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Infrared properties of three dimensional gauge theories via supersymmetric indices

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Declaration of Authorship

I, Emanuele Beratto, declare that the material presented in this dissertation is based on my own research and any result obtained by other authors has been properly acknowledged.

I hereby declare that the original contents of this dissertation have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university.

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Abstract

Dipartimento di Fisica "G. Occhialini"

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Infrared properties of three dimensional gauge theories via supersymmetric indices

by Emanuele Beratto

The thesis focuses on the study of various supersymmetric three-dimensional gauge theories, mainly with at least $\mathcal{N}=3$ supersymmetry. We range between very different theories and discuss several different aspects with the aim of validate our assumptions. Therefore, the *leitmotiv* of this work resides not so much in the topics we cover, but rather in the method that we use to obtain such results. This, in fact, consists in analysing the gauge invariant operators of the theory forming the so-called chiral ring. By having access to the chiral ring structure of the theory and to the operators forming it, we gain insight to the properties that needed to confirm or debunk our hypothesis. We will essentially use two different tools for counting and studying such chiral operators: the Hilbert series and the three-dimensional superconformal index. Thanks to the Hilbert series, we propose a quiver description for the mirror theories of the circle reduction of four-dimensional twisted $\chi(\mathfrak{a}_{2N})$ theories of class S. These mirrors are, in fact, described by "almost" star-shaped quivers containing both unitary and orthosymplectic gauge groups, along with hypermultiplets in the fundamental representation. On the other hand, by means of the superconformal index, we investigate the $\mathcal{N}=2$ preserving exactly marginal operators of the so called S-fold theories. In particular, we focus on two families of such theories, constructed by gauging the diagonal flavour symmetry of the $T(U(N))$ and $T^{[2,1^2]}_{[2,1^2]}(SU(4))$ theories. In addition, we also examine in detail the zero-form and one-form global symmetries of the Aharony-Bergman-Jafferis theories, with at least $\mathcal{N}=6$ supersymmetry, and with both orthosymplectic and unitary gauge groups. A number of dualities among all these theories are discovered and studied using the aforementioned tools.

Contents

Chapter 1

Introduction

One of the most prolific sectors of modern theoretical physics deals with the research of new *dualities*, *i.e.* some set of relations between apparently different theories. These are indeed a spectacular feature of string theory and, usually, rely on some transformation of the underlying brane systems describing the involved theories.

This phenomenon was first discovered in the $\mathcal{N}=4$ super Yang-Mills (SYM) theory in four dimensions [139], which, in the low energy regime, describes the world-volume field theory of a stack of N D3-branes on flat space. In this case, it turns out that the duality is based on the action of the $SL(2,\mathbb{Z})$ group on the Type-IIB brane setup. In fact, by considering an element M of $SL(2,\mathbb{Z})$, its action leaves the D3-branes invariant, while transforms the $(p,\ q)$ fivebranes as follows (see Section (6.2.1))

$$\begin{pmatrix} p' \\ q' \end{pmatrix} = M \begin{pmatrix} p \\ q \end{pmatrix} \tag{1.0.1}$$

From the QFT point of view, looking at the holomorphic coupling τ defined as in (3.6.16), the action of the $SL(2,\mathbb{Z})$ duality group can be described as follows

$$\tau \to \frac{a+b\tau}{c+d\tau}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \tag{1.0.2}$$

It is thus a natural consequence of (1.0.2) to think that $\mathcal{N}=4$ SYM theories with different coupling constants could actually be dual [137] and that we can create a duality web relating all these different regimes. This result has important consequences on the knowledge we can extract from a theory. Suppose, in fact, that the original theory is in its strongly-coupled regime (*i.e.* τ is initially very large). Most of the computations we could perform in such a QFT rely on perturbative expansions in τ. These, however, become more and more inaccurate as the parameter increases and we could miss some important features of the phenomena we are studying. To overcome this problem, one can then imagine to apply a so-called S-transformation

$$S \quad : \quad \tau \to \frac{1}{\tau} \tag{1.0.3}$$

The resulting theory now sits in its weakly-coupled regime, which we know how to deal with. From this simple example, it is clear how dualities can in general help in gaining insight about the strong-interacting regime of gauge theories. S-duality, indeed, played a crucial role in string theory [150], by shaping it in the unified field that we know today and making it possible for M-theory [110] to be discovered.

In this thesis, we will be interested in three-dimensional $\mathcal{N}=4$ supersymmetric field theories. As the aforementioned four-dimensional case, these theories too enjoy a rich duality web. For example, a large class of such theories can be engineered in Type IIB string theory via Hanany-Witten brane systems [107] involving finite size

D3-branes and infinite five-branes preserving eight supercharges (see Section (3.1)). From this perspective, S-duality acts by swapping D5 and NS5 branes [36, 142] and manifests field-theoretically as *mirror symmetry* [118]. This duality, relating pair of theories with non-trivial fixed point, acts by exchanging the Higgs and Coulomb branches of the moduli spaces of vacua. While the first branch is simply parametrised by non-trivial vacuum expectation values (VEVs) of the complex scalar in the $\mathcal{N} = 4$ hypermultiplets; the latter is parametrised not only by VEVs of the real scalar in the vector multiplets but also by a new type of local operators, called *monopoles*. Thanks to $\mathcal{N} = 4$ supersymmetry both these branches enjoy hyperkähler structures that are swapped under mirror symmetry. However, while the Higgs branch is classically exact (and thus can be accessed using UV knowledge only), the coulomb branch instead receives quantum corrections. The importance of mirror symmetry thus lies on this fact: it allows to "trade" the hard-to-deal-with quantum effects for classical ones, that are usually easier to treat.

These very simple examples already show us the importance of finding and studying new dualities. For some three-dimensional $\mathcal{N} = 4$ supersymmetric theories, however, the Type IIB Hanany-Witten brane setup is not always available. In such cases a new conjectured duality must then be proved (or at least strongly suggested) by other methods. The one that we will prefer in the rest of this work consists in analysing the gauge invariant operators (GIOs) forming the so-called *chiral ring* [47]. These are, in fact, the GIOs that are annihilated by all the supercharges of one chosen chirality and their VEVs parametrise the entire moduli space of the theory. Thus, having insight on the chiral ring of the theory allows us to understand its moduli space structure and, thus, its potential duality web by finding other theories with the same moduli space geometry.

We will see essentially two ways of counting and studying such chiral operators: the Hilbert series [64, 65, 105, 135, 144] and the three-dimensional superconformal index [3, 4, 32, 33, 71, 117, 122, 126]. Even if there are many ways of looking into the chiral ring of a theory, the philosophy behind all these methods is the same and consists of modifying the definition of the most common counting quantity, *i.e.* the partition function

$$\mathcal{Z} = \mathrm{Tr}\left(e^{-i\beta H}\right) = \mathrm{Tr}\left(t^{H}\right) \tag{1.0.4}$$

which counts all the operators of the theory without any distinction. Clearly, definition 1.0.4 can be refined by introducing other chemical potentials related to the mutually commuting generators of the global symmetries of the theory. Once expanded in powers of t, this refining allows us to recognise which operator is contributing at which order in the expansion, thanks to the exponents of the other chemical potentials. These are in fact nothing but the global charges of the operators. However, definition 1.0.4 can also be modified by inserting projection operators which restrict the trace and change the final result of the operation. This is exactly the case for the aforementioned Hilbert series and superconformal index.

The Hilbert series is indeed the chiral ring generating function, defined in (2.2.33) as a sort of refined partition function restricted over the chiral ring. Even if it can be computed as a whole, it is always more convenient to split its evaluation between the two different branches of the moduli space. Thus, for the Higgs branch, which is protected against quantum corrections, it can be computed using the so-called Molien formula (2.2.53); while, for the Coulomb branch, which instead receives quantum corrections, it can be evaluated using the so-called monopole formula (2.2.55). In this work, we will compute both of them for a variety of three-dimensional $\mathcal{N} = 4$

supersymmetric theories obtained by performing an S-duality transformation on a specific class of other theories. The aim of such computations is, thus, to provide non-trivial checks of our proposed new mirror dualities. In fact, once computed, the Coulomb branch Hilbert series of the mirror theory can be matched with that of the Higgs branch of the original theory and *vice versa*, as follows

$$H_{\mathcal{C}}^{\text{mirror}} = H_{\mathcal{H}} \ , \qquad H_{\mathcal{H}}^{\text{mirror}} = H_{\mathcal{C}} \tag{1.0.5}$$

This fact allows us to conceal the tough geometric study of the moduli space in a way more simple task, which amounts only to count the chiral ring operators in a graded way.

The three-dimensional superconformal index, defined in (4.2.5) as a refined partition function restricted to the $\delta = 0$ states only, counts the BPS short multiplets up to recombination. In this sense, it allows, in general, to have detailed access to both the global symmetry and supersymmetry of a given theory. The underlying reason is that, for three-dimensional superconformal field theories (SCFTs), it is possible to put various short multiplets into equivalence classes according to how they contribute to the index [146] (see also Scetion (4.5.5)). One can thus easily identify all the conserved currents for the manifest global symmetries which, according to the case, can also enhance to a bigger symmetry group. For example, considering theories with at least $\mathcal{N} = 3$ supersymmetry, the index serves as a rather simple tool to spot the presence of extra-supersymmetry charges, which gives rise to supersymmetry enhancement (see *e.g.* [74]).

In this thesis, we will use this powerful tool to investigate the operators associated with the $\mathcal{N} = 2$ preserving exactly marginal deformations in a large class of three-dimensional SCFTs, known as the S-fold theories [14, 91, 94, 154]. These theories are obtained by gauging the diagonal $U(N)$ global symmetry of the $T(U(N))$ theory [85] with Chern–Simons (CS) level k and to couple it to matter systems. Indeed, the $T(U(N))$ theory enjoys a $U(N) \times U(N)$ global symmetry, one factor acting on the Coulomb and one on the Higgs branch. As a result of this gauging along with the presence of the CS level, the description possesses $\mathcal{N} = 3$ supersymmetry. The conformal manifold, *i.e.* the space generated by exactly marginal deformations, has been a long-standing subject of study in QFTs and has indeed led to a number of new dualities [143, 145, 147]. Thus we provide a detailed study of the exactly marginal operators in some S-fold theories.

Moreover, even if the three-dimensional superconformal index contains only detailed informations about the global (zero-form) symmetries of the theory, it can also be used to better understand its global one-form symmetry. The concept of one-form symmetry (or q-form symmetry) comes from the generalisation of the standard notion of symmetry. Given a standard (zero-form) symmetry group G there is in fact an associated conserved Noether one-form current J; thus, considering now a generalised q-form symmetry group $G^{[q]}$, there will be an associated conserved Noether q-form current $J^{[q]}$ in the theory[1]. Thus, since in three spacetime dimensions, gauging a one-form symmetry yields a zero-form symmetry, and *vice versa* [87]; in many cases the superconformal index allows us to indirectly study the one-form symmetry of the original theory via the zero-form symmetry of the theory in which such a one-form symmetry is gauged. Thanks to this fact, starting from well-known dualities between

[1]If the symmetry group G is not continuous we lack the Noether current J. We can then recast the symmetry transformation law of a set of charged operators $\{\mathcal{O}^I\}$ as the action of another operator U_g on them, with g a group element of the global symmetry G (see (6.1.5)). Thanks to this approach, all the results for continuous symmetries can be translated to the discrete case.

ABJ theories with orthosymplectic and unitary gauge groups [1, 54], we try to gauge their one-form symmetries (or subgroups). Computing the refined superconformal index of both the ungauged and the gauged theories, we propose new dualities by mapping the global symmetries of one theory to the other across the duality and, thus, obtaining equal indices.

All these examples highlight not only the importance of finding new dualities, but also the relevance of the operators indexing methods which allow us to discover and confirm such dualities. Actually, without such methods, most of this work would not exist.

The rest of the thesis is organized as follows. In Chapter (2) we will provide an introduction to three-dimensional $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supersymmetric theories. For the latter, we will also describe most of the properties of the moduli space and how to compute their Hilbert series. In Chapter (3), we will introduce the Hanany-Witten brane construction along with mirror symmetry and we will define $T^{\boldsymbol{\sigma}}_{\boldsymbol{\rho}}(SU(N))$ theories and show how their Coulomb branch Hilbert series can be computed. Then, by introducing orientifold Op planes in the brane setup, we will also define $T^{\boldsymbol{\sigma}}_{\boldsymbol{\rho}}(USp'(2N))$ theories again with their Coulomb branch Hilbert series. Both these types of theories will in fact serve us later when, after reviewing the basic ideas and results of the class-S framework, we will compute the Hilbert series for some 3d mirror theories of the circle reduction of twisted $\chi(\mathfrak{a}_{2N})$ theories of class-S.

Then, in Chapter (4) we will introduce the localization procedure and compute the three-dimensional superconformal index. We will also give some explicit expressions of the index that will be needed in the subsequent sections. In Chapter (5) we will study the operators associated with the $\mathcal{N} = 2$ preserving exactly marginal deformations of some of the S-fold theories. In doing so, we will compute the superconformal index of such theories and analyse the contributions of the short multiplets order by order in the character expansion. As we will show, sometimes it will happen that supersymmetry gets enhanced to a larger group. Finally, in Chapter (6) we will talk about generalised global symmetries, focusing on the global one-form symmetries and their possible gauging. Then, after introducing ABJM and ABJ-like theories with both unitary and orthosymplectic gauge groups, we will propose new dualities between such theories involving discrete one-form symmetry gaugings. We will then compute the superconformal index for each of these theories to support the argued dualities.

We will conclude the thesis with a brief summary of the results, including some possible future directions, in Chapter (7). For ease of reading, many technical details are gathered in four appendices.

Chapter 2

$3d$ supersymmetric gauge theories

The entirety of this work has as its cornerstone three-dimensional supersymmetric gauge theories. This family of theories have been already widely studied in literature but it could nonetheless be useful to collect some known concepts that will be used later on. Thus, in this chapter, we will focus the discussion only on aspects of such $3d$ theories which are important for this thesis.

In particular, we will talk about the moduli space of vacua and its generators, *i.e.* the chiral operators. The heart of the discussion will be the introduction of the Hilbert series; one of the possible methods for indexing such chiral operators.

2.1 $3d$ $\mathcal{N}=2$ theories on flat space

The minimum possible amount of supersymmetry in three dimensions corresponds to four real supercharges grouped together into two Majorana spinors $\mathcal{Q}^I_\alpha$. Equivalently, we can consider two complex supercharges $\mathcal{Q}$ and $\overline{\mathcal{Q}}$ subject to the reality condition

$$\mathcal{Q} = \overline{\mathcal{Q}} \tag{2.1.1}$$

so that, in this way, we can include in a single complex supercharge all the minimal amount of real charges in $3d$. Moreover, this notation turns out to be very useful when considering dimensional reduction from the $4d$ case, in that the supercharges are simply inherited and constrained by (2.1.1).

The supersymmetry algebra then reads

$$\{\mathcal{Q}_\alpha, \overline{\mathcal{Q}}_\beta\} = 2\gamma^\mu_{\alpha\beta} P_\mu + 2i\epsilon_{\alpha\beta} Z \tag{2.1.2}$$

where P_μ is the momentum, Z is the real central charge and γ^μ are the Pauli matrices.

The R-symmetry group rotating these two supercharges is thus $SO(2)_R \simeq U(1)_R$ and acts as

$$\mathcal{Q} \to e^{i\alpha}\mathcal{Q}, \qquad\qquad \overline{\mathcal{Q}} \to e^{-i\alpha}\overline{\mathcal{Q}} \tag{2.1.3}$$

The relevant matter multiplets of this theory can be easily found by dimensional reduction of the $4d$ $\mathcal{N}=1$ case

Multiplet	Content		G
	$d=4$	$d=3$	
Vector (V)	A_M	A_μ	Adjoint
		$A_3 := \sigma$	
	λ	λ	
	D	D	
Chiral (Φ)	ϕ	ϕ	$\mathcal{R}$
	ψ	ψ	
	F	F	

(2.1.4)

where the $4d$ gauge vector A_M decomposes into the $3d$ gauge vector A_μ and a real scalar σ, the fermions λ, ψ respectively in the vector and chiral multiplet are Dirac spinors, the vector multiplet scalar D is real and the chiral multiplet scalars ϕ and F are both complex. Moreover, the vector multiplet transforms in the adjoint representation of the gauge group G, while the chiral multiplet transforms in a given representation $\mathcal{R}$. Observe that F and D are both auxiliary scalars, which equation of motions are related to the construction of the moduli space of vacua.

Having all these fields, we can now construct our $3d$ $\mathcal{N}=2$ Lagrangian spoiling the superspace formulation of the $4d$ $\mathcal{N}=1$ case. This can be written as a sum of several different components

$$\mathcal{L} = \mathcal{L}_{\text{SYM}} + \mathcal{L}_{\text{SCS}} + \mathcal{L}_{\text{FI}} + \mathcal{L}_{\text{Matter}} \tag{2.1.5}$$

where

1. The super Yang-Mills Lagrangian is

$$\begin{aligned} \mathcal{L}_{\text{SYM}} &= \frac{1}{g^2}\int d^2\theta \, \text{Tr}\left\{W^\alpha W_\alpha\right\} + \text{c.c} = \\ &= \text{Tr}\left\{\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\mathcal{D}_\mu\sigma\mathcal{D}^\mu\sigma - i\overline{\lambda}\gamma^\mu\mathcal{D}_\mu\lambda - i\overline{\lambda}[\sigma,\lambda] + \frac{1}{2}D^2\right\} \end{aligned} \tag{2.1.6}$$

   where W_α is the field strength superfield constructed using the $\mathcal{N}=2$ vector multiplet and $\mathcal{D}_\mu$ is the flat gauge covariant derivative.

2. The standard kinetic term for the vector multiplet can be supplemented in $3d$ by a supersymmetric Chern–Simons term

$$\mathcal{L}_{\text{SCS}} = \frac{k}{4\pi}\,\text{Tr}\left\{\varepsilon^{\mu\nu\rho}\left(A_\mu\partial_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho\right) - \overline{\lambda}\lambda + 2D\sigma\right\} \tag{2.1.7}$$

   where, for non-Abelian theories, the level k is quantised; for $SU(N)$ or $U(N)$ gauge groups it is quantised to be an integer when the trace is in the fundamental representation.

3. The Fayet-Iliopulos Lagrangian is

$$\mathcal{L}_{\text{FI}} = 2g\sum_A \xi^A \int d^2\theta d^2\overline{\theta} V_A = g\sum_A \xi^A D_A \tag{2.1.8}$$

where $A=1,\ldots,n$ labels the abelian $U(1)$ factors inside G

4. Finally, the matter Lagrangian is

$$
\begin{aligned}
\mathcal{L}_{\text{Matter}} = & \int d^2\theta d^2\overline{\theta} \sum_i \overline{\Phi}^{(i)} e^{2gV} \Phi^{(i)} + \int d^2\theta \mathcal{W}[\Phi^{(i)}] + \int d^2\overline{\theta}\overline{\mathcal{W}}[\overline{\Phi}^{(i)}] = \\
= & \overline{\mathcal{D}_\mu \phi^{(i)}} \mathcal{D}^\mu \phi^{(i)} - i\overline{\psi}^{(i)} \gamma^\mu \mathcal{D}_\mu \psi^{(i)} - i\overline{\psi}^{(i)} \sigma \psi^{(i)} - \overline{F}^{(i)} F^{(i)} + \\
& + i\sqrt{2}g \left(\overline{\phi}^{(i)} \lambda \psi^{(i)} + \overline{\psi}^{(i)} \overline{\lambda} \phi^{(i)} \right) + g\overline{\phi}^{(i)} \left(\sigma^2 + D \right) \phi^{(i)} + \\
& + \frac{\partial \mathcal{W}}{\partial \phi^{(i)}} F^{(i)} + \frac{\partial \overline{\mathcal{W}}}{\partial \overline{\phi}^{(i)}} \overline{F}^{(i)} + \frac{1}{2} \frac{\partial^2 \mathcal{W}}{\partial \phi^{(i)} \partial \phi^{(j)}} \psi^{(i)} \psi^{(j)} + \frac{1}{2} \frac{\partial^2 \overline{\mathcal{W}}}{\partial \overline{\phi}^{(i)} \partial \overline{\phi}^{(j)}} \overline{\psi}^{(i)} \overline{\psi}^{(j)}
\end{aligned}
\tag{2.1.9}
$$

where we sum over $i,j=1,\ldots,N_F$ labelling the total number of matter chiral multiplets and $\mathcal{W}$ is the superpotential, which is a gauge invariant holomorphic function of Φ with R-charge 2.

2.1.1 Topological symmetry and monopole operators

To introduce a special feature of three-dimensional gauge theories, suppose now we are studying the $3d$ Maxwell theory without supersymmetry. The 2-form field strength $F^{\mu\nu}$ satisfies both the equations of motion and the Bianchi identity

$$dF = \frac{1}{2}\partial_{[\rho}F_{\mu\nu]}dx^\mu \wedge dx^\nu \wedge dx^\rho = 0 \tag{2.1.10}$$

$$d\star F = \frac{1}{2}\epsilon^{\rho\mu\nu}\partial_{[\lambda}F_{\mu\nu]}dx^\lambda \wedge dx^\rho = 0 \tag{2.1.11}$$

where the $\star$ represents the Hodge dual operator.

In this $3d$ case, the Hodge dual of F is a 1-form current

$$J^\mu_{\text{top}} := (\star F)^\mu = \frac{1}{2}\epsilon^{\mu\rho\sigma}F_{\rho\sigma} \tag{2.1.12}$$

that is conserved in virtue of Bianchi identity

$$dJ_{\text{top}} = d\star F = 0 \tag{2.1.13}$$

The physical theory under study enjoys a symmetry not explicitly readable from the Lagrangian and, thus, the associated current cannot be thought as a standard Noëther current. We call such a symmetry an hidden symmetry. Moreover, the fields which carry a non-zero charge associated with this symmetry are not explicitly present in the Lagrangian.

This result can be generalised to each Lie group G which contains a $U(1)^k$ subgroup. Then there will be as many different currents as the number of such $U(1)$ factors. This particular hidden symmetry is called topological symmetry.

The existence of such a symmetry is related to the fact that in $d=3$ the photon is dual to a scalar φ called the dual photon. Since the photon possesses only one polarisation, it is thus natural to think of it as a simple scalar field subject to some constraints.

Let us consider the QED partition function for the gauge field

$$\mathcal{Z} = \int \mathscr{D}[A_\mu] \exp\left\{ i \int d^3x - \frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} \right\} \tag{2.1.14}$$

We can trade the measure $\mathscr{D}[A_\mu]$ for $\mathscr{D}[F_{\mu\nu}]$ since the Maxwell action depends only on the 2-form. We just have to take care of the Bianchi identity $d \star F = 0$ which is not predictable from the action only. This can be done introducing of a Lagrange multiplier φ such that now

$$\mathcal{Z} = \int \mathscr{D}[F_{\mu\nu}] \mathscr{D}[\varphi] \exp\left\{ i \int d^3x - \frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{4\pi} \varphi \epsilon^{\mu\rho\sigma} F_{\rho\sigma} \right\} \tag{2.1.15}$$

We can now integrate out $F_{\mu\nu}$ by using its new equation of motion

$$F^{\mu\nu} = -\frac{g^2}{2\pi} \epsilon^{\mu\nu\rho} \partial_\rho \varphi \tag{2.1.16}$$

This leads to the following partition function

$$\mathcal{Z} = \int \mathscr{D}[\varphi] \exp\left\{ i \int d^3x \frac{g^2}{8\pi^2} (\partial_\mu \varphi)^2 \right\} \tag{2.1.17}$$

This new scalar φ is indeed the dual photon. By virtue of this duality, the original equation of motion and Bianchi identity swap for φ

$$\begin{aligned} dJ' := d(d\varphi) = dJ_{\text{top}} = d \star F = d \star dA = 0 \\ d \star (d\varphi) = d \star J_{\text{top}} = dF = d(dA) = 0 \end{aligned} \tag{2.1.18}$$

where clearly we have identified the differential of the dual photon with the topological current of the previous dual theory:

$$J'^\mu = J^\mu_{\text{top}} = -\frac{g^2}{4\pi^2} \partial_\mu \varphi \tag{2.1.19}$$

which now realises the translation symmetry $\varphi(x) \to \varphi(x) + \alpha$.

The topological symmetry thus acts by shifting the dual photon φ of a periodic scalar α.

Now we want to look for operators possessing a non-vanishing charge under such a topological symmetry. We can notice that the conserved quantity associated to this symmetry is simply the magnetic flux

$$Q_{\text{top}} = \int d^2x J^0_{\text{top}} = \frac{1}{2\pi} \int d^2x F^{12} \tag{2.1.20}$$

Thus we need to find a "special" local operator which represents unity of magnetic flux. This can be achieved by editing the path integral and removing a single space-time point x. Then a non-trivial boundary condition for the gauge field A_μ is naturally required on any surface Σ surrounding that point. Thus we can define the monopole creation operator $V^\dagger(x)$ as the defect operator which imposes the unity of magnetic flux

$$\frac{1}{4\pi} \int_\Sigma d^2\gamma_\mu \varepsilon^{\mu\nu\rho} F_{\nu\rho} = 1 \tag{2.1.21}$$

This is not a surprise, since in quantum field theory local operators do not have to be described as polynomials in the fundamental fields [121]; they may also include disorder or defect operators, which are obtained exactly by performing the path integral with suitable singular boundary conditions. For the upcoming discussion we will closely follow two very nice reviews [61, 155].

Thus one can compute correlation functions involving the new operator $V(x)$ in the usual way inserting it in the path integral.

So, in the presence of $V^\dagger(x)$, the topological current is no longer conserved; instead it has a source

$$\partial_\mu J^\mu_{\text{top}} = \delta^3(x) \tag{2.1.22}$$

Equivalently, the monopole operator is charged under $U(1)_{\text{top}}$ so that

$$U(1)_{\text{top}}\colon V^\dagger(x) \to e^{i\alpha} V^\dagger(x) \tag{2.1.23}$$

Using the dual photon picture, where the symmetry is manifest, we can implement a monopole operator directly in the path integral by simply adding a $i\varphi(x)$ term

$$\mathcal{Z} = \int \mathscr{D}[F_{\mu\nu}]\mathscr{D}[\varphi] \exp\left\{ i \int d^3x - \frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + \frac{1}{4\pi}\varphi\varepsilon^{\mu\rho\sigma}F_{\rho\sigma} + \varphi \right\} \tag{2.1.24}$$

This ensures that the topological current has a source and thus the equation of motion for the dual photon reads exactly

$$\partial_\mu J'^\mu = \partial_\mu J^\mu_{\text{top}} = \delta^3(x) \tag{2.1.25}$$

Thus in this picture a monopole operator corresponds to [5]

$$V^\dagger(x) \sim e^{i\varphi(x)} \tag{2.1.26}$$

from which we can see that indeed the topological symmetry is realised as a translation on the dual photon

$$U(1)_{\text{top}}\colon e^{i\alpha} V^\dagger(x) \sim e^{i(\varphi(x)+\alpha)} \tag{2.1.27}$$

Let us now examine in more details the features of the monopole operator in the original theory. We will consider a generic gauge group G

By removing a point from $\mathbb{R}^3$, space-time becomes a manifold that cannot be covered completely by one single patch. Since the minimum amount of patches is two, we will parametrise the space with two emispheres

$$\begin{aligned} \mathcal{U}_N &= \{\phi \in [0, 2\pi], \theta \in [0, \frac{\pi}{2} + \epsilon]\} \\ \mathcal{U}_S &= \{\phi \in [0, 2\pi], \theta \in [0, \frac{\pi}{2} - \epsilon]\} \end{aligned} \tag{2.1.28}$$

The corresponding two gauge connection develops a Dirac monopole singularity [39] at the centre of the sphere x, *i.e.* the monopole insertion point

$$\begin{aligned} A_N(r) &\sim \frac{m}{2}\frac{1-\cos\theta}{\sin\theta} d\phi \\ A_S(r) &\sim \frac{m}{2}\frac{-1-\cos\theta}{\sin\theta} d\phi \end{aligned} \tag{2.1.29}$$

where m, a priori, is an element of the Lie algebra $\mathfrak{g}$ of the gauge group G.

The magnetic flux provided by these gauge fields must then be quantised according to Dirac quantisation. For $\theta \neq 0, \pi$ we can relate the gauge fields by a gauge transformation

$$A_N = t_{NS}^{-1} A_S t_{NS} - i t_{NS}^{-1} d t_{NS} \tag{2.1.30}$$

where t_{NS} is the transition function between the two patches of the bundle.

By requiring the transition function to be smooth and single-valued between the patches, one finds the Dirac quantisation condition [73, 99]

$$\exp\{2\pi i m\} = \mathbb{1}_G \tag{2.1.31}$$

This condition requires m to belong to the weight lattice Γ of $\widehat{G}$, the Langland dual of the group G. However, we must consider the fact that the Weyl group $\mathcal{W}_{\widehat{G}}$ acts on m as on any other weight vector and so, since we will always look at gauge invariant monopole operators only (*i.e.* the one that are invariants under the Weyl group), we should restrict to $m \in \Gamma_{\widehat{G}}/\mathcal{W}_{\widehat{G}}$.

Moreover we can always choose a gauge where in each patch m is a constant element of the Cartan subalgebra modulo the action of the Weyl group, which allows to define the magnetic charge $m = (m_1, \dots, m_r), m_a \in \mathbb{Z}, r = \text{Rank}\, G$. Thus, considering for example the case of $G = SU(N) = \widehat{G}$, modding out the action of the Weyl group $\mathcal{W}_{U(N)} = S^N$, we get that the magnetic charges satisfies $m_1 \geq m_2 \geq \ldots \geq m_r$, *i.e.* $m \in \mathbb{Z}^N/S^N$.

Considering now the $\mathcal{N} = 2$ supersymmetric case [38], in order to have a BPS monopole operator, one *a priori* has to assign similar singular behaviours to the matter fields inside the vector multiplet. It turns out that, in order for the BPS monopole to preserve the same amount of supersymmetry of an $\mathcal{N} = 2$ chiral multiplet, one must specify a boundary condition only for the real scalar σ of the form

$$\sigma \sim \frac{m}{2r} \tag{2.1.32}$$

In this supersymmetric setup, the BPS monopole operator gets the form

$$V^\dagger(x) \sim e^{i\varphi(x) + \sigma(x)} \tag{2.1.33}$$

where the dual photon φ combines with the adjoint scalar σ to form this new holomorphic operator.

2.2 3d $\mathcal{N} = 4$ theories on flat space

In the next chapters we will study superconformal indices of $\mathcal{N} = 4$ theories with eight real supercharges, satisfying

$$\begin{aligned} \{\mathcal{Q}^I_\alpha, \overline{\mathcal{Q}}^J_\beta\} &= 2\delta^{IJ}\gamma^\mu_{\alpha\beta} P_\mu + 2i\epsilon_{\alpha\beta} Z^{[I,J]} \\ \mathcal{Q}^I &= \overline{\mathcal{Q}}^I \end{aligned} \tag{2.2.1}$$

which are the generalisation of (2.1.2) and (2.1.1) with $I, J = 1, 2$.

The supercharges $\mathcal{Q}^I$ and $\mathcal{Q}^J$ transforms in the vector representation of the $SU(2)_L \times SU(2)_R \simeq SO(4)$ R-symmetry.

The previous discussion on $\mathcal{N} = 2$ three dimensional theories already encloses everything we need to construct a $\mathcal{N} = 4$ gauge theory. In fact, we can choose a particular $\mathcal{N}=2$ subalgebra inside the $\mathcal{N}=4$ one which allows us to remain in a $\mathcal{N}=2$ formulation with a specific field content and action.

More precisely, the $\mathcal{N}=4$ vector multiplets and hypermultiplets can be schematically decomposed in a $\mathcal{N}=2$ language as follows

Multiplet		Content	G
$d=3$ $\mathcal{N}=4$	$d=3$ $\mathcal{N}=2$		
Vector (V)	Vector (V)	A_μ	Adjoint
		σ	
		λ	
		D	
	Chiral (Φ)	ϕ	
		ρ	
		$\mathcal{F}$	
Hyper (H)	Chiral (χ)	H	$\mathcal{R}$
		ψ	
		F	
	Chiral $(\widetilde{\chi})$	$\widetilde{H}$	$\mathcal{R}^*$
		$\widetilde{\psi}$	
		$\widetilde{F}$	

(2.2.2)

where the $\mathcal{R}^*$ gauge representation of the $\mathcal{N}=2$ $\widetilde{\chi}$ chiral multiplet is the complex conjugate of the $\mathcal{R}$ one of the $\mathcal{N}=2$ χ chiral multiplet.

Moreover, it is interesting to observe how the various component fields combine to give representations of the $SU(2)_L \times SU(2)_R$ R-symmetry group

Multiplet	Content	$SU(2)_L \times SU(2)_R$
Vector (V)	A_μ	$(0,0)$
	$\{\sigma, \mathrm{Re}\,\phi, \mathrm{Im}\,\phi\}$	$(0,1)$
	$\{\lambda, \rho\}$	$(\frac{1}{2}, \frac{1}{2})$
	$\{D, \mathrm{Re}\,\mathcal{F}, \mathrm{Im}\,\mathcal{F}\}$	$(1,0)$
Hyper (H)	$\{H, \overline{\widetilde{H}}\}$	$(\frac{1}{2}, 0)$
	$\{\psi, \overline{\widetilde{\psi}}\}$	$(\frac{1}{2}, \frac{1}{2})$
	$\{F, \overline{\widetilde{F}}\}$	$(0,0)$

(2.2.3)

Hence, the Lagrangian for a $\mathcal{N}=4$ gauge theory can be written using the expressions we introduced for the $\mathcal{N}=2$ case. The expression is the same of (2.1.5), where now

1. The super Yang-Mills Lagrangian is

$$\begin{aligned}\mathcal{L}_{\text{SYM}} &= \frac{1}{g^2}\int d^2\theta\,\text{Tr}\left\{W^\alpha W_\alpha\right\} + \text{c.c.} + \int d^2\theta d^2\overline{\theta}\,\text{Tr}\left\{\overline{\Phi}e^{2gV}\Phi\right\} = \\ &= \text{Tr}\left\{\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\mathcal{D}_\mu\sigma\mathcal{D}^\mu\sigma - i\overline{\lambda}\gamma^\mu\mathcal{D}_\mu\lambda - i\overline{\lambda}[\sigma,\lambda] + \frac{1}{2}D^2 + \right. \\ &+ \overline{\mathcal{D}_\mu\phi}\mathcal{D}^\mu\phi + [\overline{\phi},\sigma][\sigma,\phi] - i\overline{\rho}\gamma^\mu\mathcal{D}_\mu\rho - i\overline{\rho}[\sigma,\rho] - \overline{\mathcal{F}}\mathcal{F} + \\ &\left. + i\sqrt{2}g\left(\overline{\phi}\{\lambda,\rho\} + \{\overline{\rho},\overline{\lambda}\}\phi\right) + gD[\phi,\overline{\phi}]\right\}\end{aligned} \tag{2.2.4}$$

where now everything sits in an adjoint representation of the gauge group G.

2. The Chern–Simons Lagrangian becomes

$$\mathcal{L}^{\mathcal{N}=4}_{\text{SCS}} = \frac{k}{4\pi}\,\text{Tr}\left\{A\wedge dA + \frac{2}{3}A^3 - \overline{\lambda}\lambda + 2D\sigma\right\} - \frac{k}{4\pi}\int d^2\theta\,\text{Tr}\left\{\Phi^2\right\} + \text{c.c.} \tag{2.2.5}$$

where now the second contribution makes Φ enter the superpotential in a new way.

Note that this supersymmetric Chern–Simons term preserves only $\mathcal{N}=3$ supersymmetry, *i.e.* six real supercharges rotated by an $SO(3)$ R-symmetry group.

3. The Fayet-Iliopulos Lagrangian is

$$\mathcal{L}^{\mathcal{N}=4}_{\text{FI}} = 2g\sum_A \xi^A\int d^2\theta d^2\overline{\theta}V_A = g\sum_A \xi^A D_A \tag{2.2.6}$$

4. The Matter Lagrangian is

$$\begin{aligned}\mathcal{L}^{\mathcal{N}=4}_{\text{Matter}} &= \sum_i\left\{\int d^2\theta d^2\overline{\theta}(\overline{H}^{(i)}e^{2gV}H^{(i)} + \overline{\widetilde{H}}^{(i)}e^{-2gV}\widetilde{H}^{(i)}) + \right. \\ &\left. + \int d^2\theta\sqrt{2}gH^{(i)}\Phi\widetilde{H}^{(i)} + \int d^2\overline{\theta}\sqrt{2}g\overline{\widetilde{H}}^{(i)}\overline{\Phi}\overline{H}^{(i)}\right\} = \\ &= \sum_i\left\{\left[\overline{\mathcal{D}_\mu H^{(i)}}\mathcal{D}^\mu H^{(i)} + g\overline{H}^{(i)}\left(\sigma^2 + D\right)H^{(i)} + \right.\right. \\ &\left. + i\sqrt{2}g\left(\overline{H}^{(i)}\lambda\psi^{(i)} + \overline{\psi}^{(i)}\overline{\lambda}H^{(i)}\right)\right] + \left[H^{(i)}\to-\widetilde{H}^{(i)}\right] + \\ &- i\psi^{(i)}\gamma^\mu\mathcal{D}_\mu\overline{\psi}^{(i)} - i\psi^{(i)}\sigma\overline{\psi}^{(i)} - \overline{F}^{(i)}F^{(i)} + \\ &+ \sqrt{2}g\left(2H^{(i)}\rho\widetilde{\psi}^{(i)} + 2\psi^{(i)}\rho\widetilde{H}^{(i)} + 2\psi^{(i)}\phi\widetilde{\psi}^{(i)} + \right. \\ &\left. + H^{(i)}\phi\widetilde{F}^{(i)} + F^{(i)}\phi\widetilde{H}^{(i)} + H^{(i)}\mathcal{F}\widetilde{H}^{(i)}\right) + \\ &\left. + \sqrt{2}g\left(H^{(i)}\to\overline{H}^{(i)},\Phi\to\overline{\Phi},\widetilde{H}^{(i)}\to\overline{\widetilde{H}}^{(i)}\right)\right\}\end{aligned} \tag{2.2.7}$$

where, again, we sum over $i = 1,\dots,N_F$ labelling the total number of matter hypermultiplets; however the larger amount of supersymmetry now severely constrains the form of the superpotential to be $\mathcal{W} = H^{(i)}\Phi\widetilde{H}^{(i)}$.

2.2.1 Moduli space of supersymmetric vacua

Consider a generic classical field theory. We can define an equivalence relation on the set of all vacua (*i.e.* the states of minimal energy) in the following way: two vacua Ω_1 and Ω_2 are equivalent if there exists a gauge transformation which sends Ω_1 into Ω_2. The moduli space is therefore defined to be the set $\mathcal{M}$ of all inequivalent vacua [134].

After quantisation the energy becomes a functional, namely the Hamiltonian. Therefore, we say that a state is a vacuum state if the expectation value of the Hamiltonian on such state is minimal. Since all the kinetic terms in the Hamiltonian are quadratic in the derivatives of the fields, in a vacuum field configuration, all the fields must be constants over spacetime. Then, the Lorentz invariant nature of the vacuum forces all fields apart from the scalars to be not only constant, but identical to zero in any vacuum configuration. Thus only scalar fields can assume non-vanishing vacuum expectation values and these are exactly coordinates on the moduli space $\mathcal{M}$, turning it into a differentiable manifold.

One of the most interesting aspects of supersymmetric gauge theories is the possibility of possessing non-trivial moduli spaces. This happens whenever it is possible to find non-trivial configuration of scalar fields that make their respective scalar potentials to be zero. For a generic $\mathcal{N} = 4$ supersymmetric theory, the expression of such a scalar potential can be always written as a sum of squared quantities

$$V_\varphi = \sum_{\substack{\mathcal{N}=2 \\ \text{chirals}}} |F|^2 + \frac{g^2}{2} \sum_{\substack{\mathcal{N}=2 \\ \text{vectors}}} D^2 \geq 0 \tag{2.2.8}$$

where F and D come from the integration of the auxiliary fields in all the $\mathcal{N} = 2$ chiral and vector multiplets respectively. They are the so-called F- and D-terms and their form depends on the space-time dimension and on the amount of supersymmetry. Since the scalar potential is the sum of squares of F and D-terms, the moduli space of vacua is obtained by putting these two set of expressions to zero giving rise, respectively, to F and D-term equations.

Considering our previous $\mathcal{N} = 4$ Lagrangian, *i.e.* (2.2.4)-(2.2.7), these equations read:

$$\begin{aligned}
\phi_a (T^a)_m{}^n \widetilde{H}^{(i)}_n &= 0 \\
H^{(i)n} (T^a)_n{}^m \phi_a &= 0 \\
\sqrt{2} g \sum_i H^{(i)} T^a \widetilde{H}^{(i)} - \frac{k}{2\pi} \phi_a &= 0 \\
-g[\phi, \overline{\phi}]_a - g \sum_i (\overline{H}^{(i)} T_a H^{(i)} + \overline{\widetilde{H}}^{(i)} T_a \widetilde{H}^{(i)}) - g\xi_a - \frac{k}{2\pi} \sigma_a &= 0 \\
\sigma_a (T^a)_m{}^n H^{(i)}_n &= 0 \\
\sigma_a (T^a)_m{}^n \widetilde{H}^{(i)}_n &= 0 \\
[\sigma, \phi]_a &= 0
\end{aligned} \tag{2.2.9}$$

Forgetting for now the Fayet-Iliopulos and Chern–Simons terms, the existence of two different sets of scalars, the ones in the hypermultiplets $\{H^{(i)}, \widetilde{H}^{(i)}\}$, and the ones in the vector multiplets $\{\sigma_a, \phi_a\}$, implies that the moduli space of vacua is composed of two different parts (or at most three), called *branches* and joined together at the origin:

1. Coulomb Branch (CB):

$$\begin{aligned} H^{(i)} = \widetilde{H}^{(i)} &= 0 \\ \phi_a, \sigma_a &\neq 0 \end{aligned} \tag{2.2.10}$$

   We can spoil the fact that the only remaining scalars reorganise themselves as a unique real scalar $\eta_i = (\sigma, \mathrm{Re}[\phi], \mathrm{Im}[\phi])$ transforming as a $(0,1)$ vector under the R-symmetry group $SU(2)_L \times SU(2)_R$.

   Thus, the scalar potential can be written as

$$V_\varphi(\eta_i) = \sum_{i<j} \mathrm{Tr}_G\, [\eta_i, \eta_j]^2 \geq 0 \tag{2.2.11}$$

   To obtain $V_\varphi(\eta_i) = 0$ we must then take η_i to lie in the Cartan subalgebra of the gauge group G.

   Due to spontaneous Higgsing the gauge group is then broken down to its maximal torus $U(1)^r$ with r massless vectors and $(\mathrm{Dim}\{G\} - r)$ massive ones with masses proportional to $[\eta, A_\mu^{(\text{broken})}]$.

   In addition to the η_i, there are then r massless photons. Since a photon is dual to a scalar in three space-time dimensions, there is a total of $4r$ massless scalars. They can combine together to form the scalar components of r hypermultiplets that can be thought of as living on the Coulomb branch and which VEVs can be taken as the coordinates on it. In the geometry of the problem this is reflected promoting the Coulomb branch manifold to an Hyperkähler maniofold.

   Thus, for each gauge group G_i we are considering in the theory, the classical Coulomb branch gets an additional real dimension of $4\,\mathrm{Rank}\{G_i\}$, or simply $\mathrm{Rank}\{G_i\}$ if we express it in quaternionic units:

$$\dim_{\mathbb{H}}\{\mathcal{M}_{\mathcal{C}}\} = \sum_i \mathrm{Rank}\{G_i\} \tag{2.2.12}$$

   The classical Coulomb branch has then to be quantum mechanically corrected by loop corrections and instanton effects [148, 149], but his structure always remains Hyperkähler.

2. Higgs Branch (HB):

$$\begin{aligned} H^{(i)}, \widetilde{H}^{(i)} &\neq 0 \\ \phi_a = \sigma_a &= 0 \end{aligned} \tag{2.2.13}$$

   Nothing too general can be said about this branch since its explicit form depends on both the representations of the hypermultiplets and their number of flavours. The Higgsing changes accordingly. Indeed giving VEVs to these scalars breaks G down to a certain subgroup G^{unbr}; clearly, when the number of hypermultiplets is big enough, complete breaking occurs.

   By construction, however, there are at least two complex scalars $\{H^{(i)}, \widetilde{H}^{(i)}\}$ for each $\mathcal{N} = 2$ hypermultiplet. This automatically allows us to evaluate the Higgs branch dimension in quaternionic units and conclude that it must also be an Hyperkähler manifold.

Consequently the Higgs branch dimension is given by the number of $\mathcal{N}=4$ hypermultiplets N_i charged under every gauge group G_i minus the number of gauge fields that become massive due to Higgsing $\mathrm{Dim}\{G_i\} - \mathrm{Dim}\{G_i^{\mathrm{unbr}}\}$:

$$\dim_{\mathbb{H}}\{\mathcal{M}_{\mathcal{H}}\} = \sum_i \Big[N_i - \big(\mathrm{Dim}\{G_i\} - \mathrm{Dim}\{G_i^{\mathrm{unbr}}\}\big)\Big] \tag{2.2.14}$$

where usually $\mathrm{Dim}\{G_i^{\mathrm{unbr}}\} = 0$.

It is widely believed that Higgs branch is not corrected at quantum level [11].

3. Mixed Branch (MB):

$$\begin{aligned} H^{(i)}, \widetilde{H}^{(i)} &\neq 0 \\ \phi_a, \sigma_a &\neq 0 \end{aligned} \tag{2.2.15}$$

In the most complicated cases, also mixed branches can appear. This requires to find the general solution for F and D-flatness conditions. This is challenging in general but, however, it must be stressed that also in this case the two sets of operators participate to the dynamics independently and the mixed branch is actually a product of two manifolds.

An example of a discussion of mixed branches for three-dimensional N $=4$ theories can be found in [48].

When Chern–Simons couplings are present, the story is slightly different.

In this case the Coulomb branch, *i.e.* where hypermultiplets vanish $H^{(i)} = \widetilde{H}^{(i)} = 0$, is completely lifted, since now F and D-terms would imply $\phi_a = \sigma_a = 0$. On the other hand, considering the Higgs branch where $\phi_a = \sigma_a = 0$, it is left completely unchanged with respect to the previous case where $k=0$. Thus, the presence of a Chern–Simons level can lift the Coulomb branch while the Higgs branch remains untouched.

Moreover, the two sets of scalars can acquire expectation value at the same time giving rise to mixed branches with complicated dynamics. Sometimes, the one with maximal dimension is identified with the Coulomb branch of the theory. However, all these branches are always Hyperkähler manifolds [36, 118, 148].

2.2.2 The chiral ring

We have seen that the vacuum structure of a supersymmetric theory is completely characterised by the expectation values of all the scalar fields inside the theory (let us call these collectively $\{\phi^i\}$) subject to F and D-flatness conditions. This schematically reads

$$\mathcal{M}_{\mathrm{cl}} = \big\{\langle\phi^i\rangle \mid D = F = 0\big\} \,/\, \text{gauge transf.} \tag{2.2.16}$$

where one should mod out by gauge transformations, since solutions which are related by gauge transformations are physically equivalent and describe the same vacuum state.

Generically it is not at all easy to solve the F and D-flatness conditions and find a simple parametrisation of $\mathcal{M}_{\text{cl}}$, but luckily there is another way of describing the classical moduli space which can be found in the context of algebraic geometry. The aim, naively, consists in find a new class of operators $\{\mathcal{O}^i\}$ for which the constraints due to F and D-terms and the modding by gauge transformations exchange, thus obtaining

$$\mathcal{M}_{\text{cl}} = \Big\{ \langle \mathcal{O}^i \rangle \mid \text{gauge invariant } \Big\} \,/\, \text{F and D-flatness conditions} \tag{2.2.17}$$

This leads to the concept of chiral ring [133] that we now turn to discuss. In doing so we will closely follow [9, 47]. Let us firstly define what a chiral operator is

Definition 2.2.1. Chiral operator
A chiral operator $\mathcal{O}(x)$ is a gauge invariant operator such that it is annihilated by all the supercharges of one chirality.

$$[\mathcal{Q}^I_\alpha, \mathcal{O}(x)\} = 0 \quad \forall I \tag{2.2.18}$$

For example the bottom component ϕ of a chiral multiplet Φ is annihilated by $\overline{\mathcal{Q}}^I_\alpha$ but it is not a chiral operator because it is not gauge invariant in general. However, given a gauge invariant chiral superfield its lowest component will always be a chiral operator. Therefore, in order to build gauge invariant operators, one should generally use different combinations of chiral fields enclosed in traces over the gauge group G.

One crucial property is that the VEV of any time ordered product of such operators is independent of their spacetime positions. Consider for instance the quantity

$$\begin{aligned} &\partial^1_\mu \langle 0|T\Big(\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\Big)|0\rangle = \\ &= \langle 0|T\Big(\partial^1_\mu\mathcal{O}_1(x_1)\mathcal{O}_2(x_2))\Big)|0\rangle + \delta_{\mu 0}\,\langle 0|[\mathcal{O}_1(x_1), \mathcal{O}_2(x_2)]|0\rangle\,\delta(x^0_1 - x^0_2) \end{aligned} \tag{2.2.19}$$

where we used the definition of time ordered product

$$T\Big(\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\Big) := \theta(x^0_1 - x^0_2)\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) + \theta(x^0_2 - x^0_1)\mathcal{O}_2(x_2)\mathcal{O}_1(x_1) \tag{2.2.20}$$

and the fact that

$$\partial^1_\mu\theta(x^0_1 - x^0_2) = \delta_{\mu 0}\delta(x^0_1 - x^0_2) \tag{2.2.21}$$

The second term in (2.2.19) is zero because the equal time commutator vanishes, the first is also null because, using the supersymmetry algebra

$$\begin{aligned} &\langle 0|\partial^1_\mu\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)|0\rangle \propto \\ &\propto \langle 0|[\mathcal{P}_\mu, \mathcal{O}_1(x_1)]\mathcal{O}_2(x_2)|0\rangle \propto \langle 0|[\{\mathcal{Q}^I, \overline{\mathcal{Q}}^I\}, \mathcal{O}_1(x_1)]\mathcal{O}_2(x_2)|0\rangle \end{aligned} \tag{2.2.22}$$

The idea now is to spoil the Jacobi identity and the chirality of the operators $\mathcal{O}_i$ to bring $\overline{\mathcal{Q}}^I$ to act on the vacuum state so that, if supersymmetric is not broken, it vanishes. We have that

$$[\{\mathcal{Q}^I, \overline{\mathcal{Q}}^I\}, \mathcal{O}_1(x_1)]\mathcal{O}_2(x_2) = \{\overline{\mathcal{Q}}^I, [\mathcal{Q}^I, \mathcal{O}_1(x_1)]\}\mathcal{O}_2(x_2) \tag{2.2.23}$$

and so

$$\langle 0|\{\overline{\mathcal{Q}}^I, [\mathcal{Q}^I, \mathcal{O}_1(x_1)]\}\mathcal{O}_2(x_2)|0\rangle = \langle 0|\{\overline{\mathcal{Q}}^I, [\mathcal{Q}^I, \mathcal{O}_1(x_1)]\}\mathcal{O}_2(x_2)\}|0\rangle = 0 \tag{2.2.24}$$

The manipulations above can be done for any number of chiral operators, thus we can write

$$\langle 0|T\Big(\prod_i \mathcal{O}_i(x_i)\Big)|0\rangle = \langle\prod_i \mathcal{O}_i\rangle \tag{2.2.25}$$

without specifying the positions x_i. Using this invariance, we can take a correlation function of chiral operators at distinct points, and separate the points by an arbitrarily large distance. Cluster decomposition then implies

$$\langle 0|T\Big(\prod_i \mathcal{O}_i(x_i)\Big)|0\rangle = \prod_i \langle\mathcal{O}_i\rangle \tag{2.2.26}$$

Since objects of the type $[\overline{\mathcal{Q}}^I_\alpha, \cdot\ \}$ do not contribute to the expectation values in a supersymmetric vacuum, it is natural to define an equivalence relation between chiral operators. Two chiral operators $\mathcal{O}_1(x)$ and $\mathcal{O}_2(x)$ are then equivalent if there exist a set of gauge invariant operators $X^\alpha_I(x)$ such that

$$\mathcal{O}_1(x) \sim \mathcal{O}_2(x) + [\overline{\mathcal{Q}}^I_\alpha, X^\alpha_I(x)\} \tag{2.2.27}$$

The set of equivalence classes of chiral operators forms a ring, known as the chiral ring [133].

Definition 2.2.2. Chiral ring
Given the set of all chiral operators $\mathcal{C}_0$, the chiral ring is defined to be the quotient ring of $\mathcal{C}_0$ over the equivalence relation between chiral operators

$$\mathcal{CR} = \mathcal{C}_0/\sim \tag{2.2.28}$$

A very important mathematical consequence of this construction is the existence of a map from the chiral ring $\mathcal{CR}$ to the ring of holomorphic functions over the moduli space

$$\begin{aligned} \varphi\colon\quad \mathcal{CR} &\to \mathbb{C}^\infty(\mathcal{M}) \\ \mathcal{O}(x) &\mapsto f(z^1,\dots,z^m) \end{aligned} \tag{2.2.29}$$

where $f(z^1,\dots,z^m)\colon \mathcal{M}\to\mathbb{C}$.

Moreover, when F and D-flatness conditions on chiral operators are taken into account, this map becomes bijective.

Expectation values of gauge invariant combinations of chiral operators will play a key role in the following sections when many moduli spaces of supersymmetric vacua will be discussed in details.

2.2.3 Monopole operators

We want now to discuss $\mathcal{N}=4$ BPS monopole operators. Luckily some of the results of the $\mathcal{N}=2$ holds even in this case; indeed the very definition of the BPS monopole operator is identical to the $\mathcal{N}=2$ case, *i.e.* the scalars belonging to the adjoint chiral multiplet Φ do not have to acquire any singular behaviour.

However, we have the possibility to turn on a constant background for the adjoint complex scalar ϕ_a on top of the $\mathcal{N}=2$ BPS monopole background $\{A_{N/S}, \sigma\}$, while preserving the same supersymmetry of an $\mathcal{N}=2$ chiral multiplet.

In the following we will refer to $\mathcal{N}=2$ BPS monopole operators with a background $\phi=0$ as bare monopole operators, and to $\mathcal{N}=2$ BPS monopole operators with non-vanishing $\phi\in\mathfrak{g}$ as dressed monopole operators. The Weyl group acts both on H and ϕ, and gauge invariant monopole operators are again obtained by taking invariants under the Weyl group in $\Gamma_{\widehat{G}}/\mathcal{W}_{\widehat{G}}$.

Both classes of operators take expectation values on the Coulomb branch of an $\mathcal{N}=4$ gauge theory and are needed to describe the chiral ring $\mathcal{CR}$.

Monopole operators, which classically may only be charged under the topological symmetry, can acquire nontrivial quantum numbers quantum-mechanically. The most important quantum number for our future discussion is the R-charge. For a generic non Abelian theory the R-charge of the monopole reads [18, 24, 85]

$$\Delta(m) = -\sum_{\alpha\in\Delta^+}|\alpha(m)| + \sum_i (1-\Delta_i)\sum_{\rho\in\mathcal{R}_i}|\rho(m)| \tag{2.2.30}$$

where the first sum is over the set of all positive roots $\alpha\in\Delta^+$ of the gauge group G and represents the contribution arising from the $\mathcal{N}=4$ vector multiplets, while the second sum is the contribution from the $\mathcal{N}=4$ hypermultiplets $H^{(i)}$, Δ_i is the R-charge of the i-th hypermultiplet and the internal sum runs over the weights ρ of their gauge representation $\mathcal{R}_i$. Here and in the following we will adopt the notation $\alpha(m) := \alpha\cdot m$ and the same for $\rho(m)$.

Using the nomenclature proposed by Gaiotto and Witten [85], according to the value of $\Delta(m)$ a theory can be:

- *good* if all BPS monopoles satisfy $\Delta(m) > \frac{1}{2}$;

  In this case all monopole operators will be coupled

- *ugly* if all BPS monopoles satisfy $\Delta(m) \geq \frac{1}{2}$;

  The monopoles with $\Delta(m)=\frac{1}{2}$ which saturate the bound will be free decoupled fields

- *bad* if there is one or more BPS monopoles which satisfy $\Delta(m) < \frac{1}{2}$;

  In this case $\Delta(m)$ does not correspond to the scaling dimension of the monopole operators since R-symmetry mixes with other accidental symmetries and the theory becomes non-unitary.

Moreover, the monopole gauge charge with respect to a simple group with Chern–Simons level k has the following form

$$Q_a(m) = -km_a - \sum_i\sum_{\rho\in\mathcal{R}_i}|\rho(m)|Q_a^i \tag{2.2.31}$$

where the second term represents the correction due to the hypermultiplets $H^{(i)}$ and ρ, as above, is the weight of their representations $\mathcal{R}_i$ of the gauge group.

Even the flavour charges get modified as follows

$$F_i(m) = -\sum_i \sum_{\rho\in\mathcal{R}_i} |\rho(m)| F_i \tag{2.2.32}$$

where, now, F_i is the flavour charge of the i-th hypermultiplet $H^{(i)}$.

How corrections (2.2.30), (2.2.31) and (2.2.32) arise will be clearer in Chapter (4); for now let us just continue the discussion.

2.2.4 The Hilbert Series

By exploiting the correspondence between chiral operators $\mathcal{O}_i$ and holomorphic functions stated in the previous chapter, we could simply use the generators of the chiral ring as coordinates on the moduli space.

However, it is often too hard to compute the whole chiral ring. Therefore, in this cases, we shall limit ourselves to count the different chiral operators in a graded way, so that we keep track of their different representations under global symmetries of the theory.

This can be done, for example, through the Hilbert series [64, 65, 105, 135, 144]. It is defined as follows

$$H(t,\omega,\mu) = \mathrm{Tr}\left(t^{\Delta}\prod_{a}^{\mathrm{Rank}\,G}\omega_a^{J_a}\prod_{i}^{\mathrm{Rank}\,\widehat{G}}\mu_i^{F_i}\right)_{\mathcal{CR}} \tag{2.2.33}$$

where the trace is taken over the chiral ring $\mathcal{CR}$ and is graded according to the R-charge Δ, the topological charges J_a and the Cartan generators F_i of the flavour symmetry $\widehat{G}$. t, ω_a and μ_i are the corresponding fugacities.

