\mathcal{N}(r)$ exists and is finite.

Blow-up:

set $w^\lambda(x) = \dfrac{u(\lambda x)}{\sqrt{H(\lambda)}}$, so that $\displaystyle\int_{\partial B_1} |w^\lambda|^2 dS = 1$.

$\{w^\lambda\}_{\lambda \in (0,\bar\lambda)}$ is bounded in $H^1_*(B_1) \Longrightarrow$ for any $\lambda_n \to 0^+$, $w^{\lambda_{n_k}} \rightharpoonup w$ in $H^1_*(B_1)$ along a subsequence $\lambda_{n_k} \to 0^+$, and $\int_{\partial B_1} |w|^2 dS = 1$.

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$$(E_k)\ \ \mathcal{L}_{\mathbf{A},a} w^{\lambda_{n_k}}(x) = \lambda_{n_k}^2 h(\lambda_{n_k} x) w^{\lambda_{n_k}}(x) \quad \overset{weak}{\underset{limit}{\rightsquigarrow}} \quad (E)\ \ \mathcal{L}_{\mathbf{A},a} w(x) = 0 \text{ in } B_1$$

Bootstrap and classical regularity theory $\Rightarrow$

$w^{\lambda_{n_k}} \to w$ in $C^{1,\tau}_{\rm loc}(B_1 \setminus \{0\})$, $\tau \in (0,1)$, $H^1(B_r, \mathbb{C})$, $H^1_*(B_r)$, $r \in (0,1)$.

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If $\mathcal{N}_k(r)$ the Almgren frequency function associated to (E_k) and $\mathcal{N}_w(r)$ is the Almgren frequency function associated to (E), then

$$\lim_{k\to\infty} \mathcal{N}_k(r) = \mathcal{N}_w(r) \quad \text{for all } r\in(0,1).$$

Blow-up:

By scaling $\mathcal{N}_k(r) = \mathcal{N}(\lambda_{n_k} r)$

$$\Downarrow$$

$$\mathcal{N}_w(r) = \lim_{k\to\infty} \mathcal{N}(\lambda_{n_k} r) = \gamma \quad \forall r \in (0,1)$$

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Then $\mathcal{N}_w$ is constant in $(0,1)$ and hence $\mathcal{N}_w'(r) = 0$ for any $r \in (0,1)$

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$$\left(\int_{\partial B_r} \left|\frac{\partial w}{\partial \nu}\right|^2 dS\right) \cdot \left(\int_{\partial B_r} |w|^2 dS\right) - \left(\int_{\partial B_r} \Re\left(w \frac{\partial \overline{w}}{\partial \nu}\right) dS\right)^2 = 0$$

Therefore w and $\frac{\partial w}{\partial \nu}$ are parallel as vectors in $L^2(\partial B_r, \mathbb{C})$, i.e. $\exists$ a real valued function $\eta = \eta(r)$ such that $\frac{\partial w}{\partial \nu}(r,\theta) = \eta(r) w(r,\theta)$ for $r \in (0,1)$.

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After integration we obtain

$$w(r,\theta) = e^{\int_1^r \eta(s)ds} w(1,\theta) = \varphi(r)\psi(\theta), \quad r \in (0,1), \ \theta \in \mathbb{S}^{N-1}.$$

Blow-up: $w(r,\theta)=\varphi(r)\psi(\theta)$

Rewriting equation (E) $\mathcal{L}_{\mathbf{A},a}w(x)=0$ in polar coordinates we obtain

$$\left(-\varphi''(r)-\tfrac{N-1}{r}\varphi'(r)\right)\psi(\theta)+r^{-2}\varphi(r)L_{\mathbf{A},a}\psi(\theta)=0.$$

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Let $\mu_{k_0}(\mathbf{A},a)$ be the corresponding eigenvalue $\Longrightarrow$ $\varphi(r)$ solves

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Then $\varphi(r)=c_1r^{\sigma^+}+c_2r^{\sigma^-}$ with $\sigma^\pm=-\frac{N-2}{2}\pm\sqrt{\left(\frac{N-2}{2}\right)^2+\mu_{k_0}(\mathbf{A},a)}$.

$$|x|^{\sigma^-}\psi\left(\tfrac{x}{|x|}\right)\notin H^1_*(B_1)\ \rightsquigarrow\ c_2=0,\qquad \varphi(1)=1\ \rightsquigarrow\ c_1=1$$

$$\Downarrow$$

$$w(r,\theta)=r^{\sigma^+}\psi(\theta)$$

From $\mathcal{N}_w(r)\equiv\gamma$, we deduce that $\gamma=\sigma^+$.

