Newton's Fractals on Surfaces via Bicomplex Algebra

Filippo Maggioli
maggioli@di.uniroma1.it
Sapienza - University of Rome
Italy

Daniele Baieri
baieri@di.uniroma1.it
Sapienza - University of Rome
Italy

Simone Melzi
melzismn@gmail.com
University of Milano-Bicocca
Italy

Emanuele Rodolà
rodola@di.uniroma1.it
Sapienza - University of Rome
Italy

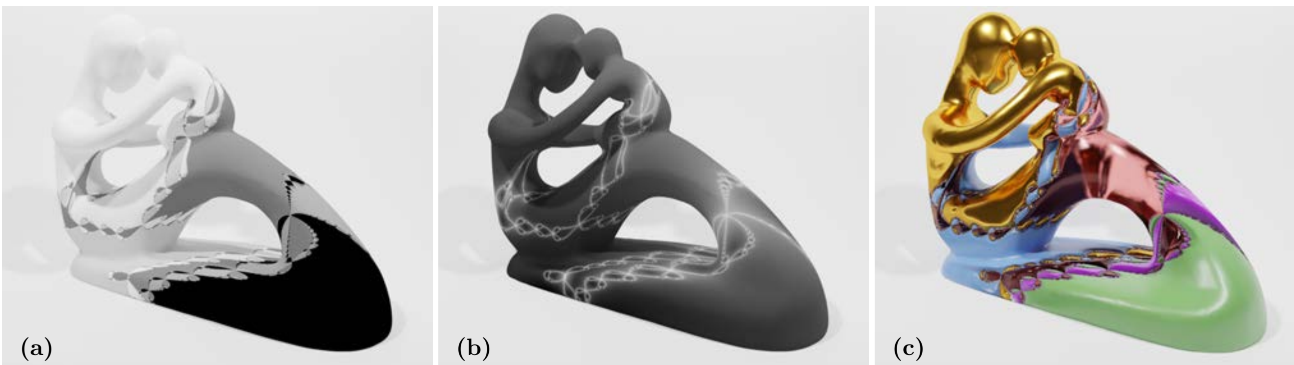

Figure 1: A bicomplex Newton's fractal on a surface (a), its convergence speed (b) and a material built over these two maps (c).

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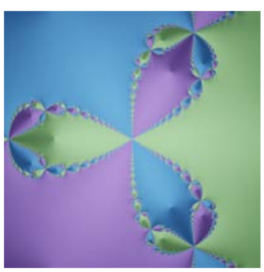
1 INTRODUCTION



The Newton-Raphson method is a well-known iterative method used to find roots of any function $f : \mathbb{C} \to \mathbb{C}$. Nearby points usually converge to the same solution. Hence one can identify regions associated with each solution, whose boundaries describe fractal patterns [Peitgen et al. 1988]. This type of pattern is known as Newton's fractal and is typically used to generate interesting visualizations and effects like the one shown in the inset figure. The usual approach is to define some polynomial $p(z)$ with roots $\{\xi_i\}_{i=1}^{n}$ and apply the Newton-Raphson method to all points on the plane. Each point is associated with an index i corresponding to the solution ξ_i it converged to and then colored with some coloration $C(i)$. Despite the beautiful visualizations that can arise, Newton's fractals can only be used to generate images.

Bicomplex numbers are a generalization of complex numbers that define a closed and commutative algebra in four dimensions [Davenport 1991, 1996]. The bicomplex field $\mathbb{BC}$ is isomorphic to the field of 2×2 matrices over $\mathbb{C}$ spanned by the following basis:

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \boldsymbol{i} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \quad \boldsymbol{j} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \boldsymbol{k} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (1)$$