This clearly reconstruct the character of the representation in which the chiral operators at a given R-charge Δ sits; considering for example a flavour symmetry group $\widehat{G}$, we get[1]

$$\sum_{\{\mathcal{O}_i\}}\prod_{i}^{\mathrm{Rank}\,\widehat{G}}\mu_i^{F_i}\Big|_{\mathcal{O}_i} = \sum_{\widetilde{\rho}\in\widetilde{\mathcal{R}}}\prod_{i}^{\mathrm{Rank}\,\widehat{G}}\mu_i^{\widetilde{\rho}_i} := \chi^{\widehat{G}}_{\widetilde{\mathcal{R}}}(\mu) \tag{2.2.34}$$

where $\widetilde{\rho}$ is the weight of the flavour representation $\widetilde{\mathcal{R}}_i$.

The gauge invariant chiral operators of the theory are 't Hooft monopole operators V_m dressed by matter fields. Thus, to compute the Hilbert series, it is useful to decompose the vector space of chiral operators in vector spaces of chiral operators of fixed magnetic charge m. For each of these subspaces there is a unique bare chiral monopole operator V_m, which however can be dressed by massless matter fields to form gauge invariants. These are the matter fields contained inside all the $\mathcal{N}=2$ chiral multiplets Φ satisfying the condition

$$\sum_a Q_a^{\Phi} m_a = 0 \tag{2.2.35}$$

where Q_a^{Φ} are the gauge electric charges of the chiral multiplet.

In order to obtain the powers of such residual matter fields, we need a function that "counts" the symmetrized products of a given set of objects. This turns out to be the plethystic exponential

[1] For all the group theory results of this section, we refer the interested reader to [158].

Definition 2.2.3. Plethystic exponential

Given a function $f\colon \mathbb{R}^n \to \mathbb{R}$ such that $f(0,0,\ldots,0) = 0$, we define the plethystic exponential of f to be

$$\mathrm{PE}[f(t_1,\ldots,t_n)] = \exp\left\{\sum_{k=1}^{\infty} \frac{f(t_1^k,\ldots,t_n^k)}{k}\right\} \tag{2.2.36}$$

Moreover it enjoys the usual "sum to product" property of the ordinary exponential, which is

$$\mathrm{PE}[f(t)+g(t)] = \mathrm{PE}[f(t)]\,\mathrm{PE}[g(t)] \tag{2.2.37}$$

The main idea is that this function keeps track of the cardinality of the set of all symmetric monomials at generic degree. More precisely, given n basic monomials $\{a_1,\ldots,a_n\}$, consider the set $S_{(n,k)}$ whose elements are all the possible symmetric monomials of degree k. In general, the coefficient of the k-th power of t in the Taylor expansion of $\mathrm{PE}[nt]$ gives the cardinality of $S_{(n,k)}$. For example

$$\mathrm{PE}[3t] = \frac{1}{(1-t)^3} = \sum_{k=0}^{\infty} \dim\{S_{(3,k)}\} t^k \tag{2.2.38}$$

where, for $k=2$

$$S_{(3,2)} = \{a_1^2, a_2^2, a_3^2, a_1a_2, a_2a_3, a_3a_1\} \tag{2.2.39}$$

Then, the grading comes from the fact that the generator of symmetrisation for n variables can be written as

$$\prod_i^n \frac{1}{(1-a_i)} = \prod_{i=1}^n \sum_{n_i=0}^{\infty} a_i^{n_i} \tag{2.2.40}$$

This also is expressible in terms of plethystic exponents as

$$\prod_i^n \frac{1}{(1-a_i)} = \prod_i^n \mathrm{PE}[a_i] = \mathrm{PE}[\sum_i^n a_i] \tag{2.2.41}$$

Taking as generators $a_i t$, we can then use t to count the degree of the symmetrisation through its exponent in the taylor expansion

$$\prod_i^n \frac{1}{(1-a_it)} = \mathrm{PE}[(\sum_i^n a_i)t] = \sum_{k=0}^{\infty} \sum_{e\in S_{(n,k)}} e(a_i) t^k \tag{2.2.42}$$

where $e(a_i)$ is an element of $S_{(n,k)}(a_i)$.

Applying this reasoning to the residual matter fields, we get that the generating function reads

$$
\begin{aligned}
&\mathrm{PE}\left[\sum_{\Phi}\delta_{\sum_a Q_a^{\Phi}m_a,0}\,t^{\Delta_\Phi}\prod_{a}^{\mathrm{Rank}\,G} z_a^{Q_a^{\Phi}}\prod_{i}^{\mathrm{Rank}\,\widehat{G}}\mu_i^{F_i^{\Phi}}\right]=\\
&=\frac{1}{\prod_{\Phi}\delta_{\sum_a Q_a^{\Phi}m_a,0}\left(1-t^{\Delta_\Phi}\prod_a^{\mathrm{Rank}\,G} z_a^{Q_a^{\Phi}}\prod_i^{\mathrm{Rank}\,\widehat{G}}\mu_i^{F_i^{\Phi}}\right)}
\end{aligned}
\tag{2.2.43}
$$

We can make this expression more compact by noticing that the operators Q_a and F_i when acting on matter fields transforming in the representation $(\mathcal{R},\widetilde{\mathcal{R}})$ of respectively the gauge group and the flavour group give the components of their weight vectors ρ_a and $\widetilde{\rho}_i$.

Thus, introducing the notations

$$
a^b := \prod_i a_i^{b_i}\ ,\quad a(b) := a\cdot b = \sum_i a_i b_i \tag{2.2.44}
$$

whenever such products are implied, equation (2.2.43) simplifies to

$$
\mathrm{PE}\left[\sum_{\Phi}\delta_{\rho(m),0}\,t^{\Delta}z^{\rho}\mu^{\widetilde{\rho}}\right]=\frac{1}{\prod_{\Phi}\delta_{\rho(m),0}\left(1-t^{\Delta}z^{\rho}\mu^{\widetilde{\rho}}\right)} \tag{2.2.45}
$$

Our strategy to compute the Hilbert series consists in evaluating (2.2.33) and (2.2.45) for all the operators which are annihilated by the supercharges of one kind, but are not necessarily gauge invariant and therefore not chiral. Subsequently, we will restrict the sum over the gauge invariant ones by means of the Molyen-Weyl projection, which consists in integrating the Hilbert series over the whole gauge group and thus, in a certain sense, averaging away all the non-gauge invariant operators in the sum.

To do so we make use of the Haar Measure of a compact Lie group[2]. By using the Weyl integral formula, the Haar measure can be computed explicitly

$$
\int d\mu_g = \frac{1}{(2\pi i)^r}\oint_{|z_1|=1}\cdots\oint_{|z_r|=1}\frac{dz_1}{z_1}\cdots\frac{dz_r}{z_r}\prod_{\alpha^+}\left(1-\prod_{k=1}^{r}z_k^{\alpha_k^+}\right) \tag{2.2.46}
$$

where the z_k variables parametrise the maximal torus of the compact Lie group group G, thus $r=\mathrm{Rank}\,G$ and α^+ are positive roots of the associated Lie algebra.

From our physical point of view, the maximal torus variable z of the Haar measure represents the fugacity associated to the gauge symmetry G. However, in the presence of dynamical magnetic charges m, the gauge group G is broken to a residual gauge group H_m of the same rank r, but given by the commutant of the magnetic charge m in G. This implies that the Haar measure should be modified accordingly. This can be achieved by simply modifying the definition (2.2.46) as follows

$$
\prod_{a=1}^{r}\oint_{|z_a|=1}\frac{dz_a}{2\pi i z_a}\prod_{\alpha\in\Delta^+}\left(1-z^{\alpha}\right)^{\delta_{\alpha(m),0}} \tag{2.2.47}
$$

where, for notational convenience, we introduced the space of positive roots Δ^+.

Then we use orthonormality of characters to project the Hilbert series over states with a given representation of gauge symmetry. Then if

[2]See footnote (1).

$$\int d\mu_g(z_i)\chi_{\mathcal{R}}(z_i)\chi_{\mathcal{R}'}(z_i) = \delta_{\mathcal{R},\mathcal{R}'} \tag{2.2.48}$$

we can project onto states with given gauge representation $\mathcal{R}$ by integrating (2.2.33) and (2.2.45) with the appropriate Haar measure and multiplying by $\chi_{\mathcal{R}}(z_i)$

Combining all these results together, since we need to project on gauge invariant states, *i.e.* state with trivial representation $\chi_{\mathcal{R}}(z_i) = 1$, the Hilbert series finally reads [63] (see also [61, (8.1)])

$$\begin{aligned} H(t,\omega,\mu) = \sum_{\{m\}} t^{\Delta(m)}\omega^m \mu^{F(m)} \prod_{a=1}^{r} \oint_{|z_a|=1} \frac{dz_a}{2\pi i z_a} \times \\ \times \prod_{\alpha\in\Delta^+} (1-z^\alpha)^{\delta_{\alpha(m),0}} \, \mathrm{PE}\left[\sum_{\Phi} \delta_{\rho(m),0} t^{\Delta} z^\rho \mu^{\widetilde{\rho}}\right] \end{aligned} \tag{2.2.49}$$

where, whenever we consider a monopole operator with $m \neq 0$, we need to use the correct expressions for its quantum numbers, *i.e.* (2.2.30), (2.2.31) and (2.2.32).

Up to now, in (2.2.49), we have considered the complete set of chiral operators, namely $\mathcal{C}_0$ in (2.2.28). This coincide with the chiral ring $\mathcal{CR}$ whenever the chiral operators are unconstrained by F-flatness conditions, *i.e.* whenever there is no superpotential $\mathcal{W} = 0$.

Whenever $\mathcal{W} \neq 0$, equation (2.2.49) must then be modified to include the F-flatness conditions and thus count only the chiral operators inside the chiral ring $\mathcal{CR}$. This can be done by including a polynomial factor $N(t,z)$ that enforces F-term equations. Clearly, according to the structure of F-term equations, $N(t,z)$ can take different expressions.

Luckily in our $3d$ $\mathcal{N} = 4$ case the superpotential is constrained to always have the schematic form $\mathcal{W} = H^{(i)}\Phi\widetilde{H}^{(i)}$. Thanks to this fact, the only relevant F-term becomes

$$F_a = \frac{\partial\mathcal{W}}{\partial\phi_a} = \sum_i \mathrm{Tr}\left(H^{(i)}(T_a^{(F)})\widetilde{H}^{(i)}\right) = 0 \tag{2.2.50}$$

which is a second order relation in the fields $\{H^{(i)},\widetilde{H}^{(i)}\}$ and carries an adjoint index of the gauge group G.

So, in (2.2.49), there shall be a factor

$$N(t,z) = \mathrm{PE}\left[-t^{\Delta_H+\Delta_{\widetilde{H}}} z^{\rho_{\mathrm{Adj}}}\right] \tag{2.2.51}$$

for each different hypermultiplet that possesses an F-term equation (2.2.50). In fact, if more than one gauge or flavour group is present, it is possible that different hypermultiplets are charged under the same gauge group. This fact can create dependences among all the F-term equations, which reduce the number of independent ones, *i.e.* the ones that generate a factor of the form (2.2.51).

Considering the group of independent F-term equations $\{F^{(\alpha)}\}$, we have that the Hilbert series for a superpotential $\mathcal{W} \neq 0$, becomes

$$
\begin{aligned}
H(t,\omega,\mu) = \sum_{\{m\}} t^{\Delta(m)}\omega^m \mu^{F(m)} \prod_{a=1}^{r} \oint_{|z_a|=1} \frac{dz_a}{2\pi i z_a} \prod_{\alpha\in\Delta^+} (1-z^\alpha)^{\delta_{\alpha(m),0}} \times \\
\times \,\mathrm{PE}\left[\sum_{\Phi} \delta_{\rho(m),0} t^{\Delta} z^\rho \mu^{\widetilde{\rho}}\right] \mathrm{PE}\left[\sum_{H^\alpha}\left(-t^{2\Delta} z^{\rho^{\mathrm{unbr}}_{\mathrm{Adj}}}\right)\right]
\end{aligned}
\tag{2.2.52}
$$

where now the $N(t,z)$ factor takes into account the breaking of the gauge group G to H_m.

Clearly this quantity is really hard to compute even for the simplest examples. Since we will be only interested in the Coulomb and Higgs branch Hilbert series separately (and we will also forget about mixed branches), it is useful to restrict and simplify the expression (2.2.52) to just one of the branches.

- Higgs branch

  Let us start from the simplest one, the Higgs branch. This amounts only to take $m_a = 0\ \forall a$ in (2.2.52) leading to the so-called Molien integral [46]

$$
\begin{aligned}
H_{\mathcal{H}}(t,\mu) = \prod_{a=1}^{r} \oint_{|z_a|=1} \frac{dz_a}{2\pi i z_a} \prod_{\alpha\in\Delta^+} (1-z^\alpha) \times \\
\times \,\mathrm{PE}\left[\sum_{\Phi} t^{\Delta} z^\rho \mu^{\widetilde{\rho}}\right] \mathrm{PE}\left[\sum_{H^\alpha}\left(-t^{2\Delta} z^{\rho_{\mathrm{Adj}}}\right)\right]
\end{aligned}
\tag{2.2.53}
$$

- Coulomb branch

  On the Coulomb branch, the expression of the plethystic exponential in (2.2.52) drastically simplifies and, after integration, give rise to a simple classical factor expressed as

$$
P_G(t,m) = \prod_{i=1}^{r} \frac{1}{1-t^{d_i(m)}} \tag{2.2.54}
$$

  which essentially counts the Casimir invariants with degree (*i.e.* R-charge) $d_i(m)$ of the residual gauge group H_m left unbroken by the magnetic flux m.

  This result is due to the fact that no hypermultiplet remains and consequently the dressing of the bare monopole operators can only occur via combinations of the complex scalar ϕ in the vector multiplet. These combinations are exactly the Casimir invariants.

  The Hilbert series (2.2.52) on the Coulomb branch takes thus the form of the so-called monopole formula [64]

$$
H_{\mathcal{C}}(t,\omega) = \sum_{\{m\}} t^{\Delta(m)}\omega^m P_G(t,m) \tag{2.2.55}
$$

  In the following we will see some explicit example of this classical factor $P_G(t,m)$.

  When dealing with the monopole formula, once (2.2.55) has been evaluated explicitly, it is useful to take the inverse of the plethystic exponential, called the plethystic logarithm

Definition 2.2.4. Plethystic logarithm

Given a function $g\colon \mathbb{R}^n \to \mathbb{R}$ such that $g(0,0,\ldots,0) = 0$, we define the Plethystic logarithm of g to be the function

$$PL[g(t_1,\ldots,t_n)] = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \log(g(t_1^k,\ldots,t_n^k)) \tag{2.2.56}$$

where $\mu(k)$ is the Möbius function defined as

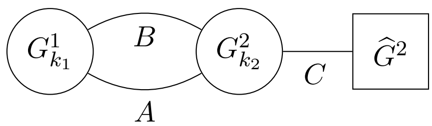
$$\mu(k) = \begin{cases} 0, & \text{if } k \text{ has one or more repeated prime factors} \\ 1, & \text{if } k = 1 \\ (-1)^n, & \text{if } k \text{ is the product of } n \text{ distinct primes} \end{cases} \tag{2.2.57}$$

This produces nothing but the exponent of the generating function of the Coulomb branch and, thus, allows to better understand the chiral generators and the relations among them. The generating function itself can be obtained taking the plethystic exponent (as in the case of the Higgs branch Hilbert series.

2.2.5 Quiver diagrams

Quivers are a class of graphs with lines connecting different nodes and in which lines starting and ending at the same node are also allowed. From a physical point of view, quivers are extremely interesting since they provide a very compact way for writing a Lagrangian.

An example of a quiver diagram could be



(2.2.58)

The rule to read off the gauge group and the matter content of a theory from a quiver graph is the following:

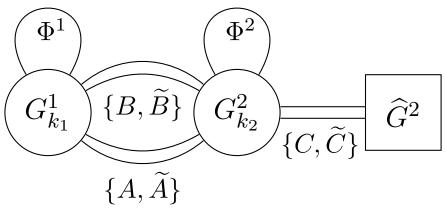
- Each circular node $G^i_{k_i}$ of the quiver diagram corresponds to a factor G^i of the total gauge group with Chern–Simons interaction of level k_i. This node thus encode the vector superfield related to the gauge group G^i;
- Each square node $\widehat{G}^i$ of the quiver diagram corresponds to a factor $\widehat{G}^i$ of the total flavour symmetry of the theory;
- Each line corresponds to a matter superfield $\{A, B, C, \ldots\}$ transforming in the bifundamental representation of the two nodes between which it stretches.

Note that these rules do not fix the Lagrangian completely. One needs to specify the superpotential, the mass terms and the Fayet-Iliopulos terms by hand, since they cannot be read off the quiver. In our case we will always assume, unless explicitly stated otherwise, that all these quantities are zero.

Clearly the content of the theory changes according to which amount of supersymmetry we choose for the quiver. In our case, for example, it can be read both in

$\mathcal{N}=2$ or $\mathcal{N}=4$ languages considering in the first case the $\mathcal{N}=2$ vector and matter multiplets, while the $\mathcal{N}=4$ in the other.

However, when viewed in a $\mathcal{N}=4$ language, the quiver (2.2.58) can also be rewritten in the $\mathcal{N}=2$ one drawing explicitly the $\mathcal{N}=2$ multiplets composing the $\mathcal{N}=4$ ones. We get



(2.2.59)

From the quiver we can easily read off the Coulomb and Higgs branch dimensions by means of (2.2.12) and (2.2.14).

Moreover, in this $\mathcal{N}=4$ case, also the superpotential can be read off the quiver; for each double line $\{H^{(\alpha)},\widetilde{H}^{(\alpha)}\}$, *i.e.* hypermultiplet, and for each gauge group connected by this line (at most two) we associate a term of the form

$$\mathcal{W}^{(\alpha)}=H^{(\alpha)}\Phi_1\widetilde{H}^{(\alpha)}-\widetilde{H}^{(\alpha)}\Phi_2H^{(\alpha)} \tag{2.2.60}$$

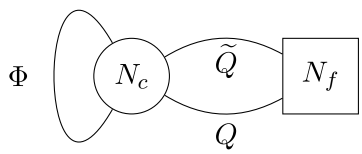
then the total superpotential $\mathcal{W}$ will just be the sum over such multiplets.

Thanks to this fact, we can also easily construct the Hilbert series (2.2.52), taking into account the correct expression for the F-terms factor (2.2.51).

Now we have all the ingredients to see our first example.

2.2.6 Example: 3d $\mathcal{N}=4$ $U(N_c)$ gauge theory with N_f fundamental hypermultiplets

We will consider as a first simple example the $U(N_c)$ gauge theory with N_f hypermultiplets $\{Q_i,\widetilde{Q}^i\}$ of gauge charge ±1, whose $\mathcal{N}=2$ quiver description is as follows



(2.2.61)

The superpotential of this theory is just

$$\mathcal{W}=\sum_i Q_i^a(\Phi)_a{}^b\widetilde{Q}_b^i \tag{2.2.62}$$

where $i=1,\ldots,N_f$ and $a,b=1,\ldots,N_c$.

The fields relevant to our discussion are the one participating in the superpotential. They have the following charges and representations (*i.e.* Dynkin labels)

Field	R	$U(1)_{\text{gauge}}$	$SU(N_c)_{\text{gauge}}$	$U(1)_{\text{flavour}}$	$SU(N_f)_{\text{flavour}}$
$(\Phi)_a{}^b$	1	0	$[1,0,\ldots,0,1]$	0	0
Q_i^a	$\frac{1}{2}$	1	$[1,0,\ldots,0]$	-1	$[0,\ldots,0,1]$
$\widetilde{Q}_a^i$	$\frac{1}{2}$	-1	$[0,\ldots,0,1]$	1	$[1,0,\ldots,0]$

(2.2.63)

Thanks to (2.2.12) and (2.2.14), we can determine the Coulomb and Higgs branch quaternionic dimensions; we have

$$\begin{aligned} \dim_{\mathbb{H}}\{\mathcal{M}_{\mathcal{H}}\} &= N_f N_c - N_c^2 \\ \dim_{\mathbb{H}}\{\mathcal{M}_{\mathcal{C}}\} &= N_c \end{aligned} \tag{2.2.64}$$

where $N_f N_c$ represents the total number of hypermultiplets $\{Q_i^a, \widetilde{Q}_a^i\}$ charged under $U(N_c)$ and we supposed $U(N_f)$ big enough to completely break the gauge group (*i.e.* $N_f \geq 2N_c$).

We can also construct the Hilbert series (2.2.53) and (2.2.53) for both branches to help us analyse their structure.

- Higgs branch

  The results of the Higgs branch analysis for such a theory are presented in [100]. We have one relevant F-term only

$$\frac{\partial \mathcal{W}}{\partial \Phi^A} = \sum_i Q_i^a (T_A)_a{}^b \widetilde{Q}_b^i = 0 \tag{2.2.65}$$

  which carries an adjoint A free index of $U(N_c)$, so that the $N(t,z)$ factor reads

$$N(t,z) = PE\left[-\left(1 + \chi^{SU(N_c)}_{[1,0,\ldots,0,1]}(z)\right) t^2\right] \tag{2.2.66}$$

  Then, the Higgs branch Hilbert series reads

$$\begin{aligned} H_{\mathcal{H}}(t,\mu) = &\int d\mu_{U(N_c)}(z) PE\left[\chi^{U(N_f)}_{[0,\ldots,0,1]}(\mu)\chi^{U(N_c)}_{[1,0,\ldots,0]}(z)t + \right. \\ &\left. +\chi^{U(N_f)}_{[1,0,\ldots,0]}(\mu)\chi^{U(N_c)}_{[0,\ldots,0,1]}(z)t - \left(1 + \chi^{SU(N_c)}_{[1,0,\ldots,0,1]}(z)\right) t^2\right] = \\ = &\int d\mu_{U(1)}(w) d\mu_{SU(N_c)}(z) PE\left[\left(\frac{q}{w}\chi^{SU(N_f)}_{[0,\ldots,0,1]}(\mu)\chi^{SU(N_c)}_{[1,0,\ldots,0]}(z) + \right.\right. \\ &\left.\left. + \frac{w}{q}\chi^{SU(N_f)}_{[1,0,\ldots,0]}(\mu)\chi^{SU(N_c)}_{[0,\ldots,0,1]}(z)\right) t - \left(1 + \chi^{SU(N_c)}_{[1,0,\ldots,0,1]}(z)\right) t^2\right] \end{aligned} \tag{2.2.67}$$

  where we used the definition of the group character (2.2.34). In the first two lines we took into account that Q_i^a and $\widetilde{Q}_a^i$ transform in opposite representations of $U(N_c)$ and $U(N_f)$. In the second two lines we splitted the representations of $U(N_{c/f})$ into $U(1)_{q/w} \times SU(N_{c/f})$ and defined the two $U(1)$s fugacities q and w of $U(N_c)$ and $U(N_f)$ respectively. In this way, the fugacities z and μ are subject to the tracelessness constraint

$$\prod_{a}^{\text{Rank}\,G} z_a = \prod_{i}^{\text{Rank}\,\widehat{G}} \mu_i = 1 \tag{2.2.68}$$

To do some explicit calculations, let us choose $N_c = 2$; in this way we get

$$\begin{aligned} H_{\mathcal{H}}(t,\mu) = &\frac{1}{(2\pi i)^2} \oint_{|z,w|=1} \frac{dw}{w}\frac{dz}{z}\frac{1-z^2}{z} \times \\ &\times PE\left[\left(\frac{q}{w}\chi^{SU(N_f)}_{[1,0,\dots,0]}(\mu) + \frac{w}{q}\chi^{SU(N_f)}_{[0,\dots,0,1]}(\mu)\right)\left(z+\frac{1}{z}\right)t - \left(2+z^2+\frac{1}{z^2}\right)t^2\right] \end{aligned} \tag{2.2.69}$$

where we have used the fact that for $SU(2)$ the fundamental and anti-fundamental representations are equal $\chi^{SU(2)}_{[1]}(z) = z + \frac{1}{z}$.
Then choosing, for example, $N_f = 4$ we get

$$\begin{aligned} H_{\mathcal{H}}(t,\mu) = \frac{1}{2\pi i} \oint_{|z,w|=1} \frac{dw}{w}\frac{dz}{z}\frac{1-z^2}{z} PE\Bigg[\Bigg(\frac{q}{w}\left(\mu_1 + \frac{\mu_2}{\mu_1} + \frac{\mu_3}{\mu_2} + \frac{1}{\mu_2}\right) + \\ + \frac{w}{q}\left(\frac{1}{\mu_1} + \frac{\mu_1}{\mu_2} + \frac{\mu_2}{\mu_3} + \mu_3\right)\Bigg)\left(z+\frac{1}{z}\right)t - \left(2+z^2+\frac{1}{z^2}\right)t^2\Bigg] \end{aligned} \tag{2.2.70}$$

To extract some information from this quantity we need to preform the integration and then expand the Hilbert series in powers of t, finding the so-called characters expansion, which is nothing but the initial trace (2.2.33). We get

$$\begin{aligned} &H_{\mathcal{H}}(t,\mu) = 1 + \chi^{SU(4)}_{[1,0,1]}(\mu)t^2 + \left(\chi^{SU(4)}_{[2,0,2]}(\mu) + \chi^{SU(4)}_{[0,2,0]}(\mu)\right)t^4 + \\ &+ \left(\chi^{SU(4)}_{[3,0,3]}(\mu) + \chi^{SU(4)}_{[1,2,1]}(\mu)\right)t^6 + \left(\chi^{SU(4)}_{[4,0,4]}(\mu) + \chi^{SU(4)}_{[2,2,2]}(\mu) + \chi^{SU(4)}_{[0,4,0]}(\mu)\right)t^8 + \ldots = \\ &= \sum_{n=0}\sum_{m=0} \chi^{SU(4)}_{[n,2m,n]}(\mu)t^{2(n+2m)} \end{aligned} \tag{2.2.71}$$

First of all we can see that the q fugacity of the $U(1)_q$ factor of the flavour symmetry $U(N_f)$ does not enter the Higgs branch Hilbert series; this is a general feature of this type of theories since, whenever we have a unitary gauge group and a unitary flavour symmetry, an overall $U(1)$ factor of the latter can always be reabsorbed into a $U(1)$ factor of the first, *e.g.* defining a new $U(1)$ gauge fugacity $u := \frac{q}{w}$.
Moreover, we can see form (2.2.71) that the Higgs branch is generated by an $SU(4)$ adjoint operator of R-charge 2. Since this gauge invariant operator must be constructed starting form the chiral fields $\{Q^a_i, \widetilde{Q}^i_a\}$, it is easy to see that this is indeed the meson

$$M_i^{\ j} = Q^a_i \widetilde{Q}^j_a \tag{2.2.72}$$

Such meson operator has to satisfy some conditions and to see them let us return to the general case. First, since M is constructed taking the product of two $N_f \times N_c$ matrices, using the property

$$\text{Rank}\, AB \leq \min\{\text{Rank}\, A, \text{Rank}\, B\} \tag{2.2.73}$$

and the fact that $N_f \geq 2N_c$, we thus get

$$\text{Rank}\, M \leq N_c \tag{2.2.74}$$

Furthermore, the F-term condition (2.2.65) imposes tracelessness and nilpotency on M

$$\text{Tr}\, M = Q_i^a \widetilde{Q}_a^i = 0 \ , \qquad M^2 = Q_i^a \widetilde{Q}_a^j Q_j^b \widetilde{Q}_b^i = 0 \tag{2.2.75}$$

We have thus found that the Higgs branch of (2.2.61) is

$$\mathcal{M}_{\mathcal{H}} = \left\{ M \in \text{GL}(N_f, \mathbb{C}) |\, \text{Rank}\, M \leq N_c, \text{Tr}\, M = 0, M^2 = 0 \right\} \tag{2.2.76}$$

which is known in the mathematical literature as a nilpotent orbit closure of $SU(N_f)$; a space parametrised by a single (co)adjoint matrix with a nilpotency condition (*e.g.* $M^2 = 0$) and potentially other relations (*e.g.* $\text{Rank}\, M \leq N_c$).

Nilpotent orbits form an important class of hyper-Kähler spaces, largely due to their simplicity, and are often associated to the branches of the moduli space.

- Coulomb branch

  The results of the Coulomb branch analysis for such a theory are presented in [64].

  First of all, as we have already seen, for $G = U(N_c)$ the magnetic charges $m = (m_1, \ldots, m_{N_c})$ satisfy $m_1 \geq m_2 \geq \ldots \geq m_{N_c}$ with $m_a \in \mathbb{Z}$, *i.e.* $m \in \mathbb{Z}^{N_c}/S^{N_c}$.

  Then we need to evaluate the monopole R-charge (2.2.30). For simplicity, whenever considering a weight of some representation $\mathcal{R}$, we will express it in the basis of weights vectors $\{\rho_1^F, \ldots, \rho_{N_c}^F\}$ of the fundamental representation. Let us start from the first term.

  Since we can write $U(N_c) = U(1) \times SU(N_c)$, the root system factorises as a disjoint union of vector spaces $\Delta_{U(N_c)} = \Delta_{U(1)} \sqcup \Delta_{SU(N_c)}$ where $\Delta_{U(1)}$ is empty.

  The root system $\Delta_{SU(N_c)}$, expressed in the basis of fundamental weights, consists of vectors in $\mathbb{R}^{N_c}$ whose entries sum to zero, *i.e.* all the $n(n-1)$ permutations of $(1, -1, 0, \ldots, 0) \in \mathbb{R}^{N_c}$. Then, the subset of positive roots is composed of all those vectors whose first non-zero entry is 1.

  Thus, the first term of $\Delta(m)$ reads

$$-\sum_{\alpha \in \Delta^+_{SU(N_c)}} |\alpha(m)| = -\sum_{a<b}^{N_c} |m_a - m_b| \tag{2.2.77}$$

  We turn now to the matter contribution, *i.e.* the second term in (2.2.30). Since we are considering N_f fundamentals, this simply reads

$$\frac{1}{2}\sum_{i=1}^{N_f}\sum_{\rho\in\mathcal{R}^F_{SU(N_c)}}|\rho(m)| = \frac{N_f}{2}\sum_{a}^{N_c}|m_a| \tag{2.2.78}$$

So the total monopole R-charge reads

$$\Delta(m) = -\sum_{a<b}^{N_c}|m_a - m_b| + \frac{N_f}{2}\sum_{a}^{N_c}|m_a| \tag{2.2.79}$$

Since $U(2)$ contains a $U(1)$ factor only, the topological symmetry possess just one fugacity ω. The topological charge thus reads

$$J_{\text{top}}(m) = \sum_{a}^{N_c} m_a \tag{2.2.80}$$

Finally, to explicitly construct the classical factor $P_{U(N_c)}(t,m)$ we associate to the magnetic flux m a partition of N_c

$$\lambda(m) = (\lambda_j(m)) \ , \qquad \sum_j \lambda_j(m) = N_c \ , \qquad \lambda_j(m) > \lambda_{j+1}(m) \tag{2.2.81}$$

which simply encodes how many of the fluxes m_i are equal when we sum over them.

For each such configuration (*i.e.* partition) of m, the residual gauge group which commutes with the monopole flux is

$$H_m = \prod_j U(\lambda_j(m)) \tag{2.2.82}$$

The classical factor is then

$$P_{U(N_c)}(t,m) = \prod_j Z^U_{\lambda_j(m)} \tag{2.2.83}$$

where

$$\begin{aligned} Z^U_k &= \prod_{i=1}^{k}\frac{1}{1-t^i} \qquad k \geq 1 \\ Z^U_0 &= 1 \end{aligned} \tag{2.2.84}$$

Now let's take $N_c = 2$ and $N_f = 4$ as in the example of the Higgs brach Hilbert series computation.

We get that

$$\Delta(m) = -|m_1 - m_2| + \frac{3}{2}(|m_1| + |m_2|) \tag{2.2.85}$$

and

$$P_{U(2)}(t,m) = \begin{cases} \frac{1}{(1-t)(1-t^2)}, & m_1 = m_2 \\ \frac{1}{(1-t)^2}, & m_1 \neq m_2 \end{cases} \tag{2.2.86}$$

So that the monopole formula reads

$$\begin{aligned} H_{\mathcal{C}}(t,\omega) &= \sum_{m_1 \geq m_2} t^{-|m_1-m_2|+\frac{3}{2}(|m_1|+|m_2|)} w^{m_1+m_2} P_{U(2)}(t,m) = \\ &= 1 + \chi^{SU(2)}_{[2]}(\omega)t^2 + \left(\chi^{SU(2)}_{[4]}(\omega) + \chi^{SU(2)}_{[2]}(\omega) + 1\right) t^4 + \dots \end{aligned} \tag{2.2.87}$$

Taking the plethystic logarithm of such expression we get

$$\mathrm{PL}\left[H_{\mathcal{C}}(t,\omega)\right] = \chi^{SU(2)}_{[2]}(\omega)(t^2+t^4) - t^6 - t^8 \tag{2.2.88}$$

which, inserted into a plethystic exponent, gives the following generating function

$$H_{\mathcal{C}}(t,\omega) = \mathrm{PE}\left[\chi^{SU(2)}_{[2]}(\omega)(t^2+t^4) - t^6 - t^8\right] \tag{2.2.89}$$

from which we can read that the Coulomb branches has 6 generators (*i.e.* two triplets under $SU(2)_{\text{top}}$) with 2 relations.

First of all, this example allows us to introduce a general feature of topological symmetry (and other symmetries in general). Indeed, even if at the classical level topological symmetries consists of $U(1)$ factors only, it can however enhance to a non-abelian group in the IR. This happens whenever we consider a "balanced" node, *i.e.* a gauge node whose charged hypermultiplets are exactly twice the number of colours $\sum_i N_f^i = 2N_c$. In such a case (as in this example) the classical topological group $U(1)^r$ actually enhances to $SU(r+1)$, thanks to the existence in the IR of a monopole operator with conformal dimension $\Delta(m) = 1$ which is the lowest component of the superconformal multiplet containing this enhanced conserved current. This is exactly the $\chi^{SU(2)}_{[2]}(\omega)t^2$ term in (2.2.89).

Let us now see in details what are all the generators of the Coulomb branch.

First of all we shall consider the classical Casimirs of the $U(2)$ gauge group, which can be written in terms of the Φ adjoint field as single trace operators $\mathrm{Tr}(\Phi^j)$ with $j = 1, 2$.

Moreover, we already know that monopoles are labeled by the magnetic charge with respect to the $U(2)$ gauge group, V_{m_1,m_2}. The whole magnetic lattice can be covered by combinations of monopoles with lowest R-charge, in our case $V_{\pm 1,0} := V_{\pm}$. These can also be dressed by Casimir invariants of the H_m residual gauge group which do not contain $U(2)$ Casimir factors, and symmetrised by the action of the Weyl group.

An explicit way to construct these operators is to go along the moduli space and diagonalise the adjoint field $\Phi = \mathrm{diag}(\phi_1, \phi_2)$ by a gauge transformation. Thus, dressing for example the $V_\pm$ monopoles we get

$$V_{\pm;(r,s)} := V_\pm \phi_1^r \phi_2^s \tag{2.2.90}$$

The monopole generators of the Coulomb branch are precisely $\{V_{\pm;(0,s)}\}$ for $s=0,1$; the other operators are not independent since various relations among them can be found.

Organising all these operators by their R-charge we can construct the operators forming the $SU(2)_{\text{top}}$ triplets in (2.2.89); these are

$$\begin{array}{llll} \chi^{SU(2)}_{[2]}(\omega)t^2 : & V_+ \ , & \mathrm{Tr}(\Phi) \ , & V_- \\ \chi^{SU(2)}_{[2]}(\omega)t^4 : & V_{+;(0,1)} \ , & \mathrm{Tr}(\Phi^2) \ , & V_{-;(0,1)} \end{array} \tag{2.2.91}$$

Everything we said about the Coulomb branch generators still holds in the general case, *i.e.* quiver (2.2.61). In this case there are $3N_c$ generators, which are

$$\mathrm{Tr}(\Phi^j) \ , \quad j=1,\dots,N_c \tag{2.2.92}$$

$$V_{\pm;(r,s)} := V_\pm \phi_1^r(\phi_2^s+\ldots+\phi_{N_c}^s) + \text{permutations} \ , \quad r=0, s=0,\dots,N_c-1 \tag{2.2.93}$$

and N_c relations.

If $N_f \neq 2N_c$ the topological symmetry does not enhance. Otherwise, if $N_f = 2N_c$, the $3N_c$ generators combines into N_c triplets of the enhanced $SU(2)_{\text{top}}$ symmetry of increasing R-charge.

Chapter 3

$3d$ mirrors of the circle reduction of twisted $\chi(\mathfrak{a}_{2N})$ theories of class-S

This chapter starts by reviewing the basic concepts behind the Type IIB string theory brane engineering of three-dimensional $\mathcal{N}=4$ supersymmetric theories. These will allow us to understand mirror symmetry and define $T^\sigma_\rho(SU(N))$ theories. Then, by introducing orientifold Op planes in the game, we will be able to define also $T^\sigma_\rho(USp'(2N))$ theories.

These two families of theories will be important for the second part of the chapter. Indeed, after briefly introducing the class-S framework, we will present our results concerning the three-dimensional mirror theories of the circle reduction of the four-dimensional $\chi(\mathfrak{a}_{2N})$ theories of class-S. It turns out that such mirror theories admit an "almost" star-shaped quiver description built by "gluing" together three different legs, each one being of the $T^\sigma_\rho(SU(N))$ or $T^\sigma_\rho(USp'(2N))$ type.

3.1 Hanany-Witten brane construction

Three dimensional $\mathcal{N}=4$ linear quivers can be realised in Type IIB string theory through Hanany-Witten brane engineering [107].

We start considering systems of D3, D5 and NS5 branes spanning the following space-time directions

Type	0	1	2	3	4	5	6	7	8	9
NS5	×	×	×	×	×	×				
D3	×	×	×				×			
D5	×	×	×					×	×	×

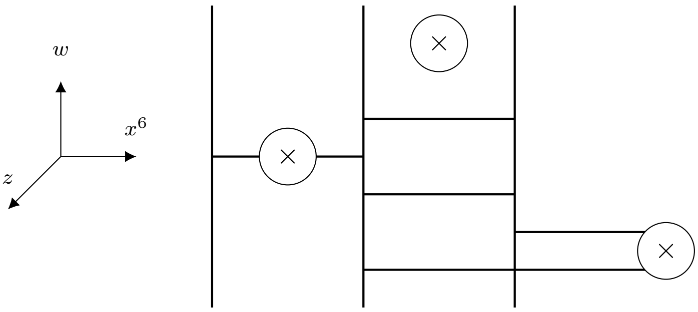
(3.1.1)

This precise configuration preserves exactly one quarter of the original 32 supersymmetries, *i.e.* 8 supercharges. Moreover the D3 branes will always have finite size L in the x^6 direction, terminating either on D5 or NS5 branes. Thanks to this, the effective quantum field theory living on the worldvolume of the D3 branes becomes three-dimensional since, despite the four dimensional nature of such branes, the string modes along x^6 become the Kaluza-Klein modes in a circle dimensional reduction of radius $r=\frac{L}{2\pi}$. Thus, our $3d$ $\mathcal{N}=4$ gauge theories will always live on the D3 branes.

Notice that the Lorentz group $SO(1,9)$ is broken by this brane setup to $SO(1,2)_{012}$, identified with the Lorentz group of the effective three-dimensional theory, and $SO(3)_{345}\times SO(3)_{789}$, which can be identified instead with R-symmetry $SU(2)_L\times SU(2)_R$.

To simplify the notation in the following, we shall define two vectors $z=(x^3,x^4,x^5)$ and $w=(x^7,x^8,x^9)$.

Graphically, a generic brane system has the following form:



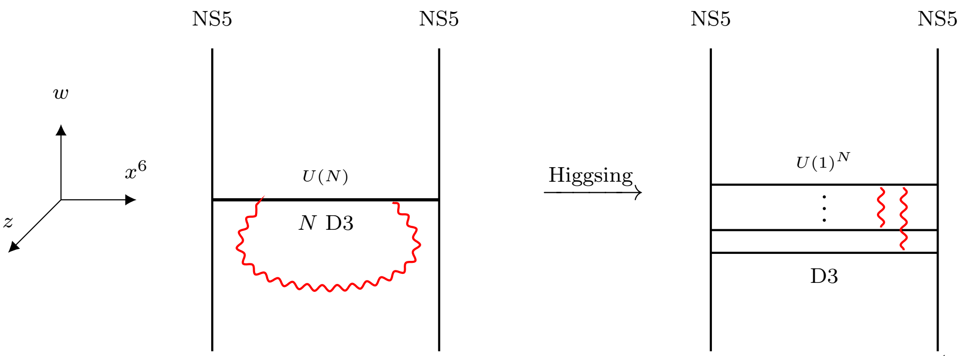
(3.1.2)

where

- Each vertical line, representing an NS5 brane, has a fixed value of x^6 and z and spans w.
- Each horizontal line, representing a D3 brane, has a fixed value of w and z and spans x^6.
- Each circle, representing a D5 brane, has a fixed value of x^6 and w and spans z.

To identify the effective $\mathcal{N} = 4$ gauge theory starting from the brane system we need to know what is the matter content arising from the spectrum of open strings stretching across the various D3 and D5 branes. Thus, the exact positions of all the branes in a given setup have important consequences on the underlying $3d$ theory. Let us then start from scratch.

A single D3 brane suspended between two NS5 brane hosts a $U(1)$ vector multiplet. Now suppose we have instead a stack of N coincident D3 branes as in Fig. (3.1.3); looking at the Chan-Paton factors of the spectrum of open strings stretching among these D3 branes, we gain a $U(N)$ vector multiplet. However, if some of the D3 branes in the stack get separated in the w direction, the respective fundamental strings acquire a non-trivial tension which is interpreted as the mass of a W-boson. This has the effect of breaking the $U(N)$ gauge group up to its maximal torus $U(1)^N$ when all the N D3 branes get separated.



(3.1.3)

This mechanism reminds of what happens on the Coulomb branch. It is thus natural to expect that the different positions w_a of the N D3 branes corresponds to the VEVs $\langle\phi_a\rangle$ of the scalars in vector multiplets: the origin of the Coulomb branch (*i.e.* $\langle\phi_a\rangle = 0$) corresponds to the case in which all the D3 branes are coincident, while separating them apart amounts to go on a generic point where some of the

VEVs are non-trivial. Thus, whenever D3 branes are free to move in the w direction, a non-trivial Coulomb branch will be present in the theory.

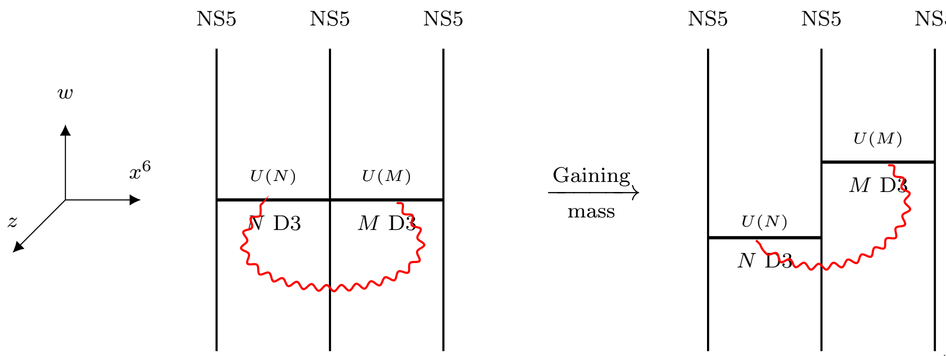
Moreover, the coupling constant of such $U(N)$ gauge theory is identified with the distance along x^6 of the two NS5 branes, *i.e.* the D3 brane extent L

$$\frac{1}{g^2} = L = |x_1^6 - x_2^6| \tag{3.1.4}$$

In this sense, bringing the NS5 branes closer until they coincide is translated as the strong coupling limit of the $3d$ QFT.

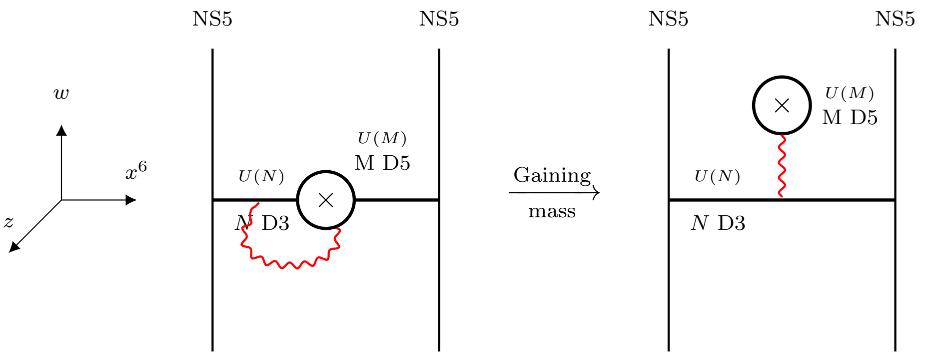
Let us now consider more than two NS5 branes and strings connecting the D3 branes in adjacent stacks as in Fig. (3.1.6). Chan-Paton factor suggest that the strings fluctuations are nothing but bi-fundamental hypermultiplets degrees of freedom with mass proportional to the branes displacement in the w direction

$$m \propto w_i^a - w_{i+1}^b \tag{3.1.5}$$



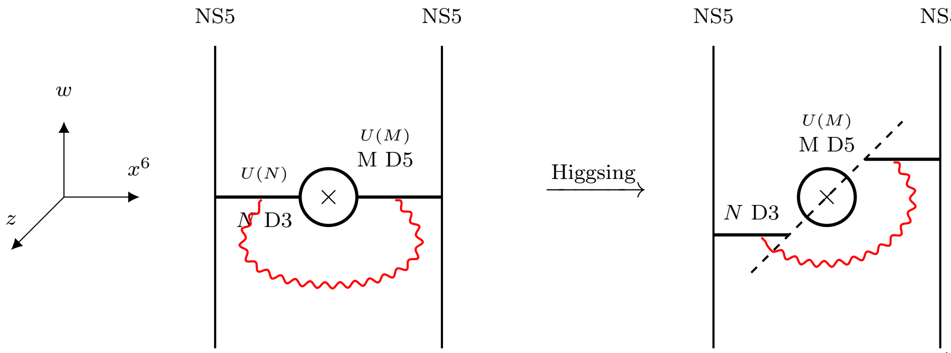
(3.1.6)

Similarly, a string that stretches between N coincident D3 and a D5 brane gives rise to an hypermultiplet in the fundamental representation of the $U(N)$ gauge group with, again, mass proportional to the w displacement between the two types of branes. If a stack of M coincident D5 branes is present (Fig. (3.1.7)), we thus gain M fundamental hypermultiplets, *i.e.* a $U(M)$ flavour symmetry.



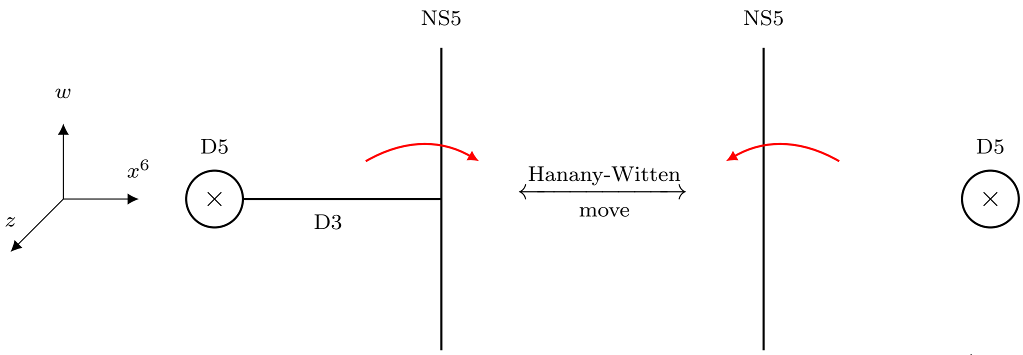
(3.1.7)

At the same time, D3 branes linking D5 branes are free to displace in the z direction, see Fig. (3.1.8). This displacement can be thought again as a mass for the W-bosons. Since we can always add an arbitrary number of D5 branes, *i.e.* arbitrarily increase the flavour group, we can in principle completely break the gauge group. This is exactly what happens on the Higgs branch of the theory giving VEVs to the scalars inside the hypermultiplets. Thus whenever a D3 brane is free to move in z direction, a non-trivial Higgs branch is present in the theory.



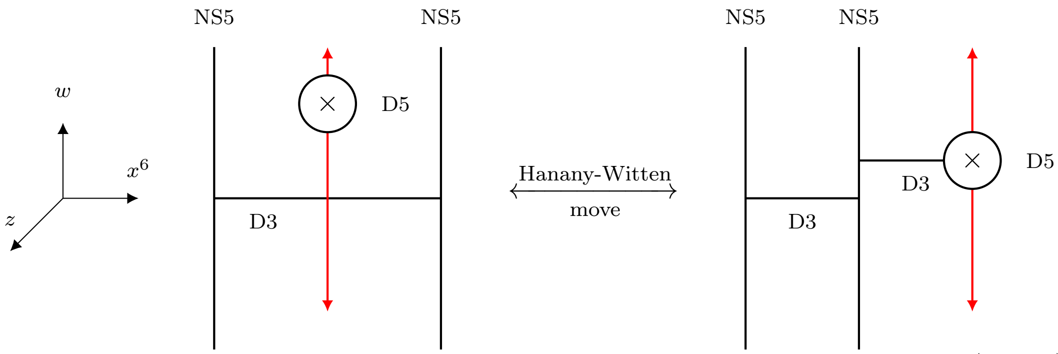
(3.1.8)

The last possibility to take into account consists of a D3 brane suspended between a D5 and an NS5 brane, as in Fig. (3.1.9). However, as proven by Hanany-Witten, this configuration can be changed into an equivalent one by moving the D5 through the NS5 brane. This move makes the D3 brane disappear and we are left with two disconnected D5 and NS5 branes. Clearly this equivalence also works in the reverse situation.



(3.1.9)

The reason of this equivalence can be better understood considering the setup of Fig. (3.1.10). As we have seen, when the D5 brane is moved on top of the D3, the mass of the corresponding hypermultiplet vanishes, providing a singularity corresponding to the appearance of a massless state. If we could move the D5 brane through the NS5 brane without the creation of a D3 brane there would be no reason why a singularity should appear. The problem is resolved conjecturing exactly the Hanany-Witten move: a new D3 brane must be created, connecting the D5 and an NS5 brane. In this setup a massless hypermultiplet can now appear whenever the original D3 inside the NS5 branes and the newly originated one are aligned along the w direction.



(3.1.10)

The mathematically rigorous explanation of the Hanany-Witten move is related to the conservation of the magnetic charge which can be assigned to either the NS5 or the D5 brane, *i.e.* the conservation law for RR and NS three-form fluxes. For definiteness, let us consider an NS5 brane and denote with L_{D5} and R_{D5} the number of D5 branes to its left and to its right respectively. Similarly, L_{D3} and R_{D3} will denote the number of D3 branes ending on the NS5 respectively from the left and from the right. Then the total magnetic charge of the given NS5 brane, dubbed "linking number" in this case, reads

$$L_{\mathrm{NS5}} = \frac{1}{2}(R_{\mathrm{D5}} - L_{\mathrm{D5}}) + (L_{\mathrm{D3}} - R_{\mathrm{D3}}) \tag{3.1.11}$$

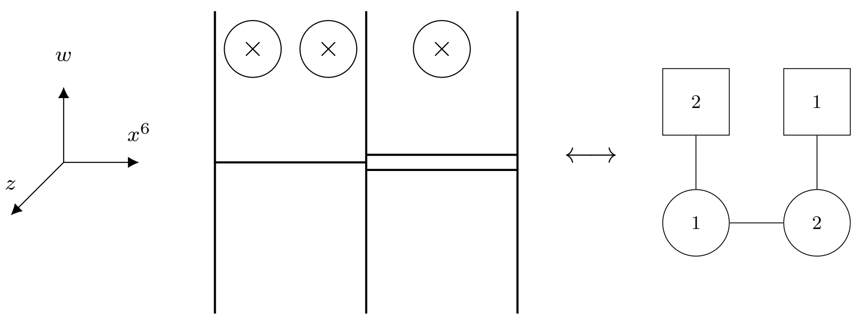
and similarly for the linking number of a given D5 brane.

Such magnetic charge must be conserved for each fivebrane after every other brane has moved. Thus, considering the Hanany-Witten move setup of Fig. (3.1.10), this is why a D3 brane must be crated when the D5 brane passes through the NS5.

Moreover, the sum of the linking numbers for all the branes involved in a certain setup has to be zero, thus constraining the form of the entire brane system.

Other constraints come from requiring unbroken supersymmetry, such as the so-called S-rule: given an NS5 brane and a D5 brane, there can be one and only one D3 brane connecting them.

Let us now take the very first example of Fig. (3.6.35) and try to read off the quiver diagram. First of all we have to use the Hanany-Witten move twice on the D5 brane to the right. In this way both the D3 branes get annihilated and the D5 brane now sits between the first two NS5 branes on the left. Separating all the D5 branes from the D3 branes and making the latter align on the same value of w, we finally get the following setup and the following associated quiver theory



(3.1.12)

Sometimes it is also useful to start from a known quiver diagram and find the underlying brane construction. The reasoning is exactly the same, but reversed.

3.2 Mirror symmetry

A crucial feature in the study of three dimensional $\mathcal{N} = 4$ theories is the so-called mirror symmetry [118], a duality (rather than a real symmetry) relating pair of theories with non-trivial fixed point that flow to the same point in the IR.

Thanks to our previous discussions, we know that, due to $\mathcal{N} = 4$ supersymmetry, both branches of the moduli space are hyper-Kähler. Given a theory with an Higgs $\mathcal{M}_{\mathcal{H}}$ and Coulomb branch $\mathcal{M}_{\mathcal{C}}$, the duality thus conjectures the existence of a mirror dual theory whose Higgs and Coulomb branch are exchanged

$$\mathcal{M}_{\mathcal{C}}^{\text{mirror}} = \mathcal{M}_{\mathcal{H}} \ , \qquad \mathcal{M}_{\mathcal{H}}^{\text{mirror}} = \mathcal{M}_{\mathcal{C}} \tag{3.2.1}$$

As a consequence, mirror symmetry also exchanges the role of the two $SU(2)$ factors of the R-symmetry group.

Moreover, the hard-to-handle quantum mechanical effects arising on the Coulomb branch appear as simple classical effects on the Higgs branch of the mirror dual theory, making mirror symmetry a very useful tools in the study of 3d $\mathcal{N} = 4$ moduli spaces.

Interestingly, this symmetry is easily realised in string theory and, in particular, in the Hanany-Witten brane setup [37, 142]. It relies in fact on the S-transformation of $SL(2,\mathbb{Z})$ duality of Type IIB superstrings, whose elements can be written as

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ , \qquad T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \tag{3.2.2}$$

The S-transformation acts non-trivially on the two types of fivebranes by exchanging them D5 $\leftrightarrow$ NS5 (and their linking numbers too), while leaving the D3 branes unchanged.

However, to obtain the correct mirror symmetry we also need to consider the spatial directions spanned by the fivebranes. Thus, we need to supplement the S-duality action with a rotation R of the spatial coordinates

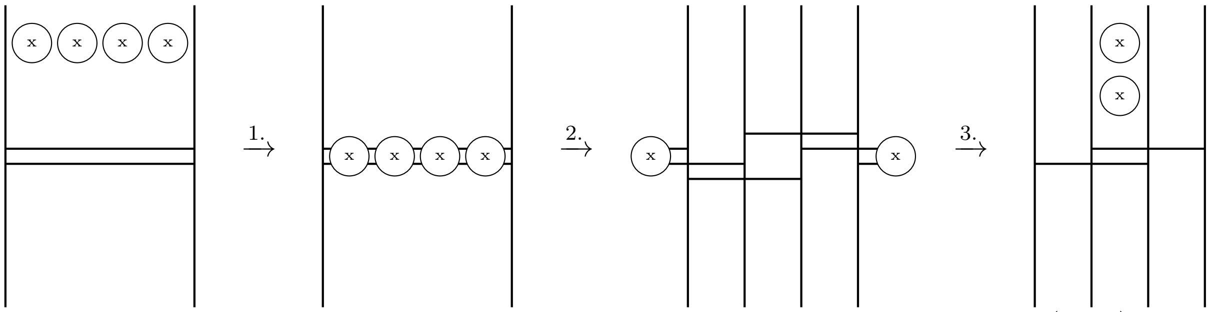
$$w \to z \ , \qquad z \to -w \tag{3.2.3}$$

In building the correct S-dual brane configuration we must perform various steps (depicted in Fig. (3.2.4)):

1. Firstly, all the D5 branes must be aligned to the D3 branes (*i.e.* $w_{\text{D5}} = w_{\text{D3}}$) and separated along the x^6 direction;

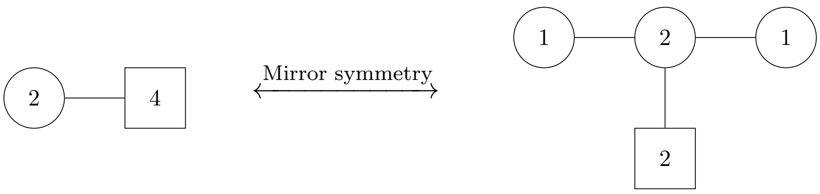
2. Then we can apply an S-transformation exchanging the D5 branes with the NS5 **but** without violating the S-rule and preserving the net number of D3 branes ending on the fivebranes;

3. Finally, the newly obtained S-dual brane configuration can be refined using a the Hanany-Witten move.



(3.2.4)

The theory on the left of Fig. (3.2.4) is nothing but the theory of the example in Section (2.2.6), $U(2)$ gauge theory with 4 fundamental hypermultiplets. On the other hand, the theory on the right is its mirror dual. The quiver description of these theories is the following



(3.2.5)

To check mirror symmetry, we can compute the dimensions of Higgs and Coulomb branches of both the theories and see if they swap.

