Information Sciences 674 (2024) 120725

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

Unifying credal partitions and fuzzy orthopartitions

Stefania Boffa *, Davide Ciucci

Dipartimento di Informatica, Sistemistica e Comunicazione, Università degli Studi di Milano-Bicocca, Viale Sarca 336, 20126 Milano, Italy

ARTICLE INFO

Keywords:
Partitions
Fuzzy orthopartitions
Generalized partitions
Credal partitions
Basic belief assignments
Bayesian bbas
Uncertainty

ABSTRACT

This work focuses on fuzzy orthopartitions and credal partitions, which are distinct mathematical models representing partitions where the membership of elements to classes is only partially known. Firstly, we show that fuzzy orthopartitions and credal partitions are special cases of generalized fuzzy orthopartitions, which we introduce in this article as a new structure for modelling partitions with uncertainty. Next, we examine the connections between credal partitions and fuzzy orthopartitions, considering that both can be seen as types of fuzzy partitions (in particular, we deal with fuzzy probabilistic and Ruspini partitions). Moreover, we find that each generalized fuzzy orthopartition corresponds to a collection of zero, one, or infinitely many credal partitions; conversely, a credal partition maps to at most one generalized fuzzy orthopartition. Finally, we identify the class of all credal partitions that coincide with fuzzy orthopartitions.

1. Introduction

In knowledge representation and data analysis, uncertainty is an inevitable and impactful aspect. Traditional models often assume a precise and definite classification of elements into distinct classes, giving rise to a partition of the objects under investigation. However, in many real-world scenarios, such clarity and precision in categorization are not always achievable due to incomplete, ambiguous, or evolving information. Recognizing this, several models have been developed to address and represent uncertainty in partitioning elements. These models include credal partitions, fuzzy probabilistic partitions, Ruspini partitions, orthopartitions and fuzzy orthopartitions. By extending the conventional notion of *crisp* partitions, these models enable a more flexible and realistic representation of knowledge, accommodating the vagueness and ambiguity that naturally arise in many practical situations.

Let us now briefly delve into the diverse methodologies developed to address the representation of uncertainty in partitions, by focusing on two main approaches: credal partitions and fuzzy orthopartitions.

Credal partitions are relevant structures in evidential clustering used to represent partitions in cases of partial knowledge concerning the membership of elements to classes [1], and they turn out to be useful in several applications (see [2–5] for some examples). Assuming that $C = \{C_1, \dots, C_n\}$ is a standard partition of a universe $U = \{u_1, \dots, u_l\}$, a credal partition is a collection $m = \{m_1, \dots, m_l\}$ of basic belief assignments (bba). Each bba m_i expresses the relationship between the element u_i and the classes of C. More precisely, let $A \subseteq C$, $m_i(A)$, called *mass of belief*, quantifies the evidence supporting the claim "u_i belongs to a class of A" [6,7]. Credal partitions generalize the so-called *fuzzy probabilistic partitions*, which are credal partitions made up of all *Bayesian bbas*, namely bbas that assign a non-zero degree only to the singletons of 2^U [8].

* Corresponding author.
E-mail addresses: stefania.boffa@unimib.it (S. Boffa), davide.ciucci@unimib.it (D. Ciucci).

https://doi.org/10.1016/j.ins.2024.120725
Received 7 December 2023; Received in revised form 15 March 2024; Accepted 7 May 2024

Available online 15 May 2024

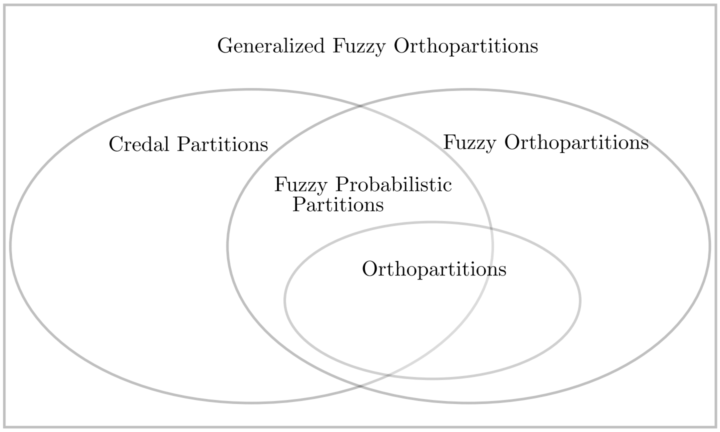

Fig. 1. The hierarchy of the models studied in this article.

Fuzzy orthopartitions have been introduced in [9,10] to model (fuzzy) Ruspini partitions [11] with partial knowledge, and are also a generalization of orthopartitions based on classical sets [12]. Mathematically, fuzzy orthopartitions are defined as collections of *Intuitionistic Fuzzy Sets* (IFSs) satisfying a specific list of axioms. Each IFS of a fuzzy orthopartition represents a class to which elements belong with a degree of [0,1] that is not precisely known, but specified in an interval. A fuzzy orthopartition where the interval is a single point is a *Ruspini partition*, which is a generalized partition where blocks are represented by fuzzy sets and the total membership degree of each element (distributed among all blocks) must be 1. Fuzzy orthopartitions generalize also the concept of orthopartitions based on classical sets [12]. Orthopartitions are formally defined as collections of orthopairs (pairs of disjoint sets of the given universe) with some properties.

Recent studies investigate orthopartitions and fuzzy orthopartitions in knowledge representation: in [9], entropy measures and operations are defined on fuzzy orthopartitions; in [13], orthopartitions are bridged with possibility theory; in [14], a class of orthopartitions is identified as special partially-defined equivalence relations, which are equivalence relations with uncertainty.

Taking into account all the above considerations, it is clear, from one side, that there is a need to represent partitions in a flexible and realistic way in order to accommodate the vagueness and ambiguity that naturally arise in many practical situations and that from the other, the landscape of models of partition with uncertainty needs a clarification and a unique framework.

This work fills in such a gap by bridging fuzzy orthopartitions and credal partitions with the introduction of a new model: generalized fuzzy orthopartition. The following considerations are the starting point to reach our goal:

- Fuzzy orthopartitions and credal partitions respectively encompass the concepts of Ruspini partitions and fuzzy probabilistic partitions. Moreover, Ruspini partitions and fuzzy probabilistic partitions mathematically coincide.
- A fuzzy orthopartition O can be seen as the collection $\mathcal{Z}_O$ of all Ruspini partitions that they could coincide with, as more knowledge about the membership class would be available. The same connection holds between a credal partition m and a class of fuzzy probabilistic partitions $\mathcal{Z}_m$. Thus, we identify a fuzzy orthopartition O with a credal partition m when the corresponding classes $\mathcal{Z}_O$ and $\mathcal{Z}_m$ coincide.

Building on these links we prove that generalized fuzzy partitions generalize both credal and fuzzy–ortho partitions and we study the relationship among all the above mentioned notions of partition with uncertainty. The hierarchy of all these models of generalized partitions is schematized in the diagram of Fig. 1. Thus, orthopartitions form a subclass of fuzzy orthopartitions; some but not all orthopartitions are special fuzzy probabilistic partitions; fuzzy probabilistic partitions can be viewed as the intersection of credal partitions and fuzzy orthopartitions; generalized fuzzy orthopartitions strictly include all the other models.

More in detail, the contributions of our work in each section are the following. The first section recalls the concepts of credal partitions and fuzzy orthopartitions (Subsections 2.1 and 2.2, respectively). Additionally, Subsection 2.3 shows that fuzzy orthopartitions and credal partitions can be respectively understood as Ruspini and fuzzy probabilistic partitions. This subsection ends by presenting a one-to-one correspondence between Ruspini and fuzzy probabilistic partitions, showing that these models syntactically coincide.

In Section 3, we introduce the novel notion of generalized fuzzy orthopartitions by relaxing the definition of fuzzy orthopartitions. Also, we prove that these models are more general than fuzzy orthopartitions, orthopartitions based on classical sets, and standard partitions.

In Section 4, we first recall the correspondence between fuzzy orthopartitions and Ruspini partitions already defined in [9] (Subsection 4.1). Subsequently, we identify credal partitions with collections of fuzzy probabilistic partitions (Subsection 4.2). Then, a generalized fuzzy orthopartition O is mapped into the class $\mathcal{F}(O)$ of all credal partitions having the same compatible fuzzy probabilistic partitions of O. Additionally, we show that $\mathcal{F}(O)$ can be empty, or made of one or infinitely many credal partitions. An analogous correspondence holds between fuzzy orthopartitions and credal partitions (Subsection 4.3).

Table 1
Models for Partitions.

Symbol of the class	Models
$\mathcal{P}$	Partitions
O'	Orthopartitions
$\mathcal{R}$	Ruspini Partitions
$\mathcal{M}^*$	Fuzzy Probabilistic Partitions
$\mathcal{O}$	Fuzzy Orthopartitions
$\mathcal{M}$	Credal Partitions
$\mathcal{O}_G$	Generalized Fuzzy Orthopartitions

Table 2
List of notations.

Notation	Definition	Explanation
$\bar{m}$	Definition 1	Basic Belief Assignment
m	Definition 2	Credal Partition
(μ_A, ν_A)	Definition 3	Intuitionistic Fuzzy Set
$h_A(u)$	Definition 3	Degree of Uncertainty
O	Definition 4	Fuzzy Orthopartition
(M, N)	Definition 5	Orthopair
B	Definition 5	Boundary Region
α	Equation (4)	Function from $\mathcal{O}'$ to $\mathcal{O}^*$
β	Equation (13)	Function from $\mathcal{P}$ to $\mathcal{O}_G^*$
π	Definition 7	Ruspini partition
f	Equation (10)	Function from $\mathcal{R}$ to $\mathcal{M}^*$
$\mathcal{F}(O)$	Definition 13	Collection of the Credal Partitions assigned to O
O_m	Definition 14	Generalized Fuzzy Orthopartition assigned to m

In reverse, Section 5 assigns a generalized fuzzy orthopartition to each credal partition. This correspondence leads to an equivalence relation on the set of all credal partitions (that is, we say that two credal partitions are equivalent if and only if they correspond to the same generalized fuzzy orthopartition). For these reasons, generalized fuzzy orthopartitions can be considered more general than credal partitions. Although credal partitions are special cases of generalized fuzzy orthopartitions, we show that not all of them can be seen as fuzzy orthopartitions. Then, we discover a necessary and sufficient condition for a credal partition to be a fuzzy orthopartition. Hence, we are able to characterize the collection of all credal partitions that are also fuzzy orthopartitions. Our results suggest interpreting generalized fuzzy orthopartitions in terms of mass function. In this way, we could obtain new models to extend the notion of partitions, and they can be understood as a credal partition with an additional level of uncertainty (see Remark 20).

In the last section, we present the conclusions and potential developments of this work.

We notice that the present paper extends the results already appeared in [15] as follows:

- The correspondence provided in [15] is generalized by dealing with credal partitions made of bbas that do not necessarily satisfy the condition of normality.
- New structures called generalized fuzzy orthopartitions are introduced to represent partitions with uncertainty and to extend the notions of both fuzzy orthopartitions and credal partitions.
- A necessary and sufficient condition is determined so that a credal partition coincides with a fuzzy orthopartition.

For the sake of clarity and in order to help the reader, let us fix here some notations. We will consider credal partitions and fuzzy orthopartitions as modelling a partially known standard partition of a finite universe $U = \{u_1, \dots, u_l\}$,[1] that we denote as $C = \{C_1, \dots, C_n\}$. We recall that C is a standard partition of U if and only if $C_1 \cup \dots \cup C_n = U$ and $C_i \cap C_j = \emptyset$ for each $i \neq j$. Moreover, in Table 1 the main classes of mathematical models used in this paper with the corresponding nomenclature are reported.

We finally provide Table 2 listing the other important symbols used in this paper (first column), the references to their definition (second column), and their brief explanation (third column).[2]

2. Preliminaries

This section consists of three main parts. Subsections 2.1 and 2.2 respectively recall the main notions of credal partitions and fuzzy orthopartitions, which are used in this article. Subsection 2.3 highlights that fuzzy orthopartitions and credal partitions are respectively generalizations of the notion of Ruspini and fuzzy probabilistic partitions. Furthermore, it mathematically identifies each Ruspini partition with a fuzzy probabilistic partition, as known in the literature.

[1] Of course, we need to suppose that $2 \leq n \leq l$.

[2] We also denote with O orthopartitions and generalized fuzzy orthopartitions that are given by Definition 6 and Definition 9.

Table 3
Definition of the elements of m of Example 1.

A	$m_1(A)$	$m_2(A)$	$m_3(A)$	$m_4(A)$
$\emptyset$	0.2	0	0	0.1
$\{C_1\}$	0	0.1	0.1	0.1
$\{C_2\}$	0.3	0.3	0	0.1
$\{C_3\}$	0	0.1	0	0.2
$\{C_1, C_2\}$	0.2	0	0.1	0.1
$\{C_1, C_3\}$	0.1	0	0	0.1
$\{C_2, C_3\}$	0	0	0.2	0.3
C	0.2	0.5	0.6	0

Finally, the last subsection highlights that fuzzy orthopartitions and credal partitions are respectively generalizations of the notion of Ruspini and fuzzy probabilistic partitions. Furthermore, it mathematically identifies each Ruspini partition with a fuzzy probabilistic partition, as known in the literature.

2.1. Credal partitions

In the sequel, we use the symbol 2^C to indicate the power set of C.

Definition 1. [6] A basic belief assignment (bba) is a function $\bar{m} : 2^C \to [0, 1]$ satisfying

$$\sum_{A \subseteq C} \bar{m}(A) = 1. \tag{1}$$

Definition 2. [1] A *credal partition* $m = \{m_1, \ldots, m_l\}$ is a collection of basic belief assignments satisfying the following property: for each $C_i \in C$, there exists $u_j \in \{u_1, \ldots, u_l\}$ such that $pl_j(\{C_i\}) > 0$, where

$$pl_j(\{C_i\}) = \sum_{\{A \in 2^C \ \mid \ \{C_i\} \cap A \neq \emptyset\}} m_j(A). \tag{2}$$

For each $i \in \{1, \ldots, l\}$, the bba m_i represents the partial knowledge regarding the relation between the element u_i of U and the classes of C: let $A \subseteq C$, $m_i(A)$ is called *mass of belief* and quantifies the evidence supporting the claim "*u_i belongs to a class of A*".

Example 1. Let $C = \{C_1, C_2, C_3\}$ be a partition of $U = \{u_1, u_2, u_3, u_4\}$, then a credal partition $m = \{m_1, m_2, m_3\}$ of U is defined by Table 3.

Therefore, $m_1(\{C_1, C_2\}) = 0.2$ means that we have some belief that u_1 belongs either to C_1 or C_2, and the weight of this belief is equal to 0.2.

A bba m is *normal* when $m(\emptyset) = 0$. This means that the element described by m surely belongs to a class of the partition C. For instance, the functions $m_2 : 2^C \to [0, 1]$ and $m_3 : 2^C \to [0, 1]$ given by Example 3 are normal bbas.

In this article, we consider credal partitions made of bbas that are not necessarily normal because we accept the open-world assumption stating that the elements of the initial universe U might belong to a class denoted with C_0 that is disjoint with $C_1, \ldots, C_n$. For instance, in the previous example, $m_1(\emptyset) = 0.2$ means that we believe with degree 0.2 that u_1 belongs to C_0 (namely, u_1 is external to C).

Let $A \subseteq C$, we say that A is a *focal set* of a bba m if and only if $m(A) > 0$. For instance, the focal sets of the bba m_1 of Example 3 are: $\emptyset$, $\{C_2\}$, $\{C_1, C_2\}$, $\{C_1, C_3\}$, and C.

In the next sections, we use the symbol $\mathcal{M}$ to denote the collection of all credal partitions of $U = \{u_1, \ldots, u_l\}$, which are related to the standard partition $C = \{C_1, \ldots, C_n\}$.

2.2. Fuzzy orthopartitions

Definition 3. [16] An *intuitionistic fuzzy set (IFS)* A of a universe U is a pair of functions $\mu_A : U \to [0, 1]$ and $\nu_A : U \to [0, 1]$ such that

$\mu_A(u) + \nu_A(u) \leq 1$ for each $u \in U$. Moreover, let $u \in U$, the value $h_A(u) = 1 - (\mu_A(u) + \nu_A(u))$ is the degree of *indeterminacy* (or *uncertainty*) of u to the IFS A.

The most common interpretation of an intuitionistic fuzzy set is the following: given $u \in U$, $\mu_A(u)$ and $\nu_A(u)$ are respectively the *membership* and *non-membership degrees* of u to the set A.

Table 4
Definition of the elements of O of Example 2.

u	$\mu_0(u)$	$\nu_0(u)$	$\mu_1(u)$	$\nu_1(u)$	$\mu_2(u)$	$\nu_2(u)$	$\mu_3(u)$	$\nu_3(u)$
u_1	0	0.1	0.1	0.3	0	0.1	0.1	0
u_2	0.2	0.4	0	0.2	0.1	0.1	0	0.5
u_3	0.2	0.4	0.1	0.4	0.2	0.5	0	0.8
u_4	0	0.5	0.3	0.2	0.2	0.4	0.3	0.2

Definition 4. [9] Let $O = \{(\mu_0, \nu_0), (\mu_1, \nu_1), \dots, (\mu_n, \nu_n)\}$ be a family of IFSs of U. Then, O is a fuzzy orthopartition of U if and only if the following properties hold for each $u \in U$:

a) $\sum_{i=0}^{n} \mu_i(u) \leq 1$,
b) $\mu_i(u) + h_j(u) \leq 1$, $\forall i \neq j$,
c) $\sum_{i=0}^{n}(\mu_i(u) + h_i(u)) \geq 1$,
d) $\forall i \in \{0, \dots, n\}$ with $h_i(u) > 0$, $\exists j \neq i$ such that $h_j(u) > 0$.

For each $i \in \{1, \dots, n\}$, the IFS (μ_i, ν_i) of a fuzzy orthopartition $O = \{(\mu_0, \nu_0), (\mu_1, \nu_1), \dots, (\mu_n, \nu_n)\}$ represents the class C_i: let $u \in U$, $\mu_i(u)$ and $\nu_i(u)$ are respectively the truth degrees to which *"u belongs to C_i"* and *"u does not belong to C_i"*. The IFS (μ_0, ν_0) refers to the possibility that the elements of U could be outside of $C_1 \cup \dots \cup C_n$, and so, it describes the relationship between the objects and the additional class that we have called C_0. Thus, $\mu_0(u)$ and $\nu_0(u)$ are respectively the *truth degrees* concerning the statements "*the object u does not belong to a class of C*" (namely, "u belongs to C_0") and "*the object u belongs to a class of C*" (namely, "u does not belong to C_0"). Let us point out that the definition of a fuzzy orthopartition differs from that provided in [9] for the presence of (μ_0, ν_0). In other words, in our previous works, we have supposed that each element of U belongs to a class of C.

