SHADOWED SETS AND RELATED ALGEBRAIC STRUCTURES

G. CATTANEO AND D. CIUCCI

Abstract. BZMV^{dM} algebras are introduced as an abstract environment to describe both shadowed and fuzzy sets. This structure is endowed with two unusual complementations: a fuzzy one $\neg$ and an intuitionistic one $\sim$. Further, we show how to define in any BZMV^{dM} algebra the Boolean sub–algebra of exact elements and to give a rough approximation of fuzzy elements through a pair of exact elements using an interior and an exterior mapping.
Then, we introduce the weaker notion of pre-BZMV^{dM} algebra. This structure still have as models fuzzy and shadowed sets but with respect to a weaker notion of intuitionistic negation $\sim_\alpha$ with $\alpha \in [0, \frac{1}{2})$. In pre-BZMV^{dM} algebras it is still possible to define an interior and an exterior mapping but, in this case, we have to distinguish between open and closed exact elements.
Finally, we see how it is possible to define α–cuts and level fuzzy sets in the pre-BZMV^{dM} algebraic context of fuzzy sets.

1. Fuzzy and Shadowed Sets

Fuzzy sets are a well known generalization of crisp (or classical, Boolean) sets, first introduced by L. Zadeh in 1965 ([28]). Since then they have been usefully applied in several environments including approximate reasoning. Further, the wide analysis of fuzzy sets from a theoretical point of view lead to the definition and study of fuzzy logic (see, for example, [11, 13, 25]). Recently Pedrycz introduced shadowed sets ([17]) as a simpler, and in his opinion more realistic, approach to vagueness than fuzzy sets. In this context we will analyze shadowed sets and their relation with fuzzy sets from an algebraic point of view.

The basic notions of fuzzy and shadowed sets are given in this section and the relation existing between them is discussed. First of all, let us introduce the definition of fuzzy set.

Definition 1.1. Let X be a set of objects, called the universe of discourse. A *fuzzy set* on X is any mapping $f : X \to [0, 1]$. In the sequel we denote the collection of all fuzzy sets on X by $[0, 1]^X$ or sometimes simply by $\mathcal{F}(X)$. Moreover, we denote by $\mathbf{k}$, for any fixed $k \in [0, 1]$, the fuzzy set $\forall x \in X$, $\mathbf{k}(x) = k$. The role of a fuzzy set is to describe vagueness: given a vague concept f on a universe X, the value $f(x)$ indicates the membership degree of the point x to the concept f. One feature of such an approach is the description of a vague concept through an exact numerical quantity. A different approach to vagueness has been proposed by Pedrycz ([17, 18, 19]). His intention was "to introduce a model which does not lend itself to precise numerical membership values but relies on basic concepts of truth values (yes - no) and on entire [open] unit interval perceived as a zone of "uncertainty." ([17]). This idea of modelling vagueness through vague (i.e., not purely numeric) information, lead him to the definition of shadowed sets.

Definition 1.2. Let X be a set of objects, called the universe. A *shadowed set* on X is any mapping $s : X \to \{0, 1, (0, 1)\}$. We denote the collection of all shadowed sets on X as $\{0, 1, (0, 1)\}^X$, or sometimes simply by $\mathcal{S}(X)$. In the sequel we will indicate $(0, 1)$ by the value $\frac{1}{2}$. This will simplify our algebraic approach from a syntactical point of view, without losing the semantic of "total uncertainty" of the value $(0, 1)$. In fact, if 1 corresponds to truth, 0 to falseness, then $\frac{1}{2}$ is halfway between true and false, i.e., it represents a really uncertain situation. From a fuzzy set it is possible to obtain a shadowed set. Let f be a fuzzy set; then, it is sufficient to fix a value $\alpha \in [0, \frac{1}{2})$ and set to 0 the membership values $f(x)$ which are less than or equal to α and set to 1 those greater than or equal to $(1 - \alpha)$. The membership values belonging to $(\alpha, 1 - \alpha)$ are those characterized by a great uncertainty or lack of knowledge and they are consequently considered the "shadow" of the induced shadowed set, i.e., they are set to $\frac{1}{2}$.
In a formal way, once fixed a value α, we can define the α–approximation function of a fuzzy set f, denoted by $s_\alpha(f)$, as the following shadowed set:

$$s_\alpha(f)(x) := \begin{cases} 0 & f(x) \leq \alpha \\ 1 & f(x) \geq 1 - \alpha \\ \frac{1}{2} & \text{otherwise} \end{cases} \tag{1}$$

In Figure 1 a fuzzy set and the induced shadowed set are represented.

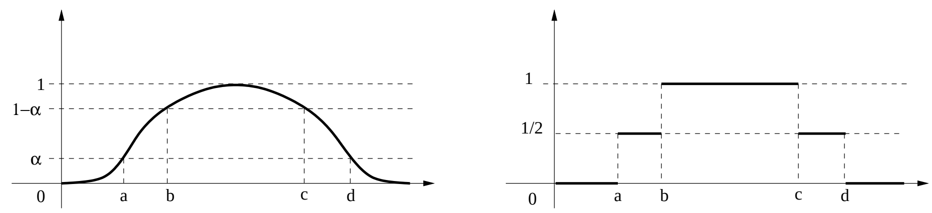

FIGURE 1. A fuzzy set and its corresponding shadowed set

2. An algebraic framework

In this section we propose BZMV^{dM} algebras ([3, 6]) as an algebraic approach to describe both fuzzy and shadowed sets.

Definition 2.1. A *de Morgan Brouwer Zadeh Many Valued* (BZMV^{dM}) algebra is a system
$\mathbf{A} = \langle A, \oplus, \neg, \sim, 0\rangle$, where A is a non empty set, $\oplus$ is a binary operation, $\neg$ and $\sim$ are unary operations, 0 is a constant, obeying the following axioms:
$(BZMV1)$ $(a \oplus b) \oplus c = (b \oplus c) \oplus a$
$(BZMV2)$ $a \oplus 0 = a$
$(BZMV3)$ $\neg(\neg a) = a$
$(BZMV4)$ $\neg(\neg a \oplus b) \oplus b = \neg(a \oplus \neg b) \oplus a$
$(BZMV5)$ $\sim a \oplus \sim\sim a = \neg 0$
$(BZMV6)$ $a \oplus \sim\sim a = \sim\sim a$
$(BZMV7)$ $\sim \neg[(\neg(a \oplus \neg b) \oplus b)] = \neg(\sim\sim a \oplus \neg \sim\sim b) \oplus \neg \sim\sim b$

On the basis of the primitive notions of a BZMV^{dM} algebra, it is possible to define the following further derived operations:

$$a \odot b := \neg(\neg a \oplus \neg b) \tag{2a}$$

$$a \vee b := \neg(\neg a \oplus b) \oplus b = a \oplus (\neg a \odot b) \tag{2b}$$

$$a \wedge b := \neg(\neg(a \oplus \neg b) \oplus \neg b) = a \odot (\neg a \oplus b) \tag{2c}$$

It is easy to prove that operations $\vee$ and $\wedge$ are the *join* and *meet* binary connectives of a *distributive lattice*, which can be considered as algebraic realization of logical disjunction and conjunction; in particular, they are idempotent operations. On the other hand the $\oplus$ and $\odot$ are not idempotent and they are the well known MV disjunction and MV conjunction connectives of the Chang approach([9, 25]). That is the following proposition holds.

Proposition 2.1. *([3]) Let $\langle A, \oplus, \neg, \sim, 0\rangle$ be a BZMV^{dM} algebra. Then the substructure $\langle A, \oplus, \neg, 0\rangle$ is an MV algebra. In other words the following conditions are satisfied:*

$(MV1)$ $(a \oplus b) \oplus c = (b \oplus c) \oplus a$
$(MV2)$ $a \oplus 0 = a$
$(MV3)$ $a \oplus \neg 0 = \neg 0$
$(MV4)$ $\neg(\neg a) = a$
$(MV5)$ $\neg(\neg a \oplus b) \oplus b = \neg(a \oplus \neg b) \oplus a$

As usual a partial order relation can be naturally induced by the lattice operations as:

$$a \leq b \text{ iff } a \wedge b = a \quad (\text{equivalently, } a \vee b = b)$$

Since it is possible to prove that $\sim 0 = \neg 0$, in the sequel we set $1 :=\sim 0 = \neg 0$. With respect to the just defined partial order relation we have that the lattice is bounded by the *least* element 0 and the *greatest* element 1, i.e.,

$$\forall a \in A, \quad 0 \leq a \leq 1$$

Let us recall that in the context of Chang MV algebras, this partial ordering can be equivalently expressed as $a \leq b$ iff $\neg a \oplus b = 1$, and according to Wajsberg algebraic formulation of Łukasiewicz implication connective in the context of MV algebras ([4, 26, 27]), this implication is defined as $a \to_L b := \neg a \oplus b$; hence, one has that $a \leq b$ iff "a implies b" is true.

The unary operation $\neg : A \mapsto A$ is a *Kleene* (or *Zadeh*) complementation (algebraic realization of a unusual (fuzzy) negation). In other words, it satisfies the properties:

(K1) $\neg(\neg a) = a$
(K2) $\neg(a \vee b) = \neg a \wedge \neg b$
(K3) $a \wedge \neg a \leq b \vee \neg b$

Let us recall that under (K1), the following conditions are equivalent.

(K2) $\neg(a \vee b) = \neg a \wedge \neg b$
(K2a) $\neg(a \wedge b) = \neg a \vee \neg b$
(K2b) $a \leq b$ implies $\neg b \leq \neg a$
(K2c) $\neg a \leq \neg b$ implies $b \leq a$

In general neither the non-contradiction law, $\forall a : \ a \wedge \neg a = 0$, nor the excluded middle law, $\forall a : \ a \vee \neg a = 1$, are satisfied by this negation.

The unary operation $\sim: A \mapsto A$ is a *Brouwer* complementation (also in this case algebraic realization of a unusual (intuitionistic) negation). In other words, it satisfies the properties:

(B1) $a \wedge \sim\sim a = a$ (equivalently, $a \leq \sim\sim a$)
(B2) $\sim (a \vee b) = \sim a \wedge \sim b$
(B3) $a \wedge \sim a = 0$

Under condition (B1), the following are equivalent.

(B2) $\sim (a \vee b) = \sim a \wedge \sim b$
(B2a) $a \leq b$ implies $\sim b \leq \sim a$

However, these latter are not equivalent to the dual contraposition law, i.e., $\sim b \leq \sim a$ implies $a \leq b$. Also the dual de Morgan law $\sim (a \wedge b) = \sim a \vee \sim b$ cannot be deduced from them. Moreover, from (B1)–(B3) the excluded middle law $\forall a,\ a \vee \sim a = 1$ cannot be derived.

