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Existence and Multiplicity of Solutions for a Bi-Non-Local Problem

Jiabin Zuo [1,*], Tianqing An [2], Alessio Fiscella [3] and Chungen Liu [1,*]

1 School of Mathematics and Information Science, Guangzhou University, Guangzhou 510006, China
2 College of Science, Hohai University, Nanjing 210098, China; antq@hhu.edu.cn
3 Dipartimento di Matematica e Applicazioni, Universita degli Studi di Milano-Bicocca, Via Cozzi 55, 20125 Milano, Italy; alessio.fiscella@unimib.it
* Correspondence: zuojiabin88@163.com (J.Z.); liucg@nankai.edu.cn (C.L.)

Abstract: The aim of this paper is to investigate the existence and multiplicity of solutions for a bi-non-local problem. Precisely, we show that the above problem admits at least a non-trivial positive energy solution by using the mountain pass theorem. Furthermore, with the help of the fountain theorem, we obtain the existence of infinitely many positive energy solutions, assuming a symmetric condition for g. The main feature and difficulty of this paper is the presence of a double non-local term involving two variable parameters.

Keywords: Kirchhoff coefficient; $p(\cdot)$-fractional Laplacian; variable exponent; variable-order

MSC: 35R11; 47G20; 35S15; 35J60

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1. Introduction

Recently, Lorenzo and Hartley in [1] came up with the fractional variable-order derivatives that are used to describe different processes of nonlinear diffusion. Indeed, the variable order problem of non-local integro-differential operators can better reflect the temperature change of an object. Therefore, a large number of researchers have begun to pay attention to fractional variable-order spaces. See [2–5] and the references therein.

In this paper, we study the following variable $s(\cdot)$-order fractional Kirchhoff type problem

$$\begin{cases} M\left(\iint_{\mathbb{R}^{2N}} \frac{1}{p(x,y)} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}} dxdy\right)(-\Delta)^{s(\cdot)}_{p(\cdot)} u(x) = \mu |u(x)|^{\overline{p}(x)-2}u(x) + g(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (1)$$

where the domain $\Omega \subset \mathbb{R}^N$ is bounded and smooth with $N > p(x,y)s(x,y)$ for any $(x,y) \in \overline{\Omega} \times \overline{\Omega}$, where μ is a real parameter, $s(\cdot): \mathbb{R}^N \times \mathbb{R}^N \to (0,1)$ and $p(\cdot): \mathbb{R}^N \times \mathbb{R}^N \to (1,\infty)$, exponent $\overline{p}(x) = p(x,x)$ for $x \in \overline{\Omega}$. Here, $(-\Delta)^{s(\cdot)}_{p(\cdot)}$ is a $p(\cdot)$-Laplace operator with fractional variable $s(\cdot)$-order, which is given by

$$(-\Delta)^{s(\cdot)}_{p(\cdot)} \varphi(x) = P.V. \int_{\mathbb{R}^N} \frac{|\varphi(x)-\varphi(y)|^{p(x,y)-2}(\varphi(x)-\varphi(y))}{|x-y|^{N+p(x,y)s(x,y)}} dy, \quad x \in \mathbb{R}^N, \quad (2)$$

along any $\varphi \in C_0^\infty(\mathbb{R}^N)$, where P.V. is the Cauchy principal value.

For the sake of convenience, we denote

$$s^- = \inf_{(x,y)\in\mathbb{R}^{2N}} s(x,y), \quad s^+ = \sup_{(x,y)\in\mathbb{R}^{2N}} s(x,y), \quad p^- = \inf_{(x,y)\in\mathbb{R}^{2N}} p(x,y), \quad p^+ = \sup_{(x,y)\in\mathbb{R}^{2N}} p(x,y),$$

$$p_s^*(x) = \frac{N\overline{p}(x)}{N-\overline{s}(x)\overline{p}(x)} \quad \text{with} \quad \overline{p}(x) = p(x,x), \quad \overline{s}(x) = s(x,x), \quad \overline{p}^- = \inf_{x\in\mathbb{R}^N} \overline{p}(x), \quad \overline{p}^+ = \sup_{x\in\mathbb{R}^N} \overline{p}(x).$$

The *continuous* function $M : \mathbb{R}^+ \to \mathbb{R}^+$ fulfills the following conditions.

(M_1) *There exist $h_2 \geq h_1 > 0$ and $\beta > 1$, with $p^+ < \beta p^-$, such that*

$$h_1 t^{\beta-1} \leq M(t) \leq h_2 t^{\beta-1} \quad \textit{for all} \quad t \in \mathbb{R}^+.$$

Furthermore, we assume the function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is *continuous* and verifies the following two conditions.

(G_1) *There exist $c_1 > 0$ and $q \in C(\overline{\Omega})$ satisfying*

$$|g(x,t)| \leq c_1 |t|^{q(x)-1}, \quad \textit{for all} \quad (x,t) \in \Omega \times \mathbb{R}$$

and

$$\beta p^+ < q^- = \inf_{x\in\overline{\Omega}} q(x) < q(x) < p_s^*(x), \quad \textit{for all} \quad x \in \Omega,$$

where β is given in (M_1) above.

(G_2) *For h_1, h_2, and β given in (M_1), there exist t_0 and $\lambda \in \left(\frac{h_2\beta(p^+)^\beta}{h_1(p^-)^{\beta-1}}, +\infty\right)$, such that*

$$0 < \lambda G(x,t) \leq t g(x,t), \quad \text{for all } t \in \mathbb{R} \text{ with } \; |t| \geq t_0, \quad \textit{and} \quad x \in \Omega,$$

where $G(x,t) = \int_0^t g(x,s)ds$.

Furthermore, we propose the following condition on the function g.

