BASIC INTUITIONISTIC PRINCIPLES IN FUZZY SET THEORIES AND ITS EXTENSIONS
(A TERMINOLOGICAL DEBATE ON ATANASSOV IFS)

GIANPIERO CATTANEO AND DAVIDE CIUCCI

Abstract. In this paper we contribute to the terminological debate about Atanassov's use of the term "Intuitionistic" in defining his structure based on orthopairs of fuzzy sets.

In particular we stress that it is defined as "intuitionistic" a negation which from one side does not satisfy a standard property of the intuitionistic Brouwer negation (contradiction law) and on the contrary asserts some principles rejected by intuitionism (strong double negation law and one of the de Morgan laws).

An algebraic Brouwer negation is studied in the context of IFS showing that it can be induced from a Heyting implication. A similar situation occurs in the case of standard Fuzzy Sets (FS). Some conditions which allow one to distinguish from the algebraic point of view FS from IFS are treated.

Finally, a particular subclass of IFS consisting of orthopairs of crisp sets (denoted by ICS) is studied, showing that shadowed sets can be algebraically identified with ICSs.

1. Introduction

The mathematical structure introduced by Atanassov in [6, 1, 2] on the basis of ortho–pairs of fuzzy sets, and called "intuitionistic fuzzy sets", recently raised a terminological debate (see [12, 13]) based on a first comment appeared several years before in [19, p.183]. Successively an explicit discussion about this terminological controversy as been published in [25], with the consequent answers in [30, 5].

The main point of discussion is that in the above quoted seminal Atanassov's papers it is constructed a particular distributive lattice equipped with a complementation $\neg p$ which does not satisfy the algebraic version of excluded middle law $\forall p : p \vee \neg p = 1$. Since this law is not accepted by intuitionistic logic, Atanassov claimed that the structure can be characterized as intuitionistic. This is explicitly asserted in the Atanassov's first widely accessible paper of 1986:

> "The definition makes clear that for the so constructed new type of fuzzy sets [i.e., ortho–pairs of standard fuzzy sets] the logical law of excluded middle is not valid, similarly to the case in intuitionistic mathematics." [1].

The crucial point with respect to this claim is that the Atanassov's complementation satisfies the algebraic version of the "strong" double negation law ($\forall p : \neg\neg p = p$, algebraically called property of involution), which is rejected by intuitionism (precisely, it is rejected that $\forall p : \neg\neg p \leq p$), and does not satisfy the contradiction law ($\forall p : p \wedge \neg p = 0$) which on the contrary is assumed to hold in intuitionistic logic. This has been recognized and stressed by one of us in a paper of 1989:

> "The remark we have to do to [Atanassov's] claim is that the only form of complementation considered by Atanassov is (...) involutive, and this law is not accepted by the intuitionistic mathematics, whereas the law of contradiction is not valid too, and this is accepted in the intuitionistic mathematics." [19].

Our point of view about this debate is that it is not at all correct to assume a term (precisely, intuitionistic), very articulated in its assumptions, from the fact that only some of its asserted principles are satisfied. It is as if in the ancient Greek mathematical tradition, after the widely accepted definition of a circle as the plane figure whose points have the same distance from a fixed point called center, someone asserts that a square is a circle from the only fact that its vertices have the same distance from the square center.

After the criticism presented in [25] on the Atanassov's adoption of the term "intuitionistic" in his original algebraic structures, a defense of his position is given by Atanassov in [5] by an "a posteriori" argument. Indeed, in [25] besides the above discussed argument about the use of "intuitionistic" term attributed to an algebraic negation which does not satisfy the great part of the accepted principles of intuitionistic logic, it is stressed that in a paper of 1984 [56] the term "intuitionistic fuzzy set theory by Takeuti and Titani is an absolute legitimate approach, in the scope of intuitionistic logic, but it has nothing to do with Atanassov's intuitionistic fuzzy sets." Denoted by A-IFS the abbreviation of "intuitionistic fuzzy sets" in Atanassov's approach and by T-IFS the abbreviation of the Takeuti and Titani use of the same term, it is argued by Atanassov that:

> "As far as publication statistics are concerned, I must note that, according to Science Citation Index, the ratio of citations of papers on A–IFS to those of T–IFS is greatly in favour of the former. So far, data is available about more than 800 papers of A–IFS by more than 150 authors from more than 30 countries. More than 25 Ph.D. or higher theses have been defended or are in preparation in at least 10 countries.
> The change of name would lead to terminological chaos, having in mind that there are other notions close to A–IFS." [5].

In this argumentation it is possible to distinguish two points: (1) the fact that the number of papers published on the A–IFS context is greater than the number of the T–IFS published ones; (2) the terminological chaos subsequent to any change of name in front of this large number of papers dedicated to A–IFS.

As to the first point, the different number of publications has nothing to do with the fact that (as we show later) the A–IFS structure based on the standard operations listed for instance on the book [4] is not a model of intuitionistic logic, and so the term intuitionistic is totally incorrect, contrary to the fact that T–IFS is an extension of Gentzen's LJ axiomatization of intuitionistic logic (plus 6 "extra" axioms), with associated algebraic semantic. In other words, whether the number of persons which work on squares, unfortunately called circles by members of a peculiar community, is numerically relevant has nothing to do with the fact that squares are squares and circles are circles, of course according to the standard tradition. At the most, this large number of papers can be considered as a good index of the interest and relevance in studying this kind of structures both from the theoretical and the applicative point of views. On the other hand, if one makes a count of the number of papers published about intuitionism, or simply to the semantic of intuitionism, it is possible to conjecture that it is enormously greater than the number of papers about A–IFS, with of course the correct use of the term intuitionism.

As to the second point, we think that a new terminology preserving the widely used acronym IFS in the Atanassov's approach would be a correct compromise avoiding the "terminological chaos" previously suggested. In other words we agree with [30] when they claim that

> "the term "intuitionistic fuzzy sets" is very unfortunate because the structure suggested by Atanassov has nothing in common with intuitionistic mathematics and logic.
> We also agree that – although many theoretical and applied paper devoted to intuitionistic fuzzy sets have appeared – this name should be changed since it is so misleading and generates superfluous polemics.
> However, taking into account the numerous papers published under this unsuitable name, we suggest that looking for the appropriate name for Atanassov's sets would be desirable as maintaining the acronym IFS."

Personally, we prefer the name of *ortho–pair* of fuzzy sets (OFS or OPFS) to denote Atanassov's pairs $\langle f, g\rangle$ of ordinary fuzzy sets $f, g \in [0,1]^X$ under the "orthogonality" relation:

$$f \leq 1 - g \quad \text{(or equivalently } \; g \leq 1 - f)$$

In this way we add another terminology to the one listed in [30]. On the other hand, for the sake of continuity with a well consolidated tradition, in this paper we will use the acronym IFS leaving the freedom of interpreting the letter "I" as any possible term different from "Intuitionistic" (for instance, following [30], incomplete, inaccurate, imperfect, indefinite, indeterminate, indistinct and so on). Another proposal could be the one of "Iper Fuzzy Sets (IFS)". where the term "Iper" (Italian version of the English "Hyper") means that we are considering not a single fuzzy set, but (precisely orthogonal) pairs of such fuzzy sets.

Finally, for the relationship between IFSs and interval-valued fuzzy sets see the section 3 of [25].

2. The role of negation in intuitionistic logic

As pointed out in the introduction, the main argument of discussion is about the term "Intuitionistic" imposed by Atanassov to his structure on the basis of an alleged intuitionistic behavior of the involved negation. The fact that the negation plays a fundamental role in intuitionistic logic is stressed by Heyting in [33, p.99]: "the main differences between classical and intuitionistic logics are in the properties of the negation." For this reason in this section we summarize the main points of the intuitionistic negation making reference to the just quoted Heyting book.

(NI-1) "In the theory of negation the *principle of excluded middle* fails. $p \vee \sim p$ demands a general method to solve every problem, or more explicitly, a general method which for any proposition p yields by specialization either a proof of p or a proof of $\neg p$.
As we do not possess such a method of construction, we have no right to assert the principle.

(NI-2) Another form of the principle is $\sim\sim p \to p$. We have met many examples of propositions for which this fails. [...]

(NI-3) Of course, the [...] formula $\sim q \to \sim p \cdot \to \cdot p \to q$ is not assertible. [...]

(NI-4) $\sim (p \wedge q) \to \sim p \vee \sim q$ cannot be asserted." [33, p.100].

If this is the "negative" part of the intuitionistic negation (in particular, the rejection of classical principles of excluded middle NI-1 and of a strong behavior of the double negation law NI-2), let us now list from the same book the accepted principles involving negation, once marked (according to Heyting) asserted formulas with symbol $\vdash$:

(I-1) "However,

$$\vdash p \to \sim\sim p \tag{1}$$

It is clear that from p it follows that it is impossible that p is impossible.

(I-2) Another important formula is

$$\vdash p \to q \cdot \to \cdot \sim q \to \sim p \tag{2}$$

[compare with the above point (NI-3), our comment].

(I-3) By substitution in (1) we find

$$\vdash \sim p \to \sim\sim\sim p \tag{3a}$$

If we substitute $\sim\sim p$ for q in (2), we find, using (1),

$$\vdash \sim\sim\sim p \to \sim p \tag{3b}$$

(3a) and (3b) show that we need never consider more than two consecutive negations.

(I-4) From $\vdash p \to p \vee q$ follows, by (2), $\vdash \sim (p \vee q) \to \sim p$; in the same way we have $\vdash \sim (p \vee q) \to \sim q$, so

$$\vdash \sim (p \vee q) \to \sim p \wedge \sim q \tag{4a}$$

The inverse formula is easily seen to be also true:

$$\vdash \sim p \wedge \sim q \to \sim (p \vee q) \tag{4b}$$

(4a) and (4b) form one of the de Morgan's equivalence.

(I-5) The other one is only half true:

$$\vdash \sim p \vee \sim q \to \sim (p \wedge q) \tag{5}$$

[compare with the above point (NI-4)].

(I-6)

$$\vdash \sim\sim (p \vee \sim p) \tag{6}$$

For $\sim (p \vee \sim p)$ would imply, by (4a), $\sim p \wedge \sim\sim p$, which is a contradiction." [33, p.100–101].