Using the above definitions, we can justify the qualification of *de Morgan* given to BZMV algebras in Definition 2.1. In fact, it can be proved (see [6]) that all the de Morgan properties can be deduced from the axioms of BZMV^{dM} algebra:

$$\neg(a \wedge b) = \neg a \vee \neg b \qquad \neg(a \vee b) = \neg a \wedge \neg b$$
$$\sim (a \wedge b) = \sim a \vee \sim b \qquad \sim (a \vee b) = \sim a \wedge \sim b$$

A third kind of complementation, called *anti-intuitionistic complementation*, can be defined in any BZMV^{dM} algebra.

Definition 2.2. Let **A** be a BZMV^{dM} algebra. The *anti-intuitionistic complementation* is the unary operation $\flat : A \mapsto A$ defined as follows:

$$\flat a := \neg \sim \neg a$$

One can easily show that $\flat$ satisfies the following conditions:

(AB1) $\flat\flat a \leq a$;
(AB2) $\flat a \vee \flat c = \flat(a \wedge c)$ [equivalently, $a \leq c$ implies $\flat c \leq \flat a$];
(AB3) $a \vee \flat a = 1$.

Besides, it is possible to define, through the interaction of the two unary operations $\neg$ and $\sim$, the two modal operators

$$\nu(a) := \sim \neg a \qquad \text{(necessity)}$$
$$\mu(a) := \neg \sim a \qquad \text{(possibility)}$$

Trivially, the following mutual definability holds:

$$\nu(a) = \neg\mu(\neg a) \quad \text{and} \quad \mu(a) = \neg\nu(\neg a)$$

That is, according to modal logic, necessity is not–possibility–not and vice versa possibility is not–necessity–not.

These modal operators turn out to have an S_5-like behavior based on a Kleene algebra, instead of on a Boolean one ([10]).

Proposition 2.2. *In any BZMV^{dM} algebra the following conditions hold:*

(1)
$$\nu(a) \leq a \leq \mu(a)$$
In other words: necessity implies actuality and actuality implies possibility (a characteristic principle of the modal system T).

(2)
$$\nu(\nu(a)) = \nu(a) \quad \mu(\mu(a)) = \mu(a)$$
Necessity of necessity is equal to necessity; similarly for possibility (a characteristic S_4-principle).

(3)
$$a \leq \nu(\mu(a))$$
Actuality implies necessity of possibility (a characteristic B-principle).

(4)
$$\mu(a) = \nu(\mu(a)) \quad \nu(a) = \mu(\nu(a))$$
Possibility is equal to the necessity of possibility; whereas necessity is equal to the possibility of necessity (a characteristic S_5-principle).

As a consequence of the above definitions we have that $\sim a = \neg\mu(a)$, that is the Brouwer complement can be interpreted as the negation of possibility or *impossibility*. Similarly, we have that $\flat a = \neg\nu(a)$, that is the anti–Brouwer complement can be interpreted as the negation of necessity or *contingency*.

An interesting strengthening of BZMV^{dM} algebras are three valued BZMV algebras, first introduced in [3].

Definition 2.3. A BZMV^3 algebra is a BZMV^{dM} algebra $\mathbf{A}$, where axiom (BZMV5) is replaced by the following condition:

(sBZMV5) $\sim a \oplus (a \oplus a) = 1$.

The importance of BZMV^3 algebras is due to the fact that, as showed in [3], they are categorically equivalent to three–valued Łukasiewicz algebras ([14]) and to MV^3 algebras ([12]), which permit an algebraic semantic characterization for a three–valued logic. Indeed, the paradigmatic model of BZMV^3 algebras is the three element set $\{0, \frac{1}{2}, 1\}$ endowed with the following operations:

$$a \oplus b = \min\{a + b, 1\}$$
$$\neg a = 1 - a$$
$$\sim a = \begin{cases} 1 & \text{if} \quad a = 0 \\ 0 & \text{otherwise} \end{cases}$$

From the point of view of modality operators, BZMV^3 algebras satisfy some stronger conditions than BZMV^{dM} algebras as can be seen in the following proposition.

Proposition 2.3. *([3]). Let $\mathbf{A}$ be BZMV^3 algebra. Then, the following conditions hold for any element $a \in A$:*

(1) $\mu(a) = a \oplus a$
(2) $\neg a \vee \mu(a) = 1$
(3) $\neg a \wedge \mu(a) = \neg a \wedge a$

2.1. **Rough approximation through exact elements.** As stated in Proposition 2.2, for any element a of a BZMV^{dM} algebra A the order chain $\nu(a) \leq a \leq \mu(a)$ holds. We are, now, interested in the elements $e \in A$ which satisfy the strongest condition $\nu(e) = e$ (equivalently, $e = \mu(e)$), i.e., in the elements which present the classical feature that actuality coincide with necessity and possibility. These elements are called *sharp* (exact, crisp) elements (in contraposition to the other elements which are *fuzzy*) and their collection is denoted by A_e. Formally,

$$A_e := \{e \in A : \nu(e) = e\} = \{e \in A : \mu(e) = e\}$$

This is not the only way to define sharp elements. In fact, since in general $a \wedge \neg a \neq 0$ (equivalently, $a \vee \neg a \neq 1$) it is possible to consider as *Kleene sharp* (K-sharp) the elements which satisfy the non contradiction (or equivalently the excluded middle) law with respect to the Kleene negation:

$$A_{e,\neg} := \{e \in A : e \wedge \neg e = 0\} = \{e \in A : e \vee \neg e = 1\}$$

Alternatively, considering the Brouwer negation we have that, in general, the double negation law does not hold since in general $a \leq a^{\sim\sim}$ (see the (B1), which is equivalent to $a^{\flat\flat} \leq a$). So, we can introduce a further definition of *Brouwer sharp* (B-sharp) elements:

$$A_{e,\sim} := \{e \in A : \sim\sim e = e\} = \{e \in A : \flat\flat e = e\}$$

Finally, as said before $\oplus$ is not an idempotent operator, i.e., in general $a \oplus a \neq a$ (equivalently, $a \odot a \neq a$). So the $\oplus$-*sharp* elements are defined as:

$$A_{e,\oplus} = \{e \in A : e \oplus e = e\} = \{e \in A : e \odot e = e\}$$

However, it can be proved that all these notions are equivalent among them ([3, 6]). Formally, let $\mathbf{A}$ be a $BZMV^{dM}$ algebra, then

$$A_e = A_{e,\sim} = A_{e,\oplus} = A_{e,\neg}$$

Consequently, we simply talk of *sharp* elements and use the symbol A_e to denote their collection. Let us recall the following result of [6, 3].

Proposition 2.4. *Let $\mathbf{A} = \langle A, \oplus, \neg, \sim, 0\rangle$ be a BZMV^{dM} algebra. Then the collection A_e of all its sharp elements satisfies the following properties:*

(1) *The lattice connectives coincide with the MV ones:*

$$\forall e, f \in A_e, \quad e \wedge f = e \odot f \quad \textit{and} \quad e \vee f = e \oplus f$$

(2) *The two negation connectives (Kleene and Brouwer) coincide:*

$$\forall e \in A_e, \quad \neg a = \sim a$$

(3) *The structure $\mathbf{A}_e = \langle A_e, \wedge, \vee, \neg, 0\rangle$ is a Boolean lattice (algebra), which is the largest BZMV^{dM} sub algebra of $\mathbf{A}$ that is at the same time a Boolean algebra with respect to the same operations $\wedge(=\odot)$, $\vee(=\oplus)$, and $\neg(=\sim)$.*

The modal operators of necessity and possibility turn out to be respectively a topological interior and topological closure operator, and they can be used to define a rough approximation through sharp elements.

Proposition 2.5. *([7, 3]) Let $\langle A, \oplus, \neg, \sim, 0\rangle$ be a BZMV^{dM} algebra. Then the map $\nu : A \to A$ such that $\nu(a) := \sim \neg a$ is a* (interconnected additive) topological interior *operator, i.e., the following conditions hold:*

(I_0)	$1 = \nu(1)$	*(normalized)*
(I_1)	$\nu(a) \leq a$	*(decreasing)*
(I_2)	$\nu(a) = \nu(\nu(a))$	*(idempotent)*
(I_3)	$\nu(a \wedge b) = \nu(a) \wedge \nu(b)$	*(multiplicativity)*
(I_4)	$\nu(a) \vee \nu(b) = \nu(a \vee b)$	*(additivity)*
(I_5)	$a \wedge \nu(\neg a) = 0$	*(interconnected)*

Properties $(I_0) - (I_3)$ qualify the mapping I as a topological interior operator ([1, 22]), whereas properties $(I_4), (I_5)$ are specific conditions satisfied in this particular case. In the context of topological interior operators the elements which coincide with their interior are called *open* elements and the collection of all these open elements is denoted as:

$$\mathbb{O}(A) = \{a \in A : a = \nu(a)\}$$

But in the structure of BZMV^{dM} trivially it is $\mathbb{O}(A) = A_e$.

Proposition 2.6. *([7, 3]) Let $\langle A, \oplus, \neg, \sim, 0\rangle$ be a BZMV^{dM} algebra. Then the map $\mu : A \to A$ such that $\mu(a) := \neg \sim a$ is a* (interconnected multiplicative) topological closure *operator, i.e., the following conditions hold:*

(C_0)	$0 = \mu(0)$	*(normalized)*
(C_1)	$a \leq \mu(a)$	*(increasing)*
(C_2)	$\mu(a) = \mu(\mu(a))$	*(idempotent)*
(C_3)	$\mu(a) \vee \mu(b) = \mu(a \vee b)$	*(additivity)*
(C_4)	$\mu(a \wedge b) = \mu(a) \wedge \mu(b)$	*(multiplicativity)*
(C_5)	$a \wedge \neg\mu(a) = 0$	*(interconnected)*

Properties $(C_0)-(C_3)$ qualify the mapping C as a topological closure operator ([1, 22]), whereas properties $(C_4), (C_5)$ are specific conditions satisfied in this particular case. In the context of topological closure operators the elements which coincide with their closure are called *closed* elements and the collection of all these closed elements is denoted as:

$$\mathbb{C}(A) = \{a \in A : a = \mu(a)\}$$

But in the structure of BZMV^{dM} it is trivial to see that $\mathbb{C}(A) = A_e$.

In a generic structure equipped with an interior and a closure operation, an element which is both open and closed is said to be *clopen* and the collection of all such clopen elements is denoted by $\mathbb{CO}(A)$. However, in a BZMV^{dM} algebra the set of open elements coincide with the set of closed elements and with the set of sharp elements and thus we have that:

$$\mathbb{CO}(A) = \mathbb{O}(A) = \mathbb{C}(A) = A_e$$

This subset of clopen elements is not empty, indeed both $0, 1 \in \mathbb{CO}(\mathcal{A})$.