It is easy to understand that Axiom (a) captures that the classes described by $(\mu_0, \nu_0), (\mu_1, \nu_1), \dots, (\mu_n, \nu_n)$ must be disjoint and Axiom (c) is a covering requirement. However, let us focus on the interpretation of Axioms (b) and (d) of Definition 4, which is not intuitive. They are necessary so that a fuzzy orthopartition O made of classical sets (i.e. $\mu_i(u), \nu_i(u) \in \{0, 1\}$) can be seen as an orthopartition according to Definition 6. Thus, Axioms (b) and (d) of Definition 4 are respectively generalizations of Axioms (b) and (d) of Definition 6, which have the following meaning. Axiom (b) of Definition 6 means that if the object u belongs to the class C_i ($u \in M_i$), then we cannot be uncertain that it may belong to another class ($u \notin B_j$ for each $i \neq j$). Axiom (d) of Definition 6 means that if the object u could belong to the class C_i ($u \in B_i$) and u surely does not belong to the other classes ($u \in N_j$ for each $j \neq i$), then u must belong to C_i ($u \in M_i$). However, we will show at the end of Subsection 2.3 that Axioms (b) and (d) of Definition 6 can be omitted to obtain models of partitions with uncertainty.

Example 2. Let $C = \{C_1, C_2, C_3\}$ be a partition of $U = \{u_1, u_2, u_3, u_4\}$, then a fuzzy orthopartition $O = \{(\mu_0, \nu_0), (\mu_1, \nu_1), (\mu_2, \nu_2), (\mu_3, \nu_3)\}$ of U is defined by Table 4.

Then, $\mu_2(u_4) = 0.2$ and $\nu_2(u_4) = 0.4$ respectively are the truth degrees of the claims *"u_4 belongs to C_2"* and *"u_4 does not belong to C_2"*. Moreover, $\mu_0(u_2) = 0.2$ and $\nu_0(u_2) = 0.4$ respectively represent the truth degrees related to the claims *"u_2 does not belong to a class of C"* and *"u_2 belongs to a class of C"*.

In this article, we use the symbol $\mathcal{O}$ to denote the collection of all fuzzy orthopartitions of $U = \{u_1, \dots, u_l\}$, which are related to the standard partition $C = \{C_1, \dots, C_n\}$.

All axioms of Definition 4 are fundamental to consider fuzzy orthopartitions as an extension of the concept of orthopartitions, which are special collections of orthopairs.

Definition 5. [17] An orthopair (M, N) of U is a pair of disjoint subsets of U, i.e. $M, N \subseteq U$ and $M \cap N = \emptyset$.

M and N are respectively called *lower approximation* and *impossibility domain* of (M, N). Also, the set defined by $B = U \setminus (M \cup N)$ is the *boundary region* of (M, N).

The orthopair (M, N) represents a set containing all elements of M and being disjoint from N with certainty.

Definition 6. [12] A collection $O = \{(M_0, N_0), (M_1, N_1), \dots, (M_n, N_n)\}$ of orthopairs of U is an *orthopartition* of U if and only if

a) $M_i \cap M_j = \emptyset$, $\forall\, i \neq j$;
b) $M_i \cap B_j = M_j \cap B_i = \emptyset$, $\forall\, i \neq j$;
c) the union of $M_0 \cup M_1 \cup \dots \cup M_n$ and $B_0 \cup B_1 \cup \dots \cup B_n$ covers U;
d) for each $u \in U$, if $u \in B_i$ then there exists $j \neq i$ such that $u \in B_j$.

An orthopartition is understood here as a partition $C = \{C_1, \dots, C_n\}$ where the membership class of some elements is known with certainty: when $u \in M_i$ for $i \in \{1, \dots, n\}$, we know that $u \in C_i$; whereas the membership class of the remaining elements is

Table 5
Definition of the elements of O of Example 4.

u	$\mu_0(u)$	$\nu_0(u)$	$\mu_1(u)$	$\nu_1(u)$	$\mu_2(u)$	$\nu_2(u)$
u_1	0	1	1	0	0	1
u_2	0	1	0	1	1	0
u_3	0	1	0	0	0	0

completely unknown: when $u \notin M_i$ for each $i \in \{0, \dots, n\}$. Furthermore, if $u \in N_i$, then we are sure that u does not belong to the class C_i. The orthopair (M_0, N_0) gives information about the elements of U that do not belong to a class of C: $u \in M_0$ denotes that $u \notin C_1 \cup \dots \cup C_n$ and conversely, $u \in N_0$ denotes that u belong to a class in $\{C_1, \dots, C_n\}$.[3]

Example 3. Let $C = \{C_1, C_2\}$ be a partition of $U = \{u_1, u_2, u_3\}$, then $O = \{(M_0, N_0), (M_1, N_1), (M_2, N_2)\}$ is an orthopartition of U, where $(M_0, N_0) = (\emptyset, \{u_1, u_2, u_3\})$, $(M_1, N_1) = (\{u_1\}, \{u_2\})$, and $(M_2, N_2) = (\{u_2\}, \{u_1\})$. Thus, we know that $u_1 \in C_1$ and $u_2 \in C_2$ with certainty, considering that $u_1 \in M_1$ and $u_2 \in M_2$. Concerning the object u_3, we only know that u_3 belongs to a class of C (namely $u_3 \in C_1$ or $u_3 \in C_2$) because $u_3 \in N_0$ and $u_3 \in B_1 \cap B_2$.

In [9], we proved that orthopartitions given by Definition 6 are special cases of fuzzy orthopartitions. This result is obtained by identifying orthopartitions with fuzzy orthopartitions made of Boolean functions (i.e., $\mu_0, \mu_1, \dots, \mu_n$ and $\nu_0, \nu_1, \dots, \nu_n$ assume their values in $\{0, 1\}$).

In details, denoting the collection of all orthopartitions with $\mathcal{O}'$ and putting

$$\mathcal{O}^* = \{\{(\mu_0, \nu_0), (\mu_1, \nu_1), \dots, (\mu_n, \nu_n)\} \in \mathcal{O} \mid \mu_i(u), \nu_i(u) \in \{0, 1\} \ \forall u \in U \text{ and } \forall i \in \{0, \dots, n\}\}, \tag{3}$$

we can consider the mapping $\alpha : \mathcal{O}' \to \mathcal{O}^*$ such that let $O = \{(M_0, N_0), (M_1, N_1), \dots, (M_n, N_n)\} \in \mathcal{O}'$,

$$\alpha(O) = \{(\mu_0, \nu_0), (\mu_1, \nu_1), \dots, (\mu_n, \nu_n)\}, \tag{4}$$

where $\mu_0, \mu_1, \dots, \mu_n$ and $\nu_0, \nu_1, \dots, \nu_n$ are respectively the characteristic functions of $M_0, M_1, \dots, M_n$ and $N_0, N_1, \dots, N_n$, i.e.

$$\mu_i(u) = \begin{cases} 1 & \text{if } u \in M_i, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \nu_i(u) = \begin{cases} 1 & \text{if } u \in N_i, \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$

$\forall u \in U$ and $\forall i \in \{0, \dots, n\}$.

Function α is a bijection and its inverse $\alpha^{-1} : \mathcal{O}^* \to \mathcal{O}'$ assigns $\{(M_0, N_0), \dots, (M_n, N_n)\} \in \mathcal{O}'$ to each fuzzy orthopartition $\{(\mu_0, \nu_0), \dots, (\mu_n, \nu_n)\}$ of $\mathcal{O}^*$ so that

$$M_i = \{u \in U \mid \mu_i(u) = 1\} \text{ and } N_i = \{u \in U \mid \nu_i(u) = 1\}, \tag{6}$$

$\forall i \in \{0, \dots, n\}$.

Let us underline that $O \in \mathcal{O}^*$ and $\alpha(O)$ (as well as $O \in \mathcal{O}'$ and $\alpha^{-1}(O)$) represent the same partition with uncertainty. The following is an example.

Example 4. Let $C = \{C_1, C_2\}$ be a partition of $U = \{u_1, u_2, u_3\}$, then $O = \{(\mu_0, \nu_0), (\mu_1, \nu_1), (\mu_2, \nu_2)\}$ is a fuzzy orthopartition of U, where its IFSs are defined by Table 5.

By (6), we can easily check that $\alpha^{-1}(O) = O'$, where

$$O' = \{(\emptyset, \{u_1, u_2, u_3\}), (\{u_1\}, \{u_2\}), (\{u_2\}, \{u_1\})\}. \tag{7}$$

Conversely, by (5), we get $\alpha(O') = O$. Of course, O contains the same information of O', which is described by Example 3: we certainly know that $u_1 \in C_1$ and $u_2 \in C_2$ since $\mu_1(u_1) = \mu_2(u_2) = 1$, and $u_3 \notin C_0$ given that $\nu_0(u_3) = 1$.

Remark 1. An orthopartition $\{(M_0, N_0), (M_1, N_1), \dots, (M_n, N_n)\}$ such that $B_i = \emptyset$ for each $i \in \{0, \dots, n\}$ is a standard partition, considering that the class of every element of U is precisely known.

Example 5. The orthopartition

$$O = \{(\{u_1\}, \{u_2, u_3\}), (\{u_2\}, \{u_1, u_3\}), (\{u_3\}, \{u_1, u_2\})\} \tag{8}$$

of $\{u_1, u_2, u_3\}$ is equivalent to the partition $P = \{\{u_1\}, \{u_2\}, \{u_3\}\}$, or more exactly, to the partition $\{\{u_2\}, \{u_3\}\}$, where u_1 is an outlier.

[3] As in the case of credal partitions and fuzzy orthopartitions, we consider the set C_0 containing all objects of U that are outside of each class in C. Therefore, (M_0, N_0) given in the previous definition represents the relationship between the objects of U and the class C_0.

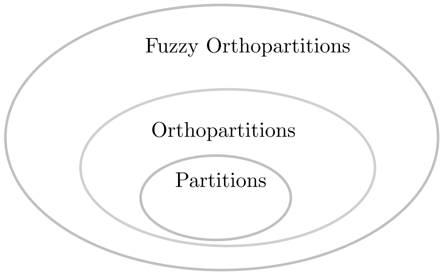

Fig. 2. The hierarchy of partitions, orthopartitions, and fuzzy orthopartitions.

Table 6
Definition of the elements of O of Example 7.

u	$\mu_0(u)$	$\nu_0(u)$	$\mu_1(u)$	$\nu_1(u)$	$\mu_2(u)$	$\nu_2(u)$
u_1	0	1	0.2	0.8	0.8	0.2
u_2	0	1	0.5	0.5	0.5	0.5
u_3	0	1	0.6	0.4	0.4	0.6

Remark 2. A fuzzy orthopartition $\{(\mu_0,\nu_0),(\mu_1,\nu_1),\ \dots,(\mu_n,\nu_n)\} \in \mathcal{O}^*$ is a partition when $h_i(u)=0$ for each $u \in U$ and for each $i \in \{0,\dots,n\}$.

Example 6. The fuzzy orthopartition $\alpha(O)$ where O is defined in Example 5, is equivalent to the partition $P=\{\{u_1\},\{u_2\},\{u_3\}\}$, or more exactly, to the partition $\{\{u_2\},\{u_3\}\}$, where u_1 is an outlier.

Summing up, partitions are special cases of orthopartitions, which are strictly included in the class of fuzzy orthopartitions. The relationship among partitions, orthopartitions, and fuzzy orthopartitions is schematized in the Euler-Venn diagram of Fig. 2.

2.3. Fuzzy orthopartitions and credal partitions as fuzzy partitions

In this section, we link Fuzzy orthopartitions with Ruspini partitions then Credal partitions with Fuzzy probabilistic partitions and finally fuzzy probabilistic partitions with Ruspini partitions.

Fuzzy orthopartitions and Ruspini partitions

Definition 7. [11] A *Ruspini partition* of U is a family $\pi=\{\pi_0,\pi_1,\dots,\pi_n\}$ of fuzzy sets on U such that

$$\pi_0(u)+\pi_1(u)+\dots+\pi_n(u)=1 \text{ for each } u \in U.$$

Let $u \in U$ and let $i \in \{0,\dots,n\}$, $\pi_i(u)$ is the truth degree to which *"u belongs to C_i"*.

Remark 3. Ruspini partitions coincide with fuzzy orthopartitions where all the degrees of uncertainty are 0. More precisely,

- a fuzzy orthopartition $\{(\mu_0,\nu_0),(\mu_1,\nu_1),\dots,(\mu_n,\nu_n)\}$ such that

  "$h_i(u)=0$ for each $u \in U$ and $i \in \{0,\dots,n\}$"

  is equivalent to the Ruspini partition $\{\mu_0,\mu_1,\dots,\mu_n\}$;
- a Ruspini partition $\{\pi_0,\pi_1,\dots,\pi_n\}$ is equivalent to the fuzzy orthopartition $\{(\pi_0,1-\pi_0),(\pi_1,1-\pi_1),\dots,(\pi_n,1-\pi_n)\}$, where $(1-\pi_i)(u)=1-\pi_i(u)$ for each $u \in U$ and for each $i \in \{0,\dots,n\}$.

Example 7. Let $O=\{(\mu_0,\nu_0),(\mu_1,\nu_1),(\mu_2,\nu_2)\}$ be the fuzzy orthopartition of the universe $\{u_1,u_2,u_3\}$, where μ_0,μ_1,μ_2 and ν_0,ν_1,ν_2 are defined by Table 6.

It is true that $h_0(u_j)=h_1(u_j)=h_2(u_j)=0$ for each $j \in \{1,2,3\}$. Also, according to Remark 3, $\{\mu_0,\mu_1,\mu_2\}$ is a Ruspini partition equivalent with O because the following equalities are true: $\mu_0(u_1)+\mu_1(u_1)+\mu_2(u_1)=0+0.2+0.8=1$, $\mu_0(u_2)+\mu_1(u_2)+\mu_2(u_2)=0+0.5+0.5=1$, and $\mu_0(u_3)+\mu_1(u_3)+\mu_2(u_3)=0+0.6+0.4=1$. Vice-versa, O can be constructed from $\{\mu_0,\mu_1,\mu_2\}$ as follows:

$$O=\{(\mu_0,1-\mu_0),(\mu_1,1-\mu_1),\dots,(\mu_n,1-\mu_n)\}. \tag{9}$$

The previous remark naturally leads to interpreting fuzzy orthopartitions as Ruspini partitions with uncertainty.

The Euler-Venn diagram of Fig. 3 represents the relation between fuzzy orthopartitions and Ruspini partitions.

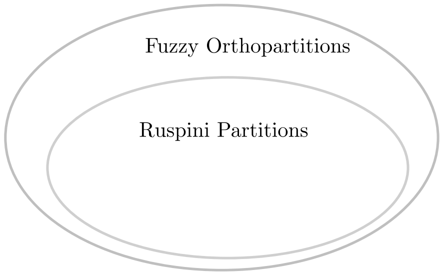

Fig. 3. Hierarchy of fuzzy ortho and Ruspini partitions.

Table 7
Definition of the elements of m of Example 8.

A	$m_1(A)$	$m_2(A)$	$m_3(A)$
$\emptyset$	0	0	0
$\{C_1\}$	0.2	0.5	0.6
$\{C_2\}$	0.8	0.5	0.4
C	0	0	0

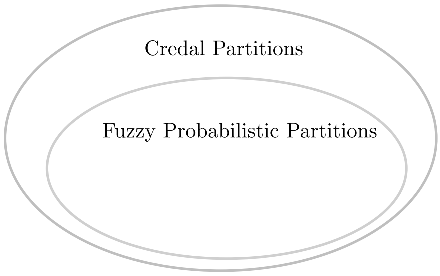

Fig. 4. The hierarchy of credal and fuzzy probabilistic partitions.

Credal partitions and fuzzy probabilistic partitions

Definition 8. [8] A fuzzy probabilistic partition $\{m_1, \dots, m_l\}$ of U is a collection of Bayesian bbas, namely for each $i \in \{1, \dots, l\}$, $\sum_{A \subseteq C} m_i(A) = 1$ and $m_i(A) = 0$ for each $A \subseteq C$ that is not empty or a singleton.

Example 8. Consider the functions m_1, m_2, m_3 defined by Table 7. Then, $m = \{m_1, m_2, m_3\}$ is a fuzzy probabilistic partition. In fact, $m_1(\emptyset) = m_2(\emptyset) = m_3(\emptyset) = 0$ and $m_1(C) = m_2(C) = m_3(C) = 0$.

Trivially, fuzzy probabilistic partitions are special cases of credal partitions. Their relationship is exhibited in the Euler-Venn diagram of Fig. 4.

Ruspini partitions and fuzzy probabilistic partitions Let us analyze the relationship between Ruspini partitions and fuzzy probabilistic partitions.

Firstly, Ruspini partitions and fuzzy probabilistic partitions are more general than standard partitions. Also, we can notice that both can be understood as mathematical tools associating a value of $[0, 1]$ to each pair made of an object of U and a class of $C \cup \{C_0\}$. Consequently, a one-to-one correspondence arises between Ruspini partitions and fuzzy probabilistic partitions.

In the sequel, we respectively denote with $\mathcal{R}$ and $\mathcal{M}^*$ the collections of all Ruspini partitions and fuzzy probabilistic partitions.

Theorem 1. *Let $f : \mathcal{R} \to \mathcal{M}^*$ be the mapping such that let $\{\pi_0, \pi_1, \dots, \pi_n\} \in \mathcal{R}$,*

$$f(\{\pi_0, \pi_1, \dots, \pi_n\}) = \{m_1, \dots, m_l\}, \tag{10}$$

where

$$m_j(\emptyset) = \pi_0(u_j) \text{ and } m_j(C_i) = \pi_i(u_j), \quad \forall\, i \in \{1, \dots, n\} \text{ and } \forall j \in \{1, \dots, l\}. \tag{11}$$

Then, f is a bijection.

Table 8
Definition of the elements of O of Example 10.

u	$\mu_0(u)$	$\nu_0(u)$	$\mu_1(u)$	$\nu_1(u)$	$\mu_2(u)$	$\nu_2(u)$	$\mu_3(u)$	$\nu_3(u)$
u_1	0	0.1	0.1	0.3	0.5	0.1	0.1	0
u_2	0.2	0.8	0	0.2	0.1	0.1	0	0.5
u_3	0.2	0.4	0.1	0.4	0.2	0.5	0	0.8
u_4	0	0.5	0.3	0.2	0.2	0.4	0.3	0.2

Proof. We can trivially verify that f is well-defined, injective, and surjective. □

Example 9. Consider the Ruspini partition $\{\mu_0, \mu_1, \mu_2\}$, where the fuzzy sets μ_0, μ_1, and μ_2 are given by Table 6. Then, we can easily verify that $f(\{\mu_0, \mu_1, \mu_2\}) = \{m_1, m_2, m_3\}$ is the fuzzy probabilistic partition represented by Table 7. For example, we get $m_1(\{C_2\}) = \mu_2(u_1) = 0.8$; therefore, we can observe that both $\{\mu_0, \mu_1, \mu_2\}$ and $\{m_1, m_2, m_3\}$ assign the value 0.2 with the object u_1 and the class C_2.

