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Risk-adjusted geometric diversified portfolios

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Abstract

In this paper, exploiting a geometric argument, a novel and intuitive approach to portfolio diversification is proposed. The risk-adjusted geometric diversified portfolio is obtained as the point that is equally distant, for a given distance, from the vertices of the simplex, as they represent the single asset portfolios, the worst portfolios in terms of diversification. The definition of risk-adjusted distance as a special case of weighted Euclidean distance permits to introduce the information on the risks of the assets composing the portfolio in a very general way. The closed form solution for the allocation problem is provided and interesting theoretical properties are proved. Further, a direct comparison with Rao's Quadratic Entropy maximization problem is outlined, thus yielding a different perspective to the use of such entropy as a diversification measure. Finally, the effectiveness of our proposal is emphasized through a comprehensive empirical out-of-sample exercise on real financial data.

Keywords Portfolio diversification · Risk-adjusted distance · Weighted Euclidean distance · Asset allocation · Rao's quadratic entropy

Mathematics Subject Classification 91G10 · 52B99

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1 Introduction

In this paper, we propose a novel asset allocation rule which takes into account the specific risk of the assets and whose strategy is based on an intuitive geometrical idea of diversification. In the following, in order to provide an intelligible overview of the paper's content, we briefly recall the fundamental concepts of risk measures and diversification, and introduce the Risk-Adjusted Distances as a special case of the well-known weighted Euclidean distances.

Risk measurement is a leading topic in financial literature. Since the release of the fundamental paper of Artzner et al. (1999), many axiomatic approaches have been proposed in order to define the general theoretical properties a function needs to verify to be considered an eligible risk measure, see for example (Rachev et al. 2008). Among the others, we recall the convex risk measures proposed in Föllmer and Schied (2002), the spectral risk measures presented in Acerbi (2002), the downside risk measures introduced in Sortino and Van der Meer (1991) and the dynamic risk measures discussed in Acciaio and Penner (2011). Each axiomatic class contains an infinite number of risk measures; as a result, the universe of possible risk measures proposed in the literature is deeply intricate and risk measures are strongly interrelated each other, see (Frittelli and Rosazza Gianin 2002).

The concept of diversification in portfolio theory is central and accountable of the popularity of Markowitz model, see (Markowitz 1952), where the idea has been first introduced. Despite its simplicity, no generally accepted unique definition of diversification is available in the literature, giving the rise to the production of many contributions on the topic (we refer to Koumou (2020) for a recent review on the topic). Several different papers deal with the asset allocation problem from the point of view of diversification; among the others, we enumerate the following contributions: in Choueifaty and Coignard (2008) and Choueifaty et al. (2013) the authors propose an allocation rule based on the maximization of the so called *diversification ratio*; in DeMiguel et al. (2009) the authors refer to the Equally Weighted Portfolio as naive diversification and compare its out-of-sample performance to alternative approaches; in Clarke et al. (2013); Maillard et al. (2010); Qian (2006) and Roncalli and Weisang (2016) the Equal Risk Contribution is proposed as the strategy that balances the risk exposure among the assets; in Meucci (2009) the author proposes to use principal component analysis to extract uncorrelated risk factors and diversify the portfolio. For an axiomatic approach to portfolio diversification measures we refer to Koumou and Dionne (2019).

The Risk-Adjusted Distances (RADs) introduced in Sect. 3.1 play a prominent role in our proposal. From a strict mathematical point of view, a RAD is simply a weighted Euclidean distance (see (Dahlquist and Björck 2003)). We observe that in the present formulation the RADs do not take into account possible dependences between assets. Indeed, though in principle it is possible to generalise our approach to handle distances defined through non-diagonal matrices and incorporating assets dependences, such strategy would have required measuring the risk between assets pairs, a topic that, as far as we know, is not well-established in the financial literature and that would have set limits in the choice of employed risk measures. In the context of portfolio theory, the RAD allows to compute the distance between investment portfolios not only in terms of difference in the allocation but also considering the risk undertaken. In the financial literature, to the best of our knowledge, this is the first attempt to conceive this kind of notion, in contrast to the case of risk-adjusted performance measures which are well-known and widely accepted by the entire community. This evidence supports the intuition behind our proposal to adjust a distance for the risk. Investors are conscious that, when evaluating the overall performance of an investment, both the return and the risk undertaken to realize that return need to be considered. The first and most famous risk-adjusted performance measure is the *Sharpe Ratio*, see (Sharpe 1966). Subsequently, more than one hundred alternative measures have been proposed, see (Caporin et al. 2014) and (Cogneau and Hübner 2009) for a comprehensive review. The majority of these proposals are defined as return over risk ratios, attempting to overcome the shortcomings of the Sharpe Ratio, which relies on the assumption of normal distribution for asset returns; we recall among the others, (Burke 1994; Dowd 2000; Farinelli and Tibiletti 2008; Kaplan and Knowles 2004; Kazemi et al. 2004; Shadwick and Keating 2002) and (Young 1991).

The main aim of this paper is to introduce and analyze, both theoretically and empirically, a novel asset allocation strategy based on RADs. We start representing the set of long-only admissible portfolios with n risky components by means of the standard simplex of $\mathbb{R}^n$, a generalization of the notion of a triangle or tetrahedron to arbitrary dimensions, see (Chalkis and Emiris 2020). In this setting, the dimension of the space stands for the number of assets, while the vertices of the simplex are the single asset portfolios, which are considered the worst portfolios in terms of diversification, as pointed out in Koumou and Dionne (2019). In order to differentiate from such maximum concentrated extreme cases, we propose to invest in the portfolio, called Risk-Adjusted Geometric Diversified Portfolio (RAGDP), equally distant from the vertices of the simplex. Geometrically, the RAGDP is represented by the circumcenter of the simplex, whose position depends on the used distance. In order to take into account the specific risk of the assets, our approach is based on the RADs, resulting in a deformation of the geometrical space. The RAGDP shows the intuitive feature to underweight the allocation on the riskier asset classes while overweighting the allocation on the less risky ones. If the Euclidean distance is used, being the simplex a regular polytope, the circumcenter corresponds to the center of gravity, so that our proposal reduces to the Equally Weighted Portfolio (EWP). We also show that the EWP is something more than a special case when the Euclidean distance is used. Indeed, it represents a limit case when the number n of assets is extremely large, that is $n \to +\infty$.

In the statistical framework a systematic approach to diversification dates back to the work of Rao (see (Rao 1982a, b; Rao and Nayak 1985; Rao 2010)) who introduced a measure of diversity through the so called Rao's Quadratic Entropy (RQE). Later on, the use of RQE has been extended to modern portfolio theory by Carmichael et al. (2015), where the RQE portfolios are defined upon a dissimilarity function that measures the differences between any two assets. In this paper, we provide a clear statement of the relationship of our approach with RQE. In particular, we prove that the RAGDP is equivalent to the maximum of RQE when the information on the risk exposure is conveniently translated into a suitable dissimilarity matrix upon which the RQE is defined. The benefit of our point of view is twofold: our approach permits to introduce the risk measurement in the diversification scheme through the RADs; further, our contribution reads as an independent alternative approach to the use of RQE as a diversification measure, allowing to highlight very interesting general properties and capable to shed new light on the concept of diversification. For instance, our proposal yields the explicit solution for the RQE maximization problem and allows to prove that when the risky assets are no more than 3 the optimal allocation is a long-only strategy. Moreover, the RAGDP viewpoint permits to provide an explicit condition among the risks of the single assets to guarantee that the RAGDP and the corresponding RQE optimal portfolio are long-only investments. Due to the equivalence between the two approaches, this permits to identify the condition for the dissimilarity matrix to give a solution belonging to the standard simplex when solving Rao's optimization problem.