Thanks to (2.2.64), we find for the first theory

$$\dim_{\mathbb{H}}\{\mathcal{M}_{\text{HB}}\} = 4 \times 2 - 4 = 4 \ , \qquad \dim_{\mathbb{H}}\{\mathcal{M}_{\text{CB}}\} = 2 \tag{3.2.6}$$

while for the mirror one, using the general formulas (2.2.12) and (2.2.14), we get

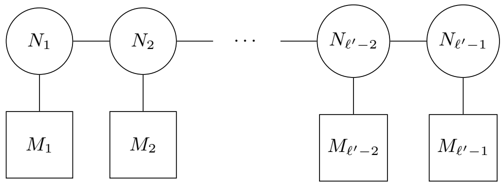
$$\begin{aligned} &\dim_{\mathbb{H}}\{\mathcal{M}^{\text{mirror}}_{\text{HB}}\} = 1 \times 2 + 2 \times 2 + 2 \times 1 - 1 - 4 - 1 = 2 \ , \\ &\dim_{\mathbb{H}}\{\mathcal{M}^{\text{mirror}}_{\text{CB}}\} = 1 + 2 + 1 = 4 \end{aligned} \tag{3.2.7}$$

Thus, we find, as expected, that the dimensions of the Higgs and Coulomb branches match between the mirror pairs as in (3.2.1). Clearly this is not a complete proof of mirror symmetry, since this matching alone does not imply that the branches, as singular spaces, coincide across mirror duality. A more detailed analysis is needed as, for example, checking the matching also between the Hilbert series. We will not go into details now as there will be lots of such examples in the following sections.

3.3 $T^\sigma_\rho(SU(N))$ theories

Among all the possible $\mathcal{N} = 4$ supersymmetric theories, a notable class surely consists of the so-called $T^{\boldsymbol{\sigma}}_{\boldsymbol{\rho}}(SU(N))$ theories [69, 85]. Such theories will be important for the discussion in the following sections so here we briefly review the crucial results.

First of all, their linear quivers are always of the form



(3.3.1)

Each quiver is formed by $\ell' - 1$ gauge nodes with gauge group $U(N_i)$ and by $\ell' - 1$ flavour symmetries with group $U(M_i)$. The value of ℓ' and the exact form of the quiver (3.3.1) are determined by two partitions of N $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_{\ell'})$ and $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_\ell)$ such that

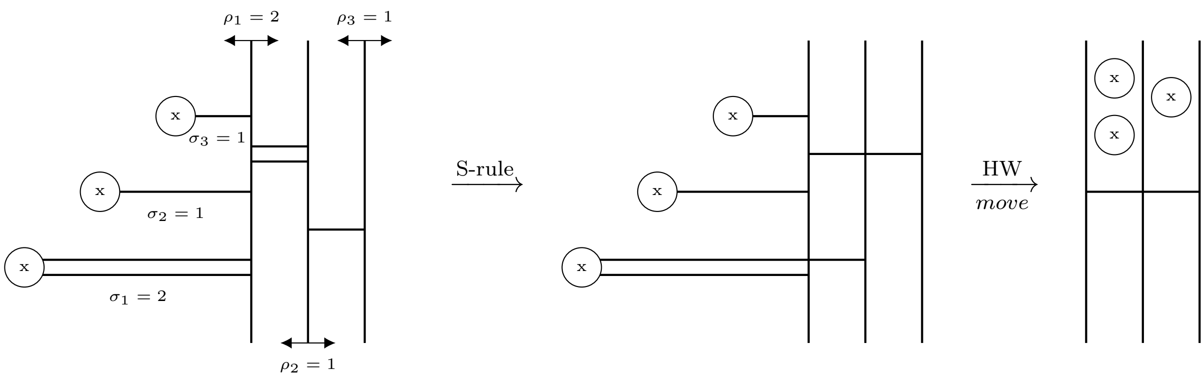
$$\sigma_1 \geq \ldots \geq \sigma_\ell > 0 \,, \qquad \rho_1 \geq \ldots \geq \rho_{\ell'} > 0 \,, \qquad \sum_{i=1}^{\ell} \sigma_i = \sum_{i=1}^{\ell'} \rho_i = N \,. \tag{3.3.2}$$

There are two ways to find out the quiver described by the two partitions $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$.

The first one [85] relies on the Hanany-Witten brane construction and the steps are the following:

1. Starting from the left, draw a Hanany-Witten brane system consisting of ℓ consecutive D5 branes (at different values of w) and then ℓ' consecutive NS5 branes along the x^6 direction;
2. Considering then the elements of the $\boldsymbol{\sigma}$ partition, σ_1 represents the total number of D3 branes ending on the more external D5 brane on the left, σ_2 the total number of D3 branes ending on the next one moving to the right and so forth;
3. Considering now the elements of the $\boldsymbol{\rho}$ partition, ρ_1 is the difference between the number of D3 branes on the left of the more internal NS5 brane and the number of D3 branes on its right. ρ_2 plays the exact same role for the next NS5 brane on the right and so on;
4. Finally, making use of the S-rule and the Hanany-Witten move, we can uniquely fix such an initial configuration and read off the quiver.

Fig. (3.3.3) is an example of this process for $T^{[2,1,1]}_{[2,1,1]}(SU(4))$ theory, which we will encounter again through this thesis.



(3.3.3)

Thanks to this construction we can easily see that under three dimensional mirror symmetry, due to the swap of D5 and NS5 branes, $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ are exchanged too. So that $T^{\boldsymbol{\sigma}}_{\boldsymbol{\rho}}(SU(N))$ is mirror dual to $T^{\boldsymbol{\rho}}_{\boldsymbol{\sigma}}(SU(N))$.

The second method [69] for building the quiver diagram of such theories relies on the fact that the two partitions $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ can be interpreted as two Young tableaux, with their elements indicating the increasing length of their rows. Moreover, it is also convenient to introduce the transpose tableau $\boldsymbol{\sigma}^T$ of the latter partition, which can be built swapping the rows and the columns of $\boldsymbol{\sigma}$. It has the following properties

$$\boldsymbol{\sigma}^T = (\widehat{\sigma}_1, \ldots, \widehat{\sigma}_{\widehat{\ell}}) \ , \qquad \widehat{\sigma}_1 \geq \ldots \geq \widehat{\sigma}_{\widehat{\ell}} > 0 \ , \qquad \sum_{i=1}^{\widehat{\ell}} \widehat{\sigma}_i = N \ . \tag{3.3.4}$$

The flavour symmetries $U(M_j)$ of the theory, with $1 \leq j \leq \ell' - 1$, are thus determined from the transpose $\boldsymbol{\sigma}^T$ as follows:

$$M_j = \widehat{\sigma}_j - \widehat{\sigma}_{j+1} \tag{3.3.5}$$

with $\widehat{\sigma}_i = 0$, for all $i \geq \widehat{\ell} + 1$. Thus $M_i = 0$ for $i \geq \widehat{\ell} + 1$, so that there are at most $\widehat{\ell}$ flavour groups.

On the other hand, the gauge symmetries $U(N_j)$, with $1 \leq j \leq \ell' - 1$, are given by

$$N_j = \sum_{k=j+1}^{\ell'} \rho_k - \sum_{i=j+1}^{\widehat{\ell}} \widehat{\sigma}_i \ . \tag{3.3.6}$$

This implies that the theories $T^{\boldsymbol{\sigma}}_{\boldsymbol{\rho}}(SU(N))$ are defined only for $\boldsymbol{\sigma}^T < \boldsymbol{\rho}$. This constraint is sometimes referred as the dominance ordering of Young tableaux.

The Higgs branch global symmetry can be easily read from the quiver diagram and, in this case, is

$$S\left(\prod_{j=1} U(M_j)\right) := \left(\prod_{j=1} U(M_j)\right) \Big/ U(1) \tag{3.3.7}$$

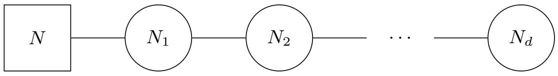
where, as in Section (2.2.6), we have modded an overall $U(1)$ that can always be reabsorbed into one of the gauge groups.

The Coulomb branch global symmetry group, on the other hand, consists at classical level of topological symmetries

$$\prod_{j=1}^{\ell'-1} U(1)^{\mathrm{Rank}\, U(N_j)} \tag{3.3.8}$$

but, as in Section (2.2.6) it can enhance to a non-abelian group in the IR if one of the gauge group happens to be balanced.

Throughout this thesis, we will focus on cases in which $\boldsymbol{\sigma} = [1^N]$, *i.e.* on theories denoted by $T_{\boldsymbol{\rho}}(SU(N))$. The corresponding quivers will always reduce to

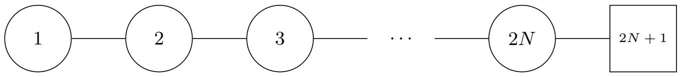


(3.3.9)

where we redefined the number of gauge nodes as $d := \ell' - 1$ for later convenience.

In the following, we mainly consider the three examples of $T_{\boldsymbol{\rho}}(SU(2N+1))$.

The partition $\boldsymbol{\rho} = [1^{2N+1}]$. We denote this theory simply by $T(SU(2N+1))$ and the corresponding quiver reads



(3.3.10)

In general $T(SU(N))$, with any generic N, is invariant under mirror symmetry, *i.e.* self-mirror. The Higgs and the Coulomb branches of this theory are both isomorphic to the closure of the maximal nilpotent orbit, dubbed nilpotent cone, of $SU(N)$, *i.e.* the same space of (2.2.76) but with the additional constraints

$$\mathrm{Tr}\left(M^p\right) = 0 \quad \forall p = 1, \dots, N \tag{3.3.11}$$

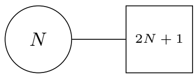
The quaternionic dimension of this space is therefore

$$\dim_{\mathbb{H}}\{\mathcal{M}\} = \frac{1}{2}\left(N^2 - N\right) = \frac{1}{2}N(N-1) \tag{3.3.12}$$

which in our particular case is $\dim_{\mathbb{H}}\{\mathcal{M}\} = 2N(2N+1)$.

The symmetries of the Higgs and Coulomb branch are thus both $SU(2N+1)$; the former is manifest as a classical flavour symmetry, whereas the latter gets enhanced in the IR from the classical topological symmetry.

The partition $\boldsymbol{\rho} = [N+1, N]$. The corresponding quiver is



(3.3.13)

which is just a simple $U(N)$ gauge theory with $2N+1$ fundamentals (see Section (2.2.6)).

3.3.1 The Coulomb branch Hilbert series

For a partition $\boldsymbol{\rho}$ of N, the Coulomb branch Hilbert series of the $T_{\boldsymbol{\rho}}(SU(N))$ theory can be computed from its quiver description using the monopole formula, described in [64].

Alternatively, one can compute this quantity using the Hall-Littlewood formula, without using the quiver description. This formula, involving the Hall-Littlewood polynomials, was firstly conjectured in [67][1].

The Hall-Littlewood formula for the Coulomb branch Hilbert series of the $T_{\boldsymbol{\rho}}(SU(N))$ theory reads

$$\begin{aligned} &H[T_{\boldsymbol{\rho}}(SU(N))](t; x_1, \dots, x_{\ell'}; n_1, \dots, n_N) \\ &= t^{\delta_{U(N)}(\boldsymbol{n})}(1-t^2)^N K_{\boldsymbol{\rho}}^{U(N)}(\boldsymbol{x}; t)\Psi^{\boldsymbol{n}}_{U(N)}(\boldsymbol{x}t^{\boldsymbol{w_\rho}}; t) \ , \end{aligned} \tag{3.3.14}$$

[1] In [152, section 6] the authors proved that the equivalence between the monopole and Hall-Littlewood formulæ in the case of maximal partition $\rho = [1^N]$ automatically implies the same equivalence for any generic partition ρ.

where the notations are as follows:

1. Recall that the integer $d+1$ is the length of the partition $\boldsymbol{\rho}$, so that $d = \ell' - 1$ is the number of gauge groups in quiver (3.3.9).

2. The Coulomb branch symmetry of the $T_{\boldsymbol{\rho}}(SU(N))$ theory is

$$\left[\prod_k U(r_k)\right] / U(1) \tag{3.3.15}$$

where r_k represents the number of times the part k appears in the partition $\boldsymbol{\rho}$ and, thus, $\sum_k r_k = d+1 = \ell'$.

The fugacities associated to this symmetry are x_1, x_2, …, x_{d+1}, subject to the constraint:

$$\prod_{i=1}^{d+1} x_i^{\rho_i} = 1 \ . \tag{3.3.16}$$

3. The power of t in the prefactor is

$$\delta_{U(N)}(\boldsymbol{n}) = \sum_{j=1}^{N} (N+1-2j) n_j \ . \tag{3.3.17}$$

4. The Hall-Littlewood polynomial associated with the group $U(N)$ is given by

$$\Psi^{\boldsymbol{n}}_{U(N)}(x_1, \dots, x_N; t) = \sum_{\sigma \in S_N} x^{n_1}_{\sigma(1)} \dots x^{n_N}_{\sigma(N)} \prod_{1 \le i < j \le N} \frac{1 - t x^{-1}_{\sigma(i)} x_{\sigma(j)}}{1 - x^{-1}_{\sigma(i)} x_{\sigma(j)}} \ . \tag{3.3.18}$$

5. $\boldsymbol{w}_r$ denotes the weights of the $SU(2)$ representation of dimension r:

$$\boldsymbol{w}_r = (r-1, r-3, \dots, 3-r, 1-r) \ . \tag{3.3.19}$$

Hence the notation $t^{\boldsymbol{w}_r}$ represents the vector

$$t^{\boldsymbol{w}_r} = (t^{(r-1)}, t^{(r-3)}, \dots, t^{-(r-3)}, t^{-(r-1)}) \ . \tag{3.3.20}$$

We abbreviate

$$\Psi^{\boldsymbol{n}}_{U(N)}(\boldsymbol{x} t^{\boldsymbol{w}_\rho}; t) := \Psi^{(n_1, \dots, n_N)}_{U(N)}(x_1 t^{\boldsymbol{w}_{\rho_1}}, x_2 t^{\boldsymbol{w}_{\rho_2}}, \dots, x_{d+1} t^{\boldsymbol{w}_{\rho_{d+1}}}; t) \ . \tag{3.3.21}$$

6. The prefactor $K^{U(N)}_{\boldsymbol{\rho}}(\boldsymbol{x}; t)$ is given by [78]

$$K_{\boldsymbol{\rho}}^{U(N)}(\boldsymbol{x};t) = \prod_{i=1}^{\text{length}(\boldsymbol{\rho}^T)} \prod_{j,k=1}^{\rho_i^T} \frac{1}{1-a_j^i \overline{a}_k^i} \ , \tag{3.3.22}$$

where $\boldsymbol{\rho}^T$ denotes the transpose of the partition $\boldsymbol{\rho}$ and

$$\begin{aligned} a_j^i &= x_j \ \ t^{\rho_j - i+1} \ , \qquad i = 1, \dots, \rho_j \\ \overline{a}_k^i &= x_k^{-1} t^{\rho_k - i+1} \ , \qquad i = 1, \dots, \rho_k \end{aligned} \tag{3.3.23}$$

For example:

- For $\boldsymbol{\rho} = [1^N]$, we have $\boldsymbol{\rho}^T = [N]$ and so

$$K_{[1^N]}^{U(N)}(\boldsymbol{x};t) = \prod_{1\le j,k \le N} \frac{1}{1-x_j x_k^{-1} t^2} = \text{PE}[t^2 \chi_{\mathbf{Adj}}^{U(N)}(\boldsymbol{x})] \ . \tag{3.3.24}$$

- For the partition $\boldsymbol{\rho} = [N+1, N]$ of $2N+1$, we have $\boldsymbol{\rho}^T = [2^N, 1]$ and so

$$K_{[N+1,N]}^{U(2N+1)}(\boldsymbol{x};t) = \text{PE}\left[t^{2N+2} + (x_1 x_2^{-1} + x_2 x_1^{-1}) \sum_{j=1}^{N} t^{2j+1} + 2\sum_{l=1}^{N} t^{2l}\right] \ . \tag{3.3.25}$$

3.4 Orientifold planes

The Hanany-Witten brane construction realising three dimensional $\mathcal{N}=4$ theories that we have discussed so far, can be enriched including additional objects provided by string theory and called orientifold p-planes and denoted by Op [75, 161].

In string theory the discrete world-sheet parity transformation acts on the two worldsheet coordinates (τ, σ) by reversing the orientation $d\sigma \wedge d\tau$ of the world-sheet and, thus, by mapping the two end points of an open string to each other.

Such concept of orientability can be made precise by defining a unitary operator Ω which implements the above discrete diffeomorphism acting on the string coordinates and thus reversing the orientation of the string. Hence, when considering a stuck of N branes on top of each other, one has to take into account also the action of Ω on the Chan-Paton factors. At the massless level, this allows to trade the standard $U(N)$ gauge group with a given subgroup defined by the action of Ω.

One can then dress the world-sheet parity transformation by additional $\mathbb{Z}_2$ symmetries of the string theory. In this way one can define more general gauge groups. These constructions are exactly the so-called orientifolds.

In particular, if one takes the additional $\mathbb{Z}_2$ symmetry to be the spatial reflection

$$\mathcal{I}_p \colon x^i \to -x^i \quad \forall i = p+1, \dots, 9 \tag{3.4.1}$$

(where for $p = 2, 3 \mod 4$ one has to include $(-1)^{F_L}$ with F_L the left-moving fermion number), then the combination $\Omega\mathcal{I}_p$ leads to orientifolds where the fixed locus of $\mathcal{I}_p$ defines exactly the Op plane introduced above.

One can think of these as non dynamical (at least at weak string coupling) defects in space-time, which carry a mass-density and preserve the same amount of supersymmetry of a parallel Dp brane.

In the presence of an Op plane a parallel physical Dp brane can split into two half Dp branes, one being the image under the orientifold involution $\Omega\mathcal{I}_p$ of the other.

When $p \leq 5$, the Op planes come in four varieties

Type	G	q
Op^-	$SO(2N)$	-2^{p-5}
Op^+	$USp(2N)$	2^{p-5}
$\widetilde{Op}^-$	$SO(2N+1)$	$\frac{1}{2} - 2^{p-5}$
$\widetilde{Op}^+$	$USp'(2N)$	2^{p-5}

(3.4.2)

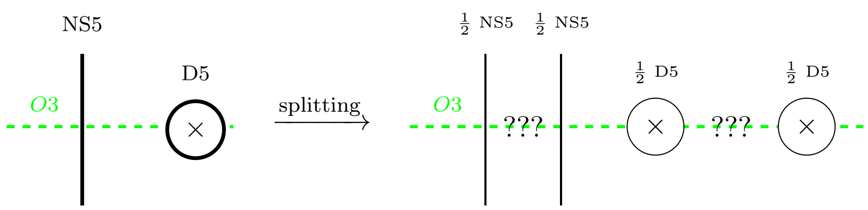
where G denotes the gauge group associated to a stack of $2N$ half Dp branes sitting on top of the Op plane, q is their fractional Dp brane charge.

Observe that half Dp branes always come in pairs. In the case of a $\widetilde{Op}^-$ plane, we get an additional half Dp brane stuck on the orientifold plane so that we shall consider as the real $\widetilde{Op}^-$ plane the bound state between the plane and this half brane. Moreover, there is no difference as Lie algebras between $USp'(2N)$ and $USp(2N)$; they however differ in a global factor when considered as gauge theories.

In Hanany-Witten brane configurations, in order to preserve supersymmetry, we can only insert $O3$ and $O5$ planes. The first must be parallel to the D3 branes, the latter to the fivebranes. We will be interested in $O3$ planes only.

3.4.1 $O3$ planes

When an $O3$ plane is introduced in the Hanany-Witten brane setup [75], we can split also the fivebranes into couples of half branes related by the orientifold involution $\Omega\mathcal{I}_3$. We can thus move the pair of half branes to touch the $O3$ plane where, in principle, they can then move freely; see Fig. (3.4.3).



(3.4.3)

Due to the conservation of the linking numbers (3.1.11) and the fact that $O3$ planes posses a D3 brane charge q, such splitting is a nontrivial dynamical process. Thus, in touching the $O3$ plane, sometimes half D3 branes are created (or annihilated) between the two half fivebranes. This different behaviour depends on which $O3$ plane is present and how many half D3 branes are already attached to the initial physical fivebrane.

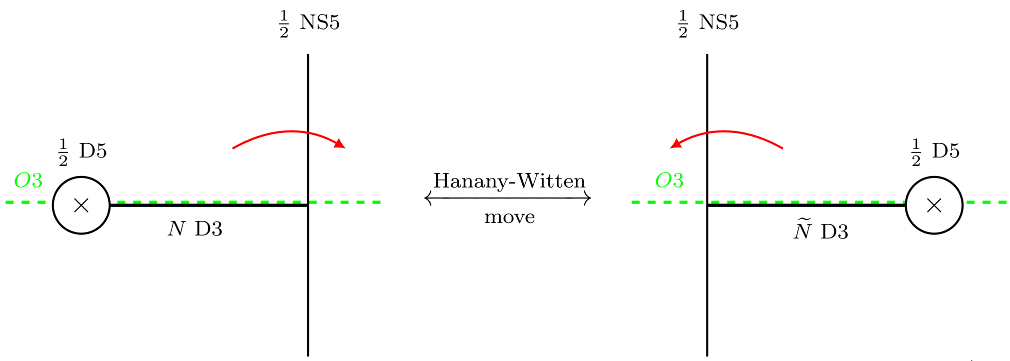
Moreover the $O3$ plane change type when it pass through a half fivebrane. To keep track of this behaviour and other properties, see the following table:

Type	G	q	S-dual	half NS5	half D5
$O3^-$	$SO(2N)$	$-\frac{1}{4}$	$O3^-$	$O3^+$	$\widetilde{O3}^-$
$O3^+$	$USp(2N)$	$\frac{1}{4}$	$\widetilde{O3}^-$	$O3^-$	$\widetilde{O3}^+$
$\widetilde{O3}^-$	$SO(2N+1)$	$\frac{1}{4}$	$O3^+$	$\widetilde{O3}^+$	$O3^-$
$\widetilde{O3}^+$	$USp'(2N)$	$\frac{1}{4}$	$\widetilde{O3}^+$	$\widetilde{O3}^-$	$O3^+$

(3.4.4)

where the fourth column displays the $O3$ planes relations under the S-duality of type IIB string theory (notice that $O3^-$ and $\widetilde{O3}^+$ are self-dual) and the last two columns contain the type change of $O3$ planes crossing a half fivebrane.

In order to keep the linking numbers invariant between a half NS5 brane and a half D5 brane connected by half D3 branes, we also need to modify the Hanany-Witten move as follows. Suppose, as in Fig. (3.4.5), that we start with N physical D3 branes connecting a half D5 brane on the left to a half NS5 brane on the right. Moving the half D5 brane along x^6 to pass through the half NS5 brane, we get $\widetilde{N}$ physical D3 branes. Thanks to S-rule we know that both $N, \widetilde{N} = 0, 1$.



(3.4.5)

Thus:

- When the charge q of the $O3$ planes at the exterior of the two half fivebranes is the same, the conservation of the linking numbers requires

$$N + \widetilde{N} = 1 \tag{3.4.6}$$

  so that there is annihilation or creation of a D3 brane in crossing.

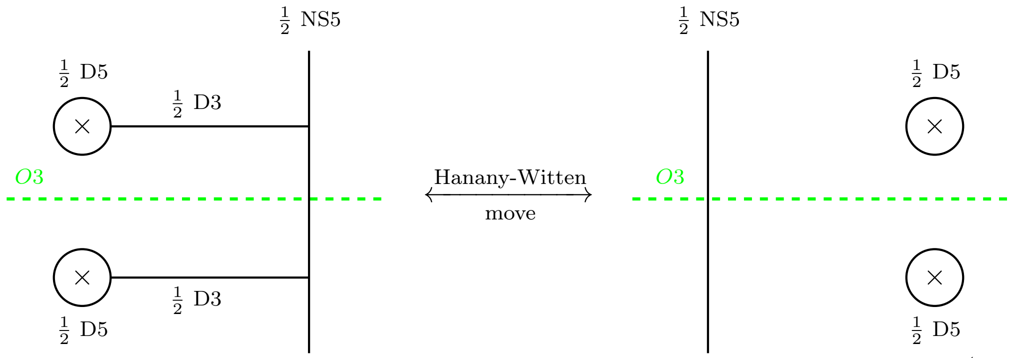
- When the charge q of the $O3$ planes is the opposite, the condition reads

$$N = \widetilde{N} = 0 \tag{3.4.7}$$

  so that there can not be any D3 brane between the two half fivebranes both before and after the crossing.

Thanks to the Hanany-Witten move, the rules of splitting physical D5 branes that end on half NS5 branes is pretty easy. We can in fact simply split the physical D5 brane and move the two half D5 branes to left of the half NS5 brane by means of

Hanany-Witten move, see Fig. (3.4.8). S-rule then prevents more than one physical D3 brane to end on the same half NS5 brane.

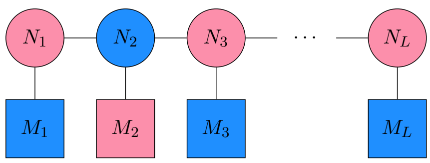


(3.4.8)

In the following, we will make large use of this trick to split the physical D5 branes attached to NS5 branes to avoid the much more complicated general rules of splitting fivebranes in other setups.

3.5 $T^\sigma_\rho(USp'(2N))$ theories

As for the $T^{\boldsymbol{\sigma}}_{\boldsymbol{\rho}}(SU(N))$ theories, the quiver diagram for $T^{\boldsymbol{\sigma}}_{\boldsymbol{\rho}}(USp'(2N))$ is also linear [69]. It however consists of alternating $(S)O/USp$ groups depicted in (3.5.1), where each red node with a label N denotes an $O(N)$ or $SO(N)$ group and each blue node with an even label $2N$ denotes a $USp(2N)$ group.



(3.5.1)

This alternating nature is due to the fact that, from the string theory perspective, quiver (3.5.1) can be realised on the worldvolume of N D3 branes parallel to an orientifold $\widetilde{O3}^+$ plane and ending on systems of half D5 branes and of half NS5 branes. Similar to the $T^{\boldsymbol{\sigma}}_{\boldsymbol{\rho}}(SU(N))$ theories, the partitions $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$ determine how the D3 branes end on the half D5 branes and on the half NS5 branes respectively. In this case both $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$ are C-partitions[2] of $2N$, of lengths ℓ and ℓ' respectively.

In quiver (3.5.1), we defined

$$L = \begin{cases} \ell' - 1 & \ell' \text{ is even} \\ \ell' & \ell' \text{ is odd} \end{cases} \tag{3.5.2}$$

and if both N_L and M_L are zero, the nodes are removed from the quiver and the length of the quiver is $L-1$.

The Hanany-Witten brane construction of such theories is subtle and the rules we have previously seen for $T^{\boldsymbol{\sigma}}_{\boldsymbol{\rho}}(SU(N))$ theories do change. Due to the presence of the $O3$ plane, we are always forced to have an even number of half D3 branes suspended between the fivebranes. This implies that, looking at the C-partitions, two things can happen:

[2] A C-partition is a partition where every odd entry must appear an even number of times.

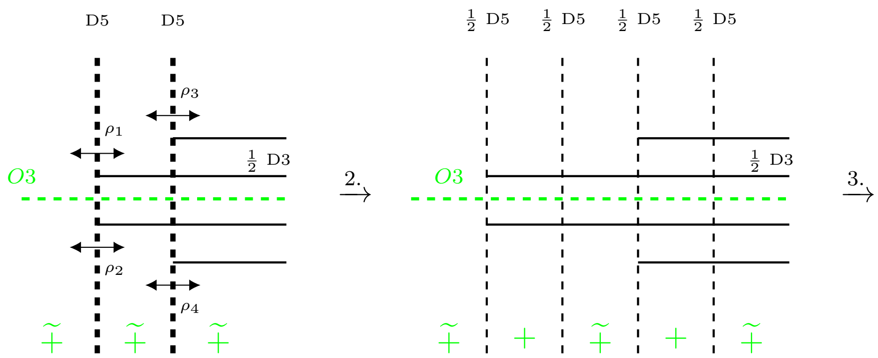
- When we encounter an even term, *i.e.* $(\ldots, 2k, \ldots)$, the respective half D3 branes are simply attached to a half fivebrane;
- Whenever we encounter a couple of odd terms, *i.e.* $(\ldots, 2k+1, 2k+1, \ldots)$, we must attach the total amount of $4k+2$ half D3 branes to the same physical fivebrane instead of two different half fivebranes.

Thus, in principle, we should pay attention to the splitting rules of every physical fivebrane sitting in the configuration.

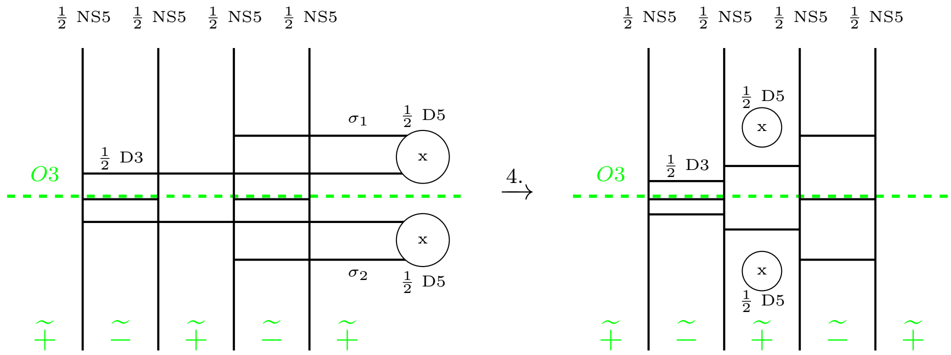
Nonetheless, we can spoil the mirror duality properties of such theories to trade $\boldsymbol{\sigma}$ for $\boldsymbol{\rho}$ and vice versa to work with D5 branes only. In this way, since by construction the physical D5 branes will always be attached to the NS5 ones, we can use the trick of Fig. (3.4.8) to freely split them. Then, by means of S-duality, we can again exchange the half D5 branes with the half NS5 and read the quiver more easily.

In Fig. (3.5.3) is depicted an example of this building technique for the $T^{[2^2]}_{[1^4]}(USp'(4))$ theory. The steps are the following:

1. By means of S-duality, we take the C-partition $\boldsymbol{\rho} = (1, 1, 1, 1)$ and consider a sequence of two physical D5 branes attached by couples of half D3 branes to some NS5 branes placed at infinity[3]. This construction is similar to the one discussed in [39], except the fact that an $O3$ plane is put into the brane system such that the semi-infinite D3 branes are on top of the $\widetilde{O3}^+$ plane.
2. We can then freely split the physical D5 branes into half branes preserving the number of half D3 branes.
3. Applying a mirror transformation, we can trade the half D5 branes for half NS5 and vice versa. Also the $O3$ plane transforms as in Table (3.4.4). In doing so, we can now consider the newly transformed D5 branes that were previously placed at infinity. Taking the C-partition $\boldsymbol{\sigma} = (2, 2)$, we can repeat the procedure outlined in steps 1. and 2..
4. Finally, using S-rule and Hanany-Wittem move, we can read off the quiver diagram.

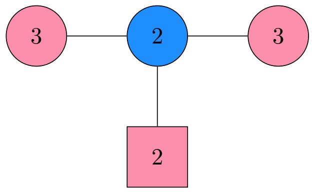


[3]Since the NS5 branes are placed at infinity, we do not mind here if they are physical of half branes.



(3.5.3)

So that, looking at the last part of Fig. (3.5.3), the quiver diagram of the $T^{[2^2]}_{[1^4]}(USp'(4))$ theory reads



(3.5.4)

The other building method of quiver (3.5.1) relies again on Young tableaux, as for the $T^{\boldsymbol{\sigma}}_{\boldsymbol{\rho}}(SU(N))$ theories [69].

The labels M_j, with $1 \leq j \leq L$, for the flavour symmetries are again determined from the transpose $\boldsymbol{\sigma}^T = (\widehat{\sigma}_1, \ldots, \widehat{\sigma}_{\widehat{\ell}})$, with $\widehat{\sigma}_1 \geq \ldots \geq \widehat{\sigma}_{\widehat{\ell}} > 0$, of $\boldsymbol{\sigma}$ as follows:

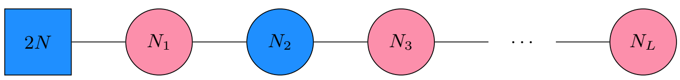
$$M_j = \widehat{\sigma}_j - \widehat{\sigma}_{j+1} \tag{3.5.5}$$

with $\widehat{\sigma}_i = 0$, for all $i \geq \widehat{\ell} + 1$.

On the other hand, the labels N_j, with $1 \leq j \leq L$, for the gauge symmetries are given by

$$N_j = \begin{cases} \left[1 + \sum_{k=j+1}^{\ell'} \rho_k\right]_{\widetilde{+}} - \left(\sum_{i=j+1}^{\widehat{\ell}} \widehat{\sigma}_i\right) & \text{for the } O/SO \text{ node}\,, \quad \text{if } \ell' \text{ is even}\,, \\ \left[\sum_{k=j+1}^{\ell'} \rho_k\right]_{-} - \left(\sum_{i=j+1}^{\widehat{\ell}} \widehat{\sigma}_i\right) & \text{for the } USp \text{ node}\,, \quad \text{if } \ell' \text{ is even}\,, \\ \left[\sum_{k=j+1}^{\ell'} \rho_k\right]_{\widetilde{+}} - \left(\sum_{i=j+1}^{\widehat{\ell}} \widehat{\sigma}_i\right) & \text{for the } O/SO \text{ node}\,, \quad \text{if } \ell' \text{ is odd}\,, \\ \left[\sum_{k=j+1}^{\ell'} \rho_k\right]_{+} - \left(\sum_{i=j+1}^{\widehat{\ell}} \widehat{\sigma}_i\right) & \text{for the } USp \text{ node}\,, \quad \text{if } \ell' \text{ is odd}\,. \end{cases} \tag{3.5.6}$$

Again, we will only focus on the case in which $\boldsymbol{\sigma} = [1^{2N}]$ and the theory in question is denoted by $T_{\boldsymbol{\rho}}(USp'(2N))$. The corresponding quiver diagram reduces to

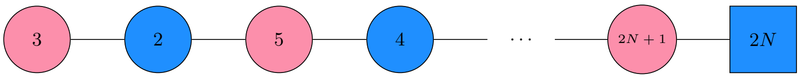


(3.5.7)

As we justify in the main text, the red circular node with a label N in this quiver should be taken as the **special orthogonal** $SO(N)$ **gauge group**.

We will mainly consider the following two examples.

The partition $\boldsymbol{\rho} = [1^{2N}]$. We denote the theory in this case by $T(USp'(2N))$ and the corresponding quiver is



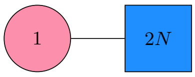
(3.5.8)

We remark this quiver is a 'bad' theory in the sense of [85]. Nevertheless, one can use this description to compute many quantities, such as the Coulomb branch dimension and the Higgs branch Hilbert series. Moreover, we can bypass the 'badness' of the quiver and compute the Coulomb branch Hilbert series using the Hall-Littlewood formula as will be explained below.

Since $\boldsymbol{\rho} = \boldsymbol{\sigma} = [1^{2N}]$, the theory is indeed self-mirror in this case. In fact, both Higgs and Coulomb branches of this theory are isomorphic to the nilpotent cone of $USp(2N)$, whose quaternionic dimension is

$$\dim_{\mathbb{H}}\{\mathcal{M}\} = \frac{1}{2}\left[\frac{1}{2}(2N)(2N+1) - N\right] = N^2 \tag{3.5.9}$$

The partition $\boldsymbol{\rho} = [2N]$. The corresponding quiver is



(3.5.10)

where the red circular node denotes the $SO(1)$ group, and so the gauge symmetry is trivial in this case. This is simply a theory of free $2N$ half hypermultiplets.

3.5.1 The Coulomb branch Hilbert series

It is possible to compute the Coulomb branch Hilbert series from the quiver description using the monopole formula [64], provided that the quiver is not a 'bad' theory in the sense of [85].

Alternatively, for a given C-partition $\boldsymbol{\rho}$ of $2N$, one can directly compute the Coulomb branch Hilbert series of $T_{\boldsymbol{\rho}}(USp'(2N))$ using a simple modification of the Hall-Littlewood formula [67], without drawing the quiver and regardless whether it is 'bad' or not.

The correct modification for the Coulomb branch Hilbert series in question is the following:

$$\begin{aligned} &H_{\mathcal{C}}[T_{\boldsymbol{\rho}}(USp'(2N))](t; \boldsymbol{x}, n_1, \ldots, n_N) \\ &= t^{\delta_{C_N}(\boldsymbol{n})}(1-t^2)^N K_{\boldsymbol{\rho}}^{C_N}(\boldsymbol{x}, t)\Psi_{C_N}^{\boldsymbol{n}}(\boldsymbol{a}(t,\boldsymbol{x}), t) \ , \end{aligned} \tag{3.5.11}$$

where the notations are as follows:

1. The power of t in the prefactor is

$$\delta_{C_N}(\boldsymbol{n}) = \sum_{j=1}^{N}(2N+2-2j)n_j \ . \tag{3.5.12}$$

2. The function $\Psi_{C_N}^{\boldsymbol{\lambda}}(\boldsymbol{x}, t)$ is the Hall-Littlewood polynomial associated with the C_N algebra and the partition $\boldsymbol{\lambda}$ is subject to $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N \geq 0$, with all $\lambda_i \in \mathbb{N}$. It is given by

$$\Psi^{\boldsymbol{\lambda}}_{C_N}(x_1,\ldots,x_N;t) = \sum_{s_1,\ldots,s_N=\pm 1}\sum_{\sigma\in S_N}\left(\prod_{i=1}^{N} x_{\sigma(i)}^{s_i\lambda_i}\frac{1-t^2x_{\sigma(i)}^{-2s_i}}{1-x_{\sigma(i)}^{-2s_i}}\right)\times \left(\prod_{1\le i<j\le N}\frac{1-t^2x_{\sigma(i)}^{-s_i}x_{\sigma(j)}^{s_j}}{1-x_{\sigma(i)}^{-s_i}x_{\sigma(j)}^{s_j}}\cdot\frac{1-t^2x_{\sigma(i)}^{-s_i}x_{\sigma(j)}^{-s_j}}{1-x_{\sigma(i)}^{-s_i}x_{\sigma(j)}^{-s_j}}\right) .$$

3. The argument $\boldsymbol{a}(t,\boldsymbol{x})$, which shall be abbreviated as $\boldsymbol{a}$, of the Hall-Littlewood polynomial can be determined by considering the decomposition

$$x^{C_N}_{\mathbf{fund}}(\boldsymbol{a}) = \sum_{j=1}^{N}(a_j+a_j^{-1}) = \sum_k x^{G_{\rho_k}}_{\mathbf{fund}}(\boldsymbol{x}_k)\chi^{SU(2)}_{[\rho_k-1]}(t) \ , \tag{3.5.13}$$

where the group G_{ρ_k} depends on the part k of the partition $\boldsymbol{\rho}$ that appears r_k times and is defined as

$$G_{\rho_k} = \begin{cases} USp(r_k) & \text{if } k \text{ is odd} \\ SO(r_k) & \text{if } k \text{ is even .}\end{cases} \tag{3.5.14}$$

For example, for $\boldsymbol{\rho}=[1^{2N}]$, we have $a_j = x_j$ for $j=1,\ldots,N$, and for $\boldsymbol{\rho}=[2N]$, we have $a_j = t^{2j-1}$ for $j=1,\ldots,N$.

4. The prefactor $K^{C_N}_{\boldsymbol{\rho}}(\boldsymbol{x},t)$ can be determined in two steps.

First of all, we need to identify the representations $\mathcal{R}_j$ of the group

$$G_{\boldsymbol{\rho}} = \prod_k G_{\rho_k} = \prod_{k \text{ odd}} USp(r_k)\times\prod_{k \text{ even}} SO(r_k) \tag{3.5.15}$$

from the following decomposition:

$$\chi^{C_N}_{\mathbf{Adj}}(\boldsymbol{a}) = \sum_{j=0,\frac{1}{2},1,\frac{3}{2},\ldots}\chi^{G_{\boldsymbol{\rho}}}_{R_j}(\boldsymbol{x}_j)\chi^{SU(2)}_{[2j]}(t) \ . \tag{3.5.16}$$

Once $\mathcal{R}_j$ are determined, the prefactor in question is then given by

$$K^{C_N}_{\boldsymbol{\rho}}(\boldsymbol{x},t) = \mathrm{PE}\left[\sum_{j=0,\frac{1}{2},1,\frac{3}{2},\ldots} t^{2j+2}\chi^{G_{\boldsymbol{\rho}}}_{\mathcal{R}_j}(\boldsymbol{x}_j)\right] . \tag{3.5.17}$$

For example, for $\boldsymbol{\rho}=[1^{2N}]$, we have

$$K^{C_N}_{[1^{2N}]}(\boldsymbol{x},t) = \mathrm{PE}\left[\chi^{C_N}_{\mathbf{Adj}}(\boldsymbol{x})t^2\right] \tag{3.5.18}$$

and for $\boldsymbol{\rho}=[2N]$, we have

$$K^{C_N}_{[1^{2N}]}(\boldsymbol{x},t) = \mathrm{PE}\left[t^4 + t^8 + \ldots + t^{4N}\right] \ . \tag{3.5.19}$$

For a given partition $\boldsymbol{\rho}$, the Coulomb branch symmetry of $T_{\boldsymbol{\rho}}(USp'(2N))$ is $G_{\boldsymbol{\rho}}$, determined by (3.5.15). In the Coulomb branch Hilbert series (3.5.11), the fugacities $\boldsymbol{x}$ are those associated with the symmetry $G_{\boldsymbol{\rho}}$, and $n_1, \ldots, n_N$ are the background magnetic fluxes associated with the flavour symmetry $USp(2N)$ of the theory.

Note that in the special case of $\boldsymbol{\rho} = [2N]$, the Hall-Littlewood polynomial is

$$\Psi^{\boldsymbol{\lambda}}_{C_N}(x_1, \ldots, x_N; t) = t^{-\sum_{j=1}^{N}(2N+1-2j)n_j}\, \mathrm{PE}[Nt^2 - t^4 - t^8 - \ldots - t^{4N}] \tag{3.5.20}$$

and so the Hilbert series (3.5.11) becomes

$$H_{\mathcal{C}}[T_{[2N]}(USp'(2N))](t; \boldsymbol{x}, n_1, \ldots, n_N) = t^{n_1+n_2+\ldots+n_N} \ . \tag{3.5.21}$$

This is indeed the Coulomb branch Hilbert series of the theory of free $2N$ half-hypermultiplets, as described in (3.5.10).

3.6 class-S primer

Throughout this chapter, we will be interested in the S^1 reduction of some four-dimensional theories of class-S. So, first of all, let us briefly review what class-S means [81].

Class-S theories are a large family of 4d $\mathcal{N} = 2$ SCFTs that may or may not admit a Lagrangian description. In building such a theory we start from a six-dimensional $\mathcal{N} = (2,0)$ superconformal field theory (SCFT) denoted by $\chi(\mathfrak{g})$, that is characterized by a simply-laced Lie algebra $\mathfrak{g}$[4]. This theory possesses 32 real supercharges organised in two multiplets $\mathcal{Q}^I$ which transforms in the spinor representations $(\mathbf{4}, \mathbf{4})$ of the Lorentz group $SO(1,5)$ and R-symmetry $SO(5)_R$ respectively.

We then compactify $\chi(\mathfrak{g})$ on a Riemann surface $C_{g,n}$ with genus g called Gaiotto curve, while preserving $4d$ $\mathcal{N} = 2$ supersymmetry. The Riemann surface can have a number n of codimension-two defects known as *punctures* at which boundary conditions must be prescribed. These boundary conditions are encoded by some data D_i with $i = 1, \ldots n$.

Each choice of simply-laced Lie algebra $\mathfrak{g}$, punctured Riemann surface $C_{g,n}$ and data D_i leads to a different $4d$ $\mathcal{N} = 2$ class-S theory that we can dub $T(\mathfrak{g}, C_{g,n}, D_i)$.

3.6.1 The partial topological twist

Performing the compactification on $C_{g,n}$ too naively would break all the symmetries apart the four-dimensional Poincare symmetry. However, there is a precise procedure, called partial topological twist [83], which preserves the four-dimensional $\mathcal{N} = 2$ supersymmetry with any Riemann surface $C_{g,n}$.

When considering a field theory on a curved background, the metric $g_{\mu\nu}$ acts as a source for the stress tensor $T^{\mu\nu}$. In the supersymmetric case, $T^{\mu\nu}$ typically belongs to the same superconformal multiplet of the R-symmetry current J^μ. The topological twist [164, 166] amounts to mixing exactly these two currents as follows

[4] A simple Lie algebra $\mathfrak{g}$ is simply-laced if all its roots have the same length. Thus, such algebras are of the ADE type.

$$T^{\mu\nu}_{\text{twist}} := T^{\mu\nu} + \partial^\mu J^\nu \tag{3.6.1}$$

before placing the theory on a non-trivial background metric.

Since (3.6.1) shifts T by total derivatives, it leaves untouched the corresponding conserved charge, *i.e.* the momentum operator. However, it has a non-trivial effect on rotations; a twisted rotation acts now by a standard rotation combined with an R-symmetry transformation.

The supercharges become either scalars or vectors under the new rotations and when we put the theory on a curved manifold, only the scalar ones are preserved.

In this sense, to obtain the correct amount of supersymmetry when placing the theory on $C_{g,n}$, we can use a *partial* topological twist, mixing only some of the R-symmetries with some of the rotation symmetries. In order to define the correct twist let us decompose our six-dimensional flat space as $\mathbb{R}^{1,5} = \mathbb{R}^{1,3} \times \mathbb{R}^2$ so that the Lorentz group becomes

$$SO(1,3) \times SO(2)_{\text{old}} \subset SO(1,5) \tag{3.6.2}$$

In this way, each supercharge $\mathcal{Q}^I$ decomposes into a pair of 4d Weyl spinors of opposite chirality under $SO(1,3)$ and of opposite charges under $SO(2)_{\text{old}}$; which means

$$\mathbf{4} \to (\mathbf{2},\mathbf{1})^{\frac{1}{2}} \oplus (\mathbf{1},\mathbf{2})^{-\frac{1}{2}} \tag{3.6.3}$$

We also decompose the R-symmetry as

$$SO(3)_R \times SO(2)_R \subset SO(5)_R \tag{3.6.4}$$

so that now each supercharge $\mathcal{Q}^I$ further decomposes into two $SO(3)_R$ spinors with opposite $SO(2)_R$ charge; namely

$$\mathbf{4} \to \mathbf{2}^{\frac{1}{2}} \oplus \mathbf{2}^{-\frac{1}{2}} \tag{3.6.5}$$

We thus define the twisted rotations on $\mathbb{R}^2$ as the diagonal $SO(2)_{\text{twist}}$ combination of the $SO(2)_{\text{old}}$ rotations and the $SO(2)_R$ R-symmetry subgroup; obtaining

$$SO(1,3) \times SO(2)_R \times SO(3)_R \times SO(2)_{\text{twist}} \subset SO(1,5) \times SO(5)_R \tag{3.6.6}$$

By construction, the $SO(2)_{\text{twist}}$ charge carried by the supercharges becomes the sum of those under $SO(2)_{\text{old}}$ and $SO(2)_R$. So we conclude that, under (3.6.6), the supercharges transform as

$$(\mathbf{4},\mathbf{4}) \to \overbrace{(\mathbf{2},\mathbf{1};\mathbf{2})^{(\frac{1}{2},\mathbf{1})}}^{Q^{\alpha A}_{z}} \oplus \overbrace{(\mathbf{2},\mathbf{1};\mathbf{2})^{(-\frac{1}{2},\mathbf{0})}}^{Q^{\alpha A}} \oplus \overbrace{(\mathbf{1},\mathbf{2};\mathbf{2})^{(\frac{1}{2},\mathbf{0})}}^{\overline{Q}^{\dot\alpha A}} \oplus \overbrace{(\mathbf{1},\mathbf{2};\mathbf{2})^{(-\frac{1}{2},-\mathbf{1})}}^{\overline{Q}^{\dot\alpha A}_{\bar z}} \tag{3.6.7}$$

where $\alpha, \dot\alpha = 1,2$ are spinor indices of $SO(1,3)$ distinguishing between the two chiralities and $A = 1,2$ is the spinor index of $SO(3)_R$. Additionally, z is the complex coordinate on $\mathbb{R}^2$ representing the $SO(2)_{\text{twist}}$ charges ± 1.

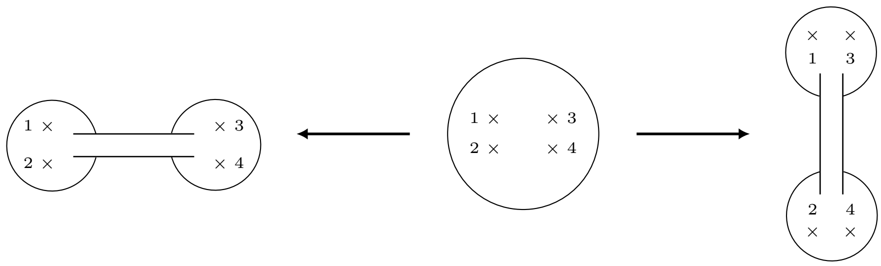
As anticipated, when deforming $\mathbb{R}^2$ to any curved Riemann surface $C_{g,n}$, only the supercharges that are scalars under $\times SO(2)_{\text{twist}}$ are preserved. These are $\{Q^{\alpha A}, \overline{Q}^{\dot\alpha A}\}$ which, in the limit where $C_{g,n}$ shrinks to a point, can be identified with the 8 supercharges of the $\mathcal{N} = 2$ supersymmetric theories on $\mathbb{R}^{1,3}$.

The R-symmetry indeed becomes

$$SU(2)_R \times U(1)_R \simeq SO(3)_R \times SO(2)_R \tag{3.6.8}$$

3.6.2 Tubes and tinkertoys

Any Riemann surface $C_{g,n}$ admits a variety of the so-called *pants decompositions* [51] that deconstruct the surface into a series of three-punctured spheres, dubbed *trinions*, glued together by connecting pairs of punctures with tubes.



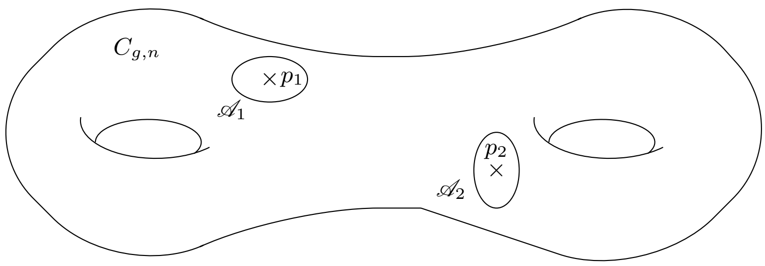
(3.6.9)

The main building blocks of the $T(\mathfrak{g}, C_{g,n}, D_i)$ class-S theories are thus:

- Theory on a sphere with two punctures, *i.e.* a tube.

  Thanks to the partial topological twist of the previous section, once compactified on $C_{g,n}$, the 4d theory depends only on the complex structure of the Gaiotto curve. This quantity can be described by the "length" and "twist" of each tube of a pants decompositions.

  Let us consider a generic surface $C_{g,n}$ with two punctures p_i of local coordinates z_i and two disks $\mathscr{A}_i$ surrounding them, as in Fig. (3.6.10).

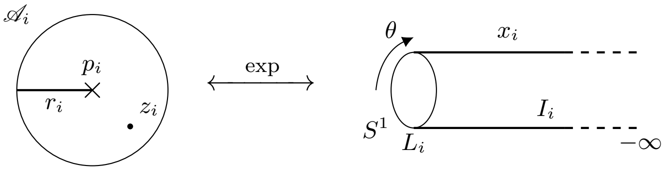


(3.6.10)

  We can trade the disks for semi-infinite cylinders thanks to the exponential map

$$\begin{aligned} e\colon I_i \times S^1 &\to \mathscr{A}_i \\ (x_i, \theta) &\mapsto z_i = e^{x_i + i\theta} \end{aligned} \tag{3.6.11}$$

  where $I_i = (-\infty, L_i]$ is a semi-infinite interval such that the radius of the disk $\mathscr{A}_i$ is $r_i = e^{L_i}$ (see Fig. (3.6.12)).



(3.6.12)

By cutting the infinite ends of such cylinders at some finite cut-off, we can then glue them together by identifying their finite ends. In doing so, we can also add a twist to the tubes before the gluing. We thus obtain a single cylinder, starting from the first puncture p_1 and ending on the second one p_2. In terms of the complex coordinates z_i, such an operation reads

$$z_1 z_2 = \eta \tag{3.6.13}$$

for some parameter η characterising the tube theory. Geometrically speaking, in fact, its modulus $|\eta|$ is related to the length-over-circumference ratio of the tube, while its phase $\mathrm{Arg}(\eta)$ indicates how the two cylinders were twisted.

These two quantities are also related to the physics of the underlying 4d theory on the tube as follows.

First of all, by compactifying the original 6d theory $\chi(\mathfrak{g})$ on the circle of length $2\pi L_5$ of the tube, we get a 5d $\mathcal{N} = 2$ SYM theory with gauge algebra $\mathfrak{g}$ and coupling $g_{5d}^2 \propto L_5$. Then, this theory is further restricted to an interval of length $L_4 \propto -\log|\eta| L_5$ which is nothing but the length of the tube. Thus, in the limit where $C_{g,n}$ shrinks to a point, we get a 4d $\mathcal{N} = 2$ gauge theory with gauge group $\mathfrak{g}$ and gauge coupling

$$\frac{1}{g_{4d}^2} \propto \frac{L_4}{g_{5d}^2} \propto -\log|\eta| \tag{3.6.14}$$

On the other hand, $\mathrm{Arg}(\eta)$ is identified with the θ angle of the four-dimensional theory. This is because the Kaluza-Klein modes along the S^1 circumference of the tube are identified with the five-dimensional instantons. These are in fact charged under the current

$$J^{\mu}_{\text{inst}} = \varepsilon^{\mu\nu\rho\sigma\lambda} \operatorname{Tr}\left(F_{\nu\rho} F_{\sigma\lambda}\right) \tag{3.6.15}$$

which also generates the rotations along S^1 that are parametrised by the twist $\mathrm{Arg}(\eta) \propto \theta$.

In this sense, the tube theory gives rise to a four-dimensional $\mathcal{N} = 2$ vector multiplet in the adjoint representation of $\mathfrak{g}$ with complexified gauge coupling

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{4d}^2} \simeq \frac{\mathrm{Arg}(\eta)}{2\pi} - \frac{4\pi i}{\log|\eta|} \tag{3.6.16}$$

- Theory on a sphere with three punctures, *i.e.* a trinion.

  Each trinion corresponds to a different class-S theory called "tinkertoy". Since such tinkertoys range in complexity from free hypermultiplets to unknown SCFTs, not much can be said in general about them.

However, each puncture turns out to be associated with a flavour symmetry of the underlying four-dimensional theory. In fact, for a given algebra $\mathfrak{g}$, there exist different types of punctures carrying a different amount of global symmetry. Thus, as we have seen, connecting two punctures by a tube amounts to gauging a diagonal combination of the two flavour symmetries by means of a 4d $\mathcal{N}=2$ vector multiplet.

In this sense, the general theory $T(\mathfrak{g}, C_{g,n}, D_i)$ can be decomposed into such tinkertoys; each decomposition corresponding to a different regime in the coupling space of the underlying 4d SCFT.

3.6.3 Punctures

As we already said in the previous paragraphs, there exist different types of punctures. To correctly classify these codimension-two defects one must understand the behaviour on the Riemann surface $C_{g,n}$ of a Lie algebra valued holomorphic one-form field $\Phi = \Phi_z dz$, called the Higgs field or Hitchin field [82]. In fact, the Casimirs $\{\mathcal{O}_j\}$ of this operator, with $j = 1, \ldots, \mathrm{Rank}\,\mathfrak{g}$, are exactly the chiral operators describing the Coulomb branch of the underlying four-dimensional theory.

To understand what are these chiral operators, we have to study the moduli space of vacua of the six-dimensional theory. which is dubbed "tensor branch". To do so, let us first consider the simplest 6d $\mathcal{N}=(2,0)$ theory, *i.e.* $\chi(U(1))$, consisting only of a free tensor multiplet. This contains:

- Five $SO(1,5)$ real scalars ϕ the $(\mathbf{5})$ of $SO(5)_R$;
- Four $SO(1,5)$ Weyl fermions λ in the $(\mathbf{4})$ of $SO(5)_R$;
- A $SO(1,5)$ 2-form B singlet under $SO(5)_R$ and with a self-dual field-strength $H = dB$.

It turns out that the tensor branch of a generic $\chi(\mathfrak{g})$ theory contains a $U(1)^{\mathrm{Rank}\,\mathfrak{g}}$ IR theory consisting of $\mathrm{Rank}\,\mathfrak{g}$ free tensor multiplets. Thus, similarly to the three-dimensional cases of Chapter (2), it is parametrised by the VEVs of the scalar components ϕ of such multiplets, belonging to the Cartan subalgebra of $\mathfrak{g}$ modulo the action of the Weyl group.

We can thus reconstruct the four-dimensional Coulomb branch from the tensor branch of the parent 6d theory, by decomposing the components of the tensor multiplet under the symmetry group

$$SO(1,3) \times SO(3)_R \times SO(2)_R \times SO(2)_{\text{old}}. \tag{3.6.17}$$

We get that:

- The real scalars decompose as

$$\begin{aligned} \phi_z := \phi_1 + i\phi_2 &\to (\mathbf{1},\mathbf{1};\mathbf{1})^{(\mathbf{0},\mathbf{1})} \\ \phi_{\overline{z}} := \phi_1 - i\phi_2 &\to (\mathbf{1},\mathbf{1};\mathbf{1})^{(\mathbf{0},\mathbf{-1})} \\ \phi_{3,4,5} &\to (\mathbf{1},\mathbf{1};\mathbf{3})^{(\mathbf{0},\mathbf{0})} \end{aligned} \tag{3.6.18}$$

- The Weyl fermions decompose as

$$\lambda_a \to (\mathbf{2},\mathbf{1};\mathbf{2})^{(\frac{1}{2},\pm\frac{1}{2})} \oplus (\mathbf{1},\mathbf{2};\mathbf{2})^{(\frac{1}{2},\pm\frac{1}{2})} \tag{3.6.19}$$

After compactifying the 6d theory on the Riemann surface $C_{g,n}$, we are left with the 4d $\mathcal{N}=2$ supercharges $\{Q^{\alpha A}, \overline{Q}^{\dot{\alpha}A}\}$ only. These transform under the symmetry group (3.6.17) as

$$Q^{\alpha A} \to (\mathbf{2},\mathbf{1};\mathbf{2})^{(\frac{1}{2},-\frac{1}{2})}\ , \qquad \overline{Q}^{\dot{\alpha}A} \to (\mathbf{1},\mathbf{2};\mathbf{2})^{(-\frac{1}{2},\frac{1}{2})} \tag{3.6.20}$$

We deduce that

$$\overline{Q}^{\dot{\alpha}A}\phi_z = 0 \tag{3.6.21}$$

because no component of λ has the appropriate $SO(2)_R$ charge $\mathbf{\frac{3}{2}}$. This is clearly similar to the definition (2.2.18) of chiral operators, apart from the gauge invariance property.