We can summarize the main points of the above discussion quoting from [27]:

> "In Heyting's logic the operations of *addition* [i.e., disjunction], *multiplication* [i.e., conjunction], *negation*, and *implication* are taken as fundamental. Since the law of double negation does not hold, it happens that these four operations are independent; no one of them is definable in terms of the other three. [...].
> Heyting's sum, product, and negation satisfy all Boolean laws [...] except the laws *of double negation* and *excluded middle*. In particular the law of *contradiction* holds."

3. The algebraic approach to intuitionistic negation

Let us translate this Heyting introduction to intuitionistic negation according to the algebraic semantic of logic, in order to better understand the uncorrect use of the term "Intuitionistic" assigned by Atanassov to his negation. First of all, let us recall that "when interpreting [complemented lattice] structures from the viewpoint of logic, it is customary to interpret the lattice elements $[a, b, \ldots]$ as *propositions*, the meet operation $[\wedge]$ as *conjunction* [and], the join operation $[\vee]$ as *disjunction* [or], and the orthocomplement $[\neg]$ as *negation* [not]." [32]. In this lattice context "it is generally agreed that the partial ordering $\leq$ of a lattice [...] can be logically interpreted as a *relation* of implication, more specifically *semantic entailment*." [31]. To be precise,

- the partial order relation of "implication" $a \leq b$ between lattice elements a and b corresponds to semantic entailment relation between formulas A and B of the object language; to be precise, it is the algebraic counterpart of the statement "$A \supset B$ is true" (or "asserted" in Heyting terminology) with respect to some implication connective $\supset$ involving formulas.

From this point of view, the *implication relation* must not be confused with an implication or conditional *operation*, i.e., a logical *connective* which like conjunction and disjunction, forms propositions out of propositions remaining at the same linguistic level. The statement "*A semantically entails B*" is not a formula in the object language, but occurs in the *metalanguage*, whereas on the contrary the statement "*A implies B*" is really a formula of the *language*.

Thus, it is of a certain interest the introduction of a binary operation $\rightarrow$ on the lattice structure playing the role of algebraic counterpart of the implication connective $\supset$ of the language, and which assigns to each pair of lattice elements a, b another lattice element $a \rightarrow b$. However,

> "since not just any binary lattice operation should qualify as a material implication, we must determine what criteria must (should) be satisfied by a lattice operation in order to be regarded as a material implication. First, it seems plausible to require that every implication operation $\rightarrow$ be related to the implication relation $(\leq)$ in such a way that if a proposition a implies (is [less or equal to]) a proposition b, then the conditional proposition $a \rightarrow b$ is universally true, and conversely. [...] Translating this into the general lattice context, we obtain $a \rightarrow b = 1$ iff $a \leq b$. Here 1 is the lattice unit element, which corresponds to the *universally true proposition*" [32].

In this just quoted paper, this condition is assumed as one of the *minimal implicative conditions*. Let us stress that if according to Birkhoff the lattice identity $\alpha = 1$ ('α is true') will be denoted by $\vdash \alpha$ [10, p.281], then we can reformulate the minimal implicative condition as follows:

$$\vdash a \rightarrow b \quad \text{iff} \quad a \rightarrow b = 1 \quad \text{iff} \quad a \leq b. \tag{7}$$

On the basis of these considerations, the intuitionistic negation in the lattice–algebraic approach is a unary mapping $\sim : \mathcal{L} \mapsto \mathcal{L}$ on a distributive lattice $\langle \mathcal{L}, \wedge, \vee, 0, 1 \rangle$ bounded by the least element 0 and the greatest element 1 with respect to the partial order relation $\leq$ of semantic entailment induced by the lattice operations. Thus, according to (7), the above points (I-1), (I-2), and (I-6) can be algebraically realized respectively by the laws:

$$(B-1) \qquad \forall a \in \mathcal{L}: \ a \leq \sim\sim a \qquad \text{(weak double negation)}$$

$$(B-2) \qquad \forall a, b \in \mathcal{L}: \ a \leq b \quad \text{implies} \sim b \leq \sim a \qquad \text{(contraposition)}$$

$$(B-3) \qquad \forall a \in \mathcal{L}: \sim\sim (a \vee \sim a) = 1 \qquad \text{(contradiction)}$$

Moreover, even if it is not explicitely formulated in the above discussed Heyting presentation of intuitionistic negation, we add the following condition:

$$(B-4) \qquad \sim 1 = 0 \qquad \text{(coherence)}$$

In this way, the structure $\langle \mathcal{L}, \wedge, \vee, \sim, 0, 1 \rangle$ can be called a *distributive lattice with Brouwer complementation*, or *Brouwer–complemented lattice* for short. Since it will be useful in the sequel to consider lattice structures with a complementation which satisfies only conditions (B-1) and (B-2), following [26] a structure of this kind is called a *distributive lattice with minimal complementation*, or shortly *min–complemented lattice*.

In the following example it is shown that the coherence condition (B–4) is independent from the other ones.

Example 3.1. Let the distributive (totally ordered) lattice based on three distinct elements $\{0, h, 1\}$ be defined by the Hasse diagram of figure 1.

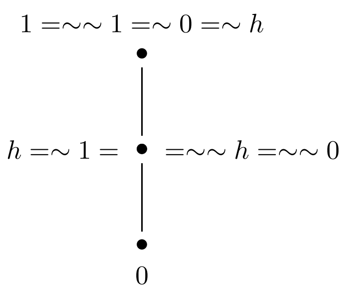


FIGURE 1. An example of a lattice in which $\sim 1 \neq 0$.

Trivially, $0 \leq h = \sim\sim 0$, $h = \sim\sim h$, $1 = \sim\sim 1$; moreover, $0 \leq h \leq 1$ implies $\sim 1 \leq \sim h = \sim 0$. Lastly, the following table shows that (B–3) holds:

x	$\sim x$	$x \vee \sim x$	$\sim\sim (x \vee \sim x)$
0	1	1	1
h	1	1	1
1	h	1	1

but $\sim 1 \neq 0$. ■

Let us now show that condition (B-3) is independent from the other ones.

Example 3.2. Let us consider the lattice of figure 2.

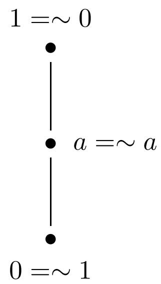


FIGURE 2. An example of a lattice in which $\sim\sim (a \vee \sim a) \neq 1$.

In this lattice conditions (B-1), (B-2) and (B-4) are satisfied, but we have that $\sim\sim (a \vee \sim a) = a \neq 1$. ■

These independencies being stated, let us prove that all the other intuitionistic principles about negation (I-3), (I-4), and (I-5) can be derived from conditions (B-1)–(B-4) of Brouwer–complemented lattices.

Proposition 3.3. *Under condition (B-1), the contraposition law (B-2) is equivalent to the following de Morgan law:*

$$\text{(dM1)} \qquad \sim (a \vee b) = \sim a \wedge \sim b \qquad \text{(intuitionistic de Morgan)}$$

which is the algebraic translation of (4a) *and* (4b) *of above point (I-4).*

Proof. First of all let us assume that the contraposition is true. From $a, b \leq a \vee b$, by contraposition, $\sim (a \vee b) \leq \sim a, \sim b$, i.e., $\sim (a \vee b)$ is a lower bound of the pair $\sim a, \sim b$. Now, let c be any lower bound of this pair, $c \leq \sim a, \sim b$, then by contraposition and (B1) $a, b \leq \sim\sim a, \sim\sim b \leq \sim c$ from which it follows that $a \vee b \leq \sim c$ and by contraposition and (B1) $c \leq \sim\sim c \leq \sim (a \vee b)$, i.e., this latter is the greatest lower bound of the pair $\sim a, \sim b$.

Vice versa, let $\sim (a \vee b) = \sim a \wedge \sim b$ be true, then if $a \leq b$ we have that $\sim b = \sim (a \vee b) = \sim a \wedge \sim b \leq \sim a$. □

Proposition 3.4. *In any min–complemented lattice the following holds.*

$$\forall a \in \mathcal{L}: \quad \sim\sim\sim a = \sim a \tag{8}$$

algebraic version of (3a) *and* (3b) *of point (I-3) which allows one to assert that "in Heyting's logic we have a law of triple negation"* [27]*. Moreover one has that*

$$\forall a, b \in \mathcal{L}: \quad \sim a \vee \sim b \leq \sim (a \wedge b) \tag{9}$$

algebraic version of the other de Morgan law of point (I-5).

Proof. Indeed, since (B-1) is true for any element of Σ, if we apply it to the element $\sim a$ we obtain $\sim a \leq \sim\sim\sim a$; on the other hand, applying the contraposition law of the Brouwer complementation to (B-1) we obtain $\sim\sim\sim a \leq \sim a$.

Applying contraposition to $a \wedge b \leq a, b$ we get $\sim a, \sim b \leq \sim (a \wedge b)$ from which it follows that $\sim a \vee \sim b \leq \sim (a \wedge b)$. □

Example 3.5. In the Brouwer–complemented lattice whose Hasse diagram is depicted in figure 3 it is not true that the dual de Morgan law with respect to (dM1) holds.

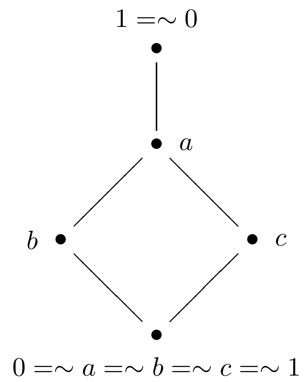


FIGURE 3. Hasse diagram relative to a Brouwer–complemented lattice for which $\sim (b \wedge c) \neq \sim b \vee \sim c$.

■

Finally, let us prove that the contradiction law expressed in the above form (B-3) is equivalent to the standard considered one. Before this proof we must note that in every min–complemented lattice one has that $\sim 0 = 1$. Indeed, this result follows from the fact that condition $0 \leq \sim 1$, by contraposition and weak double negation laws, leads to $1 \leq \sim\sim 1 \leq \sim 0$, i.e., $1 = \sim 0$.

Proposition 3.6. *In any min–complemented lattice satisfying (B-4), condition (B-3) is equivalent to the condition:*

$$\text{(B-3a)} \qquad \forall a \in \mathcal{L}: \; a \wedge \sim a = 0 \qquad \textit{(contradiction)}$$

Proof. If for every element a of the lattice one has that $a \wedge \sim a = 0$, then applying this law to the element $\sim a$ one obtains $\sim a \wedge \sim\sim a = 0$, from which, by (dM1), it follows that $\sim (a \vee \sim a) = 0$, and so $\sim\sim (a \vee \sim a) = \sim 0 = 1$.