The effectiveness of our proposal is finally highlighted through a real data out-of-sample experiment: empirical findings confirm and emphasize the goodness of our proposal, comparing its performance to the one of some very popular benchmark strategies.

The paper is organized as follows: since geometrical findings are valid in general, they are collected in a dedicated preliminary part, see Sect. 2, and then specified in the financial context of asset allocation in Sect. 3, where the RAGDP strategy is introduced and its main properties are discussed; Sect. 4 contains the comprehensive empirical study on real financial data while useful remarks and conclusive comments are summarized in Sect. 5; finally, technicalities and the complete proofs of the paper's results are detailed in Appendix A.

2 Geometry of the standard simplex

In this section we analyze some geometric properties of the standard $(n-1)$-simplex S_{n-1} of $\mathbb{R}^n$, $n \geq 1$, with respect to a generalized notion of Euclidean distance. In particular, we first recall the notion of distance defined by a symmetric positive definite matrix and the special case of weighted Euclidean distance. Next, we define the circumcenter of S_{n-1}, we provide an explicit formula for its coordinates in the case of weighted Euclidean metric and prove some properties relevant for our proposal when applied to the portfolio allocation setting.

Let $\mathcal{S}_n^+(\mathbb{R})$ be the set of $n \times n$ real symmetric positive definite matrices and $\mathcal{D}_n^+(\mathbb{R}) \subset \mathcal{S}_n^+(\mathbb{R})$ be the set of $n \times n$ real diagonal matrices with strictly positive diagonal elements.

Definition 1 Let $W \in \mathcal{S}_n^+(\mathbb{R})$. The map $\langle \cdot, \cdot \rangle_W : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that $\langle x, y \rangle_W := x^t W y$, for each $x, y \in \mathbb{R}^n$, defines an inner product on $\mathbb{R}^n$ and induces the norm $\| \cdot \|_W : \mathbb{R}^n \to \mathbb{R}$, where $\|x\|_W := \sqrt{\langle x, x \rangle_W} = \sqrt{x^t W x}$, for each $x \in \mathbb{R}^n$, and the *distance function* (*metric*) $d_W : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$ defined by

$$d_W(x, y) := \|x - y\|_W = \sqrt{(x-y)^t W (x-y)}, \qquad \forall x, y \in \mathbb{R}^n.$$

In the special case $W \in \mathcal{D}_n^+(\mathbb{R})$ then the expression of d_W, called *W-weighted Euclidean distance function (metric)*, becomes

$$d_W(x, y) = \left(\sum_{i=1}^{n} w_i (x_i - y_i)^2 \right)^{\frac{1}{2}}, \quad \forall x, y \in \mathbb{R}^n,$$

where $w_i > 0, i = 1, \ldots, n$, are the diagonal elements of W.

The following remark recalls two special cases of Definition 1.

Remark 1 Note that in the case $W = I_n$, where I_n is the identity matrix of size n, the weighted distance d_{I_n} is the standard *Euclidean distance*. Further, if W is the inverse of the covariance matrix of a given set of data, the corresponding distance yields the *Mahalanobis Distance*, see (Mahalanobis 1936).

We recall the definition of the standard simplex of $\mathbb{R}^n$ and its circumcenter, see (VanderZee et al. 2013).

Definition 2 Let $e^1, \ldots, e^n$ be the standard basis of $\mathbb{R}^n$ and let d be a distance on $\mathbb{R}^n$.

The *standard* $(n-1)$*-simplex* S_{n-1} of $\mathbb{R}^n$ is the convex hull of $e^1, \ldots, e^n$, which are called the *vertices* of S_{n-1}. Equivalently

$$S_{n-1} = \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ x_i \geq 0 \, \forall i = 0, \ldots, n \text{ and } \sum_{i=1}^{n} x_i = 1 \right\}.$$

The *circumcenter of* S_{n-1} *with respect to* d, denoted by $c_d(S_{n-1})$, is the unique point that, among all the points equidistant from the vertices of S_{n-1}, minimizes its distance from each e^i, $i = 1, \ldots, n$.

It is relevant to remark that S_{n-1} is a $(n-1)$-dimensional object of $\mathbb{R}^n$ lying on the hyperplane $\Gamma : \sum_{i=1}^n x_i = 1$, which also contains the circumcenter $c_d(S_{n-1})$. Further, unlike other classical "centers", such as the centroid and the incenter, which are always inside the simplex, the circumcenter may lie outside S_{n-1}, see (VanderZee et al. 2013).

We notice that $c_d(S_{n-1})$ may equivalently be defined as the unique solution of the following (convex) constrained minimization problem:

$$\begin{array}{ll} \text{Minimize} & \sum_{i=1}^{n} \sum_{j>i} \left(d^2(x, e^i) - d^2(x, e^j) \right)^2 \\ \text{s.t.} & \sum_{i=1}^{n} x_i = 1. \end{array} \tag{1}$$

Further, given any $W \in \mathcal{S}_n^+(\mathbb{R})$ and the associated distance function d_W, we prove that $c_{d_W}(S_{n-1})$ is the unique point at which Rao's Quadratic Entropy (see (Rao 1982a; Rao and Nayak 1985; Rao 2010)) associated to a suitable $n \times n$ dissimilarity matrix D assumes its maximum value.

Proposition 1 *Let* $W \in \mathcal{S}_n^+(\mathbb{R})$, *let* d_W *be the associated distance function on* $\mathbb{R}^n$ *and* $D = (d_{ij})$ *be the* $n \times n$ *real matrix with entries*

$$d_{ij} = \begin{cases} 0 & \text{if } i = j \\ \frac{1}{2}(e^i - e^j)^t W (e^i - e^j) & \text{if } i \neq j. \end{cases}$$

Let $H_D : \mathbb{R}^n \to \mathbb{R}$, *defined by* $H_D(x) = x^t D x$, *for each* $x \in \mathbb{R}^n$, *be Rao's Quadratic Entropy associated to* D *(* *Rao* (1982*a*); *Rao and Nayak* (1985); *Rao* (2010)). *The circumcenter* $c_{d_W}(S_{n-1})$ *is the unique solution of problem*:

$$\begin{array}{ll} \text{Maximize} & H_D(x) = x^t D x \\ \text{s.t.} & \sum_{i=1}^{n} x_i = 1 \end{array} \tag{2}$$

whose maximum value is $H_D(c_{d_W}(S_{n-1})) = d_W^2(c_{d_W}(S_{n-1}), e^i)$, $i = 1, \ldots, n$.

Proof See Appendix A. □

In the rest of the section we restrict to a given W-weighted Euclidean distance function d_W of $\mathbb{R}^n$ (see Definition 1). In this framework, Proposition 2 provides an explicit formula for the coordinates of the circumcenter $c_{d_W}(S_{n-1})$. Indeed, we underline that, in the general case of any given distance (defined starting from a generic non-diagonal matrix), the computation of the closed form solution of problem (1) proves to be much more involved than the considered case, thus encouraging the authors to address it in a future designated research.

Proposition 2 *Let d_W be a W-weighted Euclidean distance function on $\mathbb{R}^n$. The circumcenter $c_{d_W}(S_{n-1})$ of S_{n-1} has coordinates $(c_1, \dots, c_n)$ such that*

$$c_i = \frac{1}{2}\left(1 - \frac{n-2}{nw_i M_{W^{-1}}}\right), \quad \forall i = 1, \dots, n, \tag{3}$$

where $M_{W^{-1}}$ is the arithmetic mean of $w_1^{-1}, \dots, w_n^{-1}$, where $w_1, \dots, w_n$ are the diagonal elements of W.

