

**ALGEBRAIC STRUCTURES RELATED TO MANY VALUED
LOGICAL SYSTEMS
PART I: HEYTING WAJSBERG ALGEBRAS**

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ABSTRACT. A bottom-up investigation of algebraic structures corresponding to many valued logical systems is made. Particular attention is given to the unit interval as a prototypical model of these kind of structures. At the top level of our construction, Heyting Wajsberg algebras are defined and studied. The peculiarity of this algebra is the presence of two implications as primitive operators. This characteristic is helpful in the study of abstract rough approximations.

INTRODUCTION

There is a direct relationship between any logical calculus S and the class of adequate models for it, i.e., the class of algebraic structures which verify exactly the provable formulae of S . For example Boolean algebras are the algebraic counterpart of classical propositional logic and Heyting algebras correspond to intuitionistic propositional logic (see [15, pp. 380–3]). This fruitful interaction allows algebraic investigation to have a direct insight into a given calculus and conversely pure proof-theoretical techniques may contribute to pursue algebraic results. Indeed, every algebraic structure provided by join, meet and complement is an algebraic counterpart of some logical system. Precisely the Lindenbaum-Tarski algebra [CDM99] of each of these logical systems is a model of every algebraic structure we are going to introduce.

Furthermore, in our analysis each model based on the unit interval of real numbers $[0, 1]$ is prototypical, because it represents the set image of the evaluation map of each related logical calculus. The numbers of $[0, 1]$ are interpreted, after Łukasiewicz [3], as the possible truth-values which the logical sentences can be assigned to. As usually done in literature, the values 1 and 0 denote respectively truth and falsehood, whereas all the other values indicate intermediate degrees of indefiniteness.

In $[0, 1]$ models with a meet operator \wedge and its induced partial order $a \leq b$ iff $a \wedge b = a$, it is possible to define an implicative operator (i.e., residuum): $a \Rightarrow b := \sup\{c \mid a \wedge c \leq b\}$. Moreover, every residual definition of implication gives rise to an induced definition of negation: $\sim a := a \Rightarrow 0$.

Dealing with generalized meet-operator (i.e., t-norm), we have different arising definitions of implications and their related negations. Given a nilpotent t-norm as meet-operator, for instance Łukasiewicz t-norm, we have an involution (i.e., Kleene-complementation) as induced negation. On the other hand, a t-norm with non-trivial zero divisors defines a Stonean negation [17] (we denote it as Brouwer complementation).

Taking inspiration by these considerations about $[0, 1]$, we are going to study algebraic structures provided by either more than one negation or more than one implication and we will focus our attention on new added operators definable by composition of the previous ones. All these structures are algebraic counterparts of corresponding logical systems.

In particular, starting from pre-Brouwer lattices we are going to investigate bottom-up algebraic structures generalizing logical systems and their rough applications. Focusing on the relationships existing among these structures we study relations among corresponding logical calculi. A significant role in our bottom-up construction is played by BL-algebra [19] because it is an algebra able to contain the algebraic counterparts of Łukasiewicz logic, Gödel infinite-valued logic and Product logic.

At the top level of our construction we introduce Heyting Wajsberg (HW) algebras. This new structure is characterized by the presence of both and only the Gödel and Łukasiewicz implications as primitive operators. It was introduced with the aim of giving a rich and complete algebraic approach to rough sets [5, 7] and it revealed a great connection with other existing algebras related to many valued logics. Here, we will focus our attention to some of such connections, with particular reference to the substructures that can be derived in HW algebras. Further, particular attention is given to the lattice structure of any introduced algebra.

1. UNIT INTERVAL

Let us investigate in this section the real unit interval as a standard environment of algebraic model for many valued logic. For our purposes it will be interesting to deal with the totally ordered unit interval as set of truth values, treated as a numerical set equipped with an algebraic structure $([0, 1], \rightarrow_L, \rightarrow_G, 0)$ with respect to two primitive *implication* connectives (for a general treatment of implicative algebras see [26]) defined by the following equations:

$$(1a) \quad a \rightarrow_L b := \min\{1, 1 - a + b\}$$

$$(1b) \quad a \rightarrow_G b := \begin{cases} 1 & \text{if } a \leq b \\ b & \text{if } a > b \end{cases}$$

The operator \rightarrow_L is the implication connective introduced by Łukasiewicz in his infinite valued logic L_∞ , while \rightarrow_G corresponds to the implication connective introduced by Gödel in his infinite valued logic G_∞ [28]. Let us stress that the standard total order of $[0, 1]$ can also be expressed in the forms:

$$\begin{aligned} x \leq y & \text{ iff } x \rightarrow_L y = 1 \\ & \text{ iff } x \rightarrow_G y = 1 \end{aligned}$$

On the basis of these two implication connectives it is possible to introduce two connectives of negation as:

$$\begin{aligned} \neg a & := a \rightarrow_L 0 = 1 - a \\ \sim a & := a \rightarrow_G 0 = \begin{cases} 1 & \text{if } a = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

We give now an interesting list of operators which can be defined by the two primitive implication connectives:

$$\begin{aligned}
 a \vee b &:= (a \rightarrow_L b) \rightarrow_L b = \max\{a, b\} \\
 a \wedge b &:= \neg((\neg a \rightarrow_L \neg b) \rightarrow_L \neg b) = \min\{a, b\} \\
 a \oplus b &:= \neg a \rightarrow_L b = \min\{1, a + b\} \\
 a \odot b &:= \neg(a \rightarrow_L \neg b) = \max\{0, a + b - 1\} \\
 \nu(a) &:= (a \rightarrow_L 0) \rightarrow_G 0 = \begin{cases} 1 & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases} \\
 \mu(a) &:= (a \rightarrow_G 0) \rightarrow_L 0 = \begin{cases} 0 & \text{if } a = 0 \\ 1 & \text{otherwise} \end{cases}
 \end{aligned}$$

Connectives \vee and \wedge are the algebraic realizations of the logical *disjunction* connective OR and *conjunction* connective AND respectively. In some semantical interpretations, \oplus and \odot are considered as algebraic realizations of the logical connectives VEL and ET respectively, and they are also called the MV-*disjunction* and MV-*conjunction* connectives. Finally, as will be discussed in section 2.3.1, ν and μ can be considered as realizations of the modal-like connectives of *necessity* and *possibility*. Indeed, as modal operators, they satisfy the properties of a S5 system [18, 12] but, contrary to the classical case, they are based on a Kleene lattice and not on a Boolean algebra. Furthermore, they satisfy the distributive properties:

$$\begin{aligned}
 (\text{DD}_\nu) \quad & \nu(a \vee b) = \nu(a) \vee \nu(b) \\
 (\text{DD}_\mu) \quad & \mu(a \wedge b) = \mu(a) \wedge \mu(b).
 \end{aligned}$$

Let us note that $\sim a = \neg\mu(a)$, i.e., it can be interpreted as the *impossibility*, whereas necessity turns out to be the “not-possible-not” connective $\nu(a) = \neg\mu(\neg a) = \sim\neg a$. It is also possible to introduce a third negation \flat as:

$$\flat a := \neg\sim\neg a = \begin{cases} 0 & \text{if } a = 1 \\ 1 & \text{otherwise} \end{cases}$$

This \flat operator, with respect to modal properties behaves as *contingency*, since $\flat(a) = \neg\nu(a)$, i.e., it has a semantic of a “not-necessary” operator.