Let (B-3) be true. Then $\sim a \wedge \sim\sim a =$ (by (dM1)) $\sim (a \vee \sim a) =$ (by (8)) $\sim (\sim\sim (a \vee \sim a)) =$ (by (B-3)) $\sim 1 =$ (by (B-4)) 0. But from (B-1) it follows that $\sim a \wedge a \leq \sim a \wedge \sim\sim a = 0$. □

3.1. **The peculiar theoretical status of (NI-4).** Let us now investigate the role of the negative principles of intuitionistic negation, characterized by the above points (NI-1)–(NI-4), showing that in some sense (NI-1)–(NI-3) play a "stronger" role with respect to (NI-4). First of all, it is trivial to verify that any Boolean lattice is a Brouwer–complemented lattice too, and so it is interesting to characterize Boolean lattices inside Brouwer–complemented lattices.

Proposition 3.7. *Let $\langle \mathcal{L}, \wedge, \vee, \sim, 0, 1 \rangle$ be a Brouwer–complemented lattice. Then, the following are equivalent.*

(1) $\mathcal{L}$ is a Boolean lattice.
(2) $\forall a \in \mathcal{L}$, $a \vee \sim a = 1$ (excluded middle, algebraic version of (NI-1)*).*
(3) $\forall a \in \mathcal{L}$, $a = \sim\sim a$ (strong double negation, algebraic version of (NI-2)*).*
(4) $\forall a, b \in \mathcal{L}$, $\sim a \leq \sim b$ implies $b \leq a$ (dual contraposition, algebraic version of (NI-3)*).*

Proof. Trivially, in any Boolean lattice conditions (2) and (3) both hold for definition, and it is easy to prove that also condition (4) is true.

If $1 = a \vee \sim a$, then $\sim\sim a = \sim\sim a \wedge (a \vee \sim a) = (\sim\sim a \wedge a) \vee (\sim (\sim a) \wedge (\sim a)) =$ (by (B-1) and (B-3a)) a, i.e., the (3).

If the (3) holds, then $\sim (a \vee \sim a) =$ (by (dM1)) $(\sim a) \wedge \sim (\sim a) =$ (by (B-3a)) 0, from which it follows by the hypothesis and (B-4) $a \vee \sim a = \sim\sim (a \vee \sim a) = \sim 0 = 1$. Moreover, (dM1) in the particular case of the elements $\sim a$ and $\sim b$ leads to $\sim\sim a \wedge \sim\sim b = \sim (\sim a \vee \sim b)$, and making use of the (3) this result can be written as $a \wedge b = \sim (\sim a \vee \sim b)$, and so $\sim (a \wedge b) = \sim\sim (\sim a \vee \sim b) =$ (by (3)) $\sim a \vee \sim b$.

In this way we have proved the equivalence among points (1), (2) and (3). Now, from the (8), in the form $\sim a \leq \sim (\sim\sim a)$, applying the hypothesis (4) it follows $\sim\sim a \leq a$. □

On the basis of these results, it is interesting to single out those Brouwer–complemented lattices which are not Boolean, the so–called *genuine* Brouwer–complemented lattices, since they correspond to real (i.e., non Boolean) algebraic models of the intuitionistic negation. A sufficient criterium, which is satisfied in all the algebraic structures considered in this paper, consists in admitting at least one element $n \neq 1$ such that $\sim n = 0$, called *half element*. Indeed, from this condition it follows that $n \leq \sim\sim n = 1$ with $n \neq \sim\sim n$; moreover $n \vee \sim n = n \vee 0 = n \neq 1$ and so the excluded middle law does not hold and finally $\sim n \leq \sim 1$ (i.e., $0 \leq 0$) does not imply $1 \leq n$ and so the dual contraposition law does not hold. Note that in example 3.5 a, b, c are all half elements.

A peculiar comment can be done about the negative principle (N-4) whose algebraic description consists in the following *de Morgan property* dual with respect to (dM1):

$$\text{(dM2)} \qquad \sim (a \wedge b) = \sim a \vee \sim b$$

Indeed, from the Heyting point of view expressed in [33], property (dM2) "cannot be asserted" by the intuitionistic negation, but it is possible to find *genuine* Brouwer–complemented lattice structures in which (N-4) holds.

Example 3.8. In any totally ordered lattice $\mathcal{L}$ with at least three elements and bounded by 0 and 1, let us define the Brouwer genuine complementation

$$\sim a := \begin{cases} 1 & \text{if } a = 0 \\ 0 & \text{otherwise} \end{cases}$$

Let us consider two generic elements x and y supposing, without any lost in generality, that $x \leq y$. Then, from (B-2) it follows that $\sim y \leq \sim x$ and so $\sim (x \wedge y) = \sim x = \sim x \vee \sim y$. ■

Let us note that the simplest extension of classical two–valued logic toward many–valued ones consists in the introduction of a third "intermediate", or "neutral" or "indeterminate" value. Łukasiewicz developed this idea in [37] (English version: On three–valued logic, in [11, p. 87]). In such a paper he introduced a third truth value to take into account propositions which are neither true nor false, defining in this way a three–valued logic. This logic was then extended to deal with d truth values as well as with an infinite number of truth values, in particular the $\aleph_0$ and $\aleph_1$ cardinalities. For instance, one deals with the following sets of truth values, treated as numerical sets equipped with the standard total order relation induced by $\mathbb{R}$:

- $L_d = \left\{0, \frac{1}{d-1}, \frac{2}{d-1}, \ldots, \frac{d-2}{d-1}, 1\right\}$, with $d \geq 2$, for d–valued logics;
- $L_{\aleph_0} = [0,1] \cap \mathbb{Q}$, that is the set of rational values in the interval $[0,1]$, for infinite–valued logics with $\aleph_0$ truth values;
- $L_{\aleph_1} = [0,1]$, that is the set of real values in the interval $[0,1]$, for infinite–valued logics with $\aleph_1$ truth values.

The numbers of L_α, $\alpha \in \{d, \aleph_0, \aleph_1\}$, are interpreted after Łukasiewicz as the possible truth values which logical sentences can be assigned to. As usually done in literature, the values 1 and 0 denote respectively truth and falseness, whereas all the other values are used to indicate different degrees of indefiniteness.

Therefore, example 3.5 shows that condition (NI-4) is independent from the other three (NI-1)–(NI-3) and example 3.8 shows a whole class of genuine Brouwer complemented lattices in which condition (NI-4) can be verified. In other words, the theoretical status of (N-4) is different from the one of (NI-1)–(NI-3) since the algebraic version of condition (NI-4) applied to a Brouwer–complemented lattice does not necessary collapse the structure in the one of Boolean lattice (classical logic).

This says that there is a structure which is stronger than the one of Brouwer–complemented lattice. This structure, also called *(dM) Brouwer–complemented lattice*,

(i) is an algebraic model of a logic which satisfies all the principles (I-1)–(I-5) asserted by the intuitionistic negation,
(ii) moreover under the existence of at least one half element is a genuine (i.e., non Boolean) model of the intuitionistic negation since it refuses conditions (NI-1)–(NI-3),
(iii) but it does assert principle (NI-4) which "cannot be asserted" in the intuitionistic version of negation discussed by Heyting in his book [33].

This put the interesting question whether one can consider (NI-4), or its algebraic version (dM2), as an accepted principle of some intuitionistic logic. To this purpose, let us quote the following Heyting statement:

> "The word 'logic' has many different meanings. I shall not try to give a definition of intuitionistic logic [...]. Here I shall only call your attention to some formulas which express interesting methods of reasoning and show why these methods are intuitively clear within the realm of intuitionistic mathematics" [33, p.96].

As to this argument a stronger comment can be found in Rasiowa and Sikorski:

> "it is difficult for mathematicians to understand exactly the idea of intuitionists since the degree of precision in the formulation of intuitionistic ideas is far from the degree of precision to which mathematicians are accustomed in their daily work.
>
> The subject of our studies will not be intuitionism itself but intuitionistic logic (the formalization of intuitionistic logic is due to A. Heyting), which is a sort of reflection of intuitionistic ideas formulated in formalized deductive systems. A precise definition of intuitionistic logic offers no difficulty." [53, p.378–379].

Owing to this "imprecision" in the formulation of intuitionistic ideas it is possible to assume some degrees of freedom and consider the following two meta–possibilities:

(M-1) Brouwer complemented lattices without condition (dM2) are algebraic models of a particular version of intuitionistic negation, for instance the one supported by Heyting in [33].
(M-2) Brouwer complemented lattices with condition (dM2) are particular models of another version of intuitionistic negation, the one supported by Gödel in the specific context of the many–valued approach to intuitionism (see for instance [54, p.45], with the very interesting comments about the relationship between intuitionistic propositional calculus as proposed by Heyting in [33] and the Gödel approach to many–valued systems).

An interesting question is whether total ordering characterizes genuine Brouwer–complemented lattice with condition (dM2). The answer is negative, since there exist genuine Brouwer–complemented lattices satisfying (dM2) and which are not totally ordered. An example is given in figure 4.

4. The pseudo–Boolean lattice approach to intuitionistic logic: Heyting algebras

Up to now we investigated the algebraic approach to intuitionistic logic from the point of view of the negation connective only: Brouwer–complemented distributive lattice is a good algebraic axiomatization of a logic with intuitionistic negation. This according to the Heyting claim that the main difference between classical logic and intuitionistic one consists just in the very different behavior of negation, which in the intuitionistic context refuse the validity of the double negation law in its strong version, the excluded middle principle, and the dual of the contraposition law (see proposition 3.7). Anyway, in order to have a real algebraic semantic of intuitionistic logic it is necessary to introduce a lattice structure $\langle \mathcal{L}, \wedge, \vee, 0, 1 \rangle$ equipped by a primitive binary operation, denoted by $\rightarrow: \mathcal{L} \times \mathcal{L} \mapsto \mathcal{L}$, which plays the algebraic role of *implication* connective. To this

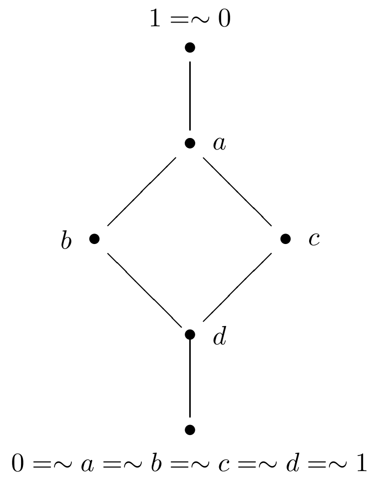


FIGURE 4. Hasse diagram relative to a (dM2) Brouwer–complemented lattice which is not totally ordered.

extent the more suitable algebraic structure is the so–called (by Birkhoff in [10, p.45]) *Brouwerian lattice*. As to this terminology there is some confusion in literature since, for instance, Rasiowa and Sikorski call this algebraic structure as *relatively pseudo–complemented lattice*, whereas "every relatively pseudo–complemented lattice with zero element is called a *pseudo–Boolean algebra*" [53, p.59]. Moreover in a note of the same page they underline that the corresponding dual lattices are called *Brouwer algebras* by McKinsey and Tarski in [39].