Given an element a of a BZMV^{dM} algebra, it is possible to give a rough approximation of a by sharp elements. In fact, $\nu(a)$ (resp., $\mu(a)$) turns out to be the best approximation from the bottom (resp., top) of a by sharp elements. To be precise, for any element $a \in A$ the following holds:

(I1) $\nu(a)$ is sharp ($\nu(a) \in A_e$).
(I2) $\nu(a)$ is an *inner* (*lower*) approximation of a ($\nu(a) \leq a$).
(I3) $\nu(a)$ is the *best* inner approximation of a by sharp elements: let $e \in A_e$ be such that $e \leq a$, then $e \leq \nu(a)$.

Analogously

(O1) $\mu(a)$ is sharp ($\mu(a) \in A_e$).
(O2) $\mu(a)$ is an *outer* (*upper*) approximation of a ($a \leq \mu(a)$).
(O3) $\mu(a)$ is the *best* outer approximation of a by sharp elements: let $f \in A_e$ be such that $a \leq f$, then $\mu(a) \leq f$.

Definition 2.4. Given a BZMV^{dM} algebra $\langle A, \oplus, \neg, \sim, 0\rangle$, the induced *rough approximation space* according to [2] is the structure $\langle A, A_e, \nu, \mu\rangle$ consisting of

- the set A of all *approximable* elements;
- the set A_e of all *sharp* (or *definable*) elements;
- the *inner approximation map* $\nu : A \to A_e$ associating to any approximable element a its best sharp inner (lower) approximation $\nu(a)$;
- the *outer approximation map* $\mu : A \to A_e$ associating to any approximable element a its best sharp outer (upper) approximation $\mu(a)$.

For any element $a \in A$, its *rough approximation* is defined as the pair of sharp elements:

$$r(a) := \langle \nu(a), \mu(a)\rangle \quad [\text{with} \quad \nu(a) \leq a \leq \mu(a)]$$

drawn in the following diagram:

$$\begin{tikzcd}
 & a \in A \arrow[dl, "\nu"'] \arrow[dr, "\mu"] \arrow[dd, "r"] & \\
\nu(a) \in A_e \arrow[dr] & & \mu(a) \in A_e \arrow[dl] \\
 & \langle \nu(a), \mu(a)\rangle &
\end{tikzcd}$$

So the map $r : A \to A_e \times A_e$ approximates an unsharp (fuzzy) element by a pair of exact ones representing its best inner and outer sharp approximation, respectively. Clearly, sharp elements are characterized by the property that they coincide with their rough approximations:

$$e \in A_e \quad \text{iff} \quad r(e) = \langle e, e\rangle.$$

An equivalent way to define a rough approximation space is to use the impossibility operator instead of the possibility one. So, given a fuzzy element its approximation is given by the map $r_i : A \to A_e \times A_e$ defined as

$$r_i(a) := \langle \nu(a), \neg\mu(a)\rangle = \langle \nu(a), \sim a\rangle$$

drawn in the following diagram:

$$\begin{tikzcd}
 & a \in A \arrow[dl, "\nu"'] \arrow[dr, "\sim"] \arrow[dd, "r_i"] & \\
\nu(a) \in A_e \arrow[dr] & & \sim a \in A_e \arrow[dl] \\
 & \langle \nu(a), \sim a \rangle &
\end{tikzcd}$$

2.2. **Fuzzy and shadowed sets.** We now come back to the concrete cases of fuzzy and shadowed sets, and we show how it is possible to give them in a canonical way the structure of BZMV^{dM} algebras.

Proposition 2.7. *Let $\mathcal{F}(X) = [0,1]^X$ be the collection of fuzzy sets on the universe X. Let us define the operators:*

$$(f \oplus g)(x) := min\{1, f(x) + g(x)\}$$
$$\neg f(x) := 1 - f(x)$$
$$\sim f(x) := \begin{cases} 1 & if \quad f(x) = 0 \\ 0 & otherwise \end{cases}$$

and the identically zero fuzzy set: $\forall x \in X$, $\mathbf{0}(x) := 0$. Then, the structure $\langle \mathcal{F}(X), \oplus, \neg, \sim, \mathbf{0} \rangle$ is a BZMV^{dM} algebra such that $\mathbf{1} = \sim \mathbf{0} = \neg \mathbf{0}$ is the identically one fuzzy set: $\forall x \in X$, $\mathbf{1}(x) = 1$.

The structure $\langle \mathcal{F}(X), \oplus, \neg, \sim, \mathbf{0} \rangle$ is not a BZMV^3 algebra as showed in the following counterexample.

Example 2.1. Let us consider the fuzzy set $\mathbf{\frac{1}{3}}$, on a fixed domain X. Then, $\sim \mathbf{\frac{1}{3}} \oplus \mathbf{\frac{1}{3}} \oplus \mathbf{\frac{1}{3}} = \mathbf{\frac{2}{3}} \neq \mathbf{1}$.

Similarly, it is possible to give the structure of BZMV^3 algebra to the collection of shadowed sets $\mathcal{S}(X) = \{0, \frac{1}{2}, 1\}^X$ on the universe X. The operations syntactically are exactly as in Proposition 2.7, but they are defined on the domain of shadowed sets.

Proposition 2.8. *Let $\mathcal{S}(X) = \{0, \frac{1}{2}, 1\}^X$ be the collection of shadowed sets on the universe X. Then, the structure $\langle \mathcal{S}(X), \oplus, \neg, \sim, \mathbf{0} \rangle$, where $\oplus, \neg, \sim, \mathbf{0}$ are defined as in Proposition 2.7, is a BZMV^3 algebra.*

Remark 1. We do agree with the claim of [18] that “shadowed sets are conceptually close to rough sets” in the meaning that they are both models of the same abstract structure of BZMV^3 algebra, “even though [their] mathematical foundations are very different. In rough sets we distinguish between three regions: the regions whose elements are fully accepted (membership value equal 1) and belonging to the concept under discussion; the regions whose elements definitely do not belong to the concept; the regions where membership grade is doubtful - these come in the form of the shadows of the introduced shadowed sets. In this sense shadowed sets narrow down a conceptual and an algorithmic gap between fuzzy sets and rough sets highlighting how these could be directly related. There is some significant difference. In rough sets the approximation space is defined in advance and the equivalence classes are kept fixed. In the concept of shadowed sets these classes are assigned dynamically".

The lattice operators induced from a BZMV^{dM} algebra according to (2b) and (2c) can be consequently defined both in $\mathcal{F}(X)$ and in $\mathcal{S}(X)$, and in these particular cases it is easy to prove the following identities:

$$(f_1 \vee f_2)(x) = \max\{f_1(x), f_2(x)\}$$
$$(f_1 \wedge f_2)(x) = \min\{f_1(x), f_2(x)\}$$

Analogously, the MV conjunction $\odot$ defined in (2a) is given by:

$$(f_1 \odot f_2)(x) = \max\{0, f_1(x) + f_2(x) - 1\}$$

Now let us consider a fuzzy set $f \in \mathcal{F}(X)$. Then, the necessity $\nu(f)$ and the possibility $\mu(f)$ of f are the two fuzzy sets defined for any point $x \in X$ respectively by the laws:

$$\nu(f)(x) = \begin{cases} 1 & f(x) = 1 \\ 0 & f(x) \neq 1 \end{cases}$$
$$\mu(f)(x) = \begin{cases} 0 & f(x) = 0 \\ 1 & f(x) \neq 0 \end{cases}$$

Trivially, a fuzzy sets $e \in \mathcal{F}(X)$ is sharp ($\forall x \in X$, $e(x) = \nu(e)(x) = \mu(e)(x)$) iff it is a Boolean valued map $e : X \mapsto \{0,1\}$, i.e., iff $e \in \{0,1\}^X$. Therefore, $\mathcal{F}_e(X) = \{0,1\}^X$. According to the general results of Proposition 2.4 (and this can be directly proved making use of the now introduced operations on $\{0,1\}^X$), for all $e_1, e_2 \in \mathcal{F}_e(X)$ one has the identities $(e_1 \wedge e_2)(x) = (e_1 \odot e_2)(x)$, $(e_1 \vee e_2)(x) = (e_1 \oplus e_2)(x)$, and $\neg e_1(x) = \sim e_1(x)$. Moreover, the structure $\langle \mathcal{F}_e(X), \wedge, \vee, \neg, \mathbf{0}\rangle$ is a Boolean algebra.

For any fuzzy set $f \in \mathcal{F}(X)$, let us introduce its *necessity domain* $A_1(f)$ and *possibility domain* $A_p(f)$ as the two subsets of the universe defined as follows:

$$A_1(f) := \{x \in X : f(x) = 1\}$$
$$A_p(f) := \{x \in X : f(x) \neq 0\}$$

For any subset A of the universe X, let us denote by χ_A the *characteristic functional* of A defined as $\chi_A(x) = 1$ if $x \in A$, and equal to 0 otherwise. Using the above introduced domains of a fuzzy set f one immediately obtains that

$$\nu(f) = \chi_{A_1(f)} \quad \text{and} \quad \mu(f) = \chi_{A_p(f)}$$

In particular, $\mathcal{F}_e(X) = \{\chi_A : A \subseteq X\}$, i.e., in the context of fuzzy sets theory the sharp (crisp) elements are the characteristic function of subsets of the universe. Moreover, the mapping $\chi : \mathcal{P}(X) \mapsto \mathcal{F}_e(X)$ associating to any subset A of X its characteristic functional $\chi_A \in \mathcal{F}_e(X)$ is a Boolean lattice isomorphism. Let us stress that characteristic functionals, as Boolean valued functions, are in particular shadowed sets and so this situation is represented in the following diagram:

$$\begin{tikzcd}
 & {[0,1]^X} \\
 & \{0,1,1/2\}^X \arrow[u, "\mathrm{id}"'] \\
A \in \mathcal{P}(X) \arrow[uur] \arrow[r, "\chi"'] & \{0,1\}^X \ni \chi_A \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

Let us now show how the abstract rough approximation of a fuzzy set f, as the pair $r_i(f) = \langle \nu(f), \sim f \rangle$, allows one to single out the 0–approximation induced shadowed set $s_0(f)$. In fact, $\nu(f)$ can be interpreted as the characteristic function of the elements which have value 1 in the induced shadowed set; and $\sim f$ as the characteristic function of the elements which have value 0. The other elements of the universe X represents the shadow of the shadowed set.