Let us stress that the two binary operations \odot and \wedge are paradigmatic examples of *continuous t-norms* [21], i.e., continuous mappings $\mathbf{t} : [0, 1] \times [0, 1] \mapsto [0, 1]$ fulfilling the following properties for all $a, b, c \in [0, 1]$:

$$\begin{aligned}
 (\text{T1}) \quad & a\mathbf{t}b = b\mathbf{t}a \quad (\text{commutativity}) \\
 (\text{T2}) \quad & (a\mathbf{t}b)\mathbf{t}c = a\mathbf{t}(b\mathbf{t}c) \quad (\text{associativity}) \\
 (\text{T3}) \quad & a \leq b \text{ implies } a\mathbf{t}c \leq b\mathbf{t}c \quad (\text{monotonicity}) \\
 (\text{T4}) \quad & a\mathbf{t}1 = 1\mathbf{t}a = a
 \end{aligned}$$

The \mathbf{t} -norm \wedge is called the Gödel \mathbf{t} -norm, and the \mathbf{t} -norm \odot is the Łukasiewicz \mathbf{t} -norm.

Let \mathbf{t} be a continuous \mathbf{t} -norm. The *implication (residuum, quasi-inverse)* operation induced by \mathbf{t} is the map $\rightarrow_{\mathbf{t}} : [0, 1] \times [0, 1] \mapsto [0, 1]$ defined for arbitrary $a, b \in [0, 1]$ as follows:

$$a \rightarrow_{\mathbf{t}} b = \sup\{c \in [0, 1] : a\mathbf{t}c \leq b\}$$

The Gödel and Łukasiewicz t -norms induce the above considered implication connectives: $a \rightarrow_{\wedge} b = a \rightarrow_G b$ and $a \rightarrow_{\odot} b = a \rightarrow_L b$ respectively.

Let \mathbf{t} be a t -norm with associated implication operation $\rightarrow_{\mathbf{t}}$. The *negation* induced from \mathbf{t} is the unary operation $\neg_{\mathbf{t}} : [0, 1] \mapsto [0, 1]$ defined as:

$$\neg_{\mathbf{t}} a := a \rightarrow_{\mathbf{t}} 0 = \sup\{c \in [0, 1] : a \mathbf{t} c = 0\}$$

It is worth noting that the negation induced by the Gödel t -norm is $a \rightarrow_G 0 = \sim a$, while the negation induced by the Łukasiewicz t -norm is $a \rightarrow_L 0 = \neg a$.

Finally, a \mathbf{t} -conorm is a mapping $\mathbf{s} : [0, 1] \times [0, 1] \mapsto [0, 1]$ fulfilling properties (T1), (T2), (T3) and the boundary condition:

$$(S4) \quad a \mathbf{s} 0 = a \text{ for all } a \in [0, 1].$$

Given a t -norm \mathbf{t} , the *dual t -conorm* is defined through the formula

$$\mathbf{s}_{\mathbf{t}}(a, b) := 1 - \mathbf{t}((1 - a), (1 - b))$$

The dual t -conorms of Łukasiewicz and Gödel t -norms, are respectively, the mappings \oplus and \vee . In the following, we will see some possible algebraic approaches of the above introduced operators. We will start from the weaker structures and we will arrive to define the notion of HW algebra (and equivalent structures) which axiomatizes both Łukasiewicz and Gödel implications.

2. LATTICE STRUCTURES

In this section, we study some notion of lattice structures enriched with a negation operator. These lattices will be the basic structure of all the algebras we will define in the following.

2.1. pre-Brouwer lattices. Let us start with a lattice structure which turns out to be the “weaker” one with respect to the interest of this paper.

Definition 2.1. A *pre-Brouwer distributive lattice* is a structure $\mathcal{A} = \langle A, \wedge, \vee, \sim, 0, 1 \rangle$ where

- (1) $\langle A, \wedge, \vee, 0, 1 \rangle$ is a distributive lattice with respect to the join and the meet operations \vee, \wedge whose induced partial ordering relation is: $a \leq b$ iff $a = a \wedge b$ (equivalently, iff $b = a \vee b$). Moreover, A is bounded by the least element 0 and the greatest element 1;
- (2) \sim is a unary operation on \mathcal{A} , called *pre-Brouwer complement*, that satisfies the following conditions:
 - (B1) $a = a \wedge \sim(\sim(a))$ (i.e., $a \leq \sim(\sim(a))$)
 - (B2) $\sim(a \vee b) = \sim a \wedge \sim b$

A *pre-Brouwer de Morgan distributive lattice* is a pre-Brouwer distributive lattice where condition (B2) is substituted by the stronger dual de Morgan law

$$(B2a) \quad \sim(a \wedge b) = \sim a \vee \sim b$$

In a pre-Brouwer lattice, under condition (B1), de Morgan law (B2) is equivalent to the weak contraposition law:

$$(B2b) \quad a \leq b \text{ implies } \sim b \leq \sim a$$

Further, as it has been observed by Dunn [14], conditions (B1) and (B2) are equivalent to the “intuitionistic contraposition” law:

$$a \leq \sim b \quad \text{iff} \quad b \leq \sim a$$

In general, the *strong contraposition law* “ $\sim a \leq \sim b$ implies $b \leq a$ ”, the *non contradiction law* “ $a \wedge \sim a = 0$ ” and the *excluded middle law* “ $a \vee \sim a = 1$ ” are not verified. Let us note that $1 = \sim 0$ and $0 = \sim 1$. Further, for any element a it holds the Brouwer condition “ $\sim a = \sim\sim a$ ”.

Example 2.1. Let us consider the pre-Brouwer distributive lattice drawn in Figure 1.

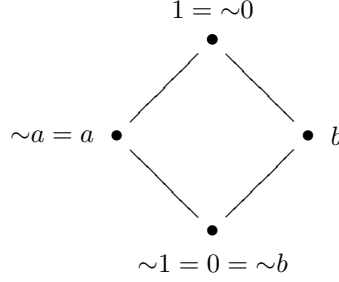


FIGURE 1. A pre-Brouwer lattice

In this example, it can be easily seen that the de Morgan law (B2a) is not satisfied: $\sim(a \wedge b) = 1 \neq 0 = \sim a \vee \sim b$. Thus, not all pre-Brouwer distributive lattices are de Morgan. Moreover, the non contradiction law is not satisfied, indeed $a \wedge \sim a = a$; nor the strong contraposition law, indeed, $\sim b \leq \sim a$ but not $a \leq b$; nor the double negation law, indeed $\sim\sim b = 1$.

2.2. Brouwer lattices.

Definition 2.2. A structure $\mathcal{A} = \langle A, \wedge, \vee, \sim, 0, 1 \rangle$ is a *Brouwer* (resp., *de Morgan Brouwer*) *distributive lattice* if it is a pre-Brouwer (resp., de Morgan pre-Brouwer) lattice satisfying the further property

$$(B3) \quad a \wedge \sim a = 0 \text{ (non contradiction law)}$$

Trivially, any Brouwer lattice is a pre-Brouwer lattice but not vice versa. For example, the pre-Brouwer lattice of Figure 1 is not a Brouwer lattice since $a \wedge \sim a = a \neq 0$.

The operation \sim is called a *Brouwer* or *intuitionistic* complement. In fact, it behaves properly with respect to the main principles of an intuitionistic negation since it satisfies neither the excluded middle law nor the double negation law, whereas it satisfies the non contradiction law (B3).

Remark 1. Sometimes, the term Brouwerian lattice (for instance, by Birkhoff [2, p. 45]) is used to mean what we call a Heyting algebra in the following.

2.3. Brouwer Zadeh lattices. By enriching Brouwer lattices with a further unary negation operator which respects Kleene’s conditions (see K1-K3 below), one obtains the so called Brouwer Zadeh lattices [10, 11]. We give now their definitions and show how it is possible to define modal-like operators and rough approximations by the interaction of the two negations [4, 6].