Following [53] we present now the equationally complete version of an abstract pseudo–Boolean lattice.

Definition 4.1. An abstract algebra $\langle \mathcal{A}, \wedge, \vee, \to, \sim \rangle$ with three binary operations $\wedge, \vee, \to$ and a unary operation $\sim$ is a *pseudo–Boolean lattice* iff the following hold:

(1) The substructure $\langle \mathcal{A}, \wedge, \vee \rangle$ is a lattice.
(2) The lattice operations $\wedge, \vee$ and the unary operation $\sim$ are linked to the implication connective by the following axioms:

$$
\begin{array}{ll}
(l_1) & a \wedge (a \to b) = a \wedge b \\
(l_2) & (a \to b) \wedge b = b \\
(l_3) & (a \to b) \wedge (a \to c) = a \to (b \wedge c) \\
(l_4) & (a \to a) \wedge b = b \\
(l_5) & \sim (a \to a) \vee b = b \\
(l_6) & a \to (\sim (a \to a)) = \sim a
\end{array}
$$

In particular, (l_5) asserts that for every element b one has that $\sim (a \to a) \leq b$ whatever be a, i.e., $\sim (a \to a)$ is the *least* element 0 of the lattice; on the other hand, (l_4) means that $b \leq (a \to a)$ whatever be a, i.e., $a \to a$ is the greatest element 1 of the lattice. Moreover, from (l_6) it follows that $\sim a = a \to 0$ for every a.

A simpler set of axioms for pseudo–Boolean algebras has been introduced by Monteiro in [43], where this structure is called *Brouwer algebra* and in which it is proved the equivalence with pseudo–Boolean lattices together with the independence of the introduced axioms. In a successive paper [44] the same structure is called *Heyting algebra*. In order to stress the semantical relevance of these structures with respect to intuitionistic logic, we adopt here the Monteiro's terminology of Heyting algebra instead of pseudo–Boolean lattice.

Definition 4.2. A *Heyting algebra* is a structure $\langle \mathcal{A}, \wedge, \vee, \to, 0 \rangle$ which satisfies the following axioms:

(H1) $a \to a = b \to b$
(H2) $(a \to b) \wedge b = b$
(H3) $a \to (b \wedge c) = (a \to c) \wedge (a \to b)$
(H4) $a \wedge (a \to b) = a \wedge b$
(H5) $(a \vee b) \to c = (a \to c) \wedge (b \to c)$
(H6) $0 \wedge a = 0$

A non–equational but more compact way to introduce Heyting algebras (pseudo–Boolean lattices) is given by the following result.

Proposition 4.3. *Let $\langle \mathcal{A}, \wedge, \vee, 0\rangle$ be a lattice with least element 0. Then, $\mathcal{A}$ is a Heyting algebra iff for any pair of elements $a, b \in \mathcal{A}$ there exists an element $a \to b \in \mathcal{A}$, called the* pseudo–complement *of a* relative *to b, such that,*

$$(I) \qquad a \wedge x \leq b \quad \textit{if and only if} \quad x \leq a \to b$$

i.e., it satisfies the pseudo–Boolean lattice condition that for every $a \in \mathcal{A}$ the set of all $x \in \mathcal{A}$ such that $a \wedge x \leq b$ contains the greatest element, denoted by $a \to b = max\{x \in \mathcal{A} : a \wedge x \leq b\}$.

Let us note that condition (I) is equivalent to the two conditions:

$$(I_1) \qquad a \wedge x \leq b \quad \text{implies} \quad x \leq (a \to b)$$
$$(C_2) \qquad a \wedge (a \to b) \leq b$$

Moreover, once introduced the lattice greatest element $1 = 0 \to 0$, the (I) applied to the case $x = 1$ leads to the minimal implicative condition (7):

$$(10) \qquad a \leq b \quad \text{if and only if} \quad a \to b = 1$$

Note that in definition 4.1 and proposition 4.3 there is no mention to the distributivity of the involved lattice. This omission is a consequence of the following result, whose proof in the context of pseudo–Boolean lattices can be found for instance in [10, p. 45], [53, p. 59].

Proposition 4.4. *Every Heyting algebra (pseudo–Boolean lattice) is necessarily distributive.*

The main result about Heyting algebras with respect to the Brouwer–complementation is the following one.

Proposition 4.5. *Let $\langle \mathcal{A}, \wedge, \vee, \to, 0\rangle$ be a Heyting algebra according to definition 4.2 and let the negation of any $a \in \mathcal{A}$ be defined as $\sim a := a \to 0$ with the further definition $1 :=\sim 0 = 0 \to 0$.*

Then the algebraic structure $\langle \mathcal{A}, \wedge, \vee, \sim, 0, 1\rangle$ is a Brouwer–complemented lattice with 0 as the least and 1 as the greatest elements of the lattice, i.e., the above conditions (B-1), (B-2), (B-3a) and (B-4) are verified (for the proof of this part see [10]*).*

Moreover, since any pseudo–Boolean lattice is in particular a pseudo–complemented distributive lattice (i.e., for any $a \in \mathcal{A}$ there exists an element $\sim a$ such that $a \wedge x = 0$ if and only if $x \leq \sim a$, which is nothing else than the (I) under the choice $b = 0$), from a result proved in [55] *(and see also* [38]*) it follows that also the further property of negation is satisfied:*

(B-5) $\quad \forall a, b \in \mathcal{A}$: $\sim\sim (a \wedge b) = \sim\sim a \wedge \sim\sim b$

Let us note that in the above quoted papers it is proved that pseudo–complemented lattices are indeed characterized as those Brouwer–complemented lattices which just satisfy this condition (B–5).

In order to stress again the deep difference between classical and intuitionistic negations, let us quote the Rasiowa and Sikorski book:

> "Also negation and disjunction are understood differently from intuitionists. The sentence $\sim a$ is considered true if the acceptance of a leads to an absurdity. With this conception of negation and implication, the tautology
>
> $$a \to \sim\sim a$$
>
> is accepted as true by intuitionists but the tautology
>
> $$\sim\sim a \to a$$
>
> is not intuitionistically true.

> The intuitionists regards a disjunction $a \vee b$ as true if one of the sentences a, b is true and there is a method by which it is possible to find out which of them is true. With this conception of the truth of disjunction, the tautology
>
> $$a \vee \sim a$$
>
> is not accepted by the intuitionist as true since there is no general method of finding out, for any given sentence a, whether a or $\sim a$ is true. Intuitionists thus reject the tautology *tertium non datur.*" [53, p. 378].

As a final result (without entering in formal details for which we refer to [53]), let us recall that the idea of treating the set of all formulas of a formalized language as an abstract algebra with operations corresponding to logical connectives was first used by A. Lindenbaum and A. Tarski. Then it is possible to prove the following.

Theorem 4.6. *Let $\mathcal{T}$ be a formalized intuitionistic theory, then the Lindembaum–Tarski algebra $\mathcal{U}(\mathcal{T})$ associated to $\mathcal{T}$ is a pseudo–Boolean algebra.*

Thus, "the metatheory of the intuitionistic logic coincides with the theory of pseudo–Boolean [Heyting] algebras in the same sense as the metatheory of classical logic coincides with the theory of Boolean algebras" [53]. Moreover, "It follows from the representation theorems [...] that the theory of pseudo–Boolean [Heyting] algebras is the theory of lattices of open subsets of a topological space. Consequently the investigation of intuitionistic logic consists in an examination of lattices of open subsets of topological spaces." [53, p.380]. Of course, if one accepts the claim that "Heyting algebras play for the intuitionistic propositional calculus the same role played by the Boolean algebras for the classical propositional calculus" [44], then the induced negation connective is not only a Brouwer complementation but it satisfies the further condition (B–5) which is not mentioned in the Heyting book quoted in section 2.

4.1. Heyting algebras with Stone condition.

As to the dual de Morgan law $\forall a, b, \sim (a \wedge b) \leq \sim a \vee \sim b$ there are examples of the topological model of intuitionistic logic in which it is not valid. For instance, in the real line $\mathbb{R}$ equipped with the standard topology, where we denote by A^o the interior of any subset A and by A^c its set theoretic complement, on the distributive lattice $\langle \mathcal{P}(\mathbb{R}), \cap, \cup \rangle$ (the power set of $\mathbb{R}$ equipped with the standard set theoretic intersection and union operations) the Brouwer complement $\sim A = (A^c)^o$ is such that $\sim ((-\infty, 0) \cap (0, +\infty)) = \mathbb{R}$ and $\sim (-\infty, 0) \cup \sim (0, +\infty) = \mathbb{R} \setminus \{0\}$.

A characterization of the dual de Morgan law (dM2) for Heyting algebras is given by the following Stone condition whose proof, proved in a more general context, can be found in [28].

Lemma 4.7. [10, p.130]. *In a Heyting algebra the* dual de Morgan property

$$\text{(dM2)} \qquad \sim (a \wedge b) = \sim a \vee \sim b$$

is equivalent to the Stone condition

$$\text{(S)} \qquad \sim a \vee \sim\sim a = 1$$

In this case the Heyting algebra is called Stone algebra *or* (S) Heyting algebra*.*

Let us note that in the context of Brouwer–complemented lattices it is possible to prove that condition (dM2) implies that Stone condition (S) is true, but in the Brouwer–complemented lattice of example 3.5 condition (S) is true, whereas (dM2) is not verified.

5. Some Algebraic Structures in I(mproved)FS and (standard) FS Theories

So far we discussed some algebraic semantics of intuitionistic logic from the abstract point of view. Let us now compare this discussion with the algebraic structures which can be induced from Atanassov's theory.