Proof See Appendix A. □

A first crucial remark on Proposition 2 regards the special cases $n = 1$ and $n = 2$ in which the circumcenter respectively coincides with the points (1) and $\left(\frac{1}{2}, \frac{1}{2}\right)$, independently of the chosen distance. While the case $n = 1$ is justified by simply observing that the standard $(n-1)$-simplex itself degenerates to be a unique point, the peculiarity of the case $n = 2$ is related to the fact that, in general, the circumcenter's computation yields coordinates that only depend on the ratio of the weights associated to the vertices and thus cannot exceed the quantity $\frac{1}{2}$ (see Proposition 3, (iv)). For this reason, in the rest of the paper, we only consider $n \geq 3$, as the cases $n = 1$ and $n = 2$, though possible and well-defined, by construction degenerate to trivial cases.

As additional remark and consequence of formula (3), we observe that as w_i increases then also c_i increases: this is intuitively justified by the fact that a larger weight associated to the ith vertex e^i brings the circumcenter closer to it.

In the following proposition we gather some useful properties on the circumcenter of S_{n-1}.

Proposition 3 *Let $W \in \mathcal{D}_n^+(\mathbb{R})$ with diagonal elements $w_i, i = 1, \dots, n$, and d_W be a W-weighted Euclidean distance function on $\mathbb{R}^n$. Let $w_{\min} = \min\{w_1, \dots, w_n\}$, $w_{\max} = \max\{w_1, \dots, w_n\}$ and $M_{W^{-1}}$ be the arithmetic mean of $w_1^{-1}, \dots, w_n^{-1}$. Let $c_{d_W}(S_{n-1}) = (c_1, \dots, c_n)$ be the circumcenter of S_{n-1} with respect to d_W. Then the following facts hold true.*

(i) *$c_i = \frac{1}{n}$, for each $i = 1, \dots, n$, if and only if $W = \alpha I_n$.*
(ii) *$\frac{1}{n} - \frac{n-2}{2n}\left(\frac{w_{\max}}{w_{\min}} - 1\right) \leq c_i \leq \frac{1}{n} + \frac{n-2}{2n}\left(1 - \frac{w_{\min}}{w_{\max}}\right)$ for each $i = 1, \dots, n$ and $W \in \mathcal{D}_n^+(\mathbb{R})$.*
(iii) *$c_i = \frac{1}{n}$, for some $i = 1, \dots, n$, if and only if $w_i = M_{W^{-1}}$; further, $c_i > \frac{1}{n}$ ($c_i < \frac{1}{n}$ respectively), for some $i = 1, \dots, n$, if and only if $w_i < M_{W^{-1}}$ ($w_i > M_{W^{-1}}$ respectively).*
(iv) *$-\frac{1}{2}(n-3) < c_i < \frac{1}{2}$, for each $i = 1, \dots, n$ and $W \in \mathcal{D}_n^+(\mathbb{R})$.*
(v) *If $n = 3$ then $c_i > 0$, for each $i = 1, \dots, n$ and $W \in \mathcal{D}_n^+(\mathbb{R})$.*
(vi) *If $n > 3$ then $c_i \geq 0$, for each $i = 1, \dots, n$, if and only if*

$$M_{W^{-1}} \geq \frac{n-2}{n} \cdot \frac{1}{w_{\min}}.$$

(vii) *Assume that* $c_i \geq 0$ *for each* $i = 1, \ldots, n$; *if* $n \to +\infty$ *then each component* c_i *goes to* 0 *as* $\frac{1}{n}$.

Proof See Appendix A. □

We do not specify any interpretation to the results of Proposition 3 in this section, leaving the comments to the contextualization in the framework of portfolio theory.

3 Risk-adjusted geometric diversified portfolio

In this section we introduce the asset allocation rule, called *Risk-adjusted geometric diversified portfolio (RAGDP)*, based on the use of the *Risk-adjusted distances (RADs)*.

3.1 Risk-adjusted distances

Let $n \geq 3$ be the number of risky assets available on a given market, let ρ be a given risk measure and ρ_i, with $i = 1, \ldots, n$, be the risk of the ith asset. We assume that $\rho_i > 0$ for each $i = 1, \ldots, n$; such assumption is reasonable considering the minimal properties required for the axiomatic definition of a risk measure, see among the others (Rachev et al. 2008). In the following definition we arrange the risks of the assets in a diagonal matrix, called *risk matrix*, that contains the risk information that will be used to "adjust" the distance.

Definition 3 (Risk-Adjusted Distance—RAD)
Let $\rho_i > 0$, with $i = 1, \ldots, n$, be the risk of the ith investment opportunity according to a given risk measure ρ. The matrix $W_\rho \in \mathcal{D}_n^+(\mathbb{R})$ whose ith diagonal element is ρ_i^{-1}, $i = 1, \ldots, n$, is called the *risk matrix* (associated to ρ) and the W_ρ-weighted Euclidean distance function d_{W_ρ} is the corresponding *Risk-adjusted distance*.

3.2 The RAGDP strategy

We restrict the feasible portfolios, as usual in practice, to the set of vectors with unitary sum, such that they are represented in $\mathbb{R}^n$ by the points of the hyperplane $\Gamma : \sum_{i=1}^n x_i = 1$. In Γ we also consider the standard $(n-1)$-simplex S_{n-1} of $\mathbb{R}^n$, see Definition 2. From a financial point of view the vertices of S_{n-1}, represented by the standard orthonormal basis $e^1, \ldots, e^n$ of $\mathbb{R}^n$, correspond to the portfolios characterized by the maximum level of concentration, that is the portfolios in which the total wealth is invested in one asset. The idea that supports the RAGDP strategy is to choose the portfolio corresponding to the point of Γ, not necessarily belonging to S_{n-1}, that is equally distant from the vertices of the simplex, trying to maximize the distance from extreme allocations. Geometrically, we look for the circumcenter of the simplex, the center of the circumscribed hypersphere. Considering that the simplex is a regular polytope, if we solve the proposed problem using the standard Euclidean distance, the solution is the Equally Weighted Portfolio (EWP), that coincides with the circumcenter and the center of gravity of the simplex. The use of alternative definitions of distance, the RADs introduced in Sect. 3, provides interesting solutions in which the circumcenter generally differs from the center of gravity, providing an allocation that is different from the EWP. The approach described so far is formalized in the following definition.

Definition 4 (Risk-adjusted geometric diversified portfolio—RAGDP) Let d_{W_ρ} be the RAD associated to a given risk measure ρ. The *Risk-adjusted geometric diversified portfolio* is represented by the portfolio $x^* = (x_1^*, \ldots, x_n^*) \in \mathbb{R}^n$ such that

$$\sum_{i=1}^{n} x_i^* = 1 \quad \text{and} \quad d_{W_\rho}(x^*, e^i) = d_{W_\rho}(x^*, e^j), \quad \forall i,j, \in \{1, \ldots, n\}.$$

Proposition 4 *Let ρ be a given risk misure, d_{W_ρ} be the associated RAD, $\rho_{\min}$, $\rho_{\max}$ and M_ρ be the minimum, the maximum and the arithmetic mean of the risks $\rho_1, \ldots, \rho_n$ of the assets. The RAGDP $x^* = (x_1^*, \ldots, x_n^*) \in \mathbb{R}^n$ satisfies the following properties.*

(i) *The RAGDP is unique and its components x_i^* satisfy*

$$x_i^* = \frac{1}{2}\left(1 - \frac{n-2}{nM_\rho}\rho_i\right), \tag{4}$$

for each $i = 1, \ldots, n$ and for any risk measure ρ.

