Pricing Asian Options in Affine Garch models

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Abstract

We derive recursive relationships for the m.g.f. of the geometric average of the underlying within some affine Garch models (namely Heston and Nandi 2000, Christoffersen, Heston and Jacobs 2006, Bellini and Mercuri, 2007, Mercuri, 2008) used for the semi-analytical valuation of geometric Asian options. Similar relationships are obtained for low order moments of the distribution of the arithmetic average of the underlying in the same models, that are used for approximate evaluation of arithmetic Asian options (Turnbull and Wakeman 1991). In both cases the accuracy of the semi-analytical procedure is assessed by means of a comparison with Montecarlo prices.

Keywords *Asian Options, Affine Garch Models, Fourier Transform, Semi-analytical Valuation, Edgeworth Series*

1 Introduction

The aim of this paper is to price a discretely monitored Asian option when the underlying asset follows an affine Garch process. Asian options are options in which the underlying variable is an average (geometric or arithmetic). The Asian options are quite popular among the derivative traders and risk managers for several reasons. First of all, Asian options smoothen possible market instabilities occurring near the expiry date. Moreover, these options provide a suitable hedge for firms - this can be the case, for instance, of commodity end-users which are financially exposed to average prices.

Several approaches have been proposed for the pricing of Asian options (see Fusai and Roncoroni 2008 for a recent review and numerical comparisons). These approaches can be broadly classified into three categories: analytical, approximation and Monte Carlo simulation.

The first approach, proposed by Carverhill and Clewlow (1990) under a geometric Brownian motion assumption, is based on the Fourier transform. These authors obtain a recursive procedure for the computation of the density of the

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arithmetic average of the underlying. Benhamou (2002) apply the same approach to some non-lognormal densities, e.g. Student t. Recently, for arithmetic Asian option under Lévy processes, Fusai and Meucci (2008) solve the valuation problem by recursive integration and derive a recursive theoretical formula for the moments. Moreover, for the Geometric Asian option, these authors provide a closed form formula in terms of the Fourier transform.

The second approach is based on approximating the true distribution of the average with a more tractable one that matches some low order moments. Following this idea, Lévy (1991) proposed to approximate an arithmetic average of lognormals with a lognormal while Turnbull and Wakeman (1992) proposed to use an Edgeworth series approximation. In this way, the authors could capture the skewness and kurtosis present in the log-returns, retaining the lognormal approximation as a special case. Recently Albrecher (2004) explored these approaches under more general Lévy processes (see Albrecher and Predota 2004). The main drawback of the approximation methods is that, in general, it is very difficult to evaluate the approximation error.

The last approach combines the Monte Carlo simulation with the analytical formula for geometric Asian options. Indeed, in order to price an arithmetic Asian option, it is possible to increase the accuracy of the Monte Carlo simulation using the geometric Asian option price as control variate, as shown by Kemma and Vorst (1990).

From an empirical point of view, it is well known that the Black-Scholes model is not able to capture some "stylized facts" such as skewness and heavy tails in the distribution of logreturns. For this reason, the Lévy processes have been introduced in finance (see among others Geman 2002, Schoutens 2003, Carr et al. 2003, for an introduction to Lévy processes with an application in finance). Although these processes are able to account for the skewness and the excess kurtosis, they are still based on an I.I.D. assumption that does not capture the dependence structure observed in real financial data.
The most common discrete time models of non I.I.D. data are Garch-like models, that are indeed very popular in Finance. For option pricing purposes, a very suitable class is constituted by the so-called affine Garch models, that yield a closed form formula for option prices, since it is possible to compute the characteristic function of log-prices and thereby options prices by inverse Fourier transform methods (see Heston and Nandi 2000 for a normal case, Christoffersen et al. 2006 for Inverse Gaussian innovations, Bellini and Mercuri 2007 for Gamma innovations and Mercuri 2008 for Tempered Stable innovations).

The contribution of this paper is two-fold. First, we provide a closed form formula for geometric Asian options when the underlying follow an affine Garch process of the above mentioned types. Second we discuss the case of the arithmetic Asian option and provide a recursive procedure for the computation of the moments of the arithmetic average that will be the basis of an approximated procedure.
This paper is organized as follow: in Section 2 we quickly review some affine Garch models present in literature. In Section 3 we obtain the valuation formula

for geometric Asian option and we check the accuracy of the proposed procedure by a comparison with Monte Carlo prices.
In the last Section we present the approximation method for the arithmetic Asian options.

2 Affine Garch Models

The aim of this section is to review the existing affine Garch models. The main feature of these models is that they yield a closed form formula for option prices in terms of Fourier transform. The models are written in an exponential form

$$S_t = S_{t-1} \exp(X_t)$$

where the logreturns X_t follow an affine Garch process.
We will write the dynamics directly under the risk neutral measure. These models may also be introduced under the objective measure, and the change of measure may be performed by means of the conditional Esscher transform, as discussed in Siu et al (2004) and Bellini - Mercuri (2007)
The first model was proposed by Heston and Nandi (HN henceforth) in which the dynamics of X_t under the martingale measure is given by:

$$\begin{cases} X_t = r - \frac{1}{2}h_t + \sqrt{h_t}Z_t \\ h_t = \alpha_0 + \alpha_1(Z_{t-1} - \lambda\sqrt{h_{t-1}})^2 + \beta_1 h_{t-1} \end{cases}$$

where Z_t are i.i.d. standard normal.

Christoffersen et al. (2006) pointed out that this model seems to be not sufficiently flexible, particularly if we consider the options with short maturities. These authors suggested that the reason is that normal innovations are not able to capture the conditional skewness and the conditional kurtosis of the logreturns. For this reason, they proposed an affine Garch model with Inverse Gaussian innovations (CHJ model henceforth), where the risk-neutral dynamics of logreturns is given by:

$$\begin{cases} X_t = r + \lambda h_t - \varepsilon_t \\ \varepsilon_t | F_{t-1} = \eta Y_t \\ Y_t \sim IG(\delta_t) \text{ with } \delta_t = \dfrac{h_t}{\eta^2} \\ h_t = \alpha_0 + \alpha_1 \varepsilon_{t-1} + \beta_1 h_{t-1} + \dfrac{\gamma h_{t-1}^2}{\varepsilon_{t-1}} \end{cases}$$