Definition 2.3. A system $\mathcal{A} = \langle A, \wedge, \vee, \neg, \sim, 0, 1 \rangle$ is a *Brouwer Zadeh* (BZ) (resp., *de Morgan Brouwer Zadeh* (BZ^{dM})) *distributive lattice* if the following conditions hold:

- (1) The substructure $\langle A, \wedge, \vee, \sim, 0, 1 \rangle$ is a distributive Brouwer (resp., de Morgan Brouwer) lattice;
- (2) The unary operation $\neg : A \mapsto A$ is a *Kleene* (or *Zadeh*) complementation.

In other words:

$$(K1) \quad \neg(\neg a) = a$$

$$(K2) \quad \neg(a \vee b) = \neg a \wedge \neg b$$

$$(K3) \quad a \wedge \neg a \leq b \vee \neg b$$

- (3) The two negations are linked by the following interconnection rule:
 (in) $\neg \sim a = \sim \sim a$

Let us note that under condition (K1), the *de Morgan* law (K2) is equivalent to each of the following properties (and consequently they are mutually equivalent among them):

$$(K2a) \quad \text{the dual de Morgan law } \neg(a \wedge b) = \neg a \vee \neg b$$

$$(K2b) \quad \text{the weak contraposition law } a \leq b \text{ implies } \neg b \leq \neg a$$

$$(K2c) \quad \text{the strong contraposition law } \neg b \leq \neg a \text{ implies } a \leq b$$

Let us note that $0 = \neg 1$ and $1 = \neg 0$. In general neither the *non-contradiction* law “ $\forall a : a \wedge \neg a = 0$ ” nor the *excluded-middle* law “ $\forall a : a \vee \neg a = 1$ ” holds with this negation, also if for some particular elements e (for instance $e = 0, 1$) it may happen that $e \wedge \neg e = 0$ and $e \vee \neg e = 1$.

Example 2.2. Let us consider the BZ^{dM} distributive lattice drawn in Figure 2.

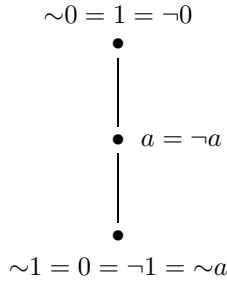


FIGURE 2. A BZ^{dM} distributive lattice

Clearly, a satisfies neither non contradiction law nor excluded middle law with respect to the Kleene negation.

Remark 2. Let us stress that the Kleene negation is also a de Morgan negation according to the usual definition [20, 13, 14] which requires only properties (K1) and (K2) and not necessarily (K3).

A third kind of complement, called *anti-intuitionistic complementation*, can be defined in any BZ lattice.

Definition 2.4. Let \mathcal{A} be a BZ lattice. The *anti-intuitionistic complement* is the unary operation $\flat : A \mapsto A$ defined as follows:

$$\flat a := \neg \sim \neg a$$

One can easily verify that \flat satisfies the following conditions:

$$(AB1) \quad \flat \flat a \leq a$$

$$(AB2) \quad ba \vee bc = b(a \wedge c) \quad [\text{equivalently, } a \leq c \text{ implies } bc \leq ba,]$$

$$(AB3) \quad a \vee ba = 1$$

If \mathcal{A} is a BZ^{dM} lattice also the dual de Morgan law is satisfied:

$$(AB4) \quad ba \wedge bc = b(a \vee c)$$

2.3.1. *Modal Operators induced in BZ lattices.* Let us consider a BZ lattice \mathcal{A} . For any element $a \in A$ we can define the following two modal-like operators

Necessity:

$$\nu(a) := \sim \neg a = bba$$

Possibility:

$$\mu(a) := \neg \nu(\neg a) = \neg \sim a = \sim \sim a$$

The name of *modal-like* operators is justified by the following facts:

- (1) The substructure $\langle A, \wedge, \vee, \neg, 0 \rangle$ is a Kleene algebra, i.e., a distributive lattice with a Kleene negation \neg , instead of a Boolean algebra;
- (2) The operators ν and μ satisfy the classical properties of an S5 modal system as it is explained in the following proposition.

Proposition 2.1. [6] *In any BZ distributive lattice the following conditions hold:*

(mod-1)

$$\nu(1) = 1.$$

That is: if a sentence is true, then also its necessity is true (necessitation rule or according to [12, p. 20] the N-principle).

(mod-2)

$$a \leq b \quad \text{implies} \quad \nu(a) \leq \nu(b)$$

if a conditional and its antecedent are both necessary, then so is the consequent (which is the characteristic K-principle of modal logic see [12, p. 7]).

(mod-3)

$$\nu(a) \leq a \leq \mu(a).$$

In other words: necessity implies actuality and actuality implies possibility (the characteristic T-principle).

(mod-4)

$$\nu(a) = \nu(\nu(a)) \quad \text{and} \quad \mu(a) = \mu(\mu(a)).$$

whatever is necessary (resp., possible) is necessarily necessary (resp., possibly possible) (the characteristic 4-principle).

(mod-5)

$$\mu(a) = \neg(\nu(\neg a))$$

which embodies the idea that what is possible is just what is not-necessarily-not (the $DF\Diamond$ -principle).

(mod-6)

$$a \leq \nu(\mu(a)).$$

Actuality implies necessity of possibility (the characteristic B-principle).

(mod-7)

$$\mu(a) = \nu(\mu(a)), \quad \nu(a) = \mu(\nu(a)).$$

Possibility is equal to the necessity of possibility; whereas necessity is equal to the possibility of necessity (the characteristic S5-principle).

Thus, we have that from a modal point of view, the structure $\langle A, \wedge, \vee, \nu, \neg, 0 \rangle$ is a “weak” S5 modal system, since it satisfies all S5 axioms but it is based on Kleene lattice instead of on a Boolean algebra.

Further, if we consider BZ^{dM} lattices, we obtain a more deviant, with respect to the above weak one, modal structure since necessity and possibility satisfy also the distributive properties:

$$\begin{aligned} (DD_\nu) \quad & \nu(a \vee b) = \nu(a) \vee \nu(b) \\ (DD_\mu) \quad & \mu(a \wedge b) = \mu(a) \wedge \mu(b) \end{aligned}$$

We note that these properties do not make sense in a classical (Boolean) environment, since under DD_ν (equivalently, DD_μ) ν and μ trivially collapse to the identity operator. However, here the underlying structure is a Kleene lattice and ν and μ are distinct non-trivial operators satisfying all S5 axioms and also DD conditions, as can be seen in the following example.

Example 2.3. Let us consider the BZ^{dM} lattice of Figure 2, then

$$\nu(a) = 0 \neq a \neq 1 = \mu(a).$$

Finally, when considering the relationship among the modal-like and the negations operators, one obtains:

$$\begin{aligned} \sim a &= \nu(\neg a) = \neg \mu(a) \\ \flat a &= \neg \nu(a) = \mu(\neg a) \end{aligned}$$

On this basis, similarly to the modal interpretation of intuitionistic logic, the Brouwer complement \sim can be interpreted as the negation of possibility or *impossibility* (also the necessity of a negation). Analogously, the anti-Brouwer complement \flat can be interpreted as the negation of necessity.

2.3.2. Rough approximation spaces in BZ lattices. We are now interested to those elements which satisfy some classical properties and which can be therefore considered as *exact*, *crisp* in contraposition to the other *fuzzy* elements. First of all, we consider those elements for which $e = \mu(e)$ (equivalently, $e = \nu(e)$), i.e., to the situations where we cannot distinguish among necessity, actuality and possibility. This picture leads one to define the substructure of all *Modal sharp* (*M-sharp*) (exact, crisp) elements, denoted by $A_{e,M}$:

$$A_{e,M} := \{e \in A : \mu(e) = e\} = \{e \in A : \nu(e) = e\}$$

However, this is not the only way to define sharp elements. In fact, we have seen that the two non-standard negations \neg and \sim do not satisfy some classical properties. Thus, we can define the *Kleene sharp* (*K-sharp*) elements as those elements which satisfy the non contradiction (or equivalently the excluded middle) law with respect to the Kleene negation:

$$A_{e,\neg} := \{e \in A : e \wedge \neg e = 0\} = \{e \in A : e \vee \neg e = 1\}$$

Alternatively, we can define the *Brouwer sharp* (*B-sharp*) elements as those elements satisfying the double negation law with respect to the Brouwer negation:

$$A_{e,B} := \{e \in A : \sim \sim e = e\} = \{e \in A : \flat \flat e = e\}$$

The relation among all these different substructures of exact elements is figured out in the following proposition.