Any bicomplex number can always be described as $z = x\mathbf{1} + y\boldsymbol{j}$, being $x, y \in \mathbb{C}$. Differently from the complex field, bicomplex algebra is isomorphic to a direct sum of two algebras over $\mathbb{C}$ [Davenport 1996]. In fact, one can construct two idempotent, zero divisors and orthogonal elements $\boldsymbol{e} = (1 + \boldsymbol{k})/2$ and $\boldsymbol{e}^{\dagger} = (1 - \boldsymbol{k})/2$ such that every bicomplex number z can be uniquely decomposed into $z = \alpha\boldsymbol{e} + \beta\boldsymbol{e}^{\dagger}$, making effectively $\{\boldsymbol{e}, \boldsymbol{e}^{\dagger}\}$ a basis for $\mathbb{BC}$ over the scalar field $\mathbb{C}$.

Most of the properties of complex numbers still hold in bicomplex algebra [Rönn 2001]. Bicomplex numbers are commutative and invertible, and the elementary functions can be easily extended [Luna-Elizarrarás et al. 2012]. The derivative of a bicomplex function $f : \mathbb{BC} \to \mathbb{BC}$ is well defined, and most of the differentiation rules still apply.

Bicomplex numbers have already been used in abstract mathematics and computer graphics applications for describing fractals [Wang and Song 2013]. However, previous work is limited to considering classic escape-time fractals, like the Mandelbrot and

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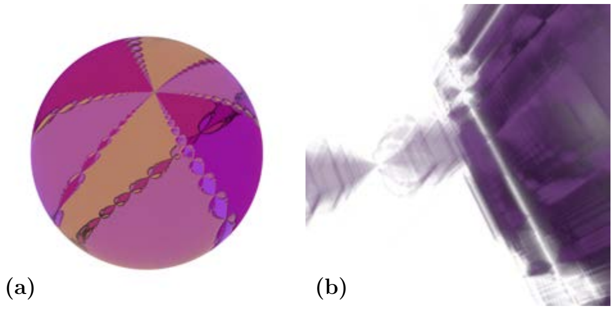

Figure 2: An example of bicomplex Newton's fractal used as decoration (a). A volumetric rendering of a region that identifies a solution of $z^3 - 1 = 0$ (b).

Figure 3: An example of bicomplex Newton's fractal used as mask for mixing different materials.

Julia sets. In this work, we show that it is possible to use bicomplex numbers to generalize even the well-known Newton's fractal. Moreover, we also provide insights for possible application in procedural texturing and volumetric rendering.

2 METHOD

Since bicomplex functions in the form $f : \mathbb{BC} \to \mathbb{BC}$ can be expanded with Taylor [Rönn 2001], and the Taylor series in $\mathbb{BC}$ converges, we can use the classical proof of the Newton-Raphson method to show that Newton's iteration converges to the root of a function: this allows us to generalize Newton's fractal to bicomplex numbers, and use it to generate 4-dimensional patterns.

In the complex plane, the roots of the polynomial $c^n - 1 = 0$ lie on the unit circle and are equispaced, and the interesting patterns arising from this particular family of polynomials are shown in most visualizations of Newton's fractal. This still holds in $\mathbb{BC}$, where the roots of $z^n - 1 = 0$ are equispaced on the unitary 4-dimensional hyper-sphere [Pogorui and Rodríguez-Dagnino 2006]. This polynomial has n^2 closed form solutions [Luna-Elizarrarás et al. 2012; Pogorui and Rodríguez-Dagnino 2006] $\beta_{ij} = \alpha_i \boldsymbol{e} + \alpha_j \boldsymbol{e}^\dagger$, with $\alpha_k = \cos(2\frac{k-1}{n}\pi) + i\sin(2\frac{k-1}{n}\pi)$.

Applying the Newton-Raphson method to solve bicomplex polynomials in the form $z^n - 1 = 0$ produces the same patterns of the complex plane, but it extends to 4 dimensions. By using the 3D coordinates of the visualized points, we can generate interesting and complex patterns, as we show in Figure 1a and 2a, while using the fourth coordinate either as a tunable parameter or as a time dependency, allowing for an additional degree of freedom.