First of all, let us consider a nonempty set of objects X, called the *universe* of the discourse. A *fuzzy set* or *generalized characteristic function* on X is any mapping $f : X \to [0, 1]$. Let us denote by $\mathcal{F}(X) = [0, 1]^X$ the collection of all such fuzzy sets. $\mathcal{F}(X)$ is a distributive lattice with respect to the meet $f \wedge g$ and joint $f \vee g$ defined for any pair of fuzzy sets $f, g \in \mathcal{F}(X)$ respectively by the pointwise laws:

$$(f_1 \wedge f_2)(x) = \min\{f_1(x), f_2(x)\} \tag{11a}$$

$$(f_1 \vee f_2)(x) = \max\{f_1(x), f_2(x)\} \tag{11b}$$

The partial order relation $\leq$ induced by the lattice structure is the pointwise one:

$$f \leq g \quad \text{iff} \quad \forall x \in X : f(x) \leq g(x)$$

With respect to this partial order, $\mathcal{F}(x)$ turns out to be bounded by the least fuzzy set $\mathbf{0}$ defined by $\forall x \in X : \mathbf{0}(x) = 0$ and the greatest fuzzy set $\mathbf{1}$ defined by $\forall x \in X : \mathbf{1}(x) = 1$. In general, for any fixed number $k \in [0,1]$ in the sequel we denote by $\mathbf{k} \in \mathcal{F}(X)$ the *constant fuzzy set* defined for every $x \in X$ by $\mathbf{k}(x) = k$.

On $\mathcal{F}(X)$ it is possible to introduce the binary relation $\perp \subseteq \mathcal{F}(X) \times \mathcal{F}(X)$ defined as:

$$f \perp g \quad \text{iff} \quad \forall x \in X : f(x) + g(x) \leq 1 \tag{12}$$

which turns out to be an *orthogonality* relation according to [18]. Indeed, one has that:

- (O-1) $f \perp g$ implies $g \perp f$ (symmetry property).
- (O-2) $f_0 \leq f$ and $f \perp g$ imply $f_0 \perp g$ (absorption property).
- (O-3) The orthogonality is *degenerate* in the sense that there exist fuzzy sets which are self–orthogonal, i.e., such that $f \perp f$; to be precise, they are all the fuzzy sets $f \in \mathcal{F}(X)$ such that $\forall x \in X$: $f(x) \leq \frac{1}{2}$.

An *ortho–pair of fuzzy sets* (also IFS according to a standard term) on the universe X is any pair of fuzzy sets $\langle f_A, g_A \rangle \in \mathcal{F}(X) \times \mathcal{F}(X)$, under the orthogonality condition $f_A \perp g_A$. The collection of all IFSs will be denoted by $\mathcal{IF}(X)$; this set is nonempty since it contains the particular elements $O := \langle \mathbf{0}, \mathbf{1} \rangle$, $I := \langle \mathbf{1}, \mathbf{0} \rangle$, and $H := \langle \frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}} \rangle$.

Then on IFS it is possible to introduce the two binary operations $\cap, \cup : \mathcal{IF}(X) \times \mathcal{IF}(X) \to \mathcal{IF}(X)$ and the unary operation $- : \mathcal{IF}(X) \to \mathcal{IF}(X)$, defined for arbitrary IFS $\langle f_A, g_A \rangle, \langle f_B, g_B \rangle$ as follows:

$$\langle f_A, g_A \rangle \cap \langle f_B, g_B \rangle := \langle f_A \wedge f_B, g_A \vee g_B \rangle \tag{13a}$$

$$\langle f_A, g_A \rangle \cup \langle f_B, g_B \rangle := \langle f_A \vee f_B, g_A \wedge g_B \rangle \tag{13b}$$

$$-\langle f_A, g_A \rangle := \langle g_A, f_A \rangle \tag{13c}$$

These operations (among a lot of other ones) are considered in Atanassov's papers [1, 2, 3] and book [4]. The obtained algebraic structure $\langle \mathcal{IF}(X), \cap, \cup, -, O, I \rangle$ is a distributive lattice with respect to the above meet $\cap$ and join $\cup$ operations, bounded by the least element O and the greatest element I, whose induced partial order relation is:

$$\langle f_A, g_A \rangle \subseteq \langle f_B, g_B \rangle \quad \text{iff} \quad \forall x \in X,\ f_A(x) \leq f_B(x) \ \text{ and } \ g_B(x) \leq g_A(x).$$

The unary operation $-$ is a *de Morgan complementation* on the distributive lattice. That is for any pair of IFSs $A = \langle f_A, g_A \rangle$ and $B = \langle f_B, g_B \rangle$ the following hold:

$$A = -(-A)$$
$$-(A \cup B) = -A \cap -B$$

In this way, IFS is a particular model of an abstract structure introduced by the following definition.

Definition 5.1. A *de Morgan lattice* is a structure $\langle \Sigma, \wedge, \vee, - \rangle$ where

- $\langle \Sigma, \wedge, \vee \rangle$ is a distributive lattice;
- $-$ is a unary operation on Σ, called *de Morgan* complementation, that satisfies the following conditions for arbitrary $a, b \in \Sigma$:
  - (D-1) $a = -(-a)$
  - (D-2) $-(a \vee b) = -a \wedge -b$

A *de Morgan algebra* is a de Morgan lattice with greatest element 1 (and so least element $0 = -1$).

A *Kleene distributive lattice (resp., algebra)* is a de Morgan lattice (resp., algebra) which satisfies the further condition:

(K) $a \wedge -a \leq b \vee -b$ (Kleene condition)

Quoting [20]: "This notion [of de Morgan lattice] has been introduced by Gr. Moisil [40, p.91] and studied by J. Kalman [34] under the name of *distributive i–lattice*. [...] If Σ has the last element 1, we shall say that Σ is a *de Morgan Algebra*. This notion has been studied by A. Bialynicki–Birula and H. Rasiowa [8, 9] under the name of *quasi–Boolean algebras*. In this case, $0 = -1$ is the first element of Σ. [...]

If the operation $-$ also verifies the condition (K) we shall say that Σ is a *Kleene lattice (algebra)*. A three–element algebra of this kind was studied by S.C. Kleene ([35], [36, p.334]) as a characteristic matrix of propositional calculus [...]. These lattices were studied by J.Kalman [34] with the name of *normal distributive i–lattices*. An important example of Kleene algebras are the N–lattices of H. Rasiowa [52]."

Let us stress that under condition (D-1) the following are equivalent:

(D-2a) $a \leq b$ implies $-b \leq -a$ (antimorphism)
(D-2b) $-a \leq -b$ implies $b \leq a$ (dual antimorphism)
(D-2c) $-a \vee -b = -(a \wedge b)$ ($\vee$ de Morgan)
(D-2) $-a \wedge -b = -(a \vee b)$ ($\wedge$ de Morgan)

Also in this case, any Boolean lattice is a de Morgan algebra and the possibility of singling out de Morgan lattices (and so also algebras) which are *genuine* resides in the sufficient condition of the existence of at least a nontrivial element h (i.e., $h \neq 0$ and $h \neq 1$), called *half element*, such that $h = -h$. Indeed, in this case the negation satisfies neither the excluded middle law ($h \wedge -h = h \neq 0$) nor the contradiction law ($h \vee -h = h \neq 1$). A similar discussion can be made in the case of Kleene structures where condition (K) assures that if a half element exists, then it is unique.

Coming back to IFS,

- the Kleene condition (K) is not valid and so IFSs are examples of de Morgan algebras which are not Kleene.
  Let us consider for instance the two IFSs $A = \langle \mathbf{0.4}, \mathbf{0.5} \rangle$ and $B = \langle \mathbf{0}, \mathbf{0.2} \rangle$. Then $A \cap -A = \langle \mathbf{0.4}, \mathbf{0.5} \rangle \not\subseteq \langle \mathbf{0.2}, \mathbf{0} \rangle = B \cup -B$.
  Furthermore, in [24] it has been proved that any de Morgan negation on IFS, whatever be its concrete definition, *cannot* satisfy the Kleene condition (K).
- Further, the negation $-$ neither satisfies the excluded middle nor the contradiction laws since IFS de Morgan algebra admits the half element $\langle \frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}} \rangle$ for which $-\langle \frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}} \rangle = \langle \frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}} \rangle$. Of course, the half element is not unique since for any $k \in [0, 1/2]$ the ortho-pair of fuzzy sets $\langle \mathbf{k}, \mathbf{k} \rangle$ is such that $-\langle \mathbf{k}, \mathbf{k} \rangle = \langle \mathbf{k}, \mathbf{k} \rangle$.

As a summary of this discussion, the IFS algebra characterized by the negation $-$ of (13c),

(P1) does not satisfy the excluded middle law, and this fits one of the intuitionistic requirements (point (N-1) of section 2);

but, and this is a real withdraw, this negation

(W1) does not satisfy the contradiction law, which on the contrary has been assumed by intuitionistic logic (point (I-6) of section 2);
(W2) asserts the strong double negation law which is rejected from intuitionistic logic (point (N-2) of section 2);
(W3) asserts the principle of dual antimorphism (dM2b), which is not accepted by intuitionistic logic (point (N-3) of section 2);
(W4) asserts the de Morgan law (dM2c), in the form $-(a \vee b) \leq -a \wedge -b$, which is refused by intuitionistic logic (point (N-4) of section 2).

It is very hard to claim that this kind of "square" is a "circle" on the fact that only a part of the intuitionistic principles about negation is verified [(P1)], contrary to the rejection of a relevant intuitionistic idea [(W1)] and the assertion of a lot of principles refused by intuitionism [(W2)–(W4)]. So, with respect to the structure $\langle \mathcal{IF}(X), \cap, \cup, -, \mathbf{0}, \mathbf{1} \rangle$ IFS is not "intuitionistic" in the sense that it is not an algebraic model of intuitionistic logic or any of its extensions.