Asymptotics at the singularity

Step 1: any $\lambda_n \to 0^+$ admits a subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$ s.t.

$$\frac{u(\lambda_{n_k}x)}{\sqrt{H(\lambda_{n_k})}} \to |x|^\gamma \psi\Big(\frac{x}{|x|}\Big)$$

weakly in $H^1(B_1)$
strongly in $H^1(B_r)$ for all $r \in (0,1)$
in $C^{1,\tau}_{\rm loc}(B_1 \setminus \{0\})$ for all $\tau \in (0,1)$

$\gamma = -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_{k_0}(\mathbf{A},a)}$, ψ eigenfunction associated to μ_{k_0}.

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Step 3: So $\lambda_{n_k}^{-\gamma} u(\lambda_{n_k}\theta) \to \sum_{i=j_0}^{j_0+m-1} \beta_i \psi_i(\theta)$ in $C^{1,\tau}(\mathbb{S}^{N-1})$

where $\{\psi_i\}_{i=j_0}^{j_0+m-1}$ is an $L^2(\mathbb{S}^{N-1})$-orthonormal basis for the eigenspace associated to μ_{k_0}.

Expanding $u(\lambda\,\theta) = \sum_{k=1}^{\infty} \varphi_k(\lambda)\psi_k(\theta)$, we compute the β_i's.

Asymptotics at the singularity

$$\beta_i = \lim_{k\to\infty} \lambda_{n_k}^{-\gamma}\varphi_i(\lambda_{n_k})$$

$$= \int_{\mathbb{S}^{N-1}} \left[R^{-\gamma}u(R\theta) + \int_0^R \frac{h(s\,\theta)u(s\,\theta)}{2\gamma+N-2}\left(s^{1-\gamma} - \frac{s^{\gamma+N-1}}{R^{2\gamma+N-2}}\right)ds\right]\overline{\psi_i(\theta)}\,dS(\theta)$$

depends neither on the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ nor on its subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$

$\Longrightarrow$ **the convergences actually hold as $\lambda \to 0^+$.**

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Theorem 1 [F.-Ferrero-Terracini, JEMS, to appear]

Let $\Omega \ni 0$ be a bounded open set in $\mathbb{R}^N$, $N \geqslant 2$, **(TC)**, **(PD)**, and $(\boldsymbol{H_0})$ hold. If $u \not\equiv 0$ weakly solves $\mathcal{L}_{\mathbf{A},a}u = h(x)\,u$ in Ω, then $\exists\, k_0 \in \mathbb{N}$, $k_0 \geqslant 1$, s. t.

$$\gamma = \lim_{r\to 0^+} \mathcal{N}_{u,h}(r) = -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_{k_0}(\mathbf{A},a)}.$$

Furthermore, for any $\tau \in (0,1)$, as $\lambda \to 0^+$,

$$\lambda^{-\gamma}u(\lambda\theta) \to \sum_{i=j_0}^{j_0+m-1} \beta_i\psi_i(\theta) \quad \text{in } C^{1,\tau}(\mathbb{S}^{N-1})$$
$$\lambda^{1-\gamma}\nabla u(\lambda\theta) \to \sum_{i=j_0}^{j_0+m-1} \beta_i\big(\gamma\psi_i(\theta)\theta + \nabla_{\mathbb{S}^{N-1}}\psi_i(\theta)\big) \quad \text{in } C^{0,\tau}(\mathbb{S}^{N-1}).$$

Asymptotics at the singularity

Corollary. Under the same assumptions as Theorem 1, let u be a weak $H^1_*(\Omega)$-solution to $\mathcal{L}_{\mathbf{A},a} u = h(x)\, u$.

(i) If $u(x) = O(|x|^k)$ as $|x| \to 0$ for all $k \in \mathbb{N}$, then $u \equiv 0$ in Ω.

(ii) If $0 < \gamma < 1$ then $u \in C^{0,\gamma}_{\mathrm{loc}}(\Omega)$.