Proposition 2.2. [9]. *Let \mathcal{A} be a BZ lattice. Then*

$$A_{e,B} = A_{e,M} \subseteq A_{e,\neg}$$

Let \mathcal{A} be a BZ^{dM} lattice. Then

$$A_{e,B} = A_{e,M} = A_{e,\neg}$$

In the case of a BZ^{dM} lattice all the subsets of exact elements coincide, thus, we can simply talk of *sharp* elements and write A_e .

As we have seen, in any BZ algebra it is possible, making use of the composition of the two negations, to introduce the modal operators, ν and μ . These operators give a rough approximation of any element $a \in A$ by M-sharp definable elements. In fact, $\nu(a)$ (resp., $\mu(a)$) turns out to be the best approximation from the bottom (resp., top) of a by M-sharp elements. To be precise, for any element $a \in A$ the following holds:

- (I1) $\nu(a)$ is M-sharp ($\nu(a) \in A_{e,M}$).
- (I2) $\nu(a)$ is an *inner (lower)* approximation of a ($\nu(a) \leq a$).
- (I3) $\nu(a)$ is the best inner approximation of a by M-sharp elements (let $e \in A_{e,M}$ be such that $e \leq a$, then $e \leq \nu(a)$).

By properties (I1)–(I3), it follows that the necessity of an element a can be expressed in the following form:

$$\nu(a) = \max\{x \in A_{e,M} : x \leq a\}$$

Analogously

- (O1) $\mu(a)$ is M-sharp ($\mu(a) \in A_{e,M}$).
- (O2) $\mu(a)$ is an *outer (upper)* approximation of a ($a \leq \mu(a)$).
- (O3) $\mu(a)$ is the best outer approximation of a by M-sharp elements (let $f \in A_{e,M}$ be such that $a \leq f$, then $\mu(a) \leq f$).

By properties (O1)–(O3), it follows that the possibility of an element a can be expressed in the following form:

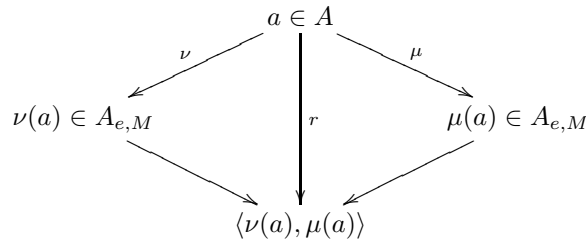
$$\mu(a) = \min\{y \in A_{e,M} : a \leq y\}$$

Definition 2.5. Given a BZ lattice \mathcal{A} the induced *rough approximation space* is the structure $\langle A, A_{e,M}, \nu, \mu \rangle$ consisting of the set A of all approximable elements, the set $A_{e,M}$ of all *definable* (or M-sharp) elements, and the *inner* (resp., *outer*) *approximation map* $\nu : A \rightarrow A_{e,M}$ (resp., $\mu : A \rightarrow A_{e,M}$).

For any element $a \in A$, its *rough approximation* is defined as the pair of M-sharp elements:

$$r(a) := \langle \nu(a), \mu(a) \rangle \quad [\text{with } \nu(a) \leq a \leq \mu(a)]$$

drawn in the following diagram:



So the map $r : A \rightarrow A_{e,M} \times A_{e,M}$ approximates an unsharp (fuzzy) element by a pair of M-sharp (crisp, exact) ones representing its inner and outer sharp approximation, respectively. Equivalently, it is possible to identify the rough approximation of a with the pair *necessity, impossibility*:

$$r_{\perp}(a) := \langle \nu(a), \neg\mu(a) \rangle = \langle \nu(a), \sim a \rangle$$

drawn in the following diagram:

$$\begin{array}{ccccc}
 & & a \in A & & \\
 & \swarrow \nu & \downarrow r_{\perp} & \searrow \sim & \\
 \nu(a) \in A_{e,M} & & & & \sim a \in A_{e,M} \\
 & \searrow & \downarrow & \swarrow & \\
 & & \langle \nu(a), \sim a \rangle & &
 \end{array}$$

Clearly, M-sharp elements are characterized by the property that they coincide with their rough approximations:

$$e \in A_{e,M} \quad \text{iff} \quad r(e) = r_{\perp}(e) = \langle e, e \rangle.$$

We remark that our interest in the study of abstract rough approximations is due to the possibility of giving an algebraic approach to concrete rough sets. The interested reader can see [4, 6].

3. RESIDUATED LATTICES

Let us consider residuated lattices [33], i.e., lattice structures $\langle A, \wedge, \vee, 0, 1 \rangle$ where it is possible to define a pair of binary operators $(*, \rightarrow)$, called respectively multiplication and residuation, such that the substructure $\langle A, *, 1 \rangle$ is a monoid and the following hold $\forall a, b, c \in A$:

$$(R1) \quad a * c \leq b \quad \text{implies} \quad c \leq a \rightarrow b$$

$$(R2) \quad a * (a \rightarrow b) \leq b$$

Trivially, (R1) and (R2) can be equivalently expressed by the following *adjointness condition*:

$$(AC) \quad c \leq (a \rightarrow b) \quad \text{iff} \quad a * c \leq b$$

As pointed out in the introduction, it is emerged by Hajek et al. works [19, 16] that the residuated lattice structure common to all logics based on the unit interval $[0, 1]$ is a Basic Logic (BL) algebra. In this section, we are going to study BL algebra and some of its enrichments. For any algebra we will introduce in the sequel, its underlying lattice structure is put in evidence.

3.1. Basic Logic algebras. As it is mentioned above, the first structure we are going to consider is the so called Basic Logic algebra. As its name suggests it is a very general structure, which can be enriched in several ways to obtain the underlying structure of many valued logical systems with different adjoint operators.

Definition 3.1. A system $\mathcal{A} = \langle A, \wedge, \vee, *, \rightarrow, 0, 1 \rangle$ is a Basic Logic (BL) algebra if $\wedge, \vee, *, \rightarrow$ are binary operators on A and $0, 1$ are constants such that:

- (1) $\langle A, \wedge, \vee, 0, 1 \rangle$ is a lattice with least element 0 and greatest element 1 with respect to the lattice ordering $a \leq b$ iff $a \wedge b = a$;
- (2) $\langle A, *, 1 \rangle$ is a monoid, i.e., the binary operation $*$ on A is commutative, associative and for all $a \in A$, $1 * a = a$;
- (3) the following properties are satisfied by all $a, b, c \in A$
 - (a) $a \wedge (b \rightarrow (a * b)) = a$
 - (b) $((a \rightarrow b) * a) \vee b = b$
 - (c) $(a \wedge b) * c = (a * c) \wedge (b * c)$
 - (d) $(a \rightarrow (a \vee b)) = 1$
 - (e) $(c \rightarrow a) \rightarrow (c \rightarrow (a \vee b)) = 1$
 - (f) $a \wedge b = a * (a \rightarrow b)$
 - (g) $(a \rightarrow b) \vee (b \rightarrow a) = 1$ (Dummett condition)

Remark 3. As proved in [19, p.50], properties (a)–(e) are equivalent to the non-equational property (AC). Thus, any BL algebra is trivially a residuated lattice satisfying also properties (f) and (g).