Precisely, in the case of $\alpha = 0$ the shadowed set $s_\alpha(f)$ defined in equation (1) can be obtained through a combination of these modal operators according to the following identity:

$$s_0(f) = \mu(f) \odot \left(\nu(f) \oplus \mathbf{\frac{1}{2}} \right) \tag{3}$$

where we recall that $\mathbf{\frac{1}{2}}$ is the fuzzy set identically equal to $\frac{1}{2}$, i.e., for all $x \in X$, $\mathbf{\frac{1}{2}}(x) := \frac{1}{2}$. Indeed (and compare with (1) applied to the particular case of $\alpha = 0$) one immediately obtains:

$$\forall x \in X, \quad [\mu(f) \odot (\nu(f) \oplus \mathbf{\frac{1}{2}})](x) = \begin{cases} 0 & f(x) = 0 \\ 1 & f(x) = 1 \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

The mapping $s_0 : \mathcal{F}(X) \to \mathcal{S}(X)$, $f \to s_0(f)$, is not a bijection nor an homomorphism between $BZMV^{dM}$ algebras, as can be seen in the following counterexample.

Example 2.2. Let us consider the fuzzy sets $f_1, f_2 : [0,1] \mapsto [0,1]$ defined as:

$$f_1(x) := \begin{cases} 0.2 & \text{if} \quad x = 0 \\ 0 & \text{otherwise} \end{cases} \qquad f_2(x) := \begin{cases} 0.3 & \text{if} \quad x = 0 \\ 0 & \text{otherwise} \end{cases}$$

So, $f_1 \neq f_2$ but

$$s_0(f_1) = s_0(f_2) = \begin{cases} 0.5 & \text{if} \quad x = 0 \\ 0 & \text{otherwise} \end{cases}$$

and this proves that s_0 is not a bijection.
Furthermore (stressing with symbols $\oplus_S$ and $\oplus_F$ the MV disjunction – also "truncated" sum – operation acting on $\mathcal{S}(X)$ and $\mathcal{F}(X)$ respectively),

$$[s_0(f_1) \oplus_S s_0(f_2)](x) = \begin{cases} 1 & f(x) = 0 \\ 0 & \text{otherwise} \end{cases} \neq \begin{cases} \frac{1}{2} & f(x) = 0 \\ 0 & \text{otherwise} \end{cases} = [s_0(f_1 \oplus_F f_2)](x)$$

and so s_0 is neither a BZMV^{dM} algebras homomorphism.

Of course, s_0 gives only the induced shadowed set in the particular case of $\alpha = 0$, and it does not capture all the possible ones that can be obtained from a fuzzy set by equation (1). In order to consider all these possibilities, it is necessary (and sufficient) to generalize the intuitionistic negation as follows:

$$\forall \alpha \in [0, \frac{1}{2}) \qquad (\sim_\alpha f)(x) := \begin{cases} 1 - f(x) & \text{if } f(x) \leq \alpha \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Clearly, this is a generalization of $\sim$, in fact when $\alpha = 0$ we obtain $\sim_0 f = \sim f$. In Figure 2, it is represented a fuzzy set and its α - impossibility (i.e., $\sim_\alpha$).

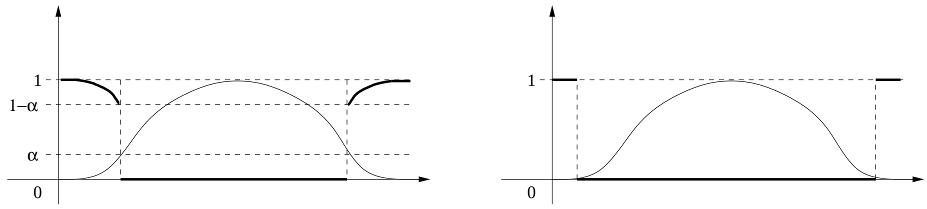

FIGURE 2. Generalized $\sim_\alpha$: at the left it is represented the case $\alpha \in (0, 1/2)$ and at the right the case $\alpha = 0$

The derived operators, μ_α and ν_α then become:

$$\mu_\alpha(f)(x) := (\neg \sim_\alpha f)(x) = \begin{cases} f(x) & f(x) \leq \alpha \\ 1 & f(x) > \alpha \end{cases} \tag{5}$$

$$\nu_\alpha(f)(x) := (\sim_\alpha \neg f)(x) = \begin{cases} f(x) & f(x) \geq (1-\alpha) \\ 0 & f(x) < (1-\alpha) \end{cases} \tag{6}$$

Let us introduce the shadowed set $s_\alpha(f)$, induced by the fuzzy set f and defined analogously to the (3):

$$s_\alpha(f) := \mu_\alpha(f) \odot \left(\nu_\alpha(f) \oplus \mathbf{\frac{1}{2}} \right)$$

This coincide with the shadowed set previously defined by the (1). Indeed, for arbitrary $x \in X$ one has:

$$[\mu_\alpha(f) \odot (\nu_\alpha(f) \oplus \mathbf{\frac{1}{2}})](x) = \begin{cases} 0 & f(x) \leq \alpha \\ 1 & f(x) \geq 1 - \alpha \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

So, given a fuzzy set f on one side we can obtain the rough approximation $r_\alpha(f) = \langle \nu_\alpha(f), \mu_\alpha(f) \rangle$ and on the other side we can induce the shadowed set $s_\alpha(f)$. The relation between the two mappings r_α and s_α is given by the mapping $\psi : \mathcal{F}(X) \times \mathcal{F}(X) \mapsto \mathcal{S}(X)$ which associates to any pair of fuzzy sets $h, k \in \mathcal{F}(X)$ the shadowed set

$$\psi(h, k) := h \odot \left(k \oplus \mathbf{\frac{1}{2}} \right) \tag{7}$$

To be precise, we have the following identity involving two mappings from $\mathcal{F}(X)$ into $\mathcal{S}(X)$:

$$\psi \circ r_\alpha = s_\alpha$$

Indeed, for an arbitrary fuzzy set $f \in \mathcal{F}(X)$ one has that:

$$\psi(r_\alpha(f)) = \psi(\langle \nu_\alpha(f), \mu_\alpha(f) \rangle) = \mu_\alpha(f) \odot (\nu_\alpha(f) \oplus \frac{\mathbf{1}}{\mathbf{2}}) = s_\alpha(f)$$

In the following diagram all the three functions, r_α, s_α and ψ are drawn, showing the relation among them:

$$\begin{tikzcd} f \arrow[r, "r_\alpha"] \arrow[dr, "s_\alpha"'] & r_\alpha(f) \arrow[d, "\psi"] \\ & s_\alpha(f) \end{tikzcd}$$

We remark that the function ψ is not a bijection as can be seen in the following counterexample.

Example 2.3. Let $\mathbf{1}$ be the identically one shadowed set on the universe $[0,1]$ and let $k : [0,1] \mapsto \{0, \frac{1}{2}, 1\}$ be the shadowed set defined for arbitrary $x \in [0,1]$ as follows:

$$k(x) = \begin{cases} 1 & \text{if } x \in [0, 1/2) \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

Then $\psi(\mathbf{1}, k) = \mathbf{1}$ and $\psi(\mathbf{1}, \mathbf{1}) = \mathbf{1}$.

Coming back to our algebraic structure, we have that, by a substitution of $\sim$ by $\sim_\alpha$, the system $\langle [0,1]^X, \oplus, \neg, \sim_\alpha, 0 \rangle$ is no more a $BZMV^{dM}$ algebra. In fact, for instance, axiom $BZMV6$ is not satisfied. Given a fuzzy set f, we have:

$$f(x) \oplus (\sim_\alpha \sim_\alpha f)(x) = \begin{cases} 1 & f(x) > \alpha \\ f(x) & f(x) \leq \alpha \end{cases} \neq \begin{cases} 1 & f(x) > \alpha \\ 0 & f(x) \leq \alpha \end{cases} = (\sim_\alpha \sim_\alpha f)(x).$$

Next section is devoted to the study of an algebrization of the structure $\langle \mathcal{F}(X), \oplus, \neg, \sim_\alpha, \mathbf{0} \rangle$ containing this new operator $\sim_\alpha$.

3. PRE-BZMV^{dM} ALGEBRAS

We, now, introduce a new algebra, which turns out to be weaker than BZMV^{dM} algebra. The advantage of this new structure is that it admits as a model the collection of fuzzy sets endowed with the operator $\sim_\alpha$.

Definition 3.1. A structure $\mathbf{A} = \langle \mathcal{A}, \oplus, \neg, \sim_w, 0 \rangle$ is a pre-BZMV^{dM} algebra, if the following are satisfied:

(1) The substructure $\langle \mathcal{A}, \oplus, \neg, 0 \rangle$ is an MV algebra, whose induced lattice operations are defined as

$$a \vee b := \neg(\neg a \oplus b) \oplus b$$
$$a \wedge b := \neg(\neg(a \oplus \neg b) \oplus \neg b)$$

and, as usual, the partial order is $a \leq b$ iff $a \wedge b = a$ (iff $\neg a \oplus b = a \to_L b = 1$).

(2) The following properties are satisfied:
   (a) $a \oplus \sim_w \sim_w a = \neg \sim_w a$
   (b) $\sim_w a \wedge \sim_w b \leq \sim_w (a \vee b)$

(c) $\sim_w a \vee \sim_w b = \sim_w (a \wedge b)$
(d) $\sim_w \neg a \leq \sim_w \neg \sim_w \neg a$

In general, it is possible to show that any BZMV^{dM} algebra is a pre-BZMV^{dM} algebra. In fact, in [6] it is shown that all axioms of definition 3.1 are true in any BZMV^{dM} algebra. In general, the vice versa does not hold.

Example 3.1. Let us define a pre–BZMV^{dM} algebra with elements $\mathcal{A} = \{0, a, b, 1\}$, and operators defined as in Table 1.

TABLE 1. Negations and $\oplus$ operators

x	$\neg x$	$\sim_w x$
0	1	1
a	b	b
b	a	0
1	0	0

$\oplus$	0	a	b	1
0	0	a	b	1
a	a	b	1	1
b	b	1	1	1
1	1	1	1	1

In Figure 3, it is drawn the Hasse diagram of the underlying lattice structure. Clearly, this structure is a pre–BZMV^{dM} algebra and not a BZMV^{dM} algebra.

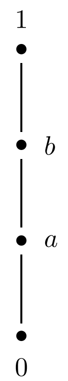

FIGURE 3. Hasse diagram of Table 1 pre–BZMV^{dM} algebra

Indeed, axiom (BZMV6) is not satisfied:

$$a \oplus \sim_w \sim_w a = a \oplus 0 = a \neq 0 = \sim_w \sim_w a$$

Proposition 3.1. *Let $\langle \mathcal{A}, \oplus, \neg, \sim_w, 0 \rangle$ be a pre-BZMV^{dM} algebra, then it is a BZMV^{dM} algebra iff the following interconnection rule holds:*

$$\forall a \in \mathcal{A},\ \sim_w \sim_w a = \neg \sim_w a.$$

Proof. We have seen that any BZMV^{dM} algebra is a fortiori a pre-BZMV^{dM} algebra. Then we have only to prove the converse.