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(i) is a *strong unique continuation property*. It extends to singular homogeneous magnetic potentials the unique continuation property proved by **Kurata** for electromagnetic potentials in the Kato class.

Further results

- Semilinear equations $\mathcal{L}_{\mathbf{A},a}u = f(x,u(x))$ in $\Omega \ni 0$ with $f : \Omega \times \mathbb{C} \to \mathbb{C}$ a Carathéodory function satisfying

$$\left|\frac{f(x,z)}{z}\right| \leqslant \begin{cases} C(1+|z|^{2^*-2}), & \text{if } N \geqslant 3, \\ C(1+|z|^{p-2}) \quad \text{for some } p>2\,, & \text{if } N=2\,, \end{cases}$$

for a.e. $x \in \Omega$ and for all $z \in \mathbb{C} \setminus \{0\}$, where $2^* = \frac{2N}{N-2}$.

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- *invariance by the Kelvin transform* $\Longrightarrow$ *asymptotics at* ∞ *for solutions in external domains*

Example: Aharonov-Bohm magnetic potentials in dim $N = 2$

$\mathbf{A}(\cos t, \sin t) = \alpha(-\sin t, \cos t)$, $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, $a(\cos t, \sin t) = a_0$, $a_0 \in \mathbb{R}$

$$\left(-i\,\nabla + \alpha\left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\right)\right)^2 u - \frac{a_0}{|x|^2} u = h\,u,$$

with $x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2$, Ω bounded, $0 \in \Omega$, and h verifying **($\boldsymbol{H_0}$)**.

Eigenvalues: $\{(\alpha - j)^2 - a_0 : j \in \mathbb{Z}\} \quad \Rightarrow \quad \mu_1(\mathbf{A}, a) = \big(\operatorname{dist}(\alpha, \mathbb{Z})\big)^2 - a_0$

If $\operatorname{dist}(\alpha, \mathbb{Z}) \neq \frac{1}{2}$, then all eigenvalues are simple and the eigenspace associated to the first eigenvalue is generated by $\psi(\cos t, \sin t) = e^{-ijt}$.

If $\operatorname{dist}(\alpha, \mathbb{Z}) = \frac{1}{2}$, then all eigenvalues have multiplicity 2.

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If $a_0 < \big(\mathrm{dist}(\alpha, \mathbb{Z})\big)^2$, $\mathrm{dist}(\alpha, \mathbb{Z}) \neq \frac{1}{2} \Rightarrow \exists\, j_0 \in \mathbb{Z}, \beta \in \mathbb{C}$ s.t.

$$\lambda^{-\sqrt{(\alpha - j_0)^2 - a_0}} u(\lambda \cos t, \lambda \sin t) \quad \xrightarrow{\lambda \to 0^+} \quad \beta e^{-ij_0 t} \quad \text{in } C^{1,\tau}(0, 2\pi)$$

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if $a_0 < \big(\mathrm{dist}(\alpha, \mathbb{Z})\big)^2$, $\mathrm{dist}(\alpha, \mathbb{Z}) = \frac{1}{2} \;\Rightarrow\; \exists\, j_0 \in \mathbb{Z}$, $\beta_1, \beta_2 \in \mathbb{C}$ s.t.

$$\lambda^{-\sqrt{(\alpha-j_0)^2 - a_0}}\, u(\lambda\cos t, \lambda \sin t) \xrightarrow{\lambda \to 0^+} \beta_1 e^{-ij_0 t} + \beta_2 e^{-i(2\alpha - j_0)t} \text{ in } C^{1,\tau}(0, 2\pi)$$

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$\mathbf{A}(\cos t, \sin t) = \alpha(-\sin t, \cos t)$, $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, $\quad a(\cos t, \sin t) = a_0$, $a_0 \in \mathbb{R}$

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Furthermore, in view of the Corollary,

if $(\operatorname{dist}(\alpha, \mathbb{Z}))^2 < 1 + a_0 \Rightarrow u \in C^{0,\gamma}_{\mathrm{loc}}(\Omega)$ with $\gamma = \sqrt{(\operatorname{dist}(\alpha, \mathbb{Z}))^2 - a_0}$

if $(\operatorname{dist}(\alpha, \mathbb{Z}))^2 \geqslant 1 + a_0 \ \Rightarrow u$ is locally Lipschitz continuous in Ω.