Remark 4. The first mathematical correspondence between fuzzy ortho and credal partitions immediately arises from Theorem 1. These two models coincide when they are Ruspini are fuzzy probabilistic partitions. Indeed, by the previous theorem, $f(\mathcal{R}) \subseteq \mathcal{M}$ (Ruspini partitions form a subclass of credal partitions) and $f^{-1}(\mathcal{M}^*) \subseteq \mathcal{O}$ (fuzzy probabilistic partitions form a subclass of fuzzy orthopartitions); furthermore, by Subsection 2.3, we know that Ruspini partitions are special fuzzy orthopartitions and fuzzy probabilistic partitions are special credal partitions.

3. Generalized fuzzy orthopartitions

In this section, we introduce a new structure called generalized fuzzy orthopartition. Then, we prove that it is a more general notion than all models described in Section 2.

Definition 9. Let $O = \{(\mu_0, \nu_0), (\mu_1, \nu_1), \ldots, (\mu_n, \nu_n)\}$ be a family of IFSs of U. Then, O is a *generalized fuzzy orthopartition* of U if and only if the following properties hold for each $u \in U$:

a) $\sum_{i=0}^{n} \mu_i(u) \leq 1$,
b) $\sum_{i=0}^{n} \mu_i(u) + h_i(u) \geq 1$.

It is easy to notice that the models given by Definition 9 are more general than fuzzy orthopartitions. Indeed, the axioms defining a generalized fuzzy orthopartition coincide with Axioms (a) and (c) of Definition 4.

We denote the collection of all generalized fuzzy orthopartitions with $\mathcal{O}_G$.

Example 10. Let $C = \{C_1, C_2, C_3\}$ be a partition of $U = \{u_1, u_2, u_3, u_4\}$, then a generalized fuzzy orthopartition $O = \{(\mu_0, \nu_0), (\mu_1, \nu_1), (\mu_2, \nu_2), (\mu_3, \nu_3)\}$ of U is defined by Table 8.

In fact, $\sum_{i=0}^{3} \mu_i(u_1) = 0 + 0.1 + 0.5 + 0.1 = 0.7 \leq 1$ and $\sum_{i=0}^{3} \mu_i(u_1) + h_i(u_1) = 0.9 + 0.7 + 0.9 + 1 = 3.5 \geq 1$. Also, we can immediately verify that Properties (a) and (b) of Definition 9 hold for u_2, u_3, and u_4.

On the other hand, it is easy to see that O is not a fuzzy orthopartition. For instance, Axiom (b) of Definition 13 is not satisfied: $\mu_2(u_1) + h_0(u_1) = 0.5 + 0.9 > 1$.

In what follows, we show that standard partitions are special cases of generalized fuzzy orthopartitions.

We use the symbol $\mathcal{P}$ to indicate the collection of all partitions of U made of $n+1$ class. Furthermore, $\mathcal{O}_G^*$ denotes the collection of all generalized fuzzy orthopartitions that are made of all Boolean functions and have all degrees of uncertainty equal to 0:

$$\mathcal{O}_G^* = \{O = \{(\mu_0, \nu_0), (\mu_1, \nu_1), \ldots, (\mu_n, \nu_n)\} \in \mathcal{O}_G \mid \forall u \in U \text{ and } \forall i \in \{0, \ldots, n\},\ \mu_i(u), \nu_i(u) \in \{0, 1\} \text{ and } h_i(u) = 0\}. \tag{12}$$

Theorem 2. *Let $\beta : \mathcal{P} \to \mathcal{O}_G^*$ be a mapping such that given $P = \{P_0, P_1, \ldots, P_n\} \in \mathcal{P}$,*

$$\beta(\{P_0, P_1, \ldots, P_n\}) = \{(\mu_0, \nu_0), (\mu_1, \nu_1), \ldots, (\mu_n, \nu_n)\}, \tag{13}$$

where

$$\mu_i(u) = \begin{cases} 1 & \text{if } u \in P_i, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \nu_i(u) = 1 - \mu_i(u), \tag{14}$$

$\forall u \in U$ and $\forall i \in \{0, \ldots, n\}$. Then, β is a bijection.

Table 9
Definition of the elements of O of Example 11.

u	$\mu_0(u)$	$\nu_0(u)$	$\mu_1(u)$	$\nu_1(u)$	$\mu_2(u)$	$\nu_2(u)$
u_1	1	0	0	1	0	1
u_2	0	1	1	0	0	1
u_3	0	1	0	1	1	0
u_4	0	1	0	1	1	0

Table 10
Definition of the elements of O of Example 12.

u	$\mu_0(u)$	$\nu_0(u)$	$\mu_1(u)$	$\nu_1(u)$	$\mu_2(u)$	$\nu_2(u)$	$\mu_3(u)$	$\nu_3(u)$
u_1	0	0	1	0	0	1	0	1
u_2	0	1	1	0	0	1	0	1
u_3	0	0	0	0	0	1	0	1
u_4	0	1	0	1	1	0	0	1

Proof. Clearly, each pair (μ_i, ν_i) defined by (14) is an IFS by considering that the following cases can occur: "$\mu_i(u) = 1$ and $\nu_i(u) = 0$" or "$\mu_i(u) = 0$ and $\nu_i(u) = 1$". More precisely, $\beta(\{P_0, P_1, \dots, P_n\})$ is a generalized fuzzy orthopartition of $\mathcal{O}^*_G$:

- Axiom (a) of Definition 9 holds, namely $\sum_{i=0}^{n} \mu_i(u) = 1$. Indeed, an element u of U belongs exactly to one class of the partition P. By (14), there exists $i \in \{0, \dots, n\}$ such that $\mu_i(u) = 1$ and $\mu_j(u) = 0$ for each $j \neq i$.
- Axiom (b) of Definition 9 holds, namely $\sum_{i=0}^{n} \mu_i(u) + h_i(u) = 1$. By using (14) again, an element $u \in U$ must belong to a class P_i of P, thus $\mu_i(u) + h_i(u) = 1$. Also, $u \notin P_j$ for each $j \neq i$. Therefore, $\mu_j(u) + h_j(u) = 0 + 0 = 0$ for each $j \neq i$.
- Finally, by (14), we can easily observe that $\mu_0, \mu_1, \dots, \mu_n$ and $\nu_0, \nu_1, \dots, \nu_n$ are Boolean functions, and all degrees of uncertainty are 0.

Let $O = \{(\mu_0, \nu_0), (\mu_1, \nu_1), \dots, (\mu_n, \nu_n)\}$ such that $O = \beta(\{P_0, P_1, \dots, P_n\})$.

Then, the function β is clearly well-defined and injective from (14): let $P = \{P_0, P_1, \dots, P_n\}$ and $P' = \{P'_0, P'_1, \dots, P'_n\}$ be partitions of $\mathcal{P}$, $P = P'$ if and only if $P_i = P'_i$ for each $i \in \{0, \dots, n\}$, which means that $\mu_i = \mu'_i$ and $\nu_i = \nu'_i$ for each $i \in \{0, \dots, n\}$, where $\beta(P) = \{(\mu_0, \nu_0), (\mu_1, \nu_1), \dots, (\mu_n, \nu_n)\}$ and $\beta(P') = \{(\mu'_0, \nu'_0), (\mu'_1, \nu'_1), \dots, (\mu'_n, \nu'_n)\}$. Therefore, $\beta(P) = \beta(P')$.

In order to show that β is a surjective function, we consider the generalized fuzzy orthopartition $O = \{(\mu_0, \nu_0), (\mu_1, \nu_1), \dots, (\mu_n, \nu_n)\} \in \mathcal{O}^*_G$. Thus, we can construct the partition $P = \{P_0, P_1, \dots, P_n\}$ such that $P_i = \{u \in U \mid \mu_i(u) = 1\}$ for each $i \in \{0, \dots, n\}$. By (14), we can immediately understand that $P \in \mathcal{P}$ and $\beta(P) = O$. □

The previous theorem determines the existence of a one-to-one correspondence between $\mathcal{P}$ and $\mathcal{O}^*_G$. In addition, as explained in the next example, $P \in \mathcal{P}$ and $\beta(P)$ contain the same information.

Example 11. Let $P = \{P_0, P_1, P_2\}$ be a partition of $U = \{u_1, u_2, u_3, u_4\}$, where $P_0 = \{u_1\}$, $P_1 = \{u_2\}$, and $P_2 = \{u_3, u_4\}$. Then, $O = \beta(P)$ is defined by Table 9.

We can see that O is a generalized fuzzy orthopartition of $\mathcal{O}^*_G$. Moreover, O contains the same information of P: u_1 is an outlier because $\mu_0(u_1) = 1$, u_2 forms the first class and $\{u_3, u_4\}$ is the second class of the partition, considering that $\mu_1(u_2) = 1$ and $\mu_2(u_3) = \mu_2(u_4) = 1$.

Generalized fuzzy orthopartitions of $\mathcal{O}_G$ made of Boolean functions can be viewed as collections of orthopairs verifying Axioms (a) and (c) of Definition 6.

Proposition 1. *Let $O = \{(\mu_0, \nu_0), (\mu_1, \nu_1), \dots, (\mu_n, \nu_n)\} \in \mathcal{O}_G$ such that μ_0, μ_1, $\dots, \mu_n$ and $\nu_0, \nu_1, \dots, \nu_n$ are Boolean. Then, $O' = \{(M_0, N_0), (M_1, N_1), \dots, (M_n, N_n)\}$ given by (6) satisfies Axioms (a) and (c) of Definition 6.*

Proof. The proof clearly follows from (6) and Definition 9. □

Remark 5. On the other hand, a generalized fuzzy orthopartition O of $\mathcal{O}_G$ made of Boolean functions is not an orthopartition according to Definition 6. More precisely, (6) transforms O into a set of orthopairs O' that could not satisfy Axioms (b) and (d) of Definition 6. The following is an example.

Example 12. Let $C = \{C_1, C_2, C_3\}$ be a partition of $U = \{u_1, u_2, u_3, u_4\}$, we consider the generalized fuzzy orthopartition $O = \{(\mu_0, \nu_0), (\mu_1, \nu_1), (\mu_2, \nu_2), (\mu_3, \nu_3)\} \in \mathcal{O}_G$, which is defined by Table 10.

Then, according to (6), the orthopairs corresponding to O' are $(M_0, N_0) = (\emptyset, \{u_2, u_4\})$, $(M_1, N_1) = (\{u_1, u_2\}, \{u_4\})$, $(M_2, N_2) = (\{u_4\}, \{u_1, u_2, u_3\})$, and $(M_3, N_3) = (\emptyset, \{u_1, u_2, u_3, u_4\})$, and they do not form an orthopartition. In fact, we can notice that Axiom (b) of Definition 6 is not satisfied because $u_1 \in B_0 \cap M_1$. Furthermore, Axiom (d) is not satisfied too because $u_1 \in B_0$ but $u_1 \notin B_1 \cup B_2 \cup B_3$.

Despite Remark 5, we can consider generalized fuzzy orthopartitions as an extension of the concept of orthopartitions. This is because we can transform a collection of orthopairs O that satisfies Axioms (a) and (c) of Definition 6 into an orthopartition that contains the same information provided by O.

Definition 10. Let $O = \{(M_0, N_0), (M_1, N_1). \dots, (M_n, N_n)\}$ be a collection of orthopairs satisfying Axioms (a) and (c) of Definition 6. We consider

$$O' = \{(M_0 \cup E_0, N_0 \cup F_0), (M_1 \cup E_1, N_1 \cup F_1), \dots, (M_n \cup E_n, N_n \cup F_n)\} \tag{15}$$

such that

$$E_i = \{u \in B_i \mid u \in N_j \ \ \forall j \neq i\} \ \text{ and } \ F_i = \{u \in B_i \mid u \in M_j \text{ with } j \neq i\}, \tag{16}$$

for each $i \in \{0, \dots, n\}$.

Proposition 2. *Let O be a collection of orthopairs satisfying Axioms (a) and (c) of Definition 6. Then, O' given by Definition 10 is an orthopartition of U.*

Proof. Let $O' = \{(M_0 \cup E_0, N_0 \cup F_0), (M_1 \cup E_1, N_1 \cup F_1), \dots, (M_n \cup E_n, N_n \cup F_n)\}$. We first show that for each $i \in \{0, \dots, n\}$, $(M_i \cup E_i, N_i \cup F_i)$ is an orthopair of U, i.e. $M_i \cup E_i$ and $N_i \cup F_i$ are disjoint. Since M_i, N_i, and B_i are mutually disjoint and $E_i, F_i \subseteq B_i$, we know that $M_i \cup E_i$ is disjoint from N_i and $N_i \cup F_i$ is disjoint from M_i. We need to verify that E_i and F_i are disjoint too. Indeed, $u \in E_i$ implies that $u \in N_j$ for each $j \neq i$. Hence, u cannot belong to M_j for each $j \neq i$, that is $u \notin F_i$. Analogously, $u \in F_i$ implies that $u \notin E_i$.

Now, let us prove that all axioms characterizing orthopartitions are satisfied by O'.

Axiom (a). Let $i \neq j$, we intend to prove that $(M_i \cup E_i) \cap (M_j \cup E_j) = \emptyset$. Since O verifies Axiom (a), $M_i \cap M_j = \emptyset$. Moreover, $M_i \cap E_j = \emptyset$ because $u \in E_i$ means that $u \in N_j$, and so $u \notin M_j$. Symmetrically, $M_j \cap E_i = \emptyset$. Finally, we can notice that $E_i \cap E_j = \emptyset$. Indeed, if $u \in E_i$, then $u \in B_i$ and $u \in N_j$. Since $B_j \cap N_j = \emptyset$ and $E_j \subseteq B_j$, we can conclude that $u \notin E_j$. Symmetrically, if $u \in E_j$, then $u \notin E_i$.

Axiom (b). Notice that the boundary region of the orthopair $(M_j \cup E_j, N_j \cup F_j)$ is $B_j \setminus (E_j \cup F_j)$. So, let $i \neq j$, we want to prove that $(M_i \cup E_i) \cap (B_j \setminus (E_j \cup F_j)) = \emptyset$. Let $u \in M_i \cup E_i$. Suppose that $u \in M_i$. If $u \in B_j \setminus (E_j \cup F_j)$, then $u \notin F_j$. This implies that $u \notin M_k$ for each $k \neq j$. Then, $u \notin M_i$, which is absurd. Suppose that $u \in E_i$ then $u \in N_j$ and so, u cannot belong to B_j. Therefore, the intersection of $M_i \cup E_i$ and $B_j \setminus (E_j \cup F_j)$ must be empty.

Axiom (c). Let $u \in U$, we need to prove that $u \in M_i \cup E_i$ or $u \in B_i \setminus (E_i \cup F_i)$ with $i \in \{0, \dots, n\}$. If "$u \in M_i$ *with* $i \in \{0, \dots, n\}$", then it is trivial that $u \in M_i \cup E_i$. Suppose that "$u \notin M_i \ \forall i \in \{0, \dots, n\}$". Since O satisfies Axiom (c) of Definition 6, $u \in B_j$ with $j \in \{0, \dots, n\}$". If $u \in N_k$ for each $k \neq j$, then $u \in E_j$, which implies that $u \in M_j \cup E_j$. Otherwise, $u \notin E_j$. Moreover, u cannot belong to F_j because we have assumed that $u \notin M_i \ \forall i \in \{0, \dots, n\}$. Lastly, $u \in B_j \setminus (E_j \cup F_j)$.

Axiom (d). Let $u \in U$ and let $i \in \{0, \dots, n\}$ such that $u \in B_i \setminus (E_i \cup F_i)$. We have to prove that $u \in B_j \setminus (E_j \cup F_j)$ with $i \neq j$. Since $u \notin F_i$, it must hold that $u \notin M_k$ for each $k \neq i$. Thus, $u \in B_k \cup N_k$ for each $k \neq i$. Since $u \notin E_i$, there exists $j \neq i$ such that $u \notin N_j$. Then, $u \in B_j$. Moreover, $u \notin E_j \cup F_j$ because we have observed that $u \notin N_i$ (hence, $u \notin E_j$) and $u \notin M_i$ and $u \notin M_k \ \forall k \neq i$ (hence, $u \notin F_j$). In the end, $u \in B_j \setminus (E_j \cup F_j)$. □

We can notice that both O and O' related to Definition 10 represent the same partition with uncertainty. Let us show an example.

Example 13. We consider the set of orthopairs $O = \{(M_0, N_0), (M_1, N_1), (M_2, N_2)\}$, where $(M_0, N_0) = (\{u_1\}, \{u_2, u_4\})$, $(M_1, N_1) = (\{u_2\}, \{u_3, u_4\})$, and $(M_2, N_2) = (\{u_4\}, \{u_2, u_3\})$. O only satisfies Axioms (a) and (c) of Definition 6. In fact, we can see that $M_0 \cap B_1 = M_0 \cap B_2 = \{u_1\}$ (Axiom (b) does not hold) and $u_3 \in B_0$ and $u_3 \in N_1 \cap N_2$ (Axiom (d) does not hold).

Despite O not being an orthopartition, it consistently represents a partition $\{C_0, C_1, C_2\}$ with uncertainty. Indeed, we know with certainty that $u_1 \in C_0$, $u_2 \in C_1$, $u_4 \in C_2$. Moreover, we can deduce that $u_3 \in C_0$ considering that u_3 cannot belong to C_1 and C_2.

Let us underline that $M_0 \cap B_1 = \{u_1\}$ does not contradict that O represents a partition. Indeed, $u_1 \in M_0$ and $u_1 \in B_1$ respectively mean that "u_1 belongs to C_0" and "u_1 could belong to C_1 or not". Thus, $u_1 \in M_0$ is not in contrast with $u_1 \in B_1$, but it specifies more precise information about the membership class of u_1. Analogously, $u_3 \in B_0$ is not in contrast with $u \notin B_1$ and $u \notin B_2$. Indeed, $u_3 \in B_0$ means that "u_3 could belong to C_0 or not", $u \notin B_1$ and $u \notin B_2$ together with $u \in N_1$ and $u \in N_2$ specifies that "u_3 belongs to C_0".

According to Definition 10, O is transformed in $O' = \{(M_0 \cup E_0, N_0 \cup F_0), (M_1 \cup E_1, N_1 \cup F_1), (M_2 \cup E_2, N_2 \cup F_2)\}$ such that $M_0 \cup E_0 = \{u_1\} \cup \{u_3\} = \{u_1, u_3\}$, $N_0 \cup F_0 = \{u_2, u_4\} \cup \emptyset = \{u_2, u_4\}$, $M_1 \cup E_1 = \{u_2\} \cup \emptyset = \{u_2\}$, $N_1 \cup F_1 = \{u_3, u_4\} \cup \{u_1\} = \{u_1, u_3, u_4\}$, $M_2 \cup E_2 = \{u_4\} \cup \emptyset = \{u_4\}$, and $N_2 \cup F_2 = \{u_2, u_3\} \cup \{u_1\} = \{u_1, u_2, u_3\}$.