With similar motivations, Bellini and Mercuri (2007) suggested a model with

Gamma innovations (BM model):

$$\begin{cases} X_t = r + \lambda h_t - \varepsilon_t \\ \varepsilon_t | F_{t-1} = \dfrac{b}{\sqrt{a}} Y_t \\ Y_t \sim Ga(ah_t, b) \\ h_t = \alpha_0 + \alpha_1 \varepsilon_{t-1} + \beta_1 h_{t-1} \end{cases}$$

Finally, Mercuri (2008) proposed an affine Garch model with Tempered Stable innovations (M model) that encompasses both the CHJ and the BM models as special cases:

$$\begin{cases} X_t = r + \lambda h_t - \varepsilon_t \\ \varepsilon_t | F_{t-1} = \dfrac{Y_t}{2\sqrt{\alpha a \left(1-\alpha\right) \left(b\right)^{(\alpha-2)/\alpha}}} \\ Y_t \sim TS\left(\alpha, ah_t, b\right) \\ h_t = \alpha_0 + \alpha_1 \varepsilon_{t-1} + \beta_1 h_{t-1} \end{cases}$$

Remark 1 *Let $s(x; \alpha, \beta, c, \delta)$ be the positively skewed α-stable density function with $\alpha \ \epsilon \ (0, 1)$. We say that $TS(\alpha, a, b)$ is a tempered stable distribution with $a \ \epsilon \ (0, 1)$, $a > 0$ and $b \geq 0$, if its density function is given by:*

$$p\left(x; \alpha, a, b\right) = \exp\left(ab\right) s(x; \alpha, 1, \frac{a}{2^\alpha \sec\left(\alpha \frac{\pi}{2}\right)}, 0) \exp\left(-\frac{1}{2} b^{1/\alpha} x\right)$$

3 Geometric Asian options

In this section we provide an analytic procedure for the pricing of a geometric Asian option where the underlying is observed at equally-spaced times (for notational simplicity we will consider unitary steps).
The payoff of the geometric Asian call option with fixed strike K is given by:

$$C\left(K, T\right) = \max\left\{G_T - K, 0\right\}$$

where

$$G_T = \left(\prod_{t=0}^{T} S_t\right)^{\dfrac{1}{T+1}} \tag{1}$$

In order to price this option, we have to evaluate the expected payoff under the martingale measure:

$$C\left(K, t\right) = e^{-r(T-t)} E_t^Q \left[\max\left\{G_T - K, 0\right\}\right] \tag{2}$$

By defining

$$Y_T := \ln\left(G_T\right)$$

we can rewrite the expected value in (2) as:

$$C\left(e^{k},t\right)=e^{-r(T-t)}\int_{k}^{+\infty}\left(e^{Y_T}-e^{k}\right)dF\left(Y_T\right) \tag{3}$$

where $k=\ln(K)$.
In a general Garch setup, this integral can not be computed analytically since the distribution of Y_T is unknown. In the next section we show how to compute in a recursive fashion the moment generating function of the variable Y_T.

3.1 Recursive computation of the m.g.f. of average logreturns Y_T

We write the m.g.f. of Y_T in exponential form

$$\varphi_t(u)=E_t\left[\exp\left(uY_T\right)\right]=$$
$$=\exp\left[\frac{u}{T+1}\sum_{h=0}^{t-1}\ln\left(S_h\right)+A\left(t:T,u\right)+B\left(t:T,u\right)h_{t+1}+C\left(t:T,u\right)\ln\left(S_t\right)\right]$$

with time-dependent coefficients $A(t:T,u)$, $B(t:T,u)$ and $C(t:T,u)$.
From the iteration property of conditional expectations we have that:

$$\begin{gathered}\varphi_t(u)=\exp\left[\frac{u}{T+1}\sum_{h=0}^{t-1}\ln\left(S_h\right)+\left(\frac{u}{T+1}+C\left(t+1:T,u\right)\right)\ln\left(S_t\right)+\right]*\\ *\exp\left[C\left(t+1:T,u\right)r+A\left(t+1:T,u\right)+\alpha_0B\left(t+1:T,u\right)+\right.\\ \left.-\tfrac{1}{2}\ln\left(1-2\alpha_1B\left(t+1:T,u\right)\right)\right]*\\ *\exp\left\{\left[C\left(t+1:T,u\right)(\lambda-\frac{1}{2})-\frac{\lambda^2}{2}+\beta_1B\left(t+1:T,u\right)+\right.\right.\\ \left.\left.+\frac{\frac{1}{2}(C\left(t+1:T,u\right)-\lambda)^2}{1-2\alpha_1B\left(t+1:T,u\right)}\right]\right\}h_{t+1}\end{gathered}$$

and by substituting and equating terms of the same order in logprice and volatility we get the following recursive system for the coefficients:

$$\begin{cases}A\left(t:T,u\right)=C\left(t+1:T,u\right)r+A\left(t+1:T,u\right)+\alpha_0B\left(t+1:T,u\right)+\\ -\frac{1}{2}\ln\left(1-2\alpha_1B\left(t+1:T,u\right)\right)\\ B\left(t:T,u\right)=C\left(t+1:T,u\right)(\lambda-\frac{1}{2})-\dfrac{\lambda^2}{2}+\beta_1B\left(t+1:T,u\right)+\\ +\dfrac{\frac{1}{2}(C\left(t+1:T,u\right)-\lambda)^2}{1-2\alpha_1B\left(t+1:T,u\right)}\\ C\left(t:T,u\right)=\frac{u}{T+1}+C\left(t+1:T,u\right)\end{cases} \tag{4}$$

The same approach may be pursued in the other affine Garch models previously introduced. In the CHJ model, we obtain the following recursive system:

$$\begin{cases} A\left(t:T,u\right)=-\dfrac{1}{2}\ln[1-2\gamma\eta^{4}B(t+1,T,u)]+C\left(t+1:T,u\right)r+ \\ +A(t+1,T,u)+\alpha_{0}B(t+1,T,u) \\ B\left(t:T,u\right)=\beta_{1}B(t+1,T,u)+C\left(t+1:T,u\right)\lambda+\dfrac{1}{\eta^{2}}+ \\ -\dfrac{1}{\eta^{2}}\sqrt{[1-2\gamma\eta^{4}B(t+1,T,u)][1-2C\left(t+1:T,u\right)\eta-2\alpha_{1}B(t+1,T,u))]} \\ C\left(t:T,u\right)=\dfrac{u}{T+1}+C\left(t+1:T,u\right) \end{cases} \tag{5}$$