Proposition 3.1. [19, p. 48] *Let $\mathcal{A} = \langle A, \wedge, \vee, *, \rightarrow, 0, 1 \rangle$ be a BL algebra. Then, the following property holds:*

$$a \leq b \text{ implies } (b \rightarrow c) \leq (a \rightarrow c).$$

In the following, some results about the completeness of BL algebras are reported.

Proposition 3.2. [19, p. 53]. *Any BL algebra is a subalgebra of the direct product of a system of linearly ordered BL algebras.*

Proposition 3.3. [19, p. 54]. *Let ϕ and ψ be well-defined terms, in the traditional way, on the language of BL algebra. An equation $\phi = \psi$ is satisfied in all linearly ordered BL algebras iff it is satisfied in all BL algebras.*

BL algebras can be extended in different ways, in particular it is possible to obtain Wajsberg algebras [31, 32] or Heyting algebras [22, 23] as shown in the following.

Definition 3.2. A structure $\mathcal{A} = \langle A, \rightarrow, \neg, 1 \rangle$ is a Wajsberg algebra if the following axioms are satisfied:

- (W1) $1 \rightarrow a = a$
- (W2) $(a \rightarrow b) \rightarrow ((b \rightarrow c) \rightarrow (a \rightarrow c)) = 1$
- (W3) $(a \rightarrow b) \rightarrow b = (b \rightarrow a) \rightarrow a$
- (W4) $(\neg a \rightarrow \neg b) \rightarrow (b \rightarrow a) = 1$

Let us remark that in any Wajsberg algebra it is possible to define an order relation as

$$a \leq b \text{ iff } a \rightarrow b = 1$$

With respect to this order relation, 1 is the maximum element and $0 := \neg 1$ the minimum element and once defined the meet and join operators in the usual manner: $a \leq b$ iff $a \wedge b = a$ iff $a \vee b = b$, the structure $\langle A, \wedge, \vee, 0, 1 \rangle$ is a distributive lattice. Further, the negation \neg is a Kleene negation, i.e., it satisfies properties (K1)–(K3) of Section 2.3.

Proposition 3.4. [30, p. 46] *A BL algebra is a Wajsberg algebra iff it satisfies also the axiom*

$$(W) \quad (a \rightarrow 0) \rightarrow 0 = a.$$

Let us remark that in any BL algebra satisfying also axiom (W), the identity $a * b = \neg(a \rightarrow \neg b)$ holds $\forall a, b \in A$ and the lattice operator \vee is linked to the residuation operator \rightarrow by the equality $a \vee b = (a \rightarrow b) \rightarrow b$.

Definition 3.3. A Gödel algebra is a BL algebra plus the axiom

$$(G) \quad \forall a \in A : a * a = a$$

Let us remark that in any Gödel algebra it holds the identity $\forall a, b \in A : a * b = a \wedge b$.

Definition 3.4. A system $\mathcal{A} = \langle A, \wedge, \vee, \rightarrow, 0 \rangle$, where A is a non empty set, $\wedge, \vee, \rightarrow$ are binary operators, 0 is an element of A , is called a *Heyting algebra* if the following axioms are satisfied:

- (H1) $a \rightarrow a = b \rightarrow b$
- (H2) $(a \rightarrow b) \wedge b = b$
- (H3) $a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c)$
- (H4) $a \wedge (a \rightarrow b) = a \wedge b$
- (H5) $(a \vee b) \rightarrow c = (a \rightarrow c) \wedge (b \rightarrow c)$
- (H6) $0 \wedge x = 0$

In [22] it is proved that Heyting algebras are equivalent to residuated lattices where the multiplication operator coincides with the lattice meet operator, i.e., $\forall a, b \in A : a \wedge b = a * b$. Further, the unary operator $\sim a := a \rightarrow 0$ is a Brouwer negation, i.e., it satisfies properties (B1)–(B3) of Section 2.2.

Proposition 3.5. *Gödel algebras are Heyting algebras satisfying also the Dummett condition*

$$(D) \quad (a \rightarrow b) \vee (b \rightarrow a) = 1$$

Proof. It is an immediate consequence of the equivalence of Heyting algebras and residuated lattices with $\wedge = *$ given by Monteiro [22]. \square

Remark 4. Let us note that the correct form of axiom (H3) is the one of Definition 3.4, as it was given in the original work [22], and not the following one:

$$(H3') \quad a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c)$$

as it has been used later by Monteiro himself [23] and other authors. Indeed, if this form is used, the underlying substructure $\langle A, \wedge, \vee \rangle$ is no more a lattice, since in general the operation \wedge is not commutative as shown in the following example.

Example 3.1. Let us consider the algebraic structure $\langle \{0, a, 1\}, \wedge, \vee, \rightarrow \rangle$ where the operators are defined according to Table 1.

TABLE 1. An example of the non correctness of axiom (H3').

\rightarrow	0	a	1
0	1	1	1
a	1	1	1
1	0	a	1

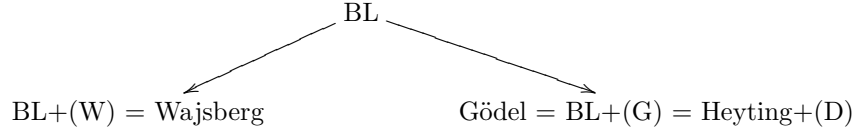
\wedge	0	a	1
0	0	0	0
a	a	a	a
1	0	a	1

\vee	0	a	1
0	0	0	1
a	0	0	1
1	1	1	1

Clearly, this structure satisfies all axioms (H1),(H2),(H3'), (H4), (H5) and (H6). However, the operators \wedge and \vee do not define a lattice. For instance, the commutative property of \wedge is not satisfied, since $a \wedge 0 \neq 0 \wedge a$ and \vee is not an idempotent operator, since $a \vee a = 0$.

Definition 3.5. A system $\mathcal{A} = \langle A, \wedge, \vee, \rightarrow, \neg \rangle$ is a *symmetric Heyting algebra* if it satisfies (H1)–(H5) and \neg is a de Morgan negation, i.e., it satisfies properties (K1) and (K2). Trivially, any symmetric Heyting algebra is a Heyting algebra once defined $0 := \neg(a \rightarrow a)$.

We can summarize the dependencies among these structures with the following diagram.



3.1.1. BL lattice structure.

Proposition 3.6. Let $\langle A, \wedge, \vee, *, \rightarrow, 0, 1 \rangle$ be a BL algebra. Then, once defined $\sim a := a \rightarrow 0$, the underlying lattice structure $\langle A, \wedge, \vee, \sim, 0 \rangle$ is a pre-Brouwer de Morgan lattice.

Proof. We have to show that the following properties hold:

- (B1) $a \wedge \sim \sim a = a$;
- (B2b) $\sim a \vee \sim b = \sim(a \wedge b)$.

(B1). Setting $b = 0$ in axiom (b) we have $\sim a * a = 0$. Setting $a = \sim b$ in axiom (a), we have $(\sim b \rightarrow (b * \sim b)) \wedge b = b$. Putting these two results together, we have $a \wedge \sim \sim a = a$.

(B2b). We prove that it holds in all linearly order BL algebras, then by Proposition 3.3 it holds in any BL algebra. Let us suppose that $a \leq b$. Then, we have $\sim(a \wedge b) = \sim a$ and on the other side $\sim b \leq \sim a$ by Proposition 3.1. Thus, $\sim a \vee \sim b = \sim a = \sim(a \wedge b)$. \square

The underlying structure is not a Brouwer lattice, because, in general, a BL algebra does not satisfy the noncontradiction law $\forall a : a \wedge \sim a = 0$, as can be seen in the following counterexample.