We thus expect the four-dimensional Coulomb branch to be parametrised by the VEVs of gauge invariant combinations of ϕ_z. These are exactly the Casimirs of $\mathfrak{g}$, *i.e.* some polynomials $P_k(\phi_z)$ of degrees d_k for $k=1,\dots,\mathrm{Rank}\,\mathfrak{g}$. For example, as we already saw in Section (2.2.6), if $\mathfrak{g}=\mathfrak{a}_N=\mathfrak{su}(N+1)$, the Casimirs are simply $\mathrm{Tr}(\phi_z^j)$ with $j=2,\dots,N+1$[5].

We thus identify

$$\langle O_j\rangle = P_j(\phi_z) \tag{3.6.22}$$

which are exactly the holomoprhic differentials of degree j encoded in the Casimirs of the Hitchin field Φ.

This Hitchin field Φ is useful in classifying the Gaiotto curve defects since its singular behaviour at the punctures p_i fully describes their data D_i.

These defects come in fact in two families:

- The so-called *regular* or *tame* punctures correspond to simple poles for the Hitchin field [52]

$$\Phi_z(z) = \frac{T}{z} + \dots \tag{3.6.23}$$

  where the puncture is located at $z=0$ and T is a semi-simple element of $\mathfrak{g}$ specifying the puncture group structure.

- The so-called *irregular* or *wild* punctures correspond to high order poles instead [156]

$$\Phi_z(z) = \frac{T}{z^{2+\frac{k}{b}}} + \dots \tag{3.6.24}$$

  with $b\in\mathbb{Z}^+, k\in\mathbb{Z}$ and $k>-b$.

Moreover, the Hitchin field Φ can possess a non-trivial monodromy condition as one circles the singualr point [52]. In fact, in the 6d $\mathcal{N}=(2,0)$ theory, one can also

[5] The Casimir for $j=1$ does exist in the unitary case only since $\mathrm{Tr}(\phi)=0$ for a $SU(N)$ valued field ϕ.

define codimension-one defects, dubbed twist lines, that connect pair of punctures (see Fig. (3.6.27)). These twist lines correspond to the discrete global symmetries associated to the outer-automorphism group $\mathrm{Out}(\mathfrak{g})$ of the Lie algebra. This fact further distinguishes the punctures types as follows:

- The so-called *untwisted* punctures have trivial monodromy condition

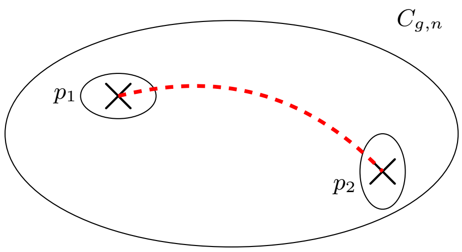
$$\Phi_z(e^{2\pi i}z) = \sigma_g \cdot \Phi_z(z) \tag{3.6.25}$$

  for some inner automorphism σ_g of the Lie algebra $\mathfrak{g}$.

- The so-called *twisted* punctures have non-trivial monodromy condition

$$\Phi_z(e^{2\pi i}z) = \sigma_g \cdot o \cdot \Phi_z(z) \tag{3.6.26}$$

  for some composition of an outer-automorphism o and inner-automorphism σ_g of $\mathfrak{g}$. These punctures can clearly be defined only when $\mathfrak{g}$ has a nontrivial outer-automorphism group $\mathrm{Out}(\mathfrak{g})$ and, due to the presence of twist line defects connecting them, they must always come in pairs.



(3.6.27)

Clearly, these different types of punctures put different constraints on the defining data D_i, which are

$$D_i := (T_i, k_i, b_i) \tag{3.6.28}$$

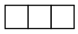
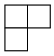
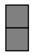
where the elements T_i belong to some subalgebras $\mathfrak{j}_i$ of $\mathfrak{g}$ according to the type of puncture. To each element T_i is then associated a different partition ρ_i which reflects the $\mathfrak{su}(2)$ embedding inside $\mathfrak{j}_i$.

In the following we will be interested in the case $\mathfrak{g} = \mathfrak{a}_{2N} = \mathfrak{su}(2N+1)$ only and we will restrict ourselves to regular defects. We are left with two types of punctures only, namely:

- The regular untwisted puncture, for which $\mathfrak{j} = \mathfrak{a}_{2N} = \mathfrak{su}(2N+1)$ and thus ρ_i is a standard partition of $2N+1$;

- The regular twisted puncture, for which $\mathfrak{j} = \mathfrak{c}_N = \mathfrak{sp}(N)$ and thus ρ_i is a C-partition of $2N$.

For example, if $N = 1$, *i.e.* $\mathfrak{a}_2 = \mathfrak{su}(3)$, the admitted partitions are listed in the following table:

Untwisted		Twisted	
ρ_i	Tableau	ρ_i	Tableau
$[1^3]$		$[1^2]_t$	
$[2,1]$		$[2]_t$	

(3.6.29)

where the principal $\mathfrak{su}(2)$ embedding partition $[\,3\,]$ does not appear in the untwisted cases as it simply does not contribute to the flavour symmetry of the trinion theory. On the other hand, for twisted punctures, even if the principal embedding $[2]_t$[6] gives likewise a flavourless puncture, it however carries a non-trivial monodromy condition.

3.6.4 Argyres-Douglas theories within class-S

Argyres-Douglas theories [10] are a special class of 4d $\mathcal{N}=2$ supersymmetric theories that are usually non-Lagrangian, possess dimensional coupling constants and have Coulomb branch operators that are of fractional scaling dimension Δ.

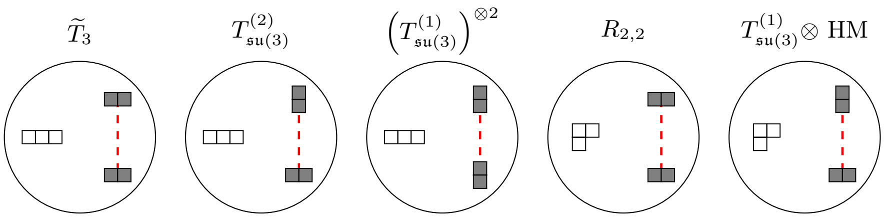
Until the introduction of twisted punctures, it was a common lore that these theories could only be obtained within the class-S framework considering, as a Riemann surfaces $C_{g,n}$, a sphere decorated with irregular punctures in two possible combinations:

- Only one (untwisted) irregular puncture;
- One (untwisted) irregular puncture and one (untwisted) regular puncture.

Indeed, the introduction of the irregular singularity on the sphere provides the desired properties for the underlying 4d theory, which cannot be obtained using only regular (untwisted) punctures.

Recently, with the introduction of twisted punctures, it has been proven by Beem and Peelaers [20] that one can consider $\chi(\mathfrak{a}_{2N})$ theories defined on trinions with regular defects only, to produce a number of different 4d Argyres-Douglas theories. These defects, however, must necessarily be of both untwisted and twisted type and they can only produce Argyres-Douglas theories whose Coulomb branch generators possess half-integer scaling dimensions.

In their paper, they focused on the $N=1$ case, for which we already listed the allowed punctures in Table (3.6.29). With these punctures the allowed trinions are the following:



(3.6.30)

[6]Throughout this work, we use the subscript t to indicate a twisted puncture.

where for each theory we have used the labels of [20], denoting the rank-n $\mathfrak{su}(3)$-instanton SCFT as $T^{(n)}_{\mathfrak{su}(3)}$. So that, in particular, $T^{(n)}_{\mathfrak{su}(3)}$ is the (A_1, D_4) Argyres-Douglas theory. Finally, HM stands for a free hypermultiplet.

Note that not all choices of punctures correspond to physical 4d SCFTs. A diagnostic to detect disallowed combinations is to check the superconformal index (see Chapter (3)); if its expression diverges, the theory is designated as bad. This is the case for the $[2,1]$, $[1^2]_t$, $[1^2]_t$ theory.

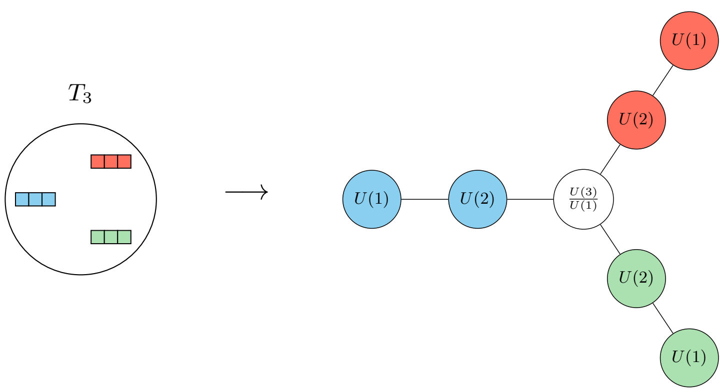
In the following we will be interested in the the $3d$ mirrors of the S^1 reduction of these twisted $\chi(\mathfrak{a}_2)$ theories (and, when possible, twisted $\chi(\mathfrak{a}_{2N})$ theories in general). The results of [42] allow us to make progress in such direction and can be used as a testing ground for our proposal. In particular, we heavily rely on the observation that the Higgs branch of the underlying 4d SCFT should match with the Coulomb branch of the 3d mirror theory of its S^1 reduction, and that the rank of the 4d SCFT (*i.e.* the complex dimension of the Coulomb branch) should match with the quaternionic Higgs branch dimension of the corresponding mirror theory. For the former, we will match the Coulomb branch Hilbert series of the 3d mirror theory with the Higgs branch Hilbert series of the 4d theory. We also study the Higgs branch Hilbert series of the mirror theory in detail. In some cases, there are more than one description of the mirror theory for a given 4d SCFT. The Hilbert series between those mirror theories are matched and we conjecture that they are dual to each other. In this way, we obtain new dual pairs between 3d $\mathcal{N}=4$ gauge theories that have not be studied elsewhere in the literature.

3.6.5 3d mirrors of $\chi(\mathfrak{a}_N)$ tinkertoys

In constructing the quiver description of the $3d$ mirrors theories of the circle reduction of twisted $\chi(\mathfrak{a}_{2N})$ theories of class-S, we were inspired by the method of [23].

In this paper the authors studied the non-Lagrangian $\chi(\mathfrak{a}_{N-1})$ theories associated to a trinion with untwisted regular punctures $\boldsymbol{\rho}_i$. By compactifying them on S^1, which leads to 3d $\mathcal{N}=4$ SCFTs when the radius tends to zero, and by applying mirror symmetry, they found out that the 3d mirror of such tinkertoy theory always has a Lagrangian description, *i.e.* a quiver.

Moreover this quiver is always a star-shaped quiver with 3 legs, each one determined by a different $T_{\boldsymbol{\rho}_i}(SU(N))$ theory (whose quiver is depicted in Fig. (3.3.9)) and with their $U(N)$ flavour nodes being commonly gauged as a central node. In this star-shaped construction, an overall $U(1)$ gauge symmetry needs to be modded out and this can be done at the central node; in which case its gauge symmetry becomes $U(N)/U(1)$, see Fig. (3.6.31).



(3.6.31)

As we have seen in the previous section, the tinkertoys of our interest are the twisted $\chi(\mathfrak{a}_{2N})$ theories associated with a sphere with one untwisted regular puncture $\boldsymbol{\rho}$ and two twisted regular punctures $\boldsymbol{\sigma}_t$ and $\boldsymbol{\lambda}_t$. Let us stress again that here $\boldsymbol{\rho}$ is a partition of $2N+1$ and $\boldsymbol{\sigma}$ and $\boldsymbol{\lambda}$ are C-partitions of $2N$ (see Table (3.6.29)).

inspired by the method of [23], we thus propose that the 3d mirror in question can obtained as follows:

1. As the 3 legs of the quiver we consider the following theories:

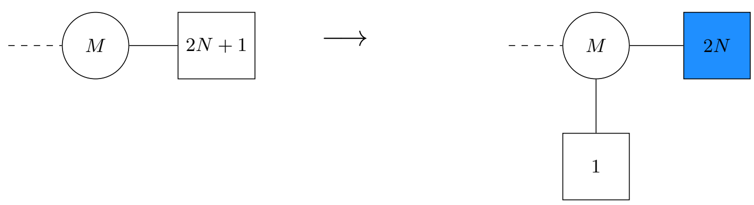
$$T_{\boldsymbol{\rho}}(SU(2N+1)) \ , \qquad T_{\boldsymbol{\sigma}}(USp'(2N)) \ , \qquad T_{\boldsymbol{\lambda}}(USp'(2N)) \ . \tag{3.6.32}$$

   The $T_{\boldsymbol{\rho}}(SU(N))$ theory was discussed in Section (3.3) and its quiver is depicted in (3.3.9); On the other hand, the $T_{\boldsymbol{\sigma}}(USp'(2N))$ theory was discussed in Section (3.5) and its quiver is depicted in (3.5.7).

2. To construct the star-shaped quiver, the $USp(2N)$ symmetry from the flavour symmetry of the theories listed in (3.6.32) are then gauged together. It thus plays the role of the central gauge node in the star-shaped quiver as mentioned in [23].

   Note that in doing this, the $USp(2N)$ flavour node of $T_{\boldsymbol{\sigma}}(USp'(2N))$ and $T_{\boldsymbol{\lambda}}(USp'(2N))$ turns into a gauge node in the star-shaped quiver in a straightforward manner. However, since the flavour node of $T_{\boldsymbol{\rho}}(SU(2N+1))$ is $U(2N+1)$, we need to decompose the bifundamental hypermultiplet between the $U(2N+1)$ flavour node and the gauge node next to it, say $U(M)$, into

   - One hypermultiplet under the $U(M)$ gauge group;
   - One bifundamental hypermultiplet between $U(M) \times USp(2N)$ (see Fig. (3.6.33)).

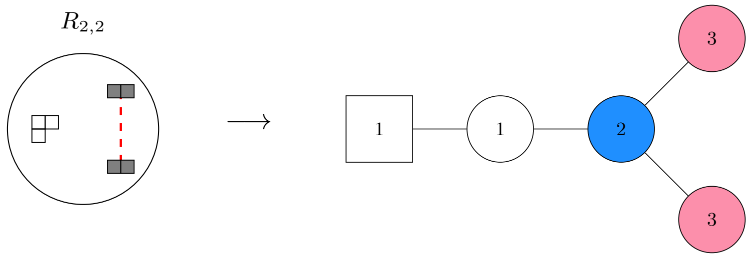


(3.6.33)

   The latter $USp(2N)$ flavour symmetry is then gauged.

3. The resulting 3d mirror quiver is an 'almost' star-shaped quiver with the central node being $USp(2N)$ and with one flavour of the fundamental hypermultiplet under the unitary group $U(M)$ located next to the central $USp(2N)$ node.

We present here, as an example, the 3d mirror theory of the S^1 reduction of the $R_{2,2}$ theory, which is also known as the C_2U_1 theory, in Fig. (3.6.34). This can be easily generalised to all the other theories of the same class depicted in Fig. (3.6.30), as demonstrated throughout the following sections. A feature of such mirror theories is that the quiver description contains unitary, symplectic and orthogonal gauge groups.



(3.6.34)

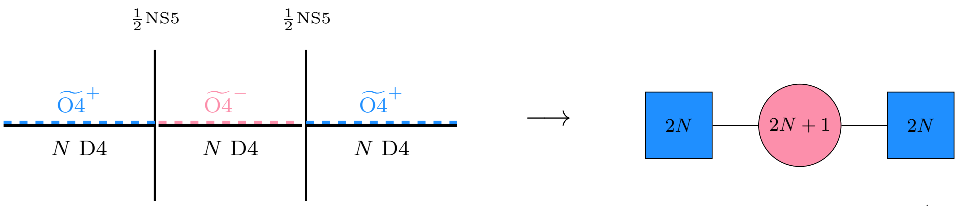
Let us now briefly comment on the motivation for using the $T_{\boldsymbol{\sigma}}(USp'(2N))$.

First of all, a simple generalisation of the previously seen $R_{2,2}$ theory, consists in considering the twisted $\chi(\mathfrak{a}_{2N})$ theory associated with a sphere with one minimal untwisted punture $[2N, 1]$ and two maximal twisted punctures $[1^{2N}]_t$. This is known as the $R_{2,2N}$ theory and was studied extensively in [53].

Each maximal twisted puncture gives rise to an $USp(2N)$ global symmetry, whereas the minimal puncture gives rise to a $U(1)$ global symmetry. Moreover, the $USp(2N)^2$ symmetry gets enhanced to $USp(4N)$, and the $R_{2,2N}$ theory has a resulting $USp(4N)\times U(1)$ global symmetry.

In [152], it was pointed out that both the $USp(2N)^2$ global symmetries carried by the maximal twisted punctures and the enhanced $USp(4N)$ flavour symmetry of the $R_{2,2N}$ theory have a global $\mathbb{Z}_2$ anomaly, introduced by Witten in [160]. This was shown within the class-S framework by turning on the mass term associated with the minimal untwisted puncture of the $R_{2,2N}$ theory. In fact, in the IR, this flows to a free theory that is described by the $SO(2N+1)$ gauge theory with $2N$ hypermultiplets in the vector representation; here it is clear that the $USp(4N)$ flavour symmetry of this theory possesses a Witten anomaly. In this sense, the $R_{2,2N}$ theory can be regarded as the ultraviolet completion of the $SO(2N+1)$ gauge theory with $2N$ flavours.

The $SO(2N+1)$ gauge theory with $2N$ flavours admits the Type IIA brane realisation [165] (see also [152, section 4]) involving an $O4$ plane, D4 branes and two half NS5 branes, in the following configuration:



(3.6.35)

with N physical D4 branes stretched between two half NS5 branes on top of the $\widetilde{O4}^-$ plane, and N other physical semi-infinite D4 brane on top of the $\widetilde{O4}^+$ plane terminating on each half NS5 brane. Note that the $O4$ plane, exactly as the $O3$ one, changes sign every time it crosses a half-NS5 brane. The $SO(2N+1)$ gauge group is realised on the D4 brane segment on top of the $\widetilde{O4}^-$ plane. The N flavours of hypermultiplets arise when two stacks of N physical D4 branes end on a half NS5-brane from opposite sides.

Indeed, the worldvolume of *each set* of semi-infinite D4 branes on top of the $\widetilde{O4}^+$ plane realises a 5d $USp(2N)$ symmetry with the discrete theta angle $\theta = \pi$ controlled by $\pi_4(USp(2N)) = \mathbb{Z}_2$ [111]. This also controls the Witten anomaly on the underlying 4d $USp(2N)$ symmetry. Since there are in total $2N$ flavours of hypermultiplets transforming under the vector representation of $SO(2N+1)$, the theory has a $USp(4N)$ flavour symmetry.

As pointed out in [152], when the two half-NS5 branes are on top of each other, the coupling of the $SO(2N+1)$ gauge group become infinite and this brane system should realise the $R_{2,2N}$ theory. Indeed, the two half-NS5 branes becomes a full phyisical NS5 brane, corresponding to the minimal untwisted puncture, and the two semi-infinite D4 branes on top of $\widetilde{O4}^+$ on each side of the brane system corresponds to each maximal twisted puncture. This picture provides a nice way of realising the Witten anomaly carried by the maximal twisted puncture.

Upon reduction on S^1, we expect that this construction corresponds to semi-infinite D3 branes on top of the $\widetilde{O3}^+$ plane. This indeed shows up in the brane configuration of the $T_{\boldsymbol{\sigma}}(USp'(2N))$ as discussed in Section (3.5) and depicted in the example of Fig. (3.5.3)[7].

In the following sections, we demonstrate our proposal for the construction of the $3d$ mirrors of the S^1 reduction of twisted $\chi(\mathfrak{a}_{2N})$ theories through a number of examples.

3.7 Twisted $\chi(\mathfrak{a}_2)$ trinions

Let us begin by examining the circle reduction of the twisted $\chi(\mathfrak{a}_2)$ theories associated with a sphere with three punctures.

3.7.1 Two copies of the (A_1, D_4) theory

The class-S description of this theory was proposed in [20] and was referred to as Theory 5 in that reference. It can be constructed by compactifying 6d (2,0) theory of the type $\chi(\mathfrak{a}_2)$ on a sphere with the following punctures:

$$[1^3]\ , \qquad [2]_t\ , \qquad [2]_t \tag{3.7.1}$$

where the subscript t indicates the twisted puncture (see Fig. (3.6.30)). Upon compactifying this theory on S^1, by adapting the prescription proposed in [23], we conjecture that the 3d mirror theory admits a 'star-shaped' quiver description constructed by 'gluing' together the following theories:

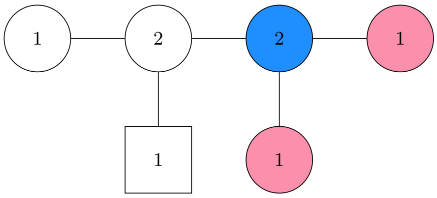
$$\begin{aligned} T_{[1^3]}(SU(3)):&\quad (U(1))-(U(2))-[U(3)]\\ T_{[2]}(USp'(2)):&\quad (SO(1))-[USp(2)]\\ T_{[2]}(USp'(2)):&\quad (SO(1))-[USp(2)] \end{aligned} \tag{3.7.2}$$

which were discussed in Sections (3.3) and (3.5).

By gluing, we mean gauging the common symmetry $USp(2)$ of the above theories, whereby it is the central node of the star-shaped quiver. Since $U(1)$ is the commutant of $USp(2)$ in $U(3)$, we should split the part $(U(2))-[U(3)]$ of $T_{[1^3]}(SU(3))$ into $[U(1)]-(U(2))-[USp(2)]$.

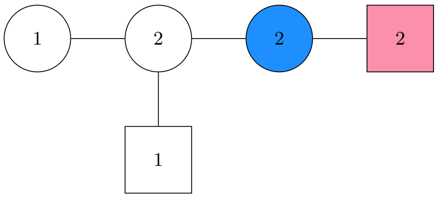
Gluing together the above theory along $USp(2)$ results in the following mirror theory

[7] In fact, we remark that the 3d $\mathcal{N}=4$ $SO(2N+1)$ gauge theory with $2N$ flavours can be written as $T_{[N^2]}(USp'(2N))$.



(3.7.3)

Note that each of the two red circular nodes denotes the $SO(1)$ group, and so the corresponding gauge symmetry is trivial. We can therefore rewrite this quiver as



(3.7.4)

where the rightmost red square node denotes the $SO(2)$ flavour symmetry.

In the following, we discuss about the Coulomb and Higgs branches of the mirror theory (3.7.3) or (3.7.4). Since upon compactification on S^1 the Higgs branch of the 4d theory is expected to be the same as that of the resulting 3d theory, it follows that the Coulomb branch of the mirror theory should match with the Higgs branch of the 4d theory, namely the product of two copies of the closure of the minimal nilpotent orbit $\overline{\text{min}}_{SU(3)}$ of $SU(3)$. Moreover, since the circle compactification of the (A_1, D_4) theory is identified with 3d $\mathcal{N} = 4$ $U(1)$ gauge theory with 3 flavours (see *e.g.* [168] and [42][8]), we expect that the Higgs branch of the mirror theory (3.7.3) or (3.7.4) should be $(\mathbb{C}^2/\mathbb{Z}_3)^2$.

Let us first comment on the enhanced Coulomb branch symmetry of quiver (3.7.4) along the line of [85]. Observe that the $U(1)$ and $U(2)$ gauge nodes in (3.7.4) are balanced. As a consequence, one expects an $SU(3)$ enhanced symmetry in the IR. Since the $USp(2)$ gauge node is also balanced, according to [85, section 5.3], this $SU(3)$ symmetry gets doubled and so the symmetry of the Coulomb branch is expected to be $SU(3)\times SU(3)$. This is in agreement with the symmetry of $(\overline{\text{min}}_{SU(3)})^2$. Subsequently we confirm such an enhanced symmetry using the Coulomb branch Hilbert series.

The quaternionic dimension Coulomb branch of (3.7.3) or (3.7.4) is

$$\dim_{\mathbb{H}}\{\mathcal{M}_{\mathcal{C}}[(3.7.3)\,\text{or}\,(3.7.4)]\} = 1 + 2 + 1 = 4\ . \tag{3.7.5}$$

This agrees with the dimension of the Higgs branch of the 4d $\mathcal{N} = 2$ theory, given by $24(c - a) = 24\left(\frac{4}{3} - \frac{7}{6}\right) = 4$, where a and c are the conformal anomalies given in (3.65) of [20]. In particular, this is equal to the dimension of $\left(\overline{\text{min}}_{SU(3)}\right)^2$. On the other hand, the quaternionic dimension of the Higgs branch of (3.7.3) or (3.7.4) is

$$\dim_{\mathbb{H}}\{\mathcal{M}_{\mathcal{H}}[(3.7.3)\,\text{or}\,(3.7.4)]\} = 2 + 2 + 4 + \frac{1}{2}(2\times 2) - (1 + 4 + 3) = 2\ . \tag{3.7.6}$$

This is in agreement with the fact that the S^1 compactification of two copies of rank-one (A_1, D_4) yields a 3d theory with two quaternionic dimensional Coulomb branch, whose mirror theory has two quaternionic dimensional Higgs branch. In particular, this is equal to the dimension of $\left(\mathbb{C}^2/\mathbb{Z}_3\right)^2$

[8] This reference studied carefully dimensional reductions for various Argyres-Douglas theories, including (A_1, D_4), by utilising the reduction of the index in [43].

Let us now study the Coulomb and Higgs branches of the mirror theory in detail using the Hilbert series. For the Coulomb branch, we present two methods in computing the Hilbert series, namely the monopole formula [64] and the Hall-Littlewood formula [67, 68]. For the Higgs branch, the Hilbert series can be computed using the Molien integral in the usual way [46] (see also [26]). All these expression were discussed in details in Sections (2.2.4), (3.3.1) and (3.3.1).

The Coulomb branch Hilbert series

The Coulomb branch Hilbert series can be computed from the monopole formula (2.2.55) and reads

$$
\begin{aligned}
&H_{\mathcal{C}}^{\text{mon}}[(3.7.3)\,\text{or}\,(3.7.4)](t;w_1,w_2) = \\
&\sum_{m\in\mathbb{Z}}\ \sum_{n_1\geq n_2>-\infty}\ \sum_{a=0}^{\infty} t^{2\Delta(m,\boldsymbol{n},a)} P_{U(1)}(t;m) P_{U(2)}(t;\boldsymbol{n}) P_{USp(2)}(t;a)\, w_1^m w_2^{n_1+n_2}\ ,
\end{aligned}
\tag{3.7.7}
$$

where we denote by m, $\boldsymbol{n} = (n_1, n_2)$ and a the magnetic fluxes associated with the gauge group $U(1)$, $U(2)$ and $USp(2)$ respectively; the function $\Delta(m,\boldsymbol{n},a)$ is the dimension of the monopole operator with magnetic fluxes $(m,\boldsymbol{n},a)$

$$
\begin{aligned}
\Delta(m,\boldsymbol{n},a) = \frac{1}{2}\sum_{i=1}^{2}\Big[|m-n_i| + |n_i| + (|n_i+a| + |n_i-a|)\Big] \\
+\frac{1}{2}\cdot\frac{1}{2}\,(2|a|+2|-a|) - |n_1-n_2| - |a-(-a)|\ ;
\end{aligned}
\tag{3.7.8}
$$

and the dressing factors are given by

$$
\begin{aligned}
P_{U(1)}(t;m) &= (1-t^2)^{-1} \\
P_{U(2)}(t;\boldsymbol{n}) &= \begin{cases} (1-t^2)^{-2} & \text{if } n_1 \neq n_2 \\ (1-t^2)^{-1}(1-t^4)^{-1} & \text{if } n_1 = n_2 \end{cases} \\
P_{USp(2)}(t;a) &= \begin{cases} (1-t^2)^{-1} & \text{if } a \neq 0 \\ (1-t^4)^{-1} & \text{if } a = 0\ . \end{cases}
\end{aligned}
\tag{3.7.9}
$$

The variables w_1 and w_2 are the topological fugacities associated with the $U(1)$ and $U(2)$ gauge group, respectively. Note that we turn off the background magnetic flux for the flavour symmetry in the above expression. Upon computing the summation, we may rewrite (3.7.7) as[9]

$$
H_{\mathcal{C}}^{\text{mon}}[(3.7.3)\,\text{or}\,(3.7.4)](t;w_1,w_2) = \left[\sum_{k=0}^{\infty} \chi^{SU(3)}_{[k,k]}(w_1,w_2)\,t^{2k}\right]^2\ .
\tag{3.7.10}
$$

[9] In this notation, the adjoint representation of $SU(3)$ is written as $\chi^{SU(3)}_{[1,1]}(w_1,w_2) = 2 + w_1 + w_1^{-1} + w_2 + w_2^{-1} + w_1 w_2 + w_1^{-1} w_2^{-1}$. In the convention where the fundamental representation of $SU(3)$ is written as $\chi^{SU(3)}_{[1,0]}(x_1,x_2) = x_1 + x_2 x_1^{-1} + x_2^{-1}$, this amounts to the change of variables $w_1 = x_1 x_2^{-2}$ and $w_2 = x_2 x_1^{-2}$.

Note that the quantity in the square bracket is the Hilbert series of the closure of the minimal nilpotent orbit $\overline{\min}_{SU(3)}$ of $SU(3)$ [26]. This result also agrees with the Hall-Littlewood limit $q \to 0$ of the Macdonald index (3.66) of [20]. It can be seen that the topological symmetry $U(1) \times U(1)$, associated with the fugacities w_1 and w_2, gets enhanced to $SU(3)$. Note, however, that this $SU(3)$ symmetry can be identified as the diagonal subgroup of $SU(3) \times SU(3)$, which is an isometry of the product $\left(\overline{\min}_{SU(3)}\right)^2$ and is also full flavour symmetry of the 4d $\mathcal{N} = 2$ theory. Indeed, the mirror theory (3.7.3) or (3.7.4) only allows for the refinement of such a diagonal subgroup in the Coulomb branch Hilbert series (3.7.7), and the rest of the full symmetry is 'hidden' in the part of quiver (3.7.4) containing the $USp(2)$ gauge group in the same way as [120]. A similar observation was made in the context of the punctures of the trinion in the class-S description of the 4d theory; see the discussion below (3.67) in [20].

Let us now discuss the Hall-Littlewood formula for computing the Coulomb branch Hilbert series. It reads

$$\begin{aligned} & H_{\mathcal{C}}^{\mathrm{HL}}[(3.7.3) \text{ or } (3.7.4)](t; y_1, y_2, y_3) = \\ & \sum_{a=0}^{\infty} t^{-2|a-(-a)|} P_{USp(2)}(t; a) \times H_{\mathcal{C}}[T_{[1^3]}(SU(3))](t; y_1, y_2, y_3; a, 0, -a) \\ & \qquad H_{\mathcal{C}}[T_{[2]}(USp'(2))](t; a) H_{\mathcal{C}}[T_{[2]}(USp'(2))](t; a) \ , \end{aligned} \tag{3.7.11}$$

where the expression for each of the above Coulomb branch Hilbert series is given in Sections (3.3) and (3.5).

We find that

$$H_{\mathcal{C}}^{\mathrm{HL}}[(3.7.3) \text{ or } (3.7.4)](t; w_1, w_2^{-1}, 1) = H_{\mathcal{C}}^{\mathrm{mon}}[(3.7.3) \text{ or } (3.7.4)](t; w_1, w_2) = (3.7.10) \ . \tag{3.7.12}$$

One of the advantages of the Hall-Littlewood formula (3.7.11) is that one only needs the information about the partitions, corresponding to the punctures of the 4d theory of class-S, and not the detailed information about the quiver of the 3d mirror theory. Moreover, this formula takes the same form as the TQFT's structure constant of the Macdonald index [78–80, 84, 132, 136] of the 4d theory; see (2.9) of [20].

The Higgs branch Hilbert series

The Higgs branch Hilbert series (2.2.53) reads

$$\begin{aligned} & H_{\mathcal{H}}[(3.7.4)](t; x, y) \\ & = \oint_{|u|=1} \frac{du}{2\pi i u} \oint_{|q|=1} \frac{dq}{2\pi i q} \oint_{|z|=1} \frac{dz}{2\pi i z} (1 - z^2) \oint_{|v|=1} \frac{dv}{2\pi i v} (1 - v^2) \times \\ & \qquad H_{\mathcal{H}}[[1]_u - [2]_{q,z}](t; u, q, z) \ H_{\mathcal{H}}[[1]_x - [2]_{q,z}](t; x, q, z) \times \\ & \qquad H_{\mathcal{H}}[[2]_{q,z} - [USp(2)]_v](t; q, z, v) \times \\ & \qquad H_{\mathcal{H}}[[USp(2)]_v - [SO(2)]_y](t; v, y) \times \\ & \qquad \mathrm{PE}\left[-2t^2 - (z^2 + 1 + z^{-2})t^2 - (v^2 + 1 + v^{-2})t^2\right] \ , \end{aligned} \tag{3.7.13}$$

where where PE denotes the plethystic exponential (2.2.36); x and y are fugacities for the $U(1)$ and the $SO(2)$ flavour symmetries respectively; and

$$
\begin{aligned}
H_{\mathcal{H}}[[1]_u - [2]_{q,z}](t,u,q,z) &= \mathrm{PE}\left[t(uq^{-1}+u^{-1}q)(z+z^{-1})\right] \\
H_{\mathcal{H}}[[2]_{q,z} - [USp(2)]_v](t,q,z,v) &= \mathrm{PE}\left[t(q^{-1}+q)(z+z^{-1})(v+v^{-1})\right] \\
H_{\mathcal{H}}[[USp(2)]_v - [SO(2)]_y](t,v,y) &= \mathrm{PE}\left[t(v+v^{-1})(y+y^{-1})\right] \ .
\end{aligned} \tag{3.7.14}
$$

Evaluating the integrals, we obtain the Hilbert series of $(\mathbb{C}^2/\mathbb{Z}_3)^2$ as expected:

$$
H_{\mathcal{H}}[(3.7.4)](t;x,y) = H[\mathbb{C}^2/\mathbb{Z}_3](t;xy)H[\mathbb{C}^2/\mathbb{Z}_3](t;xy^{-1}) \tag{3.7.15}
$$

where $H[\mathbb{C}^2/\mathbb{Z}_3](t;w)$ is the Hilbert series of $\mathbb{C}^2/\mathbb{Z}_3$ given by

$$
H[\mathbb{C}^2/\mathbb{Z}_3](t;w) = \mathrm{PE}\left[t^2+t^3(w+w^{-1})-t^6\right] \ . \tag{3.7.16}
$$

We emphasise that the $SO(2)$ symmetry in quiver (3.7.4) arises due to the proposal that each red circular node in quiver (3.7.3) is in fact $SO(1)$, and not $O(1)$. This proposal is justified by the above Higgs branch Hilbert series, since it reproduces the Hilbert series of $(\mathbb{C}^2/\mathbb{Z}_3)^2$ correctly. Note that if each red circular node in quiver (3.7.3) were taken to be $O(1)$, the quantities that carry fugacity $t^3(xy^{-1}+x^{-1}y)$, for example, would not be invariant under the $O(1)$ gauge symmetry[10]. This is also a justification to take the red circular nodes in quiver (3.5.7) to be of the special orthogonal type.

3.7.2 The (A_1, D_4) theory with a free hypermultiplet

The class-S description of this theory was proposed in [20] and was referred to as Theory 4 in that reference. It can be constructed by compactifying 6d (2,0) theory of the type $\chi(\mathfrak{a}_2)$ on a sphere with the following punctures:

$$
[2,1] \ , \qquad [1^2]_t \ , \qquad [2]_t \ . \tag{3.7.18}
$$

where the subscript t denotes the twisted puncture (see Fig. (3.6.30)).

The mirror of the 3d theory arising from compactifying such a 4d theory on a circle admits a 'star-shaped' quiver description constructed by gauging the common $USp(2)$ symmetry of the following theories [23]:

$$
\begin{aligned}
T_{[2,1]}(SU(3)): &\quad (U(1)) - [U(3)] \\
T_{[1^2]}(USp'(2)): &\quad (SO(3)) - [USp(2)] \\
T_{[2]}(USp'(2)): &\quad (SO(1)) - [USp(2)]
\end{aligned} \tag{3.7.19}
$$

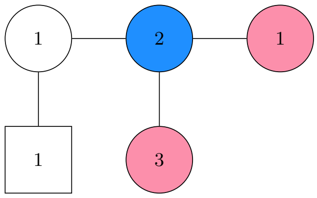
where $USp(2)$ plays the role of the central node of the star-shaped quiver. Since $U(1)$ is the commutant of $USp(2)$ in $U(3)$, we need to first rewrite the quiver for $T_{[2,1]}(SU(3))$ as $[U(1)] - (U(1)) - [USp(2)]$ and then gauge the $USp(2)$ group.

The 3d mirror theory in question is then

[10] In this case, we would have to replace the factor $H_{\mathcal{H}}[[USp(2)]-[SO(2)]](t,z,y)$ by the square of the Higgs branch Hilbert series of $[USp(2)]-(O(1))$. The latter is the Hilbert series of $\mathbb{C}^2/\mathbb{Z}_2$; see (3.32) of [26]:

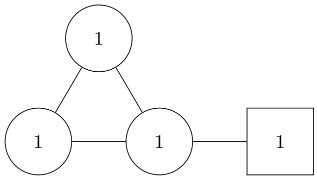
$$
H_{\mathcal{H}}[[USp(2)]-(O(1))](t;z) = \mathrm{PE}[t^2(z^2+1+z^{-2})-t^4] \ . \tag{3.7.17}
$$

The result is no longer the Hilbert series of $(\mathbb{C}^2/\mathbb{Z}_3)^2$. In particular, there is no generator of the Higgs branch at order t^3.



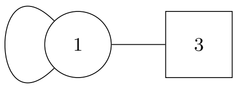
(3.7.20)

As we have proposed and justified in the previous subsection, the rightmost red circular node with the label 1 denotes the $SO(1)$ group, and the corresponding gauge symmetry is trivial. The line connecting it with the blue node thus denotes a half-hypermultiplet in the fundamental representation of the $USp(2)$ gauge group. In the following we study the Coulomb and Higgs branches of (3.7.20). Since the Higgs branch of the 4d theory is $\mathbb{C}^2 \times \overline{\text{min}}_{SU(3)}$, we expect that the Coulomb branch of the 3d mirror theory (3.7.20) is isomorphic to this space also. Moreover, similarly to the previous subsection, we also expect that the Higgs branch of (3.7.20) is isomorphic to $\mathbb{C}^2/\mathbb{Z}_3$. Due to these properties of the moduli space, we also conjecture that theory (3.7.20) is **dual** to the following quiver [36, 66, 142]:



(3.7.21)

Note that the mirror of this quiver is the well-known ADHM gauge theory for one $SU(3)$ instanton on $\mathbb{C}^2$, namely the $U(1)$ gauge theory with one adjoint and three fundamental hypermultiplets [36, 66, 142]:



(3.7.22)

The Coulomb branch of (3.7.20) is $1+1+1 = 3$ quaternionic dimensional; this is in agreement with that of $\mathbb{C}^2 \times \overline{\text{min}}_{SU(3)}$. On the other hand, the computation of the Higgs branch dimension of (3.7.20) is more subtle than the previous subsection, since the $SO(3)$ gauge group is not completely broken at a generic point on the hypermultiplet moduli space. In fact, it was argued in Footnote 7 of [69] that the Higgs branch of the theory $(SO(3))-[USp(2)]$ is the equal to that of $(O(1))-[USp(2)]$; the latter turns out to be $\mathbb{C}^2/\mathbb{Z}_2$, which is one quaternionic dimensional. The quaternionic Higgs branch dimension of (3.7.20) is therefore $(1\times1)+(1\times2)+\frac{1}{2}(2\times1)+1-(1+3) = 1$, which is equal to that of $\mathbb{C}^2/\mathbb{Z}_3$. In the following we study both branches of the moduli space in more detail using the Hilbert series.

The Coulomb branch Hilbert series

Since the $SO(3)$ gauge group in (3.7.20) has only one flavour of the hypermultiplet transforming under the vector representation, this renders quiver (3.7.20) a bad theory in the sense of [85] (see the end of Section (2.2.3)). In this case, the monopole formula diverges due to the presence of the monopole operators whose dimension is zero. Nevertheless, it is possible to compute the Coulomb branch Hilbert series using the Hall-Littlewood formula. This reads

$$
\begin{aligned}
&H_{\mathcal{C}}[(3.7.20)](t;x_1,x_2,y) = \\
&\sum_{a=0}^{\infty} t^{-2|a-(-a)|} P_{USp(2)}(t;a) \times H_{\mathcal{C}}[T_{[2,1]}(SU(3))](t;x_1,x_2;a,0,-a) \\
&\qquad H_{\mathcal{C}}[T_{[1^2]}(USp'(2))](t;y;a) H_{\mathcal{C}}[T_{[2]}(USp'(2))](t;a) \ ,
\end{aligned}
\tag{3.7.23}
$$

where the expression for each of the above Coulomb branch Hilbert series is given in Sections (3.3.1) and (3.3.1), and the fugacities x_1, x_2 have to satisfy the constraint (3.3.16):

$$
x_1^2 x_2 = 1 \ . \tag{3.7.24}
$$

Evaluating the summation, we obtain

$$
H_{\mathcal{C}}[(3.7.20)](t;x_1,x_2,y) = \mathrm{PE}\left[(y+y^{-1})t\right] \times \left[\sum_{k=0}^{\infty} \chi^{SU(3)}_{[k,k]}(\boldsymbol{u}) t^{2k}\right] \tag{3.7.25}
$$

where in this notation the character of the adjoint representation $[1,1]$ of $SU(3)$ is written as

$$
\chi^{SU(3)}_{[1,1]}(\boldsymbol{u}) = u_1 u_2 + \frac{u_1^2}{u_2} + \frac{u_1}{u_2^2} + \frac{1}{u_1 u_2} + \frac{u_2}{u_1^2} + \frac{u_2^2}{u_1} + 2 \ , \tag{3.7.26}
$$

with

$$
u_1 = (x_1 x_2^{-1})^{\frac{1}{3}} y \ , \qquad u_2 = (x_1^{-1} x_2)^{\frac{1}{3}} y \ . \tag{3.7.27}
$$

The Hilbert series (3.7.25) is indeed that of $\mathbb{C}^2 \times \overline{\mathrm{min}}_{SU(3)}$. Note that the free hypermultiplet arises from the $(SO(3)) - [USp(2)]$ part of the quiver. The can be seen from the the fact that the fugacity y associated with the $SU(2)$ symmetry of $\mathbb{C}^2$, parametrised by the expectation values of the free hypermultiplet, comes from the factor $H_{\mathcal{C}}[T_{[1^2]}(USp'(2))]$ in the Hall-Littlewood formula. It is worth pointing out that this $SU(2)$ is not manifest in the description $T_{[1^2]}(USp'(2)) : (SO(3)) - [USp(2)]$ but is enhanced in the IR; the reason for this is that the theory is self-mirror and that its flavour symmetry is $SU(2)$. Similarly, the $SU(3)$ symmetry of the space $\overline{\mathrm{min}}_{SU(3)}$ is also not manifest in quiver (3.7.20) and is enhanced in the IR. As can be seen from (3.7.27), the generators of the Cartan subalgebra of this $SU(3)$ symmetry is a linear combination of the generator of the Cartan subalgebra of $SU(2)$, which is the symmetry of $\mathbb{C}^2$, and a generator of the $U(1)$ topological symmetry in (3.7.20).

The Higgs branch Hilbert series

The Higgs branch Hilbert series can be computed as follows:

$$
\begin{aligned}
H_{\mathcal{H}}[(3.7.20)](t;w) = \oint_{|u|=1} \frac{du}{2\pi i u} \oint_{|v|=1} \frac{dv}{2\pi i v}(1-v^2)\times \\
& H_{\mathcal{H}}[[1]_u - [1]_w](t;u,w)\, H_{\mathcal{H}}[[1]_u - [USp(2)]_v](t;u,v)\times \\
& H_{\mathcal{H}}[[USp(2)]_v - (SO(3))](t;v)\times \\
& H_{\mathcal{H}}[[USp(2)]_v - (SO(1))](t;v)\times \\
& \mathrm{PE}\left[-t^2 - (v^2+1+v^{-2})t^2\right] \ ,
\end{aligned}
\tag{3.7.28}
$$

where

$$
\begin{aligned}
H_{\mathcal{H}}[[1]_u - [1]_w](t;u,w) &= \mathrm{PE}\left[(uw^{-1}+u^{-1}w)t\right] \\
H_{\mathcal{H}}[[1]_u - [USp(2)]_v](t;u,v) &= \mathrm{PE}\left[(u+u^{-1})(v+v^{-1})t\right] \\
H_{\mathcal{H}}[[USp(2)]_v - (SO(3))](t;v) &= H[\mathbb{C}^2/\mathbb{Z}_2](t;v) = \mathrm{PE}[t^2(v^2+1+v^{-2}) - t^4] \\
H_{\mathcal{H}}[[USp(2)]_v - (SO(1))](t;v) &= \mathrm{PE}[(v+v^{-1})t]
\end{aligned}
\tag{3.7.29}
$$

Note that, in the third line, we have used the fact that Higgs branch of the theory $(SO(3)) - [USp(2)]$ is isomorphic to $\mathbb{C}^2/\mathbb{Z}_2$. This has been discussed earlier.

Evaluating the integrals, we obtain the Hilbert series of $\mathbb{C}^2/\mathbb{Z}_3$ as expected:

$$
H_{\mathcal{H}}[(3.7.20)](t;w) = H[\mathbb{C}^2/\mathbb{Z}_3](t;w) = \mathrm{PE}\left[t^2 + t^3(w+w^{-1}) - t^6\right] \ . \tag{3.7.30}
$$

3.7.3 The rank-two $SU(3)$ instanton SCFT

This 4d SCFT was studied extensively in [41], where it was dubbed $\mathcal{T}_X$[11] (see also [21]). The class-S description of this theory was recently proposed in [20] and was referred to as Theory 3 or $\mathcal{T}^{(2)}_{SU(3)}$ in that reference. It can be constructed by compactifying 6d (2,0) theory of the type $\chi(\mathfrak{a}_2)$ on a sphere with the following punctures:

$$
[1^3]\ , \qquad [1^2]_t\ , \qquad [2]_t\ . \tag{3.7.31}
$$

where the subscript t denotes the twisted puncture (see Fig. (3.6.30)).

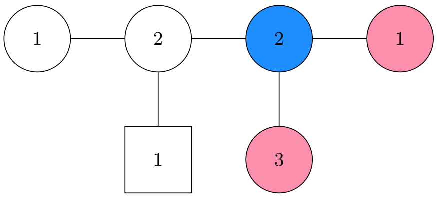
The mirror of the 3d theory arising from compactifying such a 4d theory on a circle can constructed by gauging the common $USp(2)$ symmetry of the following theories:

$$
\begin{aligned}
T_{[1^3]}(SU(3)): &\quad (U(1)) - (U(2) - [U(3)] \\
T_{[1^2]}(USp'(2)): &\quad (SO(3)) - [USp(2)] \\
T_{[2]}(USp'(2)): &\quad (SO(1)) - [USp(2)]
\end{aligned}
\tag{3.7.32}
$$

where $USp(2)$ plays the role of the central node of the star-shaped quiver.

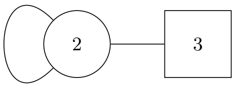
Similarly to the preceding subsections, the 3d mirror theory in question is then

[11] In fact, in [41, 44], the authors studied the $\mathcal{T}_{3,\frac{3}{2}}$ theory, which flows to a free hypermultiplet and the interacting SCFT called $\mathcal{T}_X$. The Higgs branch of the $\mathcal{T}_{3,\frac{3}{2}}$ theory is the full moduli space of two $SU(3)$ instantons on $\mathbb{C}^2$, which also includes the $\mathbb{C}^2$ factor due to the centre of the instantons. Upon decoupling the free hypermultiplet, the Higgs branch of the $\mathcal{T}_X$ theory is identified with the reduced instanton moduli space.



(3.7.33)

In the following we study the Coulomb and Higgs branches of (3.7.33). The ADHM gauge theory of the moduli space of two $SU(3)$ instantons on $\mathbb{C}^2$ is the $U(2)$ gauge theory with one adjoint and three fundamental hypermultiplets:



(3.7.34)

The Higgs branch of (3.7.34) is $\mathbb{C}^2 \times \widetilde{\mathcal{M}}_{2,SU(3)}$, where $\widetilde{\mathcal{M}}_{2,SU(3)}$ is the reduced (or centred) moduli space of two $SU(3)$ instantons on $\mathbb{C}^2$, and the Coulomb branch of (3.7.34) is the second symmetric power of $\mathbb{C}^2/\mathbb{Z}_3$ [36, 106], denoted by $\mathrm{Sym}^2(\mathbb{C}^2/\mathbb{Z}_3)$. We thus expect that the Coulomb branch of theory (3.7.33) is isomorphic to $\widetilde{\mathcal{M}}_{2,SU(3)}$ and that the Higgs branch of (3.7.33) is isomorphic to $\mathrm{Sym}^2(\mathbb{C}^2/\mathbb{Z}_3)$. Below we show that these are indeed the case.

The Coulomb branch of (3.7.33) is $1+2+1+1 = 5$ quaternionic dimensional; this is in agreement with that of $\widetilde{\mathcal{M}}_{2,SU(3)}$. On the other hand, the computation of the Higgs branch of (3.7.33) can be performed similarly to the previous subsection, *i.e.* by noting that the Higgs branch of the theory $(SO(3)) - [USp(2)]$ is the equal to $\mathbb{C}^2/\mathbb{Z}_2$ [69, Footnote 7], which is one quaternionic dimensional. The quaternionic Higgs branch dimension of (3.7.33) is therefore $(1\times2)+(2\times1)+(2\times2)+\frac{1}{2}(2\times1)+1-(1+4+3) = 2$, which is equal to that of $\mathrm{Sym}^2(\mathbb{C}^2/\mathbb{Z}_3)$. In the following we study both branches of the moduli space in more detail using the Hilbert series.

We now discuss the Coulomb branch. Since the $SO(3)$ gauge group has one flavour transforming under its vector representation, the theory is 'bad'. As a result, the monopole formula diverges. However, as in the previous subsection, we can use the Hall-Littlewood formula to compute the Coulomb branch Hilbert series

$$\begin{aligned} &H_{\mathcal{C}}[(3.7.33)](t; x_1, x_2, x_3, y) = \\ &\sum_{a=0}^{\infty} t^{-2|a-(-a)|} P_{USp(2)}(t; a) \times H_{\mathcal{C}}[T_{[1^3]}(SU(3))](t; x_1, x_2, x_3; a, 0, -a) \\ &\qquad H_{\mathcal{C}}[T_{[1^2]}(USp'(2))](t; y; a) H_{\mathcal{C}}[T_{[2]}(USp'(2))](t; a) \ , \end{aligned} \tag{3.7.35}$$

where the expression for each of the above Coulomb branch Hilbert series is given in Sections (3.3) and (3.5), and the fugacities x_1, x_2, x_3 have to satisfy the constraint (3.3.16):

$$x_1 x_2 x_3 = 1 \ . \tag{3.7.36}$$

Evaluating the summation, we obtain the Hilbert series of $\widetilde{\mathcal{M}}_{2,SU(3)}$ (see [106, (3.23)]):

$$
\begin{aligned}
&H_{\mathcal{C}}[(3.7.33)](t;x_1,x_2,x_3,y) \\
&= \mathrm{PE}\left[\left(\chi^{SU(3)}_{[1,1]}(\boldsymbol{x})+\chi^{SU(2)}_{[2]}(\boldsymbol{y})\right)t^2+\left(\chi^{SU(3)}_{[1,1]}(\boldsymbol{x})\chi^{SU(2)}_{[1]}(\boldsymbol{y})\right)t^3-t^4+\ldots\right] \ .
\end{aligned}
\tag{3.7.37}
$$

Let us now turn to the Higgs branch. The Higgs branch Hilbert series is given by

$$
\begin{aligned}
&H_{\mathcal{H}}[(3.7.33)](t;x) \\
&= \oint_{|u|=1}\frac{du}{2\pi i u}\oint_{|q|=1}\frac{dq}{2\pi i q}\oint_{|z|=1}\frac{dz}{2\pi i z}(1-z^2)\oint_{|v|=1}\frac{dv}{2\pi i v}(1-v^2)\times \\
&\qquad H_{\mathcal{H}}[[1]_u-[2]_{q,z}](t;u,q,z)\ H_{\mathcal{H}}[[1]_x-[2]_{q,z}](t;x,q,z) \\
&\qquad H_{\mathcal{H}}[[2]_{q,z}-[USp(2)]_v](t;q,z,v)\times \\
&\qquad H_{\mathcal{H}}[[USp(2)]_v-(SO(3))](t;v)\times \\
&\qquad H_{\mathcal{H}}[[USp(2)]_v-(SO(1))](t;v)\times \\
&\qquad \mathrm{PE}\left[-2t^2-(z^2+1+z^{-2})t^2-(v^2+1+v^{-2})t^2\right] \ ,
\end{aligned}
\tag{3.7.38}
$$

where the notations are as described in (3.7.14) and (3.7.29).

Evaluating the integrals, we find that

$$
H_{\mathcal{H}}[(3.7.33)](t;x)=\frac{1}{2}\left[\left(H[\mathbb{C}^2/\mathbb{Z}_3](t;x)\right)^2+H[\mathbb{C}^2/\mathbb{Z}_3](t^2;x^2)\right]
\tag{3.7.39}
$$

where the Hilbert series of $\mathbb{C}^2/\mathbb{Z}_3$ is given by (3.7.16). This is indeed the Hilbert series of $\mathrm{Sym}^2(\mathbb{C}^2/\mathbb{Z}_3)$.

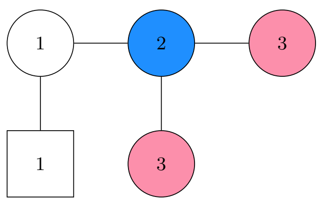
3.7.4 The $R_{2,2N}$ theory

The class-S description of the 4d $R_{2,2N}$ SCFT was proposed in [53]. This is a twisted $\chi(\mathfrak{a}_{2N})$ theory associated with a sphere with punctures:

$$
[2N,1]\ , \quad [1^{2N}]_t\ , \quad [1^{2N}]_t\ .
\tag{3.7.40}
$$

Let us first focus on the case of $N=1$. This theory is also referred to as the C_2U_1 theory in the literature and it corresponds to Theory 2 in [20] (see also Fig. (3.6.30)).

Following the procedures described in the previous subsections, we obtain the following 3d mirror theory upon reducing this theory on S^1:



(3.7.41)

The Coulomb branch of (3.7.41) is $1+1+1+1=4$ quaternionic dimensional, in agreement with the Higgs branch dimension of the 4d theory which is equal to $24(c-a)=24\left(\frac{19}{12}-\frac{17}{12}\right)=4$, where $a=\frac{17}{12}$ and $c=\frac{19}{12}$ are the conformal anomalies of the 4d theory [53]. The quaternionic dimension of the Higgs branch of (3.7.41) is $(1\times 1)+(1\times 2)+1+1-(1+3)=1$, which is in agreement with the fact that the C_2U_1 theory is a rank-one 4d theory. In the following, we use the Hilbert series to show that this Higgs branch is in fact isomorphic to $\mathbb{C}^2/\mathbb{Z}_6$.

We remark that the S^1 reduction of the C_2U_1 theory has recently been investigated in [40] using a different approach. In that reference, the theory in question was studied using the magnetic quiver with a non-simply laced edge depicted in [40, Table 2]. We will see that the Coulomb branch Hilbert series computed in that reference is in agreement with ours.

The Coulomb branch Hilbert series

The Coulomb branch Hilbert series is given by the following Hall-Littlewood formula:

$$\begin{aligned} &H_{\mathcal{C}}[(3.7.41)](t; x_1, x_2, y, z) = \\ &\sum_{a=0}^{\infty} t^{-2|a-(-a)|} P_{USp(2)}(t; a) \times H_{\mathcal{C}}[T_{[2,1]}(SU(3))](t; x_1, x_2; a, 0, -a) \\ &\qquad H_{\mathcal{C}}[T_{[1^2]}(USp'(2))](t; y; a) H_{\mathcal{C}}[T_{[1^2]}(USp'(2))](t; z; a) \ , \end{aligned} \tag{3.7.42}$$

where the expression for each of the above Coulomb branch Hilbert series is given in Sections (3.3) and (3.5) and the fugacities x_1, x_2 satisfy the relation (3.3.16):

$$x_1^2 x_2 = 1 \ . \tag{3.7.43}$$

This Hilbert series can be written concisely in a closed form in terms of the highest weight generating function (HWG) [104] as

$$\mathrm{HWG}\left[H_{\mathcal{C}}[(3.7.41)]\right] = \mathrm{PE}\left[t^2(1+\mu_1^2) + t^3(w+w^{-1})\mu_2 + t^4\mu_2^2 - t^6\mu_2^2\right] \ . \tag{3.7.44}$$

where, upon computing the power series of this expression in t, $\mu_1^{p_1}\mu_2^{p_2}$ denotes the representation $[p_1, p_2]$, whose character written in terms of y and z, of $USp(4)$. Here w is the fugacity for the $U(1)$ symmetry which can be written in terms of x_1, x_2 as

$$w = x_2 x_1^{-1} \ . \tag{3.7.45}$$

The highest weight generating function (3.7.44) is indeed in agreement with that presented in [40, Table 11, row 3 with $n = 2$].

As can be seen from the coefficient of the order t^2, the symmetry of the Coulomb branch is indeed $USp(4) \times U(1)$.