(ii) *The RAGDP coincides with the EWP, that is, $x_i^* = \frac{1}{n}$, for each $i = 1, \ldots, n$, if and only if the risks of all the investment opportunities coincide.*

(iii) *For any risk measure ρ and each $i = 1, \ldots, n$ it holds*

$$\frac{1}{n} - \frac{n-2}{2n}\left(\frac{\rho_{\max}}{\rho_{\min}} - 1\right) \le x_i^* \le \frac{1}{n} + \frac{n-2}{2n}\left(1 - \frac{\rho_{\min}}{\rho_{\max}}\right).$$

(iv) $x_i^* = \frac{1}{n}$, *for some $i = 1, \ldots, n$, if and only if $\rho_i = M_\rho$; further, $x_i^* > \frac{1}{n}$ ($x_i^* < \frac{1}{n}$ respectively), for some $i = 1, \ldots, n$, if and only if $\rho_i < M_\rho$ ($\rho_i > M_\rho$ respectively).*

(v) *The RAGDP is a long-only portfolio if and only if $n = 3$ or $n > 3$ and $M_\rho \ge \frac{n-2}{n}\rho_{\max}$*
.

(vi) *Let $n > 3$ and assume that $M_\rho \ge \frac{n-2}{n}\rho_{\max}$; if $n \to +\infty$ then the RAGDP tends to the EWP.*

Proof The results immediately follow from Propositions 2 and 3. □

Remark 2 We summarize interesting observations on RAGDP strategy.

(i) The allocation weights of the RAGDP strategy cannot exceed the value $\frac{1}{2}$ (it follows from Proposition 4, item (i)).

(ii) The RAGDP strategy never returns the maximum concentrated portfolio represented by a single asset's investment (it immediately follows from item (i)).

(iii) The RAGDP is a long-short strategy in which the allocation on one or more assets is null if and only if the corresponding risks are equal to the quantity $\left(1 + \frac{2}{n-2}\right)M_\rho$ (a consequence of Proposition 4, item (i)).

(iv) The allocation weights of the RAGDP strategy are in reverse order with respect to the assets risks (see Proposition 4, item (i)). In particular, if $\rho_i \le \rho_j$ then $x_i^* \ge x_j^*$, meaning that the RAGDP allocates more resources on less risky assets and less on riskier ones.

(v) If the risk of only one asset increases (decreases) then the RAGDP strategy allocates less (more) on that asset and more (less) over all the remaining assets. Let's consider the case in which ρ_k, for some index $k \in \{1, \dots, n\}$, increases of the positive quantity $\Delta\rho_k$ whereas all the other risks ρ_i, $i \in \{1, \dots, n\}$, $i \neq k$, remain the same. The kth component x_k^* of the RAGDP decreases of the quantity $\frac{1}{2}\frac{n-2}{n}\frac{\Delta\rho_k}{M_\rho}\frac{nM_\rho-\rho_k}{nM_\rho+\Delta\rho_k}$, whereas the generic ith component x_i^*, $i \neq k$, increases of the quantity $\frac{1}{2}\frac{n-2}{n}\frac{\rho_j}{M_\rho}\frac{\Delta\rho_k}{nM_\rho+\Delta\rho_k}$.

(vi) An immediate comparison between the RAGDP strategy x^* and the EWP can be outlined as follows: the two approaches coincide when the risks are equal over all the assets (see Proposition 4, items (ii) and (vi)) and they asymptotically coincide when $n \to +\infty$. In the remaining cases, the ith allocation weight x_i^* is greater, equal or less than $\frac{1}{n}$ if and only if the corresponding asset risk ρ_i is smaller, equal or greater than the arithmetic mean M_ρ of all the assets risks (see Proposition 4, item (iv)).

As pointed out in the introduction, the construction of RAGDP intuitively resembles the approach introduced in Carmichael et al. (2015) based on Rao's Quadratic Entropy (RQE). In order to shed light on their relationship the following result is proved.

Proposition 5 *Let d_{W_ρ} be the RAD associated to a given risk measure ρ, let $D = (d_{ij})$ be the $n \times n$ real matrix with entries*

$$d_{ij} = \begin{cases} 0 & \text{if } i = j \\ \frac{1}{2}\left(\frac{1}{\rho_i} + \frac{1}{\rho_j}\right) & \text{if } i \neq j. \end{cases}$$

and $H_D : \mathbb{R}^n \to \mathbb{R}$, defined by $H_D(x) = x^t Dx$, for each $x \in \mathbb{R}^n$, be the RQE associated to D. Then, the RAGDP coincides with the RQE optimal portfolio associated to H_D.

Proof See Appendix A. □

The above proposition clearly states the relationship between RAGDP and RQE optimal portfolio. In particular, once the information on the risk exposure is conveniently translated into the dissimilarity matrix D, the two approaches are proved to be equivalent. As a consequence, formula (4) provides the explicit solution for the RQE maximization problem and Proposition 4, item (v), yields a conditoin among the single assets' risks to guarantee that the RQE optimal portfolio is a long-only investment.

3.3 Long-only RAGDP strategy

As highlighted by the previous results, the RAGDP strategy may return a portfolio with short positions. In this section we propose a long-only RAGDP strategy based on the following idea: given a risk measure ρ and the corresponding values ρ_i, $i = 1, \dots, n$, for the n risky assets, it is always possible to define a transformation that maintains the order in terms of risk of the assets and provides a long-only allocation. This transformation, introduced in Definition 5, modifies the dispersion of the tuple $\rho_1, \dots, \rho_n$ and preserves the relative proportions in terms of risk among the assets (see Remark 3).

Definition 5 (**β -RAGDP**) Let $n > 3$, let ρ be a given risk misure, $\rho_{\max}$ and M_ρ be the maximum and the arithmetic mean of the risks $\rho_1, \dots, \rho_n$ of the assets. Assume that $\rho_{\max} > M_\rho \geq \frac{n-2}{n}\rho_{\max}$. Let $d := \frac{2}{n}\frac{\rho_{\max}}{\rho_{\max} - M_\rho}$, let β be a real number such that $0 \leq \beta \leq 1$ and $\rho_1(\beta), \dots, \rho_n(\beta)$ be defined as follows:

$$\rho_i(\beta) := \rho_{\max} - \beta d(\rho_{\max} - \rho_i), \quad \text{for each } i = 1, \dots, n. \tag{5}$$

Let $W_{\rho(\beta)} \in \mathcal{D}_n^+(\mathbb{R})$ whose ith diagonal element is $(\rho_i(\beta))^{-1}$, $i = 1, \dots, n$, and $d_{W_{\rho(\beta)}}$ be the associated RAD. The *β-RAGDP strategy* is represented by the portfolio $x^* = (x_1^*, \dots, x_n^*) \in \mathbb{R}^n$ such that

$$\sum_{i=1}^{n} x_i^* = 1 \quad \text{and} \quad d_{W_{\rho(\beta)}}(x^*, e^i) = d_{W_{\rho(\beta)}}(x^*, e^j), \quad \forall i, j, \in \{1, \dots, n\}.$$

Remark 3 We briefly point out some main peculiarities of the transformation $\rho_1(\beta), \dots, \rho_n(\beta)$ of $\rho_1, \dots, \rho_n$ introduced in Definition 5, which are obtained as direct consequences of expression (5).

(i) $\rho(\beta)_{\max} = \rho_{\max}$;
(ii) $\rho_i \leq \rho_j$ if and only if $\rho_i(\beta) \leq \rho_j(\beta)$, for each $i, j = 1, \dots, n$;
(iii) $\frac{\rho_{\max} - \rho_i(\beta)}{\rho_{\max} - \rho_j(\beta)} = \frac{\rho_{\max} - \rho_i}{\rho_{\max} - \rho_j}$, for each $i, j = 1, \dots, n$.

In the following proposition we prove that the β-RAGDP strategy effectively returns a long-only portfolio.