Axioms (BZMV1) – (BZMV4) hold in any MV algebra and so in any pre-BZMV^{dM} algebra.

(BZMV5). In any MV algebra it holds $x \oplus \neg x = 1$. So, by setting $x = \sim_w a$, we

have $\sim_w a \oplus \neg \sim_w a = 1$. Then, by hypothesis, $\sim_w a \oplus \sim_w\sim_w a = 1$.
(BZMV6). Trivially, by Axiom $a \oplus \sim_w\sim_w a = \neg \sim_w a$.
(BZMV7). In [6] it is shown that this axiom is equivalent to the following two: $\sim_w\sim_w a = \neg \sim_w a$ and $\sim_w (a \wedge b) = \sim_w a \vee \sim_w b$. These last two are exactly the hypothesis and a pre-BZMV^{dM} axiom. □

Proposition 3.2. *Let $\langle \mathcal{A}, \oplus, \neg, \sim_w, 0\rangle$ be a pre-BZMV^{dM} algebra. Then, the following properties hold:*

(1) $\sim_w 0 = \neg 0$. *In the sequel we set* $1 := \sim_w 0 = \neg 0$ *and trivially* $\forall a \in \mathcal{A}$, $0 \leq a \leq 1$,
(2) $\neg a \oplus \neg \sim_w a = 1$ *(equivalently,* $a \odot \sim_w a = 0$*),*
(3) $\sim_w a \leq \neg a$,
(4) *If* $a \leq b$ *then* $\sim_w b \leq \sim_w a$ *(contraposition law),*
(5) $\sim_w (a \vee b) = \sim_w a \wedge \sim_w b$ *($\wedge$ de Morgan law).*

Proof. (1) By Axiom (a), setting $a = 0$, we have $0 \oplus \sim_w\sim_w 0 = \neg \sim_w 0$, that is

$$\sim_w\sim_w 0 = \neg \sim_w 0 \tag{*}$$

Using again axiom (a) and setting $a = \sim_w 0$ we obtain $\sim_w 0 \oplus \sim_w\sim_w\sim_w 0 = \neg \sim_w\sim_w 0$ and by(*) $\sim_w 0 \oplus \neg \sim_w\sim_w 0 = \neg\neg \sim_w 0 = \sim_w 0$. Now, we prove that $\sim_w 0 \oplus \neg \sim_w\sim_w 0 = \neg 0$, concluding the proof.
By axiom (c), setting $b = 0$, we have, $\sim_w (a \wedge 0) = \sim_w a \vee \sim_w 0$, that is

$$\sim_w 0 = \neg(\neg \sim_w a \oplus \sim_w 0) \oplus \sim_w 0 \tag{**}$$

Now, from MV property $a \oplus \neg a = \neg 0$, we have $(\neg \sim_w a \oplus \sim_w 0) \oplus \neg(\neg \sim_w a \oplus \sim_w 0) = 1$, which can be written as $[\neg(\neg \sim_w a \sim_w 0) \oplus \sim_w 0] \oplus \neg \sim_w 0 = \neg 0$. Applying (**) we have $\sim_w 0 \oplus \neg \sim_w a = \neg 0$. And, finally, setting $x = \sim_w 0$ in the last equality, we obtain $\sim_w 0 \oplus \neg \sim_w\sim_w 0 = \neg 0$.

(2) In any MV algebra it holds $\neg a \oplus a = 1$ and consequently $\neg a \oplus a \oplus b = 1$. Setting $b = \sim_w\sim_w a$ we have $\neg a \oplus (a \oplus \sim_w\sim_w a) = 1$ and by axiom (a) $\neg a \oplus \neg \sim_w a = 1$.
(3) By definition of $\wedge$ we have $\sim_w a \wedge \neg a = \sim_w \odot(\neg \sim_w a \oplus \neg a)$, and by (2) $\sim_w a \wedge \neg a = \sim_w a \odot 1 = \sim_w a$.
(4) In any MV algebra, it holds $a \leq b$ iff $a \wedge b = a$ iff $a \vee b = b$. Now, suppose that $a \leq b$, then we have $a \wedge b = a$ and $\sim_w (a \wedge b) = \sim_w a$. Applying axiom (c) we obtain $\sim_w a \vee \sim_w b = \sim_w a$, that is $\sim_w b \leq \sim_w a$.
(5) The inequality $\sim_w a \wedge \sim_w b \leq \sim_w (a \vee b)$ is axiom (b). Here we prove $\sim_w (a \vee b) \leq a \wedge \sim_w b$. In any lattice it holds $(a \vee b) \wedge a = a$. Thus, $(\sim_w a \vee \sim_w b) \wedge \sim_w b = \sim_w b$ and by axiom (c) $\sim_w (a \wedge b) \wedge \sim_w b = \sim_w b$. Substituting b by $a \vee b$ we obtain $\sim_w (a \wedge (a \vee b)) \wedge \sim_w (a \vee b) = \sim_w (a \vee b)$ and then $\sim_w a \wedge \sim_w (a \vee b) = \sim_w (a \vee b)$. Dually, we can obtain $\sim_w b \wedge \sim_w (a \vee b) = \sim_w (a \vee b)$, and finally $\sim_w a \wedge \sim_w b \wedge \sim_w (a \vee b) = \sim_w (a \vee b)$, i.e., $\sim_w (a \vee b) \leq \sim_w a \wedge \sim_w b$.

□

So $\sim_w$ is a unary operator satisfying in particular both de Morgan laws and the contraposition law (4) of Proposition 3.2. However, it is not an intuitionistic negation, in fact, in general, it satisfies neither the non contradiction law (property B3), nor the weak double negation law (property B1), nor the Brouwer law (i.e., the following law $\forall a$, $\sim_w a = \sim_w\sim_w\sim_w a$ is not satisfied).

Example 3.2. Let us consider the algebra defined in Example 3.1. In this structure we have that

$$a \wedge \sim_w a = a,$$
$$a \wedge \sim_w\sim_w a = 0,$$
$$\sim_w a = b \neq 1 = \sim_w\sim_w\sim_w a.$$

Anyway, also in a pre-BZMV^{dM} algebra, it is possible to introduce modal operators of necessity, $\nu_w(a) := \sim_w \neg a$ and possibility $\mu_w(a) := \neg \sim_w a$. However, in this structure ν_w and μ_w do not have an S_5–like behavior but only an S_4–like one (always based on a Kleene lattice instead of on a Boolean one).

Proposition 3.3. *Let $\langle \mathcal{A}, \oplus, \neg, \sim_w, 0\rangle$ be a pre-BZMV^{dM} algebra. Then, for every $a \in \mathcal{A}$ the following properties are satisfied:*

(1) $\nu_w(a) \leq a \leq \mu_w(a)$ *[T principle]*
(2) $\nu_w(\nu_w(a)) = \nu_w(a)$ $\quad \mu_w(\mu_w(a)) = \mu_w(a)$ *[S_4 principle]*

Proof. (1) From (3) of Prop. 3.2 $\sim_w a \leq \neg a$, we have $\neg\neg a \leq \neg \sim_w a$, that is $a \leq \mu_w(a)$.
Again, from $\sim_w a \leq \neg a$, we have $\sim_w (\neg a) \leq \neg(\neg a)$, that is $\nu_w(a) \leq a$.
(2) By $\nu_w(a) \leq a$ we get $\nu_w(\nu_w(a)) \leq \nu_w(a)$. Then axiom 2(d) is $\nu_w(a) \leq \nu_w(\nu_w(a))$. So, we have $\nu(a) = \nu(\nu(a))$. By the last one and the double negation law for the fuzzy negation $\neg$, it is easily obtained that $\mu_w(a) = \mu_w(\mu_w(a))$.

□

In general the following properties are not satisfied by these weak modalities:

(3) $a \leq \nu_w(\mu_w(a))$;
(4) $\mu_w(a) = \nu_w(\mu_w(a))$;
(5) $\nu_w(a) = \mu_w(\nu_w(a))$.

This can be seen by the following counterexample.

Example 3.3. Let us consider the algebra defined in Example 3.1. Then for the particular element a the following hold:

(3) $\nu_w(\mu_w(a)) = 0 < a$
(4) $\mu_w(a) = b \neq 0 = \nu_w(\mu_w(a))$
(5) $\nu_w(b) = b \neq 1 = \mu_w(\nu_w(b))$

Even if the necessity and possibility mappings have a weaker modal behavior in pre–BZVM^{dM} algebras than in BZMV^{dM} algebras, they can still be used to define a lower and upper approximation, and it turns out that ν_w is an (additive) topological interior operator and μ_w is a (multiplicative) topological closure operator.

Proposition 3.4. *Let $\langle \mathcal{A}, \oplus, \neg, \sim_w, 0\rangle$ be a pre-BZMV^{dM} algebra. Then the map $\nu_w : \mathcal{A} \to \mathcal{A}$ such that $\nu_w(a) := \sim_w \neg a$ is an* (additive) topological interior *operator,i.e.:*

(I_0)	$1 = \nu_w(1)$	(*normalized*)
(I_1)	$\nu_w(a) \leq a$	(*decreasing*)
(I_2)	$\nu_w(a) = \nu_w(\nu_w(a))$	(*idempotent*)
(I_3)	$\nu_w(a \wedge b) = \nu_w(a) \wedge \nu_w(b)$	(*multiplicativity*)
(I_4)	$\nu_w(a) \vee \nu_w(b) = \nu_w(a \vee b)$	(*additivity*)

Proof. Trivially, $\nu_w(1) = \sim_w \neg(\neg 0) = \sim_w 0 = 1$. (I_1) and (I_2) are proved in Proposition 3.3. (I_3) and (I_4) are a direct consequence of De Morgan laws which hold for both negations $\neg$ and $\sim_w$. □

Since in general it holds that $\nu_w(a) \leq a$ it is interesting to collect all the element for which the equality holds (i.e., the elements which coincide with their interior), called *open* elements, whose collection is then defined as

$$\mathbb{O}(\mathcal{A}) = \{a \in \mathcal{A} : a = \nu_w(a)\}.$$

Proposition 3.5. *Let $\langle \mathcal{A}, \oplus, \neg, \sim_w, 0\rangle$ be a pre-BZMV^{dM} algebra. Then the map $\mu_w : \mathcal{A} \to \mathcal{A}$ such that $\mu_w(a) := \neg \sim_w a$ is a* (multiplicative) topological closure *operator. That is, the following are satisfied:*

(C_0)	$0 = \mu_w(0)$	(*normalized*)
(C_1)	$a \leq \mu_w(a)$	(*increasing*)
(C_2)	$\mu_w(a) = \mu_w(\mu_w(a))$	(*idempotent*)
(C_3)	$\mu_w(a) \vee \mu_w(b) = \mu_w(a \vee b)$	(*additivity*)
(C_4)	$\mu_w(a \wedge b) = \mu_w(a) \wedge \mu_w(b)$	(*multiplicativity*)

Proof. By duality of Proposition 3.4. □

The collection of all *closed* elements is then defined as

$$\mathbb{C}(\mathcal{A}) = \{a \in \mathcal{A} : a = \mu_w(a)\}$$

Comparing these results with the ones of Propositions 2.5, 2.6, we remark that in this case the interconnection rule does not hold for both interior and closure operators, as can be seen in the following counterexample.