$(G_3) : g(x,-t) = -g(x,t)$ *for all $(x,t) \in \Omega \times \mathbb{R}$.*

In the operator $(-\Delta)_{p(\cdot)}^{s(\cdot)}$, we suppose that *continuous* functions $s(\cdot) : \mathbb{R}^N \times \mathbb{R}^N \to (0,1)$ and $p(\cdot) : \mathbb{R}^N \times \mathbb{R}^N \to (1,\infty)$ fulfill

(H_1): $0 < s^- \leq s^+ < 1 < p^- \leq p^+$;

(H_2): *$s(\cdot)$ and $p(\cdot)$ are symmetric, i.e., $s(x,y) = s(y,x)$ and $p(x,y) = p(y,x)$ for any $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$.*

Clearly, the operator in (2) reduces to the fractional p-Laplacian $(-\Delta)_p^s$ as $p(x,y) \equiv p$ and $s(x,y) \equiv s$; see [6–8], and the references therein. In particular, we point out that An et al. in [9] studied the existence of infinitely many solutions for a class of fractional p-Laplacian equations by using the fountain theorem. They also investigated a fractional p-Laplacian system with the help of the Nehari manifold method in [10]. This type of operator has a widespread application in various sciences, such as mechanics [11], finance [12], and so on. For a Kirchhoff situation, we recall [13] where the authors construct a stationary fractional Kirchhoff problem, which is excellent pioneering work. It is worth noting that a typical non-local equation that has attracted attention is the Kirchhoff type equation, which is a physical model given by Kirchhoff [14] in 1883, i.e.,

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{3}$$

where u denotes the displacement of the string, ρ denotes the mass density, P_0 denotes the initial tension, h denotes the area of the cross section, E denotes the Young's modulus of the material, and L denotes the length of the string. For more physical phenomena described by classical Kirchhoff theory, see [15].

In addition, in the scope of ordinary differential equation research, non-local problems have also received extensive attention, and we specifically point out two excellent studies [16,17].

Very recently, great interest has been devoted to the investigation of fractional problems involving possibly variable order and variable exponent. The classic example is from Chen, Levine, and Rao [18], and it concerns applications to image restoration. We also refer to [19,20] for a multiplicity result for a problem driven by $(-\Delta)^{s(\cdot)}$, that is, operator (2) with $p(x,y) \equiv 2$. In [21–24], different approaches are described to handle a fractional operator $(-\Delta)^{s}_{p(\cdot)}$, with $s(x,y) \equiv s$. Papers [25–28] introduce variational techniques and properties for the local version of operator $(-\Delta)^{s}_{p(\cdot)}$, that is, with the integral in (2) set on Ω instead of $\mathbb{R}^N$. Finally, the authors in [29,30] try to consider problems involving a variable-order fractional operator with variable exponent $p(\cdot)$.

Motivated by the above papers, we study a new double variable order fractional Kirchhoff type problem (1). As far as we know, very few papers have studied the infinite number of solutions to such a bi-non-local equation. Indeed, in [22], the authors considered a class of fractional $p(\cdot)$-Kirchhoff type problems, such as (1) but with $s(x,y) \equiv s$, $\mu = 0$ and g of a model form. Thus, our main results stated below generalize ([22], Theorems 3.1 and 3.3) in several directions, and somehow the existence results in [21,24].

Theorem 1. *If the conditions* $(H_1) - (H_2)$, (M_1), *and* $(G_1) - (G_2)$ *hold, then, there exists* $\mu^* > 0$ *such that for any* $\mu \in (-\infty, \mu^*]$ *problem* (1) *admits a non-trivial weak solution.*

By further assuming the symmetric condition (G_3), we obtain the following multiplicity result.

Theorem 2. *If the conditions* $(H_1) - (H_2)$, (M_1), *and* $(G_1) - (G_3)$ *hold, then, for any* $\mu \in \mathbb{R}$ *problem* (1) *has infinitely many weak solutions with unbounded positive energy.*

The remaining sections are organized as follows. Section 2 introduces some lemmas and knowledge of space theory. Section 3 verifies the Palais–Smale condition. Section 4 gives the proof of Theorem 1. Section 5 proves Theorem 2.

2. Functional Analytic Setup and Preliminaries

Let

$$C_+(\overline{\Omega}) = \Big\{ r \in C(\overline{\Omega}) : 1 < r(x) \ \text{ for all } \ x \in \overline{\Omega} \Big\}.$$

For any $r \in C_+(\overline{\Omega})$ we recall the variable exponent Lebesgue space

$$L^{r(\cdot)}(\Omega) = \left\{ u : \ \text{the function } u : \Omega \to \mathbb{R} \ \text{ is measurable,} \ \ \int_\Omega |u(x)|^{r(x)} dx < \infty \right\},$$

with the norm

$$\|u\|_{r(\cdot)} = \inf \left\{ \gamma > 0 : \int_\Omega \left| \frac{u(x)}{\gamma} \right|^{r(x)} dx \le 1 \right\}.$$

Then $(L^{r(\cdot)}(\Omega), \| \cdot \|_{r(\cdot)})$ is a separable reflexive Banach space (see [31], Theorem 2.5 and Corollaries 2.7 and 2.12).

Let $\widetilde{r} \in C_+(\overline{\Omega})$ be the conjugate exponent of r, that is

$$\frac{1}{r(x)} + \frac{1}{\widetilde{r}(x)} = 1, \qquad \text{for all } x \in \overline{\Omega}.$$

Then we have the following Hölder inequality (see [31], Theorem 2.1).

Lemma 1. *Suppose that $u \in L^{r(\cdot)}(\Omega)$ and $v \in L^{\widetilde{r}(\cdot)}(\Omega)$. Then*

$$\Big|\int_\Omega uvdx\Big| \le \Big(\frac{1}{r^-} + \frac{1}{\widetilde{r}^-}\Big)\|u\|_{r(\cdot)}\|v\|_{\widetilde{r}(\cdot)} \le 2\|u\|_{r(\cdot)}\|v\|_{\widetilde{r}(\cdot)}.$$

Defining the modular functional $\rho_{r(\cdot)} : L^{r(\cdot)}(\Omega) \to \mathbb{R}$, by

$$\rho_{r(\cdot)}(u) = \int_\Omega |u(x)|^{r(x)}dx,$$

we have the next crucial result given in [32].