Note that, in this notation, the adjoint representation $[2, 0]$ of $USp(4)$ can be written as

$$\chi^{USp(4)}_{[2,0]}(\boldsymbol{u}) = \frac{u_1^2}{u_2} + \frac{u_1^2}{u_2^2} + u_1^2 + \frac{u_2^2}{u_1^2} + \frac{u_2}{u_1^2} + u_2 + \frac{1}{u_1^2} + \frac{1}{u_2} + 2 \tag{3.7.46}$$

with $u_1 = y$ and $u_2 = yz$. Recalling that the $T_{[1^2]}(USp'(2))$ theory is self-mirror, we expect the Coulomb branch symmetry of the two copies of it appearing in the quiver (3.7.41) to get enhanced in the IR to $SU(2) \times SU(2)$, corresponding to the fugacities y and z. From the above computation we see that this $SU(2) \times SU(2)$ symmetry is, in fact, further enhanced to $USp(4)$. Setting $w = 1, y = 1, z = 1$, we obtain the unrefined Hilbert series, as presented below [53, (3)] with $\tau = t^2$ and [40, Table 3, row 3]. The plethystic logarithm of the Hilbert series (3.7.42) can be obtain from the argument inside the PE in [20, (3.30)][12] by taking the limit $q \to 0$ of that expression. The generators of the moduli space and their relations were analysed in that reference.

The Higgs branch Hilbert series

The Higgs branch Hilbert series can be computed as follows:

$$
\begin{aligned}
H_{\mathcal{H}}[(3.7.41)](t;w) = \oint_{|u|=1} \frac{du}{2\pi i u} \oint_{|v|=1} \frac{dv}{2\pi i v}(1-v^2)\times \\
& H_{\mathcal{H}}[[1]_u - [1]_w](t;u,w)\, H_{\mathcal{H}}[[1]_u - [USp(2)]_v](t;u,v)\times \\
& H_{\mathcal{H}}[[USp(2)]_v - (SO(3))](t;v)\times \\
& H_{\mathcal{H}}[[USp(2)]_v - (SO(3))](t;v)\times \\
& \mathrm{PE}\left[-t^2 - (v^2+1+v^{-2})t^2\right] \ ,
\end{aligned}
\tag{3.7.47}
$$

where the notations are as in (3.7.29).

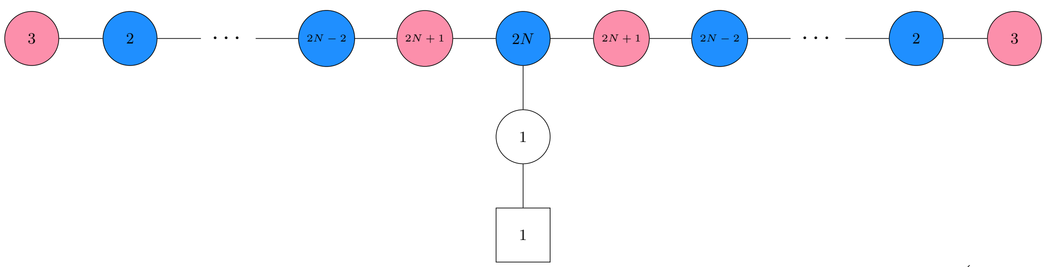
Evaluating the integrals, we obtain the Hilbert series of $\mathbb{C}^2/\mathbb{Z}_6$:

$$
H_{\mathcal{H}}[(3.7.41)](t;w) = \mathrm{PE}\left[t^2 + (w+w^{-1})t^6 - t^{12}\right] = H[\mathbb{C}^2/\mathbb{Z}_6](t;w) \ . \tag{3.7.48}
$$

The appearance of $\mathbb{C}^2/\mathbb{Z}_6$ can be understood by considering the $\mathcal{S}$-fold realization of the C_2U_1 theory [8]: In the F-theory context this model arises by probing with a $D3$ brane a background which is obtained by combining a 7-brane of type H_2 with a $\mathbb{Z}_2$ $\mathcal{S}$-fold action whose effect is to act as a sign flip on the Coulomb branch of the H_2 (or (A_1, D_4)) theory resulting in a IV^* geometry. Upon reduction to three dimensions the Coulomb branch of the (A_1, D_4) theory becomes the hyperkahler singularity $\mathbb{C}^2/\mathbb{Z}_3$ as we have seen before. We should then expect the $\mathbb{Z}_2$ $\mathcal{S}$-fold to act on this geometry, resulting therefore in a $\mathbb{C}^2/\mathbb{Z}_6$ singularity.

Mirror of the S^1 reduction of the $R_{2,2N}$ theory

We propose that the 3d mirror theory in question is



(3.7.49)

Note that the Coulomb branch of this quiver is $2N^2 + N + 1$ quaternionic dimensional, where we have used the fact that the Coulomb branch of $T_{USp'(2N)}$ is N^2 dimensional. This is in agreement with the Higgs branch dimension of the 4d theory which can be computed from $24(c-a) = 2N^2+N+1$, where the conformal anomalies are $a = \frac{14N^2+19N+1}{24}$ and $c = \frac{8N^2+10N+1}{12}$ [53]. On the other hand, the Higgs branch of quiver (3.7.49) is $2N^2 + 2N + 1 - \frac{1}{2}(2N)(2N+1) - 1 = N$, where we have used

[12] Note that the notation in [20] can be mapped to ours as follows: $t_{\text{there}} = t^2_{\text{ours}}$ and $a^3 = w$.

the fact that the Higgs branch of $T_{USp'(2N)}$ is also N^2 dimensional. This result is in agreement with the fact that the $R_{2,2N}$ theory has rank N.

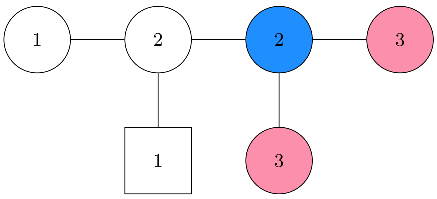
Again, we remark that there is an alternative description of the mirror theory in terms of a non-simply-laced quiver. This, together with the corresponding highest weight generating function, were given in [40, Table 11, row 3], with $n = 2N$.

3.7.5 The $\widetilde{T}_3$ or $\mathcal{T}^{(2)}_{A_2,2}$ theory

This theory was proposed and studied in [20]. It also recently appeared in [98] where it was called $\mathcal{T}^{(2)}_{A_2,2}$. It has the class-S description as a twisted $\chi(\mathfrak{a}_2)$ theory associated with the sphere with punctures (see Fig. (3.6.30)):

$$[1^3]\ , \qquad [1^2]_t\ , \qquad [1^2]_t\ . \tag{3.7.50}$$

Following the procedure described in the previous subsections, we obtain the following quiver description of the 3d mirror theory of the compactification of $\widetilde{T}_3$ on S^1:



(3.7.51)

The Coulomb branch of (3.7.52) is $1+2+1+1+1 = 6$ quaternionic dimensional, in agreement with the Higgs branch dimension of the 4d theory which is equal to $24(c-a) = 24\left(3 - \frac{11}{4}\right) = 6$, where $a = \frac{11}{4}$ and $c = 3$ are the conformal anomalies of the 4d theory, as given in (3.1) of [20]. On the other hand, the Higgs branch of (3.7.41) is $(1\times 2) + (2\times 1) + (2\times 2) + 1 + 1 - (1+4+3) = 2$, which is in agreement with the claim in [20] that $\widetilde{T}_3$ is a rank-two theory. Again, in this computation, we have used the fact that the $SO(3)$ gauge theory with one flavour has the Higgs branch isomorphic to $\mathbb{C}^2/\mathbb{Z}_2$, which is one quaternionic dimensional [69]. In the following, we investigate both branches in more detail using the Hilbert series.

As in the previous subsection, the Coulomb branch Hilbert series can be computed using the Hall-Littlewood formula:

$$\begin{aligned} &H_{\mathcal{C}}[(3.7.52)](t; x_1, x_2, x_3, y, z) = \\ &\sum_{a=0}^{\infty} t^{-2|a-(-a)|} P_{USp(2)}(t; a) \times H_{\mathcal{C}}[T_{[1^3]}(SU(3))](t; x_1, x_2, x_3; a, 0, -a) \\ &\qquad H_{\mathcal{C}}[T_{[1^2]}(USp'(2))](t; y; a) H_{\mathcal{C}}[T_{[1^2]}(USp'(2))](t; z; a)\ , \end{aligned} \tag{3.7.52}$$

where the expression for each of the above Coulomb branch Hilbert series is given in Sections (3.3) and (3.5) and the fugacities x_1, x_2, x_3 have to satisfy the constraint (3.3.16):

$$x_1 x_2 x_3 = 1\ . \tag{3.7.53}$$

The highest weight generating function of the Coulomb branch Hilbert series up to t^{12} is

$$\begin{aligned}\mathrm{PE}\Big[&t^2\left(\mu_1\mu_2+\nu^2+\sigma^2\right)+t^4\left(\mu_1\mu_2\nu\sigma+\mu_1\mu_2+1\right)\\&+t^6\left(\mu_1^3\nu\sigma+\mu_1\mu_2\nu\sigma+\mu_2^3\nu\sigma+\mu_1^3+\mu_2^3\right)\\&+t^8\left(\mu_1^3\nu\sigma+\mu_2^3\nu\sigma\right)-t^{10}\left(\mu_1^4\mu_2\nu\sigma+\mu_1\mu_2^4\nu\sigma\right)\\&-t^{12}(\mu_1^4\mu_2\nu^2\sigma^2+\mu_1^3\mu_2^3\nu^2\sigma^2+\mu_1^2\mu_2^2\nu^2\sigma^2+\mu_1\mu_2^4\nu^2\sigma^2+\mu_1^4\mu_2\nu\sigma\\&\quad+2\mu_1^3\mu_2^3\nu\sigma+\mu_1\mu_2^4\nu\sigma+\mu_2^3\mu_1^3)+\ldots\Big]\end{aligned} \tag{3.7.54}$$

where, upon computing the power series of this expression in t, $\mu_1^{p_1}\mu_2^{p_2}\nu^r\sigma^s$ denotes the representation $[p_1,p_2;r;s]$, whose character can be written as $\chi^{SU(3)}_{[p_1,p_2]}(\boldsymbol{x})\chi^{SU(2)}_{[r]}(y)\chi^{SU(2)}_{[s]}(z)$, of $SU(3)\times SU(2)\times SU(2)$. This is indeed the symmetry of the Coulomb branch of the theory. The plethystic logarithm of the Hilbert series (3.7.52) can be obtained from the argument inside the PE in [20, (3.3)] by taking the limit $q\to 0$ of that expression. The generators of the moduli space and their relations were analysed in that reference.

Now let us examine the Higgs branch. The Hilbert series can be computed in a similar way to the previous subsection; it is given by

$$\begin{aligned}H_{\mathcal{H}}[(3.7.52)](t;x)=&\oint_{|u|=1}\frac{du}{2\pi iu}\oint_{|q|=1}\frac{dq}{2\pi iq}\times\\&\oint_{|z|=1}\frac{dz}{2\pi iz}(1-z^2)\oint_{|v|=1}\frac{dv}{2\pi iv}(1-v^2)\times\\&H_{\mathcal{H}}[[1]_u-[2]_{q,z}](t;u,q,z)\ H_{\mathcal{H}}[[1]_x-[2]_{q,z}](t;x,q,z)\times\\&H_{\mathcal{H}}[[2]_{q,z}-[USp(2)]_v](t;q,z,v)\times\\&H_{\mathcal{H}}[[USp(2)]_v-(SO(3))](t;v)\times\\&H_{\mathcal{H}}[[USp(2)]_v-(SO(3))](t;v)\times\\&\mathrm{PE}\left[-2t^2-(z^2+1+z^{-2})t^2-(v^2+1+v^{-2})t^2\right]\ ,\end{aligned} \tag{3.7.55}$$

where the notations are as described in (3.7.14) and (3.7.29). Here x is the fugacity of the $U(1)$ flavour symmetry. This can be evaluated and has the following closed form:

$$\begin{aligned}&H_{\mathcal{H}}[(3.7.52)](t;x)\\&=\frac{1}{(1-t^3x)^2(1-t^3x^{-1})^2(1+t^3x)^2(1+t^3x^{-1})^2}\times\\&\quad\Big[1+t^2+2t^4+3t^6+(5+x^2+x^{-2})t^8+(6+x^2+x^{-2})t^{10}\\&\quad+(5+x^2+x^{-2})t^{12}+\ldots\text{(palindrome)}\ldots+t^{20}\Big]\ .\end{aligned} \tag{3.7.56}$$

Setting $x=1$, we obtain the following unrefined Hilbert series:

$$H_{\mathcal{H}}[(3.7.52)](t;x=1)=\frac{1-t^2+t^4+2t^6+t^8-t^{10}+t^{12}}{(1-t)^4(1+t)^4\left(1-t+t^2\right)^2\left(1+t+t^2\right)^2}\ , \tag{3.7.57}$$

where the order of the pole at $t=1$ confirms that the Higgs branch is 4 complex dimensional, or equivalently 2 quaternionic dimensional as expected. The plethystic logarithm is

$$\begin{aligned}\mathrm{PL}\left[H_{\mathcal{H}}[(3.7.52)](t;x)\right] = t^2 + t^4 + t^6\left(2x^2 + \frac{2}{x^2} + 1\right) + t^8\left(x^2 + \frac{1}{x^2} + 1\right) \\ - t^{12}\left(x^2 + \frac{1}{x^2} + 4\right) - t^{14}\left(2x^2 + \frac{2}{x^2} + 4\right) \\ - t^{16}\left(x^4 + \frac{1}{x^4} + x^2 + \frac{1}{x^2} + 2\right) + \dots\, .\end{aligned} \tag{3.7.58}$$

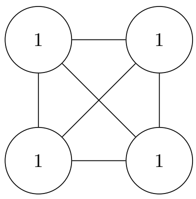
3.8 Twisted $\chi(\mathfrak{a}_2)$ theories with four punctures

In this section, we discuss the mirror theories associated with the twisted $\chi(\mathfrak{a}_2)$ theories defined on spheres with four punctures. These can be obtained by gluing two of the twisted $\chi(\mathfrak{a}_2)$ trinions in Fig. (3.6.30) with a tube theory; as we have seen, this amounts to commonly gauge the flavour symmetries of two $\mathfrak{a}_2$ punctures.

We consider two examples of such generalised class-S theories: the $\mathcal{T}_{2,\frac{3}{2},\frac{3}{2}}$ and $\mathcal{T}_{3,2,\frac{3}{2},\frac{3}{2}}$ theories. Indeed the class-S description, without an irregular puncture, of such two theories has recently been proposed in [20].

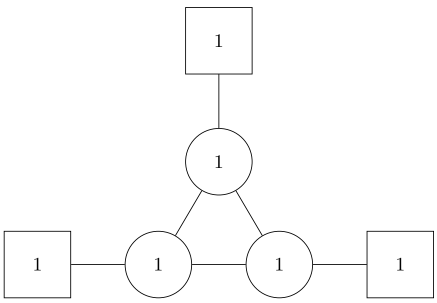
3.8.1 The $\mathcal{T}_{2,\frac{3}{2},\frac{3}{2}}$ theory

The 4d $\mathcal{N}=2$ $\mathcal{T}_{2,\frac{3}{2},\frac{3}{2}}$ SCFT was studied in [44] as an $SU(2)$ gauge theory coupled to a doublet of hypermultiplets and two copies of the (A_1, D_4) theory, where an $SU(2)$ subgroup of the $SU(3)$ global symmetry of each copy is gauged. In that reference, it was proposed that this theory is dual to another 4d $\mathcal{N}=2$ SCFT known as the $I_{4,4}$ or (A_3, A_3) theory [168]. Upon compactifying the latter on S^1, the 3d mirror theory was proposed in [170, Figure 8] (see also [25] for a derivation) to be



(3.8.1)

where an overall $U(1)$ needs to be decoupled from this quiver. Upon doing so, one obtains the following equivalent description of the above mirror theory [44, (3.3)]:



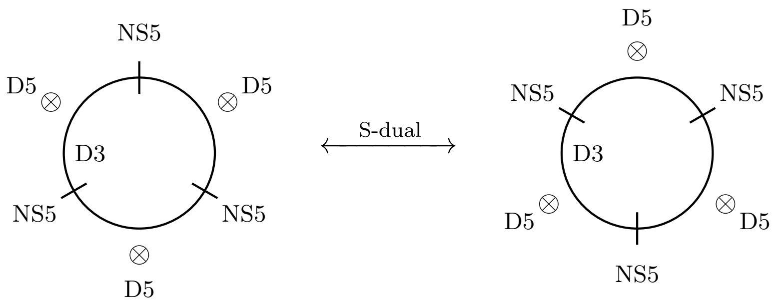
(3.8.2)

This quiver has two interesting properties:

1. It is self-mirror.

2. Both Higgs and Coulomb branches are isomorphic to the moduli space of one $SU(3)$ instanton[13] on $\mathbb{C}^2/\mathbb{Z}_3$ with the holonomy at infinity such that $SU(3)$ is broken to $U(1)^3/U(1) \cong U(1)^2$.

The first property can be understood from the Type IIB Hanany-Witten brane construction [107] involving one complete D3 brane wrapping a circle and stretching between three NS5 brane, with one D5 brane within each NS5 brane interval, as seen in Fig. (3.8.3). The mirror symmetry can be realised by an action that involves interchanging the NS5 and D5 branes, and this leaves the brane system invariant. We thus conclude that (3.8.2) is self-mirror.

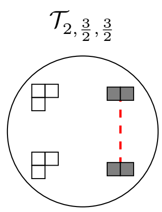


(3.8.3)

The second property follows from [35, 36, 55, 56, 70, 72, 77, 131, 135, 138, 142, 162].

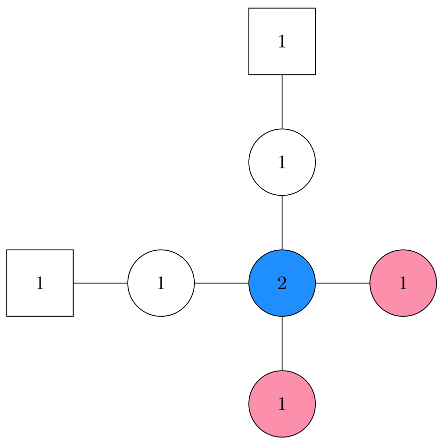
On the other hand, the $\mathcal{T}_{2,\frac{3}{2},\frac{3}{2}}$ theory can be seen within the class-S framework, as a twisted $\chi(\mathfrak{a}_2)$ theory associated with a sphere with punctures

$$[2,1]\ , \quad [2,1]\ , \quad [2]_t\ , \quad [2]_t \quad \rightarrow$$



(3.8.4)

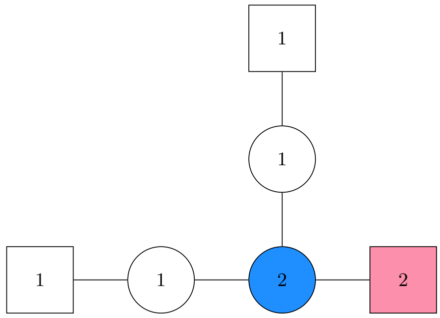
Following the procedure described in the preceding section, we obtain the 3d mirror of this theory compactified on S^1 as



(3.8.5)

Since the red circular node with the label 1 denotes $SO(1)$, this quiver can be rewritten as

[13] Strictly speaking, this should be called a $PU(3) \cong PSU(3)$ instanton.



(3.8.6)

Indeed, we conjecture that theory (3.8.2) is *dual* to theory (3.8.6). It is thus expected that the two properties discussed above also hold for theory (3.8.6). In the following we provide some non-trivial checks for these statements.

The Coulomb branch of (3.8.6) is $1+1+1=3$ quaternionic dimensional. The Higgs branch of (3.8.6) is also $1+2+\frac{1}{2}(2\times 2)+2+1-(1+3+1)=3$ quatenionic dimensional. These are also equal to the corresponding quantities of (3.8.2). The equality of the Higgs and Coulomb branch dimensions is as expected from the property that the theory is self-mirror. We now study both branches in more detail using the Hilbert series

We first consider the Higgs branch Hilbert series of (3.8.6). This is given by

$$\begin{aligned}
&H_{\mathcal{H}}[(3.8.6)](t;x,y,q) = \\
&\oint_{|u|=1}\frac{du}{2\pi i u}\oint_{|w|=1}\frac{dw}{2\pi i w}\oint_{|v|=1}\frac{dv}{2\pi i v}(1-v^2)\times \\
&\quad H_{\mathcal{H}}[[1]_u-[1]_x](t;u,x)\, H_{\mathcal{H}}[[1]_u-[USp(2)]_v](t;u,v)\times \\
&\quad H_{\mathcal{H}}[[1]_w-[1]_y](t;u,x)\, H_{\mathcal{H}}[[1]_w-[USp(2)]_v](t;w,v)\times \\
&\quad H_{\mathcal{H}}[[USp(2)]_v-[SO(2)]_q](t;v,q)\times \\
&\quad \mathrm{PE}\left[-2t^2-(v^2+1+v^{-2})t^2\right]\ ,
\end{aligned} \tag{3.8.7}$$

where the notations are as in (3.7.14) and (3.7.29). Here, x, y, q are the fugacities for each of the $U(1)$ in the $U(1)^3$ flavour symmetry of the theory. Evaluating the integrals, this can be written as

$$\begin{aligned}
&H_{\mathcal{H}}[(3.8.6)](t;x,y,q) \\
&=\mathrm{PE}\Big[3t^2+t^3\left(qx+\frac{q}{x}+\frac{1}{qx}+\frac{x}{q}+qy+\frac{q}{y}+\frac{1}{qy}+\frac{y}{q}\right) \\
&\quad +t^4\left(q^2+\frac{1}{q^2}+xy+\frac{y}{x}+\frac{x}{y}+\frac{1}{xy}\right) \\
&\quad -2t^6\left(q^2+\frac{1}{q^2}+xy+\frac{y}{x}+\frac{x}{y}+\frac{1}{xy}+2\right) \\
&\quad -3t^7\left(qx+\frac{q}{x}+\frac{1}{qx}+\frac{x}{q}+qy+\frac{q}{y}+\frac{1}{qy}+\frac{y}{q}\right)+\ldots\Big]\ .
\end{aligned} \tag{3.8.8}$$

The closed form for the unrefined Higgs branch Hilbert series, whereby $x=y=q=1$, is

$$
\begin{aligned}
&H_{\mathcal{H}}[(3.8.6)](t; x=1, y=1, q=1) \\
&= \frac{1-2t+3t^2+2t^3-2t^4+2t^5+3t^6-2t^7+t^8}{(1-t)^6(1+t)^2\,(1+t^2)\,(1+t+t^2)^2} \ .
\end{aligned} \tag{3.8.9}
$$

The order of the pole at $t=1$, which is 6, is indeed the complex dimension of the Higgs branch, and the numerator is palindromic as it should be for a Calabi-Yau variety. It can be checked using the method described in [70, 135] that this is indeed the Hilbert series of the moduli space of the instanton mentioned below (3.8.2).

We now focus on the Coulomb branch. Since theory (3.8.6) is 'good' in the sense of [85], the Coulomb branch Hilbert series can be computed using either the monopole formula or the Hall-Littlewood formula. Here we present the latter:

$$
\begin{aligned}
&H_{\mathcal{C}}[(3.8.5)\text{ or }(3.8.6)](t; x_1, x_2; y_1, y_2) = \\
&\sum_{a=0}^{\infty} t^{-2|a-(-a)|} P_{USp(2)}(t; a)\times \\
&\qquad H_{\mathcal{C}}[T_{[2,1]}(SU(3))](t; x_1, x_2; a, 0, -a)\, H_{\mathcal{C}}[T_{[2,1]}(SU(3))](t; y_1, y_2; a, 0, -a)\times \\
&\qquad H_{\mathcal{C}}[T_{[2]}(USp'(2))](t; a)\, H_{\mathcal{C}}[T_{[2]}(USp'(2))](t; a) \ .
\end{aligned} \tag{3.8.10}
$$

where the fugacities x_1, x_2 and y_1, y_2 are subject to the conditions:

$$
x_1^2 x_2 = 1 \ , \qquad y_1^2 y_2 = 1 \ . \tag{3.8.11}
$$

After imposing these conditions, we see only two $U(1)$ fugacities appear in formula (3.8.10). They represent the two $U(1)$ topological symmetries associated with each $U(1)$ gauge node in quiver (3.8.6). From description (3.8.2) and the Higgs branch computation we expect, however, that there should be three $U(1)$ global symmetries. The other $U(1)$ symmetry is indeed 'hidden' in the above Coulomb branch computation, in a similar way as described in [120]. In order to match (3.8.8) with (3.8.10), we need to unrefine one fugacity in the former:

$$
H_{\mathcal{H}}[(3.8.6)](t; x_1^3, y_1^3, q=1) = H_{\mathcal{C}}[(3.8.5)\text{ or }(3.8.6)](t; x_1, x_1^{-2}; y_1, y_1^{-2}) \ . \tag{3.8.12}
$$

It is also interesting to compare these results with the Higgs branch Hilbert series of the 4d $\mathcal{T}_{2,\frac{3}{2},\frac{3}{2}}$ theory. Recall that the Higgs branch of the (A_1, D_4) theory is the closure of the minimal nilpotent orbit of $SU(3)$, whose Hilbert series is [26]

$$
H[\overline{\text{min}}_{SU(3)}](t; \boldsymbol{u}) = \sum_{p=0}^{\infty} \chi^{SU(3)}_{[p,p]}(\boldsymbol{u}) t^{2p} \ , \tag{3.8.13}
$$

where $\boldsymbol{u} = (u_1, u_2)$ are the fugacities of $SU(3)$ such that the character of the fundamental representation $[1,0]$ is written as $u_1 + u_2 u_1^{-1} + u_2^{-1}$. We now take two copies of the (A_1, D_4) theory, gauge a common $SU(2)$ symmetry and then couple it to one flavour of the fundamental hypermultiplets. For each copy, we need to decompose representations of $SU(3)$ into those of the $SU(2)\times U(1)$ subgroup. This amounts to using the following fugacity map:

$$u_1 = x^{1/3} z \ , \qquad u_2 = x^{-1/3} z \ , \tag{3.8.14}$$

where z is the $SU(2)$ fugacity, x is the $U(1)$ fugacity, and the power $1/3$ is the normalisation of the $U(1)$ charge such that we have the following decomposition: $[1,0] \to [1]_{\frac{1}{3}} + [0]_{\frac{2}{3}}$.

The Higgs branch Hilbert series of the $\mathcal{T}_{2,\frac{3}{2},\frac{3}{2}}$ theory is then

$$\begin{aligned} & H_{\mathcal{H}}[\mathcal{T}_{2,\frac{3}{2},\frac{3}{2}}](t;x,y,q) \\ & = \oint_{|z|=1} \frac{dz}{2\pi i z}(1-z^2)\ \mathrm{PE}\left[-t^2 \chi^{SU(2)}_{[2]}(z)\right]\ \mathrm{PE}\left[(z+z^{-1})(q+q^{-1})t\right] \times \\ & \qquad H[\overline{\min}_{SU(3)}](t; x^{1/3}z, x^{-1/3}z) H[\overline{\min}_{SU(3)}](t; y^{1/3}z, y^{-1/3}z) \end{aligned} \tag{3.8.15}$$

where z is the $SU(2)$ gauge fugacity, and each of x, y and q is the $U(1)$ fugacity. Evaluating the integral, we find that

$$H_{\mathcal{H}}[\mathcal{T}_{2,\frac{3}{2},\frac{3}{2}}](t;x,y,q) = H_{\mathcal{H}}[(3.8.6)](t;x,y,q) \ , \tag{3.8.16}$$

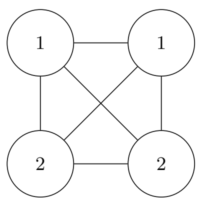
which is given by (3.8.8).

3.8.2 The $\mathcal{T}_{3,2,\frac{3}{2},\frac{3}{2}}$ theory

The 4d $\mathcal{N}=2$ $\mathcal{T}_{3,2,\frac{3}{2},\frac{3}{2}}$ SCFT was studied in [44] (see also [41]). It admits two known descriptions:

- An $SU(3)$ gauge theory coupled to two (A_1, D_4) theories with three flavours of fundamental hypermultiplets,
- an $SU(2)$ gauge theory coupled to the (A_1, D_4) theory and the $\mathcal{T}_{3,\frac{3}{2}}$ theory[14], where the Higgs branch of the latter is the full moduli space of two $SU(3)$ instantons on $\mathbb{C}^2$. These two descriptions are related by the Argyres-Seiberg duality [12].

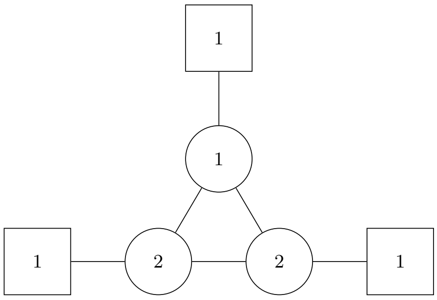
In [44], it was proposed that the $\mathcal{T}_{3,2,\frac{3}{2},\frac{3}{2}}$ theory is dual to another 4d $\mathcal{N}=2$ SCFT known as the $III^{3\times[2,2,1,1]}_{6,6}$ theory. Upon compactifying the latter on S^1, the 3d mirror theory can be obtain using the method described in [168] and the result was presented in [44, (4.3)]:



(3.8.17)

[14]Equivalently, this is an $SU(2)$ gauge theory coupled to the (A_1, D_4) theory and the $\mathcal{T}^{(2)}_{SU(3)}$ theory (see Section (3.7.3)), with a half-hypermultiplet in the doublet of the $SU(2)$ gauge group.

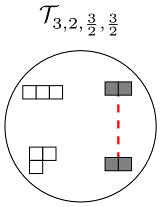
where an overall $U(1)$ needs to be decoupled from this quiver. Doing so from one of the $U(1)$ gauge node, we obtain the following equivalent quiver:



(3.8.18)

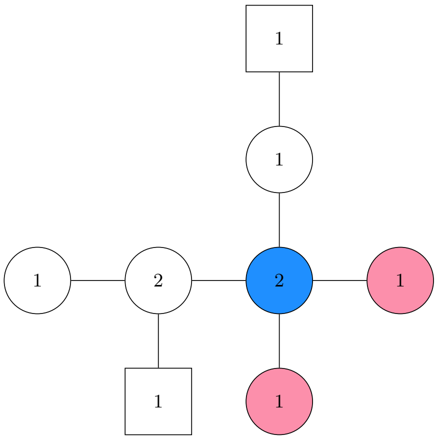
The class-S description of the $\mathcal{T}_{3,2,\frac{3}{2},\frac{3}{2}}$ theory (see [20, (5.1), (5.2)]) can be seen as a twisted $\chi(\mathfrak{a}_2)$ theory associated with the sphere with punctures

$$[1^3]\ ,\quad [2,1]\ ,\quad [2]_t\ ,\quad [2]_t \quad \rightarrow$$



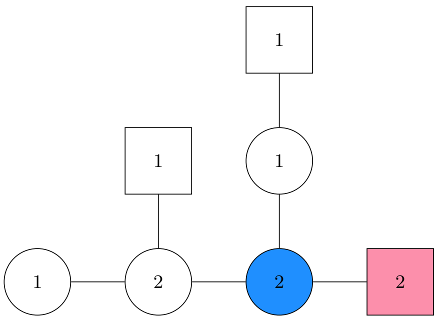
(3.8.19)

Following the procedure described in the preceding sections, we obtain the 3d mirror of this theory compactified on S^1 as



(3.8.20)

Since the red circular node denotes $SO(1)$, this quiver can be rewritten as



(3.8.21)

We conjecture that theories (3.8.18) and (3.8.21) are dual to each other. In the following we provide number of non-trivial checks.

The Coulomb branch of (3.8.21) is $1 + 2 + 1 + 1 = 5$ quaternionic dimensional. The Higgs branch of (3.8.21) is also $2 + 2 + 4 + \frac{1}{2}(2 \times 2) + 2 + 1 - (1 + 4 + 3 + 1) = 4$

quatenionic dimensional, in agreement with the fact that the 4d $\mathcal{T}_{3,2,\frac{3}{2},\frac{3}{2}}$ theory is a rank-four theory. Note that these are also equal to the corresponding quantities of (3.8.18). We now study both branches in more detail using the Hilbert series

As in the previous subsection, the theory is 'good' and so we can compute the Coulomb branch Hilbert series of (3.8.21) using either the monopole formula or the Hall-Littlewood formula. The latter reads

$$\begin{aligned} &H_{\mathcal{C}}[(3.8.21)](t;x_1,x_2,x_3;y_1,y_2) = \\ &\sum_{a=0}^{\infty} t^{-2|a-(-a)|} P_{USp(2)}(t;a)\, H_{\mathcal{C}}[T_{[1^3]}(SU(3))](t;x_1,x_2,x_3;a,0,-a) \\ &\qquad H_{\mathcal{C}}[T_{[2,1]}(SU(3))](t;y_1,y_2;a,0,-a)\times \\ &\qquad H_{\mathcal{C}}[T_{[2]}(USp'(2))](t;a)\, H_{\mathcal{C}}[T_{[2]}(USp'(2))](t;a)\ . \end{aligned} \tag{3.8.22}$$

with the following conditions on the fugacities due to (3.3.16):

$$x_1x_2x_3 = 1\ , \qquad y_1^2y_2 = 1\ . \tag{3.8.23}$$

Upon evaluating the summation, the result of (3.8.22) can be summarised as the highest weight generating function up to order t^8 as follows:

$$\begin{aligned} &\mathrm{HWG}\left[H_{\mathcal{C}}[(3.8.21)]\right] \\ &= \mathrm{PE}\Big[\,(\mu_1\mu_2+1)\,t^2 + \left(b+\frac{1}{b}\right)t^3 + (2\mu_1\mu_2+1)\,t^4 \\ &\quad + \left(b+\frac{1}{b}\right)\mu_1\mu_2 t^5 + \left(2\mu_1^3+\mu_2\mu_1+2\mu_2^3-1\right)t^6 \\ &\quad + \left(b+\frac{1}{b}\right)\left(\mu_1^3+\mu_2^3\right)t^7 + \left(\mu_1^3-2\mu_2\mu_1+\mu_2^3\right)t^8 + \dots\Big]\ , \end{aligned} \tag{3.8.24}$$

where, upon computing the power series of this expression in t, $\mu_1^{p_1}\mu_2^{p_2}$ denotes the representation $[p_1,p_2]$, whose character written in terms of x_1,x_2,x_3, of $SU(3)$. Here b is the fugacity for the $U(1)$ symmetry which can be written in terms of y_1, y_2 as

$$b = y_2y_1^{-1}\ . \tag{3.8.25}$$

As can be seen from the order t^2, the Coulomb branch symmetry of this theory is $U(3)$. This is in agreement with that of theory (3.8.21) and the flavour symmetry of the 4d theory.

The Higgs branch Hilbert series is

$$
\begin{aligned}
&H_{\mathcal{H}}[(3.8.21)](t;x,y,b) = \\
&\oint_{|u|=1} \frac{du}{2\pi i u} \oint_{|w|=1} \frac{dw}{2\pi i w} \oint_{|q|=1} \frac{dq}{2\pi i q} \times \\
&\quad \oint_{|z|=1} \frac{dz}{2\pi i z}(1-z^2) \oint_{|v|=1} \frac{dv}{2\pi i v}(1-v^2) \times \\
&\quad H_{\mathcal{H}}[[1]_u - [2]_{q,z}](t;u,q,z)\; H_{\mathcal{H}}[[1]_x - [2]_{q,z}](t;x,q,z) \times \\
&\quad H_{\mathcal{H}}[[2]_{q,z} - [USp(2)]_v](t;q,z,v)\; H_{\mathcal{H}}[[1]_w - [USp(2)]_v](t;w,v) \times \\
&\quad H_{\mathcal{H}}[[1]_w - [1]_y](t;w,y)\; H_{\mathcal{H}}[[USp(2)]_v - [SO(2)]_b](t;v,b) \times \\
&\quad \mathrm{PE}\left[-3t^2 - (v^2+1+v^{-2})t^2 - (z^2+1+z^{-2})t^2\right] \ ,
\end{aligned}
\tag{3.8.26}
$$

where the notations are as in (3.7.14) and (3.7.29). Evaluating the integrals, we obtain

$$
\begin{aligned}
&H_{\mathcal{H}}[(3.8.21)](t;x,y,b) \\
&= \mathrm{PE}\left[3t^2 + t^3\left(bx + \frac{b}{x} + \frac{1}{bx} + \frac{x}{b} + by + \frac{b}{y} + \frac{1}{by} + \frac{y}{b}\right)\right. \\
&\quad + t^4\left(b^2 + \frac{1}{b^2} + xy + \frac{y}{x} + \frac{x}{y} + \frac{1}{xy} + 1\right) \\
&\quad \left. + t^5\left(bx + \frac{b}{x} + \frac{1}{bx} + \frac{x}{b}\right) - \dots\right] .
\end{aligned}
\tag{3.8.27}
$$

The order t^2 indicates that the Higgs branch symmetry is $U(1)^3$. Setting $x = y = b = 1$, we obtain the closed form of the unrefined Hilbert series as

$$
\begin{aligned}
&H_{\mathcal{H}}[(3.8.21)](t;x=1,y=1,b=1) \\
&\frac{1}{(1-t)^8(1+t)^4(1+t^2)^2(1-t+t^2)(1+t+t^2)^3}\Big[1 - 2t + 3t^2 + 2t^3 \\
&\quad - 2t^4 + 6t^5 + 3t^6 - 2t^7 + 12t^8 - 2t^9 + \dots \text{(palindrome)} \dots + t^{16}\Big] .
\end{aligned}
\tag{3.8.28}
$$

This Higgs branch Hilbert series is in agreement with that for (3.8.18).

3.9 Twisted $\chi(\mathfrak{a}_{2N})$ theories with $N \geq 1$

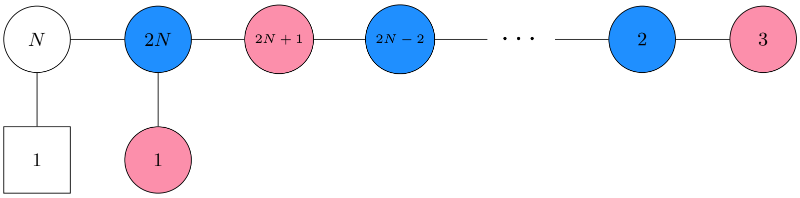
In this section, we discuss the generalisation of our results for the twisted $\chi(\mathfrak{a}_2)$ theories to the case of $\chi(\mathfrak{a}_{2N})$ with $N \geq 1$.

3.9.1 The $D_2[SU(2N+1)]$ theory with N free hypermultiplets

The class-S description (without an irregular puncture) was proposed in [20, (6.3)]. It is a twisted $\chi(\mathfrak{a}_{2N})$ theory associated with a sphere with punctures

$$
[N+1, N] \ , \qquad [1^{2N}]_t \ , \qquad [2N]_t \ .
\tag{3.9.1}
$$

For $N = 1$, this was discussed in Section (3.7.2), where the low energy theory is the (A_1, D_4) SCFT with a free hypermultiplet.. For a general N, the 3d mirror theory of the reduction of the 4d theory in question on a circle admits the following quiver description:



(3.9.2)

The Coulomb branch of (3.9.2) is $N+\sum_{j=1}^{N} 2j = N^2+2N$ quaternionic dimension. This is in agreement with the Higgs branch of the 4d theory: the Higgs branch of $D_2[SU(2N+1)]$ is $N(N+1)$ quaternionic dimensional (see Appendix (A)), and the Higgs branch of the theory of N free hypermultiplets is N quaternionic dimensional; in total we have $N^2 + 2N$ quaternionic dimensions. The Higgs branch of (3.9.2) is $N + 2N^2 + N + \dim_{\mathbb{H}} \mathcal{H}[T(USp'(2N))] - N^2 - \frac{1}{2}(2N)(2N+1) = N$, where we have used the fact that $\dim_{\mathbb{H}} \mathcal{H}[T(USp'(2N))] = N^2$. This is in agreement with the fact that $D_2[SU(2N+1)]$ is a rank N theory.

The Coulomb branch Hilbert series of (3.9.2) can be computed using the Hall-Littlewood formula as follows:

$$
\begin{aligned}
&H_{\mathcal{C}}[(3.9.2)](t; x_1, x_2, y_1, y_2, \ldots, y_N) = \\
&\sum_{n_1 \geq n_2 \geq n_N \geq 0} t^{-2\left[\sum_{j=1}^{N} |2n_j| + \sum_{1 \leq i < j \leq N}(|n_i - n_j| + |n_i + n_j|)\right]} P_{USp(2N)}(t; n_1, \ldots, n_N) \times \\
&\ H_{\mathcal{C}}[T_{[N+1,N]}(SU(2N+1))](t; x_1, x_2; n_1, n_2, \ldots, n_N, 0, -n_N, -n_{N-1}, \ldots, -n_1) \times \\
&\ H_{\mathcal{C}}[T_{[1^{2N}]}(USp'(2N))](t; y_1, y_2, \ldots, y_N; n_1, n_2, \ldots, n_N) \times \\
&\ H_{\mathcal{C}}[T_{[2N]}(USp'(2N))](t; n_1, \ldots, n_N) \ ,
\end{aligned}
\tag{3.9.3}
$$

where

$$
x_1^{N+1} x_2^{N} = 1 \ . \tag{3.9.4}
$$

Evaluating the summations, we obtain

$$
\begin{aligned}
&H_{\mathcal{C}}[(3.9.2)](t; x_1, x_2, y_1, y_2, \ldots, y_N) \\
&= \mathrm{PE}\left[\sum_{j=1}^{N} (y_j + y_j^{-1})\right] \times H[D_2[SU(2N+1)]](x_1, x_2, y_1, \ldots, y_N) \ ,
\end{aligned}
\tag{3.9.5}
$$

where the first factor is the Hilbert series of $\mathbb{H}^N \cong \mathbb{C}^{2N}$ and the second factor is as described in Appendix (A). We have tested this expression for $N = 1, 2, 3$. This confirms that the moduli space of the Coulomb branch is a product of $\mathbb{H}^N$ and that of the Higgs branch fo the $D_2[SU(2N+1)]$ theory, as expected from the 4d theory.

The Higgs branch Hilbert series, on the other hand, can be computed as follows:

$$
\begin{aligned}
&H_{\mathcal{H}}[(3.9.2)](t;x) = \int d\mu_{U(N)}(\boldsymbol{z}) \int d\mu_{USp(2N)}(\boldsymbol{v})\times \\
&\quad H_{\mathcal{H}}[[1]_x - [N]_{\boldsymbol{z}}]\, H_{\mathcal{H}}[[N]_z - [USp(2N)]_{\boldsymbol{v}}]\times \\
&\quad H_{\mathcal{C}}[T_{[1^{2N}]}(USp'(2N))](t;\boldsymbol{v};0,0,\dots,0)\, H_{\mathcal{H}}[[SO(1)] - (USp(2N))_{\boldsymbol{v}}]\times \\
&\quad \mathrm{PE}\left[-\left(\sum_{i,j=1}^{N} z_i z_j^{-1}\right) t^2 - \chi^{USp(2N)}_{[2,0,\dots,0]}(\boldsymbol{v}) t^2\right] ,
\end{aligned}
\tag{3.9.6}
$$

where we have used the fact that $T_{[1^{2N}]}(USp'(2N))$ is self-mirror and so the Higgs branch Hilbert series of such a theory can be computed from the Coulomb branch one. Here

$$
\begin{aligned}
H_{\mathcal{H}}[[1]_x - [N]_{\boldsymbol{z}}] &= \mathrm{PE}\left[\left(x^{-1}\sum_{j=1}^{N} z_j + x\sum_{j=1}^{N} z_j^{-1}\right) t\right] \\
H_{\mathcal{H}}[[N]_z - [USp(2N)]_{\boldsymbol{v}}] &= \mathrm{PE}\left[\left(\sum_{j=1}^{N} z_j + \sum_{j=1}^{N} z_j^{-1}\right)\left(\sum_{k=1}^{N} v_k + \sum_{k=1}^{N} v_k^{-1}\right) t\right] \\
\chi^{USp(2N)}_{[2,0,\dots,0]}(\boldsymbol{v}) &= N + \sum_{j=1}^{N}(v_j^2 + v_j^{-2}) + \sum_{1\le i<j\le N}(v_i v_j + v_i^{-1}v_j^{-1} + v_i v_j^{-1} + v_i^{-1} v_j) \\
d\mu_{USp(2N)}(\boldsymbol{v}) &= \prod_{j=1}^{N} v_j^{-1}(1-v_j^2) \prod_{1\le i<j\le N}(1-v_i v_j)(1-v_i v_j^{-1}) \\
d\mu_{U(N)}(\boldsymbol{z}) &= \prod_{j=1}^{N} z_j^{-1} \prod_{1\le i<j\le N}(1-z_i z_j^{-1}) .
\end{aligned}
\tag{3.9.7}
$$

In the case of $N = 2$, for example, we have

$$
\begin{aligned}
&H_{\mathcal{H}}[(3.9.2)_{N=2}](t;x) \\
&= \mathrm{PE}\left[t^2 + (x+x^{-1})t^3 + t^4 + (x+x^{-1})t^5 - t^8 - t^{10} + \dots\right] .
\end{aligned}
\tag{3.9.8}
$$

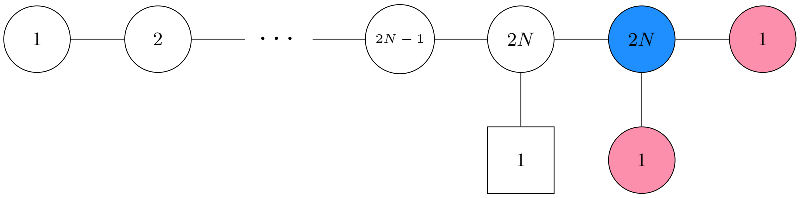
It can be checked that this is in agreement with the Higgs branch Hilbert series of (A.0.1), with $N = 2$. This is indeed the Coulomb branch Hilbert series of the S^1 reduction of the $D_2[SU(2N+1)]$ theory.

3.9.2 Two copies of the $D_2[SU(2N+1)]$ theory

The class-S description (without an irregular puncture) was proposed in [20, (6.4)]. It is a twisted $\chi(\mathfrak{a}_{2N})$ theory associated with a sphere with punctures

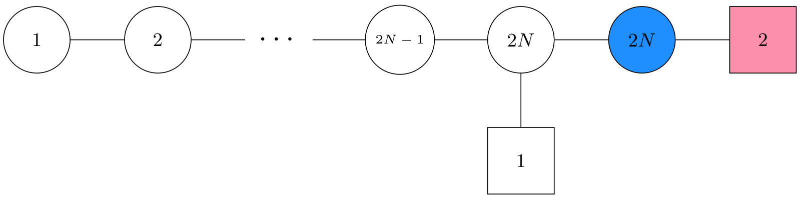
$$
[1^{2N+1}] , \qquad [2N]_t , \qquad [2N]_t . \tag{3.9.9}
$$

The 3d mirror of the reduction of this 4d theory on S^1 can be described by the following quiver:



(3.9.10)

Since the red circular node is $SO(1)$, this quiver can be rewritten as



(3.9.11)

For $N = 1$, we recover quiver (3.7.4). The Coulomb branch of (3.9.11) is $\sum_{j=1}^{2N} j + N = 2N(N+1)$, in agreement with the Higgs branch dimension of a product of two $D_2[SU(2N+1)]$. The Higgs branch of (3.9.11) is $\sum_{j=1}^{2N-1} j(j+1) + 2N + 4N^2 + 2N - \sum_{j=1}^{2N} j^2 - N(2N+1) = 2N$, in agreement with the fact that $D_2[SU(2N+1)]$ is a rank N theory.

Similarly to the case of quiver (3.7.4), we can see the enhanced Coulomb branch symmetry of quiver (3.9.11) using the observation of [85]. Since all of the $U(s)$ gauge nodes, with $s = 1, \ldots, 2N$, in (3.9.11) are all balanced, one expects an $SU(2N+1)$ enhanced symmetry in the IR. Moreover, since the $USp(2N)$ gauge node is also balanced, according to [85, section 5.3], this $SU(2N+1)$ symmetry gets doubled and so the symmetry of the Coulomb branch is expected to be $SU(2N+1) \times SU(2N+1)$. This is in agreement with the Higgs branch symmetry of a product of two $D_2[SU(2N+1)]$. Shortly we confirm this using the Coulomb branch Hilbert series.

The Coulomb branch Hilbert series of (3.9.11) can be computed using either the monopole formula or the the Hall-Littlewood formula. The latter reads

$$\begin{aligned} &H_{\mathcal{C}}[(3.9.11)](t; x_1, \ldots, x_{2N+1}) = \\ &\sum_{n_1 \geq n_2 \geq n_N \geq 0} t^{-2\left[\sum_{j=1}^{N} |2n_j| + \sum_{1 \leq i < j \leq N} (|n_i - n_j| + |n_i + n_j|)\right]} P_{USp(2N)}(t; n_1, \ldots, n_N) \times \\ &H_{\mathcal{C}}[T_{[1^{2N+1}]}(SU(2N+1))](t; x_1, \ldots, x_{2N+1}; n_1, n_2, \ldots, n_N, 0, -n_N, -n_{N-1}, \ldots, -n_1) \times \\ &H_{\mathcal{C}}[T_{[2N]}(USp'(2N))](t; n_1, \ldots, n_N)\, H_{\mathcal{C}}[T_{[2N]}(USp'(2N))](t; n_1, \ldots, n_N) \ , \end{aligned} \tag{3.9.12}$$

where

$$x_1 x_2 \cdots x_{2N+1} = 1 \ . \tag{3.9.13}$$

Evaluating the summations, we obtain

$$H_{\mathcal{C}}[(3.9.11)](t; x_1, \ldots, x_{2N+1}) = \left[H_{\mathcal{H}}[D_2[SU(2N+1)]](x_1, \ldots, x_{2N+1})\right]^2 \ , \tag{3.9.14}$$

where the Higgs branch Hilbert series $H_{\mathcal{H}}[D_2[SU(2N+1)]]$ of $D_2[SU(2N+1)]$ is given in Appendix (A). We have tested this expression for $N = 1, 2, 3$. Similarly to the remark below (3.7.10), the full Coulomb branch symmetry is expected to be

$SU(2N+1)^2$; however, it is possible to see only the diagonal subgroup $SU(2N+1)$, corresponding to the fugacities $x_1, \ldots, x_{2N+1}$ in the Hilbert series.

The Higgs branch Hilbert series, on the other hand, can be computed as follows:

$$\begin{aligned} H_{\mathcal{H}}[(3.9.11)](t;x,y) = &\int d\mu_{U(2N)}(\boldsymbol{z}) \int d\mu_{USp(2N)}(\boldsymbol{v}) \times \\ &H_{\mathcal{H}}[T_{[1^{2N}]}(SU(2N))](t;\boldsymbol{z}) \times \\ &H_{\mathcal{H}}[[1]_x - [2N]_{\boldsymbol{z}}]\, H_{\mathcal{H}}[[2N]_{\boldsymbol{z}} - [USp(2N)]_{\boldsymbol{v}}] \times \\ &H_{\mathcal{H}}[(USp(2N))_{\boldsymbol{v}} - [SO(2)]_y] \times \\ &\mathrm{PE}\left[- \left(\sum_{i,j=1}^{2N} z_i z_j^{-1} \right) t^2 - \chi^{USp(2N)}_{[2,0,\ldots,0]}(\boldsymbol{v}) t^2 \right] , \end{aligned} \tag{3.9.15}$$

where the Higgs branch Hilbert series of $T_{[1^{2N}]}(SU(2N))$ is given by [105, (3.4)]

$$H_{\mathcal{H}}[T_{[1^{2N}]}(SU(2N))](t;\boldsymbol{z}) = \mathrm{PE}\left[t^2 \sum_{i,j=1}^{2N} z_i z_j^{-1} \right] \prod_{p=1}^{2N} (1 - t^{2p}) \ . \tag{3.9.16}$$

For example, in the case of $N = 2$, we obtain

$$H_{\mathcal{H}}[(3.9.11)_{N=2}](t;x,y) = H_{\mathcal{H}}[(3.9.2)_{N=2}](t;xy) H_{\mathcal{H}}[(3.9.2)_{N=2}](t;xy^{-1}) \ , \tag{3.9.17}$$

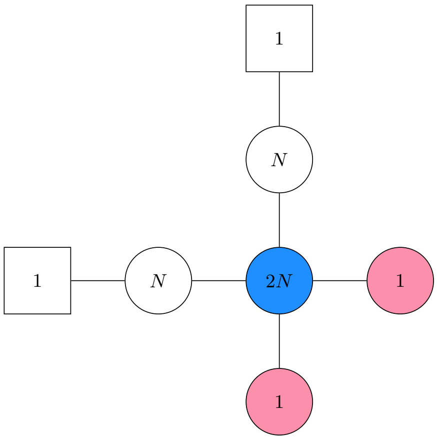
where $H_{\mathcal{H}}[(3.9.2)_{N=2}](t;x)$, which is the Coulomb branch Hilbert series of the S^1 reduction of the $D_2[SU(2N+1)]$ theory, is given by (3.9.8).

3.9.3 A sphere with punctures $[N+1, N]$, $[N+1, N]$, $[2N]_t$, $[2N]_t$

We study a $USp(2N)$ gauge theory coupled to one flavour of the fundamental hypermultiplets and two copies of the $D_2[SU(2N+1)]$ theory, where a $USp(2N)$ subgroup of the $SU(2N+1)$ global symmetry of each copy is gauged. The class-S description (without an irregular puncture) was proposed in [20, (6.5)]. It is a twisted $\chi(\mathfrak{a}_{2N})$ theory associated with a sphere with punctures

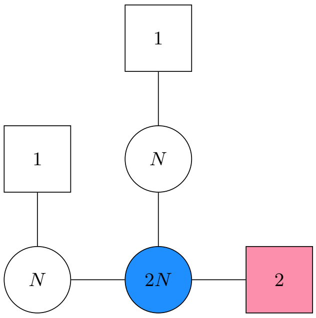
$$[N+1, N] \ , \qquad [N+1, N] \ , \qquad [2N]_t \ , \qquad [2N]_t \ . \tag{3.9.18}$$

The case of $N = 1$ was studied in Section (3.8.1). The 3d mirror of the reduction of this 4d theory on S^1 can be described by the following quiver:



(3.9.19)

Since the red circular node denotes $SO(1)$, this quiver can be rewritten as



(3.9.20)

The quaternionic dimension of the Higgs branch dimension of the 4d theory is $2N(N+1) + 2N - \frac{1}{2}(2N)(2N+1) = 3N$, where $N(N+1)$ is the Higgs branch dimension of the $D_2[SU(2N+1)]$ theory. The Coulomb branch of quiver (3.9.20) is $N + N + N = 3N$ quaternionic dimensional, in agreement with that of the Higgs branch of the 4d theory. The Higgs branch of quiver (3.9.20) is $N + 2N^2 + 2N^2 + N + 2N - \left[N^2 + N^2 + \frac{1}{2}(2N)(2N+1)\right] = 3N$. This is in agreement with the fact that each copy of the $D_2[SU(2N+1)]$ theory is of rank N and the $USp(2N)$ gauge group has rank N, and so in total we have $3N$ dimensional Coulomb branch as expect. Observe that the Coulomb and Higgs branches of (3.9.20) have the same dimension. Indeed, as we shall discuss below, theory (3.8.21) is **self-mirror** for any N, where the case of $N = 1$ was indeed self-mirror as shown in Section (3.8.1).