Proposition 6 *The β-RAGDP returns a long-only portfolio.*

Proof See Appendix A. □

Remark 4 We gather some observations regarding the parameter β used in the β-RAGDP strategy. The possible values of β range in [0, 1], which guarantee to obtain a long-only portfolio. Within such interval, increasing β yields an increase in the differences among the modified tuple of the assets' risks $\rho_1(\beta), \dots, \rho_n(\beta)$ and, consequently, a growing dispersion in the tuple of weights of the portfolio strategy. In particular, in the special extreme case $\beta = 0$, all the risks of the assets are equal to the maximum of the original risks $\rho_{\max}$, so that the β-RAGDP strategy coincides with the EWP (see Remark 2, item (vi)). On the other hand, the opposite extreme case $\beta = 1$ yields the least concentrated long-only RAGDP strategy. From the equality $\rho(\beta)_{\max} = \left(1 + \frac{2}{n-2}\right) M_{\rho(\beta)}$ (see (7), Appendix A) and Remark 2, item (iii), it follows that the allocation on the most risky asset is null, so that the β-RAGDP strategy results in a $n - 1$ assets portfolio. Note that the long-only case corresponding to $\beta = 1$ will be employed in the empirical application (see Sect. 4).

3.4 Comparison with EWP

We end the section with a comparison of the in-sample variance of RAGDP, β-RAGDP and EWP strategies.

Proposition 7 *Let d_{W_ρ} be the RAD associated to a given risk measure ρ, let $\rho_{\max}$ and M_ρ be the maximum and the arithmetic mean of the risks $\rho_1, \dots, \rho_n$ of the assets. Let $\beta \in [0, 1]$ and* Var(RAGDP), Var(β-RAGDP) *and* Var(EWP) *be the in-sample variance of RAGDP, β-RAGDP and EWP respectively. Let V be the covariance matrix, $\mathbf{1}_n = (1, \dots, 1)^t \in \mathbb{R}^n$ and denote $M_\rho \mathbf{1}_n - \rho$ by D_ρ. Then*

$$\text{(i)} \qquad \mathrm{Var(RAGDP)} \leq \mathrm{Var(EWP)} \Longleftrightarrow nD_\rho^t V D_\rho + 2(M_\rho^2 \mathbf{1}_n^t V \mathbf{1}_n - \rho^t V \rho) \leq 0,$$

$$\text{(ii)} \qquad \mathrm{Var}(\beta\text{-RAGDP}) \leq \mathrm{Var(EWP)} \Longleftrightarrow \begin{cases} 0 \leq \beta \leq \min\{1, \overline{\beta}\} & \text{if } \overline{\beta} > 0 \\ \beta = 0 & \text{if } \overline{\beta} \leq 0 \end{cases}$$

where

$$\overline{\beta} = \frac{1}{d} \frac{4\rho_{\max} D_\rho^t V \mathbf{1}_n}{4\rho_{\max} D_\rho^t V \mathbf{1}_n - [nD_\rho^t V D_\rho + 2(M_\rho^2 \mathbf{1}_n^t V \mathbf{1}_n - \rho^t V \rho)]}.$$

Proof See Appendix A. □

4 Empirical results

In this section we provide a comprehensive empirical study to evaluate the out-of-sample performance of the following portfolio strategies.

- *EWP strategy*. The EWP is the portfolio that allocates the equal proportion to each asset, that is the constant quantity $\frac{1}{n}$.
- *ERC strategy*. The ERC is the portfolio that equalizes the risk contribution of each asset; for more details see (Maillard et al. 2010). In the rest of the section we refer to $\mathrm{ERC_d}$ and ERC to identify the ERC portfolio in the special case of diagonal covariance matrix and in the general case respectively.
- *RAGDP strategy*. We refer to the RAGDP as the portfolio whose components are expressed by formula (4) and the variances of the assets are used to define the RAD; we denote this strategy by $\mathrm{RAGDP^V}$. To fully evaluate the RAGDP, we also implement the strategy for different very common risk measures: standard deviation (StDev), mean absolute deviation (MAD), Value at risk (V@R) computed at a significance level of 5%, maximum drawdown (MDD); we denote the corresponding strategies by $\mathrm{RAGDP^{StDev}}$, $\mathrm{RAGDP^{MAD}}$, $\mathrm{RAGDP^{V@R}}$ and $\mathrm{RAGDP^{MDD}}$. In Sect. 3.3 the β-RAGDP strategy, a long-only version of the RAGDP depending on an additional parameter β, has been introduced. In the application we will also consider such strategies, setting $\beta = 1$ and using the aforementioned risk measures. The corresponding strategies will be denoted by $\mathrm{RAGDP_1^V}$, $\mathrm{RAGDP_1^{StDev}}$, $\mathrm{RAGDP_1^{MAD}}$, $\mathrm{RAGDP_1^{V@R}}$ and $\mathrm{RAGDP_1^{MDD}}$.
- *GMV strategy*. The Global Minimum Variance portfolio is the solution to the classical Markowitz problem, the vertex of the efficient frontier in the mean-variance plane; we denote by GMV and $\mathrm{GMV_{lo}}$ the long-short and long-only strategies respectively, see (Constantinides and Malliaris 1995).

The analysis is performed through a rolling-window exercise: given a T observations dataset of asset returns, we set the estimation window length equal to w_e. Then, starting from $w_e + 1$, the previous w_e observations are used to calculate the portfolio on the base of the given strategy while observation $w_e + 1$ is used to calculate the out-of-sample return of the portfolio. This out-of-sample return is then reduced taking into account the transaction costs, that are introduced in a proportional way and set equal to 50 basis points per transaction as assumed in DeMiguel et al. (2009). The described process iteratively continues dropping the first return in the dataset and adding the return of the subsequent period, resulting in a $T - w_e$ length series of out-of-sample returns. The performance criteria used to compare the effectiveness of the considered allocation strategies are principally calculated on the base of the out-of-sample returns or, alternatively, considering the variation in the allocation of a portfolio after rebalancing at the end of the period. We now enumerate and, when necessary, briefly explain, the performance criteria taken into account.

- *Average Return* (AR): arithmetic mean of the out-of-sample returns.
- *Standard Deviation* (StDev): standard deviation of the out-of-sample returns.
- *Sharpe Ratio* (SR): ratio of the previous quantities, see (Sharpe 1966).
- *Average Portfolio TurnOver* (TO): a measure of the stability over time of the allocation strategy defined by

$$\mathrm{TO} = \frac{1}{T - w_e} \sum_{t=1}^{T-w_e} \sum_{i=1}^{n} |x_{i,t+1} - x_{i,(t+1)^-}|,$$

  where $x_{i,t+1}$ is the share invested in the ith asset at the beginning of the trading period that ranges from $t + 1$ to $t + 2$, while $x_{i,(t+1)^-}$ is the allocation on the ith asset at the end of the previous trading period from t to $t + 1$, resulting from the initial allocation $x_{i,t}$ in combination to the variation of the prices in the period. The TO quantifies the average amount of portfolio rebalancing that is necessary to implement a given strategy, providing an immediate information about the impact of transaction costs on the performance of a strategy.
- *Average Leverage* (L): a measure of the leverage of a portfolio defined as:

$$\mathrm{L} = \frac{1}{T - w_e} \sum_{t=1}^{T-w_e} \sum_{i=1}^{n} |x_{i,t}|$$

  We note that in the case of long-only portfolios $\mathrm{L} = 1$ while for long-short portfolios $\mathrm{L} > 1$. The average leverage puts the accent on the macroscopic differences among long-only and long-short portfolios: in particular, long-short portfolios benefit of a potential extra diversification opportunity based on the negative weights that are able to create artificial negative correlations among the assets. In mean-variance analysis this translates in a reduced risk. Taking into account the leverage of a portfolio is needful to highlight the presence of further sources of risk that potentially remain hidden in the mean-variance approach.
- *Value at Risk* (V@R): the well-known risk measure of losses calculated at 0.01 level, see (Jorion 2006).