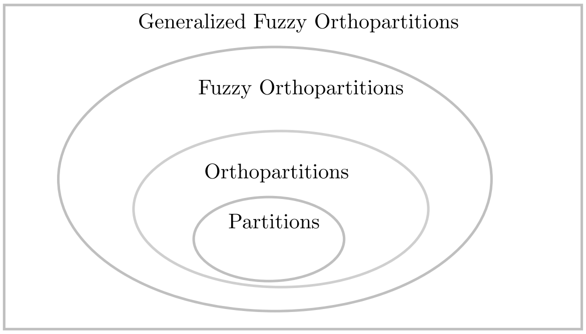

Fig. 5. The hierarchy of the models defined in Subsections 2.2 and 3.

See that $O' = \{(\{u_1,u_3\},\{u_2,u_3\}),(\{u_2\},\{u_1,u_3,u_4\}),(\{u_4\},\{u_1,u_2,u_3\})\}$ exactly contains the same information of O. More exactly, both O and O' represent the partition $\{\{u_1,u_3\},\{u_2\},\{u_4\}\}$. The advantage to consider O' instead of O is that O' does not contain redundant or hidden information.

In general, a collection of orthopairs O that exclusively satisfies Axioms (a) and (c), carries redundancy and hidden information. These problems are solved when O is transformed into the orthopartition O'. So, we can think that O' arises eliminating the redundancy from O and making clear the information hidden in O. Practically, Definition 10 constructs O' starting from O as follows.

- When $u \in M_i$ and $u \in B_j$ (Axiom (b) of Definition 6 does not hold for O), the redundant information is given by $u \in B_j$: of course, we already know that u belongs to the class C_i from $u \in M_i$. Then, u is moved from B_j to the impossibility domain of the orthopair representing C_j.
- When $u \in B_i$ and $u \notin B_j$ $\forall j \neq i$ (Axiom (b) of Definition 6 does not hold for O), the information about the membership class of u is hidden in O. Indeed, if there exists $j \neq i$ such that $u \in M_j$, then we are sure that u belongs to C_j. Therefore, in order to construct O', we need to move u from B_i to the impossibility domain of the orthopair representing C_i. If $u \in N_j$ for each $j \neq i$, then we are sure that u belongs to C_i, and so, u passes from B_i to the lower approximation of the orthopair representing C_i.

We can conclude that collections of orthopairs satisfying Axioms (a) and (c) of Definition 6 and orthopartitions are equivalent and both are consistent models (i.e., they do not contain contradictions) to represent partitions in presence of partial knowledge.

The relationship among the models defined in Subsections 2.2 and 3 is exhibited in the Euler-Venn diagram Fig. 5.

4. From fuzzy orthopartitions to credal partitions

The main goal of this section is achieved in Subsection 4.3, where we construct a class of credal partitions assigned to a given fuzzy orthopartition. In order to do this, we need to show that

- each fuzzy orthopartition can be viewed as a collection of Ruspini partitions (Subsection 4.1);
- each credal partition can be viewed as a collection of fuzzy probabilistic partitions (Subsection 4.2).

4.1. Compatible Ruspini partitions

A fuzzy orthopartition corresponds to a collection of Ruspini partitions. Their definition is formulated by thinking that fuzzy orthopartitions approximate Ruspini partitions when the truth degree of elements of the initial universe is not exactly known. In fact, starting from a fuzzy orthopartition, the truth degree of the statement "*the object u_j belongs to the class C_i*" is uncertain, considering that we only know that it is a value of the interval $[\mu_i(u_j), 1-\nu_i(u_j)]$. If knowledge about the membership classes of the elements increases in such a way that it is no longer partial, we can precisely determine the degree assigned with u_j and C_i, and obtain a Ruspini partition. All Ruspini partitions that potentially could coincide with a fuzzy orthopartition are formally defined as follows.

Definition 11. Let O be a fuzzy orthopartition of U. We say that a Ruspini partition π is *compatible* with O if and only if

$$\mu_i(u_j) \leq \pi_i(u_j) \leq 1-\nu_i(u_j), \text{ for each } i \in \{0,\dots,n\} \text{ and } j \in \{1,\dots,l\}.$$

We use the symbol C_O to indicate the collection of all Ruspini partitions compatible with the fuzzy orthopartition O.

Example 14. Consider the fuzzy orthopartition O defined by Table 4. According to Definition 11, a Ruspini partition $\pi = \{\pi_0,\pi_1,\pi_2,\pi_3\}$ is compatible with O if and only if each $u_j \in \{u_1,u_2,u_3,u_4\}$ satisfies the following inequalities:

Table 11
Definition of the elements of $\{\pi_0, \pi_1, \pi_2, \pi_3\}$ of Example 14.

u	$\pi_0(u)$	$\pi_1(u)$	$\pi_2(u)$	$\pi_3(u)$
u_1	0.3	0.3	0.3	0.1
u_2	0.2	0.1	0.3	0.4
u_3	0.4	0.2	0.3	0.1
u_3	0.1	0.3	0.4	0.2

$$\begin{cases} \mu_0(u_j) \leq \pi_0(u_j) \leq \mu_0(u_j) + h_0(u_j), \\ \mu_1(u_j) \leq \pi_1(u_j) \leq \mu_1(u_j) + h_1(u_j), \\ \mu_2(u_j) \leq \pi_2(u_j) \leq \mu_2(u_j) + h_2(u_j), \\ \mu_3(u_j) \leq \pi_3(u_j) \leq \mu_3(u_j) + h_3(u_j). \end{cases} \tag{17}$$

Then, it is easy to verify that an example of Ruspini partition $\{\pi_0, \pi_1, \pi_2, \pi_3\}$ compatible with O is defined by Table 11.

For example, $\pi_0(u_1) = 0.3$ is between $\mu_0(u_1) = 0$ and $\mu_0(u_1) + h_0(u_1) = 0 + 0.9 = 0.9$, $\pi_1(u_1) = 0.3$ is between $\mu_1(u_1) = 0.1$ and $\mu_1(u_1) + h_1(u_1) = 0.1 + 0.6 = 0.7$, and so on.

Remark 6. For each credal partition $O = \{(\mu_0, \nu_0), (\mu_1, \nu_1), \dots, (\mu_n, \nu_n)\}$, there exists at least a Ruspini partition $\pi = \{\pi_0, \pi_1 \dots, \pi_n\}$ that is compatible with O. In fact, according to Definition 11,

- $|\mathcal{Z}_O| = 1$, when O is a Ruspini partition (all the degrees of uncertainty are equal to 0);
- $|\mathcal{Z}_O| = \infty$, otherwise. This is because if the degree of uncertainty $h_i(u_j)$ is not 0 for some $i \in \{0, \dots, n\}$ and $j \in \{1, \dots, l\}$, then we can choose in infinite ways a precise truth value into the interval $[\mu_i(u_j), 1 - \nu_i(u_j)]$.

Of course, Definition 11 can be also extended to generalized fuzzy orthopartitions given by Definition 9.

4.2. *Compatible fuzzy probabilistic partitions*

A credal partition can be seen as a fuzzy probabilistic partition under condition of uncertainty. As such, a credal partition can represent several fuzzy probabilistic partitions, once the uncertainty is solved. The meaning of this correspondence is analogous to that held between fuzzy orthopartitions and Ruspini partitions explained in Subsection 4.1, and it is discussed in the following remark.

Remark 7. Imagine a dynamic situation, where knowledge about the membership class of the elements is partial and increases over time so that credal partitions become fuzzy probabilistic partitions. In this context, a credal partition $\{m_1, \dots, m_l\}$ is transformed in $\{m'_1, \dots, m'_l\}$ such that

- m'_i is a Bayesian bba,
- $m'_i(\emptyset)$ belongs to the interval $[m_i(\emptyset), 1 - m_i(C)]$,
- $m'_i(\{C_j\})$ belongs to the interval $[m_i(\{C_j\}), \sum_{\{A \mid C_j \in A\}} m_i(A)]$, for each $j \in \{1, \dots, n\}$.

Therefore, if $A = \{C'_1, \dots, C'_k\}$ where $C'_1, \dots, C'_k$ belong to C and $k \geq 2$ (i.e., A is not a singleton), then $m_i(A)$ is distributed among the masses of belief concerning $C'_1, \dots, C'_k$, i.e., the degrees $m'_i(C'_1), \dots, m'_i(C'_k)$ supporting the propositions "u_i belongs to C'_1", …, "u_i belongs to C'_k". Moreover, the limit cases $m'_i(\{C_j\}) = m_i(\{C_j\})$ and $m'_i(\{C_j\}) = \sum_{\{A \mid C_j \in A\}} m_i(A)$ respectively occur when

- "if $C_j \subset A$ and u_i belongs to A, then u_i belongs to $A \setminus \{C_j\}$" and
- "if $C_j \subset A$ and u_i belongs to A, then u_i belongs to C_j".

Furthermore, concerning the outliers, we get the limit cases $m'_i(\emptyset) = m_i(\emptyset)$ and $m'_i(\emptyset) = 1 - m_i(C)$ respectively when

- "u_i belongs to C_0" (i.e., "u_i is outside of C") and
- "u_i does not belong to C_0" (i.e., "u_i belongs to a class of C").

In what follows, we formally define the class of all fuzzy probabilistic partitions assigned with a given credal partition according to the previous remark.

Definition 12. Let m be a credal partition of U. We say that a fuzzy probabilistic partition m' is *compatible* with m if and only if for each $j \in \{1, \dots, l\}$,

Table 12
Definition of the elements of m' of Example 15.

A	$m'_1(A)$	$m'_2(A)$	$m'_3(A)$	$m'_4(A)$
$\emptyset$	0.3	0.2	0.4	0.1
$\{C_1\}$	0.3	0.1	0.2	0.3
$\{C_2\}$	0.3	0.3	0.3	0.4
$\{C_3\}$	0.1	0.4	0.1	0.2

(a) $m_j(\emptyset) \leq m'_j(\emptyset) \leq 1 - m_j(C)$,
(b) $m_j(\{C_i\}) \leq m'_j(\{C_i\}) \leq \sum_{\{A \mid C_i \in A\}} m_j(A)$, for each $i \in \{1, \dots, n\}$.

We use the symbol $\mathcal{Z}_m$ to indicate the collection of all fuzzy probabilistic partitions compatible with a given credal partition m.

Example 15. Consider the credal partition $m = \{m_1, m_2, m_3, m_4\}$ defined by Table 3 of Example 1. By Definition 12, a fuzzy probabilistic partition $m' = \{m'_1, m'_2, m'_3, m'_4\}$ is compatible with m if and only if for each bba $m'_i \in m'$ the following inequalities are satisfied:

$$\begin{cases} m_i(\emptyset) \leq m'_i(\emptyset) \leq 1 - m_i(C), \\ m_i(\{C_1\}) \leq m'_i(\{C_1\}) \leq m_i(\{C_1\}) + m_i(\{C_1, C_2\}) + m_i(\{C_1, C_3\}) + m_i(C), \\ m_i(\{C_2\}) \leq m'_i(\{C_2\}) \leq m_i(\{C_2\}) + m_i(\{C_1, C_2\}) + m_i(\{C_2, C_3\}) + m_i(C), \\ m_i(\{C_3\}) \leq m'_i(\{C_3\}) \leq m_i(\{C_3\}) + m_i(\{C_1, C_3\}) + m_i(\{C_2, C_3\}) + m_i(C). \end{cases} \tag{18}$$

In addition, since m_i is a bba, we have $m'_i(\emptyset) + m'_i(\{C_1\}) + m'_i(\{C_2\}) + m'_i(\{C_3\}) = 1$.

Therefore, m'_1, m'_2, m'_3, and m'_4 represent bbas of a fuzzy partition compatible with m if and only if the following hold:

$$\begin{cases} 0.2 \leq m'_1(\emptyset) \leq 0.8, \\ 0 \leq m'_1(\{C_1\}) \leq 0.5, \\ 0.3 \leq m'_1(\{C_2\}) \leq 0.7, \\ 0 \leq m'_1(\{C_3\}) \leq 0.3, \\ m'_1(\emptyset) + \sum_{i=1}^{3} m'_1(\{C_i\}) = 1. \end{cases} \quad \begin{cases} 0 \leq m'_2(\emptyset) \leq 0.5, \\ 0.1 \leq m'_2(\{C_1\}) \leq 0.6, \\ 0.3 \leq m'_2(\{C_2\}) \leq 0.8, \\ 0.1 \leq m'_2(\{C_3\}) \leq 0.6, \\ m'_2(\emptyset) + \sum_{i=1}^{3} m'_2(\{C_i\}) = 1. \end{cases} \tag{19}$$

$$\begin{cases} 0 \leq m'_3(\emptyset) \leq 0.4, \\ 0.1 \leq m'_3(\{C_1\}) \leq 0.8, \\ 0 \leq m'_3(\{C_2\}) \leq 0.9, \\ 0 \leq m'_3(\{C_3\}) \leq 0.8, \\ m'_3(\emptyset) + \sum_{i=1}^{3} m'_3(\{C_i\}) = 1. \end{cases} \quad \begin{cases} 0.1 \leq m'_4(\emptyset) \leq 1, \\ 0.1 \leq m'_4(\{C_1\}) \leq 0.3, \\ 0.1 \leq m'_4(\{C_2\}) \leq 0.5, \\ 0.2 \leq m'_4(\{C_3\}) \leq 0.6, \\ m'_4(\emptyset) + \sum_{i=1}^{3} m'_4(\{C_i\}) = 1. \end{cases} \tag{20}$$

For example, the bbas m'_1, m'_2, m'_3, and m'_4 defined by Table 12 form a fuzzy probabilistic partition m' compatible with m.

Remark 8. The class of fuzzy probabilistic partitions compatible with a given credal partition is non-empty. In fact, each fuzzy probabilistic partition $m' = \{m'_j \mid j \in \{1, \dots, l\}\}$ compatible with $m = \{m_j \mid j \in \{1, \dots, l\}\}$ can be obtained by choosing for each $j \in \{1, \dots, l\}$ the values $x_0, x_1, \dots, x_n$ so that

$$m'_j(\emptyset) = m_j(\emptyset) + x_0 \text{ and } m'_j(\{C_i\}) = m_j(\{C_i\}) + x_i \ \forall i \in \{1, \dots, n\}, \tag{21}$$

where

$$x_0 \leq 1 - (m_j(\emptyset) + m_j(C)) \text{ and } x_i \leq \sum_{\{A \mid \{C_i\} \subset A\}} m_j(A) \ \forall i \in \{1, \dots, n\}, \tag{22}$$

from items (a) and (b) of Definition 12.

Moreover, since each m'_j must be a bba, we need to require that

$$(m_j(\emptyset) + x_0) + (m_j(\{C_1\}) + x_1) + \dots + (m_j(\{C_n\}) + x_n) = 1. \tag{23}$$

This is always possible for the following reasons. Firstly $m_j(\emptyset) + m_j(\{C_1\}) + \dots + m_j(\{C_n\}) \leq 1$ because m_i is a bba. Secondly, when $x_0, x_1, \dots, x_n$ are the greatest ones according to (22) (i.e., $x_0 = 1 - (m_j(\emptyset) + m_j(C))$ and $x_i = \sum_{\{A \subseteq C \mid \{C_i\} \subset A\}} m_j(\{C_i\})$, we get

$$(m_j(\emptyset) + x_0) + (m_j(\{C_1\}) + x_1) + \dots + (m_j(\{C_n\}) + x_n) =$$

$$1 - m_j(C) + \sum_{\{A \subseteq C \mid C_1 \in A\}} m_j(\{C_1\}) + \ldots + \sum_{\{A \subseteq C \mid C_n \in A\}} m_j(\{C_n\}) \geq 1 \quad (24)$$

because m_i is a bba.

In the sequel, we use the notions of compatible Ruspini and fuzzy probabilistic partitions to establish a bridge between fuzzy orthopartitions and credal partitions.

4.3. From a fuzzy orthopartition to a class of credal partitions

In this subsection, we mainly associate a given generalized fuzzy orthopartition O to a class of credal partitions $\mathcal{F}(O)$. Then, we show how to compute the credal partitions associated with O and we discover that $\mathcal{F}(O)$ can be empty, a singleton or made up of an infinite number of credal partitions.

Let $O \in \mathcal{O}_G$, we intend to find the class of all credal partitions $\mathcal{F}(O)$ such that $f(\mathcal{Z}_O) = \mathcal{Z}_m$ (equivalently, $f^{-1}(\mathcal{Z}_m) = \mathcal{Z}_O$) for each $m \in \mathcal{F}(O)$.[4] Thus, we associate with O the collection of all credal partitions that have the same compatible fuzzy partitions of O. This result is exhibited by Theorem 3.

Formally, the class of credal partitions corresponding to a given fuzzy orthopartition is defined as follows. Recall that we denote the collection of all credal partitions of l bbas with $\mathcal{M}$.

Definition 13. Let O be a generalized fuzzy orthopartition of U. Then, we put

$$\mathcal{F}(O) = \{m \in \mathcal{M} \mid m_j(\emptyset) = \mu_0(u_j), m_j(C) = \nu_0(u_j),\ m_j(\{C_i\}) = \mu_i(u_j), \text{ and } \sum_{\{A \mid C_i \in A\}} m_j(A) = \mu_j(u_i) + h_j(u_i)\ \forall\, i \in \{1, \ldots, n\} \text{ and } \forall\, j \in \{1, \ldots, l\}\}. \quad (25)$$

Of course, Definition 13 and all results presented in this subsection also hold when O is a fuzzy orthopartition given by Definition 4.

Remark 9. Since $m \in \mathcal{F}(O)$ is a credal partition according to Definition 2, using (2) and (25), a necessary but not sufficient condition for O to be $\mathcal{F}(O) \neq \emptyset$ is the following:

$$\text{for each } i \in \{1, \ldots, n\}, \text{ there exists } u_j \in U \text{ so that } \mu_i(u_j) + h_i(u_j) > 0. \quad (26)$$

Of course, considering that $\mu_i(u) + h_i(u) > 0$ if and only if $\nu_i(u) < 1$, Property (26) is not satisfied by O when there exists a class C_i that is certainly empty (namely, $\nu_i(u) = 1$ for each $u \in U$).

Thus, in the rest of this section, we assume that a given generalized fuzzy orthopartition verifies Property (26).

First of all, let us focus on $\mathcal{F}(O)$ when $n = 2$.

As shown in the following proposition, Definition 13 associates a given generalized fuzzy orthopartition O composed of three IFSs (namely, $n = 2$) with at most one credal partition.

Proposition 3. *Let $O = \{(\mu_0, \nu_0), (\mu_1, \nu_1), (\mu_2, \mu_2)\}$ be a generalized fuzzy orthopartition of U, we consider the following property:*

$$\text{“}h_i(u) = \nu_0(u)\text{” and “}\mu_0(u) + \mu_1(u) + \mu_2(u) + \nu_0(u) = 1\text{”} \quad (27)$$

$\forall\, i \in \{1, 2\}$ and $\forall\, u \in U$. Then,

$$|\mathcal{F}(O)| = \begin{cases} 1 & \text{if Property (27) is satisfied by } O, \\ 0 & \text{otherwise.} \end{cases} \quad (28)$$

Proof. By Definition 13, a credal partition $m = \{m_1, \ldots, m_l\}$ is associated to $O = \{(\mu_0, \nu_0), (\mu_1, \nu_1), (\mu_2, \nu_2)\}$, when

$$m_j(\emptyset) = \mu_0(u_j), m_j(\{C_1\}) = \mu_1(u_j), m_j(\{C_2\}) = \mu_2(u_j), \text{ and } m_j(C) = \nu_0(u_j) \quad (29)$$

for each $j \in \{1, \ldots, l\}$. Therefore, since the values assumed by μ_0, μ_1, μ_2 and ν_0, ν_1, ν_2 are uniquely determined, we are sure that $|\mathcal{F}(O)| \leq 1$. In particular, $|\mathcal{F}(O)| = 1$ when

(a) $m_1, \ldots, m_l$ are bbas,

[4] Recall that the function f is defined by Theorem 1.