Proposition 1. *Suppose that $u \in L^{r(\cdot)}(\Omega)$ and $\{u_j\} \subset L^{r(\cdot)}(\Omega)$. Then*

(1) $\|u\|_{r(\cdot)} < 1(resp. = 1, > 1) \Leftrightarrow \rho_{r(\cdot)}(u) < 1(resp. = 1, > 1)$,
(2) $\|u\|_{r(\cdot)} < 1 \Rightarrow \|u\|^{r^+}_{r(\cdot)} \le \rho_{r(\cdot)}(u) \le \|u\|^{r^-}_{r(\cdot)}$,
(3) $\|u\|_{r(\cdot)} > 1 \Rightarrow \|u\|^{r^-}_{r(\cdot)} \le \rho_{r(\cdot)}(u) \le \|u\|^{r^+}_{r(\cdot)}$,
(4) $\lim\limits_{j\to\infty} \|u_j\|_{r(\cdot)} = 0(\infty) \Leftrightarrow \lim\limits_{j\to\infty} \rho_{r(\cdot)}(u_j) = 0(\infty)$,
(5) $\lim\limits_{j\to\infty} \|u_j - u\|_{r(\cdot)} = 0 \Leftrightarrow \lim\limits_{j\to\infty} \rho_{r(\cdot)}(u_j - u) = 0$.

The variable order fractional Sobolev spaces with variable exponent is defined by

$$W^{s(\cdot),p(\cdot)}(\Omega) = \left\{u \in L^{\overline{p}(\cdot)}(\Omega) : \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\gamma^{p(x,y)}|x-y|^{N+p(x,y)s(x,y)}}dxdy < \infty, \text{ for some } \gamma > 0\right\}$$

with the norm $\|u\|_{s(\cdot),p(\cdot)} = \|u\|_{\overline{p}(\cdot)} + [u]_{s(\cdot),p(\cdot)}$, where

$$[u]_{s(\cdot),p(\cdot)} = \inf\left\{\gamma > 0 : \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\gamma^{p(x,y)}|x-y|^{N+p(x,y)s(x,y)}}dxdy < 1\right\}.$$

We define the new variable order fractional Sobolev spaces with variable exponent (see more details in reference [29]):

$$X = \left\{u : \mathbb{R}^N \to \mathbb{R} : u|_\Omega \in L^{\overline{p}(\cdot)}(\Omega), \iint_Q \frac{|u(x)-u(y)|^{p(x,y)}}{\gamma^{p(x,y)}|x-y|^{N+p(x,y)s(x,y)}}dxdy < \infty, \text{ for some } \gamma > 0\right\},$$

where $Q := \mathbb{R}^{2N}\backslash(\Omega^c \times \Omega^c)$. The space X is endowed with the norm

$$\|u\|_X = \|u\|_{\overline{p}(\cdot)} + [u]_X,$$

where

$$[u]_X = \inf\left\{\gamma > 0 : \iint_Q \frac{|u(x)-u(y)|^{p(x,y)}}{\gamma^{p(x,y)}|x-y|^{N+p(x,y)s(x,y)}}dxdy < 1\right\}.$$

We notice that the norms $\|\cdot\|_{s(\cdot),p(\cdot)}$ and $\|\cdot\|_X$ are not the same because $\Omega \times \Omega \subset Q$ and $\Omega \times \Omega \ne Q$. This makes $W^{s(\cdot),p(\cdot)}(\Omega)$ not sufficient for studying the kind of problem like (1).

For this, we set our Banach space workspace as

$$X_0 = \left\{u \in X : u = 0 \text{ } a.e. \text{ in } \mathbb{R}^N\backslash\Omega\right\},$$

which is separable and reflexive (see [30], Proposition 3.7), with respect to the norm

$$\begin{aligned}\|u\|_{X_0} &= \inf\left\{\gamma > 0 : \iint_Q \frac{|u(x)-u(y)|^{p(x,y)}}{\gamma^{p(x,y)}|x-y|^{N+p(x,y)s(x,y)}}dxdy < 1\right\} \\ &= \inf\left\{\gamma > 0 : \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{\gamma^{p(x,y)}|x-y|^{N+p(x,y)s(x,y)}}dxdy < 1\right\},\end{aligned}$$

where the last equality is a consequence of the fact that $u = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$.

We are ready to introduce an embedding theorem for X_0, given in ([29], Theorem 2.5).

Lemma 2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain. Let $p(\cdot)$ and $s(\cdot)$ satisfy $(H_1)-(H_2)$, such that $N > p(x,y)s(x,y)$ for all $(x,y) \in \overline{\Omega} \times \overline{\Omega}$. Then for any $r \in C_+(\overline{\Omega})$ with $1 < r(x) < p_s^*(x)$ for $x \in \overline{\Omega}$, there exists a positive constant $C_r = C_r(N, s, p, r, \Omega)$ such that*

$$\|u\|_{r(x)} \leq C_r \|u\|_{X_0} \tag{4}$$

for any $v \in X_0$. Furthermore, the embedding $X_0 \hookrightarrow L^{r(\cdot)}(\Omega)$ is compact.