The dataset is composed by the daily returns from January 3, 2000 to September 17, 2020 of the ten sectors portfolios of the S &P index obtained using the Global Industry Classification Standard (GICS): Energy, Material, Industrials, Consumer-Discretionary, Consumer-Staples, Healthcare, Financials, Information-Technology, Telecommunications, and

Table 1 Performance criteria for each investment strategy ($w_e = 60$)

	AR	StDev	SR	TO	L	V@R
GMV_{lo}	−0.000089	0.0101	−0.0088	0.01920	1	0.0301
EWP	0.000139	0.0118	0.0117	0.00056	1	0.0345
ERC_d	0.000152	0.0111	0.0137	0.00112	1	0.0328
ERC	0.000148	0.0113	0.0131	0.00055	1	0.0332
$RAGDP_1^V$	0.000180	0.0107	0.0169	0.00233	1	0.0320
$RAGDP_1^{StDev}$	0.000178	0.0106	0.0168	0.00250	1	0.0316
$RAGDP_1^{MAD}$	0.000180	0.0106	0.0170	0.00282	1	0.0319
$RAGDP_1^{V@R}$	0.000157	0.0106	0.0149	0.00323	1	0.0321
$RAGDP_1^{MDD}$	0.000145	0.0109	0.0133	0.00279	1	0.0325
GMV	0.000089	0.0085	0.0104	0.02489	2.34	0.0255
$RAGDP^V$	0.000382	0.0137	0.0280	0.00829	1.95	0.0436
$RAGDP^{StDev}$	0.000223	0.0100	0.0223	0.00396	1.24	0.0295
$RAGDP^{MAD}$	0.000230	0.0100	0.0230	0.00456	1.25	0.0295
$RAGDP^{V@R}$	0.000196	0.0101	0.0195	0.00636	1.34	0.0296
$RAGDP^{MDD}$	0.000195	0.0108	0.0180	0.00695	1.53	0.0313

Table 2 Performance criteria for each investment strategy ($w_e = 120$)

	AR	StDev	SR	TO	L	V@R
GMV_{lo}	0.000035	0.0101	0.0035	0.01318	1	0.0296
EWP	0.000139	0.0118	0.0117	0.00055	1	0.0343
ERC_d	0.000154	0.0111	0.0139	0.00072	1	0.0328
ERC	0.000147	0.0113	0.0130	0.00054	1	0.0335
$RAGDP_1^V$	0.000179	0.0107	0.0167	0.00136	1	0.0323
$RAGDP_1^{StDev}$	0.000178	0.0106	0.0168	0.00143	1	0.0315
$RAGDP_1^{MAD}$	0.000176	0.0106	0.0165	0.00158	1	0.0316
$RAGDP_1^{V@R}$	0.000168	0.0106	0.0159	0.00210	1	0.0313
$RAGDP_1^{MDD}$	0.000149	0.0109	0.0136	0.00183	1	0.0325
GMV	0.000146	0.0083	0.0177	0.01179	2.11	0.0257
$RAGDP^V$	0.000364	0.0124	0.0293	0.00468	1.87	0.0379
$RAGDP^{StDev}$	0.000224	0.0099	0.0227	0.00212	1.22	0.0291
$RAGDP^{MAD}$	0.000231	0.0099	0.0233	0.00238	1.23	0.0291
$RAGDP^{V@R}$	0.000205	0.0100	0.0206	0.00362	1.28	0.0296
$RAGDP^{MDD}$	0.000174	0.0106	0.0164	0.00390	1.44	0.0321

Utilities. The window length is set equal to $w_e = 60$ or $w_e = 120$ that, using daily data, is equivalent respectively to 3 or 6 working months.

Tables 1 and 2 collect the values of the performance criteria for each considered investment strategy when an estimation rolling window of length $w_e = 60$ and $w_e = 120$ is considered. Within each table, we arranged the long-only and long-short portfolios into separate sub-tables. While the Sharpe Ratios balances the return on the risk undertaken permitting to directly compare any portfolio strategy in terms of risk-adjusted performances, we underline the necessity to consider the leverage to fully compare the results of the strategies. Since the leverage of a portfolio impacts both on the magnitude of the returns and their standard deviation, it can happen that standard performance indicators like the Sharpe Ratio do not highlight the leverage. For this reason we decided to separate long-only portfolios from long-short portfolios to fully show how the performances of the allocation strategies are influenced by the leverage.

An initial analysis highlights that the results of Tables 1 and 2 do not significantly differ from a qualitative point of view. This suggests that the length of w_e only has a mild effect on the performances of the considered strategies and that comments and remarks are common to both cases.

We consider first the long-only strategies. An important evidence is the fact that the RAGDP strategies show the best performances in term of out-of-sample Sharpe Ratio. This implies that the proposed strategies are able to control the risk without penalizing the returns. In fact, all the considered RAGDP require to implement active investment portfolio strategies as showed by their higher level of TO with respect to EWP, ERC_d and ERC; the extra investment activity with the related increased impact of the transaction costs is able then to produce good results both in terms of average (net) returns and standard deviation. On the opposite, the EWP, ERC_d and ERC strategies mainly base their competitiveness on the stability in time of the allocation and the subsequent low impact of transaction costs.

We note that the considerations on the risk of the strategies made on the base of the standard deviation are still valid if we refer to the V@R. Further, the results obtained by the RAGDP strategies appear to be robust with respect to the use of different risk measures to evaluate the results. In our opinion this reflects the fact that the proposed allocation strategy highly depends on the ordering of the portfolio's assets induced by the values of the single asset's risk, allocating more on the less risky asset, as explained in the theoretical part of the paper, see Remark 2 item (iv).

One specific comment is needed for the GMV strategies, including both the long-only and the long-short cases. By our results, the global minimum variance portfolios show to be very competitive in reducing the risk with respect to the other strategies. Despite of this, the performance in terms of Sharpe Ratios is very poor because the risk reduction is obtained without preserving the average return; in this case, the active investment strategy requires a frequent and severe rebalancing of the portfolio, as testified by the level of TO, that strongly impacts the average return through the transaction costs. An effective representation of the magnitude of the TO is given by Fig. 1 which reports the daily variations of each asset's weights in the whole duration (left panel) and in the time period from September 03, 2001 to October 29, 2001 (right panel) for the EWP, $\text{RAGDP}_1^{\text{V}}$ and GMV_{lo} strategies. The shorter period of two months has been arbitrarily chosen in order to better visualize the time evolution of the weights for the strictly active investment strategies. The GMV_{lo} portfolio shows a highly unstable behavior, since both the portfolio's weights and its composition in terms of asset classes dramatically change. The figure provides a graphical immediate intuition on the impact of transaction costs when comparing active and passive investment strategies.