Example 3.2. Let us consider the BL algebra, with support $\{0, a, b, 1\}$ and operators defined in Table 2.

TABLE 2. An example of $\{0, a, b, 1\}$ BL algebra with $*$, \rightarrow operators and derived negation \sim .

*	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

\rightarrow	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

x	$\sim x$
0	1
a	a
b	0
1	0

In figure 3, it is drawn the Hasse diagram of the underlying lattice structure.

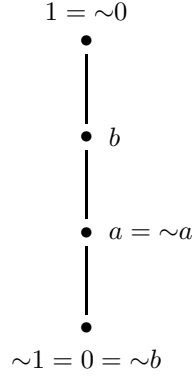


FIGURE 3. Hasse diagram of the lattice structure underlying the BL algebra of Table 2.

In this structure we have that $a \wedge \sim a = a$. So, \sim is not a Brouwer negation and furthermore it is not a de Morgan negation, since $\sim \sim b = 1 \neq b$.

Further, we have that in all BL algebras the Kleene condition is satisfied by the \sim negation.

Proposition 3.7. *Let $\langle A, \wedge, \vee, *, \rightarrow, 0, 1 \rangle$ be a BL algebra. Then, for all $a, b \in A$ the condition*

$$(K3) \quad a \wedge \sim a \leq b \vee \sim b$$

holds.

Proof. We prove that (K3) holds in all linearly order BL algebras. Then, by Proposition 3.3, (K3) holds in any BL algebra. Let us suppose that $a \leq b$, then, trivially, $a \wedge \sim a \leq a \leq b \leq b \vee \sim b$. On the other hand, if $b \leq a$ then by Proposition 3.1, we have $\sim a \leq \sim b$ and finally, $a \wedge \sim a \leq \sim a \leq \sim b \leq b \vee \sim b$. \square

3.2. Strict Basic Logic algebras. Let us consider BL algebras based on the unit interval $[0, 1]$ where $*$ is a triangular norm without non-trivial zero divisors [21], i.e., verifying:

$$\forall a, b \in [0, 1] : x * y = 0 \quad \text{iff} \quad a = 0 \text{ or } b = 0.$$

So, the Gödel algebra $\langle [0, 1], \wedge, \vee, \rightarrow_G, 0, 1 \rangle$ is an example of this kind of structure, whereas the Wajsberg algebra $\langle [0, 1], \rightarrow_L, \neg, 1 \rangle$ is not. Following [16], a new axiom can be introduced in BL algebras in order to obtain as models exactly these kind of structures.

Definition 3.6. A structure $\mathcal{A} = \langle A, \wedge, \vee, *, \rightarrow, 0, 1 \rangle$ is a *Strict Basic Logic* (SBL) algebra if it is a BL algebra and satisfies the further axiom

$$(a * b) \rightarrow 0 = (a \rightarrow 0) \vee (b \rightarrow 0)$$

Proposition 3.8. [16]. *Let \mathcal{A} be a Gödel algebra then \mathcal{A} is a SBL algebra.*

Proposition 3.9. [16]. *Let ϕ and ψ be well-defined terms, in the traditional way, on the language of SBL algebra. An equation $\phi = \psi$ is satisfied in all linearly ordered SBL algebras iff it is satisfied in all SBL algebras.*

3.2.1. *SBL lattice structure.*

Lemma 3.1. *Let \mathcal{A} be a SBL algebra and define $\sim a := a \rightarrow 0$. Then, for all $a \in \mathcal{A}$, the condition*

$$a \wedge \sim a = 0$$

holds.

Proof. By Definition 3.6 and de Morgan properties $\sim(a * b) = \sim a \vee \sim b = \sim(a \wedge b)$ and by BL axiom (f) $\sim(a * b) = \sim(a * (a \rightarrow b))$. Now, by the property $x * \sim x = 0$ we have

$$0 = [a * (a \rightarrow b)] * \sim[a * (a \rightarrow b)] = [a * (a \rightarrow b)] * \sim(a * b).$$

Finally, setting $b = \sim a$, we have the thesis. \square

Proposition 3.10. *Let $\langle A, \wedge, \vee, *, \rightarrow, 0, 1 \rangle$ be a SBL algebra. Once defined $\sim a := a \rightarrow 0$, the structure $\langle A, \wedge, \vee, \sim, 0 \rangle$ is a de Morgan Brouwer lattice.*

Proof. Since any SBL algebra is a BL algebra and hence a pre-Brouwer lattice, we have only to prove that $a \wedge \sim a = 0$ which is just Lemma 3.1. \square

3.3. SBL_{\neg} algebras. Finally, a new unary operator is added to SBL algebras. The axioms characterizing this new operator are developed in order to obtain a de Morgan negation i.e., a negation satisfying the double negation law, “ $\forall a : \neg\neg a = a$ ” and the contraposition law “ $\forall a, b : \neg a \leq \neg b$ implies $b \leq a$ ”; and in order to define a unary operator ν as the algebraic counterpart of the Baaz operator Δ [1].

Definition 3.7. A system $\mathcal{A} = \langle A, \wedge, \vee, *, \rightarrow, \neg, 0, 1 \rangle$ is a SBL_{\neg} algebra if

- (1) $\langle A, \wedge, \vee, *, \rightarrow, 0, 1 \rangle$ is a SBL algebra
- (2) \neg is a unary operator such that, once defined $\sim a = a \rightarrow 0$ and $\nu(a) = \sim\neg a$, the following are satisfied
 - (SBL $_{\neg}$ 1) $\neg\neg a = a$
 - (SBL $_{\neg}$ 2) $\sim a \leq \neg a$
 - (SBL $_{\neg}$ 3) $\nu(a \rightarrow b) = \nu(\neg b \rightarrow \neg a)$
 - (SBL $_{\neg}$ 4) $\nu(a) \vee \nu(\neg a) = 1$
 - (SBL $_{\neg}$ 5) $\nu(a \vee b) \leq \nu(a) \vee \nu(b)$
 - (SBL $_{\neg}$ 6) $\nu(a) * (\nu(a \rightarrow b)) \leq \nu(b)$

Proposition 3.11. [16]. *Let ϕ and ψ be well-defined terms, in the traditional way, on the language of SBL_{\neg} algebra. An equation $\phi = \psi$ is satisfied in all linearly ordered SBL_{\neg} algebras iff it is satisfied in all SBL_{\neg} algebras.*

 3.3.1. *SBL_{\neg} lattice structure.*

Proposition 3.12. *Let $\langle A, \wedge, \vee, *, \rightarrow, \neg, 0, 1 \rangle$ be a SBL_{\neg} algebra. Then, once defined $\sim a := a \rightarrow 0$, the structure $\langle A, \wedge, \vee, \neg, \sim, 0 \rangle$ is BZ^{dM} lattice.*

Proof. Since any SBL_{\neg} algebra is a SBL algebra and hence a Brouwer de Morgan lattice, we have only to prove that

- (K1) $\neg\neg a = a$
- (K2) $\neg(a \vee b) = \neg a \wedge \neg b$
- (K3) $a \wedge \neg a \leq b \vee \neg b$
- (in) $\neg\sim a = \sim\sim a$

(K1) It is an axiom of SBL_{\neg} algebras.

(K2) It has been proved in [16].

(K3) Similar to Proposition 3.7.

(in) By $\sim a \leq \neg a$, we get $\sim a = \sim a \wedge \neg a$, and applying this to $\nu(a)$: $\sim\nu(a) = \sim\nu(a) \wedge \neg\nu(a)$. By axioms (a) and (b), we have $\neg\nu(a) = \neg\nu(a) \wedge [\nu(a) \rightarrow (\neg\nu(a) * \nu(a))] = \neg\nu(a) \wedge \sim\nu(a)$. So, we have that $\sim\nu(a) = \neg\nu(a)$, which, once applied to $\neg a$, gives the thesis. \square

4. HW ALGEBRAS

At the top level of our hierarchy, we introduce and study the new structure of Heyting Wajsberg (HW) algebra [5, 7]. Its originality consists in the presence of two implication connectives as primitive operators.