Table 13
Definition of the elements of O of Example 16.

u	$\mu_0(u)$	$\nu_0(u)$	$\mu_1(u)$	$\nu_1(u)$	$\mu_2(u)$	$\nu_2(u)$
u_1	0.4	0.2	0.3	0.5	0.1	0.7
u_2	0.1	0.5	0.2	0.3	0.2	0.3
u_3	0.2	0.1	0.3	0.6	0.4	0.5

Table 14
Definition of the elements m of Example 16.

A	$m_1(A)$	$m_2(A)$	$m_3(A)$
$\emptyset$	0.4	0.1	0.2
$\{C_1\}$	0.3	0.2	0.3
$\{C_2\}$	0.1	0.2	0.4
C	0.2	0.5	0.1

Table 15
Definition of the elements of O' of Example 16.

u	$\mu'_0(u)$	$\nu'_0(u)$	$\mu'_1(u)$	$\nu'_1(u)$	$\mu'_2(u)$	$\nu'_2(u)$
u_1	0.2	0.2	0.3	0.5	0.1	0.7
u_2	0.1	0.5	0.3	0.3	0.2	0.3
u_3	0.2	0.1	0.2	0.6	0.4	0.5

(b) $\mu_1(u_j) + h_1(u_j) = m_j(\{C_1\}) + m_j(C)$ and $\mu_2(u_j) + h_2(u_j) = m_j(\{C_2\}) + m_j(C)$ $\forall j \in \{1, \dots, l\}$ (see Definition 13).

Otherwise, $|\mathcal{F}(O)| = 0$.

Clearly, $m_1, \dots, m_l$ are bbas if and only if $\sum_{A \in 2^C} m_j(A) = 1$ for each $j \in \{1, \dots, l\}$. Since $C = \{C_1, C_2\}$, $\sum_{A \in 2^C} m_j(A) = m_j(\emptyset) + m_j(\{C_1\}) + m_j(\{C_2\}) + m_j(C)$. By (29), $m_j(\emptyset) + m_j(\{C_1\}) + m_j(\{C_2\}) + m_j(C) = \mu_0(u) + \mu_1(u) + \mu_2(u) + \nu_0(u)$, which must be equal to 1. Hence, item (a) is equivalent to $\mu_0(u) + \mu_1(u) + \mu_2(u) + \nu_0(u) = 1$.

Moreover, by (29), it is easy to see that item (b) is equivalent to $\mu_i(u) + h_i(u) = \mu_i(u) + \nu_0(u)$ $\forall u \in U$. □

By Proposition 3, a generalized fuzzy orthopartition that represents $C = \{C_1, C_2\}$ is a credal partition when the membership degree related to $u \in C_0$ (expressing how much u is external to the classes of C) coincides with the degrees of uncertainty of u to C_1 and C_2, and the sum of the membership degrees related to $u \in C_0$, $u \in C_1$, $u \in C_2$, and $u \in C$ is exactly 1.

Example 16. Consider the generalized fuzzy orthopartition $O = \{(\mu_0, \nu_0), (\mu_1, \nu_1), (\mu_2, \nu_2)\}$ of $U = \{u_1, u_2, u_3\}$ defined by Table 13.

We can notice that O satisfies Property (27): $h_1(u_1) = \nu_0(u_1) = 0.2$, $\mu_0(u_1) + \mu_1(u_1) + \mu_2(u_1) + \nu_0(u_1) = 1$, and so on. Then, according to the previous proposition, $\mathcal{F}(O)$ is made of one credal partition $m = \{m_1, m_2, m_3\}$, which is defined by Table 14.

On the other hand, we can consider the generalized fuzzy orthopartition O' defined by Table 15 and notice that Property (27) is not satisfied. For instance, $\mu'_0(u_2) + \mu'_1(u_2) + \mu'_2(u_2) + \nu'_0(u_2) = 0.1 + 0.3 + 0.2 + 0.5 = 1.1 > 1$ and $h'_1(u_3) = 0.2$ is different from $\nu'_0(u_3) = 0.1$. Hence, by the previous proposition, we know that $\mathcal{F}(O) = \emptyset$.

Remark 10. What happens when O is composed of Boolean functions? In this case, by Proposition 1, O is a collection of orthopairs $\{(M_0, N_0), (M_1, N_1), (M_2, N_2)\}$ given by (6) and satisfying Properties (a) and (c) of Definition 6. Thus, O can be identified with a credal partition (namely, Property (27) is satisfied by O) if and only if one of the following conditions holds for each $u \in U$:

(a) $u \in N_0$ and $u \in B_1 \cap B_2$ (we know that u certainly belongs to a class of C, but we do not have any information about the specific class of u).
(b) $u \in M_0$ (we know that u is certainly an outlier).
(c) $u \in B_0$, $u \in M_i$ with $i \in \{1, 2\}$, and $u \in N_j$ with $j \neq i$ (we know that u certainly belongs to C_i).

However, when $\{(M_0, N_0), (M_1, N_1), (M_2, N_2)\}$ is an orthopartition according to Definition 6, only items (a) and (b) can occur. In fact, the last case contradicts Property (b) of Definition 6.

Now, focusing on generalized fuzzy orthopartitions made of more than 3 IFSs (namely, $n \geq 3$), we show how to find the class of the corresponding credal partitions w.r.t. Definition 13.

By (25), in order to determine a bba m_j of a credal partition m in $\mathcal{F}(O)$, we respectively set $m_j(\{C_1\}), \dots, m_j(\{C_n\})$, $m_j(\emptyset)$, and $m_j(C)$ to $\mu_1(u_j), \dots, \mu_n(u_j)$, $\mu_0(u_j)$, and $\nu_0(u_j)$; then, it remains to find the value of every $m_j(A)$ with $2 \leq |A| < n$ so that

Table 16
Definition of the elements of O' of Example 17.

u	$\mu_0(u)$	$\nu_0(u)$	$\mu_1(u)$	$\nu_1(u)$	$\mu_2(u)$	$\nu_2(u)$	$\mu_3(u)$	$\nu_3(u)$
u_1	0.3	0.1	0	0.7	0.1	0.5	0.1	0.5
u_2	0.1	0.2	0	0.3	0	0.3	0.1	0.5
u_3	0.1	0.2	0	0.3	0.1	0.5	0	0.3
u_4	0.4	0.3	0	0.5	0	0.5	0	0.5

1. $\sum_{\{A \mid C_1 \in A\}} m_j(A) = \mu_1(u_j) + h_1(u_j)$, …, $\sum_{\{A \mid C_n \in A\}} m_j(A) = \mu_n(u_j) + h_n(u_j)$ (from (25)) and
2. $\sum_{A \subseteq C} m_j(A) = 1$ (m_j must be a bba).

Therefore, if we consider a variable x^j_A for each $\emptyset \subset A \subset C$, then the values of $\{m_j(A) \mid 2 \leq |A| < n\}$ form a solution of the following system:

$$S_j = \begin{cases} \mu_1(u_j) + \sum_{\{A \mid \{C_1\} \subset A \subset C\}} x^j_A + \nu_0(u_j) = \mu_1(u_j) + h_1(u_j), \\ \vdots \\ \mu_n(u_j) + \sum_{\{A \mid \{C_n\} \subset A \subset C\}} x^j_A + \nu_0(u_j) = \mu_n(u_j) + h_n(u_j), \\ \mu_0(u_j) + \mu_1(u_j) + \ldots + \mu_n(u_j) + \sum_{\{A \mid 2 \leq |A| < n\}} x^j_A + \nu_0(u_j) = 1. \end{cases} \tag{30}$$

Of course, S_j can be rewritten as

$$S_j = \begin{cases} \sum_{\{A \mid \{C_1\} \subset A \subset C\}} x^j_A = h_1(u_j) - \nu_0(u_j), \\ \vdots \\ \sum_{\{A \mid \{C_n\} \subset A \subset C\}} x^j_A = h_n(u_j) - \nu_0(u_j), \\ \sum_{\{A \mid 2 \leq |A| < n\}} x^j_A = 1 - (\mu_0(u_j) + \mu_1(u_j) + \ldots + \mu_n(u_j)) - \nu_0(u_j). \end{cases} \tag{31}$$

From now on, let $A = \{C_{i_1}, \ldots, C_{i_k}\}$ be a subset of C, we use the symbol $x^j_{i_1 \ldots i_k}$ to indicate the variable assigned to A and u_j. For instance, in Example 2, x^2_{12} is the variable corresponding to $\{C_1, C_2\}$ and u_2.

It is easy to notice that S_j is a linear system with $n+1$ equations and $2^n - n - 2$ variables by considering that

- 2^n is the number of subsets of C (the variables of S_j are associated with subsets of C, that is $A \mapsto x^j_A$);
- n is the cardinality of C ($m_j(\{C_1\}), \ldots, m_j(\{C_n\})$ are already determined by $\mu_1(u_j), \ldots, \mu_n(u_j)$, respectively);
- 2 is the cardinality of $\{\emptyset, C\}$ ($\mu_j(\emptyset)$ and $\mu_j(C)$ are already determined by $\mu_0(u_j)$ and $\nu_0(u_j)$).

Since S_j is a system of linear equations, it has zero, one or an infinite number of solutions. Then, we can immediately deduce that given $O \in \mathcal{O}_G$, the class $\mathcal{F}(O)$ can be empty, or made of one or infinitely many credal partitions.

In the rest of this subsection, we deeply investigate the cardinality of $\mathcal{F}(O)$.

Let us consider the case $n = 3$.

Remark 11. If $n = 3$ then, S_j is made of 4 equations and 3 variables, then it has a unique solution or none. As a consequence, let $O = \{(\mu_0, \nu_0), (\mu_1, \nu_1), (\mu_2, \nu_2), (\mu_3, \nu_3)\}$, then $\mathcal{F}(O) = \emptyset$ or $\mathcal{F}(O) = \{m\}$.

Example 17. Consider the fuzzy orthopartition O of Example 2. Then, the system

$$S_1 = \begin{cases} x^1_{12} + x^1_{13} = 0.5; \\ x^1_{12} + x^1_{23} = 0.8; \\ x^1_{13} + x^1_{23} = 0.8; \\ x^1_{12} + x^1_{23} + x^1_{13} = 0.7, \end{cases} \tag{32}$$

associated with m_1 has no solution. Therefore, $\mathcal{F}(O) = \emptyset$.

On the other hand, Table 16 defines a fuzzy orthopartition O' so that $|\mathcal{F}(O')| = 1$.

Indeed, the linear systems related to u_1, u_2, u_3, u_4 are respectively the following and all of them have a unique solution.

$$S_1 = \begin{cases} x^1_{12} + x^1_{13} = 0.2; \\ x^1_{12} + x^1_{23} = 0.3; \\ x^1_{13} + x^1_{23} = 0.3; \\ x^1_{12} + x^1_{23} + x^1_{13} = 0.4. \end{cases} \qquad S_2 = \begin{cases} x^2_{12} + x^2_{13} = 0.5; \\ x^2_{12} + x^2_{23} = 0.5; \\ x^2_{13} + x^2_{23} = 0.2; \\ x^2_{12} + x^2_{23} + x^2_{13} = 0.6. \end{cases} \tag{33}$$

Table 17
Definition of the elements of m' of Example 17.

A	$m'_1(A)$	$m'_2(A)$	$m'_3(A)$	$m'_4(A)$
$\emptyset$	0.3	0.1	0.1	0.4
$\{C_1\}$	0	0	0	0
$\{C_2\}$	0.1	0	0.1	0
$\{C_3\}$	0.1	0.1	0	0
$\{C_1, C_2\}$	0.1	0.4	0.1	0.1
$\{C_1, C_3\}$	0.1	0.1	0.4	0.1
$\{C_2, C_3\}$	0.2	0.1	0.1	0.1
C	0.1	0.2	0.2	0.3

$$S_3 = \begin{cases} x^3_{12} + x^3_{13} = 0.5; \\ x^3_{12} + x^3_{23} = 0.2; \\ x^3_{13} + x^3_{23} = 0.5; \\ x^3_{12} + x^3_{23} + x^3_{13} = 0.6. \end{cases} \qquad S_4 = \begin{cases} x^4_{12} + x^4_{13} = 0.2; \\ x^4_{12} + x^4_{23} = 0.2; \\ x^4_{13} + x^4_{23} = 0.2; \\ x^4_{12} + x^4_{23} + x^4_{13} = 0.3. \end{cases} \tag{34}$$

Therefore, $\mathcal{F}(O') = \{m'\}$, where $m' = \{m'_1, m'_2, m'_3, m'_4\}$ is defined by Table 17.

Remark 12. Let us notice that $\mathcal{F}(O)$ can be empty also when each system in $\{S_1, \dots, S_l\}$ is consistent but the solution of at least one of them is negative: the values that we find must represent masses of belief, hence, they have to belong to $[0, 1]$.

Now, we examine the case $n \geq 4$.

Proposition 4. *Let O be a generalized fuzzy orthopartition such that $|O| \geq 5$ (namely, $n \geq 4$). Then, the system S_j given by (30) has an infinite number of solutions, for each $j \in \{1, \dots, l\}$.*

Proof. We use some results of linear algebra [18]. First of all, let us prove that S_j is consistent. By *Rouché–Capelli* theorem, S_j is consistent if and only if "*the rank of its coefficient matrix A_n denoted with $rank(A_n)$ is equal to the rank of its augmented matrix*". This is always true when $rank(A_n)$ is maximum, namely $rank(A_n) = n + 1$ because S_j has $n + 1$ equations and $2^n - n - 2$ variables, and $n + 1 < 2^n - n - 2$, for each $n \geq 4$.

By induction, we want to prove that $rank(A_n) = n + 1$ for each $n \geq 4$.

(Base case) Let us show that $rank(A_n) = n + 1$ for $n = 4$. In this case, we have

$$S_j = \begin{cases} x_{12} + x_{13} + x_{14} + x_{123} + x_{124} + x_{134} = a; \\ x_{12} + x_{23} + x_{24} + x_{123} + x_{234} + x_{124} = b; \\ x_{13} + x_{23} + x_{34} + x_{123} + x_{134} + x_{234} = c; \\ x_{14} + x_{24} + x_{34} + x_{124} + x_{234} + x_{134} = d; \\ x_{12} + x_{13} + x_{14} + x_{23} + x_{24} + x_{34} + x_{123} + x_{124} + x_{134} + x_{234} = e, \end{cases} \tag{35}$$

where a, b, c, d, and e are given as in (31).

Thus, the coefficient matrix A_4 of S_j is the following and $rank(A_4) = 5$.

$$A_4 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \tag{36}$$

(Induction step) Suppose that $rank(A_n) = n + 1$ for $n > 4$. Let us prove that $rank(A_{n+1}) = n + 2$.

By the inductive hypothesis, $rank(A_n) = n + 1$. Then, we can find a minor A^*_n of A_n of order $n + 1$ such that its determinant $det(A^*_n)$ is not zero. So, the matrix A^*_n can be written as

$$A^*_n = \begin{pmatrix} a_{11} & \dots & a_{1(n+1)} \\ \vdots & & \vdots \\ a_{(n+1)1} & \dots & a_{(n+1)(n+1)} \end{pmatrix} \tag{37}$$

where its elements are the coefficients of the variables of S_j given by (31), which belong to $\bigcup_{i=1}^{n} \{x^j_A \mid \{C_i\} \subset A \subset \{C_1, \dots, C_n\}\}$.

Now, let us consider the matrix A_{n+1}. Observe that the system related to A_{n+1} can be constructed starting from S_j as follows.

Table 18
Definition of the elements of O of Example 18.

$\mu_0(u_1)$	$\nu_0(u_1)$	$\mu_1(u_1)$	$\nu_1(u_1)$	$\mu_2(u_1)$	$\nu_2(u_1)$	$\mu_3(u_1)$	$\nu_3(u_1)$	$\mu_4(u_1)$	$\nu_4(u_1)$
0.1	0	0.1	0.2	0	0.5	0	0.4	0	0.7

- for each $i \in \{1,\dots,n\}$, $\sum_{\{A \mid \{C_i,C_{n+1}\}\subseteq A\subset\{C_1,\dots,C_n,C_{n+1}\}\}} x^j_A$ is added to the first member of Equation i.
- $\sum_{\{A \mid C_{n+1}\subset A\subseteq\{C_1,\dots,C_n,C_{n+1}\}\}} x^j_A$ is added to the first member of Equation $n+1$.
- Equation

$$\sum_{\{A \mid \{C_{n+1}\}\subset A\subset\{C_1,\dots,C_n,C_{n+1}\}\}} x^j_A = h_{n+1}(u_j) - \nu_0(u_j) \tag{38}$$

is added to S_j.

As a consequence, A_{n+1} can be obtained adding one row and several columns to A_n, which regard the variables of

$$\{x^j_A \mid C_{n+1} \subset A \subset \{C_1,\dots,C_n,C_{n+1}\}\}. \tag{39}$$

Therefore, a minor A^*_{n+1} of A_{n+1} of order $n+2$ is

$$A^*_{n+1} = \begin{pmatrix} a_{11} & \dots & a_{1(n+1)} & c_1 \\ \vdots & & \vdots & \vdots \\ a_{(n+1)1} & \dots & a_{(n+1)(n+1)} & c_{n+1} \\ 0 & \dots & 0 & 1 \end{pmatrix} \tag{40}$$

where $c_1,\dots,c_{n+1}$ are the coefficients related to one of the variables in $\{x^j_A \mid C_{n+1} \subset A \subset \{C_1,\dots,C_n,C_{n+1}\}\}$. By the properties of the determinants of matrices, we know that $det(A^*_{n+1}) = det(A^*_n)$, considering the last row of A^*_{n+1}: the $(n+2,n+2)$ entry is 1 and all other entries are 0. Since $det(A^*_{n+1}) \neq 0$, $rank(A_{n+1}) = n+2$.