A situation similar to the one just investigated about IFS happens also in the case of standard FS theory. Indeed, if in the distributive lattice $\langle \mathcal{F}(X), \wedge, \vee, \mathbf{0}, \mathbf{1} \rangle$ with respect to the lattice operations (11a) and (11b) one adds the further unary operator

$$\forall x \in X : \quad \neg f(x) := 1 - f(x)$$

than one obtains that the structure $\langle \mathcal{F}(X), \wedge, \vee, \neg, \mathbf{0}, \mathbf{1} \rangle$ satisfies the two conditions (D-1) and (D-2) of de Morgan algebra, but furthermore

- the Kleene condition (K) is verified and so FSs are examples of Kleene algebras. Indeed, for every pair of fuzzy sets $f, g \in \mathcal{F}(X)$ and any $x \in X$ trivially $f(x) \wedge \neg f(x) \leq 1/2 \leq g(x) \vee \neg g(x)$.
- Further, the negation $\neg$ neither satisfies the excluded middle nor the contradiction laws since FS Kleene algebra admits the half element $\frac{\mathbf{1}}{\mathbf{2}}$ for which $\neg \frac{\mathbf{1}}{\mathbf{2}} = \frac{\mathbf{1}}{\mathbf{2}}$. Let us recall that the existence of the half element is a sufficient condition for the genuineness of the structure, but this does not prevent to have other elements which do not satisfy both the excluded middle and the contradiction laws. Indeed, for instance for all $k \in (0, 1)$ we have $\mathbf{k} \vee \neg \mathbf{k} = \max(\mathbf{k}, \neg \mathbf{k}) \neq \mathbf{1}$ and $\mathbf{k} \wedge \neg \mathbf{k} = \min(\mathbf{k}, \neg \mathbf{k}) \neq \mathbf{0}$.

As a consequence, all the above points (P1) and (W1)–(W4) can be applied not only to the IFS case but also to the present FS case, with the important difference that FS are models of Kleene (i.e., de Morgan plus (K)) algebras whereas IFS are models of de Morgan algebras only.

From one point of view, the more correct one, points (W1)–(W4) forbid to state that $\neg$ is an algebraic version of the intuitionistic negation, similarly to the case of IFS with respect to the complementation (13c). But from another point of view, if according to Atanassov's meta–principle one defines as "Intuitionistic Fuzzy Set" any structure involving fuzzy sets and equipped with a complementation in which only "the logical law of excluded middle is not valid, similarly to the case in intuitionistic mathematics" [1], then owing to point (P1) there is no reason to deny also to standard fuzzy set theory the term of Intuitionistic Fuzzy Set.

5.1. **The Heyting algebra of IFS.** As proposed in [23, p. 64] (relatively to the unit interval $[0, 1]$ and extended by us to $\mathcal{F}(X)$ on [13]) it is possible to define an intuitionistic implication also on $\mathcal{IF}(X)$ as follows. Let $A = \langle f_A, g_A \rangle$ and $B = \langle f_B, g_B \rangle$ be two IFSs, then for x ranging on X:

$$(\langle f_A, g_A \rangle \Rightarrow \langle f_B, g_B \rangle)(x) := \begin{cases} (1, 0) & \text{if } f_A(x) \leq f_B(x) \\ & \text{and } g_A(x) \geq g_B(x) \\ (1 - g_B(x), g_B(x)) & \text{if } f_A(x) \leq f_B(x) \\ & \text{and } g_A(x) < g_B(x) \\ (f_B(x), 0) & \text{if } f_A(x) > f_B(x) \\ & \text{and } g_A(x) \geq g_B(x) \\ (f_B(x), g_B(x)) & \text{if } f_A(x) > f_B(x) \\ & \text{and } g_A(x) < g_B(x) \end{cases} \tag{14}$$

The structure $\langle \mathcal{IF}(X), \cap, \cup, \Rightarrow, \langle \mathbf{0}, \mathbf{1} \rangle \rangle$ is a Heyting algebra. The Brouwer negation induced by the implication connective $\Rightarrow$ in the usual manner $\sim \langle f_A, g_A \rangle = \langle f_A, g_A \rangle \Rightarrow \langle \mathbf{0}, \mathbf{1} \rangle$ is the following one defined whatever be $x \in X$ by the law:

$$\sim \langle f_A, g_A \rangle (x) = \begin{cases} (1, 0) & \text{if} \quad g_A(x) = 1 \\ (0, 1) & \text{if} \quad g_A(x) \neq 1 \end{cases} \tag{15a}$$

Indeed, if in (14) one set $f_B(x) = 0$ and $g_B(x) = 1$, then the first case produces the result $(1, 0)$ for $f_A(x) = 0$ and $g_A(x) = 1$, but condition $g_A(x) = 1$ implies that necessarily $f_A(x) = 0$ by the orthogonality condition (12) on IFS. The second and forth cases lead to the pair $(0, 1)$ corresponding to the conditions $g_A(x) \neq 1$ with $f_A(x) = 0$ and $f_A(x) \neq 0$ respectively, i.e., it happens when $g_A(x) \neq 1$. The third case is impossible since it should happen for $g_A(x) = 1$ and $f_A(x) \neq 0$, contrary to the orthogonality condition on IFS.

To be more explicit, this is a "compact" form of the real formula:

$$\sim \langle f_A, g_A\rangle = \langle h_A, k_A\rangle \qquad \text{with} \qquad \begin{aligned} h_A(x) &= \begin{cases} 1 & \text{if} \quad g_A(x) = 1 \\ 0 & \text{if} \quad g_A(x) \neq 1 \end{cases} \\ k_A(x) &= \begin{cases} 0 & \text{if} \quad g_A(x) = 1 \\ 1 & \text{if} \quad g_A(x) \neq 1 \end{cases} \end{aligned} \tag{15b}$$

In this way, if one introduces the subset of the universe $A_1(g_A) := \{x \in X : g_A(x) = 1\}$ and denotes by $\chi_B : X \mapsto \{0,1\}$ the characteristic (*crisp*) functional of the subset B of X defined as $\chi_B(x) = 1$ if $x \in B$ and $= 0$ otherwise, then (15b) can be rewritten as

$$\sim \langle f_A, g_A\rangle = \left\langle \chi_{A_1(g_A)}, \chi_{A_1(g_A)^c} \right\rangle \tag{15c}$$

from which it follows that $\sim\sim \langle f_A, g_A\rangle = \left\langle \chi_{A_1(g_A)^c}, \chi_{A_1(g_A)} \right\rangle$ with $f_A(x) \leq \chi_{A_1(g_A)^c}(x)$ and $\chi_{A_1(g_A)} \leq g_A(x)$, i.e., $\langle f_A, g_A\rangle \leq \sim\sim \langle f_A, g_A\rangle$.

If we adopt the Atanassov's meta–attitude discussed in the introduction which consists in appending the term "Intuitionistic" to any algebraic structure containing a complementation which satisfies all the principles required by intuitionistic negation, then

- we would conclude that $\mathcal{IF}(X)$ is *simultaneously* an intuitionistic fuzzy set environment (with respect to negation $\sim$) and a non–intuitionistic fuzzy set environment (with respect to negation $-$).

5.2. **The Heyting algebra of standard FS.** On the other hand, let us consider the collection $\mathcal{F}(X)$ of all standard fuzzy sets, then the lattice structure with respect to the meet and join operations (13) can be equipped with the implication connective defined for every $x \in X$ by the following law, extension to FS of the many–valued Gödel implication defined for instance on $[0,1]$ (see [54, p.44]):

$$(f_1 \to f_2)(x) := \begin{cases} 1 & f_1(x) \leq f_2(x) \\ f_2(x) & \text{otherwise} \end{cases} \tag{16}$$

The structure $\langle \mathcal{F}(X), \wedge, \vee, \to, \mathbf{0}\rangle$ is a Heyting algebra whose Brouwer–complement, defined as usual as $\sim f = f \to \mathbf{0}$, is explicitly given whatever be $x \in X$ by the formula (compare with the example 3.8):

$$\sim f(x) = \begin{cases} 1 & \text{if } f(x) = 0 \\ 0 & \text{otherwise} \end{cases} \tag{17a}$$

Making use of the characteristic-function notion and introducing the subset of the universe $A_0(f) := \{x \in X : f(x) = 0\}$, this formula assumes the following compact form (and compare with (15c)):

$$\sim f = \chi_{A_0(f)} \tag{17b}$$

If in the FS case we adopt a modified version of the Atanassov's meta–attitude of considering as "Intuitionistic" any algebraic structure satisfying all the principles of intuitionistic logic (i.e., Heyting algebras), another kind of chaos rises since

- both standard FSs and Atanassov's IFSs might rightfully be *simultaneously* called Intuitionistic Fuzzy Sets.

5.3. **Heyting algebraic embedding of FS into IFS.** It is worth noting that if one introduces the subset of IFS

$$\mathcal{IF}(X)^* := \{\langle f, \neg f\rangle : f \in \mathcal{F}(X)\}$$

then the implication connective (14) assumes the form

$$(\langle f_1, \neg f_1\rangle \Rightarrow \langle f_2, \neg f_2\rangle)(x) = \begin{cases} (1,0) & \text{if} \quad f_1(x) \leq f_2(x) \\ (f_2(x), 1 - f_2(x)) & \text{otherwise} \end{cases}$$

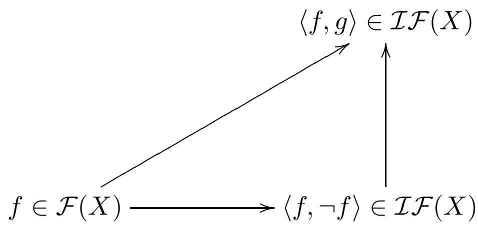
which corresponds to the compact form

$$\langle f_1, \neg f_1\rangle \Rightarrow \langle f_2, \neg f_2\rangle = \langle f_1 \to f_2, \neg(f_1 \to f_2)\rangle$$

From this one obtains the Brouwer–complement

$$\sim \langle f, \neg f\rangle = \langle f, \neg f\rangle \Rightarrow \langle \mathbf{0}, \mathbf{1}\rangle = \langle \sim f, \neg(\sim f)\rangle = \text{(17b)} =$$
$$= \left\langle \chi_{A_0(f)}, \chi_{A_0(f)^c}\right\rangle = \left\langle \chi_{A_1(\neg f)}, \chi_{A_1(\neg f)^c}\right\rangle$$

In this way $\langle \mathcal{IF}(X)^*, \wedge, \vee, \Rightarrow, \langle \mathbf{0}, \mathbf{1}\rangle\rangle$ turns out to be a sub–Heyting algebra of the IFS Heyting algebra. Moreover, the Heyting algebras $\langle \mathcal{F}(X), \wedge, \vee, \rightarrow, \mathbf{0}\rangle$ and $\langle \mathcal{IF}(X)^*, \wedge, \vee, \Rightarrow, \langle \mathbf{0}, \mathbf{1}\rangle\rangle$ are isomorphic by the one to one and onto correspondence $f \rightarrow \langle f, \neg f\rangle$ and so FS can be considered as a sub–Heyting algebra of IFS according to the following diagram:



5.4. **The Stone extra condition characterizing IFS and FS.** Let us stress that the intuitionistic fuzzy logic presented by Takeuti and Titani in [56] consists of the axioms and inference rules of intuitionistic logic together with extra axioms and inference rules characterizing the structure of $[0, 1]$. So it is of a certain interest to investigate whether some extra property of the algebraic semantic based on Heyting algebras holds in the concrete structures of FS ($\mathcal{F}(X)$) and IFS ($\mathcal{IF}(X)$). This extra property exists and can be formulated in terms of the underlying Brouwer–complemented lattice structure. As we have seen in proposition 3.3, whereas the de Morgan principles (4a) and (4b) of (I-4) are realized in such a structure, only one of the dual de Morgan principle, precisely (9) of proposition 3.4, holds. Now, it is easy to prove that in the particular cases of both Heyting algebra structures $\mathcal{F}(X)$ and $\mathcal{IF}(X)$ the dual de Morgan principle (dM2) of lemma 4.7 holds, which is equivalent to the Stone condition (S).