In BM model, we have:

$$\begin{cases} A\left(t:T,u\right)=C\left(t+1:T,u\right)r+A\left(t+1:T,u\right)+B\left(t+1:T,u\right)\omega \\ B\left(t:T,u\right)=\beta B\left(t+1:T,u\right)+C\left(t+1:T,u\right)\lambda+ \\ -a\log\left(1+\dfrac{C\left(t+1:T,u\right)}{\sqrt{a}}-\dfrac{\alpha_{1}}{\sqrt{a}}B\left(t+1:T,u\right)\right) \\ C\left(t:T,u\right)=\dfrac{u}{T+1}+C\left(t+1:T,u\right) \end{cases} \tag{6}$$

and in M model we get:

$$\begin{cases} A\left(t,T,u\right)=C\left(t+1:T,u\right)r+A\left(t+1,T,u\right)+\alpha_{0}B\left(t+1,T,u\right) \\ B\left(t,T,\phi\right)=C\left(t+1:T,u\right)\lambda+\beta_{1}B\left(t+1,T,u\right)+ \\ +ab\left[1-\left(1-\dfrac{\left(\alpha_{1}B\left(t+1,T,u\right)-C\left(t+1:T,u\right)\right)}{\sqrt{\alpha ba\left(1-\alpha\right)}}\right)^{\alpha}\right] \\ C\left(t:T,u\right)=\dfrac{u}{T+1}+C\left(t+1:T,u\right) \end{cases} \tag{7}$$

see the Appendix for the complete derivations.

Remark 2 *The recursive system, in BM model, can be obtained as a special case of the M model by setting* $b=1, a=\dfrac{a_1}{\alpha}$ *and computing the limit for* $\alpha \to 0^+$. *Moreover it is possible to recover the CHJ model by imposing* $\alpha=\dfrac{1}{2}, a=\dfrac{1}{\eta^2}, b=1$ *and considering the following variance dynamics*

$$h_t=\alpha_0+\alpha_1\varepsilon_{t-1}+\beta_1 h_{t-1}$$

where ε_t *is conditionally distributed as* $IG\left(\dfrac{h_t}{\eta^2}\right)$.

In all cases the recursive relations have to be numerically implemented with the terminal conditions:

$$\begin{cases} A\left(T:T,u\right)=0 \\ B\left(T:T,u\right)=0 \\ C\left(T:T,u\right)=\dfrac{u}{T+1} \end{cases}$$

3.2 Option pricing

In order to price a geometric Asian call option, we follow the approach proposed in Carr and Madan (1998).
Let $C\left(e^k,t\right)$ the call price a time t, we introduce a dumping parameter $\delta>0$ (usually chosen between 1.5 and 2) such that the quantity:

$$c_\delta\left(e^k,t\right):=e^{\delta k}C\left(e^k,t\right)$$

adimts the Fourier transform with respect to the logarithm of the strike-price:

$$\begin{aligned}\mathcal{F}\left[c_\delta\left(e^k,t\right)\right](\gamma) &= \int_{-\infty}^{+\infty} e^{i\gamma k}c_\delta\left(e^k,t\right)dk = \\ &= \frac{e^{-r(T-t)}\varphi_t\left(\gamma-\delta i-i\right)}{\delta^2+\delta-\gamma^2+i\left(2\delta+1\right)\gamma}\end{aligned}$$

where $\varphi_t(\cdot)$ is the characteristic function given the information a time t.
The option price may then be obtained by Fourier inversion:

$$C\left(e^k,t\right)=\frac{e^{-\delta k}}{\pi}\int_0^{+\infty} e^{i\gamma k}\mathcal{F}\left[c_\delta\left(e^k,t\right)\right](\gamma)\,d\gamma \tag{8}$$

In the following tables we compare the option prices obtained by the Fourier approach with the Monte Carlo prices:

Insert table 1,2,3,4 about here

We consider the parameters reported in Mercuri (2008). These parameters are calibrated on S&P500 closing option prices (from 06/22/06 to 07/10/06) hence they allows a comparison between the different models. In all cases the results from the recursive procedure are in good accordance with Monte Carlo prices. Moreover we note that the higher prices are achieved in HN model while the lower ones in M model and this difference increases for options out-of-the money. One possible explanation is the presence of a single tail in conditional distributions, with the exception of the HN model.

4 Arithmetic Asian options

In this section we deal with the case of an arithmetic Asian option, that is, an option written on the arithmetic average of an asset price or financial index.
We will follow the approximation approach that is based on replacing the true distribution of the arithmetic average with a more tractable distribution matching some low order moments (see Fusai and Roncoroni 2008 for a recent review). Recently Albrecher (2004) extends this method to exponential Lévy processes. The idea is to compute the moments of the true distribution by means of a recursive procedure, much in the spirit of the preceding section.

We have to evaluate the following expected values:

$$\mu_n = E_0^Q\left[\left(\frac{1}{T+1}A_T\right)^n\right] \tag{9}$$

where $n = 1, ..., N$, (usually $N = 4$) and the variable A_T is defined as:

$$A_T := \sum_{t=0}^{T+1} S_t$$

we can rewrite A_T using the log-returns

$$\begin{aligned} A_T &= S_0\left[1 + \exp\left(X_1\right) + ... + \exp\left(X_1 + ... + X_T\right)\right] = \\ &= S_0\left[1 + \exp\left(X_1\right)\left[...\left[1 + \exp\left(X_T\right)\right]\right]\right] \end{aligned}$$

and by a straightforward calculation we have

$$A_T^n = S_0^n \sum_{j_1=0}^{n} \binom{n}{j_1} \exp\left(j_1 X_1\right)\left[...\left[1 + \exp\left(X_T\right)\right]\right]^{j_1}$$

and with repeated applications of the binomial theorem we get:

$$A_T^n = S_0^n \sum_{j_1=0}^{n} \sum_{j_2=0}^{j_1} ... \sum_{j_T=0}^{j_{T-1}} E_0^Q\left[\exp\left(j_1 X_1 + j_2 X_2 + ... + j_T X_T\right)\right] \tag{10}$$