Example 3.4. Both interconnection rules are equal to the de Morgan property $a \wedge \sim_w a = 0$, which in general does not hold in pre–BZMV^{dM} algebras as showed in Example 3.2.

Further, in general, in a pre–BZMV^{dM} algebra, the subsets of $\mathcal{A}$ of open and closed elements do not coincide

$$\mathbb{C}(\mathcal{A}) \neq \mathbb{O}(\mathcal{A})$$

neither one is a subset of the other, as showed in the following example.

Example 3.5. We again consider the algebra defined in Example 3.1. Then, we have that a is a closed element, i.e., $\mu_w(a) = a$, but it is not an open element, indeed, $\nu_w(a) = 0$. Viceversa, b is an open element, i.e., $\nu_w(b) = b$, but not a closed one, since $\mu_w(b) = 1$.

So, it is worthwhile to consider also the set of all *clopen* elements, i.e., elements which are both closed and open:

$$\mathbb{CO}(\mathcal{A}) = \mathbb{C}(\mathcal{A}) \cap \mathbb{O}(\mathcal{A})$$

Also in the context of pre–BZMV^{dM} it is possible to have several definitions of sharp elements. As in BZMV^{dM} we consider the following definitions of exact elements.

The *Kleene sharp* (K-sharp) elements which satisfy the non contradiction (or equivalently the excluded middle) law with respect to the Kleene negation:

$$\mathcal{A}_{e,\neg} := \{e \in \mathcal{A} : e \wedge \neg e = 0\} = \{e \in \mathcal{A} : e \vee \neg e = 1\}$$

The *Brouwer sharp* (B-sharp) elements which satisfy the double negation law with respect to the negation $\sim_w$:

$$\mathcal{A}_{e,\sim_w} := \{e \in \mathcal{A} : \sim_w \sim_w e = e\}$$

The $\oplus$-*sharp* elements which satisfy the idempotent properties with respect the operations $\oplus$ and $\odot$:

$$\mathcal{A}_{e,\oplus} = \{e \in \mathcal{A} : e \oplus e = e\} = \{e \in \mathcal{A} : e \odot e = e\}$$

But in pre—BZMV^{dM} algebras we also consider the *Brouwer–0 sharp* elements which satisfy the non contradiction law with respect to the negation $\sim_w$:

$$\mathcal{A}_{e,\sim_w,0} := \{e \in \mathcal{A} :\ e \wedge \sim_w e = 0\}$$

Finally, we define as *Brouwer–1 sharp* the elements which satisfy the excluded middle law with respect to the negation $\sim_w$:

$$\mathcal{A}_{e,\sim_w,1} := \{e \in \mathcal{A} :\ e \vee \sim_w e = 1\}.$$

The relations among all these collections of exact sets are analyzed in the following propositions.

Lemma 3.1. *In any pre–BZMV^{dM} algebra* **A** *the following properties hold:*

(1) $\sim_w a = \sim_w \neg \sim_w a$
(2) $\sim_w a \wedge \sim_w \sim_w a = 0$
(3) $(a \odot \neg b) \oplus (a \wedge b) = a$

Proof. (1) It follows trivially by the property $\nu_w(\nu_w(a)) = \nu_w(a)$.

(2) Applying axiom (a) to $\neg \sim_w a$ we have $\neg \sim_w a \oplus \sim_w \sim_w \neg \sim_w a = \neg \sim_w \neg \sim_w a$ and by property (1) of this lemma $\neg \sim_w a \oplus \sim_w \sim_w a = \neg \sim_w a$. Now, by $\wedge$ definition $\sim a \wedge \sim_w \sim_w a = \sim_w a \odot (\neg \sim_w a \oplus \sim_w \sim_w a) = (\sim_w a) \odot \neg(\sim_w a) = 0$.

(3) In any MV algebra it holds $a \odot b \leq a$, i.e., $(a \odot b) \wedge a = (a \odot b)$ and by $\wedge$ definition:

$$a \odot (\neg a \oplus (a \odot b)) = a \odot b \qquad (*)$$

By lattice properties it holds $(a \wedge b) \vee a = a$ which by $\vee$ definition is $(a \wedge b) \oplus (\neg(a \wedge b) \odot a) = a$. By de Morgan properties and $\vee$ definition we have $a = (a \wedge b) \oplus ((\neg a \vee \neg b) \odot a) = (a \wedge b) \oplus (a \odot (\neg a \oplus (a \odot \neg b)))$. Finally, applying Equation (*), we have the thesis.

□

Proposition 3.6. *Let* $\mathbf{A}$ *be a pre–BZMV*^{dM} *algebra. Then,*

$$\mathbb{CO}(\mathcal{A}) = \mathcal{A}_{e,\sim_w} = \mathcal{A}_{e,\sim_w,1} = \mathcal{A}_{e,\neg} = \mathcal{A}_{e,\oplus} \subseteq \mathbb{O}(\mathcal{A}) \subseteq \mathcal{A}_{e,\sim_w,0}$$

Proof. $\mathbb{CO}(\mathcal{A}) \subseteq \mathcal{A}_{e,\sim_w}$.
Let $e \in \mathbb{CO}(\mathcal{A})$. Then $\neg \sim_w e = e = \neg\neg e$, so $\sim_w e = \neg e$ and finally $\sim_w\sim_w e = \sim_w \neg e = e$, that is $e \in \mathcal{A}_{e,\sim_w}$.
$\mathcal{A}_{e,\sim_w} \subseteq \mathcal{A}_{e,\sim_w,1}$.
Let us suppose that $e = \sim_w\sim_w e$. Then $e\wedge \sim_w e = \sim_w\sim_w e\wedge \sim_w e = 0$ by Lemma 3.1(2). Thus, $\sim 0 = \sim (e\wedge \sim_w e) = \sim_w e\vee \sim_w\sim_w e = \sim_w e \vee e$. that is $e \in \mathcal{A}_{e,\sim_w,1}$.
$\mathcal{A}_{e,\sim_w,1} \subseteq \mathcal{A}_{e,\neg}$.
Let us suppose that $e \in \mathcal{A}_{e,\sim_w,1}$, i.e., $e\vee \sim_w e = 1$. Then, considering that $\sim_w e \leq \neg e$, we have $1 = e\vee \sim_w e \leq e \vee \neg e$, that is $e\neg\neg e = 1$ and $e \in \mathcal{A}_{e,\neg}$.
$\mathcal{A}_{e,\neg} = \mathcal{A}_{e,\oplus}$.
It holds in any MV algebra ([9]).
$\mathcal{A}_{e,\neg} \subseteq \mathbb{CO}(\mathcal{A})$.
Let us suppose that $e \in \mathcal{A}_{e,\neg}$. Since we already now that $\nu_w(e) \leq e \leq \mu_w(e)$ in order to prove that e is clopen it is sufficient to show that $\mu_w(e) \leq \nu_w(e)$. By $e \vee \neg e = 1$ we have $\sim_w e\wedge \sim_w \neg e = 0$. Now, setting $a = \sim_w \neg e$ and $b = \sim_w e$ in Lemma 3.1(3) we obtain $(\sim_w \neg e \odot \neg \sim_w e) \oplus (\sim_w \neg e\wedge \sim_w e) = \sim_w \neg e$ that is

$$(*) \qquad \sim_w \neg e \odot \neg \sim_w e = \sim_w \neg e$$

By $e \wedge \neg e = 0$ we have $\sim_w e\vee \sim_w \neg e = 1$, that is $\sim_w e \oplus (\sim_w \neg e \odot \neg \sim_w a) = 1$ and by Equation (*) $\sim_w e\oplus \sim_w \neg e = 1$. Now, by lattice properties we have $(\sim_w \neg e) \vee (\sim_w \neg e \wedge \neg \sim_w e) = \sim_w \neg e$, that is $\sim_w \neg e \vee (\neg \sim_w e \odot (\sim_w e\oplus \sim_w \neg e)) = \sim_w \neg e$. So, $\sim_w \neg e \vee \neg \sim_w e = \sim_w \neg e$, i.e., $\neg \sim_w e \leq \sim_w \neg e$.
$\mathbb{CO}(\mathcal{A}) \subseteq \mathbb{O}(\mathcal{A})$. Trivial.
$\mathbb{O}(\mathcal{A}) \subseteq \mathcal{A}_{e,\sim_w,0}$.
Let $e \in \mathbb{O}(\mathcal{A})$. By axiom (a) applied to $\neg e$ we have $\neg e\oplus \sim\sim \neg e = \neg \sim \neg e$ and considering that e is an open element $\neg e\oplus \sim e = \neg e$. Now, by definition of $\wedge$, $e\wedge \sim e = e \odot (\neg e\oplus \sim e) = e \odot \neg e = 0$, i.e., $e \in \mathcal{A}_{e,\sim_w,0}$. □

We will prove now that the collection of clopen elements is a Boolean algebra. To be precise, the following proposition holds.

Proposition 3.7. *Let* $\mathbf{A} = \langle \mathcal{A}, \oplus, \neg, \sim_w, 0\rangle$ *be a pre–BZMV*^{dM} *algebra. Then the collection* $\mathbb{CO}(\mathcal{A})$ *of all its clopen elements satisfies the following properties:*

(1) *The lattice connectives coincide with the MV ones:*

$$\forall e, f \in \mathbb{CO}(\mathcal{A}), \quad e \wedge f = e \odot f \quad \textit{and} \quad e \vee f = e \oplus f$$

(2) *The two negation connectives (Kleene and Brouwer) coincide:*

$$\forall e \in \mathbb{CO}(\mathcal{A}), \quad \neg e = \sim_w e$$

(3) *The structure* $\mathbf{A}_e = \langle \mathbb{CO}(\mathcal{A}), \wedge, \vee, \neg, 0\rangle$ *is a Boolean lattice (algebra), which is the largest pre–BZMV*^{dM} *sub algebra of* $\mathbf{A}$ *that is at the same time a Boolean algebra with respect to the same operations* $\wedge(=\odot)$, $\vee(=\oplus)$, *and* $\neg(=\sim)$.