We note that $\|\cdot\|_{X_0}$ and $\|\cdot\|_X$ are equivalent norms on X_0. We define the fractional modular functional $\varrho_{p(\cdot)}^{s(\cdot)} : X_0 \to \mathbb{R}$, by

$$\varrho_{p(\cdot)}^{s(\cdot)}(u) = \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}}dxdy.$$

Then, similar to Proposition 1, we get

Proposition 2. *([30], Lemmas 3.4 and 3.5). Suppose that $u \in X_0$ and $\{u_j\} \subset X_0$. Then*

(1) $\|u\|_{X_0} < 1 (resp. = 1, > 1) \Leftrightarrow \varrho_{p(\cdot)}^{s(\cdot)}(u) < 1 (resp. = 1, > 1)$,

(2) $\|u\|_{X_0} < 1 \Rightarrow \|u\|_{X_0}^{p^+} \leq \varrho_{p(\cdot)}^{s(\cdot)}(u) \leq \|u\|_{X_0}^{p^-}$,

(3) $\|u\|_{X_0} > 1 \Rightarrow \|u\|_{X_0}^{p^-} \leq \varrho_{p(\cdot)}^{s(\cdot)}(u) \leq \|u\|_{X_0}^{p^+}$,

(4) $\lim_{j\to\infty} \|u_j\|_{X_0} = 0(\infty) \Leftrightarrow \lim_{j\to\infty} \varrho_{p(\cdot)}^{s(\cdot)}(u_j) = 0(\infty)$,

(5) $\lim_{j\to\infty} \|u_j - u\|_{X_0} = 0 \Leftrightarrow \lim_{j\to\infty} \varrho_{p(\cdot)}^{s(\cdot)}(u_j - u) = 0.$

A function $u \in X_0$ is a weak solution of problem (1), if

$$\begin{aligned} M\Big(\delta_{p(\cdot)}(u)\Big) &\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+p(x,y)s(x,y)}}dxdy \\ &= \mu\int_\Omega |u(x)|^{\overline{p}(x)-2}u(x)\phi(x)dx + \int_\Omega g(x,u)\phi dx,\end{aligned} \tag{5}$$

for all $\phi \in X_0$, where

$$\delta_{p(\cdot)}(u) = \iint_{\mathbb{R}^{2N}} \frac{1}{p(x,y)} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}}dxdy.$$

Considering the following functional associated with problem (1), defined by $\mathcal{I}_\mu : X_0 \to \mathbb{R}$

$$\mathcal{I}_\mu(u) = \widetilde{M}\Big(\delta_{p(\cdot)}(u)\Big) - \mu \int_\Omega \frac{1}{\overline{p}(x)}|u(x)|^{\overline{p}(x)}dx - \int_\Omega G(x,u)dx,$$

where $\widetilde{M}(t) = \int_0^t M(\tau)d\tau$. Clearly, it follows from the continuity of M that $\mathcal{I}_\mu$ is well defined and of class C^1 on X_0. Furthermore, we have

$$\begin{aligned}\langle \mathcal{I}_\mu'(u), \phi\rangle = \quad & M\Big(\delta_{p(\cdot)}(u)\Big) \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+p(x,y)s(x,y)}}dxdy \\ & -\mu \int_\Omega |u(x)|^{\overline{p}(x)-2}u(x)\phi(x)dx - \int_\Omega g(x,u)\phi(x)dx,\end{aligned}$$

for all u, $\phi \in X_0$. Hence, the weak solutions of problem (1) are the critical points of $\mathcal{I}_\mu$. If such a weak solution exists and is non-trivial, then μ is an eigenvalue of problem (1).

We conclude this section by presenting a technical result that is useful in studying the compactness of $\mathcal{I}_\mu$. The proof of this proposition is similar to ([26], Lemma 4.2) and working on X_0.

Proposition 3. *We consider the following functional $\mathcal{A} : X_0 \to X_0^*$, with X_0^* the dual space of X_0, such that*

$$\langle \mathcal{A}(u), \phi\rangle = \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+p(x,y)s(x,y)}}dxdy,$$

for any u, $\phi \in X_0$. Then:

(i) The operator $\mathcal{A}$ is bounded and strictly monotone;

(ii) $\mathcal{A}$ is a mapping of type (S_+), that is, if $u_j \rightharpoonup u \in X_0$ and $\limsup_{j\to\infty} \mathcal{A}(u_j)(u_j - u) \leq 0$, then $u_j \to u \in X_0$;

(iii) $\mathcal{A} : X_0 \to X_0^*$ is a homeomorphism.

Throughout this paper, for simplicity, we use $\{c_i,\ i \in \mathbb{N}\}$ to denote different non-negative or positive constants. In addition, we denote with c^+ and c^-, respectively, the positive part and negative part of a number $c \in \mathbb{R}$.

3. Palais–Smale Condition

We now recall the definition of the Palais–Smale condition. We say that $\mathcal{I}_\mu$ fulfills the Palais–Smale condition at the level $c \in \mathbb{R}$ if any sequence $u_j \subset X_0$ fulfilling

$$\mathcal{I}_\mu(u_j) \to c \text{ and } \mathcal{I}_\mu'(u_j) \to 0 \text{ in } X_0^* \text{ as } j \to \infty, \tag{6}$$

possesses a convergent subsequence in X_0.

Lemma 3. *Suppose that (M_1), $(G_1) - (G_2)$, and $(H_1) - (H_2)$ hold. Then for any $\mu \in \mathbb{R}$ the functional $\mathcal{I}_\mu$ fulfills the Palais–Smale condition for any $c \in \mathbb{R}$.*

Proof. Let $\mu \in \mathbb{R}$. Suppose a sequence $\{u_j\} \subset X_0$ verifying (6). We argue in two steps.

Step 1. We first show that the sequence $\{u_j\} \subset X_0$ is bounded. For this end, by (M_1), (G_2), Propositions 1 and 2, and Lemma 2, we get