In the Heston-Nandi model it is possible to compute recursively the quantity:

$$\varphi_0\left(j_1, ..., j_T\right) = E_0^Q\left[\exp\left(j_1 X_1 + j_2 X_2 + ... + j_T X_T\right)\right]$$

indeed by defining:

$$\begin{gathered} \varphi_t\left(j_1, ..., j_T\right) = \exp\left[j_1 X_1 + j_2 X_2 + ... + A\left(t : T, j_{t+1}, ..., j_T\right) + \right. \\ \left. + B\left(t : T, j_{t+1}, ..., j_T\right) h_{t+1}\right] \end{gathered}$$

We have by the iteration property of conditional expectations:

$$\begin{gathered} \varphi_t\left(j_1, ..., j_T\right) = E_t^Q\left[\varphi_{t+1}\left(j_1, ..., j_T\right)\right] = \\ = * \exp\left[j_1 X_1 + ... + j_t X_t + j_{t+1} r + A\left(t+1 : T, j_{t+2}, ..., j_T\right) + \right. \\ \left. + \alpha_0 B\left(t+1 : T, j_{t+2}, ..., j_T\right) - \tfrac{1}{2}\ln\left(1 - 2\alpha_1 B\left(t+1 : T, j_{t+2}, ..., j_T\right)\right)\right] * \\ * \exp\left[\left(\lambda j_{t+1} - \frac{j_{t+1}}{2} - \frac{\lambda^2}{2} + \beta_1 B\left(t+1 : T, j_{t+2}, ..., j_T\right) + \right.\right. \\ \left.\left. + \frac{\frac{1}{2}\left(\lambda - j_{t+1}\right)^2}{1 - 2\alpha_1 B\left(t+1 : T, j_{t+2}, ..., j_T\right)}\right) h_{t+1}\right] \end{gathered}$$

Hence we obtain the following recursive system for the coefficients:

$$\left\{\begin{array}{l} A\left(t:T,,j_{t+1},...,j_{T}\right)=j_{t+1}r+A\left(t+1:T,j_{t+2},...,j_{T}\right)+ \\ +\alpha_{0}B\left(t+1:T,j_{t+2},...,j_{T}\right)-\dfrac{1}{2}\ln\left(1-2\alpha_{1}B\left(t+1:T,j_{t+2},...,j_{T}\right)\right) \\ B\left(t:T,,j_{t+1},...,j_{T}\right)=\lambda j_{t+1}-\dfrac{j_{t+1}}{2}-\dfrac{\lambda^{2}}{2}+\beta_{1}B\left(t+1:T,j_{t+2},...,j_{T}\right)+ \\ +\dfrac{\frac{1}{2}\left(\lambda-j_{t+1}\right)^{2}}{1-2\alpha_{1}B\left(t+1:T,j_{t+2},...,j_{T}\right)} \end{array}\right. \tag{11}$$

As for the recursive determination of the m.g.f. of the geometric average, this approach works also in the other considered affine Garch models.
In CHJ, BM, M model we obtain respectively in equations (12), (13) , (14):

$$\left\{\begin{array}{l} A\left(t:T,j_{t+2},...,j_{T}\right)=-\dfrac{1}{2}\ln[1-2\gamma\eta^{4}B(t+1,T,j_{t+2},...,j_{T})]+j_{t+1}r+ \\ +A(t+1,T,j_{t+2},...,j_{T})+\alpha_{0}B(t+1,T,j_{t+2},...,j_{T}) \\ B\left(t:T,j_{t+2},...,j_{T}\right)=\beta_{1}B(t+1,T,j_{t+2},...,j_{T})+j_{t+1}\lambda+\dfrac{1}{\eta^{2}}-\dfrac{1}{\eta^{2}}* \\ *\sqrt{[1-2\gamma\eta^{4}B(t+1,T,j_{t+2},...,j_{T})][1-2j_{t+1}\eta-2\alpha_{1}B(t+1,T,j_{t+2},...,j_{T}))]} \end{array}\right. \tag{12}$$

$$\left\{\begin{array}{l} A\left(t:T,j_{t+2},...,j_{T}\right)=j_{t+1}r+A\left(t+1:T,j_{t+2},...,j_{T}\right)+ \\ +\alpha_{0}B\left(t+1:T,j_{t+2},...,j_{T}\right) \\ B\left(t:T,u\right)=\beta B\left(t+1:T,j_{t+2},...,j_{T}\right)+j_{t+1}\lambda+ \\ -a\log\left(1+\dfrac{j_{t+1}}{\sqrt{a}}-\dfrac{\alpha_{1}}{\sqrt{a}}B\left(t+1:T,j_{t+2},...,j_{T}\right)\right) \end{array}\right. \tag{13}$$

$$\left\{\begin{array}{l} A\left(t,T,j_{t+2},...,j_{T}\right)=j_{t+1}r+A\left(t+1,T,j_{t+2},...,j_{T}\right)+ \\ +\alpha_{0}B\left(t+1,T,j_{t+2},...,j_{T}\right) \\ B\left(t,T,j_{t+2},...,j_{T}\right)=j_{t+1}\lambda+\beta_{1}B\left(t+1,T,j_{t+2},...,j_{T}\right)+ \\ +ab\left[1-\left(1-\dfrac{\left(\alpha_{1}B\left(t+1,T,j_{t+2},...,j_{T}\right)-j_{t+1}\right)}{\sqrt{\alpha ba\left(1-\alpha\right)}}\right)^{\alpha}\right] \end{array}\right. \tag{14}$$

Remark 3 *As in geometric case, the recursive systems* (12) *and* (13) *can be obtained as special cases of M model by imposing the same condition introduced in* section *3.1*

In order to check the accuracy of the proposed procedures, we compare the first fourth moments of the sum variable in the above models obtained by recursive systems and by the Monte Carlo simulation respectively. As in previous section we use the calibrated parameters reported on Mercuri (2008).

We report the results in the tables 5, 6, 7 and 8:

Insert table 5,6,7,8 about here

Notice that, as expected, the first moments are equal for all models. For the other moments, the lower values are obtained in M model while the higher ones in HN model. However, the Monte Carlo simulations confirm the correctness of recursive systems.

Finally, we study the approximations of the arithmetic option price by means of Edgeworth series expansions. In order to capture the skewness and kurtosis of the average variable, we follow the approach proposed in Turnbull and Wakeman (1991). These authors adopt a fourth-order Edgeworth series expansion as approximation of the true variable.