3 IMPLEMENTATION

We implement the algorithm as a pixel shader, where each thread computes Newton's iteration on the given point. This gives us the possibility to fully exploit GPU computing.

We optimize our iteration scheme using ad-hoc algebraic manipulations for the specific polynomial in use:

$$z_{k+1} = z_k - \frac{f(z_k)}{\frac{\partial f(x_k)}{\partial z}} = z_k - \frac{z_k^n - 1}{n\, z_k^{n-1}} = \frac{1}{n}\left((n-1)z_k + z_k^{1-n}\right) \quad (2)$$

This way we avoid bicomplex division, which usually requires 64 FLOPs. We also take advantage of the decomposition in the basis $\{\boldsymbol{e}, \boldsymbol{e}^\dagger\}$ to efficiently compute powers: in fact, since both $\boldsymbol{e}$ and $\boldsymbol{e}^\dagger$ are idempotent, and since $\boldsymbol{e}\boldsymbol{e}^\dagger = 0$, then we have $z^n = (\alpha\boldsymbol{e} + \beta\boldsymbol{e}^\dagger)^n = \alpha^n\boldsymbol{e} + \beta^n\boldsymbol{e}^\dagger$ (for any $n \in \mathbb{Z}$), and complex powers can be computed efficiently using polar coordinates.

We exploit the decomposition to represent bicomplex numbers in the whole shader. Other than simplifying and speeding up the computation, this representation also makes it easier to compute which bicomplex solution a number converged to. If a number converged to some $z = \alpha\boldsymbol{e} + \beta\boldsymbol{e}^\dagger$, we can reduce the problem to find to which complex solution α and β converged.

A smooth gradient representing the speed of convergence typically enriches Newton's fractals using it, for example, as a heightmap for additional detail in the texture. An option for computing it is

$$t = P - \sum_{k=0}^{P} \frac{1}{1 + \exp(\delta_k + \theta) - \exp(\theta)} \quad (3)$$

where $\delta_k = |z_{k+1} - z_k|$ is the magnitude of each step, P is the number of iterations and θ is a tunable parameter. The same approach can be generalized to a bicomplex setting, as we show in Figure 1b.

4 RESULTS AND CONCLUSIONS

Bicomplex numbers offer an instrument for generalizing Newton's fractal to 4 dimensions. This type of fractal offers a new possibility for procedural generation of 4D textures. We have shown that bicomplex Newton's fractals generate interesting and complex patterns, similar to their complex counterpart. Our results prove that these patterns can fit all the most common applications, from decorating surfaces (Figure 1c) and masking materials (Figure 3), to volumetric rendering of fractal regions (Figure 2b).

REFERENCES

C.M. Davenport. 1991. *A Commutative Hypercomplex Calculus with Applications to Special Relativity.* Eigenverl.

C.M. Davenport. 1996. *A Commutative Hypercomplex Algebra with Associated Function Theory.* Birkhauser Boston Inc., USA, 213–227.

M.E. Luna-Elizarrarás, M. Saphiro, D.C. Struppa, and A. Vajiac. 2012. Bicomplex Numbers and their Elementary Functions. *Cubo (Temuco)* 14 (00 2012), 61 – 80.

H.O. Peitgen, D. Saupe, Y. Fisher, M.F. Barnsley, Association for Computing Machinery Special Interest Group on Graphics, M. McGuire, B.B. Mandelbrot, R.L. Devaney, and R.F. Voss. 1988. *The Science of Fractal Images.* Springer New York.

A.A. Pogorui and R.M. Rodríguez-Dagnino. 2006. On the set of zeros of bicomplex polynomials. *Complex Variables and Elliptic Equations* 51, 7 (2006), 725–730.

S. Rönn. 2001. Bicomplex algebra and function theory. arXiv:math/0101200 [math.CV]

X.-Y. Wang and W. Song. 2013. The generalized M–J sets for bicomplex numbers. *Nonlinear Dynamics* 72 (2013), 17–26.