$$\begin{aligned}\lambda \mathcal{I}_\mu(u_j) - \langle \mathcal{I}_\mu'(u_j), u_j\rangle = \quad & \lambda \widetilde{M}\Big(\delta_{p(\cdot)}(u_j)\Big) - M\Big(\delta_{p(\cdot)}(u_j)\Big) \iint_{\mathbb{R}^{2N}} \frac{|u_j(x)-u_j(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}}dxdy \\ & -\mu \int_\Omega \Big(\frac{\lambda}{\overline{p}(x)} - 1\Big)|u_j|^{\overline{p}(x)}dx - \int_\Omega (\lambda G(x,u_j) - g(x,u_j)u_j)dx \\ & \geq \frac{\lambda h_1}{\beta}\Big(\delta_{p(\cdot)}(u_j)\Big)^\beta - h_2\Big(\delta_{p(\cdot)}(u_j)\Big)^{\beta-1}\Big(\varrho_{p(\cdot)}^{s(\cdot)}(u_j)\Big) - \mu^+ \int_\Omega \Big(\frac{\lambda}{\overline{p}(x)} - 1\Big)|u_j|^{\overline{p}(x)}dx \\ & \geq \frac{\lambda h_1}{\beta (p^+)^\beta}\Big(\varrho_{p(\cdot)}^{s(\cdot)}(u_j)\Big)^\beta - \frac{h_2}{(p^-)^{\beta-1}}\Big(\varrho_{p(\cdot)}^{s(\cdot)}(u_j)\Big)^\beta - \mu^+\Big(\frac{\lambda}{p^-} - 1\Big)\varrho_{\overline{p}(\cdot)}(u_j)\end{aligned}$$

$$\geq \left(\frac{\lambda h_1}{\beta(p^+)^\beta} - \frac{h_2}{(p^-)^{\beta-1}}\right)\min\left\{\|u_j\|_{X_0}^{\beta p^-}, \|u_j\|_{X_0}^{\beta p^+}\right\} - \mu^+\left(\frac{\lambda}{p^-} - 1\right)\max\left\{(C_{\overline{p}}\,\|u_j\|_{X_0})^{\overline{p}^-}, (C_{\overline{p}}\,\|u_j\|_{X_0})^{\overline{p}^+}\right\},$$

and recall that $\lambda > p^+ \geq \overline{p}(x) \geq p^-$ for $x \in \overline{\Omega}$, by (G_2). Thus from (6), there exists $\sigma_\mu > 0$ such that as $j \to \infty$, there holds

$$\lambda c + \sigma_\mu \|u_j\|_{X_0} + o(1) \geq \left(\frac{\lambda h_1}{\beta(p^+)^\beta} - \frac{h_2}{(p^-)^{\beta-1}}\right)\min\left\{\|u_j\|_{X_0}^{\beta p^-}, \|u_j\|_{X_0}^{\beta p^+}\right\} - \mu^+\left(\frac{\lambda}{p^-} - 1\right)\max\left\{(C_{\overline{p}}\,\|u_j\|_{X_0})^{\overline{p}^-}, (C_{\overline{p}}\,\|u_j\|_{X_0})^{\overline{p}^+}\right\},$$

which implies that $\{u_j\}$ is bounded in X_0, as $1 < p^- \leq \overline{p}^- \leq \overline{p}^+ \leq p^+ < \beta p^- \leq \beta p^+$ by (G_1).

Step 2. We will show that $\{u_j\}$ converges strongly in X_0. In view of Lemma 2 and the reflexivity of X_0, that there exists a subsequence, still denoted by $\{u_j\}$, and $u \in X_0$ such that

$$u_j \rightharpoonup u \ \text{ in } \ X_0, \qquad u_j \to u \ \text{ in } \ L^{r(\cdot)}(\Omega), \qquad u_j(x) \to u(x) \ \text{ a.e. in } \ \Omega, \tag{7}$$

for any $r \in C_+(\overline{\Omega})$, with $1 < r(x) < p_s^*(x)$ for $x \in \overline{\Omega}$. Using the Hölder inequality (Lemma 1) and (7) with $r \equiv \overline{p}$, from $p^+ < \beta p^+ < p_s^*(x)$ for $x \in \overline{\Omega}$, by (G_1), we get

$$\left|\int_\Omega |u_j|^{\overline{p}(x)-2}u_j(u_j-u)dx\right| \leq \int_\Omega |u_j|^{\overline{p}(x)-1}|u_j-u|dx \leq 2\||u_j|^{\overline{p}(x)-1}\|_{\frac{\overline{p}(x)}{\overline{p}(x)-1}}\|u_j-u\|_{\overline{p}(x)} \to 0 \ \text{ as } \ j \to \infty.$$

Therefore,

$$\lim_{j\to\infty}\int_\Omega |u_j|^{\overline{p}(x)-2}u_j(u_j-u)dx = 0. \tag{8}$$

According to (G_1), (7) with $r \equiv q$ and the Hölder inequality (Lemma 1) , we have

$$\left|\int_\Omega g(x,u_j)(u_j-u)dx\right| \leq c_1\int_\Omega |u_j|^{q(x)-1}|u_j-u|dx \leq 2c_1\||u_j|^{q(x)-1}\|_{\frac{q(x)}{q(x)-1}}\|u_j-u\|_{q(x)} \to 0 \ \text{ as } \ j \to \infty,$$

which implies that

$$\lim_{j\to\infty}\int_\Omega g(x,u_j)(u_j-u)dx = 0. \tag{9}$$

By virtue of (6), we get

$$\langle \mathcal{I}'_\mu(u_j), u_j - u\rangle \to 0. \tag{10}$$

As $\{u_j\}$ is bounded in X_0, and in view of Proposition 2, passing to subsequence, if necessary, we suppose that

$$\delta_{p(\cdot)}(u_j) \to \kappa \geq 0, \ \text{ as } \ j \to \infty.$$

If $\kappa = 0$, then $\{u_j\}$ strongly converges to $u = 0$ in X_0 and the proof is complete. If $\kappa > 0$, in view of the function M is continuous, we know

$$M\left(\delta_{p(\cdot)}(u_j)\right) \to M(\kappa) > 0 \ \text{ as } \ j \to \infty.$$

Thus, it follows from (M_1) that

$$0 < c_2 < M\left(\delta_{p(\cdot)}(u_j)\right) < c_3 \ \text{ as } \ j \to \infty. \tag{11}$$

From (8)–(11), we obtain

$$\lim_{j\to\infty}\iint_{\mathbb{R}^{2N}} \frac{|u_j(x)-u_j(y)|^{p(x,y)-2}\Big(u_j(x)-u_j(y)\Big)\Big((u_j(x)-u_j(y))-(u(x)-u(y))\Big)}{|x-y|^{N+p(x,y)s(x,y)}}dxdy = 0.$$

Now together with (7), we have

$$u_j \rightharpoonup u \in X_0, \qquad \limsup_{j\to\infty} \mathcal{A}(u_j)(u_j-u) \le 0.$$

Therefore, $\mathcal{A}$ is a mapping of type (S_+), which implies that $u_j \to u$ in X_0 from Proposition 3. This concludes the proof of the Palais–Smale compactness condition. □

4. Proof of Theorem 1

The next two lemmas verify the mountain pass geometry of $\mathcal{I}_\mu$.