Let $f_{\log}\left(y; m, v^2\right)$ the lognormal density where the parameters m and v^2 match the mean and the variance of the variable A_T:

$$
\begin{aligned}
m &= 2\log\left(E_0^Q\left[A_T\right]\right) - \frac{1}{2}\log\left(E_0^Q\left[A_T^2\right]\right) \\
v^2 &= \log\left(E_0^Q\left[A_T^2\right]\right) - 2\log\left(E_0^Q\left[A_T\right]\right)
\end{aligned}
$$

The fourth-order Edgeworth approximation $f_{edg}\left(y; m, v^2\right)$ is given by:

$$
f_{edg}\left(y; m, v^2\right) = f_{\log}\left(y; m, v^2\right) + \sum_{i=1}^{4} \frac{k_i}{i!} \frac{\partial^i f_{\log}\left(y; m, v^2\right)}{\left(\partial y\right)^i} + e\left(y\right)
$$

where k_i is the difference in the ith cumulant[1] between the exact distribution and approximate distribution, namely $k_i = \chi_i\left(f\right) - \chi_i\left(f_{\log}\left(y; m, v^2\right)\right)$, with

$$
\begin{aligned}
\chi_1\left(f\right) &= E_0^Q\left[A_T\right] \\
\chi_2\left(f\right) &= E_0^Q\left[\left(A_T - E_0^Q\left[A_T\right]\right)^2\right] \\
\chi_3\left(f\right) &= E_0^Q\left[\left(A_T - E_0^Q\left[A_T\right]\right)^3\right] \\
\chi_4\left(f\right) &= E_0^Q\left[\left(A_T - E_0^Q\left[A_T\right]\right)^4\right] - 3\chi_2\left(F\right)
\end{aligned}
$$

[1] The cumulants of random variable X with distribution function F are defined by:

$$
\chi_i\left(F\right) = \left[\frac{\partial^i \ln\left(E\left(e^{tX}\right)\right)}{\left(\partial t\right)^i}\right]_{t=0}, \quad i = 1, 2, ...
$$

Therefore the approximate Asian option price is given by:

$$c_{edg}(K,0) = e^{-rT}\frac{S_0}{T+1}\left[e^{m+\frac{v^2}{2}}N\left(\frac{m+v^2-\log\left(\frac{nK}{S_0}\right)}{v}\right)+\right.$$
$$\left.-\frac{(T+1)K}{S_0}N\left(\frac{m-\log\left(\frac{nK}{S_0}\right)}{v}\right)\right]+$$
$$+e^{-rT}\frac{S_0}{T+1}\left[-\frac{k_3}{3!}\frac{\partial f_{\log}\left(y;m,v^2\right)}{\partial y}+\frac{k_4}{4!}\frac{\partial^2 f_{\log}\left(y;m,v^2\right)}{(\partial y)^2}\right]_{y=\frac{nK}{S_0}}$$

If we consider the first two cumulants we obtain the approximation proposed by Lévy (1992). In the following tables, we compare the option prices obtained by fourth-order Edgeworth series expansion with Monte Carlo simulation:

Insert table 9,10,11,12 about here

Notice that the approximate formula seems to work better in HN model. This result may be due to the conditionally normal assumption for innovations. Generally, the approximate formula seems to be consistent with the Monte Carlo prices for options in the money.

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5 Appendix

Derivation of conditional m.g.f. in HN model for the Y_T variable

Given the formula

$$\varphi_t(u) = E_t\left[\exp\left(uY_T\right)\right] \tag{15}$$

$$= \exp\left[\frac{u}{T+1}\sum_{h=0}^{t-1}\ln\left(S_h\right) + A\left(t:T,u\right) + B\left(t:T,u\right)h_{t+1} + C\left(t:T,u\right)\ln\left(S_t\right)\right]$$

we suppose that the relation (15) holds a time $t+1$ and by iteration property of the conditional expected value we compute the conditional m.g.f a time t:

$$\varphi_t(u) = E_t\left[\varphi_{t+1}(u)\right] =$$
$$= \exp\left[\frac{u}{T+1}\sum_{h=0}^{t-1}\ln(S_h) + \left(\frac{u}{T+1} + C(t+1:T,u)\right)\ln(S_t)\right] *$$
$$* \exp\left[\alpha_0 B(t+1:T,u) + rC(t+1:T,u) + A(t+1:T,u)\right] *$$
$$* \exp\left[\left(\beta_1 B(t+1:T,u) - \frac{1}{2}C(t+1:T,u)\right)h_{t+1}\right] *$$
$$* E_t\left[\exp\left[\alpha_1 B(t+1:T,u)(Z_t - \lambda\sqrt{h_t})^2 + C(t+1:T,u)\sqrt{h_t}Z_t\right]\right]$$

$$\varphi_t(u) = E_t\left[\varphi_{t+1}(u)\right]$$
$$= \exp\left[\frac{u}{T+1}\sum_{h=0}^{t-1}\ln(S_h) + \left(\frac{u}{T+1} + C(t+1:T,u)\right)\ln(S_t)\right] *$$
$$* \exp\left[\alpha_0 B(t+1:T,u) + rC(t+1:T,u) + A(t+1:T,u)\right] *$$
$$* \exp\left[\left(\beta_1 B(t+1:T,u) - \frac{1}{2}C(t+1:T,u) + \lambda C(t+1:T,u) + \right.\right.$$
$$\left.\left. - \frac{C^2(t+1:T,u)}{4\alpha_1 B(t+1:T,u)}\right)h_{t+1}\right] *$$
$$* E_t\left[\exp\left[\alpha_1 B(t+1:T,u)(Z_t - \left(\lambda - \frac{C(t+1:T,u)}{2\alpha_1 B(t+1:T,u)}\right)\sqrt{h_{t+1}})^2\right]\right]$$

using the moment generating function of the non central Chi-square

$$E[\exp(a(z+b)^2)] = \exp(-\frac{1}{2}\ln(1-2a) + \frac{ab^2}{1-2a})$$

we obtain

$$\varphi_t(u) = E_t\left[\varphi_{t+1}(u)\right]$$
$$= \exp\left[\frac{u}{T+1}\sum_{h=0}^{t-1}\ln(S_h) + \left(\frac{u}{T+1} + C(t+1:T,u)\right)\ln(S_t)\right] *$$
$$* \exp\left[\alpha_0 B(t+1:T,u) + rC(t+1:T,u) + A(t+1:T,u) + \right.$$
$$\left. - \frac{1}{2}\ln(1 - 2\alpha_1 B(t+1:T,u))\right] *$$
$$* \exp\left[\left(\beta_1 B(t+1:T,u) + \left(\lambda - \frac{1}{2}\right)C(t+1:T,u) + \right.\right.$$
$$\left.\left. - \frac{\lambda^2}{2} + \frac{\frac{1}{2}(\lambda - C(t+1:T,u))^2}{1 - 2\alpha_1 B(t+1:T,u)}\right)h_{t+1}\right]$$

Derivation of conditional m.g.f. in CHJ model for the Y_T variable.