So, we have proved that S_j is consistent for each $n \geq 4$. Then, since S_j has more variables than equations, it has infinitely many solutions. □

On one hand, S_j has an infinite number of solutions for $n \geq 4$ from Proposition 4; on the other hand, the solutions of S_j are not always positive. Clearly, when the solutions of S_j are negative they cannot represent masses of belief. For this reason, we need to require that S_j is subjected to the following constraints:

$$x^j_A \geq 0 \quad \forall\, 2 \leq |A| < n.$$

Consequently, supposing that $n \geq 4$, we can say that

- $\mathcal{F}(O) = \emptyset$ if and only if there exists $j \in \{1,\dots,l\}$ such that S_j is "inconsistent (i.e., it has no solution)" or "consistent but with only negative solutions (i.e., if $s_1,\dots,s_k$ form a solution of S_j, then there exists $s \in \{s_1,\dots,s_k\}$ such that $s < 0$)".
- $\mathcal{F}(O)$ is infinite if and only if "$S_1,\dots,S_l$ are consistent and have infinitely many solutions, which are non-negative (i.e., for each $S_j \in \{S_1,\dots,S_l\}$, there exists a solution $s_1,\dots,s_k$ of S_j such that $s_1,\dots,s_k \geq 0$)".

In the sequel, we provide an example of generalized fuzzy orthopartition corresponding to a class of infinitely many credal partitions.

Example 18. Let O be the generalized fuzzy orthopartition of $\{u_1,u_2,u_3,u_4,u_5,u_6\}$ defined by Table 18. For convenience, we suppose that $(\mu_i(u_1),\nu_i(u_1)) = (\mu_i(u_j),\nu_i(u_j))$ for each $i \in \{0,1,2,3,4\}$ and for each $j \in \{2,3,4,5,6\}$.

In order to determine the credal partitions of $\mathcal{F}(O)$, we need to find the positive solutions of the system

$$S_1 = \begin{cases} x_{12}+x_{13}+x_{14}+x_{123}+x_{124}+x_{134} = 0.7; \\ x_{12}+x_{23}+x_{24}+x_{123}+x_{234}+x_{124} = 0.5; \\ x_{13}+x_{23}+x_{34}+x_{123}+x_{134}+x_{234} = 0.6; \\ x_{14}+x_{24}+x_{34}+x_{124}+x_{234}+x_{134} = 0.3; \\ x_{12}+x_{13}+x_{14}+x_{23}+x_{24}+x_{34}+x_{123}+x_{124}+x_{134}+x_{234} = 0.8. \end{cases} \tag{41}$$

S_1 has an infinite number of solutions. One of them is represented by

$$x_{12} = x_{134} = 0.2,\ x_{13} = x_{14} = x_{23} = x_{24} = x_{124} = x_{234} = 0,\ x_{34} = 0.1,\ \text{and}\ x_{123} = 0.3. \tag{42}$$

Table 19
Definition of the elements of m_i and m'_i of Example 18.

A	$m_i(A)$	$m'_i(A)$
$\emptyset$	0.1	0.1
$\{C_1\}$	0.1	0.1
$\{C_2\}$	0	0
$\{C_3\}$	0	0
$\{C_4\}$	0	0
$\{C_1, C_2\}$	0.2	0
$\{C_1, C_3\}$	0	0.3
$\{C_1, C_4\}$	0	0
$\{C_2, C_3\}$	0	0
$\{C_2, C_4\}$	0	0
$\{C_3, C_4\}$	0.1	0
$\{C_1, C_2, C_3\}$	0.3	0.2
$\{C_1, C_2, C_4\}$	0	0.2
$\{C_1, C_3, C_4\}$	0.2	0
$\{C_2, C_3, C_4\}$	0	0.1
$\{C_1, C_2, C_3, C_4\}$	0	0

Table 20
Definition of O w.r.t. u_1 in Example 19.

$\mu_0(u_1)$	$\nu_0(u_1)$	$\mu_1(u_1)$	$\nu_1(u_1)$	$\mu_2(u_1)$	$\nu_2(u_1)$	$\mu_3(u_1)$	$\nu_3(u_1)$	$\mu_4(u_1)$	$\nu_4(u_1)$
0.1	0.3	0.2	0.6	0.1	0.6	0.3	0.6	0	0.7

Also, a second one is given by

$$x_{234} = 0.1,\ x_{12} = x_{14} = x_{23} = x_{24} = x_{34} = x_{134} = x_{134} = 0,\ x_{13} = 0.3, \text{ and } x_{123} = x_{124} = 0.2. \tag{43}$$

Since the proposed solutions of S_1 are positive, they can represent the values assumed by bbas. Lastly, two credal partitions of $\mathcal{F}(O)$ are $m = \{m_1, \dots, m_6\}$ and $m' = \{m'_1, \dots, m'_6\}$ such that $m_1 = m_2 = m_3 = m_4 = m_5 = m_6$ and $m'_1 = m'_2 = m'_3 = m'_4 = m'_5 = m'_6$, where m_i and m'_i are defined by Table 19.

The next propositions find some properties of O for which $\mathcal{F}(O) = \emptyset$.

Proposition 5. *Let O be a generalized fuzzy orthopartition with $n \geq 3$. If there exists $i \in \{1, \dots, n\}$ and $u_j \in U$ such that $h_i(u_j) < \nu_0(u_j)$, then $\mathcal{F}(O) = \emptyset$.*

Proof. Let us consider (31). We can observe that if $h_i(u_j) < \nu_0(u_j)$ then $\sum_{\{A \mid \{C_i\} \subset A \subset C\}} x^j_A < 0$. Then, it is trivial that among all values forming a solution of S_j at least one of them must be negative. □

Example 19. We consider a generalized fuzzy orthopartition $O = \{(\mu_0, \nu_0), (\mu_1, \nu_1), (\mu_2, \nu_2), (\mu_3, \nu_3), (\mu_4, \nu_4)\}$ of $\{u_1, \dots, u_{10}\}$. Let us focus on the element u_1, then $\mu_0(u_1)$, $\mu_1(u_1)$, …, $\mu_4(u_1)$ and $\nu_0(u_1), \nu_1(u_1), \dots, \nu_4(u_1)$ are defined by Table 20.

The system corresponding to u_1 is

$$S_1 = \begin{cases} x_{12} + x_{13} + x_{14} + x_{123} + x_{124} + x_{134} = -0.1; \\ x_{12} + x_{23} + x_{24} + x_{123} + x_{234} + x_{124} = 0; \\ x_{13} + x_{23} + x_{34} + x_{123} + x_{134} + x_{234} = -0.2; \\ x_{14} + x_{24} + x_{34} + x_{124} + x_{234} + x_{134} = 0; \\ x_{12} + x_{13} + x_{14} + x_{23} + x_{24} + x_{34} + x_{123} + x_{124} + x_{134} + x_{234} = 0. \end{cases} \tag{44}$$

We can observe that $\mu_1(u_1) + h_1(u_1) = 0.4$, which is less than $\mu_1(u_1) + \nu_0(u_1) = 0.2 + 0.3 = 0.5$. Then, S_1 only has solutions including negative values. In fact, $x_{12} + x_{13} + x_{14} + x_{123} + x_{124} + x_{134} = -0.1$ implies that $x < 0$ for some $x \in \{x_{12}, x_{13}, x_{14}, x_{123}, x_{124}, x_{134}\}$. Therefore, in agreement with the previous proposition, $\mathcal{F}(O) = \emptyset$.

Remark 13. The condition $h_i(u) < \nu_0(u)$ given by the previous proposition is equivalent to $u \in N_i \cap N_0$, when O is made of Boolean functions (recall that each (M_i, N_i) is defined by (6)). As a consequence, $\mathcal{F}(O) = \emptyset$ in case $N_i \cap N_0 = \emptyset$ for some $i \in \{1, \dots, n\}$. Moreover, if O is an orthopartition according to Definition 6, $\mathcal{F}(O) = \emptyset$ whenever we know the class of at least an element that is not an outlier (see that if $u \in M_i$ then $u \in N_0$ and more in general $u \in N_j$ for each $j \neq i$ from Axiom (b) of Definition 6).

Table 21
Definition of O w.r.t. u_1 in Example 21.

$\mu_0(u_1)$	$\nu_0(u_1)$	$\mu_1(u_1)$	$\nu_1(u_1)$	$\mu_2(u_1)$	$\nu_2(u_1)$	$\mu_3(u_1)$	$\nu_3(u_1)$	$\mu_4(u_1)$	$\nu_4(u_1)$
0.1	0.3	0.2	0.2	0.1	0.3	0.3	0.3	0	0.4

Example 20. Consider the orthopartition $\{(M_0, N_0), (M_1, N_1), (M_2, N_2)\} = \{(\emptyset, \{a, b\}), (\{a\}, \{b\}), (\{b\}, \{a\})\}$ of the universe $\{a, b, c\}$. Then, it is equivalent to the fuzzy orthopartition $\{(\mu_0, \nu_0), (\mu_1, \nu_1), (\mu_2, \nu_2)\}$ by means of (14). Let us focus on the element a, we know that a certainly belongs to C_1. Then, we get $h_1(a) = h_2(a) = 0$ and $\nu_0(a) = 1$. Consequently, according to the previous remark, $\mathcal{F}(O) = \emptyset$.

Proposition 6. *Let O be a generalized fuzzy orthopartition with $n \geq 3$. If there exist $i \in \{1, \ldots, n\}$ and $u_j \in U$ such that $h_i(u_j) > 1 - (\mu_0(u_j) + \mu_1(u_j) + \ldots + \mu_n(u_j))$, then $\mathcal{F}(O) = \emptyset$.*

Proof. Focus on (31) and consider the i-th equation and the last equation of S_j. They are respectively

$$\sum_{\{A \mid \{C_i\} \subset A \subset C\}} x_A^j = h_i(u_j) - \nu_0(u_j) \tag{45}$$

and

$$\sum_{\{A \mid 2 \leq |A| < n\}} x_A^j = 1 - (\mu_0(u_j) + \mu_1(u_j) + \ldots + \mu_n(u_j)) - \nu_0(u_j). \tag{46}$$

We can notice that (46) can be written as

$$\sum_{\{A \mid \{C_i\} \subset A \subset C\}} x_A^j + \sum_{\{A \mid C_i \notin A \wedge 2 \leq |A| < n\}} x_A^j = 1 - (\mu_0(u_j) + \mu_1(u_j) + \ldots + \mu_n(u_j)) - \nu_0(u_j). \tag{47}$$

Then, (45) and (47) imply that

$$\sum_{\{A \mid C_i \notin A \wedge 2 \leq |A| < n\}} x_A^j = \nu_i(u_j) - (\mu_0(u_j) + \mu_1(u_j) + \ldots + \mu_n(u_j)). \tag{48}$$

By hypothesis, $1 - h_i(u_j) - (\mu_0(u_j) + \mu_1(u_j) + \ldots + \mu_n(u_j)) < 0$. Since $(1 - h_i(u_j)) = (\mu_i(u_j) + \nu_i(u_j))$, $(\mu_i(u_j) + \nu_i(u_j)) - (\mu_0(u_j) + \mu_1(u_j) + \ldots + \mu_n(u_j)) < 0$. Then, $\nu_i(u_j) - (\mu_0(u_j) + \mu_1(u_j) + \ldots + \mu_n(u_j)) < 0$. Hence, the thesis clearly holds. Indeed, let $\{m_j(A) \mid 2 \leq A < n\}$ be a solution of S_j, then there exists $m_j(A)$ with $C_i \notin A$ such that $m_j(A) < 0$. □

Example 21. We consider a generalized fuzzy orthopartition $O = \{(\mu_0, \nu_0), (\mu_1, \nu_1), (\mu_2, \nu_2), (\mu_3, \nu_3), (\mu_4, \nu_4)\}$ of $\{u_1, \ldots, u_{10}\}$ and we focus on the element u_1. Then, Table 21 defines O w.r.t. u_1.

By Table 21, we can see that $h_1(u_1) = 0.6$ and $1 - (\mu_0(u_1) + \mu_1(u_1) + \mu_2(u_1) + \mu_3(u_1) + \mu_4(u_1)) = 1 - 0.7 = 0.3$. So, in agreement with the previous proposition, it must be true that $\mathcal{F}(O) = \emptyset$.

In fact, the system corresponding to u_1 is

$$S_1 = \begin{cases} x_{12} + x_{13} + x_{14} + x_{123} + x_{124} + x_{134} = 0.3; \\ x_{12} + x_{23} + x_{24} + x_{123} + x_{234} + x_{124} = 0.3; \\ x_{13} + x_{23} + x_{34} + x_{123} + x_{134} + x_{234} = 0.1; \\ x_{14} + x_{24} + x_{34} + x_{124} + x_{234} + x_{134\}} = 0.3; \\ x_{12} + x_{13} + x_{14} + x_{23} + x_{24} + x_{34} + x_{123} + x_{124} + x_{134} + x_{234} = 0.1 \end{cases} \tag{49}$$

By the first and last equations of S_1, we get $x_{23} + x_{24} + x_{34} + x_{234} = -0.2$. Then, if $\{m_1(A) \mid 2 \leq A < n\}$ is a solution of S_1, then at least a value of $\{m_1(\{C_2, C_3\}), m_1(\{C_2, C_4\}), m_1(\{C_3, C_4\}), m_1(\{C_2, C_3, C_4\})\}$ must be negative.

Remark 14. The condition $h_i(u) > 1 - (\mu_0(u) + \mu_1(u) + \ldots + \mu_n(u))$ given by the previous proposition is equivalent to "$u \in B_i$ and $u \in M_j$ with $j \neq i$", in case O is made of Boolean functions (recall that the orthopair (M_i, N_i) is defined by (6)). In addition, when O is an orthopartition according to Definition 6, $u \in B_j \cap M_j$ cannot occur due to Axiom (b).

As explained in Subsection 4.2, a generalized fuzzy orthopartition O is seen as the class C_O made of all its compatible Ruspini partitions. The next theorem shows that the same set of Ruspini partitions is equivalent (by means of the function f given by Theorem 1) to the set of fuzzy probabilistic partitions compatible with a given credal partition $m \in \mathcal{F}(O)$.

Theorem 3. *Let O be a generalized fuzzy orthopartition of U, and let $m \in \mathcal{F}(O)$. Then, $\mathcal{Z}_m = f(\mathcal{Z}_O)$ (equivalently, $\mathcal{Z}_O = f^{-1}(\mathcal{Z}_m)$).*

Proof. If $m' \in \mathcal{Z}_m$ then $m_i(\emptyset) \leq m_i'(\emptyset) \leq 1 - m_i(C)$ and $m_i(\{C_j\}) \leq m_i'(\{C_j\}) \leq \sum_{\{A|C_j \in A\}} m_i(A)$ from Definition 12. Furthermore, $m \in \mathcal{F}(O)$ implies that $m_i(\emptyset) = \mu_0(u_i)$, $m_i(C) = \nu_0(u_i)$, $m_i(\{C_j\}) = \mu_j(u_i)$ and $\sum_{\{A|C_j \in A\}} m_i(A) = \mu_j(u_i) + h_j(u_i)$ from Definition 13. Then, $m' \in \mathcal{Z}_O$ clearly follows from Definition 11. So, we can conclude that $\mathcal{Z}_m \subseteq f(\mathcal{Z}_O)$.

The case $f(\mathcal{Z}_O) \subseteq \mathcal{Z}_m$ is symmetric and omitted. □

Example 22. Consider O and $m \in \mathcal{F}(O)$ defined in Example 18. According to Theorem 3, O and m have the same compatible fuzzy probabilistic partitions: $f(\mathcal{Z}_O) = \mathcal{Z}_m$.

For example, consider m^* such that $m_i^*(\emptyset) = 0.7$, $m_i^*(\{C_1\}) = 0.3$ and $m_i^*(\{C_2\}) = m_i^*(\{C_3\}) = m_i^*(\{C_4\}) = 0$.

Then, we can immediately check that $m^* \in f(\mathcal{C}_O) \cap \mathcal{C}_m$. For instance, we get $m_i(\emptyset) \leq 0.7 \leq 1 - m_i(C)$ and $\mu_0(u_i) \leq 0.7 \leq \mu_0(u_i) + h_0(u_i)$, considering that $m_i(\emptyset) = \mu_0(u_i) = 0.1$ and $1 - m_i(C) = \mu_0(u_i) + h_0(u_i) = 1$.

We close this subsection with the following remark, which compares the correspondence introduced here with that of [15].

Remark 15. In [15], we have studied the correspondence between fuzzy orthopartitions and credal partitions made of normal bbas, where we considered orthopartitions without the IFS (μ_0, ν_0) (equivalently, we supposed the absence of outliers). In that case, the value $m_i(C)$ is unknown, then the system S_i for finding m_i includes an additional variable, which is associated with $m_i(C)$.

5. From a credal partition to a fuzzy orthopartition

In this section, we explain how to associate a generalized fuzzy orthopartition to a given credal partition. The meaning of such correspondence is dual to that exhibited in Subsection 4.3: given a credal partition m, we intend to consider a generalized fuzzy orthopartition O such that $f(\mathcal{Z}_O) = \mathcal{Z}_m$ (equivalently, $f^{-1}(\mathcal{Z}_m) = \mathcal{Z}_O$).

Principally, we achieve the following goals:

- we assign a generalized fuzzy orthopartition O_m to each credal partition m (Definition 14 and Theorem 4);
- we show that in general credal partitions are not fuzzy orthopartitions by means of Definition 14 (Example 23 and Remark 16);
- we determine a class of credal partitions that coincide with fuzzy orthopartitions w.r.t. Definition 14 (Theorem 5);
- we discuss the connection between ortho and credal partitions made of Boolean functions (Remark 19);
- we use our results to interpret generalized fuzzy orthopartitions in terms of mass functions (Remark 20).

The following definition assigns a generalized fuzzy orthopartition to each credal partition.

Definition 14. Let $m \in \mathcal{M}$. Then, we consider

$$O_m = \{(\mu_0, \nu_0), (\mu_1, \nu_1), \ldots, (\mu_n, \nu_n)\} \tag{50}$$

such that let $u_i \in \{u_1, \ldots, u_l\}$,

(a) $\mu_0(u_i) = m_i(\emptyset)$ and $\nu_0(u_i) = m_i(C)$;
(b) $\mu_j(u_i) = m_i(\{C_j\})$ and $\nu_j(u_i) = 1 - \sum_{\{A \mid C_j \in A\}} m_i(A)$, for each $C_j \in \{C_1, \ldots, C_n\}$.

The next theorem shows that O_m assigned to $m \in \mathcal{M}$ by Definition 14 is a generalized fuzzy orthopartition.

Theorem 4. *Let $m \in \mathcal{M}$, then O_m is a generalized fuzzy orthopartition.*

Proof. First of all, we need to verify that each $(\mu_j, \nu_j) \in O_m$ is an IFS. By Definition 14, for each $j \in \{1, \ldots, n\}$, we write

$$\mu_j(u_i) + \nu_j(u_i) = m_i(C_j) + 1 - \sum_{\{A \mid C_j \in A\}} m_i(A),$$

which is equal to

$$m_i(C_j) + 1 - m_i(C_j) - \sum_{\{A \mid \{C_j\} \subset A\}} m_i(C_j) = 1 - \sum_{\{A \mid \{C_j\} \subset A\}} m_i(C_j).$$

Since m_i is a bba, $0 \leq \sum_{\{A \mid \{C_j\} \subset A\}} m_i(C_j) \leq 1$. Hence, we can deduce that

$$0 \leq \mu_j(u_i) + \nu_j(u_i) \leq 1.$$

Consequently, by Definition 3, (μ_j, ν_j) is an IFS for each $j \in \{1, \ldots, n\}$.