Lemma 4. *Suppose that* (M_1), (G_1), *and* $(H_1)-(H_2)$ *hold. Then, there exist numbers* $\rho > 0$, $\mu^* = \mu^*(\rho) > 0$ *and* $\alpha = \alpha(\rho) > 0$ *such that* $\mathcal{I}_\mu(u) \ge \alpha > 0$ *for any* $u \in X_0$ *with* $\|u\|_{X_0} = \rho$, *and for any* $\mu \in (-\infty, \mu^*]$.

Proof. Let $u \in X_0$ be such that $\|u\|_{X_0} = \rho \in \big(0, \min\{1, 1/C_{\overline{p}}, 1/C_q\}\big)$, with $C_{\overline{p}}$ and C_q given in Lemma 2. In view of (G_1), Propositions 1 and 2, and Lemma 2, we have that

$$\begin{aligned}
\mathcal{I}_\mu(u) &\ge \widetilde{M}\Big(\delta_{p(\cdot)}(u)\Big) - \mu^+\int_\Omega \tfrac{1}{\overline{p}(x)}|u(x)|^{\overline{p}(x)}dx - c_1\int_\Omega \tfrac{1}{q(x)}|u(x)|^{q(x)}dx \\
&\ge \tfrac{h_1}{\beta}\Big(\delta_{p(\cdot)}(u)\Big)^\beta - \tfrac{\mu^+}{p^-}\rho_{\overline{p}(\cdot)}(u) - \tfrac{c_1}{q^-}\rho_{q(\cdot)}(u) \\
&\ge \tfrac{h_1}{\beta(p^+)^\beta}\|u\|_{X_0}^{\beta p^+} - \tfrac{\mu^+}{p^-}C_{\overline{p}}^{\overline{p}^-}\|u\|_{X_0}^{p^-} - \tfrac{c_1}{q^-}C_q^{q^-}\|u\|_{X_0}^{q^-} \\
&= \rho^{\beta p^+}\left(\frac{h_1}{\beta(p^+)^\beta} - \frac{c_1 C_q^{q^-}}{q^-}\rho^{q^- - \beta p^+}\right) - \frac{\mu^+ C_{\overline{p}}^{\overline{p}^-}}{p^-}\rho^{p^-}.
\end{aligned}$$

Let us consider

$$\widetilde{\rho} = \left(\frac{h_1}{2\beta(p^+)^\beta}\cdot\frac{q^-}{c_1 C_q^{q^-}}\right)^{\frac{1}{q^- - \beta p^+}} \quad \text{and} \quad \mu^* = \frac{p^-}{C_{\overline{p}}^{\overline{p}^-}}\cdot\frac{h_1}{4\beta(p^+)^\beta}\rho^{\beta p^+ - p^-}.$$

Then, for any $u \in X_0$ with $\|u\|_{X_0} = \rho \in \big(0, \min\{1, 1/C_{\overline{p}}, 1/C_q, \widetilde{\rho}\}\big)$ and all $\mu \in (-\infty, \mu^*]$, since $\beta p^+ < q^-$ by (G_1) we have

$$\begin{aligned}
\mathcal{I}_\mu(u) &\ge \rho^{\beta p^+}\left(\frac{h_1}{\beta(p^+)^\beta} - \frac{c_1 C_q^{q^-}}{q^-}\widetilde{\rho}^{q^- - \beta p^+}\right) - \frac{2\mu^+ C_{\overline{p}}^{\overline{p}^-}}{p^-}\rho^{p^-} \\
&= \frac{h_1}{2\beta(p^+)^\beta}\rho^{\beta p^+} - \frac{\mu^+ C_{\overline{p}}^{\overline{p}^-}}{p^-}\rho^{p^-} \ge \frac{h_1}{4\beta(p^+)^\beta}\rho^{\beta p^+} = \alpha > 0,
\end{aligned}$$

concluding the proof. □

Lemma 5. *Suppose that* (M_1), (G_2), *and* $(H_1)-(H_2)$ *hold. Then, for any* $\mu \in \mathbb{R}$ *there exists* $u \in X_0$ *with* $\|u\|_{X_0} > \rho$, *where* $\rho > 0$ *is given in Lemma 4, such that* $\mathcal{I}_\mu(u) < 0$.

Proof. Let $\mu \in \mathbb{R}$. By (G_2), we have that for all $D > 0$, there exists $C_D > 0$ such that

$$G(x,t) \ge D|t|^\lambda - C_D, \text{ for all } (x,t) \in \Omega\times\mathbb{R}. \tag{12}$$

Take $\varphi \in C_0^\infty(\mathbb{R}^N)$, with $\varphi > 0$. Let $t > 1$. From (12) and (M_1) we get

$$\begin{aligned}\mathcal{I}_\mu(t\varphi) &= \widetilde{M}\Big(\delta_{p(\cdot)}(t\varphi)\Big) - \mu \int_\Omega \frac{1}{\overline{p}(x)}|t\varphi|^{\overline{p}(x)}dx - \int_\Omega G(x,t\varphi)dx \\ &\leq \frac{h_2}{\beta(p^-)^\beta} t^{\beta p^+}\left(\varrho_{p(\cdot)}^{s(\cdot)}(\varphi)\right)^\beta - \frac{\mu^-}{p^+}t^{p^-}\int_\Omega |\varphi|^{\overline{p}(x)}dx - Dt^\lambda \int_\Omega |\varphi|^\lambda dx + C_D|\Omega|.\end{aligned}$$