As in HN model, we suppose that the relation (15) holds a time $t+1$ and therefore, by iteration property of the conditional expected value, we compute the conditional m.g.f a time t:

$$\varphi_t(u) = \exp\left[\frac{u}{T+1}\sum_{h=0}^{t-1}\ln(S_h) + \left(\frac{u}{T+1} + C(t+1:T,u)\right)\ln(S_t)\right] *$$
$$* \exp[C(t+1:T,u)(X_t + r + \lambda h_{t+1}) + A(t+1,T,u) +$$
$$+ B(t+1,T,u)(\alpha_0 + \beta_1 h_{t+1})] *$$
$$E_t[\exp(\varepsilon_{t+1}(C(t+1:T,u)\eta + \alpha_1 B(t+1,T,u)) + \frac{B(t+1,T,u)\gamma h_{t+1}^2}{\varepsilon_{t+1}}))]$$

the expected value in above formula can be calculated using the generalized moment generating function of Inverse Gaussian with δ degree of freedom :

$$E[\exp(\theta y + \frac{\phi}{y})] = \frac{\delta}{\sqrt{(\delta^2 - 2\phi)}}\exp(\delta - \sqrt{(\delta^2 - 2\phi)(1-2\theta)}) \tag{16}$$

Remembering that $\delta_t = \frac{h_t}{\eta^2}$ we get

$$E_t[\exp(\varepsilon_{t+1}(C(t+1:T,u)\eta + \alpha_1 B(t+1,T,u)) + \frac{B(t+1,T,u)\gamma h_{t+1}^2}{\varepsilon_{t+1}}))] =$$
$$= \frac{\frac{1}{\eta^2}}{\sqrt{\frac{1}{\eta^4} - 2B(t+1,T,u)\gamma}}\exp\left[\frac{h_{t+1}}{\eta^2} + \right.$$
$$\left. -h_{t+1}\sqrt{[\frac{1}{\eta^4} - 2B(t+1,T,u)\gamma]\cdot[1 - 2(C(t+1:T,u)\eta + \alpha_1 B(t+1,T,u))]}\right]$$

from which the recursive relation can be recovered.

Derivation of conditional m.g.f. in BM model for the Y_T variable.

Following the same approach proposed in HN model, we obtain for the conditional moment generating function:

$$\varphi_t(u) = \exp\left[\frac{u}{T+1}\sum_{h=0}^{t-1}\ln(S_h) + \left(\frac{u}{T+1} + C(t+1:T,u)\right)\ln(S_t)\right] *$$
$$* \exp[C(t+1:T,u)(X_t + r + \lambda h_{t+1}) + A(t+1,T,u) +$$
$$+ B(t+1,T,u)(\alpha_0 + \beta_1 h_{t+1})] *$$
$$E_t[\exp(\varepsilon_{t+1}(C(t+1:T,u) + \alpha_1 B(t+1,T,u))]$$

hence, using the moment generating function of Gamma variable, we get

$$\varphi_t(u) = \exp\left[\frac{u}{T+1}\sum_{h=0}^{t-1}\ln(S_h) + \left(\frac{u}{T+1} + C(t+1:T,u)\right)\ln(S_t)\right] *$$
$$*\exp[C(t+1:T,u)(X_t + r + \lambda h_{t+1}) + A(t+1,T,u) +$$
$$+B(t+1,T,u)(\alpha_0 + \beta_1 h_{t+1})]\left(1 + \frac{\theta + \alpha_1 B(t+1,T,\theta)}{\sqrt{a}}\right)^{-ah_{t+1}}$$

Derivation of conditional m.g.f. in BM model for the Y_T variable.

For this model, the first step is given by:

$$\varphi_t(u) = \exp\left[\frac{u}{T+1}\sum_{h=0}^{t-1}\ln(S_h) + \left(\frac{u}{T+1} + C(t+1:T,u)\right)\ln(S_t)\right] *$$
$$*\exp[C(t+1:T,u)(X_t + r + \lambda h_{t+1}) + B(t+1,T,u)(\alpha_0 + \beta_1 h_{t+1}) +$$
$$+A(t+1,T,u)]E_t\left[\exp\left(\frac{(\alpha_1 B(t+1,T,u) - C(t+1:T,u))}{2\sqrt{\alpha a(1-\alpha)(b)^{(\alpha-2)/\alpha}}}Z_{t+1}\right)\right]$$

recalling that the moment generating function of Tempered Stable distribution with parameters (α, a, b) is given by:

$$E[\exp(\theta X)] = \exp\left[ab\left[1 - \left(1 - 2\theta b^{-1/\alpha}\right)^{\alpha}\right]\right]$$

we get

$$\varphi_t(u) = \exp\left[\frac{u}{T+1}\sum_{h=0}^{t-1}\ln(S_h) + \left(\frac{u}{T+1} + C(t+1:T,u)\right)\ln(S_t)\right] *$$
$$*\exp[C(t+1:T,u)(X_t + r + \lambda h_{t+1}) + B(t+1,T,u)(\alpha_0 + \beta_1 h_{t+1}) +$$
$$+A(t+1,T,u)]\exp\left[ah_{t+1}b\left[1 - \left(1 - \frac{(\alpha_1 B(t+1,T,u) - C(t+1:T,u))}{\sqrt{\alpha ba(1-\alpha)}}\right)^{\alpha}\right]\right]$$

from which the recursive system for the coeffecients in (15).

$T = 60$						
N		$K = 0.95$	$K = 0.975$	$K = 1$	$K = 1.025$	$K = 1.05$
500		0,05240	0,03251	0,01782	0,006530	0,00301
1000		0,05492	0,03430	0,01773	0,008011	0,00266
1500		0,05283	0,03364	0,01767	0,006770	0,00271
Semian. price		0,05336	0,03288	0,01733	0,007599	0,00272
$T = 90$						
N		$K = 0.95$	$K = 0.975$	$K = 1$	$K = 1.025$	$K = 1.05$
500		0,05278	0,03555	0,02209	0,01163	0,005018
1000		0,05390	0,03500	0,02024	0,01162	0,004647
1500		0,05660	0,03501	0,02147	0,01111	0,005278
Semian. price		0,05587	0,03656	0,02152	0,01125	0,005176

Tab.1: Comparison between Monte Carlo and semianalytical formula for geometric Asians options in the HN model with parameters: $\alpha_0 = 4.23 * 10^{-5}, \alpha_1 = 2.8 * 10^{-5}, \beta_1 = 4.86 * 10^{-1}, \lambda = 4.67$.