Finally, we consider the long-short RAGDP strategies. They are competitive with respect to the alternative long-only strategies and, in general, superior for all the criteria: their Average Portfolio TurnOver is small enough not permitting the transaction costs to erode the returns. Their risk, both measured through the Standard Deviation and

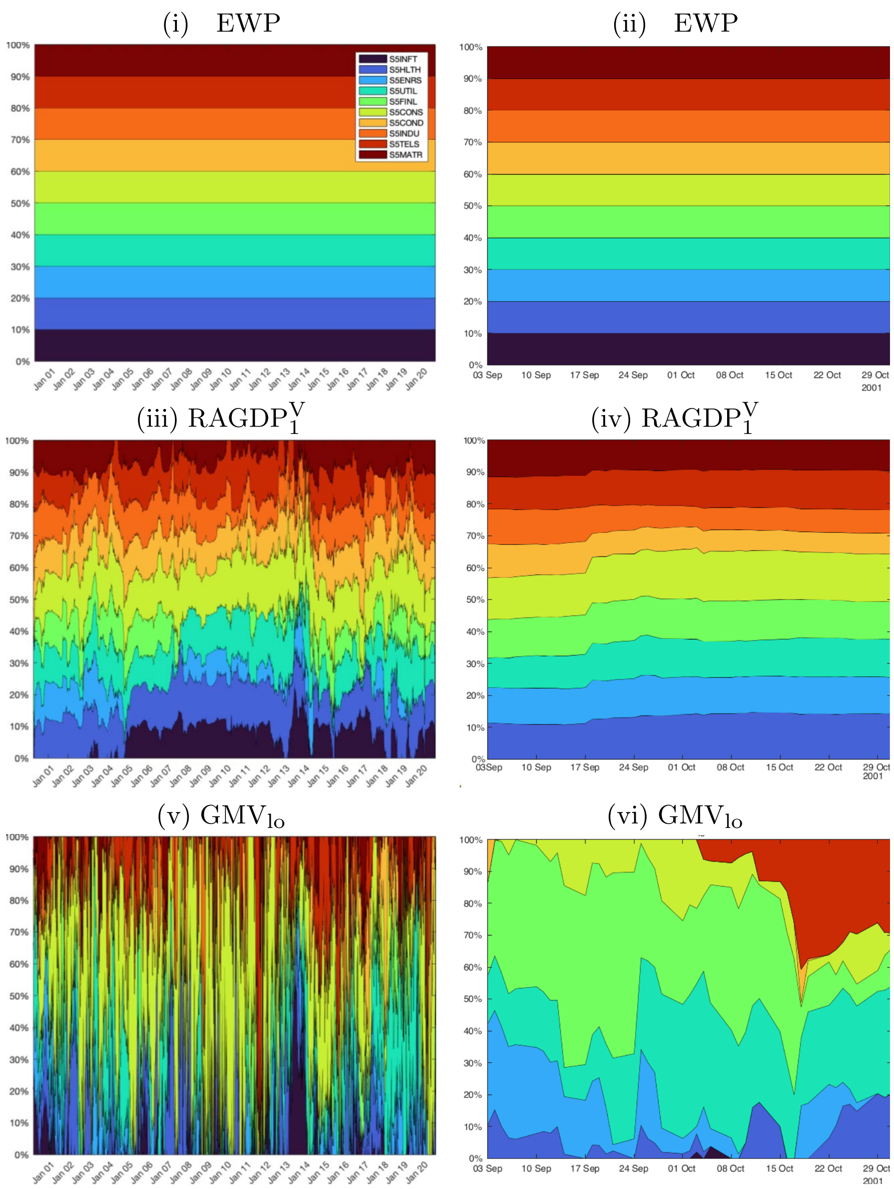

Fig. 1 Daily variations of the portfolio assets weights in the whole duration (left panel) and in the shorter time period of two months (right panel) for the EWP, $\mathrm{RAGDP}_1^{\mathrm{V}}$ and $\mathrm{GMV}_{\mathrm{lo}}$ strategies

the Value at Risk, is generally lower than the one of the other considered approaches, while the Average Return is higher, resulting in a significant higher Sharpe Ratio. As a general conclusive remark on the empirical experiment, Tables 1 and 2 highlight how the strategies based on geometric diversification constitute a competitive alternative to well-known allocation strategies.

5 Conclusions

In this paper a novel allocation strategy based on the intuitive idea of geometric diversification is presented. Theoretical results and properties of the proposed allocation strategy are derived together with the closed form solution in the general case of RADs. The approach permits to include any given risk measure to define a geometric diversified portfolio adjusted for that specific risk measure, thus taking advantage of the huge number of risk measures proposed in the literature. Further, if compared to portfolios constructed using Entropy-based diversification methods defined upon existing risk measures, geometric diversified portfolios appear to be extremely intuitive. Under suitable assumptions, a direct comparison with RQE optimal portfolios is provided; this makes the RAGDP an alternative approach to the use of Rao's Quadratic Entropy as a maximum diversification measure, thus yielding a new formulation and an explicit solution to the entropy maximization problem. Moreover, the empirical out-of-sample exercise provides the ultimate proof supporting the effectiveness of the proposal, showing that RAGDPs are a competitive alternative to the other investment strategies. Such empirical evidence is promising and suggests a fruitful employ of the proposed novel approach even in practice. As further research, the authors plan to investigate a much more general case of Risk-Adjusted Distances, in which the risk information carried by the risk matrix could also consider the amount of risk associated to each assets' pair.

A Proofs of the paper's results

This is a technical appendix giving the complete proofs of the results stated in the paper.

Proof of Proposition 1 The proof is based on (Pavoine et al. 2005, Proof of Proposition 1) which has been adapted here to the case of a general distance d_W on $\mathbb{R}^n$. By definition of D the $n \times n$ real matrix $(\sqrt{d_{ij}})$ is Euclidean so that d_{ij} is a conditionally negative definite function (see (Rao 1982b; Rao and Nayak 1985; Critchley and Fichet 1997; Pavoine et al. 2005)). Using the expression of d_{ij} and simply denoting the circumcenter $c_{d_W}(S_{n-1})$ by c, Rao's Quadratic Entropy $H_D(x)$ can be rewritten as follows:

$$H_D(x) = x^t D x = \sum_{i,j=1}^{n} d_{ij} x_i x_j = \frac{1}{2} \sum_{i,j=1}^{n} x_i x_j (e^i - e^j)^t W (e^i - e^j)$$
$$= \frac{1}{2} \sum_{i,j=1}^{n} x_i x_j ((e^i - c) - (e^j - c))^t W ((e^i - c) - (e^j - c)).$$

Letting $z^k = e^k - c$, for each $k = 1, \dots, n$, the previous expression becomes

$$\begin{aligned} H_D(x) =& \frac{1}{2}\sum_{i,j=1}^{n} x_i x_j((z^i)^t W z^i + (z^j)^t W z^j - 2(z^i)^t W z^j) \\ =& \frac{1}{2}\sum_{i,j=1}^{n} \left(2x_i x_j (z^i)^t W z^i - 2x_i x_j (z^i)^t W z^j\right) \\ =& \sum_{i,j=1}^{n} \left(x_i x_j (z^i)^t W z^i - x_i x_j (z^i)^t W z^j\right). \end{aligned}$$

Further, letting R be the distance of c from any point e^k, $k = 1, \dots, n$, and using the constraint $\sum_{i=1}^{n} x_i = 1$, we have:

$$\begin{aligned} H_D(x) &= \sum_{i=1}^{n} x_i (z^i)^t W z^i - \sum_{i,j=1}^{n} x_i (e^i - c)^t W x_j (e^j - c) \\ &= \sum_{i=1}^{n} x_i R^2 - \left(\sum_{i}^{n} x_i e^i - c\right)^t W \left(\sum_{j=1}^{n} x_j e^j - c\right) \\ &= \sum_{i=1}^{n} x_i R^2 - (x-c)^t W (x-c) = R^2 - d_W^2(x,c), \end{aligned}$$

so the result follows.