Since $h_2 \geq h_1$ and $p^+ \geq p^-$, we get $\lambda > \beta p^+ \geq \beta p^- > p^-$, we deduce that $\mathcal{I}_\mu(t\varphi) \to -\infty$ as $t \to \infty$. Then for $t > 1$ sufficiently large, we can let $u = t\varphi$ such that $\|u\|_{X_0} > \rho$ and $\mathcal{I}_\mu(u) < 0$. □

Proof of Theorem 1. According to Lemmas 3–5, considering also that $\mathcal{I}_\mu(0) = 0$, our functional $\mathcal{I}_\mu$ fulfills all conditions of the mountain pass theorem. Thus, problem (1) has a non-trivial weak solution. □

5. Proof of Theorem 2

The proof of Theorem 2 is based on the application of the fountain theorem, which can be found in [33]. For this, as the real Banach space X_0 is reflexive and separable, there exist $\{w_i\} \subset X_0$ and $\{w_i^*\} \subset X_0^*$ such that

$$X_0 = \overline{\operatorname{span}\{w_i : i \in \mathbb{N}^+\}}, \qquad X_0^* = \overline{\operatorname{span}\{w_i^* : i \in \mathbb{N}^+\}}$$

and

$$\langle w_i^*, w_j\rangle = \begin{cases} 1, i = j\,, \\ 0, i \neq j\,. \end{cases}$$

$$X_0^i = \operatorname{span}\{w_i\}, \qquad Y_j = \bigoplus_{i=1}^{j} X_0^i, \qquad Z_j = \overline{\bigoplus_{i=j}^{\infty} X_0^i}, \quad j = 1, 2, \ldots$$

Now we are ready to introduce the fountain theorem.

Theorem 3. *([33]) Consider an even functional $I \in C^1(X_0, \mathbb{R})$. Assume that for every $j \in \mathbb{N}$, there exist $\rho_j > \gamma_j > 0$ such that*

(I_1) $a_j := \max\limits_{u\in Y_j, \|u\|_{X_0}=\rho_j} I(u) \leq 0,$

(I_2) $b_j := \inf\limits_{u\in Z_j, \|u\|_{X_0}=\gamma_j} I(u) \to +\infty,\ j \to \infty,$

(I_3) *I fulfills the Palais–Smale condition for every $c > 0$.*

Then I has an unbounded sequence of critical values.

Lemma 6. *Suppose that (H_1)–(H_2) hold. Let $r \in C_+(\overline{\Omega})$, with $1 < r(x) < p_s^*(x)$ for any $x \in \overline{\Omega}$, and denote*

$$\xi_j := \sup\Big\{\|u\|_{r(\cdot)} :\ \ u \in Z_j,\ \ \|u\|_{X_0} = 1\Big\}. \tag{13}$$

Then, $\xi_j \to 0$ as $j \to \infty$

Proof. By definition of Z_j we have $Z_{j+1} \subset Z_j$ and so $0 < \xi_{j+1} \leq \xi_j$ for any $j \in \mathbb{N}$. Thus $\xi_j \to \xi \geq 0$ as $j \to \infty$. Moreover, by (13) there exists $v_j \in Z_j$ such that

$$\|u_j\|_{X_0} = 1, \quad 0 \leq \xi_j - \|u_j\|_{r(\cdot)} < \frac{1}{j}. \tag{14}$$

As $\{u_j\}$ is bounded in X_0, there exists a subsequence of $\{u_j\}$, still denoted by u_j, such that $u_j \rightharpoonup u$ in X_0 and $\langle w_i^*, u\rangle = \lim\limits_{j\to\infty}\langle w_i^*, u_j\rangle = 0$ for $i \in \mathbb{N}^+$. Hence we have $u = 0$.

Furthermore, by Lemma 2 we obtain that $u_j \to 0$ in $L^{r(\cdot)}(\Omega)$. Therefore, by (14) we have $\xi_j \to 0$ as $j \to \infty$. □

Proof of Theorem 2. We know that the functional $\mathcal{I}_\mu$ fulfills the Palais–Smale condition by Lemma 3. In what follows, we show that $\mathcal{I}_\mu$ satisfies all conditions of Theorem 3, step by step.

In view of (G_1) and (G_2), there exist two positive numbers c_4 and c_5 such that

$$|G(x,t)| \geq c_4|t|^\lambda - c_5|t|, \text{ for all } (x,t) \in \Omega \times \mathbb{R}. \tag{15}$$

For $u \in Y_j$, with $\|u\|_{X_0} > 1$, by (15) and (M_1), we get

$$\begin{aligned}\mathcal{I}_\mu(u) &= \widetilde{M}\Big(\delta_{p(\cdot)}(u)\Big) - \mu\int_\Omega \tfrac{1}{\overline{p}(x)}|u|^{\overline{p}(x)}dx - \int_\Omega G(x,u)dx \\ &\leq \tfrac{h_2}{\beta(p^-)^\beta}\Big(\varrho^{s(\cdot)}_{p(\cdot)}(u)\Big)^\beta - \tfrac{\mu^-}{p^+}\rho_{\overline{p}(\cdot)}(u) - c_4\int_\Omega |u|^\lambda dx + c_5\int_\Omega |u|dx.\end{aligned}$$

On a finite dimensional space Y_j all the norms are equivalent, so there are three positive constants c_6, c_7, and c_8 such that

$$\|u\|^{\overline{p}^-}_{\overline{p}(\cdot)} \geq c_6\|u\|^{\overline{p}^-}_{X_0}, \qquad \|u\|^\lambda_\lambda \geq c_7\|u\|^\lambda_{X_0}, \qquad \|u\|_1 \geq c_8\|u\|_{X_0}.$$