It is clear that (μ_0, ν_0) is an IFS too. In fact, by Definition 14, $\mu_0(u_i) + \nu_0(u_i) = m_i(\emptyset) + m_i(C)$, which is less than or equal to 1 because m_i is a bba.

Then, we intend to prove that the axioms of Definition 9 hold for O_m.

Table 22
Definition of the elements of O_m of Example 23.

u	$\mu_0(u)$	$\nu_0(u)$	$\mu_1(u)$	$\nu_1(u)$	$\mu_2(u)$	$\nu_2(u)$	$\mu_3(u)$	$\nu_3(u)$
u_1	0.2	0.2	0	0.5	0.3	0.3	0	0.7
u_2	0	0.5	0.1	0.4	0.3	0.2	0.1	0.4
u_3	0	0.6	0.1	0.2	0	0.1	0	0.2
u_4	0.1	0	0.1	0.7	0.1	0.5	0.2	0.4

a) Let $u_i \in U$. By hypothesis, m_i is a bba. Then, $m_i(\emptyset) + m_i(\{C_1\}) + \ldots + m_i(\{C_n\}) \leq 1$. Thus, $\mu_0(u_i) + \mu_1(u_i) + \ldots + \mu_n(u_i) \leq 1$ by considering that $\mu_0(u_i) = m_i(\emptyset)$ and $\mu_j(u_i) = m_i(\{C_j\})$ for each $C_j \in \{C_1, \ldots, C_n\}$. Then, Property (a) of Definition 9 holds for O_m.

b) Let $u_i \in U$. Then, $(1-\nu_0(u_i)) + (1-\nu_1(u_i)) + \ldots + (1-\nu_n(u_i))$ is equal to

$$(1 - m_i(C)) + \sum_{\{A \mid C_1 \in A\}} m_i(A) + \ldots + \sum_{\{A \mid C_n \in A\}} m_i(A) \tag{51}$$

due to Definition 14. We can observe that $C \in \{A \mid C_i \in A\}$ for each $C_i \in \{C_1, \ldots, C_n\}$. As a consequence, we can rewrite (51) as

$$(1 - m_i(C)) + \left(\sum_{\{A \mid \{C_1\} \subseteq A \subset C\}} m_i(A) + m_i(C) \right) + \sum_{\{A \mid C_2 \in A\}} m_i(A) + \ldots + \sum_{\{A \mid C_n \in A\}} m_i(A) = 1 + \sum_{\{A \mid \{C_1\} \subseteq A \subset C\}} m_i(A) + \sum_{\{A \mid C_2 \in A\}} m_i(A) + \ldots + \sum_{\{A \mid C_n \in A\}} m_i(A). \tag{52}$$

Of course (52) is greater than or equal to 1, considering that each value of $\{m_i(A) \mid A \in 2^C\}$ appears among the addends of (52) at least once and m_i is a bba. Then, Property (b) of Definition 9 holds for O_m. □

Example 23. Let us consider the credal partition of Example 1. Then, O_m is defined by Table 22.

Indeed, by Definition 14, we can easily verify that

- $\mu_0(u_1) = m_1(\emptyset) = 0.2$ and $\nu_0(u_1) = m_1(C) = 0.2$;
- $\mu_1(u_1) = m_1(\{C_1\}) = 0$ and $\nu_1(u_1) = 1 - \{m_1(\{C_1\}) + m_1(\{C_1, C_2\}) + m_1(\{C_1, C_3\}) + m_1(C)\} = 1 - \{0 + 0.2 + 0.1 + 0.2\} = 0.5$;
- $\mu_2(u_1) = m_2(\{C_2\}) = 0.3$ and $\nu_2(u_1) = 1 - \{m_1(\{C_2\}) + m_1(\{C_1, C_2\}) + m_1(\{C_2, C_3\}) + m_1(C)\} = 1 - \{0.3 + 0.2 + 0 + 0.2\} = 0.3$;
- $\mu_3(u_1) = m_2(\{C_3\}) = 0$ and $\nu_3(u_1) = 1 - \{m_1(\{C_3\}) + m_1(\{C_1, C_3\}) + m_1(\{C_2, C_3\}) + m_1(C)\} = 1 - \{0 + 0.1 + 0 + 0.2\} = 0.7$.

Analogously, we can calculate the values assumed by $\mu_0, \mu_1, \mu_2, \mu_3$ and $\nu_0, \nu_1, \nu_2, \nu_3$ on u_2, u_3, and u_4.

Clearly, O_m is a generalized fuzzy orthopartition of $\{u_1, u_2, u_3, u_4\}$:

Axiom (a) of Definition 9 holds for O_m: $\sum_{i=0}^{3} \mu_i(u_1) = 0.2 + 0 + 0.3 + 0 = 0.5 \leq 1$, $\sum_{i=0}^{3} \mu_i(u_2) = 0 + 0.1 + 0.3 + 0.1 = 0.5 \leq 1$, $\sum_{i=0}^{3} \mu_i(u_3) = 0 + 0.1 + 0 + 0 = 0.1 \leq 1$, and $\sum_{i=0}^{3} \mu_i(u_4) = 0.1 + 0.1 + 0.1 + 0.2 = 0.5 \leq 1$;

Axiom (b) of Definition 9 holds for O_m $\sum_{i=0}^{3}(\mu_i(u_1) + h_i(u_1)) = 0.8 + 0.5 + 0.7 + 0.3 = 2.3 \geq 1$, $\sum_{i=0}^{3}(\mu_i(u_2) + h_i(u_2)) = 0.5 + 0.6 + 0.8 + 0.6 = 2.5 \geq 1$, $\sum_{i=0}^{3}(\mu_i(u_3) + h_i(u_3)) = 0.4 + 0.8 + 0.9 + 0.8 = 2.9 \geq 1$, and $\sum_{i=0}^{3}(\mu_i(u_4) + h_i(u_4)) = 1 + 0.3 + 0.5 + 0.6 = 2.4 \geq 1$.

Remark 16. In general, O_m is not a fuzzy orthopartition according to Definition 4. Indeed, we can see that O_m given by Table 22, does not satisfy Axiom (b) of Definition 4. For example, we get $h_0(u_4) = 0.9$ and $\mu_3(u_4) = 0.2$, hence $\mu_3(u_4) + h_0(u_4) > 1$.

In order to provide an example of credal partition m' so that Axiom (d) of Definition 4 does not hold for $O_{m'}$, we suppose that $C = \{C_1, C_2\}$ and $U = \{u_1, u_2, u_3\}$. Then, we take into account $m' = \{m'_1, m'_2, m'_3\} \in \mathcal{M}$ such that $m'_1 = m'_2 = m'_3$ and m'_1 is defined as follows:

$$m'_1(\emptyset) = 0.4, m'_1(\{C_1\}) = 0.4, m'_1(\{C_2\}) = 0.2, \text{ and } m'_1(A) = 0 \text{ for each } A \in \{\{C_1, C_2\}, \{C_1, C_3\}, \{C_2, C_3\}, C\}. \tag{53}$$

By Definition 14, $O_{m'} = \{(\mu'_0, \nu'_0), (\mu'_1, \nu'_1), (\mu'_2, \nu'_2)\}$, where for each $u_j \in U$,

$$\mu'_0(u_j) = 0.4, \nu'_0(u_j) = 0, \mu'_1(u_j) = 0.4, \nu'_1(u_j) = 0.6, \mu'_2(u_j) = 0.2, \text{ and } \nu'_2(u_j) = 0.8. \tag{54}$$

So, we can notice that $h'_0(u_j) > 0$, but $h'_1(u_j) = h'_2(u_j) = 0$. So, the latter statement proves that Axiom (d) of Definition 4 is not satisfied by $O_{m'}$.

In the sequel, we determine the class of all credal partitions of $\mathcal{M}$ that can be identified with a generalized fuzzy orthopartition satisfying Axiom (b) of Definition 4.

Lemma 1. *Let $m \in \mathcal{M}$, O_m satisfies Axiom (b) of Definition 13 if and only if $m_i(\{C_j\}) \leq m_i(\emptyset) + m_i(C)$ $\forall\, i \in \{1, \ldots, l\}$ and $\forall\, j \in \{1, \ldots, n\}$.*

Proof. (⇐). By hypothesis, let $i \in \{1,\dots,n\}$ and $j \in \{1,\dots,l\}$, $\mu_j(u_i) + h_0(u_i) \leq 1$. Hence, by Definition 14, $\mu_j(u_i) = m_i(\{C_j\})$ and $h_0(u_i) = 1 - m_i(\emptyset) - m_i(C)$. Then, $m_i(\{C_j\}) + 1 - m_i(\emptyset) - m_i(C) \leq 1$, which implies that $m_i(\{C_j\}) \leq m_i(\emptyset) + m_i(C)$.

(⇒). We need to prove that $\mu_j(u_i) + h_k(u_i) \leq 1$ for each $j,k \in \{0,1,\dots,n\}$ such that $j \neq k$. We separately consider three cases:

1. Suppose that $j,k \in \{1,\dots,n\}$, then $\mu_j(u_i) = m_i(\{C_j\})$ and $h_k(u_i) = \sum_{\{A \in 2^C \mid \{C_j\} \subset A\}} m_i(A)$. Since m_i is a bba, $\sum_{A \in 2^C} m_i(A) = 1$. As a consequence, we get $m_i(\{C_j\}) + \sum_{\{A \in 2^C \mid \{C_j\} \subset A\}} m_i(A) \leq 1$.
2. Suppose that $j = 0$ and $k \in \{1,\dots,n\}$, then $\mu_0(u_i) = m_i(\emptyset)$ and $h_k(u_i) = \sum_{\{A \in 2^C \mid \{C_j\} \subset A\}} m_i(A)$. Analogously to the previous case, $m_i(\emptyset)+$ $\sum_{\{A \in 2^C \mid \{C_j\} \subset A\}} m_i(A) \leq 1$ because m_i is a bba.
3. Suppose that $j \in \{1,\dots,n\}$ and $k = 0$, then $\mu_j(u_i) = m_i(\{C_j\})$ and $h_0(u_i) = 1 - m_i(\emptyset) - m_i(C)$. Then, $\mu_j(u_i) + h_0(u_i) = m_i(\{C_j\}) + 1 - m_i(\emptyset) - m_i(C)$. Also, by hypothesis, $m_i(\{C_j\}) \leq m_i(\emptyset) + m_i(C)$. Hence, $m_i(\{C_j\}) + 1 - m_i(\emptyset) - m_i(C) \leq 1$. □

The next lemma provides the class of all credal partitions of $\mathcal{M}$ that can be identified with a generalized fuzzy orthopartition satisfying Axiom (d) of Definition 4.

Lemma 2. *Let $m \in \mathcal{M}$. Then, O_m satisfies Axiom (d) of Definition 4 if and only if for each $u_i \in U$, "$m_i(\emptyset) = 1$" or "there exists $A \in 2^C$ such that $|A| \geq 2$ and $m_i(A) > 0$".*

Proof. (⇒). Let $u_i \in U$, assuming that Axiom (d) is satisfied, we can separately analyze two complementary cases:

1. "$h_j(u) = 0\ \forall j \in \{0,\dots,n\}$" and
2. "there exists $j,k \in \{0,\dots,n\}$ such that $j \neq k$" and $h_j(u_i), h_k(u_i) > 0$.

1. First of all, $h_0(u_i) = 0$ means that $m_i(\emptyset) + m_i(C) = 1$. Then, since m_i is a bba,
$$m_i(A) = 0 \text{ for each } A \in 2^C \setminus \{\emptyset, C\}. \tag{55}$$
Let $j \in \{1,\dots,n\}$, then $\mu_j(u_i) = m_i(\{C_j\}) = 0$ from (55). Moreover, $h_j(u_i) = 0$ by hypothesis. Since $\nu_j(u_i) = 1 - (\mu_j(u_i) + h_j(u_i))$, it must be true that $\nu_j(u_i) = 1$. By Definition 14, $\sum_{\{A \mid C_j \in A\}} m_i(A) = 0$. As a consequence, $m_i(C) = 0$, and finally, $m_i(\emptyset) = 1$.
2. If $j,k \in \{0,\dots,n\}$ and $j \neq k$, then $j \in \{1,\dots,n\}$ or $k \in \{1,\dots,n\}$. Suppose that $j \in \{1,\dots,n\}$, then $h_j(u_i) > 0$. Consequently, it is true the inequality $\sum_{\{A \mid \{C_j\} \subset A\}} m_i(A) > 0$. This implies the existence of $A \in 2^C$ such that $|A| \geq 2$ (because it strictly includes $\{C_j\}$) and $m_i(A) > 0$.

(⇐). If $m_i(\emptyset) = 1$, then $m_i(A) = 0$ for each $A \neq \emptyset$. Thus, $\mu_j(u_i) = m_i(\{C_j\}) = 0$ and $\nu_j(u_i) = 1 - \sum_{\{A \mid \{C_j\} \subseteq A\}} m_i(A) = 1 - 0 = 1$. Hence, $h_j(u_i) = 0$ for each $j \in \{1,\dots,n\}$.

If $h_j(u_i) > 0$ with $j \in \{1,\dots,l\}$ then $\sum_{\{A \mid \{C_j\} \subset A\}} m_i(A) > 0$. So, we can consider $|\tilde{A}| \geq 2$ containing C_i so that $m_i(\tilde{A}) > 0$. Let $C_k \in \tilde{A}$, it is easy to understand that $h_k(u_i) > 0$.

If $h_0(u) > 0$, then $1 - m_i(\emptyset) - m_i(C) > 0$. Hence, $m_i(\emptyset) + m_i(C) < 1$. Thus, we know that $m_i(\emptyset) < 1$. By hypothesis, there exists $A \in 2^C$ such that $|A| \geq 2$ and $m_i(A) > 0$. Then, let $C_j \in A$, we can immediately observe that $h_j(u_i) = \sum_{\{A \mid \{C_j\} \subset A\}} m_i(A) > 0$. □

The following theorem shows a sufficient and necessary condition for a credal partition to correspond to a fuzzy orthopartition w.r.t. Definition 14.

Theorem 5. *Let $m \in \mathcal{M}$, O_m is a fuzzy orthopartition if and only if one of the following properties holds for each $m_i \in m$:*

(a) $m_i(\emptyset) + m_i(C) = 1$;
(b) $m_i(\{C_j\}) \leq m_i(\emptyset) + m_i(C)\ \forall j \in \{1,\dots,n\}$ *and there exists* $|A| \geq 2$ *such that* $m_i(A) > 0$.

Proof. Firstly, notice that $m_i(\emptyset) + m_i(C) = 1$ implies that $m_i(\{C_j\}) = 0$ for each $j \in \{1,\dots,n\}$. Hence, the inequality $m_i(\{C_j\}) \leq m_i(\emptyset) + m_i(C)$ trivially holds.

(⇒). Let $m \in \mathcal{M}$ such that O_m satisfies Axioms (b) and (d) of Definition 4. By Lemma 2, $m_i(\emptyset) + m_i(C) = 1$ or there exists $|A| \geq 2$ such that $m_i(A) > 0$. Furthermore, by Lemma 1, $m_i(\{C_j\}) \leq m_i(\emptyset) + m_i(C)$.

(⇐). By Theorem 4, O_m is a generalized fuzzy orthopartition. Then, Axioms (a) and (c) of Definition 13 are satisfied from O_m. The other axioms are clearly consequences of Lemmas 1 and 2. □

According to the previous theorem, a credal partition can be identified with a fuzzy orthopartition when it is composed by special bbas: "non-zero masses of belief are assigned only to $\emptyset$ and C", or "the mass of belief of each individual class is less than or equal to the sum of the masses of belief of $\emptyset$ and C and each bba is not Bayesian".

Table 23
Definition of the elements of m of Example 24.

A	$m_1(A)$	$m_2(A)$	$m_3(A)$	$m_4(A)$
$\emptyset$	0.2	0.2	0.4	1
$\{C_1\}$	0	0.1	0.1	0
$\{C_2\}$	0	0.3	0	0
C	0.8	0.4	0.5	0

Example 24. Consider the credal partition defined by Table 23.

Then, $O_m = \{(\mu_0, \nu_0), (\mu_1, \nu_1), (\mu_2, \nu_2)\}$ is defined as follows:

- $(\mu_0(u_1), \nu_0(u_1)) = (0.2, 0.8), (\mu_1(u_1), \nu_1(u_1)) = (0.2, 0), (\mu_2(u_1), \nu_2(u_1)) = (0.2, 0)$,
- $(\mu_0(u_2), \nu_0(u_2)) = (0.2, 0.4), (\mu_1(u_2), \nu_1(u_2)) = (0.1, 0.5), (\mu_2(u_2), \nu_2(u_2)) = (0.3, 0.3)$,
- $(\mu_0(u_3), \nu_0(u_3)) = (0.4, 0.5), (\mu_1(u_3), \nu_1(u_3)) = (0.1, 0.4), (\mu_2(u_3), \nu_2(u_3)) = (0, 0.5)$,
- $(\mu_0(u_4), \nu_0(u_4)) = (1, 0), (\mu_1(u_4), \nu_1(u_4)) = (0, 1), (\mu_2(u_4), \nu_2(u_4)) = (0, 1)$.

We can view that m_1 and m_4 verify Property (a) of Theorem 5, while m_2 and m_3 verify Property (b) of Theorem 5.

Then, according to Theorem 5, O_m is a fuzzy orthopartition, hence it satisfies all axioms of Definition 4.

Remark 17. Recall that, despite Theorem 5, fuzzy orthopartitions and credal partitions coincide when they are respectively Ruspini partitions and fuzzy probabilistic partitions (see Subsection 2.3).

Remark 18. Assume that $m = \{m_1, \dots, m_l\}$ is made of Boolean functions. Then, by the previous theorem,

O_m is a fuzzy orthopartition if and only if "$m_i(A) = 1$ with $|A| \neq 1$".

By (6), O_m is equivalent to the collection of orthopairs $\{(M_0, N_0), (M_1, N_1), \dots, (M_n, N_n)\}$, which is an orthopartition if and only if $\forall u_j \in U$ one of the following holds:

- $u_j \in M_0$ and $u_j \in N_i$ for each $i \in \{1, \dots, n\}$, when $m_j(\emptyset) = 1$;
- $u_j \in N_0$ and $u_j \in B_i$ for each $i \in \{1, \dots, n\}$, when $m_j(C) = 1$;
- $u_j \in B_0$, $u_j \in B_i$ for each $C_i \in A$, and $u_j \in N_i$ for each $C_i \notin A$, when $m_j(A) = 1$.

In the sequel, we connect fuzzy orthopartitions and credal partitions provided by Definitions 13 and 14.