Let us first examine the Higgs branch. The Higgs branch Hilbert series of (3.9.20) is given by

$$
\begin{aligned}
&H_{\mathcal{H}}[(3.9.20)](t; x, y, q) = \\
&\int d\mu_{U(N)}(\boldsymbol{u}) \int d\mu_{U(N)}(\boldsymbol{w}) \int d\mu_{USp(2N)}(\boldsymbol{v}) \times \\
&\qquad H_{\mathcal{H}}[[N]_{\boldsymbol{u}} - [1]_x](t; u, x)\, H_{\mathcal{H}}[[N]_{\boldsymbol{u}} - [USp(2)]_v](t; u, v) \times \\
&\qquad H_{\mathcal{H}}[[N]_{\boldsymbol{w}} - [1]_y](t; u, x)\, H_{\mathcal{H}}[[N]_{\boldsymbol{w}} - [USp(2)]_v](t; w, v) \times \\
&\qquad H_{\mathcal{H}}[[USp(2)]_v - [SO(2)]_q](t; v, q) \times \\
&\qquad \mathrm{PE}\left[- \left(\sum_{i,j=1}^{N} u_i u_j^{-1} \right) t^2 - \left(\sum_{i,j=1}^{N} w_i w_j^{-1} \right) t^2 - \chi^{USp(2N)}_{[2,0,\ldots,0]}(\boldsymbol{v}) t^2 \right] ,
\end{aligned}
\tag{3.9.21}
$$

Let us compute the integrals in the case of $N = 2$, we obtain

$$
\begin{aligned}
&H_{\mathcal{H}}[(3.9.20)_{N=2}](t;x,y,q) = \\
&\mathrm{PE}\left[3t^2 + t^3\left(qx + \frac{q}{x} + \frac{1}{qx} + \frac{x}{q} + qy + \frac{q}{y} + \frac{1}{qy} + \frac{y}{q}\right)\right. \\
&\quad + t^4\left(5 + q^2 + \frac{1}{q^2} + xy + \frac{y}{x} + \frac{x}{y} + \frac{1}{xy}\right) \\
&\quad + 2t^5\left(qx + \frac{q}{x} + \frac{1}{qx} + \frac{x}{q} + qy + \frac{q}{y} + \frac{1}{qy} + \frac{y}{q}\right) \\
&\quad \left. + t^6\left(3 + q^2 + \frac{1}{q^2} + xy + \frac{y}{x} + \frac{x}{y} + \frac{1}{xy}\right) + \ldots\right] .
\end{aligned}
\tag{3.9.22}
$$

On the other hand, the Higgs branch Hilbert series of the 4d theory is given by

$$
\begin{aligned}
&H_{\mathcal{H}}[\text{4d theory}](t;x,y,q) = \\
&\int d\mu_{USp(2N)}(\boldsymbol{z})\ \mathrm{PE}\left[-t^2\chi^{USp(2N)}_{[2,0,\ldots,0]}(\boldsymbol{z})\right]\ \mathrm{PE}\left[\chi^{USp(2N)}_{[1,0,\ldots,0]}(\boldsymbol{z})(q+q^{-1})t\right] \times \\
&H_{\mathcal{H}}[D_2[SU(2N+1)]](t; x^{\frac{1}{3}}(z_1,\ldots,z_N), x^{\frac{1}{3}}(z_1^{-1},\ldots,z_N^{-1}), x^{\frac{2N}{3}})\times \\
&H_{\mathcal{H}}[D_2[SU(2N+1)]](t; y^{\frac{1}{3}}(z_1,\ldots,z_N), y^{\frac{1}{3}}(z_1^{-1},\ldots,z_N^{-1}), y^{\frac{2N}{3}})\ ,
\end{aligned}
\tag{3.9.23}
$$

where each of x, y, q is a $U(1)$ fugacity and $\boldsymbol{z} = (z_1, \ldots, z_N)$ are the $USp(2N)$ gauge fugacity. The expression for $H_{\mathcal{H}}[D_2[SU(2N+1)]]$ is given in Appendix (A). Here, under the decomposition $SU(2N+1) \supset SU(2N) \times U(1) \supset USp(2N) \times U(1)$, we have

$$
\begin{aligned}
SU(2N+1) &\ \rightarrow\ USp(2N) \times U(1) \\
[1,0,\ldots,0] &\ \rightarrow\ [1,0,\ldots,0]_{\frac{1}{3}} + [0,\ldots,0]_{\frac{2N}{3}}\ .
\end{aligned}
\tag{3.9.24}
$$

If we write the character of the fundamental representation $[1,0,\ldots,0]$ of $SU(2N+1)$ as $\sum_{j=1}^{2N+1} u_j$ (with $\prod_{j=1}^{2N+1} u_j = 1$) and that of the fundamental representation $[1,0,\ldots,0]$ of $USp(2N)$ as $\sum_{j=1}^{N}(v_j + v_j^{-1})$, then a fugacity map is

$$
u_k = \begin{cases} q^{\frac{1}{3}} v_k\ , & k = 1,2,\ldots,N \\ q^{\frac{1}{3}} v_{k-N}^{-1}\ , & k = N+1, N+2\ldots, 2N \\ q^{\frac{2N}{3}}\ , & k = 2N+1\ , \end{cases}
\tag{3.9.25}
$$

where q is the fugacity for the $U(1)$ symmetry. Since theory (3.9.20) is self-mirror, its Higgs branch Hilbert series can be equated to that of the 4d theory as follows:

$$
H_{\mathcal{H}}[\text{4d theory}](t;x,y,q) = H_{\mathcal{H}}[(3.9.20)](t;x,y,q)\ .
\tag{3.9.26}
$$

The Coulomb branch Hilbert series of (3.9.20) can be computed using either the monopole formula or the the Hall-Littlewood formula. The latter reads

$$
\begin{aligned}
&H_{\mathcal{C}}[(3.9.20)](t; x_1, x_2, y_1, y_2) = \\
&\sum_{n_1 \geq n_2 \geq n_N \geq 0} t^{-2[\sum_{j=1}^{N} |2n_j| + \sum_{1 \leq i < j \leq N}(|n_i - n_j| + |n_i + n_j|)]} P_{USp(2N)}(t; n_1, \ldots, n_N) \times \\
&H_{\mathcal{C}}[T_{[N+1,N]}(SU(2N+1))](t; x_1, x_2; n_1, n_2, \ldots, n_N, 0, -n_N, -n_{N-1}, \ldots, -n_1) \times \\
&H_{\mathcal{C}}[T_{[N+1,N]}(SU(2N+1))](t; y_1, y_2; n_1, n_2, \ldots, n_N, 0, -n_N, -n_{N-1}, \ldots, -n_1) \times \\
&H_{\mathcal{C}}[T_{[2N]}(USp'(2N))](t; n_1, \ldots, n_N)\, H_{\mathcal{C}}[T_{[2N]}(USp'(2N))](t; n_1, \ldots, n_N) \ ,
\end{aligned}
\tag{3.9.27}
$$

where

$$
x_1^{N+1} x_2^{N} = y_1^{N+1} y_2^{N} = 1 \ . \tag{3.9.28}
$$

Taking into account these constraints on the fugacities, we see that there are only two $U(1)$ fugacities that are manifest in the Hilbert series (3.9.27), whereas the full Coulomb branch symmetry is $U(1)^3$. This phenomenon is similar to what we have encountered in Section (3.8.1). In order to match (3.9.21) with (3.9.27), we need to unrefine one fugacity in the former:

$$
\begin{aligned}
&H_{\mathcal{C}}[(3.9.27)](t; x^{N}, x^{-N-1}, y^{N}, y^{-N-1}) \\
&= H_{\mathcal{H}}[(3.9.20)](t; x^{2N+1}, y^{2N+1}, q = 1) \\
&= H_{\mathcal{H}}[\text{4d theory}](t; x^{2N+1}, y^{2N+1}, q = 1) \ ,
\end{aligned}
\tag{3.9.29}
$$

where we have checked this relation for $N = 1, 2$.

Chapter 4

The superconformal index

This chapter will be entirely dedicated to the study of the 3d supersymmetric index, which we shall refer to as the *index* for brevity, in the framework of $3d\ \mathcal{N}=2$ superconformal field theories (SCFTs).

This supersymmetric invariant quantity is nothing but the supersymmetric partition function on $S^2 \times S^1$ with a peculiar periodic boundary condition for the fields along S^1 and counts the number of BPS operators. Since it is also an RG flow invariant, it can be easily computed in the UV and can nonetheless give interesting insights about the IR fixed point. Moreover, whenever the theory admits a superconformal point, it is referred as superconformal index and thus counts the BPS short multiplets up to recombination. As we will see, signals of supersymmetry (or global symmetry) enhancement can be quite easily found by computing the index.

In the following, we will review the basic ideas behind the definition of the index and we will see how to explicitly compute this quantity throughout a technique called supersymmetric localization.

4.1 $3d\ \mathcal{N}=2$ superconformal algebra

Since we are interested in computing the 3d supersymmetric index for $3d\ \mathcal{N}=2$ superconformal field theories (SCFTs), let us briefly review their superconformal algebra and multiplets in the notation adopted by [59].

In three dimensions the Lorentz group is $SO(3) \sim SU(2)$, thus its representations are denoted by $[j]$ where j is the integer-valued Dynkin label of $\mathfrak{su}(2)$ and is linked to the third component of the angular momentum j_3 by $j = 2j_3$.

Thus the conformal group is just $SO(3,2)$ and in a given representation, since in radial quantization the dilaton D can be identified with the Hamiltonian H of the system, we will always diagonalise D. Its eigenvalue Δ is called the scaling dimension and thus we write $[j]_\Delta$ to denote the $SO(3,2)$ representations.

The $\mathcal{N}=2$ superconformal algebra is thus $\mathfrak{osp}(2|4)$ which R-symmetry is $SO(2)_R \sim U(1)_R$. Including the eigenvalue R of the R-symmetry, a generic representation is then denoted by

$$[j]_\Delta^{(R)} \quad \text{with} \quad j \in \mathbb{N}, \ \ \Delta, R \in \mathbb{R} \tag{4.1.1}$$

The 4 real supercharges of $3d\ \mathcal{N}=2$ theories are recast into two independent $\mathcal{Q}$ and $\widetilde{\mathcal{Q}}$ supercharges transforming as

$$\mathcal{Q} \in [1]_{\frac{1}{2}}^{(1)}\ , \quad \mathcal{Q} \in [1]_{\frac{1}{2}}^{(-1)} \tag{4.1.2}$$

since

$$[D,\mathcal{Q}]=\frac{1}{2}\mathcal{Q}\ ,\quad [D,\widetilde{\mathcal{Q}}]=\frac{1}{2}\widetilde{\mathcal{Q}} \tag{4.1.3}$$

Thus the 4 real supercharges are explicitly

$$\mathcal{Q}^{+}_{\pm}:=[\pm 1]^{(1)}_{\frac{1}{2}}\ ,\quad \mathcal{Q}^{-}_{\pm}:=[\pm 1]^{(-1)}_{\frac{1}{2}} \tag{4.1.4}$$

Requiring the non-negativity of the norm of the fields contained in a superconformal multiplet, we get the following unitarity bounds and shortening conditions for $\mathcal{Q}$

Name	Superconformal primary	Unitarity bound	First conformal primary null state
L	$[j]^{(R)}_{\Delta}$	$\Delta>\frac{1}{2}j-R+1$	-
A_1	$[j]^{(R)}_{\Delta}$, $j\geq 1$	$\Delta=\frac{1}{2}j-R+1$	$[j-1]^{(R-1)}_{\Delta+\frac{1}{2}}$
A_2	$[0]^{(R)}_{\Delta}$	$\Delta=-R+1$	$[0]^{(R-2)}_{\Delta+1}$
B_1	$[0]^{(R)}_{\Delta}$	$\Delta=-R$	$[1]^{(R-1)}_{\Delta+\frac{1}{2}}$

(4.1.5)

and analogous ones for $\widetilde{\mathcal{Q}}$

Name	Superconformal primary	Unitarity bound	First conformal primary null state
$\overline{L}$	$[j]^{(R)}_{\Delta}$	$\Delta>\frac{1}{2}j+R+1$	-
$\overline{A}_1$	$[j]^{(R)}_{\Delta}$, $j\geq 1$	$\Delta=\frac{1}{2}j+R+1$	$[j-1]^{(R+1)}_{\Delta+\frac{1}{2}}$
$\overline{A}_2$	$[0]^{(R)}_{\Delta}$	$\Delta=R+1$	$[0]^{(R+2)}_{\Delta+1}$
$\overline{B}_1$	$[0]^{(R)}_{\Delta}$	$\Delta=R$	$[1]^{(R+1)}_{\Delta+\frac{1}{2}}$

(4.1.6)

Then the $\mathcal{N}=2$ superconformal multiplets must obey both $\mathcal{Q}$ and $\widetilde{\mathcal{Q}}$ unitarity bounds and shortening conditions. Not all the possible choices are however mutually compatible and the consistent ones are:

$$L\overline{L}\ ,\quad L\overline{A}_1\ ,\quad L\overline{A}_2\ ,\quad L\overline{B}_1\ ,\quad A_1\overline{A}_1\ ,\quad A_2\overline{A}_2\ ,\quad A_2\overline{B}_1\ ,\quad B_1\overline{B}_1 \tag{4.1.7}$$

and their barred counterparts.

The $\mathcal{N}=2$ $L\overline{L}$ long multiplet is displayed in figure (4.1.8).

$$
\begin{tikzcd}
 & & \boxed{[j]^{(R)}_{\Delta},\quad \Delta > \frac{1}{2}j - R + 1} \arrow[dl] \arrow[dr] & & \\
 & \boxed{\mathcal{Q}^{-}_{\pm} \colon [j \pm 1]^{(R-1)}_{\Delta+\frac{1}{2}}} \arrow[dl] \arrow[dr] & & \boxed{\mathcal{Q}^{+}_{\pm} \colon [j \pm 1]^{(R+1)}_{\Delta+\frac{1}{2}}} \arrow[dl] \arrow[dr] & \\
\boxed{\mathcal{Q}^{-}_{+}\mathcal{Q}^{-}_{-} \colon [j]^{(R-2)}_{\Delta+1}} \arrow[dr] & & \boxed{\mathcal{Q}^{+}_{\pm}\mathcal{Q}^{-}_{\mp} \colon 2[j]^{(R)}_{\Delta+1} \oplus \mathcal{Q}^{+}_{\pm}\mathcal{Q}^{-}_{\pm} \colon [j \pm 2]^{(R)}_{\Delta+1}} \arrow[dl] \arrow[dr] & & \boxed{\mathcal{Q}^{+}_{+}\mathcal{Q}^{+}_{-} \colon [j]^{(R+2)}_{\Delta+1}} \arrow[dl] \\
 & \boxed{\mathcal{Q}^{-}_{+}\mathcal{Q}^{-}_{-}\mathcal{Q}^{+}_{\pm} \colon [j \pm 1]^{(R-1)}_{\Delta+\frac{3}{2}}} \arrow[dr] & & \boxed{\mathcal{Q}^{+}_{+}\mathcal{Q}^{+}_{-}\mathcal{Q}^{-}_{\pm} \colon [j \pm 1]^{(R+1)}_{\Delta+\frac{3}{2}}} \arrow[dl] & \\
 & & \boxed{\mathcal{Q}^{-}_{+}\mathcal{Q}^{-}_{-}\mathcal{Q}^{+}_{+}\mathcal{Q}^{+}_{-} \colon [j]^{(R)}_{\Delta+2}} & &
\end{tikzcd}
\tag{4.1.8}
$$

The superconformal primary is the topmost state $[j]^{(R)}_{\Delta}$ while all the other states are the conformal primaries. The arrows show schematically the action of the 4 real supercharges inside $\mathcal{Q}$ and $\widetilde{\mathcal{Q}}$ on such highest weight states. This is the most generic situation for $\Delta > \frac{1}{2}j - R + 1$. For small values of Δ we would get short multiplets, in which some of the conformal primaries would be missing according to the first conformal primary null states of Tables (4.1.5) and (4.1.6).

By acting with the translation operator and Lorentz generators on each conformal primary state we can then get the infinite tower of descendants, *i.e.* the non-highest weight states in the conformal representation.

4.2 The Witten index

In a theory with interactions, the partition function usually cannot be computed so easily due to the presence of the interacting terms in the Lgrangian. Nevertheless, if the theory possesses supersymmetry, the 3d supersymmetric index can be computed exactly even when interactions are turned on.

The starting observation that lead Witten [163] to define such a quantity is that all the supersymmetric states with energy $E \neq 0$ are two-fold degenerate: one state is indeed bosonic and the other fermionic. This is simply a consequence of supersymmetry, for which the Hamiltonian operator H always commutes with the supercharges

$$[\mathcal{Q}, H] = [\overline{\mathcal{Q}}, H] = 0 \tag{4.2.1}$$

Thus, given any Hamiltonian eigenstate $|\psi\rangle$ with eigenvalue $E \neq 0$, if we consider the state $|\phi_\psi\rangle := \overline{\mathcal{Q}}\,|\psi\rangle$ obtained by acting on $|\psi\rangle$ with the supersymmetry charges, then

$$H\,|\phi_\psi\rangle = H\overline{\mathcal{Q}}\,|\psi\rangle = \overline{\mathcal{Q}}H\,|\psi\rangle = E\overline{\mathcal{Q}}\,|\psi\rangle = E\,|\phi_\psi\rangle \tag{4.2.2}$$

This implies that both $|\psi\rangle$ and $|\phi_\psi\rangle$ posses the same Hamiltonian eigenvalue $E \neq 0$, which means that the spectrum is paired whenever the energy is different from zero. This pairing of non-zero energy states is a very robust phenomenon, solely based on supersymmetry and is thus expected to hold even when the theory possesses interactions that preserve it.

Therefore, as we turn on the interactions, the spectrum of the theory obviously changes; the excited states of the original free theory can become zero energy states for the new interacting theory (or vice versa) but, thanks to supersymmetry, they always move in couples so that, in the end, the new excited states are still paired.

Thus, slightly modifying the definition of partition function, we can building an invariant quantity under supersymmetric deformations called Witten index and defined as follows

$$I = \mathrm{Tr}\left((-1)^F x^H \prod_i^{\mathrm{Rank}\,\widehat{G}} \mu_i^{F_i}\right) \tag{4.2.3}$$

where F is the fermionic number, counting the number of fermions in a given state. μ_i are the fugacities with respect to the Cartan generators $\{F_i\}$ of the global symmetry $\widehat{G}$ of the theory, keeping track of the different representations of the states of the theory under $\widehat{G}$. One can always get the unrefined index by setting $\mu_i = 1\ \forall i$.

The insertion of the $(-1)^F$ operator inside the partition function allows for cancellations of bosonic and fermionic contributions at non-zero energy. Thus the Witten index gets contributions only from zero energy states of the supersymmetric theory and thus it does not depend on x.

Moreover, since this quantity is invariant under smooth supersymmetric deformations of the theory, it can be computed for any interacting theory by simply forgetting about all the interactions and setting their contributions to zero.

Let us now turn our attention to the superconformal case [32, 33].

In a SCFT the Witten index is defined as everything else in radial qunatization, thus the Hamiltonian of the system is replaced by the dilatation operator D. In radial qunatization, this operator does not commute with the supercharges (see (4.1.3)), but we can always define a new operator δ which both commutes with D and preserves the aforementioned pairing property. Then, in refining the index, we shall turn on only the fugacities for those mutually commuting operators that also commute with δ; in this way the cancellation between the supersymmetric states with $\delta \neq 0$ still holds. This implies that we will be considering only the maximal tori and Cartan generators of the superconformal group.

In the $3d\ \mathcal{N}=2$ superconformal case, selecting one of the real supercharges of the theory $\mathcal{Q}$, we can define the pairing operator as

$$\delta := \{\mathcal{Q}, \overline{\mathcal{Q}}\} = \Delta - R - j_3 \tag{4.2.4}$$

where Δ is the scaling dimension (*i.e.* the eigenvalue of the Hamiltonian D in radial quantization), R the $SO(2)_R$ R-charge and j_3 the third component of the angular momentum.

The only commuting operator with δ in $Osp(2|4)$ is thus $\Delta + j_3$.

Thus, in this setup the Witten index becomes the so-called superconformal index that reads

$$\mathcal{I} = \mathrm{Tr}\left((-1)^F x_1^{\Delta-R-j_3} x_2^{\Delta+j_3} \prod_i^{\mathrm{Rank}\,\widehat{G}} \mu_i^{F_i}\right) \tag{4.2.5}$$

where, since, by construction, only states with $\delta = 0$ contribute to the trace, the index will not depend on x_1.

It is well known that the partition function of a theory can be explicitly computed as the path integral of the same theory defined on the background curved space

S^3; the same thing happens with the superconformal index for which, however, the background is $S^2 \times S^1$ [117, 126]. Moreover, in some favourable cases, such path integral only receives contributions from particular constant field configurations and it can be reduced to a finite-dimensional ordinary integral. The technique behind this enormous simplification is called supersymmetric localization and will be the main topic of the following sections.

4.3 Supersymmetric localization

The computations we will present in the next chapters have their foundations in the localization procedure, so it is useful to review the main ideas behind this powerful technique. We will mainly follow [62], more details can be found in [140, 141].

Let us consider a theory with a fermionic symmetry generated by a Grassmann odd charge $\mathscr{Q}$. We will focus on BPS observables $\mathcal{O}_{BPS}$, *i.e.* gauge invariant $\mathscr{Q}$-closed operators which are preserved by the supercharge $\mathscr{Q}\mathcal{O}_{BPS} = 0$. Note that we are not interested in all the other properties of $\mathcal{O}_{BPS}$. In this sense it may be both a local or a non-local operator, or even a product of such operators; moreover it could also be a line operator, as a supersymmetric Wilson loop, or a surface operator and so on.

Suppose now we want to compute the expectation value of a $\mathscr{Q}$-exact observable $\mathcal{V} = \mathscr{Q}\mathcal{O}$; this however vanishes due to

$$\langle \mathcal{V} \rangle = \langle \mathscr{Q}\mathcal{O} \rangle = \int_{\mathscr{F}} [\mathcal{D}X] (\mathscr{Q}\mathcal{O}) e^{-S[X]} = \int_{\mathscr{F}} [\mathcal{D}X] \mathscr{Q}(\mathcal{O} e^{-S[X]}) = 0 \tag{4.3.1}$$

where, the first equality holds due to the fact that the action is $\mathscr{Q}$-closed, *i.e.* $\mathscr{Q}S[X] = 0$, and the latter produces integral of a total derivative in field space, which is zero provided that there are no boundary terms.

Then, any path integral with some insertions of such BPS $\mathscr{Q}$-closed observables only depends on the $\mathscr{Q}$-cohomology class $[\mathcal{O}_{BPS}]$ of the BPS operators

$$\langle \mathcal{O}_{BPS} + \mathscr{Q}\mathcal{O} \rangle = \langle \mathcal{O}_{BPS} \rangle \tag{4.3.2}$$

for any gauge invariant operator $\mathcal{O}$.

Suppose now $\mathcal{O}_{BPS} = S[X]$; such a deformation by a $\mathscr{Q}$-exact observable does not change the path integral (up to a boundary contribution). This idea is the starting point of localization and, to understand how this technique works, let us compute $\langle \mathcal{O}_{BPS} \rangle$.

Firstly we require that the path integral is well-defined. This can be easily achieved by placing our QFT on a compact manifold.

Since the expectation value $\langle \mathcal{O}_{BPS} \rangle$ only depends on the $\mathscr{Q}$-cohomology class $[\mathcal{O}_{BPS}]$, we may add to the classical action $S[X]$ the $\mathscr{Q}$-variation of a fermionic functional $\mathcal{V}_F$ without changing the quantity; in fact

$$\langle \mathcal{O}_{BPS} \rangle_{S+\mathscr{Q}\mathcal{V}_F} = \int_{\mathscr{F}} [\mathcal{D}X] \mathcal{O}_{BPS} e^{-S[X]-t\mathscr{Q}\mathcal{V}_F} = \langle \mathcal{O}_{BPS} e^{-t\mathscr{Q}\mathcal{V}_F} \rangle_S = \langle \mathcal{O}_{BPS} \rangle \quad \forall t \tag{4.3.3}$$

If we choose $\mathcal{V}_F$ in a clever way, the fact that (4.3.3) holds for any value of the parameter t can help us in evaluating $\langle \mathcal{O}_{BPS} \rangle$. We can in fact try to simplify the path integral by taking a particular limit of t. Indeed, assuming that the bosonic part of the deformation term $\mathscr{Q}\mathcal{V}_F|_B$ is positive semi-definite and $t \geq 0$, we can then evaluate $\langle \mathcal{O}_{BPS} \rangle$ by taking the limit

$$\langle \mathcal{O}_{BPS} \rangle = \lim_{t \to +\infty} \int_{\mathscr{F}} [\mathcal{D}X] \mathcal{O}_{BPS} e^{-S[X] - t\mathscr{Q}\mathcal{V}_F} \tag{4.3.4}$$

In the limit (4.3.4), the integrand is dominated by the saddle points of the so-called localising action $S_{loc}[X] = \mathscr{Q}\mathcal{V}_F$.

In a given supersimmetric QFT, there is always a canonical choice for the localizing Lagrangian density, which is

$$\mathcal{L}_{loc} = \mathscr{Q} \sum_{\{\Psi\}} ((\overline{\mathscr{Q}\Psi})\Psi + \overline{\Psi}(\overline{\mathscr{Q}\overline{\Psi}})) \tag{4.3.5}$$

where the sum runs over all the fermions of the theory $\{\Psi\}$.

Considering the bosonic and fermionic parts of $\mathcal{L}_{loc}$

$$\mathcal{L}_{loc}|_B = \sum_{\{\Psi\}} (|\mathscr{Q}\Psi|^2 + |\mathscr{Q}\overline{\Psi}|^2) \tag{4.3.6}$$

$$\mathcal{L}_{loc}|_F = \sum_{\{\Psi\}} ((\mathscr{Q}\overline{\mathscr{Q}\Psi})\Psi + \overline{\Psi}(\mathscr{Q}\overline{\mathscr{Q}\overline{\Psi}})) \tag{4.3.7}$$

it is easy to see that this localizing Lagrangian density possesses a saddle points space which is nothing but the so-called BPS locus

$$\mathscr{F}_{\mathscr{Q}} = \{X \in \mathscr{F} \mid \Psi = 0, \ \ \mathscr{Q}\Psi = 0\} \tag{4.3.8}$$

which is made of BPS configurations, where all the fermions of the theory and their $\mathscr{Q}$-variations are set to zero.

In evaluating the path integral (4.3.4), we will use a method similar to the semi-classical approximation with $\hbar = \frac{1}{t}$.

Firstly we expand all the fields X of the theory about the saddle point configurations

$$X = X_0 + \frac{1}{\sqrt{t}} \delta X \tag{4.3.9}$$

Then, by substituting (4.3.9) into the action, we get the semiclassical expansion

$$S[X] = S[X_0] + \frac{1}{2} \int \int \left. \frac{\delta^2 S_{loc}}{\delta X^2} \right|_{X = X_0} (\delta X)^2 := S[X_0] + \frac{1}{2} \int \int \delta_X^2 S[X_0] (\delta X)^2 \tag{4.3.10}$$

where all higher orders in t vanish in the $t \to +\infty$ limit because they are weighted by negative powers of t.

Finally, we can integrate out the fluctuations δX normal to the localisation locus $\mathscr{F}_{\mathscr{Q}}$; this is easily done since their contribution gives a gaussian integral.

- For fermionic fields, the integration just gives the standard determinant $\det \left[\delta_X^2 S[X_0] \big|_F \right]$ of the fermionic part of $\delta_X^2 S[X_0]$.
- For the bosonic fields, on the other hand, it can be carried out by transforming the deformations δX to diagonalize the bosonic part of $\delta_X^2 S[X_0]$. In this way,

the bosonic Gaussian path integral is reduced to an infinite product of standard Gaussian integrals which return the inverse of the determinant det $\left[\delta_X^2 S[X_0]\big|_B\right]$.

Thus we are left with the ratio of the determinants of the operators appearing at quadratic orders in the bosonic and fermionic fluctuations respectively, which is called 1-loop super-determinant

$$\mathrm{S}\det\left[\delta_X^2 S[X_0]\right] := \frac{\det\left[\delta_X^2 S[X_0]\big|_B\right]}{\det\left[\delta_X^2 S[X_0]\big|_F\right]} \tag{4.3.11}$$

Finally, putting all together, we get the so-called localisation formula

$$\langle \mathcal{O}_{BPS}\rangle = \int_{\mathscr{F}_{\mathscr{Q}}} [\mathcal{D}X_0]\; \mathcal{O}_{BPS}|_{X=X_0}\, e^{-S[X_0]} \frac{1}{\mathrm{S}\det\left[\delta_X^2 S[X_0]\right]} \tag{4.3.12}$$

Note that there is some freedom in deriving the supersymmetric localisation formula (4.3.12):

- We can use any of the multiple conserved supercharges $\{\mathcal{Q}^I\}$ to define BPS observables and perform localisation technique;
- Once we have chosen a localising supercharge $\mathcal{Q}^I = \mathscr{Q}$, we can still have the freedom to suitably choose the fermionic operator $\mathcal{V}_F$, which do not necessarily need to be the canonical one (4.3.5).

These different choices clearly affect the localisation locus $\mathscr{F}_{\mathscr{Q}}$ and, therefore, the 1-loop super-determinant. However, the result is always the same when the integration in (4.3.12) is carried out.

As anticipated in the previous section, the superconformal index (4.2.5) can be computed thanks to localization as the path integral of a theory defined on the compact curved background $S^2 \times S^1$ [117, 126]. Thus, in the following sections, we will review how to define 3d $\mathcal{N}=2$ theories on three manifolds.

4.4 3d $\mathcal{N}=2$ theories on a three manifold

So far, we only discussed 3d $\mathcal{N}=2$ theories on flat space. To make use of the localization technique, however, we need to understand how such theories behave when defined on compact curved spaces.

In general, by naively substituting flat metric and ordinary derivatives with curved metric and covariant derivatives, one does not obtain a supersymmetric theory on a curved space. Indeed there are two different strategies to define rigid supersymmetry on a curved space:

- By correcting flat space supersymmetry by trial and error;
- By non-linearly coupling the supersymmetric QFT to supergravity and then take a rigid limit which makes supergravity non-dynamical.

We will follow the latter strategy, originally outlined in [76], briefly reviewing its main ideas. Thus, to define a supersymmetric QFT on a curved space $\mathcal{M}$

1. We couple the theory to supergravity, so that the flat metric is replaced with a supergravity multiplet, typically containing a fluctuating metric $g_{\mu\nu}$, the gravitino ψ_μ and other auxiliary fields;

2. We take a rigid limit (*i.e.* $G_N \to 0$) where dynamical supergravity decouples: all the bosonic fields $\{\phi_B\}$ inside the supergravity multiplet are fixed to background values while all the fermions $\{\phi_F\}$ are set to zero.

Since, thanks to point 2., the supersymmetric variations δ_ϵ of the bosonic fields inside the supergravity multiplet automatically vanish

$$\langle \delta_\epsilon \phi_B \rangle \sim \langle \phi_F \rangle = \langle \psi_\mu \rangle = 0 \tag{4.4.1}$$

requiring such a background multiplet to be supersymmetric invariant only amounts to set

$$\langle \delta_\epsilon \phi_F \rangle = \langle \delta_\epsilon \psi_\mu \rangle = 0 \tag{4.4.2}$$

which leads to the famous generalised Killing spinor equations for the supersymmetry parameters ϵ. Since we are interested in 3d $\mathcal{N} = 2$ theories, in solving such equations, we will require that our supersymmetric field theories preserve four real supercharges and a $U(1)_R$ R-symmetry [57, 127].

Recall from Section (2.1) that, for 3d $\mathcal{N} = 2$ theories, we can consider as supercharges two real Majorana spinors $\mathcal{Q}^I_\alpha$ or, equivalently, a complex spinor $\mathcal{Q}$ and its conjugate $\overline{\mathcal{Q}}$, related by a reality condition (2.1.1). Since in the following we will work in Euclidean signature, this latter notation is more convenient, because spinors and scalars which are conjugate in Minkowskian signature are complexified and treated as independent. So, we will highlight the fact that they must actually thought as independent by a tilde, *e.g.* $\mathcal{Q}$ and $\widetilde{\mathcal{Q}}$.

Let us now consider the $4d$ $\mathcal{N} = 1$ gravity supermultiplet and dimensionally reduce it to the $3d$ $\mathcal{N} = 2$ case

Multiplet	Content	
	$d = 4$	$d = 3$
Gravity ($\mathscr{H}$)	h_{MN}	$h_{\mu\nu}$
		$h_{4\mu} := C_\mu$
		$h_{44} := \mathrm{Re}\{H\}$
	ψ_M	ψ_μ
	A_M	A^R_μ
		$A_4 := \mathrm{Im}\{H\}$

(4.4.3)

where $h_{\mu\nu}$ is the three-dimensional metric, C_μ is a gauge field often dualised into a conserved vector $V^\mu = i\epsilon^{\mu\rho\sigma}\partial_\rho C_\sigma$ and the scalar H is Hodge dual to its field strength $H = \frac{i}{2}\epsilon^{\mu\rho\sigma}\partial_\mu B_{\rho\sigma}$. Finally, A^R_μ is the R-symmetry gauge field.

In the rigid limit, the supersymmetry condition (4.4.2) can thus be explicitly written as [57]

$$\begin{aligned}
\left(\mathcal{D}_\mu - iA^R_\mu\right)\epsilon &= \left(-iV_\mu - \frac{H}{2}\gamma_\mu - \frac{1}{2}\varepsilon_{\mu\rho\sigma}V^\rho\gamma_\sigma\right)\epsilon \\
\left(\mathcal{D}_\mu + iA^R_\mu\right)\widetilde{\epsilon} &= \left(+iV_\mu - \frac{H}{2}\gamma_\mu + \frac{1}{2}\varepsilon_{\mu\rho\sigma}V^\rho\gamma_\sigma\right)\widetilde{\epsilon}
\end{aligned} \tag{4.4.4}$$

where the supersymmetry parameters ϵ and $\widetilde{\epsilon}$ becomes Killing spinors of R-charge 1 and -1 respectively, as highlighted by the different signs in front of A^R_μ. Here $\mathcal{D}_\mu$ is the linearised version of the curved covariant derivative acting on spinors

$$\mathcal{D}_\mu := \partial_\mu - \frac{i}{4}\omega_{\mu ab}\varepsilon^{abc}\gamma_c = \partial_\mu - \frac{i}{2}\varepsilon^{\lambda\rho\sigma}\partial_\lambda h_{\rho\mu}\gamma_\sigma \tag{4.4.5}$$

with spin connection $\omega_{\mu ab}$.

In the following, we will find suitable background fields $\{h_{\mu\nu}, A_\mu, V_\mu, H\}$ for a given curved manifold $\mathcal{M}$, such that there will exist at least one non-trivial solution to the Killing spinor equations (4.4.4).

4.4.1 The three-sphere S^3

Let us first consider the three-sphere S^3 with radius r. One can check that the background given by

$$A^R_\mu = V_\mu = 0 \ , \qquad H = -\frac{i}{r} \tag{4.4.6}$$

possesses 4 linearly independent killing spinors, splitting into 2 Killing spinors ϵ^I and 2 Killing spinors $\widetilde{\epsilon}^I$ solving

$$\mathcal{D}_\mu\epsilon^I = \frac{i}{2r}\gamma_\mu\epsilon^I \tag{4.4.7}$$

$$\mathcal{D}_\mu\widetilde{\epsilon}^I = \frac{i}{2r}\gamma_\mu\widetilde{\epsilon}^I \tag{4.4.8}$$

Since, thus, every supersymmetry parameters can be identified with a different killing spinor, the round S^3 is a maximally supersymmetric background for a $3d$ $\mathcal{N} = 2$ supersymmetric theory.

Viewing S^3 as the $SU(2)$ group manifold, we can choose a frame with vielbein such that

$$\mathcal{D}_\mu = \partial_\mu + \frac{i}{2r}\gamma_\mu \quad \to \quad \partial_\mu\epsilon^I = \partial_\mu\widetilde{\epsilon}^I = 0 \tag{4.4.9}$$

so that the Killing spinor equations are solved by constant spinors in this particular frame.

The supersymmetry transformations $\delta = \delta_\epsilon + \delta_{\widetilde{\epsilon}}$ of a vector multiplet on S^3 are [57, 159]

$$\begin{aligned}
\delta A_\mu &= -i(\epsilon\gamma^\mu\overline{\lambda} + \widetilde{\epsilon}\gamma^\mu\lambda) \\
\delta\sigma &= -\epsilon\overline{\lambda} + \widetilde{\epsilon}\lambda \\
\delta\lambda &= \left(i(D + \frac{\sigma}{r}) - \frac{i}{2}\varepsilon^{\rho\sigma\lambda}F_{\rho\sigma}\gamma_\lambda - i(\mathcal{D}_\mu\sigma)\gamma^\mu\right)\epsilon \\
\delta\overline{\lambda} &= \widetilde{\epsilon}\left(-i(D + \frac{\sigma}{r}) - \frac{i}{2}\varepsilon^{\rho\sigma\lambda}F_{\rho\sigma}\gamma_\lambda + i(\mathcal{D}_\mu\sigma)\gamma^\mu\right) \\
\delta D &= i\mathcal{D}_\mu(\epsilon\gamma^\mu\overline{\lambda} - \widetilde{\epsilon}\gamma^\mu\lambda) + \frac{1}{r}(\epsilon\overline{\lambda} - \widetilde{\epsilon}\lambda) + i(\epsilon[\sigma, \overline{\lambda}] - \widetilde{\epsilon}[\sigma, \lambda])
\end{aligned} \tag{4.4.10}$$

where $F_{\rho\sigma}$ is the field strength of A_μ and $\mathcal{D}_\mu = \nabla_\mu - iA_\mu$ is the curved gauge covariant derivative.

On the other hand, the supersymmetry transformations of the charged chiral multiplet are

$$
\begin{aligned}
\delta\phi &= \sqrt{2}\epsilon\psi \\
\delta\overline{\phi} &= \sqrt{2}\widetilde{\epsilon}\overline{\psi} \\
\delta\psi &= \sqrt{2}\epsilon F + \sqrt{2}i(\sigma - i\frac{R}{r})\widetilde{\epsilon}\phi - \sqrt{2}i\widetilde{\epsilon}\gamma^\mu\mathcal{D}_\mu\phi \\
\delta\overline{\psi} &= \sqrt{2}\widetilde{\epsilon}\overline{F} - \sqrt{2}i(\sigma + i\frac{R}{r})\epsilon\overline{\phi} + \sqrt{2}i\gamma^\mu\epsilon\mathcal{D}_\mu\overline{\phi} \\
\delta F &= -\sqrt{2}i(\sigma - i\frac{R-2}{r})\widetilde{\epsilon}\psi + 2i\epsilon\lambda\overline{\phi} - \sqrt{2}i\mathcal{D}_\mu(\widetilde{\epsilon}\gamma^\mu\psi) \\
\delta\overline{F} &= \sqrt{2}i(\sigma + i\frac{R-2}{r})\epsilon\overline{\psi} - 2i\widetilde{\epsilon}\overline{\lambda}\phi + \sqrt{2}i\mathcal{D}_\mu(\overline{\psi}\gamma^\mu\epsilon)
\end{aligned}
\tag{4.4.11}
$$

where R is the R-charge of the multiplet.

The most generic Lagrangian (2.1.5) for such a background gets modified as follows:

1. The original super Yang-Mills Lagrangian (2.1.6), becomes

$$
\begin{aligned}
\mathcal{L}_{\text{SYM}} = \operatorname{Tr}\bigg\{ &\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\mathcal{D}_\mu\sigma\mathcal{D}^\mu\sigma - i\overline{\lambda}\gamma^\mu\mathcal{D}_\mu\lambda + \\
&-i\overline{\lambda}[\sigma,\lambda] + \frac{1}{2}(D + \frac{\sigma}{r})^2 + \frac{\lambda\overline{\lambda}}{2r}\bigg\}
\end{aligned}
\tag{4.4.12}
$$

   where now $\mathcal{D}_\mu$ is the curved gauge covariant derivative.

2. The supersymmetric Chern–Simons term $\mathcal{L}_{\text{SCS}}$ does not get modified by the curved background and happens to be equal to the flat one (2.1.7).

3. The flat Fayet-Iliopulos Lagrangian (2.1.8), becomes

$$
\mathscr{L}_{\text{FI}} = i\sum_A \xi^A\left(D - \frac{\sigma}{r}\right)_A \tag{4.4.13}
$$

   where $A = 1, \ldots, \operatorname{Rank} G$ labels the abelian $U(1)$ factors inside G

4. Finally, the matter Lagrangian (2.1.9) gets modified as follows

$$
\begin{aligned}
\mathcal{L}_{\text{matter}} = &\overline{\mathcal{D}_\mu\phi}\mathcal{D}^\mu\phi - i\psi\gamma^\mu\mathcal{D}_\mu\overline{\psi} - \overline{F}F + \\
&+ i\sqrt{2}\left(\overline{\phi}\lambda\psi + \overline{\psi\lambda}\phi\right) - i\overline{\psi}\left(\sigma - i\frac{2R-1}{2r}\right)\psi + \\
&+ \overline{\phi}\left(D + \sigma^2 - i\frac{2R-1}{r}\sigma - R\frac{R-2}{r^2}\right)\phi
\end{aligned}
\tag{4.4.14}
$$

   where R is the R-charge of the chiral multiplet. For the sake of simplicity we suppressed the flavour indices i, j, turned off the superpotential $\mathcal{W} = 0$ and redefined the vector multiplet fields to reabsorb the coupling constant g.

Since both Lagrangians $\mathcal{L}_{\text{SYM}}$ (4.4.12) and $\mathcal{L}_{\text{matter}}$ (4.4.14) with no superpotential are $\mathcal{Q}$- and $\widetilde{\mathcal{Q}}$-exact expressions, namely

$$
\epsilon\widetilde{\epsilon}\mathcal{L}_{\text{SYM}} = \delta_\epsilon\delta_{\widetilde{\epsilon}}\operatorname{Tr}\left\{\overline{\lambda}\lambda - 2D\sigma\right\} \tag{4.4.15}
$$

$$\epsilon\widetilde{\epsilon}\mathcal{L}_{\text{matter}} = \delta_\epsilon\delta_{\widetilde{\epsilon}}\left(\overline{\psi}\psi - 2i\overline{\phi}\sigma\phi + 2\frac{R-1}{r}\overline{\phi}\phi\right) \tag{4.4.16}$$

the most common choice for the localising supercharge is $\mathscr{Q} = \mathcal{Q} + \widetilde{\mathcal{Q}}$. In fact, since our localising action must be the $\mathscr{Q}$-variation of a $\mathscr{Q}$-exact observable, both $\mathcal{L}_{\text{SYM}}$ and $\mathcal{L}_{\text{matter}}$ can act as localising Lagrangians.

Alternatively one can always make the canonical choice (4.3.5), leading again to $\mathcal{L}_{\text{SYM}}$ (4.4.12) for the vector multiplet and to

$$\epsilon\widetilde{\epsilon}\mathcal{L}_{loc} = \delta_\epsilon\delta_{\widetilde{\epsilon}}\left(\overline{\psi}\psi - 2i\overline{\phi}\sigma\phi\right) \tag{4.4.17}$$

for the chiral multiplet.

As we have seen, the saddle points of these actions must coincide with the BPS configurations (4.3.8) for the localising supercharge $\mathscr{Q}$ [123].

For the vector multiplet this reads

$$\lambda = 0 \tag{4.4.18}$$

$$\delta\lambda = \left(i(D+\frac{\sigma}{r}) - \frac{i}{2}\varepsilon^{\mu\nu\rho}F_{\mu\nu}\gamma^\rho - i\mathcal{D}_\mu\sigma\gamma^\mu\right)\epsilon = 0 \tag{4.4.19}$$

The vanishing of $\delta\lambda$ thus requires

$$\frac{1}{2}\varepsilon_{\mu\nu\rho}F^{\mu\nu} = -\mathcal{D}_\rho\sigma \tag{4.4.20}$$

$$D = -\frac{\sigma}{r} \tag{4.4.21}$$

By acting on the first equation with a covariant derivative and using the Bianchi identity (2.1.11) of the field strength $F_{\mu\nu}$, we get

$$\varepsilon^{\mu\nu\rho}\mathcal{D}_\rho F_{\mu\nu} = 0 = -\mathcal{D}_\rho(\mathcal{D}^\rho\sigma) \tag{4.4.22}$$

This implies that the scalar σ must be covariantly constant, namely

$$D_\rho\sigma = 0 \tag{4.4.23}$$

and thus, by means of (4.4.20), that $F_{\mu\nu} = 0$.

We may then pick a gauge in which $A_\mu = 0$, so that the gauge covariant derivative $\mathcal{D}_\mu$ becomes just the standard curved derivative ∇_μ. In this way, (4.4.23) is satisfied when $\sigma = \sigma_0$, with σ_0 a fixed element in the Lie-algebra $\mathfrak{g}$ of the gauge group G.

Hence (4.4.21) implies

$$D = -\frac{\sigma_0}{r} \tag{4.4.24}$$

On the other hand, for the chiral multiplets we have

$$\psi = \overline{\psi} = 0 \tag{4.4.25}$$

$$\delta\psi = \epsilon F \tag{4.4.26}$$

$$\delta\overline{\psi} = -\sqrt{2}i(\sigma_0 + i\frac{R}{r})\epsilon\overline{\phi} + \sqrt{2}i\gamma^\mu\epsilon\mathcal{D}_\mu\overline{\phi} = 0 \tag{4.4.27}$$

The vanishing of $\delta_\epsilon\psi$ and $\delta_\epsilon\overline{\psi}$ thus require

$$F = 0 \tag{4.4.28}$$
$$-\sqrt{2}i(\sigma_0 + i\frac{R}{r})\overline{\phi} + \sqrt{2}i\gamma^\mu\mathcal{D}_\mu\overline{\phi} = 0 \tag{4.4.29}$$

With the same reasoning as for the vector multiplet (but a little more work), the solution of (4.4.29) requires simply $\phi = 0$.

In the end, the saddle points of the localising action (2.1.5) involve vanishing chiral multiplets and are determined by a constant background for the real scalar σ in the vector multiplet. Since all matter fields vanish, the precise choice of the superpotential $\mathcal{W}$ does not matter, provided it ensures superconformal invariance on the quantum level.

4.4.2 $S^2 \times S^1$ curved background

Since we are interested in evaluating the index (4.2.5), we will now consider the curved background $S^2 \times S^1$. As we will see, this is the correct background for evaluating the index with the localizing procedure. In the following we will adopt the same notations of [117]; thus, let r and βr be the radius of S^2 and the period of S^1, respectively. According to the case, we will use coordinates x^a with $a = 1, 2$ for S^2 and x^3 for S^1 separately or x^μ with $\mu = 1, \ldots, 3$ when referring to $S^2 \times S^1$ collectively.

Luckily most of the results of the previous section for the S^3 background still hold with minor modifications. Indeed, before considering the compact space $S^2 \times S^1$, let us consider $S^2 \times R$ insted. We will compactify the "time" direction R to S^1 later on.

The four linearly independent Killing spinors on this non-compact background split into two couples satisfying

$$\mathcal{D}_\mu\epsilon^I = -\frac{1}{2r}\gamma_a\gamma_3\epsilon^I \tag{4.4.30}$$
$$\mathcal{D}_\mu\widetilde{\epsilon}^I = \frac{1}{2r}\gamma_a\gamma_3\widetilde{\epsilon}^I \tag{4.4.31}$$

if for S^3 we could use any holomorphic or anti-holomorphic supersymmetry as localizing supercharge $\mathscr{Q}$, in computing the index (4.2.5) we should use only the anti-holomorphic ones. This is due to the fact that the index encodes the spectrum of BPS operators $\mathcal{O}_{BPS}$ for which $\overline{\mathcal{Q}}\mathcal{O}_{BPS} = 0$ only.

In this case, the localizing supercharge $\mathscr{Q}$ should then be a component of $\overline{\mathcal{Q}}$. With such a choice we only preserve the $\widetilde{\epsilon}$ supersymmetry parameter, thus letting $\epsilon = 0$.

If we now consider two linearly independent Killing spinors $\widetilde{\epsilon}^1$ and $\widetilde{\epsilon}^2$, they form a doublet of the $SO(3)$ isometry of S^2 and thus we can assume they have j_3 eigenvalues $\pm\frac{1}{2}$, respectively. Our localising supercharge $\mathscr{Q}$ will be the one associated to $\widetilde{\epsilon}^1$.

In compactifying the "time" direction R, since the Killing equation (4.4.31) implies

$$\widetilde{\epsilon}^1(x^3) \propto e^{\frac{x^3}{2r}} \tag{4.4.32}$$

the $\widetilde{\epsilon}^1$ supersymmetry parameter cannot satisfy the standard periodic boundary condition along S^1. Instead, it satisfies

$$\widetilde{\epsilon}^1(x^3 + \beta r) = e^{\frac{\beta}{2}}\widetilde{\epsilon}^1(x^3) \tag{4.4.33}$$

where the extra factor $e^{\frac{\beta}{2}}$ represents the insertion of a twist operator.

By using the Killing spinor quantum numbers

$$R(\tilde{\epsilon}^1) = -1 \ , \quad j_3(\tilde{\epsilon}^1) = \frac{1}{2} \ , \quad F_i(\tilde{\epsilon}^1) = 0 \tag{4.4.34}$$

we can rewrite the twisted periodicity condition (4.4.33) as

$$\tilde{\epsilon}^1(x^3+\beta r) = e^{-\beta_1(R+j_3)} e^{\beta_2 j_3} e^{\alpha_i F_i} \tilde{\epsilon}^1(x^3) \tag{4.4.35}$$

where β_1, β_2 , and α_i are real parameters satisfying $\beta = \beta_1 + \beta_2$.

For consistency, the same boundary condition (4.4.33) should be imposed on all the fields $\{X\}$ of the theory

$$X(x^3+\beta r) = e^{-\beta_1(R+j_3)} e^{\beta_2 j_3} e^{\alpha_i F_i} X(x^3) \tag{4.4.36}$$

Then it is easy to see that the path integral over $S^2 \times S^1$ with exactly this twisted boundary condition gives the index (4.2.5) when we define

$$x_1 := e^{-\beta_1} \ , \quad x_2 := e^{-\beta_2} \ , \quad \mu_i := e^{-\alpha_i} \tag{4.4.37}$$

The standard choice for the localizing Lagrangians on such a background with such Killing spinors are

$$\epsilon\tilde{\epsilon}\mathcal{L}^{\text{loc}}_{\text{SYM}} = \delta_\epsilon \delta_{\tilde{\epsilon}} \operatorname{Tr}\left\{-\frac{1}{2}\overline{\lambda}\lambda\right\} \tag{4.4.38}$$

$$\epsilon\tilde{\epsilon}\mathcal{L}^{\text{loc}}_{\text{matter}} = \delta_\epsilon \delta_{\tilde{\epsilon}} \left(-\frac{i}{2}\overline{\phi}F\right) \tag{4.4.39}$$

which explicitly give

$$\begin{aligned}
\epsilon\tilde{\epsilon}\mathcal{L}^{\text{loc}}_{\text{SYM}} &= \frac{1}{2}F^{\mu\nu}F_{\mu\nu} - \varepsilon_{\mu\rho\sigma}F^{\rho\sigma}\mathcal{D}_\mu\sigma + \mathcal{D}_\mu\sigma\mathcal{D}^\mu\sigma + \frac{1}{r^2}\sigma^2 + \\
&+ \frac{2\delta^{\mu,3}}{r}\sigma(\epsilon_{\mu\rho\sigma}F^{\rho\sigma} - \mathcal{D}_\mu\sigma) + D^2 - 2\overline{\lambda}\gamma^\mu\mathcal{D}_\mu\lambda - 2\overline{\lambda}[\sigma,\lambda] - \frac{1}{r}\overline{\lambda}\gamma^3\lambda \qquad (4.4.40)\\
\epsilon\tilde{\epsilon}\mathcal{L}^{\text{loc}}_{\text{matter}} &= -\overline{\phi}\mathcal{D}_\mu\mathcal{D}^\mu\phi + \overline{\phi}\sigma^2\phi + i\overline{\phi}D\phi + \\
&- \overline{\psi}\gamma^\mu\mathcal{D}_\mu\psi - \overline{\psi}\sigma\psi - \sqrt{2}(\overline{\psi}\overline{\lambda})\phi - \sqrt{2}\overline{\phi}(\lambda\psi) + \overline{F}F + \\
&+ \frac{1-2R}{r}(\overline{\phi}\mathcal{D}_3\phi + \frac{1}{2}\overline{\psi}\gamma^3\psi) + R\frac{1-R}{r^2}\overline{\phi}\phi \qquad (4.4.41)
\end{aligned}$$

If we define a vector V_μ as

$$V_\mu = \frac{1}{2}\varepsilon_{\mu\rho\sigma}F^{\rho\sigma} - \mathcal{D}_\mu\sigma - \frac{\delta_{\mu,3}}{r}\sigma \tag{4.4.42}$$

then the localizing Lagrangian for the vector multiplet (4.4.40) heavily simplifies to

$$\epsilon\tilde{\epsilon}\mathcal{L}^{\text{loc}}_{\text{SYM}} = V_\mu V^\mu + D^2 - 2\overline{\lambda}\gamma^\mu\mathcal{D}_\mu\lambda - 2\overline{\lambda}[\sigma,\lambda] - \frac{1}{r}\overline{\lambda}\gamma^3\lambda \tag{4.4.43}$$

Repeating the same reasoning of the previous section for S^3, one can find that the path integral localizes around $V_\mu = 0$ which implies the following monopole background solutions

$$A^0_\mu dx^\mu = \frac{l}{\beta r} dx^3 + m B_a dx^a \ , \quad \sigma_0 = \frac{m}{2r} := \frac{s}{r} \tag{4.4.44}$$

with all the other fields vanishing. Here, after an appropriate change of variables, the Wilson line l around S^1, the magnetic charge m of the Dirac monopole B_i and $s := \frac{m}{2}$ all take values in the Cartan part of the Lie algebra $\mathfrak{g}$ of the gauge group G.

4.5 The superconformal index

As we have already anticipated, the index (4.2.5) is just the $S^2 \times S^1$ localized path integral

$$\mathcal{I} = \int_{\mathscr{F}_Q} [\mathcal{D}X_0] e^{-S[X_0]} \frac{1}{\operatorname{S}\det\left[\frac{\delta^2 S_{loc}[X_0]}{\delta X_0^2}\right]} \tag{4.5.1}$$

where $\mathscr{F}_Q$ is the BPS locus (4.3.8), $\{X_0\}$ are the values of the fields at the saddle points configurations (4.4.44) and $\operatorname{S}\det$ stands for the super-determinant (4.3.11) of the localising action constructed form the Lagrangians (4.4.42) and (4.4.41).

Using the results of the previous section, the above formula becomes

$$\mathcal{I} = \sum_{\{m_a\}} \int \left(\prod_a^{\operatorname{Rank} G} dl_a\right) \det J e^{-S[l_a, m_a]} \frac{\det \Delta_{loc}[l_a, m_a]|_F}{\det \Delta_{loc}[l_a, m_a]|_B} \tag{4.5.2}$$

where J is the Jacobian of the appropriate variables change for the Wilson line l and the monopole magnetic charge m of (4.4.44) to lie in the Cartan of the gauge group G. Here, for simplicity, we defined $\Delta_{loc} := \frac{\delta^2 S_{loc}[X_0]}{\delta X_0^2}$.

The easiest way to compute the super-determinants of these operators Δ_{loc} is diagonalize them and then take the product of all their possible eigenvalues [117, 126]. In this section we will see how to do so.

4.5.1 Chiral multiplet contribution

To obtain the chiral multiplet contribution to the Δ_{loc} operator in (4.5.2), we need to substitute the field expansions about the saddle points (4.3.9) in the localizing Lagrangian (4.4.41). Considering the quadratic terms only, that are the ones that contribute to Δ_{loc}, we get for the chiral multiplet

$$\begin{aligned}\epsilon\widetilde{\epsilon}\mathcal{L}^{\text{loc}}_{\text{matter}} \sim & -\overline{\phi}\mathcal{D}^\mu\mathcal{D}_\mu\phi + \frac{1}{r^2}\overline{\phi}s^2\phi + R\frac{1-R}{r^2}\overline{\phi}\phi + \frac{1-2R}{r}\overline{\phi}\mathcal{D}_3\phi + \\ & + \frac{1}{r}\overline{\psi}s\psi - \overline{\psi}\gamma^\mu\mathcal{D}_\mu\psi + \frac{1-2R}{2r}\overline{\psi}\gamma^3\psi + \overline{F}F\end{aligned} \tag{4.5.3}$$

Integrating out the auxiliary field F, we get a constant factor that can simply be dropped.

The scalar field ϕ contribution Let us consider the scalar field ϕ first; performing the derivatives, we get the operator

$$\Delta^\phi_{loc} = -\mathcal{D}_3\mathcal{D}_3 - \mathcal{D}_i\mathcal{D}_i + \frac{s^2}{r^2} + R\frac{1-R}{r^2} + \frac{1-2R}{r}\mathcal{D}_3 \tag{4.5.4}$$

Now we need to diagonalize Δ^ϕ_{loc}.

First of all, whenever a $\mathcal{D}_3$ operator appears inside a Δ_{loc} operator, it should be understood as its eigenvalue. By taking the twisted boundary condition (4.4.36) into account and the form of the gauge field (4.4.44) along S^1, the eigenvalues of $\mathcal{D}_3$ are thus given by

$$\mathcal{D}_3 = \frac{1}{\beta r}\Big(2\pi i n - i\rho(l) - (R+j_3)\beta_1 + j_3\beta_2 + \widetilde{\rho}(\alpha)\Big) \quad \text{with} \quad n \in \mathbb{Z} \tag{4.5.5}$$

where ρ and $\widetilde{\rho}$ are respectively the weight vectors of the gauge and flavour representations $(\mathcal{R}, \widetilde{\mathcal{R}})$ of the field and we made use of the same notation of (2.2.44) (see Section (2.2.4)).

The hard part of diagonalizing the Δ_{loc} operators thus comes from the gauge covariant derivative $\mathcal{D}_i$ and its combinations. However, on the S^2 space, the gauge covariant derivative acting on a generic field X becomes

$$\mathcal{D}_i = \nabla_i^{(S^2)} - i\rho(m)B_i = \partial_i - i\sigma\omega_i - i\rho(m)B_i \tag{4.5.6}$$

where $\nabla_i^{(S^2)}$ is the curved covariant derivative on S^2 with spin connection ω_i and σ its spin.

Thanks to the fact that on S^2 the Dirac monopole B_i is related to the spin connection ω_i by

$$B_i = \frac{1}{2}\omega_i \tag{4.5.7}$$

we can rewrite the covariant derivative as

$$\mathcal{D}_i = \nabla_i^{\text{eff}} := \partial_i - i\sigma_{\text{eff}}\omega_i \quad \text{with} \quad \sigma_{\text{eff}} = \sigma + \frac{1}{2}\rho(m) = \sigma + \rho(s) \tag{4.5.8}$$

We have thus transformed the monopole contribution $\rho(m)$ into a spin shifting $\rho(s)$. In this way we get an effective curved covariant derivative ∇_i^{eff} without a gauge component, but with modified effective spin σ_{eff} induced by the monopole background.

The diagonalization process of such effective curved covariant derivative ∇_i^{eff} and its combinations is well known in the literature and can be performed thanks to harmonic expansion [116]. Indeed, on S^2 we can always expand a generic field X with spin σ into the so-called generalised spin σ_{eff} harmonics $\Omega^{\sigma_{\text{eff}}}$ whose exact form and properties depend on the original spin σ of the field [167]. These, by construction, are in fact the eigenfunctions of the combinations of ∇_i^{eff} appearing in the kinetic terms of the Lagrangian on the S^2 background.

For example, the scalar field ϕ with $\sigma = 0$ can be expanded using the generalised spin σ_{eff} *spherical* harmonics

$$Y_{j,j_3}^{\sigma_{\text{eff}}}(\theta,\phi)\ , \quad j \geq |\sigma_{\text{eff}}|\ , \quad -j \leq j_3 \leq j \tag{4.5.9}$$

where (θ, ϕ) are the standard angular coordinates of S^2.

These happen to be the eigenfunctions of the $\nabla_i^{\text{eff}}\nabla_{\text{eff}}^i$ operator in (4.5.4), for which

$$\nabla_i^{\text{eff}}\nabla_{\text{eff}}^i Y_{j,j_3}^{\sigma_{\text{eff}}} = -\frac{1}{r^2}\Big(j(j+1) - \sigma_{\text{eff}}^2\Big)Y_{j,j_3}^{\sigma_{\text{eff}}} \tag{4.5.10}$$

Thus the operator Δ_{loc}^{ϕ} in (4.5.4) diagonalizes as

$$\Delta_{loc}^{\phi} = \frac{1}{r^2}(j + R + r\mathcal{D}_3)(j + 1 - R - r\mathcal{D}_3) \tag{4.5.11}$$

where $\mathcal{D}_3$ should be understood to be its eigenvalue (4.5.5).