Proof. Trivially, if $e \in \mathbb{CO}(\mathcal{A})$, then $e = \neg \sim_w e$ and by $\neg$ double negation law $\neg e = \sim_w e$. The remaining of the proposition holds for any MV algebra, as proved by Chang in [9], thus also for any pre–BZMV^{dM} algebra. □

The converse of the above properties, i.e., $\mathbb{O}(\mathcal{A}) \subseteq \mathcal{A}_{e,\neg}$ and $\mathcal{A}_{e,\sim_w,0} \subseteq \mathbb{O}(\mathcal{A})$ are not valid and the two sets $\mathcal{A}_{e,\sim_w,0}$ and $\mathbb{C}(\mathcal{A})$ are incomparable as can be seen in the following counterexamples.

Example 3.6. Let us consider Example 3.1. Then, we have that a is a closed element, i.e., $a = \mu_w(a)$, but it does not satisfy the non contradiction law with respect to negation $\sim_w$: $a \wedge \sim_w a = a$. Thus, we have that

$$\mathbb{C}(\mathcal{A}) \not\subseteq \mathcal{A}_{e,\sim,0}.$$

Further, b is an open element, i.e., $\nu_w(b) = b$, but it does not satisfy the non contradiction law for the Kleene negation: $b \wedge \neg b = a$. So,

$$\mathbb{O}(\mathcal{A}) \not\subseteq \mathcal{A}_{e,\neg}.$$

Example 3.7. In Table 2 it is introduced another example of pre–BZMV^{dM} algebra, whose Hasse diagram is given in Figure 4.

TABLE 2. Negations and $\oplus$ operators

x	$\neg x$	$\sim_w x$
0	1	1
a	a	0
1	0	0

$\oplus$	0	a	1
0	0	a	1
a	a	1	1
1	1	1	1

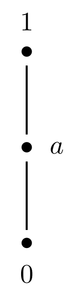

FIGURE 4. Hasse diagram of Table 2 pre–BZMV^{dM} algebra

In this algebra we have that the element a satisfies the non contradiction law with respect to the negation $\sim_w$: $a \wedge \sim_w a = 0$, but it is not a closed or an open element:

$$\mu_w(a) = 1 \neq a$$
$$\nu_w(a) = 0 \neq a$$

Thus this proves that

$$\mathcal{A}_{e,\sim_w,0} \not\subseteq \mathbb{C}(\mathcal{A})$$
$$\mathcal{A}_{e,\sim_w,0} \not\subseteq \mathbb{O}(\mathcal{A})$$

The above considerations lead to the definition of an abstract approximation space generated by a pre-BZMV^{dM} algebra.

Definition 3.2. Let $\mathbf{A}$ be a pre-BZMV^{dM} algebra. The induced rough approximation space is the structure $\langle \mathcal{A}, \mathbb{O}(\mathcal{A}), \mathbb{C}(\mathcal{A}), \nu_w, \mu_w \rangle$, where

- $\mathcal{A}$ is the set of approximable elements;
- $\mathbb{O}(\mathcal{A}) \subseteq \mathcal{A}$ is the set of *innerdefinable* elements, such that 0 and $1 \in \mathbb{O}(\mathcal{A})$;
- $\mathbb{C}(\mathcal{A}) \subseteq \mathcal{A}$ is the set of *outerdefinable* elements, such that 0 and $1 \in \mathbb{C}(\mathcal{A})$;
- $\nu_w : \mathcal{A} \to \mathbb{O}(\mathcal{A})$ is the *inner approximation map*;
- $\mu_w : \mathcal{A} \to \mathbb{C}(\mathcal{A})$ is the *outer approximation map*.

For any element $a \in \mathcal{A}$, its *rough approximation* is defined as the pair:

$$r_w(a) := \langle \nu_w(a), \mu_w(a) \rangle \quad [\text{with} \quad \nu_w(a) \leq a \leq \mu_w(a)]$$

drawn in the following diagram:

$$\begin{tikzcd}
& a \in \mathcal{A} \arrow[dl, "\nu_w"'] \arrow[dr, "\mu_w"] \arrow[dd, "r_w"] & \\
\nu_w(a) \in \mathbb{O}(\mathcal{A}) \arrow[dr] & & \mu_w(a) \in \mathbb{C}(\mathcal{A}) \arrow[dl] \\
& \langle \nu_w(a), \mu_w(a) \rangle &
\end{tikzcd}$$

This approximation is the best approximation by open (resp. closed) elements that it is possible to define on a pre-BZMV^{dM} structure, i.e., there hold properties similar to (I1)–(I3) and (O1)–(O3), the only difference is that here we have to distinguish between open-exact and closed-exact elements.

3.1. **Fuzzy Sets.** The collection of all fuzzy sets can be equipped with a structure of pre-BZMV^{dM} algebra, according to the following result.

Proposition 3.8. *Let $\mathcal{F}(X)$ be the collection of fuzzy sets based on the universe X and let $\alpha \in [0, \frac{1}{2})$. Once defined the standard $\oplus$ and $\neg$ operators on $\mathcal{F}$, and the $\sim_\alpha$ negation as in Equation* (4)*, then the structure $\mathbb{F}_\alpha = \langle \mathcal{F}(X), \oplus, \neg, \sim_\alpha, \mathbf{0} \rangle$ is a pre-BZMV^{dM} algebra, which is not a BZMV^{dM} algebra.*

We now give an example of the fact that $\mathbb{F}_\alpha$ is not a BZMV^{dM} algebra.

Example 3.8. Let us consider the structure $\mathbb{F}_{0.4}$ with $X = \mathbb{R}$, and define the fuzzy set $f(x) = 0.3$ for all $x \in \mathbb{R}$. Then, $\sim_\alpha f(x) \oplus \sim_\alpha \sim_\alpha f(x) = 0.7$ for all x. So, axiom (BZMV5) is not satisfied.

In this context the modal operators of necessity ν_w and possibility μ_w are defined as in Equations (5), (6). As expected they do not satisfy the B and S_5 principles of Proposition 2.2.

Example 3.9. Let us consider the algebra $\mathbb{F}_{0.4}$, with $X = [0, 1]$, and define the fuzzy set

$$f(x) = \begin{cases} 0.3 & \text{if} \quad x < \frac{1}{2} \\ 0.7 & \text{otherwise} \end{cases}$$

We have:

$$\mu_\alpha(f(x)) = \begin{cases} 0.3 & x < \frac{1}{2} \\ 1 & x \geq \frac{1}{2} \end{cases} \neq \begin{cases} 0 & x < \frac{1}{2} \\ 1 & x \geq \frac{1}{2} \end{cases} = \nu_\alpha(\mu_\alpha(f(x))).$$

$$\nu_\alpha(f(x)) = \begin{cases} 0 & x < \frac{1}{2} \\ 0.7 & x \geq \frac{1}{2} \end{cases} \neq \begin{cases} 0 & x < \frac{1}{2} \\ 1 & x \geq \frac{1}{2} \end{cases} = \mu_\alpha(\nu_\alpha(f(x))).$$

Finally, $f(x)$ is incomparable with $\nu_\alpha(\mu_\alpha(f(x)))$.

In the pre–BZMV^{dM} algebraic context of fuzzy sets discussed in Proposition 3.8, the collection of closed and open elements are respectively:

$$\mathbb{C}(\mathcal{F}(X)) = \{f \in \mathcal{F}(X) : f(x) > \alpha \quad \text{iff} \quad f(x) = 1\}$$
$$\mathbb{O}(\mathcal{F}(X)) = \{f \in \mathcal{F}(X) : f(x) < 1 - \alpha \quad \text{iff} \quad f(x) = 0\}$$

The clopen sets are the 0–1 valued fuzzy sets, $\mathbb{CO}(\mathcal{F}(X)) = \{0,1\}^X$.

Example 3.10. In the universe $[0,1]$, once set $\alpha = 0.4$, an example of open element is the fuzzy set

$$f_1(x) = \begin{cases} 0 & \text{if} \quad x < \frac{1}{2} \\ 0.7 & \text{otherwise} \end{cases}$$

and an example of closed set is the fuzzy set

$$f_2(x) = \begin{cases} 0.3 & \text{if} \quad x < \frac{1}{2} \\ 1 & \text{otherwise} \end{cases}$$

The fuzzy sets f_1 and f_2 are drawn in Figure 5.

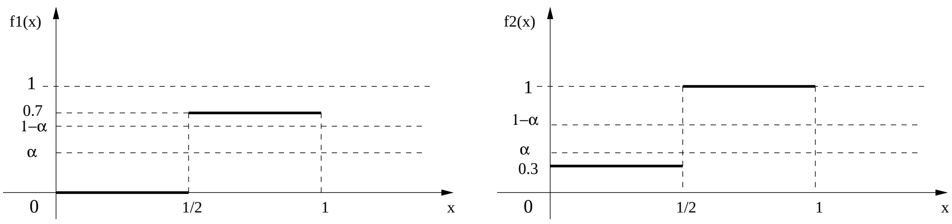

FIGURE 5. Example of open fuzzy set, f_1, and closed fuzzy set, f_2.

When considering the mapping $s_\alpha : \mathcal{F}(X) \mapsto \mathcal{S}(X)$, we see that it satisfies the following properties

$$s_\alpha(f) = s_\alpha(s_\alpha(f)) \qquad (\textit{idempotent})$$
$$f_1 \leq f_2 \quad \text{implies} \quad s_\alpha(f_1) \leq s_\alpha(f_2) \quad (\textit{monotone})$$

Further, if f is a shadowed set, i.e., a fuzzy set which assumes only three values, $f : X \mapsto \{0, \frac{1}{2}, 1\}$, then it also holds

$$s_\alpha(f) = f$$

Finally, we remark that we can also enrich the collection of shadowed sets $\{0, \frac{1}{2}, 1\}^X$ with the operation $\sim_\alpha$, in order to equip it with a pre-BZMV^{dM} structure. However, it can be easily proved that in this case $\sim_\alpha$ is equal to $\sim_0$ for all $\alpha \in [0, \frac{1}{2})$ and so we again obtain a $BZMV^3$ algebra.

3.2. **Level fuzzy sets and α–cuts.** Shadowed sets are an approximation of fuzzy sets through a less precise construct. As pointed out also in [17], in literature other methods of approximating a fuzzy set are known. In particular, we consider level fuzzy sets and α–cuts.