Theorem 6. *Let O be a generalized fuzzy orthopartition of U and let $m \in \mathcal{F}(O)$. Then, $O_m = O$.*

Proof. The thesis clearly follows from both Definitions 13 and 14. □

In other words, we can start from a generalized fuzzy orthopartition O, consider $\mathcal{F}(O)$ and obtain O again by applying Definition 14 to any credal partition in $\mathcal{F}(O)$. Lastly, it is important to notice that the previous theorem allows us to rewrite Equation (25) as

$$\mathcal{F}(O) = \{m \in \mathcal{M} \mid O_m = O\}$$

and see O_m as a fuzzy orthopartition of U verifying $f(\mathcal{C}_{O_m}) = \mathcal{C}_m$. This result is dual to that provided by Theorem 3) and it is formalized as follows:

Theorem 7. *Let m be a credal partition of U, then $\mathcal{Z}_m = f(\mathcal{Z}_{O_m})$ (equivalently, $f^{-1}(\mathcal{Z}_m) = \mathcal{Z}_{O_m}$).*

Proof. The thesis immediately follows from Theorems 3 and 6. □

Remark 19. Let us focus on the correspondence between credal partitions and orthopartitions based on Definition 4, using the results obtained above. So, we consider the collection $\mathcal{M}'$ of all credal partitions made of Boolean bbas, then each $m \in \mathcal{M}'$ is equivalent to O_m, which can be seen as a collection of orthopairs from (6). Then, the following is true.

- By Theorem 4, O_m is a collection of orthopairs verifying Axioms (a) and (c) of Definition 6. As shown in subsection 5, the vice-versa is not always true.
- By Remark 18, O_m is an orthopartition if and only if $m_i(\{C_j\}) = 0$ for each $i \in \{1, \dots, l\}$ and $j \in \{1, \dots, m\}$.
- On the other hand, by Proposition 2, O_m can be transformed in an orthopartition O'_m, which is equivalent to O_m.

By the previous consideration, we deduce that orthopartitions are more general than credal partitions composed of Boolean bbas.

Let us complete our results with some corollaries.

Corollary 1. *Let O_m be the generalized fuzzy orthopartition of $m \in \mathcal{M}$, then O_m satisfies Property* (26).

Proof. The thesis clearly follows from Remark 9 and the definition of O_m. □

Corollary 2. *Let O_m be the generalized fuzzy orthopartition of $m \in \mathcal{M}$.*

- *If $n = 2$ or $n = 3$, then $|\mathcal{F}(O_m)| = 1$;*
- *If $n \geq 4$, then $|\mathcal{F}(O_m)| = \infty$.*

Proof. The thesis follows from the results obtained in Subsection 4.3. □

As a consequence, if we confine to credal partitions representing partitions with 2 o 3 classes, i.e., $n \in \{2, 3\}$, we can prove that different credal partitions correspond to different fuzzy orthopartitions.

Corollary 3. *Let $m, m' \in \mathcal{M}$ such that $m \neq m'$. If $n \in \{2, 3\}$, then $O_m \neq O_{m'}$.*

Proof. The thesis follows from Corollary 2. □

However, when $n \geq 4$, two different credal partitions could be associated with the same fuzzy orthopartition. The following is an example.

Example 25. We consider the credal partitions $m = \{m_1, \dots, m_6\}$ and $m' = \{m'_1, \dots, m'_6\}$ of $U = \{u_1, \dots, u_6\}$ associated to $C = \{C_1, C_2, C_3, C_4\}$ such that $m_2 = m'_2$, $m_3 = m'_3$, $m_4 = m'_4$, $m_5 = m'_5$, and $m_6 = m'_6$, while $m_1 \neq m'_1$ and are defined by Table 19.

We can observe that $O_m = \{(\mu_0, \nu_0), (\mu_1, \nu_1), (\mu_2, \nu_2), (\mu_3, \nu_3), (\mu_4, \nu_4)\}$ and $O_{m'} = \{(\mu'_0, \nu'_0), (\mu'_1, \nu'_1), (\mu'_2, \nu'_2), (\mu'_3, \nu'_3), (\mu'_4, \nu'_4)\}$ coincide. Of course, we get $(\mu_i(u), \nu_i(u)) = (\mu'_i(u), \nu'_i(u))$ for each $u \in \{u_2, u_3, u_4, u_5, u_6\}$. Concerning u_1, we can easily check that

- $(\mu_0(u_1), \nu_0(u_1)) = (\mu'_0(u_1), \nu'_0(u_1)) = (0.1, 0)$,
- $\mu_1(u_1) = \mu'_1(u_1) = 0.1$,
- $\nu_1(u_1) = 1 - \{m_1(\{C_1\}) + m_1(\{C_1, C_2\}) + m_1(\{C_1, C_3\}) + m_1(\{C_1, C_4\}) + m_1(\{C_1, C_2, C_3\}) + m_1(\{C_1, C_2, C_4\}) + m_1(\{C_1, C_3, C_4\}) + m_1(\{C_1, C_2, C_3, C_4\})\} = 1 - \{0.1 + 0.2 + 0 + 0 + 0.3 + 0 + 0.2\} = 1 - 0.8 = 0.2$,
- $\nu'_1(u_1) = 1 - \{m'_1(\{C_1\}) + m'_1(\{C_1, C_2\}) + m'_1(\{C_1, C_3\}) + m'_1(\{C_1, C_4\}) + m'_1(\{C_1, C_2, C_3\}) + m'_1(\{C_1, C_2, C_4\}) + m'_1(\{C_1, C_3, C_4\}) + m'_1(\{C_1, C_2, C_3, C_4\})\} = 1 - \{0.1 + 0 + 0.3 + 0 + 0.2 + 0.2 + 0\} = 1 - 0.8 = 0.2$,

The other values are similarly calculated to prove that $O_m = O_{m'}$.

By Corollary 3, we can consider an equivalence relation R on $\mathcal{M}$ so that

"let $m, m' \in \mathcal{M},\ m\ R\ m'$ if and only if $O_m = O_{m'}$".

Remark 20. Let $O \in \mathcal{O}_G$ such that $\mathcal{F}(O) \neq \emptyset$. Let us attach a new semantics to O by supposing to deal with masses instead of truth degrees. Thus, O can be understood as a credal partition with an additional level of uncertainty: some masses are known and others are not but need to satisfy particular conditions. O corresponds to one of the credal partitions of $\mathcal{F}(O)$ once all masses are determined. Let $m = \{m_1, \dots, m_l\} \in \mathcal{F}(O)$, what can we say about the masses of m? Surely, for each $u_j \in \{u_1, \dots, u_l\}$ and $i \in \{0, \dots, n\}$, the mass of "u_j belongs to C_i" is known because it coincides with $\mu_i(u_j)$; the mass of "u_j belongs to C" is known because it coincides with $\nu_0(u_j)$. The remaining masses are unknown; on the other hand, for each $u_j \in \{u_1, \dots, u_l\}$, the mass $m_j(\tilde{A})$ where $2 \leq |\tilde{A}| < |C|$ must verify the condition

$$h_k(u_j) = \sum_{\{A \mid \{C_k\} \subset A\}} m_j(A), \tag{56}$$

for each $C_k \in \tilde{A}$.

The following is an illustrative example.

Example 26. Let S be a set of students of secondary school and let $\mathcal{D}$ be a dataset containing the information of a survey of such students. Suppose that starting from $\mathcal{D}$, we can determine the degree course among "$Science$", "$Management$", and "$Engineering$",

Table 24
Definition of the elements of O.

$\mu_{Others}(\bar{s})$	0.1	$\nu_{Others}(\bar{s})$	0.1
$\mu_{Science}(\bar{s})$	0.1	$\nu_{Science}(\bar{s})$	0.6
$\mu_{Management}(\bar{s})$	0.2	$\nu_{Management}(\bar{s})$	0.6
$\mu_{Engineering}(\bar{s})$	0.3	$\nu_{Engineering}(\bar{s})$	0.5

which is close to the inclinations of each student of S. In particular, we can extract from $\mathcal{D}$ a generalized fuzzy orthopartition O of S composed of three clusters $C_{Science}, C_{Management}$, and $C_{Engineering}$ together with an additional one C_{Others} to deal with the students who don't have the aptitude for any of such degree courses. Using the notation of fuzzy orthopartitions, we set

$$C_j = (\mu_j, \nu_j), \text{ where } j \in \{Others, Science, Management, Engineering\}.$$

Now, let us assume that the generalized fuzzy orthopartition O w.r.t. a student $\bar{s} \in S$ is defined by Table 24.

Then, according to Remark 20, O can be viewed as a generalized credal partition. This means that let $x \in \{Science, Management, Engineering\}$,

$\mu_x(\bar{s})$ is the mass that "$\bar{s}$ can attend the degree course in x."

Moreover, $\mu_{Others}(\bar{s})$ and $\nu_{Others}(\bar{s})$ are respectively the masses that

"$\bar{s}$ cannot attend any of the courses among Science, Management, and Engineering"

and

"$\bar{s}$ can attend a degree course among Science, Management, Engineering".

However, the masses related to the other subsets of $\{C_{Science}, C_{Management}, C_{Engineering}\}$ are unknown and by (56), they must verify the following constraints:

$$h_{Science}(\bar{s}) = m_{\bar{s}}(\{C_{Science}, C_{Management}\}) + m_{\bar{s}}(\{C_{Science}, C_{Engineering}\}) + m_{\bar{s}}(\{C_{Science}, C_{Management}, C_{Engineering}\}), \quad (57)$$

$$h_{Management}(\bar{s}) = m_{\bar{s}}(\{C_{Science}, C_{Management}\}) + m_{\bar{s}}(\{C_{Management}, C_{Engineering}\}) + m_{\bar{s}}(\{C_{Science}, C_{Management}, C_{Engineering}\}), \quad (58)$$

$$h_{Engineering}(\bar{s}) = m_{\bar{s}}(\{C_{Science}, C_{Engineering}\}) + m_{\bar{s}}(\{C_{Management}, C_{Engineering}\}) + m_{\bar{s}}(\{C_{Science}, C_{Management}, C_{Engineering}\}), \quad (59)$$

where $h_{Science}(\bar{s}) = 0.5$, $h_{Management}(\bar{s}) = 0.2$, and $h_{Engineering}(\bar{s}) = 0.2$ from Table 24.

Supposing that $\mathcal{F}(O)$ is non-empty, O can specialize in a special credal partition of $\mathcal{F}(O)$, once we have more information about students (for instance, after an interview with the students). As an example, O can become the credal partition of $\mathcal{F}(O)$ including a bba $m_{\bar{s}}$ defined as follows.

Some values of $m_{\bar{s}}$ are derived by Table 24:

- $m_{\bar{s}}(\{C_{Others}\}) = \mu_{Others}(\bar{s}) = 0.1$,
- $m_{\bar{s}}(\{C_{Science}\}) = \mu_{Science}(\bar{s}) = 0.1$,
- $m_{\bar{s}}(\{C_{Management}\}) = \mu_{Management}(\bar{s}) = 0.2$,
- $m_{\bar{s}}(\{C_{Engineering}\}) = \mu_{Engineering}(\bar{s}) = 0.3$,
- $m_{\bar{s}}(\{C_{Science}, C_{Management}, C_{Engineering}\}) = 0.1$,

The remaining values of $m_{\bar{s}}$ satisfy Equations (57), (58), and (59):

- $m_{\bar{s}}(\{C_{Science}, C_{Management}\}) = 0.1$,
- $m_{\bar{s}}(\{C_{Management}, C_{Engineering}\}) = 0$,
- $m_{\bar{s}}(\{C_{Science}, C_{Engineering}\}) = 0.1$.

6. Conclusions and future directions

We explored the links between fuzzy orthopartitions and credal partitions unifying them by introducing the concept of generalized fuzzy orthopartition. In short, both fuzzy-ortho and credal partitions are

- more general than fuzzy probabilistic partitions and
- specific instances of generalized fuzzy orthopartitions.

Among all credal partitions, we identified those that coincide with fuzzy orthopartitions. Our approach is mainly based on the idea that fuzzy-ortho and credal partitions can be seen as classes of fuzzy probabilistic partitions. Indeed, a fuzzy orthopartition, as well as a credal partition, is meant as a fuzzy probabilistic partition with a higher degree of uncertainty.

In the future, we plan to deepen our understanding of fuzzy-ortho and credal partitions using the findings provided in this article. For instance, we will transfer the operations given on fuzzy orthopartitions in [9] to credal partitions: let $\otimes$ be an operation on $\mathcal{O}_G$, we will determine the formula and the meaning of $\otimes_{\mathcal{M}}$ so that $m \otimes_{\mathcal{M}} m' = m^*$ if and only if $O_m \otimes O_{m'} = O_{m^*}$, for each $m, m' \in \mathcal{M}$. Additionally, we will compare the existing measures of uncertainty in both settings. After that, we may extend our work by comparing fuzzy orthopartitions with other generalized partitions, which are called *three-way fuzzy partitions* and introduced in [19]. Also, we intend to define some measures based on Definition 9 to capture how much a given set of IFSs is close to being a generalized orthopartition.

From a more practical perspective, two aspects deserve attention:

- the application of fuzzy orthopartitions to clustering. Boolean orthopartitions are strictly related to rough and three-way clustering [12] and they have been showed useful to define evaluation measures of different soft clustering approaches [20]. We would like to define a similar relationship between fuzzy orthopartitions and intuitionistic fuzzy clusterings. Indeed, several clustering algorithms for intuitionistic fuzzy sets are already proposed in the literature (see [21–25] for some example). Considering that fuzzy orthopartitions are special collections of intuitionistic fuzzy sets, we plan to determine the conditions under which the result of a intuitionistic fuzzy clustering is also a fuzzy orthopartition. The final goal is to define new evaluation measures on intuitionistic fuzzy clusterings.
- the construction of fuzzy orthopartitions from credal partitions. Starting from an existing method to generate credal partitions from data (see [26,1] for some examples), we can transform the obtained credal partitions into fuzzy orthopartitions using Definition 14 or more generally we could extend those methods to build a generic fuzzy orthopartition directly from the given data set.

Of course, both the previous points will produce methods to extract fuzzy orthopartitions from data.

CRediT authorship contribution statement

Stefania Boffa: Writing – review & editing, Writing – original draft, Methodology, Formal analysis, Conceptualization. **Davide Ciucci:** Writing – review & editing, Writing – original draft, Supervision, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgements

This work was partially supported by the MUR under the grant “Dipartimenti di Eccellenza 2023-2027” of the Department of Informatics, Systems and Communication of the University of Milano-Bicocca, Italy.

References

[1] Thierry Denœux, M-H. Masson, Evclus: evidential clustering of proximity data, IEEE Trans. Syst. Man Cybern., Part B, Cybern. 34 (1) (2004) 95–109.
[2] Linqing Huang, Zhunga Liu, Quan Pan, Jean Dezert, Evidential combination of augmented multi-source of information based on domain adaptation, Sci. China Inf. Sci. 63 (2020) 1–14.
[3] Zhun-Ga Liu, Lin-Qing Huang, Kuang Zhou, Thierry Denoeux, Combination of transferable classification with multisource domain adaptation based on evidential reasoning, IEEE Trans. Neural Netw. Learn. Syst. 32 (5) (2020) 2015–2029.
[4] Zuo-wei Zhang, Zhe Liu, Arnaud Martin, Zhun-ga Liu, Kuang Zhou, Dynamic evidential clustering algorithm, Knowl.-Based Syst. 213 (2021) 106643.
[5] Zuowei Zhang, Liangbo Ning, Zechao Liu, Qingyu Yang, Weiping Ding, Mining and reasoning of data uncertainty-induced imprecision in deep image classification, Inf. Fusion 96 (2023) 202–213.
[6] Glenn Shafer, A Mathematical Theory of Evidence, Princeton University Press, 1976.
[7] Philippe Smets, Robert Kennes, The transferable belief model, Artif. Intell. 66 (2) (1994) 191–234.
[8] James C. Bezdek, James Keller, Raghu Krisnapuram, Nikhil Pal, Fuzzy Models and Algorithms for Pattern Recognition and Image Processing, vol. 4, Springer Science & Business Media, 1999.
[9] Stefania Boffa, Davide Ciucci, Logical entropy and aggregation of fuzzy orthopartitions, Fuzzy Sets Syst. 455 (2023) 77–101.
[10] Stefania Boffa, Davide Ciucci, Fuzzy orthopartitions and their logical entropy, in: CEUR Workshop Proceedings, WILF, 2021, p. 3074.
[11] Enrique H. Ruspini, A new approach to clustering, Inf. Control 15 (1) (1969) 22–32.
[12] Andrea Campagner, Davide Ciucci, Orthopartitions and soft clustering: soft mutual information measures for clustering validation, Knowl.-Based Syst. 180 (2019) 51–61.

[13] Stefania Boffa, Davide Ciucci, Orthopartitions and possibility distributions, Fuzzy Sets Syst. 455 (2023) 53–76.
[14] Stefania Boffa, Andrea Campagner, Davide Ciucci, Partially-defined equivalence relations: relationship with orthopartitions and connection to rough sets, Inf. Sci. 657 (2024) 119941.
[15] Stefania Boffa, Davide Ciucci, A correspondence between credal partitions and fuzzy orthopartitions, in: Belief Functions: Theory and Applications: 7th International Conference, BELIEF 2022, Paris, France, October 26–28, 2022, Proceedings, Springer, 2022, pp. 251–260.
[16] Krassimir Atanassov, Intuitionistic fuzzy sets, in: Intuitionistic Fuzzy Sets, Springer, 1999, pp. 1–137.
[17] Davide Ciucci, Orthopairs and granular computing, Granul. Comput. 1 (3) (2016) 159–170.
[18] Igor R. Shafarevich, Alexey O. Remizov, Igor R. Shafarevich, Alexey O. Remizov, Matrices and determinants, in: Linear Algebra and Geometry, 2013, pp. 25–77.
[19] Xue Rong Zhao, Yiyu Yao, Three-way fuzzy partitions defined by shadowed sets, Inf. Sci. 497 (2019) 23–37.
[20] Andrea Campagner, Davide Ciucci, Thierry Denoeux, A general framework for evaluating and comparing soft clusterings, Inf. Sci. 623 (2023) 70–93.
[21] Zeshui Xu, Jian Chen, Junjie Wu, Clustering algorithm for intuitionistic fuzzy sets, Inf. Sci. 178 (19) (2008) 3775–3790.
[22] Zeshui Xu, Junjie Wu, Intuitionistic fuzzy c-means clustering algorithms, J. Syst. Eng. Electron. 21 (4) (2010) 580–590.
[23] Dimitris K. Iakovidis, Nikos Pelekis, Evangelos Kotsifakos, Ioannis Kopanakis, Intuitionistic fuzzy clustering with applications in computer vision, in: Advanced Concepts for Intelligent Vision Systems: 10th International Conference, ACIVS 2008, Juan-les-Pins, France, October 20-24, 2008. Proceedings 10, Springer, 2008, pp. 764–774.
[24] Tamalika Chaira, A novel intuitionistic fuzzy c means clustering algorithm and its application to medical images, Appl. Soft Comput. 11 (2) (2011) 1711–1717.
[25] Dante Mújica-Vargas, Superpixels extraction by an intuitionistic fuzzy clustering algorithm, J. Appl. Res. Technol. 19 (2) (2021) 140–152.
[26] Zhun-ga Liu, Quan Pan, Jean Dezert, Grégoire Mercier, Credal c-means clustering method based on belief functions, Knowl.-Based Syst. 74 (2015) 119–132.