Definition 4.1. A system $\mathcal{A} = \langle A, \rightarrow_L, \rightarrow_G, 0 \rangle$ is a *Heyting Wajsberg (HW) algebra* if A is a non empty set, $0 \in A$ and $\rightarrow_L, \rightarrow_G$ are binary operators, such that, once defined

- (2a) $1 := \neg 0$
- (2b) $\neg a := a \rightarrow_L 0$
- (2c) $\sim a := a \rightarrow_G 0$
- (2d) $a \wedge b := \neg((\neg a \rightarrow_L \neg b) \rightarrow_L \neg b)$
- (2e) $a \vee b := (a \rightarrow_L b) \rightarrow_L b$

the following are satisfied:

- (HW1) $a \rightarrow_G a = 1$
- (HW2) $a \rightarrow_G (b \wedge c) = (a \rightarrow_G c) \wedge (a \rightarrow_G b)$
- (HW3) $a \wedge (a \rightarrow_G b) = a \wedge b$
- (HW4) $(a \vee b) \rightarrow_G c = (a \rightarrow_G c) \wedge (b \rightarrow_G c)$
- (HW5) $1 \rightarrow_L a = a$
- (HW6) $a \rightarrow_L (b \rightarrow_L c) = \neg(a \rightarrow_L c) \rightarrow_L \neg b$
- (HW7) $\neg \sim a \rightarrow_L \sim \sim a = 1$
- (HW8) $(a \rightarrow_G b) \rightarrow_L (a \rightarrow_L b) = 1$

Lemma 4.1. *In any HW algebra the following properties hold*

- (L1) $a \rightarrow_L a = 1$
- (L2) $a \rightarrow_L b = \neg b \rightarrow_L \neg a$
- (L3) $\neg \neg a = a$
- (L4) $a \wedge 1 = a; 1 \wedge a = a$
- (L5) $1 \rightarrow_G c = c$
- (L6) $a \wedge b = b \wedge a$

Proof. (L1) Setting $b = a$ in (HW8) and using (HW1), we get $1 \rightarrow_L (a \rightarrow_L a) = 1$. Now, by (HW5) we have the thesis.
(L2) Setting $a = 1$ in (HW6), we have $1 \rightarrow_L (b \rightarrow_L c) = \neg(1 \rightarrow_L c) \rightarrow_L \neg b$, then by (HW5) we derive the thesis.
(L3) First, by definition of \neg and (HW5), we can derive that $\neg 1 = 0$. Now, applying (L2) to (HW5), we get $\neg a \rightarrow_L \neg 1 = a$. Finally, by $\neg 1 = 0$ and definition of \neg , we get $\neg \neg a = a$.

(L4) By \wedge definition:

$$\begin{aligned}
 a \wedge 1 &= \neg((\neg a \rightarrow_L \neg 1) \rightarrow_L \neg 1) && (L2), (L3) \\
 &= \neg(1 \rightarrow_L \neg(1 \rightarrow_L a)) && (HW5) \\
 &= \neg(1 \rightarrow_L \neg a) && (HW5), (L3) \\
 &= \neg\neg a = a
 \end{aligned}$$

Similarly, we get $1 \wedge a = a$.

(L5) Setting $a = b = 1$ in (HW2), we get $1 \rightarrow_G (1 \wedge c) = (1 \rightarrow_G c) \wedge (1 \rightarrow_G 1)$. Then, by (HW1) and (L4) we have the thesis.

(L6) Simply by setting $a = 1$ in (HW2) and using (L5). \square

Proposition 4.1. *Let \mathcal{A} be a HW algebra. Then, the following equations are mutually equivalent:*

$$\begin{aligned}
 (3a) \quad & a \rightarrow_L b = 1 \\
 (3b) \quad & a \rightarrow_G b = 1 \\
 (3c) \quad & a \wedge b = a
 \end{aligned}$$

Proof. $a \rightarrow_L b = 1$ implies $a \wedge b = a$. By definition of \wedge :

$$\begin{aligned}
 a \wedge b &= \neg[(\neg b \rightarrow_L \neg a) \rightarrow_L \neg a] && (L2), (L3) \\
 &= \neg[a \rightarrow_L \neg(a \rightarrow_L b)] && \text{Hypothesis} \\
 &= \neg(a \rightarrow_L \neg 1) && (L2), (HW5), (L3) \\
 &= \neg\neg a = a
 \end{aligned}$$

$b \wedge a = a$ implies $a \rightarrow_G b = 1$. Setting $c = a$ in axiom (HW2) we get:

$$\begin{aligned}
 a \rightarrow_G (b \wedge a) &= (a \rightarrow_G b) \wedge (a \rightarrow_G a) && \text{Hypothesis, (HW1)} \\
 a \rightarrow_G a &= (a \rightarrow_G b) \wedge 1 && (HW1), (L4) \\
 1 &= a \rightarrow_G b
 \end{aligned}$$

$a \rightarrow_G b = 1$ implies $a \rightarrow_L b = 1$. Using (HW8) and hypothesis, one has $1 \rightarrow_L (a \rightarrow_L b) = 1$. Then by axiom (HW5), $a \rightarrow_L b = 1$. \square

Proposition 4.2. *Let \mathcal{A} be a HW algebra. The binary operator \leq defined as*

$$a \leq b \text{ iff } a \rightarrow_L b = 1 \text{ (equivalently, iff } a \rightarrow_G b = 1 \text{ iff } a \wedge b = a)$$

is a partial order on A .

Proof. $a \leq a$ is property (L1).

Let us suppose that $a \leq b$ and $b \leq a$. By (HW3) and $a \rightarrow_G b = 1$ we get $a = a \wedge b$. Dually, by (HW3) and $b \rightarrow_G a = 1$ we have $b = b \wedge a$ and by (L6) we have the thesis.

Finally, let us suppose that $a \leq b$ and $b \leq c$. By (HW2) and (L4), we obtain $1 = (a \rightarrow_G c)$, that is $a \leq c$. \square

The following result is now quite trivial.

Proposition 4.3. *The concrete structure $\langle [0, 1], \rightarrow_L, \rightarrow_G, 0 \rangle$ based on the real unit interval, where the operators \rightarrow_L and \rightarrow_G are the ones introduced in equations (1a) and (1b), satisfies HW axioms. As its name suggests, an HW algebra is a pasting of a Heyting and a Wajsberg algebra as stated in the following propositions.*

Proposition 4.4. *Let \mathcal{A} be a HW algebra. Then, by defining \wedge and \vee as in Definition 4.1, we have that $\langle A, \rightarrow_G, \wedge, \vee, \neg \rangle$ is a symmetric Heyting algebra according to Definition 3.4.*

Proof. Axioms (H1), (H3), (H4) and (H5) are axioms (HW1)–(HW4). We prove (H2). By definition of \wedge and setting $b = 1$ in (HW4) we have:

$$1 \rightarrow_G c = \neg\{\neg(a \rightarrow_G c) \rightarrow_L \neg(1 \rightarrow_G c)\} \rightarrow_L \neg(1 \rightarrow_G c)$$

By (L5) and \wedge definition we obtain $c = (a \rightarrow_G c) \wedge c$, i.e., (H2).