Taking the product of all the possible eigenvalues (4.5.11), we obtain the scalar field contribution to the super-determinat

$$\det \Delta^{\phi}_{loc} = \prod_{\rho\in\mathcal{R}} \prod_{j=|\rho(s)|}^{\infty} \prod_{j_3=-j}^{j} \prod_{n=-\infty}^{\infty} (j+R+r\mathcal{D}_3)(j+1-R-r\mathcal{D}_3) \tag{4.5.12}$$

The fermionic field ψ contribution Next, we need to consider the fermionic field ψ contribution to Δ_{loc}, that is the matrix operator

$$\Delta^{\psi}_{loc} = \gamma^{\mu}\mathcal{D}_{\mu} - \frac{1-2R}{2r}\gamma_3 + \frac{s}{r} = \begin{pmatrix} \mathcal{D}_3 - \frac{1-2R}{2r} + \frac{s}{r} & \mathcal{D}_+ \\ \mathcal{D}_- & -\mathcal{D}_3 + \frac{1-2R}{2r} + \frac{s}{r} \end{pmatrix} \tag{4.5.13}$$

where we defined $\mathcal{D}_{\pm} = \mathcal{D}_1 \pm i\mathcal{D}_2$.

This can again be diagonalised by expanding the fermionic field ψ into generalised spin σ_{eff} *monopole* harmonics $\Psi^{\sigma_{\text{eff}}}_{j,j_3}$. Since the upper and lower components of the spinor ψ have the effective spins $\sigma_{\text{eff}} = \rho(s) \mp \frac{1}{2}$ respectively, we get

$$\Psi^{\rho(s)-\frac{1}{2}}_{j,j_3}(\theta,\phi) \propto \begin{pmatrix} Y^{\rho(s)-\frac{1}{2}}_{j,j_3}(\theta,\phi) \\ -Y^{\rho(s)+\frac{1}{2}}_{j,j_3}(\theta,\phi) \end{pmatrix} \tag{4.5.14}$$

where we made use of the generalised spin σ_{eff} spherical harmonics in(4.5.9).

Indeed, these are eigenfunctions for the Δ^{ψ}_{loc} matrix operator (4.5.13) which diagonalizes as follows

$$\Delta^{\psi}_{loc}\Psi^{\rho(s)-\frac{1}{2}}_{j,j_3} = \begin{pmatrix} \frac{1}{r}\left(j+\frac{1}{2}\right) + \mathcal{D}_3 - \frac{1-2R}{2r} & 0 \\ 0 & -\left[-\frac{1}{r}\left(j+\frac{1}{2}\right) + \mathcal{D}_3 - \frac{1-2R}{2r}\right] \end{pmatrix} \Psi^{\rho(s)-\frac{1}{2}}_{j,j_3} \tag{4.5.15}$$

where the factor $\frac{1}{r}\left(j+\frac{1}{2}\right)$ comes from the eigenvalue of the S^2 operator $\gamma^i\mathcal{D}_i + \frac{s}{r}$ for which $\Psi^{\sigma_{\text{eff}}}_{j,j_3}$ are eigenfunctions.

Thus to obtain the determinant of Δ^{ψ}_{loc} we firstly have to take its matrix determinant. Here we will denote the matrix determinant as $\det_\star$ to distinguish it from the determinant of the differential operator. The result, however, differs according to the value of the total angular momentum j.

- If $j \geq |\rho(s)| + \frac{1}{2}$, both the $Y^{\rho(s)\mp\frac{1}{2}}_{j,j_3}$ components of $\Psi^{\sigma_{\text{eff}}}_{j,j_3}$ exist and thus the matrix determinant is

$$\det_{\star} \Delta_{\psi} = \frac{1}{r^2}(j+R+r\mathcal{D}_3)(j+1-R-r\mathcal{D}_3) \tag{4.5.16}$$

- In the extremal case $j = |\rho(s)| - \frac{1}{2}$, only one of the $Y^{\rho(s)\mp\frac{1}{2}}_{j,j_3}$ components exists, according to the value of $\rho(s) \lessgtr 0$. In this case only one eigenvalue remains, leading to

$$\det_{\star} \Delta_{\psi} = \frac{1}{r}(j+R+r\mathcal{D}_3) \tag{4.5.17}$$

Combining together these results, the full operator determinant reads

$$\det \Delta_\psi = \frac{1}{r^2} \prod_{\rho \in \mathcal{R}} \left\{ \prod_{j=|\rho(s)|-\frac{1}{2}}^{\infty} \prod_{j_3=-j}^{j} \prod_{n=-\infty}^{\infty} (j + R + r\mathcal{D}_3) \times \right. \\ \left. \times \prod_{j=|\rho(s)|+\frac{1}{2}}^{\infty} \prod_{j_3=-j}^{j} \prod_{n=-\infty}^{\infty} (j + 1 - R - r\mathcal{D}_3) \right\} \tag{4.5.18}$$

where $\mathcal{D}_3$ should be understood to be its eigenvalue (4.5.5).

Luckily, looking at the two determinants (4.5.12) and (4.5.2), one can see that the two factors are exactly the same. Thus, in order to find the closed expressions for such determinants, we just need to look at the factors once.

Let us start with the factor $(j + R + r\mathcal{D}_3)$. The explicit form of this eigenvalue is

$$\beta(j+R+r\mathcal{D}_3) = 2\pi i n - i\rho(l) + (j - j_3)\beta_1 + (j+R+j_3)\beta_2 + \widetilde{\rho}(\alpha) := 2\pi i n + z \tag{4.5.19}$$

where we have defined the following recurring quantity

$$z := -i\rho(l) + (j - j_3)\beta_1 + (j + R + j_3)\beta_2 + \widetilde{\rho}(\alpha) \tag{4.5.20}$$

Carrying out the product over n first, one obtains

$$2 \prod_{n=-\infty}^{\infty} (\pi i n + \frac{z}{2}) = 2\sinh\frac{z}{2} = e^{\frac{z}{2}}(1 - e^{-z}) = e^{\frac{z}{2}} \exp\left\{ -\sum_{n=1}^{\infty} \frac{1}{n} e^{-nz} \right\} \tag{4.5.21}$$

All the other products in the two determinants (4.5.12) and (4.5.2), can now be turned into a summation over the exponents. Indeed, if we commonly recast the expressions for ϕ and ψ as

$$\prod_{\dots} \prod_{n=-\infty}^{\infty} (2\pi i n + z)^{(-1)^{F+1}} \tag{4.5.22}$$

where the first product represents all the other products apart from the one with respect to n and F is the fermionic number of the respective field; then we get

$$\prod_{\dots} \prod_{n=-\infty}^{\infty} (2\pi i n + z)^{(-1)^{F+1}} = \exp\left\{ -\sum_{\dots} (-1)^F \frac{z}{2} \right\} \exp\left\{ \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\dots} (-1)^F e^{-nz} \right\} \tag{4.5.23}$$

The second term in (4.5.23) is nothing but the plethystic exponent (2.2.36) of the function

$$f(e^{il_a}, x_1, x_2, \mu_i) := \sum_{\dots} (-1)^F e^{-z} \quad \text{where} \quad e^{-z} = e^{i\rho(l)} x_1^{j-j_3} x_2^{j+R+j_3} \mu^{\widetilde{\rho}} \tag{4.5.24}$$

which is called "letter index" from the fact that it represents the index of an elementary excitation, also known in the literature as a letter. Here again we made use of the notations of (2.2.44).

For a moment let us just forget about the first term in (4.5.23), we will cover it later. Thus, the letter indices for ϕ and ψ are

$$f_\phi(e^{il_a}, x_1, x_2, \mu_i) = \sum_{\rho\in\mathcal{R}} e^{i\rho(l)} x_2^R \mu^{\tilde\rho} \sum_{j=|\rho(s)|}^{\infty} \sum_{j_3=-j}^{j} (x_1x_2)^j \left(\frac{x_2}{x_1}\right)^{j_3} \tag{4.5.25}$$

$$f_\psi(e^{il_a}, x_1, x_2, \mu_i) = -\sum_{\rho\in\mathcal{R}} e^{i\rho(l)} x_2^R \mu^{\tilde\rho} \sum_{j=|\rho(s)|}^{\infty} \sum_{j_3=-j}^{j-1} (x_1x_2)^j \left(\frac{x_2}{x_1}\right)^{j_3} \tag{4.5.26}$$

$$\tag{4.5.27}$$

so that the total index reads

$$f_{\phi+\psi}(e^{il_a}, x_1, x_2, \mu_i) = \sum_{\rho\in\mathcal{R}} e^{i\rho(l)} x_2^R \mu^{\tilde\rho} \left[x_2^{2|\rho(s)|} \sum_{j=0}^{\infty} x_2^{2j}\right] = \sum_{\rho\in\mathcal{R}} e^{i\rho(l)} \mu^{\tilde\rho} \frac{x_2^{2|\rho(s)|+R}}{1-x_2^2} \tag{4.5.28}$$

since every term in the sum over the angular momenta j, j_3 of $f_\psi(e^{il_a}, x_1, x_2, \mu_i)$ cancels the respective one in the same sum of $f_\phi(e^{il_a}, x_1, x_2, \mu_i)$ apart from the term with $j_3 = j$.

Repeating exactly the same reasoning for the other eigenvalue $(j + 1 - R - r\mathcal{D}_3)$ in the two determinants (4.5.12) and (4.5.2), one can find that the letter indices read

$$f_\phi(e^{il_a}, x_1, x_2, \mu_i) = \sum_{\rho\in\mathcal{R}} e^{-i\rho(l)} x_2^{-R} \mu^{-\tilde\rho} \sum_{j=|\rho(s)|}^{\infty} \sum_{j_3=-j}^{j} (x_1x_2)^{j+1} \left(\frac{x_1}{x_2}\right)^{j_3} \tag{4.5.29}$$

$$f_\psi(e^{il_a}, x_1, x_2, \mu_i) = -\sum_{\rho\in\mathcal{R}} e^{-i\rho(l)} x_2^{-R} \mu^{-\tilde\rho} \sum_{j=|\rho(s)|}^{\infty} \sum_{j_3=-j-1}^{j} (x_1x_2)^{j+1} \left(\frac{x_1}{x_2}\right)^{j_3} \tag{4.5.30}$$

$$\tag{4.5.31}$$

so that the total index is

$$\begin{aligned} f_{\phi+\psi}(e^{il_a}, x_1, x_2, \mu_i) &= -\sum_{\rho\in\mathcal{R}} e^{-i\rho(l)} x_2^{-R} \mu^{-\tilde\rho} \left[x_2^{2|\rho(s)|} \sum_{j=0}^{\infty} x_2^{2(j+1)}\right] = \\ &= -\sum_{\rho\in\mathcal{R}} e^{-i\rho(l)} \mu^{-\tilde\rho} \frac{x_2^{2|\rho(s)|-2-R}}{1-x_2^2} \end{aligned} \tag{4.5.32}$$

since now every term in the sum over the angular momenta j, j_3 of $f_\psi(e^{il_a}, x_1, x_2, \mu_i)$ cancels the respective one in the sum of $f_\phi(e^{il_a}, x_1, x_2, \mu_i)$ and only the term with $j_3 = -j - 1$ remains.

By summing the two contributions (4.5.28) and (4.5.32) we obtain the letter index for one chiral multiplet Φ; then, the letter index accounting for every chiral multiplet $\{\Phi\}$ in the theory is just the sum

$$f_\Phi(e^{il}, x_2, \mu_i) = \sum_{\{\Phi\}} \sum_{\rho \in \mathcal{R}_\Phi} \left[e^{i\rho(l)} \mu^{\tilde{\rho}} \frac{x_2^{2|\rho(s)|+R^{(\Phi)}}}{1-x_2^2} - e^{-i\rho(l)} \mu^{-\tilde{\rho}} \frac{x_2^{2|\rho(s)|+2-R^{(\Phi)}}}{1-x_2^2} \right] \quad (4.5.33)$$

As previously said this quantity does not depend on the variable x_1 which is consistent with the fact that only BPS states contribute to the index (4.2.5).

Then, to obtain the correct contribution to the localized formula (4.5.2), let us stress that we need to consider the plethystic exponential of such letter index, as in (4.5.23), namely

$$\mathrm{PE}\left[f_\Phi(e^{il_a}, x_2, \mu_i) \right] = \exp\left\{ \sum_{n=1}^{\infty} \frac{1}{n} f_\Phi(e^{inl_a}, x_2^n, \mu_i^n) \right\} \quad (4.5.34)$$

4.5.2 Vector multiplet contribution

Now we shall turn our attention to the vector multiplet contribution to the localized index (4.5.2).

Since the saddle points (4.4.44) of the localizing Lagrangian (4.4.40) are obtained by setting $V_\mu = 0$; to find the vector multiplet contribution we must consider the following fluctuations around those saddle points

$$A_\mu = A_\mu^0 + \delta A_\mu \quad \sigma = \sigma^0 + \delta\sigma \quad (4.5.35)$$

Moreover, when considering the localizing Lagrangian (4.4.40), we clearly have the complication of fixing the gauge. This is done with the standard Fadeev-Popov method, by introducing ghost fields c and $\bar{c}$. The gauge-fixing Lagrangian on the $S^2 \times S^1$ background is

$$\epsilon\tilde{\epsilon}\mathcal{L}_{\mathrm{gf}} = -\bar{c}\mathcal{D}^i\mathcal{D}_i c - (\mathcal{D}_i A^i)^2 := -\bar{c}\mathcal{D}^i\mathcal{D}_i c + V_{\mathrm{gf}}V_{\mathrm{gf}} \quad (4.5.36)$$

Then the gauge-fixed localizing Lagrangian turns out to be

$$\epsilon\tilde{\epsilon}\mathcal{L}_{\mathrm{SYM}}^{\mathrm{loc}} = V_\mu V^\mu - \bar{c}\mathcal{D}^i\mathcal{D}_i c + V_{\mathrm{gf}}V_{\mathrm{gf}} + D^2 - 2\bar{\lambda}\gamma^\mu\mathcal{D}_\mu\lambda - 2\bar{\lambda}[\sigma, \lambda] - \frac{1}{r}\bar{\lambda}\gamma^3\lambda \quad (4.5.37)$$

Using the definitions (4.5.35), we can now substitute the field expansions about the saddle points (4.3.9) in the localizing Lagrangian (4.5.37). Considering the quadratic terms only, we get

$$\epsilon\tilde{\epsilon}\mathcal{L}_{\mathrm{SYM}}^{\mathrm{loc}} \simeq V_\mu V^\mu - \bar{c}\mathcal{D}^i\mathcal{D}_i c + V_{\mathrm{gf}}V_{\mathrm{gf}} + D^2 - 2\bar{\lambda}\gamma^\mu\mathcal{D}_\mu\lambda - 2\frac{1}{r}\bar{\lambda}s\lambda - \frac{1}{r}\bar{\lambda}\gamma^3\lambda \quad (4.5.38)$$

Again, we can simply integrate out the auxiliary field D, getting a constant factor that can simply be dropped.

The ghosts c and $\bar{c}$ contribution By expanding the ghost fields c and $\bar{c}$ into the generalised spin σ_{eff} spherical harmonics $Y_{j,j_3}^{\sigma_{\mathrm{eff}}}$ of (4.5.9) and by looking to the eigenvalue equation (4.5.10), we can easily see that their contribution is exactly

$$\det \Delta^c_{loc} = \prod_{\alpha\in\Delta} \prod_{j=|\alpha(s)|}^{\infty} \prod_{j_3=-j}^{j} \prod_{n=-\infty}^{\infty} \left(j(j+1) - \alpha(s)^2\right) \tag{4.5.39}$$

where now, due to the fact that the vector multiplet is in the adjoint representation of he gauge group G, the weight vector ρ simply becomes the root α of the Lie algebra $\mathfrak{g}$. Here Δ stands for the root space.

The gaugino λ contribution The contribution of the gaugino λ is exactly the same as the one of the fermionic field ψ of the chiral multiplet Φ, Eq. , apart from the fact that the R charge is zero and that the representation $\mathcal{R}$ is the adjoint; thus we get

$$\begin{aligned}\det \Delta_\lambda = \frac{1}{r^2} \prod_{\alpha\in\Delta} \Bigg\{ & \prod_{j=|\alpha(s)|-\frac{1}{2}}^{\infty} \prod_{j_3=-j}^{j} \prod_{n=-\infty}^{\infty} (j + r\mathcal{D}_3) \times \\ & \times \prod_{j=|\alpha(s)|+\frac{1}{2}}^{\infty} \prod_{j_3=-j}^{j} \prod_{n=-\infty}^{\infty} (j + 1 - r\mathcal{D}_3) \Bigg\}\end{aligned} \tag{4.5.40}$$

The vector contribution The only contribution left is the one of the fluctuations of the gauge vector δA_μ and the scalar $\delta\sigma$, which reads

$$\Delta^v_{loc} = V_\mu V^\mu + V_{\text{gf}} V_{\text{gf}} \tag{4.5.41}$$

For the sake of simplicity, we should redefine a vector $v_M := (\delta\sigma,\ \delta A_\mu)$ in a $4d$ $\mathcal{N} = 1$ notation. Then, we have

$$V_\mu = \begin{pmatrix} -\mathcal{D}^{(0)}_1 & -i\alpha(s) & -\mathcal{D}^{(0)}_3 & \mathcal{D}^{(0)}_2 \\ -\mathcal{D}^{(0)}_2 & \mathcal{D}^{(0)}_3 & -i\alpha(s) & -\mathcal{D}^{(0)}_1 \\ -\mathcal{D}^{(0)}_3 - \frac{1}{r} & -\mathcal{D}^{(0)}_2 & \mathcal{D}^{(0)}_1 & -i\alpha(s) \end{pmatrix} \begin{pmatrix} \delta\sigma \\ \delta A^1 \\ \delta A^2 \\ \delta A^3 \end{pmatrix} \tag{4.5.42}$$

$$V_{\text{gf}} = \begin{pmatrix} 0 & \mathcal{D}^{(0)}_1 & \mathcal{D}^{(0)}_2 & 0 \end{pmatrix} \begin{pmatrix} \delta\sigma \\ \delta A^1 \\ \delta A^2 \\ \delta A^3 \end{pmatrix} \tag{4.5.43}$$

where we have split

$$-\mathcal{D}_\mu\sigma = -\mathcal{D}^{(0)}_\mu \delta\sigma + i[\delta A_\mu, \sigma^{(0)}] = -\mathcal{D}^{(0)}_i \delta\sigma - \mathcal{D}_3\delta\sigma - i\alpha(s)\delta A_\mu \tag{4.5.44}$$

To easily compute the matrix determinant of the Δ^v_{loc} operator, we can expand $\delta\sigma$ into generalised spin σ_{eff} spherical harmonics $Y^{\rho(s)}_{j,j_3}$ (4.5.9), while each component of the vector fluctuations δA_μ can be in turn expanded into the so-called generalised spin σ_{eff} *vector* harmonics $C_{\mu}{}^{\rho(s)+\lambda}_{j,j_3}$ with intrinsic spin $\lambda = 0, \pm 1$ and defined as follows [157]

$$\begin{aligned} C_{\mu\, j,j_3}^{\;\rho(s)+1} &= \frac{1}{\sqrt{2(\mathcal{J}^2+\rho(s))}}\left(\mathcal{D}_\mu + \frac{i}{|r|}\varepsilon_{\mu\rho\sigma}r^\rho\mathcal{D}^\sigma\right)Y_{j,j_3}^{\rho(s)} \\ C_{\mu\, j,j_3}^{\;\rho(s)} &= \frac{r_\mu}{|r|}Y_{j,j_3}^{\rho(s)} \\ C_{\mu\, j,j_3}^{\;\rho(s)-1} &= \frac{1}{\sqrt{2(\mathcal{J}^2-\rho(s))}}\left(\mathcal{D}_\mu - \frac{i}{|r|}\varepsilon_{\mu\rho\sigma}r^\rho\mathcal{D}^\sigma\right)Y_{j,j_3}^{\rho(s)} \end{aligned} \tag{4.5.45}$$

where we also defined

$$\mathcal{J}^2 = j(j+1) - \rho(s)^2 \tag{4.5.46}$$

Thus we get

$$A_\mu = \sum_{j,j_3}\sum_{\lambda} a_\lambda^{j,j_3} C_{\mu\, j,j_3}^{\;\alpha(s)+\lambda} \tag{4.5.47}$$

$$\sigma = \sum_{j,j_3} b^{j,j_3}\frac{Y_{j,j_3}^{\alpha(s)}}{|r|} \tag{4.5.48}$$

Then it is convenient to chose a new basis

$$\{x^\mu = x^1, x^2, x^3\} \quad\to\quad \{x^\pm = \frac{1}{\sqrt{2}}(x^1+x^2), x^3\} \tag{4.5.49}$$

Thanks to this change of basis and to the properties of the harmonics, we can recast the Δ^v_{loc} operator in (4.5.41) as

$$\Delta^v_{loc} = V_M V^M \tag{4.5.50}$$

where we have defined

$$V_M = \begin{array}{c} Y_{j,j_3}^{\alpha(s)}/r \;\times \\ C_{+\, j,j_3}^{\;\alpha(s)+1} \;\times \\ C_{-\, j,j_3}^{\;\alpha(s)-1} \;\times \\ C_{3\, j,j_3}^{\;\alpha(s)} \;\times \end{array} \begin{pmatrix} 0 & -s_+ & -s_- & 0 \\ -s_+ & -i\alpha(s) - ir\mathcal{D}_3 & 0 & is_+ \\ -s_- & 0 & -i\alpha(s)+ir\mathcal{D}_3 & -is_- \\ -r\mathcal{D}_3 - 1 & -is_+ & is_- & -i\alpha(s) \end{pmatrix} \begin{pmatrix} b \\ a_+ \\ a_- \\ a_0 \end{pmatrix} \tag{4.5.51}$$

with

$$s_\pm := \sqrt{\frac{\mathcal{J}^2 \pm \alpha(s)}{2}} \tag{4.5.52}$$

and $\mathcal{J}$ as in (4.5.46) with the generic weight vector ρ being a root α of the adjoint representation.

As in the fermionic case, in evaluating the matrix determinant of such operator, we have now to consider three different cases according to the value of the total angular momentum j:

- If $j \geq |\alpha(s)|+1$, then all the three components of the vector harmonics $C_{\mu\, j,j_3}^{\;\alpha(s)+\lambda}$ exist and thus the matrix determinant reads

$$\det_{\star} \Delta^v_{loc} = (j(j+1) - \alpha(s)^2)(j - D_3)(j+1+D_3) \tag{4.5.53}$$

- If $j = |\alpha(s)|$ there is no a_- mode and the operator becomes

$$V_M = \begin{matrix} Y^{\alpha(s)}_{j,j_3}/r & \times \\ C_{+\,j,j_3}^{\alpha(s)+1} & \times \\ C_{3\,j,j_3}^{\alpha(s)} & \times \end{matrix} \begin{pmatrix} 0 & -s_+ & 0 \\ -s_+ & -i\alpha(s) - i\mathcal{D}_3 & is_+ \\ -\mathcal{D}_3 - 1 & -is_+ & -i\alpha(s) \end{pmatrix} \begin{pmatrix} b \\ a_+ \\ a_0 \end{pmatrix} \tag{4.5.54}$$

so that the matrix determinant now reads

$$\det_{\star} \Delta^v_{loc} = \pm i(j(j+1) - \alpha(s)^2)(j+1+D_3) \tag{4.5.55}$$

where we used the fact that in this particular case $|\alpha(s)| = j$.

- If $j = |\alpha(s)| - 1$, the only mode left is a_+ and the vector collapses into

$$V_+ = C_{+\,j,j_3}^{\alpha(s)+1} \left(-i\alpha(s) - iD_3\right) a_+ \tag{4.5.56}$$

In this case the matrix determinant is simply the eigenvalue

$$\det_{\star} \Delta^v_{loc} = \pm i(j+1+D_3) \tag{4.5.57}$$

where again we used the fact that $|\alpha(s)| = j+1$.

Confronting the determinant arising from the ghost fields c and $\overline{c}$ (4.5.39) and the determinants coming from the vector field v_M (4.5.53) and (4.5.55), we see that they share the common eigenvalue $(j(j+1) - \rho(s)^2)$ for $j \geq |\alpha(s)|$. Thus, since the ghost fields are fermionic, their contribution exactly cancels the respective contribution of the vector field in $\det_\star \Delta^v_{loc}$.

Taking into account this fact and combining all the results, we get the following operator determinant

$$\det \Delta^v_{loc} = \prod_{\alpha\in\Delta} \left\{ \prod_{j=|\alpha(s)|-1}^{\infty} \prod_{j_3=-j}^{j} \prod_{n=-\infty}^{\infty} (j+1+r\mathcal{D}_3) \right\} \left\{ \prod_{j=|\alpha(s)|+1}^{\infty} \prod_{j_3=-j}^{j} \prod_{n=-\infty}^{\infty} (j - r\mathcal{D}_3) \right\} \tag{4.5.58}$$

Following the same procedure adopted with the chiral multiplet, one can show that for the eigenvalue $(j+1+r\mathcal{D}_3)$ the letter indices for the gaugino λ and the gauge-fixed vector v_M are then

$$f_v(e^{il_a}, x_1, x_2) = \sum_{\alpha\in\Delta} e^{i\alpha(l)} x_2 \sum_{j=|\alpha(s)|-1}^{\infty} \sum_{j_3=-j}^{j} (x_1x_2)^j \left(\frac{x_2}{x_1}\right)^{j_3} \tag{4.5.59}$$

$$f_\lambda(e^{il_a}, x_1, x_2) = -\sum_{\alpha\in\Delta} e^{i\alpha(l)} x_2 \sum_{j=|\alpha(s)|-1}^{\infty} \sum_{j_3=-j}^{j+1} (x_1x_2)^j \left(\frac{x_2}{x_1}\right)^{j_3} \tag{4.5.60}$$

$$\tag{4.5.61}$$

so that, when summing them together, only the $j_3 = j + 1$ term of $f_\lambda(e^{il_a}, x_1, x_2)$ contributes to the index.

Thus the total letter index reads

$$f_{v+\lambda}(e^{il_a}, x_1, x_2) = -\sum_{\alpha\in\Delta} e^{-i\alpha(l)} x_2 \left[x_2^{2(|\alpha(s)|-1)} \sum_{j=0}^{\infty} x_2^{2j+1} \right] = -\sum_{\alpha\in\Delta} e^{-i\alpha(l)} \frac{x_2^{|\alpha(m)|}}{1-x_2^2} \tag{4.5.62}$$

Similarly, from the other eigenvalue $(j - r\mathcal{D}_3)$ we get

$$f_v(e^{il_a}, x_1, x_2) = \sum_{\alpha\in\Delta} e^{-i\alpha(l)} x_2^{-1} \sum_{j=|\alpha(s)|+1}^{\infty} \sum_{j_3=-j}^{j} (x_1 x_2)^{j+1} \left(\frac{x_1}{x_2}\right)^{j_3} \tag{4.5.63}$$

$$f_\lambda(e^{il_a}, x_1, x_2) = -\sum_{\alpha\in\Delta} e^{-i\alpha(l)} x_2^{-1} \sum_{j=|\alpha(s)|+1}^{\infty} \sum_{j_3=-j+1}^{j} (x_1 x_2)^{j+1} \left(\frac{x_1}{x_2}\right)^{j_3} \tag{4.5.64}$$

$$\tag{4.5.65}$$

so that, when summing them together, only the $j_3 = -j$ term of $f_A(e^{il_a}, x_1, x_2)$ will contribute.

Thus

$$f_{v+\lambda}(e^{il_a}, x_1, x_2) = \sum_{\alpha\in\Delta} e^{-i\alpha(l)} x_2^{-1} \left[x_2^{2(|\alpha(s)|+1)+1} \sum_{j=0}^{\infty} x_2^{2j} \right] = \sum_{\alpha\in\Delta} e^{-i\alpha(l)} \frac{x_2^{|\alpha(m)|+2}}{1-x_2^2} \tag{4.5.66}$$

Finally, by summing up the two contributions (4.5.62) and (4.5.66), we obtain the letter index for the vector multiplet V, namely

$$f_V(e^{il_a}, x_2) = \sum_{\alpha\in\Delta} \left[-e^{i\alpha(l)} x_2^{|\alpha(m)|} \right] \tag{4.5.67}$$

However, since there should be no net contribution to the super-determinant from modes which do not feel the magnetic flux m, generally one should write

$$f_V(e^{il_a}, x_2) = \sum_{\alpha\in\Delta} -e^{i\alpha(l)} \left[x_2^{|\alpha(m)|} - \delta_{\alpha(m),0} \right] \tag{4.5.68}$$

where we subtracted from (4.5.67) the contribution from the modes possessing $\alpha(m) = 0$.

Thus, the contribution of a vector multiplet to the localized formula (4.5.2) is just the plethystic exponential

$$\mathrm{PE}\left[f_V(e^{il_a}, x_2) \right] = \exp\left\{ \sum_{n=1}^{\infty} \frac{1}{n} f_V(e^{inl_a}, x_2^n) \right\} \tag{4.5.69}$$

4.5.3 The monopole charges corrections

Up to now we purposely forgot about the second factor in (4.5.23). This prefactor exists for both the chiral and vector multiplets and thus its explicit form changes according to the value of the variable z and the eigenvalue under consideration.

To show how to compute this quantity, let us focus on the eigenvalue $(j+R+r\mathcal{D}_3)$ of the chiral multiplet. Then the prefactor reads

$$\exp\left\{-\sum_{\dots}(-1)^F\frac{z}{2}\right\}=\exp\left\{-\sum_{\dots}(-1)^F\frac{1}{2}\left[-i\rho(l)+(j-j_3)\beta_1+(j+R+j_3)\beta_2+\widetilde{\rho}(\alpha)\right]\right\} \tag{4.5.70}$$

Since this quantity is formally divergent, we need to correctly regularize it [126]. The most general regularization would be the insertion inside the index (4.2.5) of a factor

$$e^{il'}x_1'^{-(j-j_3)}x_2'^{-(j+R+j_3)}\mu'^{-\widetilde{\rho}} \tag{4.5.71}$$

where, for the sake of simplicity, we dubbed these parameters as the chemical potentials (apart from a prime); however they are merely regulators and should be taken to 1 when the computation of the trace is ruled out.

The same process must be done for each other eigenvalue of both the chiral and the vector multiplets.

The prefactor (4.5.70) is formally very similar to the letter index (4.5.24) and in fact, once regularized as above for each eigenvalue, it turns out to be

$$-\sum_{\dots}(-1)^F\frac{z}{2}=\frac{1}{2}\lim_{e^{il'_a},x'_1,x'_2,\mu'_i\to 1}(\partial_{e^{il'_a}}+\beta_1\partial_{x'_1}+\beta_2\partial_{x'_2}+\alpha^i\partial_{\mu'_i})f_{tot}(e^{il'_a},x'_1,x'_2,\mu'_i) \tag{4.5.72}$$

where, since x_1 disappears from the letter indices, its contribution vanishes. Here, the function $f_{tot}(e^{il'_a},x'_1,x'_2,\mu'_i)$ is just the total letter index made up by summing together the chiral multiplet contribution (4.5.33) and the vector multiplet one (4.5.68).

In the end, this regularization process gives [117]

$$\exp\left\{-\sum_{\dots}(-1)^F\frac{z}{2}\right\}=e^{ib_0(l)}x_2^{\epsilon_0}\prod_i^{\mathrm{Rank}\,\widehat{G}}\mu_i^{q_0^i} \tag{4.5.73}$$

where we have defined

- The zero-point contributions to the energy

$$\epsilon_0=\frac{1}{2}\lim_{e^{il'_a},x'_2,\mu'_i\to 1}\partial_{x'_2}f_{tot}(e^{il'_a},x'_2,\mu'_i)=\sum_{\{\Phi\}}(1-R^{(\Phi)})\sum_{\rho\in R_\Phi}|\rho(s)|-\sum_{\alpha\in\Delta}|\alpha(s)| \tag{4.5.74}$$

- The zero-point contributions to the flavour charges

$$\begin{aligned}q_0^i&=\frac{1}{2}\lim_{e^{il'_a},x'_2,\mu'_i\to 1}\partial_{\mu'_i}f_{tot}(e^{il'_a},x'_2,\mu'_i)=\\&=-\sum_{\{\Phi\}}\sum_{\rho\in R_\Phi}F_i\left[\frac{1}{2(x_2-1)}+\left(\frac{1}{4}+|\rho(s)|\right)+\mathcal{O}(x_2-1)\right]\end{aligned} \tag{4.5.75}$$

where it is clear that we need some regularization.

However, since this quantity does not depend on the chiral multiplets R-charges $R^{(\Phi)}$, it is plausible that after an appropriate regularization q_0^i does not depend on x_2. Thus we can neglect any term inside (4.5.75) apart from

$$q_0^i \simeq -\sum_{\{\Phi\}} \sum_{\rho \in R_\Phi} |\rho(s)| F_i \tag{4.5.76}$$

- The 1-loop correction to Chern–Simons terms

$$b_0(l) = \frac{1}{2} \lim_{e^{il'_a}, x'_2, \mu'_i \to 1} \partial_{e^{il'}} f_{tot}(e^{il'_a}, x'_2, \mu'_i) = \ldots = -\sum_{\{\Phi\}} \sum_{\rho \in R_\Phi} |\rho(s)| \rho(l) \tag{4.5.77}$$

which is obtained by the exact same reasoning of q_0^i.

These are exactly the corrections to the monopole quantum numbers of equations (2.2.30), (2.2.31) and (2.2.32) that we already encountered in Chapter (2).

Up to now, the contribution of the chiral and vector multiplets to the superdeterminant of the localized index (4.5.2) reads

$$\prod_{\ldots} \prod_{n=-\infty}^{\infty} (2\pi i n + z)^{(-1)^{F+1}} = e^{ib_0(l)} x_2^{\epsilon_0} \mu^{q_0} \, \mathrm{PE}\left[f_{tot}(e^{il_a}, x_2, \mu_i) \right] \tag{4.5.78}$$

4.5.4 The localizing formula

Before writing down the full localizing formula for the $3d$ superconformal index (4.5.2), we need to explicitly compute the Vandermonde determinan $\det J$ of the change of variables for the Wilson line l and the monopole magnetic charge m to lie in the Cartan of the gauge group G. This reads

$$\det J = \prod_{\alpha \in \Delta, \alpha(m)=0} 2i \sin \frac{\alpha(l)}{2} \tag{4.5.79}$$

where, because the gauge group G is broken by the magnetic flux m (see Section (2.2.4)), the product is restricted over the roots for the unbroken gauge group H_m only, namely the ones that satisfy $\alpha(m) = 0$.

Performing the same algebraic manipulations of (4.5.21) and below, we can rewrite the Vandermonde determinan as

$$\det J = \mathrm{PE}\left[f'(e^{inl_a}) \right] = \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} f'(e^{inl_a}) \right] \tag{4.5.80}$$

where we defined

$$f'(e^{il_a}) := \prod_{\alpha \in \Delta, \alpha(m)=0} \left[-e^{-i\alpha(l)} \right] \tag{4.5.81}$$

It is easy to see that the above determinant can be combined with the vector letter index to give

$$\det J \, \mathrm{PE}\left[f_V(e^{il_a}, x_2) \right] = \mathrm{PE}\left[\sum_{\alpha \in \Delta} \left[-e^{i\alpha(l)} x_2^{|\alpha(m)|} \right] \right] \tag{4.5.82}$$

In addition to this Jacobian factor arising from the fixing of continuous gauge symmetries, we need a statistical factor associated with the Weyl group $\mathcal{W}_m$ of the unbroken gauge group H_m. For example, suppose the gauge group G is broken by the monopoles magnetic charge m to a subgroup $H_m = \otimes_k U(N_k)$, then the statistical factor reads

$$\mathrm{Sym} = \prod_k N_k! = \prod_{j=1}^{\mathrm{Rank}\,G} \left[\sum_{k\geq j}^{\mathrm{Rank}\,G} \delta_{m_j,m_k} \right] \tag{4.5.83}$$

Thus we should include a factor $\frac{1}{\mathrm{Sym}}$ in the definition of the integration measure of the localized formula (4.5.2).

Taking into account everything we have achieved so far, the complete localized formula for the superconformal index reads [117, 126]

$$\mathcal{I} = \sum_{\{m_a\}} \frac{1}{\mathrm{Sym}} \int \left(\prod_a^{\mathrm{Rank}\,G} \frac{dl_a}{2\pi} \right) e^{ik\pi\,\mathrm{Tr}\{ls\}} e^{ib_0(l)} x_2^{\epsilon_0} \mu^{q_0}\, \mathrm{PE}\left[f_{tot}(e^{il_a}, x_2, \mu_i) \right] \tag{4.5.84}$$

where the total letter index reads

$$f_{tot}(e^{il_a}, x_2, \mu_i) = -\sum_{\alpha\in\Delta} e^{i\alpha(l)} x_2^{|\alpha(m)|} + \sum_{\Phi} \sum_{\rho\in\mathcal{R}} \left[e^{i\rho(l)} \mu^{\tilde{\rho}} \frac{x_2^{2|\rho(s)|+R}}{1-x_2^2} - e^{-i\rho(l)} \mu^{-\tilde{\rho}} \frac{x_2^{2|\rho(s)|+2-R}}{1-x_2^2} \right] \tag{4.5.85}$$

and the contribution of the action evaluated at the localizing locus $V_\mu = 0$ (see (4.4.44)) is simply

$$S[X_0] = -2ik\,\mathrm{Tr}\{ls\} \tag{4.5.86}$$

which comes from the Chern–Simons part (2.1.7).

It is then convenient to introduce new complex variables $z_a = e^{il_a}$ which run over unit circles in the complex plane and rewrite the plethystic exponential $\mathrm{PE}\left[f_{tot}(e^{il_a}, x_2, \mu_i)\right]$ in a more compact way [122].

To do so, we will make use of the q-Pochhammer product $(w;q)_n$, defined as follows

$$(w;q)_n := \prod_{j=0}^{n-1} (1 - wq^j) \tag{4.5.87}$$

where n could be either finite or infinite.

Specifically, let us firs consider the plethystic exponential of the letter index for a single chiral field Φ

$$\mathrm{PE}\left[f_\Phi(z_a, x_2, \mu_i)\right] = \prod_{\rho\in\mathcal{R}} \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} \left(z^{n\rho} \mu^{n\tilde{\rho}} \frac{x_2^{n(2|\rho(s)|+R)}}{1-x_2^{2n}} - z^{-n\rho} \mu^{-n\tilde{\rho}} \frac{x_2^{n(2|\rho(s)|+2-R)}}{1-x_2^{2n}} \right) \right] \tag{4.5.88}$$

We shall rewrite the denominator as a geometric series, namely

$$\frac{1}{1-x_2^{2n}} = \sum_{k=0}^{\infty} x_2^{2kn} \tag{4.5.89}$$

Then, by interchanging the order of the two summations over n and k, we get

$$
\begin{aligned}
&\mathrm{PE}\left[f_\Phi(z_a, x_2, \mu_i)\right] = \\
&= \prod_{\rho\in\mathcal{R}} \exp\left\{\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n}\left[\left(z^\rho \mu^{\widetilde{\rho}} x_2^{2|\rho(s)|+R+2k}\right)^n - \left(z^{-\rho}\mu^{-\widetilde{\rho}} x_2^{2|\rho(s)|+2-R+2k}\right)^n\right]\right\} = \\
&= \prod_{\rho\in\mathcal{R}}\prod_{k=0}^{\infty} \exp\left[-\ln\left(1 - z^\rho \mu^{\widetilde{\rho}} x_2^{2|\rho(s)|+R+2k}\right) + \ln\left(1 - z^{-\rho}\mu^{-\widetilde{\rho}} x_2^{2|\rho(s)|+2-R+2k}\right)\right] = \\
&= \prod_{\rho\in\mathcal{R}}\prod_{k=0}^{\infty} \frac{1 - z^{-\rho}\mu^{-\widetilde{\rho}} x_2^{2|\rho(s)|+2-R+2k}}{1 - z^\rho \mu^{\widetilde{\rho}} x_2^{2|\rho(s)|+R+2k}} = \prod_{\rho\in\mathcal{R}} \frac{(z^{-\rho}\mu^{-\widetilde{\rho}} x_2^{2|\rho(s)|+2-R}; x^2)_\infty}{(z^\rho \mu^{\widetilde{\rho}} x_2^{2|\rho(s)|+R}; x^2)_\infty}
\end{aligned}
\tag{4.5.90}
$$

where in the second line we recognised the series expansion of $\ln(1-x)$ and for the last equality we used the q-Pochhammer symbol definition (4.5.87).

Considering instead the plethystic exponential of the vector multiplet letter index, we have

$$
\begin{aligned}
\mathrm{PE}\left[f_V(z_a, x_2)\right] &= \mathrm{PE}\left[\sum_{\alpha\in\Delta}\left(-z^\alpha x_2^{|\alpha(m)|}\right)\right] = \\
&= \prod_{\alpha\in\Delta}\exp\left\{-\sum_{n=1}^{\infty}\frac{1}{n}\left(z^\alpha x_2^{|\alpha(m)|}\right)^n\right\} = \prod_{\alpha\in\Delta}\left(1 - z^\alpha x_2^{|\alpha(m)|}\right)
\end{aligned}
\tag{4.5.91}
$$

Then, using the definitions of the monopole charges corrections ϵ_0 (4.5.74), q_0^i (4.5.76) and $b_0(a)$ (4.5.77), we can finally rewrite the index (4.5.84) in a more compact way as

$$
\mathcal{I} = \sum_{\{m_a\}}\frac{1}{\mathrm{Sym}}\left(\prod_a^{\mathrm{Rank}\,G}\oint\frac{dz_a}{2\pi z_a} z_a^{k m_a}\right) Z_V(\{\boldsymbol{z},\boldsymbol{m}\}; x)\prod_\Phi Z_\Phi(\{\boldsymbol{z},\boldsymbol{m}\},\boldsymbol{\mu}; x) \tag{4.5.92}
$$

where we have defined

$$
Z_V(\{\boldsymbol{z},\boldsymbol{m}\}; x) := \prod_{\alpha\in\Delta} x^{-\frac{|\alpha(m)|}{2}}\left(1 - z^\alpha x_2^{|\alpha(m)|}\right) \tag{4.5.93}
$$

$$
Z_\Phi(\{\boldsymbol{z},\boldsymbol{m}\},\boldsymbol{\mu}; x) := \prod_{\rho\in\mathcal{R}}\left(x^{(1-R)} z^{-\rho}\mu^{-\widetilde{\rho}}\right)^{\frac{|\rho(m)|}{2}} \frac{(z^{-\rho}\mu^{-\widetilde{\rho}} x_2^{|\rho(m)|+2-R}; x^2)_\infty}{(z^\rho\mu^{\widetilde{\rho}} x_2^{|\rho(m)|+R}; x^2)_\infty} \tag{4.5.94}
$$

and $x = x_2$. In addition, we made use of the fact that $m = 2s$ and introduced the shorthand notation

$$
\begin{aligned}
\{\boldsymbol{z},\boldsymbol{m}\} &= \{(z_1,\cdots,z_{\mathrm{Rank}\,G}),(m_1,\cdots,m_{\mathrm{Rank}\,G})\} \\
\boldsymbol{\mu} &= \left(\mu_1,\cdots,\mu_{\mathrm{Dim}\,\widetilde{\mathcal{R}}}\right)
\end{aligned}
\tag{4.5.95}
$$

It is easy to note that z_a and μ_i in the chiral multiplet contribution (4.5.94) always appear on a very similar footing. This is because they are of the same nature: both are variables parametrizing the maximal tori of their respective symmetry groups. In fact, we can think of the chemical potentials μ_i as the S^1 Wilson lines for fixed background gauge fields which couple to the global symmetries of the theory. Thus, we can also introduce a new magnetic flux variable n_i for such background fields. At the index level, n_i will appear on the same footing of the gauge flux variable m_a, leading to the following replacement

$$\rho(m) \to \rho(m) + \widetilde{\rho}(n) \tag{4.5.96}$$

Thus, the generalized superconformal index becomes [71, 122]

$$\mathcal{I}(\{\boldsymbol{\mu}, \boldsymbol{n}\}; x) = \sum_{\{m_a\}} \frac{1}{\text{Sym}} \left(\prod_a^{\text{Rank}\, G} \oint \frac{dz_a}{2\pi z_a} z_a^{km_a} \right) Z_V(\{\boldsymbol{z}, \boldsymbol{m}\}; x) \prod_\Phi Z_\Phi(\{\boldsymbol{z}, \boldsymbol{m}\}, \{\boldsymbol{\mu}, \boldsymbol{n}\}; x) \tag{4.5.97}$$

where now

$$\begin{aligned} Z_\Phi(\{\boldsymbol{z}, \boldsymbol{m}\}, \{\boldsymbol{\mu}, \boldsymbol{n}\}; x) := \prod_{\rho \in \mathcal{R}} \prod_{\widetilde{\rho} \in \widetilde{\mathcal{R}}} \left(x^{(1-R)} z^{-\rho} \mu^{-\widetilde{\rho}} \right)^{\frac{|\rho(m) + \widetilde{\rho}(n)|}{2}} \times \\ \times \frac{(z^{-\rho} \mu^{-\widetilde{\rho}} x_2^{|\rho(m) + \widetilde{\rho}(n)| + 2 - R}; x^2)_\infty}{(z^{\rho} \mu^{\widetilde{\rho}} x_2^{|\rho(m) + \widetilde{\rho}(n)| + R}; x^2)_\infty} \end{aligned} \tag{4.5.98}$$

where now $\widetilde{\mathcal{R}}$ is the representation of the flavour symmetry group for the chiral multiplet Φ.

In this way we can also gauge the flavour symmetries, by simply integrating and summing over respectively the μ_i and the n_i variables and by introducing the appropriate contribution for their vector multiplets (if the global symmetry is non Abelian).

Moreover, according to the cases, there are additional fugacities and magnetic fluxes one can turn on in the index (4.5.97) .

For example, since we are working in $3d$, we can consider the topological symmetry current J^μ_{top} (2.1.12). To determine the contribution of this symmetry to the generalized index, we need to repeat the procedure we used above for flavour symmetries and couple the symmetry current J^μ_{top} to a background vector multiplet V_{top}. At the index level, this amounts to include a new $\mathcal{N} = 2$ BF term, which can be thought of as an off-diagonal Chern–Simons term. Thus, combining these classical Chern–Simons and BF contributions we get a factor

$$Z_{\text{cl}}(\{\boldsymbol{z}, \boldsymbol{m}\}, \{\omega, \nu\}) := \prod_a^{\text{Rank}\, G} \omega^{m_a} z_a^{km_a + \nu} \tag{4.5.99}$$

where ω and ν are respectively the continuous fugacity and the discrete flux for the background vector multiplet V_{top}.

Thus the final expression of the index reads

$$\mathcal{I}(\{\boldsymbol{\mu},\boldsymbol{n}\};x)=\sum_{\{m_a\}}\frac{1}{\mathrm{Sym}}\left(\prod_a^{\mathrm{Rank}\,G}\oint\frac{dz_a}{2\pi z_a}\right)\times \\ \times Z_{\mathrm{cl}}(\{\boldsymbol{z},\boldsymbol{m}\},\{\omega,\nu\})Z_V(\{\boldsymbol{z},\boldsymbol{m}\};x)\prod_\Phi Z_\Phi(\{\boldsymbol{z},\boldsymbol{m}\},\{\boldsymbol{\mu},\boldsymbol{n}\};x) \tag{4.5.100}$$

Clearly, if we have more than one gauge group or more than one flavour symmetry group, we should include all their contribution leading to more Z_{cl}, Z_V and Z_Φ factors inside (4.5.100) with different fugacities.

4.5.5 Superconformal multiplets and the index

Now that we have a straightforward analytic way to compute the index, we shall go back to the very initial definition of the index as a trace (4.2.5).

Once evaluated explicitly, we can in fact expand the localizing formula (4.5.100) for the index in a power series in x setting for convenience all the background magnetic fluxes for the global symmetries to zero. We thus get

$$\mathcal{I}(x,\mu,n=0)=\sum_{p=0}^{\infty}\chi_p(\mu)x^p \tag{4.5.101}$$

where $\chi_p(\mu)$ is the character of a certain representation of the global symmetry of the theory whose fugacity is exactly μ.

The index keeps track of the superconformal short multiplets, up to recombination. This feature makes the reconstruction of the whole content of short multiplets from the index an extremely hard task. Nevertheless, one may classify the equivalence classes of the multiplets according to their contribution to each order of x in the power series (4.5.101). To do this, we will closely follow [146] (see also [19] for the 4d counterpart). Thus first we must consider all the possible short multiplets of 3d $\mathcal{N}=2$ superconformal theories and see if they indeed satisfy the pairing constraint

$$\delta=\Delta-R-j_3=0 \tag{4.5.102}$$

Looking at (4.2.5), these multiplets will then contribute to the index at order $p=\Delta+j_3$.

In studying such contributions it is useful to define the modified index as follows

$$\widetilde{\mathcal{I}}(x,\mu,n=0)=(1-x^2)\left[\mathcal{I}(x,\mu,n=0)-1\right] \tag{4.5.103}$$

where all of the terms up to order $p=2$ in the modified index $\widetilde{\mathcal{I}}$ are equal to those in the original index $\mathcal{I}$.

Then we can define the index contribution as

$$I(p,j):=(-1)^j\frac{x^p}{1-x^2}\quad\text{with}\quad p=\Delta+j_3 \tag{4.5.104}$$

Thus, listing all the possible $\mathcal{N}=2$ superconformal short multiplets in the notation adopted by [59], we get

Multiplet	Contributing conformal primary	Contribution to $\mathcal{I}$
$L\overline{A}_1[j\geq 1]^{(R>0)}_{\frac{j}{2}+R+1}$	$[j+1]^{(R+1)}_{\frac{j}{2}+R+\frac{3}{2}}$	$I(j+r+2,j+1)$
$L\overline{A}_2[0]^{(R>0)}_{R+1}$	$[1]^{(R+1)}_{R+\frac{1}{2}}$	$I(r+2,1)$
$A_1\overline{L}[j\geq 1]^{(R<0)}_{\frac{j}{2}-R+1}$	-	0
$A_2\overline{L}[0]^{(R<0)}_{-R+1}$	-	0
$L\overline{B}_1[0]^{(R>0)}_{R}$	$[0]^{(R)}_{R}$	$I(r,0)$
$B_1\overline{L}[0]^{(R<0)}_{-R}$	-	0
$A_1\overline{A}_1[j\geq 1]^{(0)}_{\frac{j}{2}+1}$	$[j+1]^{(1)}_{\frac{j}{2}+\frac{3}{2}}$	$I(j+2,j+1)$
$A_2\overline{A}_2[0]^{(0)}_{1}$	$[1]^{(1)}_{\frac{3}{2}}$	$I(2,1)$
$A_2\overline{B}_1[0]^{(\frac{1}{2})}_{\frac{1}{2}}$	$[0]^{(\frac{1}{2})}_{\frac{1}{2}}$	$I(\frac{1}{2},0)$
$B_1\overline{A}_2[0]^{(-\frac{1}{2})}_{\frac{1}{2}}$	$[1]^{(\frac{1}{2})}_{1}$	$I(\frac{3}{2},1)$
$B_1\overline{B}_1[0]^{(0)}_{0}$	$[0]^{(0)}_{0}$	$I(0,0)$

(4.5.105)

Then the only $\mathscr{N}=2$ multiplets that can non-trivially contribute to the modified index (4.5.103) at order x^p for $p\leq 2$ are as follows

Multiplet	Contribution to I	Type
$A_2\overline{B}_1[0]^{(\frac{1}{2})}_{\frac{1}{2}}$	$+x^{\frac{1}{2}}$	Free fields
$B_1\overline{A}_2[0]^{(-\frac{1}{2})}_{\frac{1}{2}}$	$-x^{\frac{3}{2}}$	Free fields
$L\overline{B}_1[0]^{(1)}_{1}$	$+x$	Relevant operators
$L\overline{B}_1[0]^{(2)}_{2}$	$+x^2$	Marginal operators
$A_2\overline{A}_2[0]^{(0)}_{1}$	$-x^2$	Conserved currents

(4.5.106)

Indeed, the $A_2\overline{B}_1$ and $B_1\overline{A}_2$ multiplets are free fields and their combination form a free chiral multiplet.

The multiplets of the $L\overline{B}_1$ type are chiral fields and thus they contribute with relevant operators for $p<2$ and marginal operators for $p=2$.

Finally the $A_2\overline{A}_2$ multiplet is just the standard conserved current multiplet.

So, the coefficient of x^2 in the index counts the number of marginal operators minus the number of conserved currents.

However, since the S-fold theory has at least $\mathcal{N}=3$ supersymmetry, we consider the contribution from $\mathcal{N}=3$ multiplets to the $\mathcal{N}=2$ index. In particular, the relevant $\mathcal{N}=3$ current multiplets and their decomposition to $\mathcal{N}=2$ multiplets are [74]

Type	$\mathcal{N}=3$ multiplet	Decomposition into $\mathcal{N}=2$ multiplets
Flavour current	$B_1[0]_1^{(2)}$	$L\overline{B}_1[0]_1^{(1)} + B_1\overline{L}[0]_{-1}^{(1)} + A_2\overline{A}_2[0]_1^{(0)}$
Extra SUSY-current	$A_2[0]_1^{(0)}$	$A_2\overline{A}_2[0]_1^{(0)} + A_1\overline{A}_1[1]_{\frac{3}{2}}^{(0)}$
Stress tensor	$A_1[1]_{\frac{3}{2}}^{(0)}$	$A_1\overline{A}_1[1]_{\frac{3}{2}}^{(0)} + A_1\overline{A}_1[2]_2^{(0)}$

(4.5.107)

where it should be noted that the multiplets $A_1\overline{A}_1[1]_{\frac{3}{2}}^{(0)}$ and $A_1\overline{A}_1[2]_2^{(0)}$ contribute to $\widetilde{\mathcal{I}}$ as $+x^3$ and $-x^4$ respectively [146, Table 2].

From these last two tables, we see that orders x and x^2 of the index contain the following information:

$$\begin{aligned} \text{Order } x: \quad & \mathcal{N}=3 \text{ flavour currents ;} \\ \text{Order } x^2: \quad & (\mathcal{N}=2 \text{ preserving exactly marginal operators}) \\ & -(\mathcal{N}=3 \text{ flavour currents}) - (\mathcal{N}=3 \text{ extra SUSY-currents}) \ . \end{aligned} \tag{4.5.108}$$

Since in the following we will be interested in the supersymmetry enhancement phenomenon, this instructs us to focus only on the operators of the SCFTs with R-charge up to 2. Those with R-charge 1 are in correspondence with the $\mathcal{N}=3$ flavour currents. While, at order x^2, the information of the $\mathcal{N}=2$ marginal operators leads to the precise information of the $\mathcal{N}=3$ extra SUSY-current and, hence, the amount of (enhanced) supersymmetry of the corresponding SCFT.

4.6 Expressions of superconformal indices

In this section, we summarise the expressions for the different factors Z_{cl} (4.5.99), Z_V (4.5.93) and Z_Φ (4.5.98) of the superconformal indices for some theories that we will discuss later on in the rest of this work. We follow the convention adopted in [3, 4].

4.6.1 Gauge Groups

We will now write down the classical factors Z_{cl} (4.5.99) and the factors Z_V (4.5.93) for all the gauge groups of interest for this work.

$U(N)_k$ gauge group

Let us start by considering a $U(N)_k$ gauge group with Chern–Simons level k and a $U(1)$ topological symmetry.

The classical factor Z_{cl} (4.5.99) takes the simple form

$$Z_{\text{cl}}^{U(N)}(\{\boldsymbol{z},\boldsymbol{m}\},\{\omega,\nu\}) = \prod_{a=1}^{N} z_a^{km_a+\nu}\omega^{m_a} \tag{4.6.1}$$

where $\{\omega,\nu\}$ are the fugacity and the respective magnetic flux associated with the $U(1)_\omega$ topological symmetry.

Turning now our attention to the factor Z_V (4.5.93), we have explicitly

$$
\begin{aligned}
Z_V^{U(N)}(\{\boldsymbol{z},\boldsymbol{m}\}) = \prod_{a<b}^{N} & x^{-|m_a-m_b|}(1-(-1)^{m_a-m_b} z_a z_b^{-1} x^{|m_a-m_b|})\times \\
& \times (1-(-1)^{m_a-m_b} z_a^{-1} z_b x^{|m_a-m_b|})
\end{aligned}
\tag{4.6.2}
$$

$Usp(2N)_k$ gauge group

In the following we will consider also $USp(2N)_k$ gauge groups, for which we use a different normalisation of the CS level and the topological symmetry is absent:

$$
Z_{\text{cl}}^{USp(2N)}(\{\boldsymbol{z},\boldsymbol{m}\}) = \prod_{a=1}^{N} z_a^{2km_a} \tag{4.6.3}
$$

$$
Z_V^{USp(2N)}(\{\boldsymbol{z},\boldsymbol{m}\}) = \prod_{a=1}^{N} x^{-2|m_a|}(1-(-1)^{2m_a} z_a^2 x^{2|m_a|})(1-(-1)^{2m_a} z_a^{-2} x^{2|m_a|}) \tag{4.6.4}
$$

$SO(2N+\epsilon)_k$ gauge group

For compactness we denoted together the odd and the even case by $SO(2N+\epsilon)_k$ for $\epsilon = 0, 1$. When dealing with orthogonal groups we must pay a little attention.

In fact $SO(2N+\epsilon)_k$ gauge theories with $N > 1$ possess a topological discrete $\mathbb{Z}_2^{\mathcal{M}}$ symmetry but also a charge conjugation discrete $\mathbb{Z}_2^{\mathcal{C}}$ symmetry. In the following we will call ζ and χ the fugacities for respectively the $\mathbb{Z}_2^{\mathcal{M}}$ and $\mathbb{Z}_2^{\mathcal{C}}$ symmetries. We thus have the conditions $\zeta^2 = \chi^2 = 1$.

In the simplest case of $SO(2)_k \simeq U(1)_k$ the topological symmetry is simply $U(1)_\zeta$.

The classical factor Z_{cl} (4.5.99) thus takes the form

$$
\mathcal{Z}_{\text{cl}}^{SO(2N+\epsilon)}(\{\boldsymbol{z},\boldsymbol{m}\}) = \prod_{a=1}^{N} z_a^{2km_a}\zeta^{m_a} \tag{4.6.5}
$$

where we use the same normalisation of the CS level as in (4.6.3).