Thus, both IF and IFS are Heyting algebras satisfying the extra condition (dM2), or equivalently (S), i.e., they are *(S) Heyting algebras*. Making use of (10) in proposition 4.3, condition (dM2) can be formulated also as $\sim (a \wedge b) \rightarrow \sim a \vee \sim b = 1$, algebraic version of (NI-4) of section 2. Summarizing:

- The IFS structure $\langle \mathcal{IF}(X), \cap, \cup, \Rightarrow, O\rangle$ and FS structure $\langle \mathcal{F}(X), \wedge, \vee, \rightarrow, \mathbf{0}\rangle$ generate the two negations (15a) and (17a) which both satisfy all the intuitionistic principles of negation (I-1)–(I-6), moreover they satisfy the further principle (NI-4), which cannot be asserted in the Heyting approach to intuitionistic negation outlined in [33] (see section 2) and the principle (B–5) which is hidden in the same Heyting negation.

Note that if one considers L–fuzzy sets, i.e., mappings from the universe X with values on a bounded lattice $\langle L, \wedge, \vee, 0, 1\rangle$, with L not totally ordered, then both the implication connectives (14) and (16) are not well defined, contrary to the fact that the Brouwer complementations (15a) and (17a) can be applied to this more general case.

6. ICS Algebra and Shadowed Sets

In this subsection we investigate a particular subclass of the class $\mathcal{IF}(X)$ of all IFSs on the universe X. To this aim, let us consider the collection, denoted by $\mathcal{IC}(X)$, of all ortho–pairs $\langle \chi_{A_1}, \chi_{A_0}\rangle$ of characteristic functions of subsets A_1, A_0 of X. Trivially $\mathcal{IC}(X) \subseteq \mathcal{IF}(X)$ and

$$\chi_{A_1} \perp \chi_{A_0} \quad \text{iff} \quad A_1 \cap A_0 = \emptyset$$

Therefore, $\mathcal{IC}(X)$ consists of pairs of *crisp* sets, denoted by (ICS), sharp realizations of two subsets A_1 and A_0 under the orthogonality condition of their disjointness. So we can identify ICSs with pairs of mutually disjoint subsets of X:

$$\langle \chi_{A_1}, \chi_{A_0}\rangle \quad \longleftrightarrow \quad \langle A_1, A_0\rangle \tag{18}$$

and we denote their collection also by

$$\mathcal{IC}(X) := \{\langle A_1, A_0\rangle \in \mathcal{P}(X)\times\mathcal{P}(X) : A_1 \cap A_0 = \emptyset\}$$

The subset A_1 (resp., A_0) is the *certainty* (resp., *impossibility*) *domain* of the involved ICS $\langle A_1, A_0\rangle$.

The operations (13) in this particular case assume the form

$$\langle A_1, A_0\rangle \cap \langle B_1, B_0\rangle = \langle A_1 \cap B_1, A_0 \cup B_0\rangle$$
$$\langle A_1, A_0\rangle \cup \langle B_1, B_0\rangle = \langle A_1 \cup B_1, A_0 \cap B_0\rangle$$
$$-\langle A_1, A_0\rangle = \langle A_0, A_1\rangle$$

$\langle \mathcal{IC}(X), \cap, \cup\rangle$ is a distributive lattice whose induced partial order relation is

$$\langle A_1, A_0\rangle \subseteq \langle B_1, B_0\rangle \quad \text{iff} \quad A_1 \subseteq B_1 \text{ and } B_0 \subseteq A_0$$

This lattice is bounded by the least element $O = \langle\emptyset, X\rangle$ and the greatest element $I = \langle X, \emptyset\rangle$, moreover there exists the half element $\frac{I}{2} = \langle\emptyset,\emptyset\rangle$.

- *The complementation* $-$ *is a Kleene complementation since* $\langle A_1, A_0\rangle \cap - \langle A_1, A_0\rangle = \langle\emptyset, A_1 \cup A_0\rangle$ *and* $\langle B_1, B_0\rangle \cup - \langle B_1, B_0\rangle = \langle B_1 \cup B_0, \emptyset\rangle$*, and so*

$$\langle A_1, A_0\rangle \cap - \langle A_1, A_0\rangle \subseteq \langle\emptyset,\emptyset\rangle \subseteq \langle B_1, B_0\rangle \cup - \langle B_1, B_0\rangle$$

The restriction to the present case (i.e., to ortho–pairs of subsets and their crisp representations according to the identification (18)) of the implication (14) produces the ICS implication:

$$\langle A_1, A_0\rangle \Rightarrow \langle B_1, B_0\rangle = \langle (A_1^c \cap B_0^c) \cup A_0 \cup B_1,\ A_0^c \cap B_0\rangle \tag{19}$$

Let us note that this ICS implication connective has been introduced in the context of rough set theory by Pagliani in [47, proposition 3.8] several years before the contribution [23] in the context of the unit interval $[0, 1]$, and surprisingly it results to be just the restriction of this latter to the case of crisp ortho–pairs. A review of all the results concerning the Pagliani's approach to implication connectives in rough set algebras can be found in [48], and a contribution of ours is in [15].

The complementation $\sim$ induced by (15c) in the present case assumes the form (equivalently obtained from (19) by $\sim \langle A_1, A_0\rangle := \langle A_1, A_0\rangle \Rightarrow \langle\emptyset, X\rangle$):

$$\sim \langle A_1, A_0\rangle = \langle A_0, (A_0)^c\rangle$$

Making use of this Brouwer complementation as primitive connective together with the above Kleene negation $-$, the ICS implication connective (19) assumes the equational form:

$$\langle A_1, A_0\rangle \Rightarrow \langle B_1, B_0\rangle = \big(\neg \sim \neg \langle A_1, A_0\rangle \cap \neg \sim \langle B_1, B_0\rangle\big) \cup \sim \langle A_1, A_0\rangle \cup \langle B_1, B_0\rangle$$

We can summarize all that saying:

- *The ICS structure* $\langle \mathcal{IC}(X), \cap, \cup, \Rightarrow, \langle\emptyset, X\rangle\rangle$ *is a (S) Heyting algebra with the (unique) half element* $\langle\emptyset,\emptyset\rangle$ *(genuineness of the structure).*

From this discussion it follows that we do agree with the following comment of [25]:

> "It is worth pointing out that [the Atanassov's IFS approach] may be seen as a fuzzification of the ideas of sub–definite set, introduced some year before by Narin'yani [45] who separately handles the (ordinary) set A_1 of elements known as belonging to the sub–definite set and the (ordinary) set A_0 of elements known as not belonging to it, with the condition $A_1 \cap A_0 = \emptyset$ (together with some constraints about the cardinalities of A_1 and A_0).
> Such a condition is extended to the two membership functions f_A and g_A, which for IFS are supposed to satisfy the constraint (12)."

To the best of our knowledge, the concept of ICS has been introduced for the first time by M. Yves Gentilhomme in [29] (see also [42]) in an equivalent way with respect to the one described here. Indeed, in these papers pairs of ordinary subsets of the universe X of the kind $\langle A_1, A_p\rangle$, under the condition $A_1 \subseteq A_p$ are considered. Of course, once introduced the following operations on the Gentilhomme version of ICS:

$$\begin{aligned}\langle A_1, A_p\rangle \cap \langle B_1, B_p\rangle &= \langle A_1 \cap B_1, A_p \cap B_p\rangle \\ \langle A_1, A_p\rangle \cup \langle B_1, B_p\rangle &= \langle A_1 \cup B_1, A_p \cup B_p\rangle \\ -\langle A_1, A_p\rangle &= \langle (A_p)^c, (A_1)^c\rangle \\ \sim \langle A_1, A_p\rangle &= \langle (A_p)^c, (A_p)^c\rangle\end{aligned}$$

the mapping $\langle A_1, A_0\rangle \to \langle A_1, (A_0)^c\rangle$ institute a one-to-one and onto correspondence which allows one to identify the two approaches.

The pairs $\langle A_1, A_0\rangle$ where also considered in [19]. In this context they are called *classical preclusive propositions* and analyzed from the point of view of algebraic rough approximations. Indeed, to any ICS pair $\langle A_1, A_0\rangle$, it is possible to assign a fuzzy set $f := \frac{1}{2}(\chi_{A_1} + \chi_{A_0^c})$ such that its rough approximation $r(f) := \langle f_i(x), f_e(x)\rangle$ where

$$f_i(x) := \begin{cases} 1 & f(x) = 1 \\ 0 & \text{otherwise} \end{cases} \qquad f_e(x) := \begin{cases} 1 & f(x) = 0 \\ 0 & \text{otherwise} \end{cases}$$

coincides with the starting pair, i.e., $r(f) = \langle f_i, f_e\rangle = \langle A_1, A_0\rangle$.

Later on, Çoker [22] introduced in an independent way the so called "intuitionistic set" as a weakening of IFS to classical sets, whose definition exactly coincides with ICS.

6.1. **ICSs and shadowed sets.** Let us consider the three-valued lattice of numbers from the real unit interval $\{0, u, 1\}$ totally ordered with respect to the order chain $0 \le u \le 1$. If one interprets the real value u as the interval $(0, 1)$, this choice corresponds to a different approach to vagueness proposed by Pedrycz in [49, 50, 51]. His intention was "to introduce a model which does not lend itself to precise numerical membership values but relies on basic concepts of truth values (yes - no) and on entire unit interval perceived as a zone of "uncertainty." [49]. This idea of modelling vagueness through vague (i.e., not purely numeric) information, lead him to the following definition of shadowed sets.