$T = 60$						
N		$K = 0.95$	$K = 0.975$	$K = 1$	$K = 1.025$	$K = 1.05$
500		0,05303	0,02847	0,00737	0	0
1000		0,05158	0,02803	0,00738	0	0
1500		0,05205	0,02826	0,00685	0	0
Semian. price		0,05244	0,02857	0,00715	4,15*10^{-7}	1,95*10^{-7}
$T = 90$						
N		$K = 0.95$	$K = 0.975$	$K = 1$	$K = 1.025$	$K = 1.05$
500		0,05347	0,03006	0,00965	0	0
1000		0,05371	0,02990	0,00972	0	0
1500		0,05429	0,03054	0,00960	2,66*10^{-8}	0
Semian. price		0,05367	0,03025	0,00954	1,33*10^{-8}	8,62*10^{-9}

Tab.2: Comparison between Monte Carlo and semianalytical formula for geometric Asians options in CHJ model with parameters: $\alpha_0 = 17.131 * 10^{-6}, \alpha_1 = 0.515 * 10^{-4}, \beta_1 = 0.017$, $\lambda = 110.648$, $\eta = 0.033, \gamma = 0.033$.

$T = 60$						
N		$K = 0.95$	$K = 0.975$	$K = 1$	$K = 1.025$	$K = 1.05$
500		0,05461	0,03226	0,013074	0,00204	6,83*10^{-6}
1000		0,05398	0,03138	0,013292	0,00194	7,43*10^{-6}
1500		0,05251	0,03004	0,012827	0,00184	9,95*10^{-6}
Semian. price		0,05319	0,03090	0,012490	0,00187	8,26*10^{-7}
$T = 90$						
N		$K = 0.95$	$K = 0.975$	$K = 1$	$K = 1.025$	$K = 1.05$
500		0,05591	0,03314	0,0147	0,00406	3,66*10^{-4}
1000		0,05688	0,03397	0,0152	0,00438	3,45*10^{-4}
1500		0,05542	0,03395	0,0161	0,00426	3,03*10^{-4}
Semian. price		0,05495	0,03350	0,0158	0,00445	3,39*10^{-4}

Tab.3: Comparison between Monte Carlo and semianalytical formula for geometric Asians options in the BM model with parameters:
$\lambda = 68.220, \alpha_0 = 25.77 * 10^{-6}, \alpha_1 = 67.051 * 10^{-4}, \beta = 0.7341 * 10^{-3}, a = 4721.8$

$T = 60$						
N		$K = 0.95$	$K = 0.975$	$K = 1$	$K = 1.025$	$K = 1.05$
500		0,05136	0,02866	0,01011	0,001801	2,58*10^{-5}
1000		0,05299	0,02908	0,01091	0,001578	9,95*10^{-6}
1500		0,05180	0,03006	0,01082	0,001750	2,57*10^{-5}
Semian. price		0,05176	0,02905	0,01090	0,001543	9,69*10^{-6}
$T = 90$						
N		$K = 0.95$	$K = 0.975$	$K = 1$	$K = 1.025$	$K = 1.05$
500		0,05237	0,03203	0,01350	0,00363	2,79*10^{-4}
1000		0,05339	0,03217	0,01429	0,00436	3,48*10^{-4}
1500		0,05295	0,03228	0,01463	0,00412	4,70*10^{-4}
Semian. price		0,05312	0,03155	0,01431	0,00393	3,89*10^{-4}

Tab.4: Comparison between Monte Carlo and semianalytical formula for geometric Asians options in the M model with parameters:
$\lambda = 147.29, \alpha_0 = 3.021 * 10^{-6}, \beta = 0.828, \alpha_1 = 7.946 * 10^{-4}$,
$k = 0.33$, $a = 12411, b = 3.599$.

$T = 15$					
N		I nth	II nth	III nth	IV nth
500		1,0009	1,005	1,0035	1,0061
1000		1,0004	1,0018	1,0037	1,0027
1500		0,99991	0,9999	1,0038	1,0047
Semian.		1,0005	1,0013	1,0024	1,0039
$T = 30$					
N		I nth	II nth	III nth	IV nth
500		1,0006	1,0004	1,003	1,0023
1000		1,0014	1,0012	1,0057	1,0121
1500		1,0004	1,0015	1,0081	1,006
Semian.		1,001	1,0027	1,0053	1,0087

Tab.5: Comparison between Monte Carlo and semianalytical procedure for the nth moments in the HN model with parameters:
$\alpha_0 = 4.23 * 10^{-5}, \alpha_1 = 2.8 * 10^{-5}, \beta_1 = 4.86 * 10^{-1}, \lambda = 4.67$.

$T = 15$					
N		I nth	II nth	III nth	IV nth
500		1,0003	0,9998	1,0001	1,0022
1000		1,0006	1,0015	1,0019	1,0033
1500		1,0003	1,0012	1,0017	1,0021
Semian.		1.0005	1,0010	1,0016	1,0022
$T = 30$					
N		I nth	II nth	III nth	IV nth
500		1,0014	1,0021	1,0054	1,0063
1000		1,0017	1,0016	1,0033	1,0064
1500		1,0009	1,0020	1,0045	1,0052
Semian.		1,001	1,0021	1,0034	1,0048

Tab.6: Comparison between Monte Carlo and semianalytical procedure for the nth moments in CHJ model with parameters:
$\alpha_0 = 17.131 * 10^{-6}, \alpha_1 = 0.515 * 10^{-4}, \beta_1 = 0.017, \lambda = 110.648, \eta = 0.033,$
$\gamma = 0.033$.