Level fuzzy sets are obtained from a fuzzy set by setting to zero the membership functions below a certain threshold value γ ([21]). Formally, let f be a fuzzy set on the domain X, a γ – level fuzzy set of f is defined as

$$\forall x \in X \quad f_l^{\gamma}(x) := \begin{cases} f(x) & \text{if} \quad f(x) \geq \gamma \\ 0 & \text{otherwise} \end{cases}$$

Level fuzzy sets can be easily obtained in the context of the pre–BZMV^{dM} algebra $\mathbb{F}_\alpha$. Indeed, the necessity ν_α, as defined in Equation (6), of a fuzzy set f is a $1-\gamma$ level fuzzy set:

$$f_l^{\gamma} = \nu_{(1-\gamma)}(f)$$

Of course, in this context, α (equivalently γ) is not limited to $[0, \frac{1}{2})$ but can belong to the whole interval $[0, 1]$.

Example 3.11. Let $X = [0, 1]$ and f the fuzzy set defined as $f(x) = x$. If we set $\gamma = 0.6$ than the 0.6 – level fuzzy set of f is:

$$\forall x \in [0,1] \quad f_l^{0.6}(x) = \begin{cases} x & \text{if} \quad x \geq 0.6 \\ 0 & \text{otherwise} \end{cases} = \nu_{0.4}(f)(x)$$

Further, also the negation defined by Radecki ([21]) on level fuzzy sets:

$$\forall x \in X \quad \neg_R(f_l^{\gamma}(x)) := \begin{cases} 0 & f(x) \geq \gamma \text{ and } f(x) > 1-\gamma \\ 1 - f(x) & \gamma \leq f(x) \leq 1-\gamma \\ 1 & f(x) < \gamma \end{cases}$$

can be recovered in a similar way:

$$\neg_R(f_l^{\gamma}) = \nu_{(1-\gamma)}(\neg f_l^{\gamma})$$

We remark that this equation is well defined for all $\gamma \in [0, 1]$, and if $\gamma > 0.5$ then $\neg_R(f_l^{\gamma}(x)) \in \{0, 1\}$.

Example 3.12. Let us consider the fuzzy set f of Example 3.11, and $\gamma = 0.6$. Then for all $x \in X$

$$\neg_R(f_l^{0.6})(x) = \begin{cases} 0 & x \geq 0.6 \\ 1 & x < 0.6 \end{cases} = \nu_{0.4}\left(\begin{cases} 1 - x & \text{if} \quad x \geq 0.6 \\ 1 & \text{otherwise} \end{cases} \right) = \nu_{0.4}(\neg f_l^{0.6}(x))$$

Similarly, we can also define α–cuts in the algebra $\mathbb{F}_\alpha$. We recall that an α–cut (resp., strong α–cut) is obtained from a fuzzy set by setting to 1 the membership values greater than or equal to (resp., greater than) a fixed value α and to 0 the other ones ([11]). Formally, let f be a fuzzy set defined over the domain X and $\alpha \in [0, 1]$ an α–cut of f is:

$$\forall x \in X \quad f_c^{\alpha}(x) := \begin{cases} 1 & \text{if} \quad f(x) \geq \alpha \\ 0 & \text{otherwise} \end{cases}$$

and a strong α–cut is defined as:

$$\forall x \in X \quad f_s^\alpha(x) := \begin{cases} 1 & \text{if} \quad f(x) > \alpha \\ 0 & \text{otherwise} \end{cases}$$

So, in the context of the algebra $\mathbb{F}_\alpha$, we have that $\mu_\alpha(\nu_\alpha(f))$ is a strong α–cut of f and $\nu_\alpha(\mu_\alpha(f))$ is a $(1-\alpha)$ – cut of f:

$$f_s^\alpha = \mu_\alpha(\nu_\alpha(f))$$
$$f_c^\alpha = \nu_{(1-\alpha)}(\mu_{(1-\alpha)}(f))$$

We remark that also in this case the parameter α can range in whole unit interval $[0,1]$.

Example 3.13. Let f be the fuzzy set of Example 3.11 and $\alpha = 0.6$. Then

$$\forall x \in [0,1] \quad f_c^{0.6}(x) = \begin{cases} 1 & x \geq 0.6 \\ 0 & x < 0.6 \end{cases} = \mu_{0.6}(\nu_{0.6}(f))$$

$$\forall x \in [0,1] \quad f_s^{0.6}(x) = \begin{cases} 1 & x > 0.6 \\ 0 & x \leq 0.6 \end{cases} = \nu_{0.4}(\mu_{0.4}(f))$$

Let us stress that there is another way to obtain α–cuts through a different definition of the generalized intuitionistic negation, let us call it $\approx_\alpha$. Once fixed $\alpha \in [0,1]$, let us define for all $x \in X$

$$\approx_\alpha f(x) := \begin{cases} 1 & \text{if} \quad f(x) \leq \alpha \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

In Figure 6 it is drawn a fuzzy set f and $\approx_\alpha (f)$ for a fixed value of α.

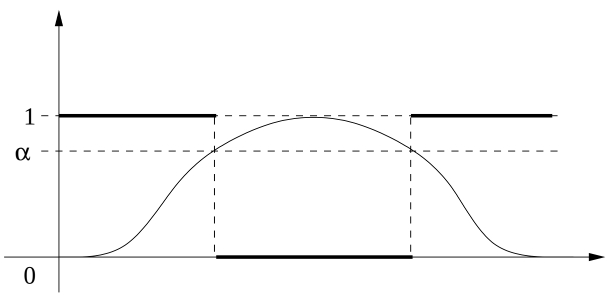

FIGURE 6. Example of a fuzzy set, and its $\approx_\alpha$ impossibility.

Of course, when $\alpha = 0$, we have that $\sim_0 f = \sim f$, that is the negation $\approx_\alpha$ is a generalization of $\sim$. This non standard negation satisfies the properties

(B2)	$\approx_\alpha (f \vee g) = \approx_\alpha f \wedge \approx_\alpha g$	($\wedge$ de Morgan)
(B2a)	$\approx_\alpha (f \wedge g) = \approx_\alpha f \vee \approx_\alpha g$	($\vee$ de Morgan)
(B2b)	$f \leq g$ implies $\approx_\alpha g \leq \approx_\alpha f$	(contraposition)

In general, these properties are not equivalent. This behaviour is due to the non validity of the double negation law $\approx_\alpha\approx_\alpha f = f$, neither in its weak form $f \leq \approx_\alpha\approx_\alpha f$.

Example 3.14. Let $f = \mathbf{\frac{1}{2}}$ and $\alpha = 0.7$. Then $\approx_{0.7}\approx_{0.7} f = \mathbf{0} < f$.

The following further properties are also satisfied by $\approx_\alpha$:

$$\approx_\alpha f = \approx_\alpha\approx_\alpha\approx_\alpha f \quad \text{(Brouwer law)}$$
$$\approx_\alpha\approx_\alpha f = \neg \approx_\alpha f \quad \text{(interconnection rule)}$$

On the other hand, in general, the following are not valid.

$$f \wedge \approx_\alpha f = 0 \quad \text{(non contradiction)}$$
$$\approx_\alpha f \leq \neg f \quad \text{(normalization)}$$

Example 3.15. Let $f = \mathbf{\frac{1}{2}}$ and $\alpha = 0.7$. Then, $f \wedge \approx_{0.7} f = f$ and $\neg f = f < \mathbf{1} = \approx_{0.7} f$.

Let X be our universe of discourse. If we define the necessity $\Box_\alpha$ and the possibility $\Diamond_\alpha$ of a fuzzy set $f \in [0,1]^X$ as

$$\forall x \in X \quad \Box_\alpha(f)(x) := (\approx_\alpha \neg f)(x) = \begin{cases} 1 & f(x) \geq (1-\alpha) \\ 0 & f(x) < (1-\alpha) \end{cases}$$
$$\forall x \in X \quad \Diamond_\alpha(f)(x) := (\neg \approx_\alpha f)(x) = \begin{cases} 1 & f(x) > \alpha \\ 0 & f(x) \leq \alpha \end{cases}$$

it turns out that they do not have a modal behavior. In fact, they satisfy the S_4 and S_5 modal principles

$$(S_4) \quad \Box_\alpha(\Box_\alpha(f)) = \Box_\alpha(f) \quad \Diamond_\alpha(\Diamond_\alpha(f)) = \Diamond_\alpha(f)$$
$$(S_5) \quad \Box_\alpha(\Diamond_\alpha(f)) = \Diamond_\alpha(f) \quad \Diamond_\alpha(\Box_\alpha(f)) = \Box_\alpha(f)$$

but in general, it is not true that $\Box_\alpha(f) \leq f$ and $f \leq \Diamond_\alpha(f)$.

Example 3.16. Let us set for instance $f = \mathbf{0.8}$ and $\alpha = 0.4$. Then, $f < \mathbf{1} = \Box_{0.4}(f)$. Further, let $g = \mathbf{0.4}$ and $\alpha = 0.8$. Then, $\mathbf{0} = \Diamond_{0.8}(g) < g$.

However, the operator $\Box_\alpha$ is an α–cut and $\Diamond_\alpha$ is a strong α–cut. More precisely, let f be a fuzzy set, then

$$\Box_\alpha(f) = f_c^\alpha$$
$$\Diamond_\alpha(f) = f_s^\alpha$$

4. Conclusions

In this paper shadowed sets have been analyzed from the algebraic point of view. As a first result we have seen that the collection of all shadowed sets of a given universe turns out to be a BZMV^3 algebra, once properly defined the involved operations. This is a stronger structure than the BZMV^{dM} algebraic structure the collection of all fuzzy sets on the same universe is a model of. Moreover, it is possible to algebraically define a mapping which, once given a fuzzy set, returns the particular induced 0–approximation shadowed set. In order to generalize such a mapping to the case of a generic α–approximation of fuzzy sets, it is necessary to introduce the new structure of pre-BZMV^{dM} algebra. It was also shown that the collection of fuzzy sets with a generalized notion of intuitionistic negation is a model of pre-BZMV^{dM} algebras.

A possible development of the present work is a deeper theoretical analysis of pre-BZMV^{dM} algebras, which involves the study of the independence of its axioms, the proof of a representation and a completeness theorem. On the other hand, it would also be interesting to analyze the implications of such a structure in an application context.

As a final conclusion, let us stress that Variable Precision Rough Set Model of Ziarko [29], that is similar to shadowed set approach in terms of stratification of the uncertainty region, can be naturally equipped with a structure of BZMV^3 algebra, as we shall discuss in a forthcoming paper.

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Dipartimento di Informatica, Sistemistica e Comunicazione, Università di Milano–Bicocca, Via Bicocca degli Arcimboldi 8, I–20126 Milano (Italy)
E-mail address: {cattang,ciucci}@disco.unimib.it