Definition 6.1. Let X be a set of objects, called the universe. A *shadowed set* on X is any mapping $s : X \to \{0, u, 1\}$. We denote the collection of all shadowed sets on X as $\{0, u, 1\}^X$, or sometimes simply by $\mathcal{S}(X)$.

Let us introduce for any ICS $\langle A_1, A_0\rangle$ its *uncertainty domain* $A_u = X \setminus (A_1 \cup A_0)$. Then the mapping $\mathcal{IC}(X) \mapsto \mathcal{S}(X)$ defined by the law

$$\langle A_1, A_0\rangle \to \chi_{A_1} + u \cdot \chi_{A_u} = \begin{cases} 1 & \text{if } x \in A_1 \\ u & \text{if } x \in A_u \\ 0 & \text{if } x \in A_0 \end{cases}$$

is a one-to-one and onto correspondence which allows one to identify ICS and shadowed sets. In this way all the algebraic structures of ICSs are automatically inherited by shadowed sets, in particular the one of Kleene algebra and the one of (S) Heyting algebra.

If for any fuzzy set $f \in \mathcal{F}(X)$ one introduces its *necessity domain* $A_1(f) := \{x \in X : f(x) = 1\}$ and *uncertainty domain* $A_u(f) := \{x \in X : f(x) \neq 0, 1\}$, then the mapping $\mathcal{F}(X) \mapsto \mathcal{S}(X)$ defined by the law

$$f \to \chi_{A_1(f)} + u \cdot \chi_{A_u(f)} = \begin{cases} 1 & \text{if } f(x) = 1 \\ u & \text{if } f(x) \neq 0, 1 \\ 0 & \text{if } f(x) = 0 \end{cases}$$

is an onto correspondence which institutes a surjective morphism from the logical–algebraic structures (for instance (S) Heyting algebra) of fuzzy sets $\mathcal{F}(X)$ onto the corresponding logical-algebraic structures of shadowed sets $\mathcal{S}(X)$. For a more detailed investigation about these algebraic arguments, see our contribution [14].

6.2. **The three–valued extra condition characterizing ICS, and not FS and IFS.** As the Kleene condition (K) of the negation $-$ distinguishes both FS and ICS as different algebraic structures with respect to IFS, in this subsection we want to investigate another condition which allows one to distinguish ICS as different from FS and IFS, the so called *three–value condition*:

(3) $a \vee \sim a = a \vee -a.$

Indeed, it is easy to prove that inside $\mathcal{IC}(X)$ whatever be the pair $a = \langle A_1, A_0 \rangle$ it is $a \vee \sim a = \langle A_1, A_0 \rangle \vee \langle A_0, (A_0)^c \rangle = \langle A_1 \cup A_0, \emptyset \rangle$ and $a \vee -a = \langle A_1, A_0 \rangle \vee \langle A_0, A_1 \rangle = \langle A_1 \cup A_0, \emptyset \rangle$, i.e., the condition (3) is satisfied.

On the other hand, inside $\mathcal{F}([0,5])$, with $[0,5]$ the closed interval in $\mathbb{R}$, if one considers the fuzzy set $f = (1/2) \cdot \chi_{(1,2)} + \chi_{[2,3]} + (1/3) \cdot \chi_{(3,4)}$, then $f \vee \sim f = \chi_{[0,1]} + (1/2) \cdot \chi_{(1,2)} + \chi_{[2,3]} + (1/3) \cdot \chi_{(3,4)} + \chi_{[4,5]}$ and $f \vee -f = \chi_{[0,1]} + (1/2) \cdot \chi_{(1,2)} + \chi_{[2,3]} + (2/3) \cdot \chi_{(3,4)} + \chi_{[4,5]}$, i.e. the condition (3) is not satisfied.

Finally, inside $\mathcal{IF}(X)$ the IFS $a = \langle \mathbf{0.3}, \mathbf{0.6} \rangle$ defined for any element $x \in X$ as the pair of constant fuzzy sets $a(x) = \langle 0.3, 0.6 \rangle$ is such that $a \vee \sim a = \langle \mathbf{0.3}, \mathbf{0.6} \rangle$ and $a \vee -a = \langle \mathbf{0.6}, \mathbf{0.3} \rangle$ and so also in this case condition (3) is not satisfied.

7. Conclusions and final comments

We can summarize the different behavior of FS, IFS and ICS with respect to some relevant algebraic conditions in the following table, where (D) stays for a de Morgan lattice which is not Kleene (see definition 5.1):

	FS	IFS	ICS
(K) for $-$	yes	no, but (D)	yes
(S) for $\sim$	yes	yes	yes
(3) for $\sim$	no	no	yes

So, FS differs from IFS for condition (K), FS differs from ICS for condition (3), IFS differs from ICS for both conditions (K) and (3). Let us recall that in [16, 17] a BK lattice (i.e., a distributive lattice equipped with a Brouwer $\sim$ and a Kleene $-$ complementations linked by the interconnection rule $- \sim a = \sim\sim a$) which satisfies the further conditions (S) and (3) has been called a *three–valued BZ lattice*. In the same paper it is shown that these structures are categorically equivalent to *three–valued Łukasiewicz algebras* introduced in [20] (see also [21]), based on the *modal possibility operator* defined for every element a by $\mu(a) := - \sim a$. Thus, ICS are particular models of three–valued BZ lattices (or, equivalently, three–valued Łukasiewicz algebras). Recalling that from the algebraic point of view ICS cannot be distinguished from shadowed sets, also these latter are particular models of three–valued BZ lattices. A contribution of ours about suitable algebras for shadowed sets, with the relation between fuzzy sets and shadowed sets can be found in [14]. A deeper discussion involving algebraic arguments about ICS (and so also shadowed sets) will be treated in a forthcoming paper of ours.

As to the terminological debate about intuitionism we have proved that

(i) Atanassov's IFS structure based on the "negation" $-$ of (13c) is *not* an algebraic model of intuitionistic negation owing to the following drawbacks:
   (a) the contradiction law accepted by intuitionistic logic is not satisfied by $-$;
   (b) the negation $-$ asserts the principle of double negation which is refused by intuitionistic logic;
   (c) the negation $-$ asserts the de Morgan law denoted in section 5 by (D-2c), also this refused by intuitionistic logic.

(ii) All the FS, IFS and ICS (= shadowed sets) algebraic structures based on the "negation" $\sim$ induced by the corresponding Heyting algebra are genuine (non Boolean) models of the intuitionistic negation.

(iii) Furthermore in all these cases principle (dM2), equivalently (S), is satisfied, which is not asserted by Heyting intuitionistic negation discussed in [33].

Let us remark that there exists a possible terminological chaos if one persists in the (Atanassov's) meta–attitude to append the term "intuitionistic" (resp., "non–intuitionistic") to algebraic structures which are (resp., are not) models of this logic: indeed, IFS are simultaneously "intuitionistic fuzzy sets" and "non–intuitionistic fuzzy sets" depending if one considers Atanassov's original approach with Kleene negation $-$ of (13c) or the Brouwer negation connective $\sim$ of (15a). A kind of contradiction.

From another point of view, in agreement with this meta–attitude, not only Atanassov's "Ortho-pairs of Fuzzy Sets" can be called *intuitionistic fuzzy sets* with respect to the Brouwer negation (15a), but there is no reason to refuse that also standard "Fuzzy Sets" equipped with the Brouwer negation (17a) can be called *intuitionistic fuzzy sets*. In this case we have that the same term should denote two quite different algebraic structures. A real confusion.

In this case the problem resides exactly on the fact that "if we decide that the name 'IFS' must be reserved for T–IFS [or other "intuitionistic" algebraic structure], as it was offered in [25], then for each construction of this kind that satisfies intuitionistic logic axiomatic we have to invent new name" [5]. As underlined before, in the present paper we denote FS for standard "Fuzzy Sets" with behavior (K) and (S), IFS for "Ortho-pairs of Fuzzy Sets" with behavior (S), and ICS for "Ortho-pairs of Crisp Sets" with the three-valued behavior (K), (S) and (3).

Let us conclude making some remarks to the following Atanassov's answer given in [5]:

> "In their paper Takeuti and Titani consider propositions (variables) valuated into the range $[0, 1]$, i.e., using fuzzy values, for which we can define such a degree of membership function that satisfies the axiom of intuitionistic logic."

In this statement there is a kind of confusion between syntax and semantic. It is not correct to assert that one can define such a degree of membership function on $[0, 1]$ in such a way that it satisfies the axiom of intuitionistic logic (our interpretation of the above Atanassov's statement). On the contrary, it is correct to say that the *semantic* of Takeuti and Titani intuitionistic fuzzy logic is based on $[0, 1]$ equipped with suitable algebraic operations introduced in [56, p.851]. Indeed, in [56] "Intuitionistic Fuzzy Logic" is constructed as a real "logical system", with a clear syntax, i.e., formalized language plus "axioms and inference rules [which] are those of intuitionistic logic (Gentzen's LJ) together with the extra axioms and inference rules which characterize the structure of $[0, 1]$." The semantical aspect is based on the fact that "propositional variables are interpreted as variables ranging over the closed interval $[0, 1]$ of real numbers, and propositional constants are interpreted as real numbers in $[0, 1]$." For this reason T–Intuitionistic Fuzzy Logic is defined as $[0, 1]$*–valued logic*, and not with a spurious terminology attributed to an algebraic $[0, 1]$ model of this system (see section 5 of [56] for a semantic based on fuzzy sets $[0, 1]^X$ instead of on the interval $[0, 1]$).

Summarizing, Intuitionistic Fuzzy Logic of [56] is a real logical system with something more than an intuitionistic logic axiomatic since it assumes Gentzen's LJ axioms of intuitionistic logic plus 6 extra axioms and some extra inference rules. Its semantic is based on the real unit interval $[0, 1]$ equipped with suitable operations. So in the case of TT it is correct to say that it deals with a real *formalized logic* of *intuitionistic* type, whose semantic is based on the *fuzzy* interval $[0, 1]$, i.e., we have to do with an *intuitionistic fuzzy logic*.

For completeness, in [5] it is announced that "in [7], [...] T. Trifonov and I defined the operations 'implication' and 'negation' on these propositions in such a way that they would as well satisfy the axioms discussed in [56]". But, at the moment we have not been able to obtain this Notes IFS.

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Dipartimento di Informatica, Sistemistica e Comunicazione, Università di Milano–Bicocca, Via Bicocca degli Arcimboldi 8, I–20126 Milano (Italy)

E-mail address: {cattang,ciucci}@disco.unimib.it