Consequently, from the above inequalities and Propositions 1 and 2, for any $u \in Y_j$ with $\|u\|_{X_0} > \max\Big\{1, c_6^{-1/\overline{p}^-}\Big\}$, we have

$$\mathcal{I}_\mu(u) \leq \frac{h_2}{\beta(p^-)^\beta}\|u\|^{\beta p^+}_{X_0} - \frac{\mu^-}{p^+}c_6\|u\|^{p^-}_{X_0} - c_4\,c_7\|u\|^\lambda_{X_0} + c_5\,c_8\|u\|_{X_0}.$$

Since $\lambda > \beta p^+ > p^- > 1$ by (G_2), by choosing $\rho_j > \max\Big\{1, c_6^{-1/\overline{p}^-}\Big\}$ large enough, we get

$$a_j := \max_{u \in Y_j, \|u\|_{X_0} = \rho_j} \mathcal{I}_\mu(u) \leq 0.$$

Therefore, the condition (I_1) of Theorem 3 holds.

According to (G_1), (13), and Propositions 1 and 2, we get for any $u \in Z_j$ with $\|u\|_{X_0} > 1$

$$\begin{aligned}\mathcal{I}_\mu(u) &\geq \widetilde{M}\Big(\delta_{p(\cdot)}(u)\Big) - \mu^+\int_\Omega \tfrac{1}{\overline{p}(x)}|u(x)|^{\overline{p}(x)}dx - c_1\int_\Omega \tfrac{1}{q(x)}|u|^{q(x)}dx \\ &\geq \tfrac{h_1}{\beta}\Big(\delta_{p(\cdot)}(u)\Big)^\beta - \tfrac{\mu^+}{p^-}\rho_{\overline{p}(\cdot)}(u) - \tfrac{c_1}{q^-}\rho_{q(\cdot)}(u) \\ &\geq \tfrac{h_1}{\beta(p^+)^\beta}\|u\|^{\beta p^-}_{X_0} - \tfrac{\mu^+}{p^-}\max\Big\{\|u\|^{\overline{p}^-}_{\overline{p}(\cdot)}, \|u\|^{\overline{p}^+}_{\overline{p}(\cdot)}\Big\} - \tfrac{c_1}{q^-}\max\Big\{\|u\|^{q^-}_{q(\cdot)}, \|u\|^{q^+}_{q(\cdot)}\Big\} \\ &\geq \tfrac{h_1}{\beta(p^+)^\beta}\|u\|^{\beta p^-}_{X_0} - \tfrac{\mu^+}{p^-}\max\Big\{(\xi_j\|u\|_{X_0})^{\overline{p}^-}, (\xi_j\|u\|_{X_0})^{\overline{p}^+}\Big\} - \tfrac{c_1}{q^-}\max\Big\{(\xi_j\|u\|_{X_0})^{q^-}, (\xi_j\|u\|_{X_0})^{q^+}\Big\} \\ &\geq \tfrac{h_1}{\beta(p^+)^\beta}\|u\|^{\beta p^-}_{X_0} - \frac{\mu^+\xi_j^{p^-}}{p^-}\|u\|^{p^+}_{X_0} - \frac{c_1\xi_j^{q^-}}{q^-}\|u\|^{q^+}_{X_0}.\end{aligned} \tag{16}$$

We can suppose $\xi_j < 1$ for j sufficiently large, in view of Lemma 6. Let us define

$$\gamma_j := \left(\frac{c_1\beta(q^-)^{\beta-1}}{h_1}\cdot\xi_j^{q^-}\right)^{\frac{1}{\beta p^- - q^+}},$$

then since $\gamma_j \to +\infty$ as $j \to +\infty$ by Lemma 6 and the fact that $q^+ \geq q^- > \beta p^+ \geq \beta p^-$ by (G_1), we can assume that $\gamma_j > 1$ for j larger. Hence, by (16) applied for any $u \in Z_j$ with $\|u\|_{X_0} = \gamma_j$, we obtain

$$\begin{aligned}\mathcal{I}_\mu(u) &\geq \frac{h_1}{\beta}\left(\frac{1}{(p^+)^\beta} - \frac{1}{(q^-)^\beta}\right)\gamma_j^{\beta p^-} - \frac{\mu^+ \xi_j^{p^-}}{p^-}\gamma_j^{p^+} \\ &= \gamma_j^{p^+}\left[\frac{h_1}{\beta}\left(\frac{1}{(p^+)^\beta} - \frac{1}{(q^-)^\beta}\right)\gamma_j^{\beta p^- - p^+} - \frac{\mu^+ \xi_j^{p^-}}{p^-}\right] \to +\infty\end{aligned}$$

as $j \to \infty$, by Lemma 6, as also $p^+ < \beta p^-$ by (M_1) and $q^- > \beta p^+ > p^+$ by (G_1). Thus, the condition (I_2) of Theorem 3 holds. So for j large enough, $b_j > 0$. Theorem 3.5 of [33] implies then the existence of a sequence $\{u_n\} \subset X_0$ fulfilling

$$\mathcal{I}_\mu(u_n) \to c_j \text{ and } \mathcal{I}_\mu'(u_n) \to 0 \text{ in } X_0^* \text{ as } n \to \infty. \tag{17}$$

It follows from the condition (I_3) that c_j is a critical value of $\mathcal{I}_\mu$. According to $c_j \geq b_j$ and $b_j \to +\infty, j \to \infty$, the proof of Theorem 2 is complete, considering that $\mathcal{I}_\mu$ is even by (G_3). □

6. Conclusions

In this work, the existence of a solution is obtained by the mountain pass lemma, and the existence of infinitely many solutions with positive energy to Equation (1) is established by using the fountain theorem. We consider a class of complex bi-nonlocal problems, which improves the previous results. In order to overcome the difficulties arising from such problems, we use more sophisticated analytical techniques. This kind of equation has a wide range of applications in many fields, and interested readers may refer to the thin obstacle problem [34,35], ultra-relativistic limits of quantum mechanics [11], finance [12] and so on.

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