□

Proof of Proposition 2 In the case $n = 1$ by Definition 2 we trivially get that the unique coordinate of the circumcenter is equal to 1; therefore, for the rest of the proof, we let $n \geq 2$. Let $i \in \{1, \dots, n\}$; according to Definition 2 the circumcenter $c_{d_W}(S_{n-1})$ must satisfy the following system of equations:

$$\begin{cases} d_W(c, e^j) = d_W(c, e^i), & \forall j = 1, \dots, n, j \neq i \\ \sum_{i=1}^{n} c_i = 1. \end{cases} \tag{6}$$

Using Definition 1, system (6) can also be rewritten as follows:

$$\begin{cases} \sum_{k \neq j} w_k c_k^2 + w_j (c_j - 1)^2 = \sum_{k \neq i} w_k c_k^2 + w_i (c_i - 1)^2, & \forall j = 1, \dots, n, j \neq i \\ \sum_{i=1}^{n} c_i = 1 \end{cases}$$

that is

$$\begin{cases} c_j = \frac{1}{2}\left(1 - \frac{w_i}{w_j}\right) + 2\frac{w_i}{w_j} c_i, & \forall j = 1, \dots, n, j \neq i \\ \sum_{i=1}^{n} c_i = 1 \end{cases}$$

from which we get

$$\frac{1}{2}\sum_{j=1, j\neq i}^{n}\left(1 - \frac{w_i}{w_j}\right) + 2\sum_{j=1, j\neq i}^{n} \frac{w_i}{w_j} c_i + c_i = 1$$

and consequently

$$c_i = \frac{1}{2}\left(1 - \frac{n-2}{w_i \sum_{j=1}^{n} \frac{1}{w_j}}\right),$$

so that the result follows.

□

Proof of Proposition 3

(i) We assume that $c_i = \frac{1}{n}$, for each $i = 1, \ldots, n$; then, from formula (3), it follows

$$M_{W^{-1}} = \frac{1}{w_i},$$

that is all the diagonal elements w_i, $i = 1, \ldots, n$, are equal to a positive real value α, thus $W = \alpha I_n$. On the other hand, suppose that $w_i = \alpha$, $i = 1, \ldots, n$; then, $M_{W^{-1}} = \frac{1}{\alpha}$ and consequently (3) yields

$$c_i = \frac{1}{2}\left(1 - \frac{n-2}{n}\right) = \frac{1}{n}, \qquad \text{for each } i = 1, \ldots, n.$$

(ii) The inbetweeness property of the arithmetic mean yields $\frac{1}{w_{\max}} \leq M_{W^{-1}} \leq \frac{1}{w_{\min}}$, and consequently $w_{\min} \leq \frac{1}{M_{W^{-1}}} \leq w_{\max}$. Using such inequality in expression (3) we obtain

$$\frac{1}{2}\left(1 - \frac{n-2}{n}\frac{w_{\max}}{w_i}\right) \leq c_i \leq \frac{1}{2}\left(1 - \frac{n-2}{n}\frac{w_{\min}}{w_i}\right)$$

which yields

$$\frac{1}{n} - \frac{n-2}{2n}\left(\frac{w_{\max}}{w_{\min}} - 1\right) \leq c_i \leq \frac{1}{n} + \frac{n-2}{2n}\left(1 - \frac{w_{\min}}{w_{\max}}\right).$$

(iii) It is an immediate consequence of formula (3).

(iv) Let $i \in \{1, \ldots, n\}$. From formula (3) it is straightforward to get $c_i < \frac{1}{2}$. Further $nM_{W^{-1}} > \frac{1}{w_i}$, from which, using again (3), we get:

$$c_i = \frac{1}{2}\left(1 - \frac{n-2}{nw_i M_{W^{-1}}}\right) > -\frac{1}{2}(n-3).$$

(v) The assertion immediately follows from item (iv).

(vi) Using formula (3) the condition $c_i \geq 0$, for each $i = 1, \ldots, n$, can equivalently be rewritten as follows:

$$M_{W^{-1}} \geq \frac{n-2}{n} \cdot \frac{1}{w_i},$$

from which, computing the maximum of its right-hand side, the result follows.

(vii) From item (vi) and the inbetweeness property of the arithmetic mean we have

$$\frac{n-2}{n}\cdot\frac{1}{w_{\min}}\leq M_{W^{-1}}\leq\frac{1}{w_{\min}}$$

from which, if $n\to+\infty$, then $M_{W^{-1}}\to\frac{1}{w_{\min}}$. Finally, applying item (i), the result follows.

□

Proof of Proposition 5 From Carmichael et al. (2015), if the each function d_{ij} is conditionally negative definite, the RQE optimal portfolio is the unique solution of the constrained optimization problem (2). Consequently, by Definition 2 and 4 and Proposition 1 the result follows.

□

Proof of Proposition 6 Let $\rho(\beta)_{\max}$ and $M_{\rho(\beta)}$ be the maximum and the arithmetic mean of $\rho_1(\beta),\ldots,\rho_n(\beta)$ and $d=\frac{2}{n}\frac{\rho_{\max}}{\rho_{\max}-M_\rho}$ as given in Definition 5. By Remark 3, item (i), $\rho(\beta)_{\max}=\rho_{\max}$; further

$$\begin{aligned}M_{\rho(\beta)}=&\frac{1}{n}\sum_{i=1}^{n}\rho_i(\beta)=\frac{1}{n}\sum_{i=1}^{n}\left(\rho_{\max}-\beta d(\rho_{\max}-\rho_i)\right)\\=&\rho_{\max}\left(1-\frac{2}{n}\beta\frac{\rho_{\max}-M_\rho}{\rho_{\max}-M_\rho}\right)=\rho_{\max}-\beta d(\rho_{\max}-M_\rho).\end{aligned}\tag{7}$$

Using in the above expression the assumption $\beta\leq 1$ we get

$$M_{\rho(\beta)}\geq\rho_{\max}-\frac{2}{n}\rho_{\max}=\frac{n-2}{n}\rho_{\max}=\frac{n-2}{n}\rho(\beta)_{\max}$$

from which, by Proposition 4-(v), the result follows.

□

Proof of Proposition 7 We rewrite the RAGDP solution x^* (see formula (4)) as follows:

$$x^*=\frac{1}{n}\mathbf{1}_n+\frac{n-2}{n}\left(\mathbf{1}_n-\frac{\rho}{M_\rho}\right)$$

and, by some computations, we express the quantity Var(RAGDP) $=x^{*t}Vx^*$ as follows:

$$\begin{aligned}x^{*t}Vx^*=&\frac{1}{n^2}\mathbf{1}_n^tV\mathbf{1}_n+\\&+\frac{n-2}{4n^2M_\rho^2}\left[n(M_\rho\mathbf{1}_n-\rho)^tV(M_\rho\mathbf{1}_n-\rho)+2(M_\rho^2\mathbf{1}_n^tV\mathbf{1}_n-\rho^tV\rho)\right],\end{aligned}$$

from which item (i) immediately follows.

To prove item (ii), using formula (5), we express

$$\begin{aligned}\rho(\beta) =&(1-\beta d)\rho_{\max}+\beta d\rho\\ M_\rho(\beta) =&(1-\beta d)\rho_{\max}+\beta d M_\rho\\ D_\rho(\beta) =&M_\rho(\beta)\mathbf{1}_n-\rho(\beta)=\beta d D_\rho.\end{aligned}$$

By some computations, the condition expressed in item (i) with $\rho=\rho(\beta)$, $M_\rho=M_\rho(\beta)$ and $D_\rho=D_\rho(\beta)$ given above yields:

$$\beta d[(-4\rho_{\max}D_\rho^t V\mathbf{1}_n+nD_\rho^t VD_\rho+2(M_\rho^2\mathbf{1}_n^t V\mathbf{1}_n-\rho^t V\rho))\beta d+4\rho_{\max}D_\rho^t V\mathbf{1}_n]\le 0$$

from which item (ii) follows.

□

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Declarations

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