$T = 15$					
N		I nth	II nth	III nth	IV nth
500		1,0021	1,0014	1,0007	1,003
1000		1,0001	1,0023	1,0006	1,0023
1500		1,0003	1,0009	1,0028	1,0031
Semian.		1.0005	1.0011	1.0019	1.0029
$T = 30$					
N		I nth	II nth	III nth	IV nth
500		1,0009	1,0067	1,0088	1,0063
1000		1,0008	1,0037	1,0039	1,0064
1500		1,002	1,0032	1,0074	1,0067
Semian.		1,002	1,0024	1,0041	1,0063

Tab.7: Comparison between Monte Carlo and semianalytical procedure for the nth moments in the BM model with parameters:
$\lambda = 68.220, \alpha_0 = 25.77 * 10^{-6}, \alpha_1 = 67.051 * 10^{-4}, \beta = 0.7341 * 10^{-3}, a = 4721.8$

$T = 15$					
N		I nth	II nth	III nth	IV nth
500		1,0013	1,0014	1,0012	1,0026
1000		1,0001	1,0012	1,0018	1,0031
1500		1,0003	1,0011	1,0025	1,0031
Semian.		1,0005	1,0009	1,0014	1,0018
$T = 30$					
N		I nth	II nth	III nth	IV nth
500		1,0005	1,0004	1,0044	1,0055
1000		1,0008	1,0023	1,0029	1,0044
1500		1,0005	1,0015	1,0029	1,0032
Semian.		1,001	1,002	1,0029	1,0039

Tab.8: Comparison between Monte Carlo and semianalytical procedure for the nth moments in the M model with parameters:
$\lambda = 147.29, \alpha_0 = 3.021 * 10^{-6}, \beta = 0.828, \alpha_1 = 7.946 * 10^{-4}$,
$k = 0.33$, $a = 12411, b = 3.599$.

$T=15$						
N		$K=0.95$	$K=0.975$	$K=1$	$K=1.025$	$K=1.05$
500		0,05040	0,02541	0,00660	0,00103	1,20*10^{-4}
1000		0,05041	0,02605	0,00830	0,00123	8,33*10^{-5}
1500		0,05092	0,02628	0,00831	0,00124	5,38*10^{-5}
Semian. price		0,05042	0,02615	0,00767	$9,02*10^{-4}$	$3,8*10^{-5}$
$T=30$						
N		$K=0.95$	$K=0.975$	$K=1$	$K=1.025$	$K=1.05$
500		0,05268	0,02977	0,01089	0,00291	6,2924*10^{-4}
1000		0,05168	0,02752	0,01179	0,00332	6,4368*10^{-4}
1500		0,05156	0,02897	0,01150	0,00337	7,2596*10^{-4}
Semian. price		0,05121	0,02850	0,01166	0,00318	5,5477*10^{-4}

Tab.9: Comparison between Monte Carlo and approximate formula for arithmetic Asians options in the HN model with parameters: $\alpha_0 = 4.23 * 10^{-5}, \alpha_1 = 2.8 * 10^{-5}, \beta_1 = 4.86 * 10^{-1}, \lambda = 4.67$.

$T=15$						
N		$K=0.95$	$K=0.975$	$K=1$	$K=1.025$	$K=1.05$
500		$0,0505$	$0,0254$	$0,0023$	0	0
1000		$0,0504$	$0,0256$	$0,00246$	0	0
1500		$0,0507$	$0,0260$	$0,00243$	0	0
Semian. price		$0,0504$	$0,0263$	$0,00350$	$9,45*10^{-4}$	$7,12*10^{-9}$
$T=30$						
N		$K=0.95$	$K=0.975$	$K=1$	$K=1.025$	$K=1.05$
500		$0,0513$	$0,0267$	$0,00430$	0	0
1000		$0,0513$	$0,0263$	$0,00435$	0	0
1500		$0,0514$	$0,0266$	$0,00421$	0	0
Semian. price		$0,0509$	$0,0304$	$0,00540$	$0,00261$	$4,44*10^{-5}$

Tab.10: Comparison between Monte Carlo and approximate formula for arithmetic Asians options in CHJ model with parameters: $\alpha_0 = 17.131 * 10^{-6}, \alpha_1 = 0.515 * 10^{-4}, \beta_1 = 0.017, \lambda = 110.648$, $\eta = 0.033$, $\gamma = 0.033$.

$T = 15$						
N		$K = 0.95$	$K = 0.975$	$K = 1$	$K = 1.025$	$K = 1.05$
500		0,0512	0,0264	0,00543	1,01*10^{-6}	0
1000		0,0501	0,0270	0,00529	7,32*10^{-6}	0
1500		0,0504	0,0264	0,00520	4,07*10^{-6}	0
Semian. price		0,0504	0,0274	0,00330	3,99*10^{-4}	2,67*10^{-5}
$T = 30$						
N		$K = 0.95$	$K = 0.975$	$K = 1$	$K = 1.025$	$K = 1.05$
500		0,0503	0,0274	0,00838	2,59*10^{-5}	0
1000		0,0508	0,0279	0,00795	8,14*10^{-5}	0
1500		0,0520	0,0282	0,00833	3,17*10^{-5}	0
Semian. price		0,0516	0,0289	0,00660	1,33*10^{-4}	2,08*10^{-4}

Tab.11: Comparison between Monte Carlo and approximate formula for arithmetic Asians options in the BM model with parameters:
$\lambda = 68.220, \alpha_0 = 25.77 * 10^{-6}, \alpha_1 = 67.051 * 10^{-4}, \beta = 0.7341 * 10^{-3}, a = 4721.8$

$T = 15$						
N		$K = 0.95$	$K = 0.975$	$K = 1$	$K = 1.025$	$K = 1.05$
500		0,05304	0,02670	0,00362	0	0
1000		0,05220	0,02539	0,00411	0	0
1500		0,05210	0,02664	0,00451	0	0
Semian. price		0,05040	0,02542	4,66*10^{-4}	0	0
$T = 30$						
N		$K = 0.95$	$K = 0.975$	$K = 1$	$K = 1.025$	$K = 1.05$
500		0,05130	0,02762	0,00729	0	0
1000		0,05308	0,02787	0,00811	5,18*10^{-5}	0
1500		0,05102	0,02877	0,00599	2,32*10^{-5}	0
Semian. price		0,05086	0,02592	9,76*10^{-4}	0	0

Tab.12: Comparison between Monte Carlo and approximate formula for arithmetic Asians options in the M model with parameters:
$\lambda = 147.29, \alpha_0 = 3.021 * 10^{-6}, \beta = 0.828, \alpha_1 = 7.946 * 10^{-4}$,
$k = 0.33$, $a = 12411, b = 3.599$.