(K1) is property (L3). (K2) follows from \wedge, \vee definitions and (K1). \square

Proposition 4.5. *Let \mathcal{A} be a HW algebra. Then $\langle A, \rightarrow_L, \neg, 1 \rangle$ is a Wajsberg algebra according to Definition 3.2.*

Proof. Axiom (W1) is axiom (HW5). Axiom (W2) can be derived by the lattice property $b \wedge a \leq b \vee c$ and Proposition 4.1 as follows:

$$\begin{aligned} 1 &= b \wedge a \rightarrow_L b \vee c && \text{Def. } \wedge, \vee \\ &= \neg((\neg b \rightarrow_L \neg a) \rightarrow_L \neg a) \rightarrow_L ((b \rightarrow_L c) \rightarrow_L c) && (L2) \\ &= \neg((a \rightarrow_L b) \rightarrow_L \neg a) \rightarrow_L ((b \rightarrow_L c) \rightarrow_L c) && (HW6) \\ &= (a \rightarrow_L b) \rightarrow_L \neg((b \rightarrow_L c) \rightarrow_L c) \rightarrow_L \neg a && (HW6) \\ &= (a \rightarrow_L b) \rightarrow_L ((b \rightarrow_L c) \rightarrow_L (a \rightarrow_L c)) \end{aligned}$$

By definition of \vee , axiom (W3) is equivalent to $a \vee b = b \vee a$ which holds in any lattice (and hence in any Heyting algebra). Finally, axiom (W4) can be derived by property (L2). \square

When considering the underlying lattice structure of HW algebras, the following result can be proved.

Proposition 4.6. *Let \mathcal{A} be a HW algebra. Then, once defined \wedge, \vee, \sim, \neg and 1 as in Definition 4.1, the structure $\langle A, \wedge, \vee, \neg, \sim, 0, 1 \rangle$ is a BZ^{dM} lattice.*

Proof. Due to Theorems 4.4 and 4.5, the only properties that still need a proof are

$$(in) \quad \neg \sim a = \sim \sim a$$

$$(dM) \quad \sim(a \wedge b) = \sim a \vee \sim b$$

The first one follows from (HW7) and (HW8). Indeed, by (HW7) we have $\neg \sim a \leq \sim \sim a$. By (HW8) with $b = 0$ we get $\sim a \leq \neg a$ which applied to $\sim a$ gives $\sim \sim a \leq \neg \sim a$ and so (in) is verified.

Let us prove the second one. As proved in [25], in a pseudo complemented lattice [2, p. 125] and hence, in any Heyting algebra, the Stone condition

$$(S) \quad \sim a \vee \sim \sim a = 1$$

implies the de Morgan property (dM). So, it is sufficient to prove that (S) holds in any HW algebra. Let $a \in A$ and define $y = \sim a$. So, by (in) we get $\sim y = \neg y$. Using the non contradiction law for the Brouwer negation $y \wedge \sim y = 0$ (it holds in any pseudo complemented lattice [27, p. 60] hence in any Heyting algebra [22] and

by Theorem 4.4 in any HW algebra), we derive $y \wedge \neg y = 0$ and then $y \vee \neg y = 1$. Finally, we get $y \vee \sim y = 1$, that is $\sim a \vee \sim \sim a = 1$. \square

Hence, as we did in BZ lattices (see Definition 2.4), also in any HW algebra \mathcal{A} it is possible to define the anti-intuitionistic negation as $\flat(a) := \neg \sim \neg a$. Further, also the modal-like operators of necessity $\nu(a) := \sim \neg a$ and possibility $\mu(a) := \neg \sim a$ satisfy all the properties of Section 2.3.1. Finally, in any HW algebra it is possible to induce an abstract approximation space in the sense of Definition 2.5.

Remark 5. It is well known that in any lattice it holds the *duality principle* [27], i.e., (\wedge, \vee) and $(0, 1)$ are pair of dual operators: by replacing an operator with the dual one in a true statement, we again obtain a true statement. Monteiro [23] proved that a duality principle holds also for symmetric Heyting algebras, i.e., Heyting algebras with a de Morgan negation \neg . This result can be extended in a natural way to HW algebras defining the dual operators of the Gödel and Łukasiewicz implications respectively as $a \leftarrow_G b := \neg(\neg a \rightarrow_G \neg b)$ and $a \leftarrow_L b := \neg(\neg a \rightarrow_L \neg b)$.

4.1. Monteiro implication. The two primitive implications, Łukasiewicz and Gödel, are not the only one definable in HW algebras. Here we focus our attention on a third kind of implication, called “faible” in Monteiro [23]. This operator is strictly linked to modalities and to the anti-intuitionistic negation.

Definition 4.2. For any pair a, b of elements of a HW algebra the Monteiro implication is defined as:

$$a \rightarrow_M b := \mu(\neg a) \vee b$$

It can be shown that the following hold.

Proposition 4.7. *In any HW algebra the Monteiro implication satisfies the following properties:*

- (M1) $a \rightarrow_M a = 1$
- (M2) $a \rightarrow_M (b \rightarrow_M a) = 1$
- (M3) $a \rightarrow_M (b \rightarrow_M c) = (a \rightarrow_M b) \rightarrow_M (a \rightarrow_M c)$
- (M4) $((a \rightarrow_M b) \rightarrow_M a) \rightarrow_M a = 1$
- (M5) $1 \rightarrow_M a = a$
- (M6) $a \rightarrow_M 0 = \flat a$

Proof. These properties hold in all Heyting algebras with a Kleene negation \neg satisfying also the Stone condition (S) of \sim [23], hence they are valid also in HW algebras. \square

We remark that, as stated in (M6), the Monteiro implication permits to introduce independently the anti-intuitionistic negation.

Since we are greatly interested in the unit interval, it is natural to consider how behaves this kind of implication in $[0, 1]$ and if it can be defined as the residuation of a t -norm. The following proposition is an answer to these questions.

Proposition 4.8. *Let us consider the Monteiro implication on the unit interval $[0, 1]$. It does not exist a t -norm \mathbf{t} such that \rightarrow_M is the residuation of \mathbf{t} , i.e., such that for all $a, b \in [0, 1]$*

$$a \rightarrow_M b = \sup\{c \in [0, 1] : \mathbf{t}c \leq b\}$$

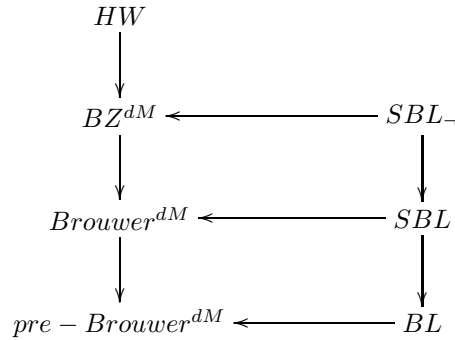
Proof. Once considered on the unit interval, the Monteiro implication reduces to

$$a \rightarrow_M b = \begin{cases} b & \text{if } a = 1 \\ 1 & \text{otherwise} \end{cases}$$

Now, suppose that there exists a t -norm \mathbf{t} generating \rightarrow_M . Then, for all $a \neq 1$ and for all b it should hold that $a\mathbf{t}1 \leq b$. But if \mathbf{t} is a t -norm then it must hold $a\mathbf{t}1 = a$, and consequently $a \leq b$, which, clearly, is not satisfied by all pairs $\langle a, b \rangle \in [0, 1) \times [0, 1]$. \square

5. CONCLUSIONS

The original structure of HW algebra has been introduced. Some weakening of HW algebras have been considered and their lattice structure has been studied. In the following diagram the relationship existing among all the discussed algebraic structures is summarized.



In the second part of this work [8], we will complete this diagram introducing some other algebras equivalent to HW algebras. This result will also enable us to show that SBL_{\neg} algebras are a substructure of HW algebras and that any HW algebra satisfies the Dummett condition for the Gödel implication.

An open problem about HW algebras is the proof of the independence of its axioms. We are able to prove that all axioms but (HW7) are independent from the others. In the case of (HW7) we cannot say if it depends or not from the others.

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