Moreover, if these discrete symmetries are coupled to background gauge fields and these latter are made dynamical, the $SO(2N+\epsilon)_k$ gauge theory is transformed into a new gauge theory with a different gauge group. All possible gaugings of these discrete symmetries and the respective gauge groups are depicted in Fig. (4.6.6) for a generic $SO(N)$.

$$
\begin{tikzcd}
 & Spin(N) \arrow[dr, "\mathcal{C}"] & \\
SO(N) \arrow[ur, "\mathcal{M}"] \arrow[r, "\mathcal{CM}"] \arrow[dr, "\mathcal{C}"'] & O^-(N) \arrow[r, "\mathcal{M}"] & Pin(N) \\
 & O^+(N) \arrow[ur, "\mathcal{M}"'] &
\end{tikzcd}
\tag{4.6.6}
$$

In the following, we will be interested in $O^+(N)$ gauge groups and thus we will need to gauge the discrete charge conjugation symmetry $\mathbb{Z}_2^{\mathcal{C}}$, *i.e.* to sum over $\chi = \pm 1$.

Let us consider the case $\chi = +1$ first, the factor Z_V (4.5.93) can be collectively written as

$$\begin{aligned}
Z_V^{SO(2N+\epsilon)}(\{\boldsymbol{z},\boldsymbol{m}\},\chi=+1) = &\left(\prod_{a=1}^{N} x^{-|m_a|}(1-(-1)^{m_a} z_a x^{|m_a|})(1-(-1)^{m_a} z_a^{-1} x^{|m_a|})\right)^{\epsilon} \\
&\times \prod_{a<b}^{N} x^{-|m_a+m_b|-|m_a-m_b|} \times (1-(-1)^{m_a+m_b} z_a z_b x^{|m_a+m_b|}) \\
&\times (1-(-1)^{m_a-m_b} z_a^{-1} z_b^{-1} x^{|m_a-m_b|})(1-(-1)^{m_a-m_b} z_a z_b^{-1} x^{|m_a-m_b|}) \\
&\times (1-(-1)^{m_a-m_b} z_a^{-1} z_b x^{|m_a-m_b|})
\end{aligned} \tag{4.6.7}$$

while for $\chi = -1$ we have separately [4, 113, 115]

$$\begin{aligned}
Z_V^{SO(2N)}(\{\boldsymbol{z},\boldsymbol{m}\};\chi=-1) = &\prod_{a=1}^{N-1} x^{-2|m_a|}(1-(-1)^{2m_a} z_a^2 x^{2|m_a|})(1-(-1)^{2m_a} z_a^{-2} x^{2|m_a|}) \\
&\times \prod_{a<b}^{N-1} x^{-|m_a+m_b|-|m_a-m_b|} \times (1-(-1)^{m_a+m_b} z_a z_b x^{|m_a+m_b|}) \\
&\times (1-(-1)^{m_a-m_b} z_a^{-1} z_b^{-1} x^{|m_a-m_b|})(1-(-1)^{m_a-m_b} z_a z_b^{-1} x^{|m_a-m_b|}) \\
&\times (1-(-1)^{m_a-m_b} z_a^{-1} z_b x^{|m_a-m_b|}) \,,
\end{aligned} \tag{4.6.8}$$

$$\begin{aligned}
Z_V^{SO(2N+1)}(\{\boldsymbol{z},\boldsymbol{m}\};\chi=-1) = &\prod_{a=1}^{N} x^{-|m_a|}(1+(-1)^{m_a} z_a x^{|m_a|})(1+(-1)^{m_a} z_a^{-1} x^{|m_a|}) \\
&\times \prod_{a<b}^{N} x^{-|m_a+m_b|-|m_a-m_b|} \times (1-(-1)^{m_a+m_b} z_a z_b x^{|m_a+m_b|}) \\
&\times (1-(-1)^{m_a-m_b} z_a^{-1} z_b^{-1} x^{|m_a-m_b|})(1-(-1)^{m_a-m_b} z_a z_b^{-1} x^{|m_a-m_b|}) \\
&\times (1-(-1)^{m_a-m_b} z_a^{-1} z_b x^{|m_a-m_b|}) \,.
\end{aligned} \tag{4.6.9}$$

In particular, the expression for $\chi = -1$ in the $SO(2N)_k$ case is obtained by setting $z_N = 1$, $z_N^{-1} = -1$ and $m_N = 0$ in the one for $\chi = +1$, while the expression for generic χ in the $SO(2N+1)$ can also be written compactly as

$$\begin{aligned}
Z_V^{SO(2N+1)}(\{\boldsymbol{z},\boldsymbol{m}\};\chi) = &\prod_{a=1}^{N} x^{-|m_a|}(1-(-1)^{m_a}\chi z_a x^{|m_a|})(1-(-1)^{m_a}\chi z_a^{-1} x^{|m_a|}) \\
&\times \prod_{a<b}^{N} x^{-|m_a+m_b|-|m_a-m_b|} \times (1-(-1)^{m_a+m_b} z_a z_b x^{|m_a+m_b|}) \\
&\times (1-(-1)^{m_a-m_b} z_a^{-1} z_b^{-1} x^{|m_a-m_b|})(1-(-1)^{m_a-m_b} z_a z_b^{-1} x^{|m_a-m_b|}) \\
&\times (1-(-1)^{m_a-m_b} z_a^{-1} z_b x^{|m_a-m_b|}) \,.
\end{aligned} \tag{4.6.10}$$

4.6.2 Matter fields

Since we will be interested in $\mathcal{N} = 4$ supersymmetric theory, in order to write down the superconformal index (4.5.100), we need to decompose each $\mathcal{N} = 4$ multiplet into

$\mathcal{N} = 2$ ones according to Table (2.2.2). Indeed the functions Z_V (4.5.93) and Z_Φ (4.5.98) are valid for $\mathcal{N} = 2$ vector and chiral multiplets respectively.

Moreover, in the chiral multiplets contribution Z_Φ, we can turn on an additional d fugacity corresponding to the axial symmetry $U(1)_d := U(1)_L - U(1)_R$, where $U(1)_L$ and $U(1)_R$ are the Cartan subalgebras of the $SU(2)_L$ and $SU(2)_R$ parts of the $\mathcal{N} = 4$ R-symmetry $SU(2)_L \times SU(2)_R$ (see Section (2.2)). In the following we do not turn on the background magnetic flux for $U(1)_d$.

Thus the contribution Z_Φ of the chiral fields can be modified as follows

$$Z_\Phi(\{\boldsymbol{z},\boldsymbol{m}\},\{\boldsymbol{\mu},\boldsymbol{n}\},d;x) := \prod_{\rho\in\mathcal{R}}\prod_{\widetilde{\rho}\in\widetilde{\mathcal{R}}}\left(x^{(R-1)}z^{\rho}\mu^{\widetilde{\rho}}d\right)^{-\frac{|\rho(m)+\widetilde{\rho}(n)|}{2}}\times$$
$$\times\frac{((-1)^{\rho(m)+\widetilde{\rho}(n)}z^{-\rho}\mu^{-\widetilde{\rho}}d^{-1}x_2^{|\rho(m)+\widetilde{\rho}(n)|+2-R};x^2)_\infty}{((-1)^{\rho(m)+\widetilde{\rho}(n)}z^{\rho}\mu^{\widetilde{\rho}}d\,x_2^{|\rho(m)+\widetilde{\rho}(n)|+R};x^2)_\infty} \tag{4.6.11}$$

In the following, we will only have two types of chiral fields: the bifundamental chirals forming the $\mathcal{N} = 4$ hypermultiplets and the adjoint chirals inside the $\mathcal{N} = 4$ vector multiplets. For both these types of chirals we will rewrite the contribution Z_Φ (4.6.11) in a more explicit way, making use of the simpler contribution of a chiral transforming under a single $U(1)$ symmetry with fugacity and flux $\{z, m\}$ respectively

$$Z_{\text{chir}}\left(\{z,m\},d;R;x\right) := (x^{R-1}zd)^{-\frac{|m|}{2}}\prod_{j=0}^{\infty}\frac{1-(-1)^m z^{-1}d^{-1}x^{|m|+2-R+2j}}{1-(-1)^m zdx^{|m|-R+2j}}\,. \tag{4.6.12}$$

Unitary case

Let us consider the unitary case first. The contributions of the chiral fields in this case are as follows:

- The two $\mathcal{N} = 2$ bifundamental chirals stretched between two $U(N)$ and $U(M)$ symmetry groups contribute

$$\begin{aligned}
&Z_\Phi^{U(N)\times U(M)}(\{\boldsymbol{z},\boldsymbol{m}\},\{\boldsymbol{w},\boldsymbol{n}\},d;x)\\
&=\prod_{i=1}^{N}\prod_{j=1}^{M}Z_{\text{chir}}\left(\{z_iw_j^{-1},m_i-n_j\},d^{-1};\frac{1}{2};x\right)Z_{\text{chir}}\left(\{z_i^{-1}w_j,m_j-m_i\},d^{-1};\frac{1}{2};x\right)=\\
&=\prod_{i=1}^{N}\prod_{j=1}^{M}\left[(z_iw_j^{-1}x^{R-1}d^{-1})^{-\frac{1}{2}(|m_i-n_j|)}\frac{((-1)^{m_i-n_j}z_i^{-1}w_jdx^{2-r+|m_i-n_j|}d^{-1};x^2)}{((-1)^{m_i-n_j}z_iw_j^{-1}d^{-1}x^{r+|m_i-n_j|}d;x^2)}\right].
\end{aligned} \tag{4.6.13}$$

  where $\{\boldsymbol{z},\boldsymbol{m}\}$ and $\{\boldsymbol{w},\boldsymbol{n}\}$ are the fugacities and the corresponding magnetic fluxes for $U(N)$ and $U(M)$ respectively.

  If one of the groups is of the $SU(N)$ type, the fugacities and the corresponding background fluxes must then be subject to the conditions

$$\prod_{i=1}^{N} z_i = 1 \ , \qquad \sum_{i=1}^{N} m_i = 0 \ . \tag{4.6.14}$$

- The $\mathcal{N}=2$ adjoint chiral in the $\mathcal{N}=4$ vector multiplet associated to a $U(N)$ gauge group contributes

$$Z_{\varphi}^{U(N)}(\{\boldsymbol{z},\boldsymbol{m}\},d;x) = \prod_{i,j=1}^{N} Z_{\text{chir}}\left(\{z_i z_j^{-1}, m_i - m_j\}, d^2; 1; x\right) \tag{4.6.15}$$

Orthosymplectic case

In this case, for the sake of simplicity, let us turn off the axial fugacity $d=1$.

- For the hypers in the bifundamental of $SO(2N+\epsilon)\times USp(2M)$ for $\epsilon = 0,1$ we thus get

$$\begin{aligned} &Z_{\Phi}^{SO(2N+\epsilon)\times USp(2M)}(\{\boldsymbol{z},\boldsymbol{m}\},\{\boldsymbol{w},\boldsymbol{n}\};\chi=+1;x) = \\ &= \left[\prod_{j=1}^{M} Z_{\text{chir}}\left(\{w_j^{-1},-n_j\};\frac{1}{2};x\right) Z_{\text{chir}}\left(\{w_j,n_j\};\frac{1}{2};x\right)\right]^{\epsilon} \\ &\times \prod_{i=1}^{N}\prod_{j=1}^{M} Z_{\text{chir}}\left(\{z_i w_j, m_i+n_j;\}\frac{1}{2};x\right) Z_{\text{chir}}\left(\{z_i w_j^{-1}, m_i-n_j\};\frac{1}{2};x\right) \\ &\times Z_{\text{chir}}\left(\{z_i^{-1} w_j, m_i-n_j\};\frac{1}{2};x\right) Z_{\text{chir}}\left(\{z_i^{-1} w_j^{-1}, -m_i-n_j\};\frac{1}{2};x\right) \ , \end{aligned} \tag{4.6.16}$$

where, as previously, $\{\boldsymbol{z},\boldsymbol{m}\}$ and $\{\boldsymbol{w},\boldsymbol{n}\}$ are the fugacities and the corresponding fluxes for the left node and the right node respectively.

The last expression holds only for $\chi=+1$. The correct contribution for $\chi=-1$ for the even case $\epsilon=0$ is obtained by setting $z_N=1$, $z_N^{-1}=-1$ and $m_N=0$.

For the odd case, *i.e.* when $\epsilon=1$, we have instead a compact expression for generic χ

$$\begin{aligned} &Z_{\Phi}^{SO(2N+1)\times USp(2M)}(\{\boldsymbol{z},\boldsymbol{m}\},\{\boldsymbol{w},\boldsymbol{n}\};\chi;x) = \\ &= \prod_{j=1}^{M} Z_{\text{chir}}\left(\{\chi w_j^{-1},-n_j\};\frac{1}{2};x\right) Z_{\text{chir}}\left(\{\chi^{-1} w_j,n_j\};\frac{1}{2};x\right) \\ &\times \prod_{i=1}^{N}\prod_{j=1}^{M} Z_{\text{chir}}\left(\{z_i w_j, m_i+n_j\};\frac{1}{2};x\right) Z_{\text{chir}}\left(\{z_i w_j^{-1}, m_i-n_j\};\frac{1}{2};x\right) \\ &\times Z_{\text{chir}}\left(\{z_i^{-1} w_j, m_i-n_j\};\frac{1}{2};x\right) Z_{\text{chir}}\left(\{z_i^{-1} w_j^{-1}, -m_i-n_j\};\frac{1}{2};x\right) \ , \end{aligned} \tag{4.6.17}$$

- For the adjoint chiral we get instead the following expressions

$$
\begin{aligned}
&Z_{\varphi}^{USp(2N)}(\{\boldsymbol{z},\boldsymbol{m}\};x) = \\
&\quad \times \prod_{i,j=1}^{N} Z_{\text{chir}}\left(\{z_i z_j^{-1}, m_i - m_j\}, d^2; 1; x\right) \\
&\quad \times \prod_{i\leq j}^{N} Z_{\text{chir}}\left(\{z_i z_j, m_i + m_j\}, d^2; 1; x\right) Z_{\text{chir}}\left(\{z_i^{-1} z_j^{-1}, -m_i - m_j\}, d^2; 1; x\right)
\end{aligned}
\tag{4.6.18}
$$

$$
\begin{aligned}
&Z_{\varphi}^{SO(2N+\epsilon)}(\{\boldsymbol{z},\boldsymbol{m}\};\chi=1;x) = \\
&\quad \times \left[\prod_{i=1}^{N} Z_{\text{chir}}\left(\{z_i^{-1}, -m_i\}, d^2; 1; x\right) Z_{\text{chir}}\left(\{z_i, m_i\}, d^2; 1; x\right)\right]^{\epsilon} \\
&\quad \times \prod_{i<j}^{N} Z_{\text{chir}}\left(\{z_i z_j, m_i + m_j\}, d^2; 1; x\right) Z_{\text{chir}}\left(\{z_i z_j^{-1}, m_i - m_j\}, d^2; 1; x\right) \\
&\quad \times Z_{\text{chir}}\left(\{z_i^{-1} z_j, -m_i + m_j\}, d^2; 1; x\right) Z_{\text{chir}}\left(\{z_i^{-1} z_j^{-1}, -m_i - m_j\}, d^2; 1; x\right)
\end{aligned}
\tag{4.6.19}
$$

where again the last expression holds only for $\chi = +1$. The correct contribution with $\chi = -1$ for $\epsilon = 0$ is obtained by setting $z_N = 1$, $z_N^{-1} = -1$ and $m_N = 0$.

For $\epsilon = 1$, we have instead a compact expression for generic χ

$$
\begin{aligned}
&Z_{\varphi}^{SO(2N+1)}(\{\boldsymbol{z},\boldsymbol{m}\};\chi=1;x) = \\
&\quad \times \prod_{i=1}^{N} Z_{\text{chir}}\left(\{\chi z_i^{-1}, -m_i\}, d^2; 1; x\right) Z_{\text{chir}}\left(\{\chi^{-1} z_i, m_i\}, d^2; 1; x\right) \\
&\quad \times \prod_{i<j}^{N} Z_{\text{chir}}\left(\{z_i z_j, m_i + m_j\}, d^2; 1; x\right) Z_{\text{chir}}\left(\{z_i z_j^{-1}, m_i - m_j\}, d^2; 1; x\right) \\
&\quad \times Z_{\text{chir}}\left(\{z_i^{-1} z_j, -m_i + m_j\}, d^2; 1; x\right) Z_{\text{chir}}\left(\{z_i^{-1} z_j^{-1}, -m_i - m_j\}, d^2; 1; x\right)
\end{aligned}
\tag{4.6.20}
$$

Chapter 5

Marginal operators and supersymmetry enhancement in 3d S-fold SCFTs

The main goal of this chapter is to investigate through the use of the superconformal index the operators associated with the $\mathcal{N}=2$ preserving exactly marginal deformations[1] in a large class of 3d superconformal field theories (SCFTs) with at least $\mathcal{N}=3$ supersymmetry, known as the 3d S-fold theories [14, 88, 89, 91, 93–97, 154].

Indeed, the space generated by such exactly marginal deformations, also known as the conformal manifold, has been a long-standing subject of study in QFTs and, when considering SCFTs, such conformal manifolds have several rich structures. For example, as demonstrated in [101, 129, 130], conformal manifolds of 4d $\mathcal{N}=1$ and 3d $\mathcal{N}=2$ SCFTs can be described by a symplectic quotient of the space of marginal couplings by the complexified continuous global symmetry group[2]. Moreover, for 4d $\mathcal{N}=2$ SCFTs, as shown by several recent findings *e.g.* [143, 145, 147], the study of conformal manifolds has led to a number of intriguing dualities; these include 4d $\mathcal{N}=1$ weakly coupled Lagrangian descriptions of several strongly coupled 4d $\mathcal{N}=2$ SCFTs. These provide solid motivation for studying exactly marginal operators in the SCFTs and in this chapter we will see lots of examples.

We will start by introducing the $T(U(N))$ and $T^{[2,1^2]}_{[2,1^2]}(SU(4))$ theories, along with their indices. Then, by gauging the diagonal flavour symmetries of the two, we will consider the 3d S-fold theories obtained with such building blocks by adding a Chern–Simons level and fundamental hypermultiplets. The marginal operators are thus studied by means of the superconformal index.

5.1 Index of $T^{\sigma}_{\rho}(SU(N))$ theories

One of the important by-products of the detailed study of the exactly marginal operators in S-fold theories is that we can extract the information of the conserved currents, which include $\mathcal{N}=3$ flavour currents and $\mathcal{N}=3$ extra SUSY-currents (see (4.5.108) and below). As we have seen, from the latter, we can determine whether supersymmetry gets enhanced at the fixed point, and if so we can also deduce the amount of supersymmetry of the SCFT. This heavily relies on the superconformal index [3, 4, 32, 33, 71, 117, 122, 126] of the S-fold theory in question, thus in the following sections we will be interested in the index expressions of such S-fold theories.

[1]It should be noted that, for any 3d $\mathcal{N}=3$ SCFT, there is no $\mathcal{N}=3$ preserving marginal deformation [58]. This statement also holds for $\mathcal{N}\geq 3$.

[2]See also [34] for the conformal manifold of 3d $\mathcal{N}=2$ Chern–Simons–matter theories.

Before doing so, however, it is important to better understand the index expressions of theories of the $T_{\boldsymbol{\rho}}^{\boldsymbol{\sigma}}(SU(N))$ family. The most general quiver diagram for such theories is depicted in Fig. (3.3.1). Indeed, as we will see later, S-fold theories can be constructed by some gaugings of this linear theories.

5.1.1 The $T(SU(N))$ and $T(U(N))$ theories

We can start by considering $T_{\boldsymbol{\rho}}^{\boldsymbol{\sigma}}(SU(N))$ for $\boldsymbol{\sigma} = \boldsymbol{\sigma} = [1^N]$. In Fig. (3.3.10) we already draw the quiver for $T(SU(2N+1))$.

The different contributions to the superconformal index are:

- The couples of $\mathcal{N}=2$ bifundamental chirals forming the $\mathcal{N}=4$ hypermultiplets stretched between the $U(j)$ and the $U(j+1)$ gauge groups, whose contribution is (4.6.13);
- The adjoint $\mathcal{N}=2$ chirals, with contributions (4.6.15), and the $\mathcal{N}=2$ vector multiplets, with contributions (4.6.2), forming the $\mathcal{N}=4$ vector multiplets;
- The classical contribution (4.6.1) coming from each topological symmetry.

Putting all together, the index of such a theory reads

$$
\begin{aligned}
&\mathcal{I}_{T(SU(N))}(\{(\omega_1\dots,\omega_N),(\nu_1,\dots,\nu_N)\},\{(\mu_1\dots,\mu_N),(n_1,\dots,n_N)\},d;x) \\
&= \sum_{m_1^{(1)}\in\mathbb{Z}}\ \sum_{m_1^{(2)},m_2^{(2)}\in\mathbb{Z}} \cdots \sum_{m_1^{(N-1)},\dots,m_{N-1}^{(N-1)}\in\mathbb{Z}}\ \prod_{j=1}^{N-1}\frac{1}{j!}\prod_{k=1}^{j}\oint\frac{dz_k^{(j)}}{2\pi i z_k^{(j)}}\,\omega_j^{m_k^{(j)}}\left(z_k^{(j)}\right)^{\nu_j}\times \\
&\prod_{j=1}^{N-1} Z_{\Phi}^{U(j)\times U(j+1)}(\{\boldsymbol{z}^{(j)},\boldsymbol{m}^{(j)}\},\{\boldsymbol{z}^{(j+1)},\boldsymbol{m}^{(j+1)}\},d;x)\ Z_{\varphi}^{U(j)}(\{\boldsymbol{z}^{(j)},\boldsymbol{m}^{(j)}\},d;x)\times \\
&Z_{\Phi}^{U(N-1)\times SU(N)}(\{\boldsymbol{z}^{(N-1)},\boldsymbol{m}^{(N-1)}\},\{\boldsymbol{\mu},\boldsymbol{n}\},d;x)\prod_{j=1}^{N} Z_V^{U(j)}(\{\boldsymbol{z}^{(j)},\boldsymbol{m}^{(j)}\};x)\ ,
\end{aligned}
\tag{5.1.1}
$$

where $\{(\omega_1\dots,\omega_N),(\nu_1,\dots,\nu_N)\}$, and $\{(\mu_1\dots,\mu_N),(n_1\dots,n_N)\}$ are the fugacities and the corresponding fluxes for the (enhanced) $SU(N)$ topological simmetries and the $SU(N)$ flavour symmetries respectively. These, being of the $SU(N)$ type, are both subject to the conditions (4.6.14).

It is interesting to point out that the index of $T(SU(N))$ satisfies the following property

$$
\begin{aligned}
&\mathcal{I}_{T(SU(N))}(\{\boldsymbol{\mu},\boldsymbol{n}\},\{\boldsymbol{\omega},\boldsymbol{\nu}\},d;x) \\
&= \frac{\omega_N^{n_1+\cdots+n_N}(\mu_1\cdots\mu_N)^{\nu_N}}{\mu_N^{\nu_1+\cdots+\nu_N}(\omega_1\cdots\omega_N)^{n_N}}\times\mathcal{I}_{T(SU(N))}(\{\boldsymbol{\omega},\boldsymbol{\nu}\},\{\boldsymbol{\mu},\boldsymbol{n}\},d^{-1};x)
\end{aligned}
\tag{5.1.2}
$$

where, upon imposing the conditions (4.6.14), the prefactor indicated in red is equal to unity.

The index of the $T(U(N))$ theory is defined as follows:

$$
\begin{aligned}
&\mathcal{I}_{T(U(N))}(\{\boldsymbol{\omega},\boldsymbol{\nu}\},\{\boldsymbol{\mu},\boldsymbol{n}\},d;x) \\
&= \omega_N^{n_1+\cdots+n_N}(\mu_1\cdots\mu_N)^{\nu_N}\times\mathcal{I}_{T(SU(N))}(\{\boldsymbol{\omega},\boldsymbol{\nu}\},\{\boldsymbol{\mu},\boldsymbol{n}\},d;x)
\end{aligned}
\tag{5.1.3}
$$

where we do *not* impose the conditions (4.6.14) in this definition. Since $T(U(N))$ is a product of $T(SU(N))$ and $T(U(1))$ [85], where $T(U(1))$ is an almost empty theory containing only the mixed Chern–Simons term, we regard the blue factor as the index of the $T(U(1))$ theory[3]. It follows from (5.1.2) that the index of $T(U(N))$ satisfies

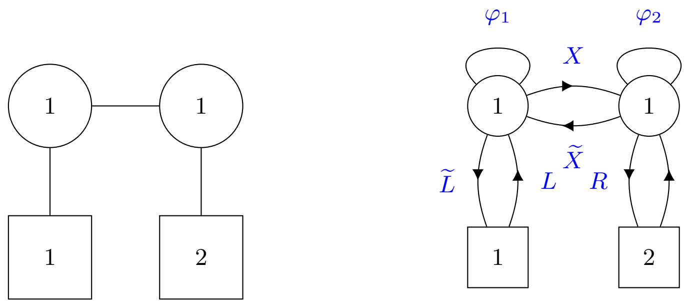
$$\mathcal{I}_{T(U(N))}(\{\boldsymbol{\mu},\boldsymbol{n}\},\{\boldsymbol{\omega},\boldsymbol{\nu}\},d;x) = \mathcal{I}_{T(U(N))}(\{\boldsymbol{\omega},\boldsymbol{\nu}\},\{\boldsymbol{\mu},\boldsymbol{n}_{\boldsymbol{\mu}}\},d^{-1};x)\ . \tag{5.1.4}$$

Upon setting the background fluxes to zero, $\boldsymbol{\nu}=\boldsymbol{n}=\boldsymbol{0}$, the indices of $T(U(N))$ and $T(SU(N))$ are equal. In the following, we will be interested in the power series of such indices up to order x^2.

It is important to note that both $T(SU(N))$ and $T(U(N))$ possess an $SU(N)_H \times SU(N)_C$ Higgs and Coulomb branch symmetry whose fugacities are $\{\boldsymbol{\mu},\boldsymbol{n}\}$ and $\{\boldsymbol{\omega},\boldsymbol{\nu}\}$ respectively.

5.1.2 The $T_{[2,1^2]}^{[2,1^2]}(SU(4))$ theory

The $T_{[2,1^2]}^{[2,1^2]}(SU(4))$ is a 3d $\mathcal{N}=4$ SCFT that admits a Lagrangian description in terms of a linear quiver [85] (see also Fig. (3.3.3))



(5.1.5)

The index of this theory can be computed from the quiver description (5.3.91) as

$$\begin{aligned}
&\widehat{\mathcal{I}}_{(5.3.91)}(\{\omega_1,\nu_1\},\{\omega_2,\nu_2\},\{a,n_a\},\{(b_1,b_2),(n_{b_1},n_{b_2})\},d;x)\\
&= \sum_{m_1,m_2\in\mathbb{Z}} \oint \frac{dz_1}{2\pi z_1}\oint \frac{dz_2}{2\pi i z_2}\omega_1^{m_1} z_1^{\nu_1}\omega_1^{m_2} z_2^{\nu_2}\times\\
&\quad Z_{L,\widetilde{L}}(\{z_1,m_1\},\{a,n_a\},d;x) Z_{R,\widetilde{R}}(\{z_2,m_2\},\{\boldsymbol{b},\boldsymbol{n_b}\},d;x)\times\\
&\quad Z_{X,\widetilde{X}}(\{z_1,m_1\},\{z_2,m_2\},d;x) Z_{\varphi_1}(\{z_1,m_1\},d;x) Z_{\varphi_2}(\{z_2,m_2\},d;x)\ ,
\end{aligned} \tag{5.1.6}$$

where $\{\omega_1,\nu_1\},\{\omega_2,\nu_2\}$ are the topological fugacities and the corresponding fluxes for each $U(1)$ gauge group, $\{z_1,m_1\}$, $\{z_2,m_2\}$ are gauge fugacities and fluxes for each $U(1)$ gauge group. The fugacities and the corresponding background fluxes for the $U(1)$ and $U(2)$ flavour symmetries are denoted by $\{a,n_a\}$ and $\{(b_1,b_2),(n_{b_1},n_{b_2})\} = \{\boldsymbol{b},\boldsymbol{n_b}\}$ respectively. The fugacity d corresponds to the axial symmetry, as described above.

Looking at Eqs. (4.6.13) and (4.6.15), the contributions of the chiral fields in the theory are as follows:

[3]The importance of this contact term for the $T(U(N))$ theory at the level of the S_b^3 partition function was already noticed in [45, (3.26)] and in [7, (4.6)].

$$\begin{aligned}
Z_{L,\widetilde{L}}(\{z_1,m_1\},\{a,n_a\},d;x) &= Z_{\Phi}^{U(1)\times U(1)}(\{z_1,m_1\},\{a,n_a\},d;x)\\
Z_{R,\widetilde{R}}(\{z_2,m_2\},\{\boldsymbol{b},\boldsymbol{n_b}\},d;x) &= Z_{\Phi}^{U(1)\times U(2)}(\{z_2,m_2\},\{(b_1,b_2),(n_{b_1},n_{b_2})\},d;x)\\
Z_{X,\widetilde{X}}(\{z_1,m_1\},\{z_2,m_2\},d;x) &= Z_{\Phi}^{U(1)\times U(1)}(\{z_1,m_1\},\{z_2,m_2\},d;x)\\
Z_{\varphi_1}(\{z_1,m_1\},d;x) &= Z_{\varphi}^{U(1)}(\{z_1,m_1\},d;x)\\
Z_{\varphi_2}(\{z_2,m_2\},d;x) &= Z_{\varphi}^{U(1)}(\{z_2,m_2\},d;x)
\end{aligned} \tag{5.1.7}$$

Setting the background magnetic fluxes to zero, $\nu_1 = \nu_2 = n_a = n_{b_1} = n_{b_2} = 0$, and setting $d = 1$, we obtain the following series expansion of $\widehat{\mathcal{I}}_{(5.3.91)}$ in x:

$$\begin{aligned}
&\widehat{\mathcal{I}}_{(5.3.91)}(\{\omega_1,0\},\{\omega_2,0\},\{a,0\},\{(b_1,b_2),(0,0)\},d=1;x)\\
&= 1 + x\left(\frac{b_1}{b_2}+\frac{b_2}{b_1}+\omega_1+\frac{1}{\omega_1}+4\right)\\
&\quad + x^{\frac{3}{2}}\left(\frac{a}{b_1}+\frac{a}{b_2}+\frac{b_1}{a}+\frac{b_2}{a}+\omega_1\omega_2+\omega_2+\frac{1}{\omega_2}+\frac{1}{\omega_1\omega_2}\right)\\
&\quad + x^2\Big(\frac{b_1\omega_1}{b_2}+\frac{b_1}{b_2\omega_1}+\frac{b_2}{b_1\omega_1}+\frac{b_2\omega_1}{b_1}\\
&\qquad +\frac{b_1^2}{b_2^2}+\frac{2b_1}{b_2}+\frac{b_2^2}{b_1^2}+\frac{2b_2}{b_1}+\omega_1^2+2\omega_1+\frac{2}{\omega_1}+\frac{1}{\omega_1^2}+2\Big)+\ldots\ .
\end{aligned} \tag{5.1.8}$$

Since the $T^{[2,1^2]}_{[2,1^2]}(SU(4))$ theory is self-mirror, the Higgs and Coulomb branch symmetries are equal, each of which is $\left(\frac{U(2)\times U(1)}{U(1)}\right)$. We shall then rewrite $\widehat{\mathcal{I}}_{(5.3.91)}$ in such a way that the fugacities and the correponding background fluxes of such symmetries appear on equal footing. For this purpose, we make the following reparametrisation:

$$\begin{aligned}
&\omega_1 = w_1 w_2^{-1}\ , && \omega_2 = w_2\ , && b_1 = af_1\ , && b_2 = af_2\\
&\nu_1 = n_{w_1} - n_{w_2}\ , && \nu_2 = n_{w_2}\ , && n_{b_1} = n_a + n_{f_1}\ , && n_{b_2} = n_a + n_{f_2}\ .
\end{aligned} \tag{5.1.9}$$

Let us also define

$$\begin{aligned}
&\mathcal{I}_{(5.3.91)}(\{\boldsymbol{w},\boldsymbol{n_w}\},\{\boldsymbol{f},\boldsymbol{n_f}\},\{a,n_a\},d)\\
&:= \widehat{\mathcal{I}}_{(5.3.91)}(\{w_1w_2^{-1},n_{w_1}-n_{w_2}\},\{w_2,n_{w_2}\},\{a,n_a\},\\
&\qquad \{(af_1,af_2),(n_a+n_{f_1},n_a+n_{f_2})\},d)\ .
\end{aligned} \tag{5.1.10}$$

The function $\mathcal{I}_{(5.3.91)}$ has the following properties:

$$\begin{aligned}
&\mathcal{I}_{(5.3.91)}(\{\boldsymbol{w},\boldsymbol{n_w}\},\{\boldsymbol{f},\boldsymbol{n_f}\},\{a,n_a\},d;x)\\
&= w_1^{n_a}a^{n_{w_1}}\mathcal{I}_{(5.3.91)}(\{\boldsymbol{w},\boldsymbol{n_w}\},\{\boldsymbol{f},\boldsymbol{n_f}\},\{1,0\},d;x)\ .
\end{aligned} \tag{5.1.11}$$

and

$$\begin{aligned}
&\mathcal{I}_{(5.3.91)}(\{\boldsymbol{w},\boldsymbol{n_w}\},\{\boldsymbol{f},\boldsymbol{n_f}\},\{a,n_a\},d;x)\\
&= \left(\frac{w_1^{n_a}a^{n_{w_1}}}{f_1^{n_a}a^{n_{f_1}}}\right)\mathcal{I}_{(5.3.91)}(\{\boldsymbol{f},\boldsymbol{n_f}\},\{\boldsymbol{w},\boldsymbol{n_w}\},\{a,n_a\},d^{-1};x)\ .
\end{aligned} \tag{5.1.12}$$

If we define

$$\begin{aligned}&\widehat{I}_{(5.3.91)}(\{\boldsymbol{w},\boldsymbol{n_w}\},\{\boldsymbol{f},\boldsymbol{n_f}\},\{a,n_a\},d;x)\\&:=f_1^{n_a}a^{n_{f_1}}\times\mathcal{I}_{(5.3.91)}(\{\boldsymbol{w},\boldsymbol{n_w}\},\{\boldsymbol{f},\boldsymbol{n_f}\},\{a,n_a\},d;x)\ ,\end{aligned} \tag{5.1.13}$$

then the identity (5.1.12) implies that the index $\widehat{I}_{(5.3.91)}$ satisfies the following condition

$$\widehat{I}_{(5.3.91)}(\{\boldsymbol{w},\boldsymbol{n_w}\},\{\boldsymbol{f},\boldsymbol{n_f}\},\{a,n_a\},d;x)=\widehat{I}_{(5.3.91)}(\{\boldsymbol{f},\boldsymbol{n_f}\},\{\boldsymbol{w},\boldsymbol{n_w}\},\{a,n_a\},d^{-1};x)\ . \tag{5.1.14}$$

Note that the prefactor indicated in blue in (5.1.13) indicates a mixed Chern–Simons term, similarly to the $T(U(N))$ theory[4].

For simplicity, in the following, we will focus on the case $\{a,n_a\}=\{1,0\}$ and define

$$I_{(5.3.91)}(\{\boldsymbol{w},\boldsymbol{n_w}\},\{\boldsymbol{f},\boldsymbol{n_f}\},d;x):=\widehat{I}_{(5.3.91)}(\{\boldsymbol{w},\boldsymbol{n_w}\},\{\boldsymbol{f},\boldsymbol{n_f}\},\{1,0\},d;x)\ , \tag{5.1.15}$$

so that the index satisfies the following property:

$$I_{(5.3.91)}(\{\boldsymbol{w},\boldsymbol{n_w}\},\{\boldsymbol{f},\boldsymbol{n_f}\},d;x)=I_{(5.3.91)}(\{\boldsymbol{f},\boldsymbol{n_f}\},\{\boldsymbol{w},\boldsymbol{n_w}\},d^{-1};x)\ . \tag{5.1.16}$$

The series expansion of $I_{(5.3.91)}$ in x when $\boldsymbol{n_w}=\boldsymbol{n_f}=(0,0)$ is as follows:

$$\begin{aligned}&I_{(5.3.91)}(\{\boldsymbol{w},\boldsymbol{0}\},\{\boldsymbol{f},\boldsymbol{0}\},d;x)\\&=1+x\left[d^{-2}\left(\frac{f_1}{f_2}+\frac{f_2}{f_1}+2\right)+d^2\left(\frac{w_2}{w_1}+\frac{w_1}{w_2}+2\right)\right]+\\&\quad+x^{3/2}\left[d^{-3}\left(f_1+f_2+\frac{1}{f_2}+\frac{1}{f_1}\right)+d^3\left(w_1+w_2+\frac{1}{w_1}+\frac{1}{w_2}\right)\right]\\&\quad+x^2\Big[\frac{f_1w_2}{f_2w_1}+\frac{f_1w_1}{f_2w_2}+\frac{f_2w_2}{f_1w_1}+\frac{f_2w_1}{f_1w_2}+d^{-4}\Big(\frac{f_1^2}{f_2^2}+\frac{f_2^2}{f_1^2}+\frac{2f_1}{f_2}+\frac{2f_2}{f_1}+3\Big)\\&\qquad+d^4\Big(\frac{w_1^2}{w_2^2}+\frac{w_2^2}{w_1^2}+\frac{2w_1}{w_2}+\frac{2w_2}{w_1}+3\Big)-4\Big]+\dots\ .\end{aligned} \tag{5.1.17}$$

Setting $\{a,n_a\}=\{1,0\}$ amounts to modding out the $U(1)$ factor in the numerator of the symmetry $\frac{U(2)\times U(1)}{U(1)}$ by the $U(1)$ in the denominator; the result is then identified with the $U(2)$ symmetry for the Higgs or the Coulomb branch.

It is convenient to rewrite the index (5.1.17) by setting

$$w_1=bu\ ,\quad w_2=bu^{-1}\ ,\quad f_1=qh\ ,\quad f_2=qh^{-1} \tag{5.1.18}$$

so that

[4] The importance of contact terms for the $T_\rho^\sigma[SU(N)]$ theory at the level of the S_b^3 partition function was noticed in [114], see for example equation (2.56) of that reference for the case of $\sigma=[2,1^2]$ and $\rho=[1^4]$.

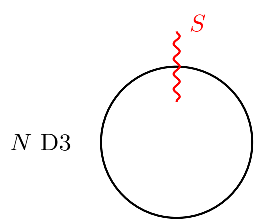
$$
\begin{aligned}
&I_{(5.3.91)}(\{(bu,bu^{-1}),\mathbf{0}\},\{(qh,qh^{-1}),\mathbf{0}\},d;x)\\
&= 1 + x\left[d^2\left(1+\chi^{SU(2)}_{[2]}(u)\right) + d^{-2}\left(1+\chi^{SU(2)}_{[2]}(h)\right)\right]\\
&\quad + x^{\frac{3}{2}}\left[d^3(b+b^{-1})\chi^{SU(2)}_{[1]}(u) + d^{-3}(q+q^{-1})\chi^{SU(2)}_{[1]}(h)\right]\\
&\quad + x^2\Big[d^4\left(1+\chi^{SU(2)}_{[2]}(u)+\chi^{SU(2)}_{[4]}(u)\right) + d^{-4}\left(1+\chi^{SU(2)}_{[2]}(h)+\chi^{SU(2)}_{[4]}(h)\right)\\
&\qquad + \chi^{SU(2)}_{[2]}(u)\chi^{SU(2)}_{[2]}(h) - \left(\chi^{SU(2)}_{[2]}(h)+1\right) - \left(\chi^{SU(2)}_{[2]}(u)+1\right) - 1\Big] + \dots
\end{aligned}
\tag{5.1.19}
$$

where the blue terms denote the contribution of the $U(2)\times U(2)$ global symmetry of the theory and the brown term -1 denotes the contribution of the $U(1)_d$ axial symmetry.

5.2 S-Fold theories

Now we have all the elements to discuss S-fold theories, so we start with the so-called pure S-fold theories.

These theories can be realised through Hanany-Witten construction by considering D3 branes wrapping a circle with the presence of $SL(2,\mathbb{Z})$ duality walls [13, 85, 102]. These are surfaces passing through which the system undergoes an S-duality transformation and thus give rise to a local $SL(2,\mathbb{Z})$ action on the worldvolume theory of D3 branes. The brane setup is depicted in Fig. (5.2.1) where we used a red wiggle line to denote the S-duality wall.



(5.2.1)

Using the same nomenclature proposed in [14], an S-duality wall is also called S-fold, or S-flip if we want to stress the absence of Chern–Simons terms.

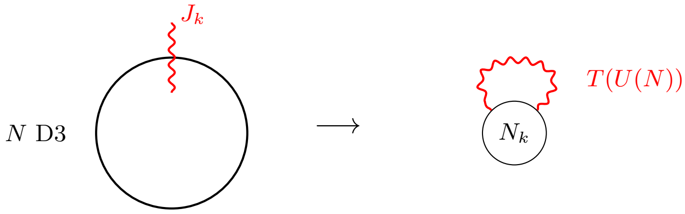
Elements of $SL(2,\mathbb{Z})$ can be always written as combination of the two S and T generators (3.2.2); Chern–Simons levels can be turned on considering walls for more general elements in $SL(2,\mathbb{Z})$, namely:

$$
J_k = -ST^k = \begin{pmatrix} k & 1 \\ -1 & 0 \end{pmatrix} \tag{5.2.2}
$$

For a duality wall associated with such an element J_k of $SL(2,\mathbb{Z})$, the corresponding theory can be described by the gauging of the diagonal $U(N)$ global symmetry of the $T(U(N))$ theory [85] with Chern–Simons level k [14, 88, 89, 91, 94, 154][5]. The situation is depicted in Fig. (5.2.3) where we used a red wiggle line to denote

[5] We emphasise that the S-fold theories considered in [88, 89, 91, 154] were constructed by gauging the diagonal $SU(N)$ global symmetry of the $T(SU(N))$ theory. These theories were studied in the context of 3d-3d correspondence. However, the gauge groups of the theories studied in [14] were taken to be of the unitary type. Without any further hypermultiplets added to the theory, it was shown in [97] that the index of these two families of theories are equal. In this work, we take the gauge groups to be of the unitary type.

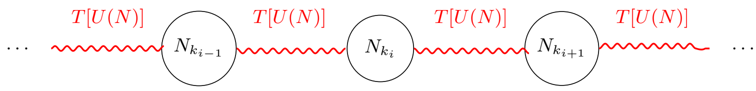
both the duality wall in the brane setup and the $T(U(N))$ theory with gauged global symmetries in the quiver diagram.



(5.2.3)

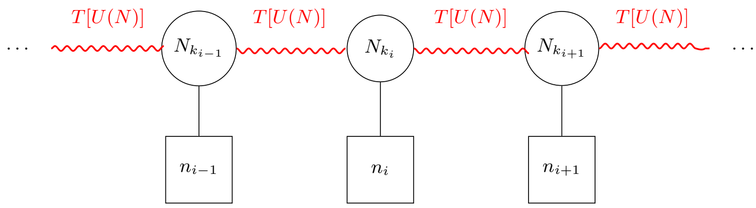
As a result of this gauging along with the presence of the CS level, the description possesses $\mathcal{N} = 3$ supersymmetry. However, at the infrared (IR) fixed point, it was shown that for $k \geq 3$ supersymmetry gets enhanced to $\mathcal{N} = 4$ in the case of $N = 2$ [88, 97] and in the large N limit [14].

This result can be generalised to the S-fold theories associated with multiple duality walls whose description can be written in terms of a 'quiver diagram' with multiple $U(N)_{k_i}$ gauge nodes, possibly with non-zero CS levels, connected by $T(U(N))$ links [14].



(5.2.4)

In addition to the pure S-fold theories, we may couple hypermultiplets to $U(N)_{k_i}$ gauge groups in the former. In terms of the brane configuration, this could be viewed as adding D5 and/or NS5 branes to the aforementioned brane system in the same way as described in [107] (see also Section (3.1)).



(5.2.5)

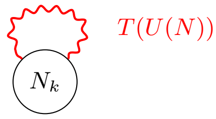
The resulting theories were investigated in [14] for vanishing CS levels and in [97] for general CS levels. Some of the latter were shown to exhibit supersymmetry enhancement (even up to $\mathcal{N} = 5$) and have interesting dualities that can be regarded a generalisation of 3d mirror symmetry, discovered in [118]. We shall henceforth refer to the pure S-fold theories, constructed as described above, and those coupled to hypermultiplets collectively as S-fold theories with $T(U(N))$ building block.

As we have seen in the previous section, similarly to the $T(U(N))$ theory, the $T^{[2,1^2]}_{[2,1^2]}(SU(4))$ is also self-mirror. Thus, as done with the $T(U(N))$ theory, we can form other S-fold theories by gauging the diagonal symmetry $G = \left(\frac{U(2)\times U(1)}{U(1)}\right)$ of $T^{[2,1^2]}_{[2,1^2]}(SU(4))$, possibly with a CS level. As before, we may also couple hypermultiplets to the diagonal symmetry G. In principle, this construction can be applied to a more general $T^{\boldsymbol{\rho}}_{\boldsymbol{\rho}}(SU(N))$ theory. However, due to various technicalities in the computation, we restrict ourselves to $N = 4$ and $\boldsymbol{\rho} = [2, 1^2]$. We shall henceforth refer to these theories collectively as S-fold theories with the $T^{[2,1^2]}_{[2,1^2]}(SU(4))$ building block.

As done with the linear quivers, in the following sections we will summarize the index expressions for such S-fold building blocks.

5.2.1 The $T(U(N))$ theory as a building block

Let us now commonly gauge the diagonal subgroup of the $SU(N)_H$ Higgs and $SU(N)_C$ Coulomb branch symmetries of the $T(U(N))$ theory and obtain the following theory



(5.2.6)

The index of theory (5.2.6) reads

$$\begin{aligned}\mathcal{I}_{(5.2.6);k,N}(\{\omega,\nu\},d;x) = &\sum_{m_1,m_2\dots,m_N\in\mathbb{Z}} \frac{1}{N!}\prod_{j=1}^{N}\oint \frac{dz_j}{2\pi i z_j}\,\omega^{m_j} z_j^{km_j+\nu}\times\\ &Z_V^{U(N)}(\{(z_1,z_2,\dots,z_N),(m_1,m_2,\dots,m_N)\};x)\times\\ &\mathcal{I}_{T(U(N))}(\{(z_1,z_2\dots,z_N),(m_1,m_2,\dots,m_N)\},\\ &\qquad\{(z_1^{-1},z_2^{-1}\dots,z_N^{-1}),(-m_1,-m_2,\dots,-m_N)\},d;x)\ ,\end{aligned} \tag{5.2.7}$$

where now $\{\omega,\nu\}$ are the fugacity and the respective flux for the topological symmetry associated to the newly introduced gauge group $U(N)$. In the following we will turn off ν by setting $\nu=0$. Note in the $T(U(N))$ contribution $\mathcal{I}_{T(U(N))}$ the convention that, when gauging the Higgs and Coulomb branch symmetries of the $T(U(N))$ theory, they come in opposite way z_j and z_j^{-1} (also m_j and $-m_j$) for $j=1,\dots,N$. In the notation of [14], this corresponds to the $U(N)_-=\mathrm{diag}(U(N)\times U(N)^\dagger)$ choice of gauging the Higgs and Coulomb branch symmetries of $T(U(N))$. Another choice of gauging corresponds to the index

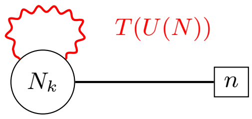
$$\begin{aligned}\widehat{\mathcal{I}}_{(5.2.6);k,N}(\{\omega,\nu\},d) = &\sum_{m_1,m_2\dots,m_N\in\mathbb{Z}} \frac{1}{N!}\prod_{j=1}^{N}\oint \frac{dz_j}{2\pi i z_j}\,\omega^{m_j} z_j^{km_j+\nu}\times\\ &Z_V^{U(N)}(\{(z_1,z_2,\dots,z_N),(m_1,m_2,\dots,m_N)\};x)\times\\ &\mathcal{I}_{T(U(N))}(\{(z_1,z_2\dots,z_N),(m_1,m_2,\dots,m_N)\},\\ &\qquad\{(z_1,z_2\dots,z_N),(m_1,m_2,\dots,m_N)\},d;x)\ ,\end{aligned} \tag{5.2.8}$$

where in the notation of [14], this choice corresponds to the $U(N)_+=\mathrm{diag}(U(N)\times U(N))$ type of gauging. It follows from (5.1.3) and from the fact that the index of $T(SU(N))$ is invariant under inversion of the $SU(N)$ fugacities because of the Weyl group of $SU(N)$, that the indices corresponding to these two types of gauging are related by the flipping of the sign of k together with the sign of the background topological flux n up to the change of variables $z_i\to z_i^{-1}$:

$$\mathcal{I}_{(5.2.6);k,N}(\{\omega,\nu\},d;x) = \widehat{\mathcal{I}}_{(5.2.6);-k,N}(\{\omega,-\nu\},d;x)\ . \tag{5.2.9}$$

For definiteness, we use the convention of (5.2.7), namely the $U(N)_-$ type of gauging, throughout the rest of this chapter.

We can add n hypermultiplets in the fundamental representation of $U(N)$ to theory (5.2.6) and obtain



(5.2.10)

Since the axial $U(1)_d$ symmetry is broken by the presence of such fundamental hypermultiplets, we set $d = 1$.

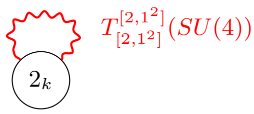
The index of theory (5.2.10) is thus

$$\begin{aligned}
\mathcal{I}_{(5.2.10);k,N,n}(\omega; \{\boldsymbol{\mu}, \boldsymbol{n}\}) = \sum_{m_1, m_2 \ldots, m_N \in \mathbb{Z}} \frac{1}{N!} \prod_{j=1}^{N} \oint \frac{dz_j}{2\pi i z_j} \, \omega^{m_j} z_j^{k m_j} \times \\
Z_V^{U(N)}(\{(z_1, z_2, \ldots, z_N), (m_1, m_2, \ldots, m_N)\}; x) \times \\
\mathcal{I}_{T(U(N))}(\{(z_1, z_2 \ldots, z_N), (m_1, m_2, \ldots, m_N)\}, \\
\{(z_1^{-1}, z_2^{-1} \ldots, z_N^{-1}), (-m_1, -m_2, \ldots, -m_N)\}, d = 1; x) \times \\
Z_\Phi^{U(N) \times SU(n)}(\{(z_1, z_2 \ldots, z_N), (m_1, m_2, \ldots, m_N)\}, \\
\{(\mu_1, \mu_2, \ldots, \mu_n), (n_1, n_2, \ldots, n_n)\}; x) \; .
\end{aligned} \tag{5.2.11}$$

In the above expression, we turned off the background flux for the topological symmetry. In the following, we will also set the background flavour magnetic fluxes to zero, $\boldsymbol{n} = \boldsymbol{0}$.

5.2.2 The $T_{[2,1^2]}^{[2,1^2]}(SU(4))$ theory as a building block

We now consider the following theory



(5.2.12)

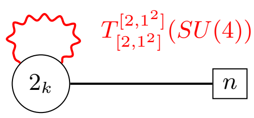
formed by gauging the diagonal subgroup of the Higgs and Coulomb branch $U(2)$ symmetries of $T_{[2,1^2]}^{[2,1^2]}(SU(4))$.

Its index reads

$$\begin{aligned}
\mathcal{I}_{(5.2.12)}(k; \{\omega, \nu\}; x) = \sum_{m_1, m_2 \in \mathbb{Z}} \frac{1}{2!} \left[\prod_{j=1}^{2} \oint \frac{dz_j}{2\pi i z_j} \, w^{m_j} z_j^{k m_j + n} \right] Z_V^{U(2)}(\{(z_1, z_2), (m_1, m_2)\}; x) \times \\
I_{(5.3.91)}(\{(z_1, z_2), (m_1, m_2)\}, \{(z_1^{-1}, z_2^{-1}), (-m_1, -m_2)\}, d = 1; x) \; ,
\end{aligned} \tag{5.2.13}$$

where now $\{\omega, \nu\}$ are the fugacity and the respective flux for the topological symmetry and the contribution $Z_V^{U(2)}$ of the $U(2)$ vector multiplet is given by (4.6.2). In the following we will turn off ν by setting $\nu = 0$. The axial symmetry $U(1)_d$ is broken and so we set $d = 1$ in the above expression.

Similarly to the case of $T(U(N))$, we can couple n flavours of the fundamental hypermultiplets to the $U(2)$ gauge group of theory (5.2.12). This results in theory



(5.2.14)

whose index is

$$
\begin{aligned}
&\mathcal{I}_{(5.2.12);k,n}(\omega;\{\boldsymbol{\mu},\boldsymbol{n}\};x) \\
&= \sum_{m_1,m_2\in\mathbb{Z}} \frac{1}{2!}\left[\prod_{j=1}^{2}\oint \frac{dz_j}{2\pi i z_j}\,\omega^{m_j} z_j^{km_j}\right] Z_V^{U(2)}(\{(z_1,z_2),(m_1,m_2)\};x)\times \\
&\quad I_{(5.3.91)}(\{(z_1,z_2),(m_1,m_2)\},\{(z_1^{-1},z_2^{-1}),(-m_1,-m_2)\},d=1;x)\times \\
&\quad Z_\Phi^{U(2)\times SU(n)}(\{(z_1,z_2),(m_1,m_2)\}\{(\mu_1,\mu_2,\ldots,\mu_n),(n_1,n_2,\ldots,n_n)\};x)\ .
\end{aligned}
\tag{5.2.15}
$$

where we turned off the background magnetic flux for the topological symmetry in the above expression. In the following, we will also set the background fluxes for the flavour symmetries to zero, $\boldsymbol{n}=0$, and use the fugacity map:

$$
\mu_1 = qh_1, \quad \mu_2 = qh_2h_1^{-1}\ , \quad \mu_3 = qh_3h_2^{-1}\ , \quad \ldots\ , \quad \mu_n = qh_{n-1}^{-1}\ , \tag{5.2.16}
$$

where $h_1,\ldots,h_n$ are the fugacities of the $SU(n)$ flavour symmetry and q is the fugacity for the $U(1)$ flavour symmetry.

5.3 Marginal operators of the S-fold theories

The problem of enumerating all marginal operators in 3d S-fold SCFTs becomes more complicated as the number of the operators with R-charges up to 2 increases. This is partly due to the fact that not all gauge invariant quantities that one can possibly write down are independent from each other. They may be subject to various relations. Some of these relations can actually be derived from the effective superpotential of the theory. However, as we shall see in the subsequent sections, several S-fold theories contain gauge invariant monopole operators and dressed monopole operators in the spectrum, whose existence is indicated by the index. There can also be relations between these operators that cannot be obtained from the effective superpotential. In this case, we conjecture the form of such relations based on the index and in analogue of those known in other 3d $\mathcal{N}=4$ gauge theories presented in Appendix (B). In this regard, the S-fold theories with the $T_{[2,1^2]}^{[2,1^2]}(SU(4))$ building block are much more complicated than those with the $T(U(N))$ building block. In this work, we thus only present preliminary results for the former theories.

5.3.1 S-fold theories with the $T(U(N))$ building block

In this section, we consider S-folds theories whose building block is the $T(U(N))$ theory. Several aspects of a number of such theories with $N=2$ were studied in [97]. We will focus on the cases of $N=2$ and $N=3$ (except in subsection (5.3.1) where we discuss only the case of $N=2$) and analyse the operators with R-charge up to two in detail.

The $T(U(N))$ theory

We briefly discuss some important aspects of the $T(U(N))$ theory in Section (5.1.1). The indices for $N=2$, 3 can be obtained from (5.1.3) and the explicit result is as follows:

$$
\begin{aligned}
N=2: \quad & 1+x\left(d^2\chi^{SU(2)}_{[2]}(\omega)+d^{-2}\chi^{SU(2)}_{[2]}(\mu)\right)+x^2\Big[d^4\chi^{SU(2)}_{[4]}(\omega)+d^{-4}\chi^{SU(2)}_{[4]}(\mu) \\
& \quad -\left(\chi^{SU(2)}_{[2]}(\omega)+\chi^{SU(2)}_{[2]}(\mu)\right)-1\Big]+\ldots \\
N\geq 3: \quad & 1+x\left(d^2\chi^{SU(N)}_{[1,0,\ldots,0,1]}(\boldsymbol{\omega})+d^{-2}\chi^{SU(N)}_{[1,0,\ldots,0,1]}(\boldsymbol{\mu})\right)+x^2\Big[d^4\chi^{SU(N)}_{[2,0,\ldots,0,2]}(\boldsymbol{\omega}) \\
& +d^4\chi^{SU(N)}_{[0,1,0,\ldots,0,1,0]}(\boldsymbol{\omega})+d^4\chi^{SU(N)}_{[1,0,\ldots,0,1]}(\boldsymbol{\omega})+(d\to d^{-1},\boldsymbol{\omega}\to\boldsymbol{\mu}) \\
& +\chi^{SU(N)}_{[1,0,\ldots,0,1]}(\boldsymbol{\omega})\chi^{SU(N)}_{[1,0,\ldots,0,1]}(\boldsymbol{f}) \\
& -\left(\chi^{SU(N)}_{[1,0,\ldots,0,1]}(\boldsymbol{\omega})+\chi^{SU(N)}_{[1,0,\ldots,0,1]}(\boldsymbol{\mu})\right)-1\Big]+\ldots
\end{aligned}
\tag{5.3.1}
$$

where the term -1 highlighted in brown is the contribution of the axial $U(1)_d$ symmetry.

Since we shall make extensive use of $\mathcal{N}=3$ supersymmetry in subsequent discussion, it is instructive to view the result from the perspective of the $\mathcal{N}=3$ index, where d is set to unity. The terms at order x are the contribution of the $\mathcal{N}=3$ $SU(N)\times SU(N)$ flavour currents and these terms appear again as negative terms at order x^2 and the term -1 highlighted in brown is the contribution of the $\mathcal{N}=3$ extra SUSY-current (see (4.5.108) and below). This is as expected since the theory has $\mathcal{N}=4$ supersymmetry. We also point out the absence of the term $\chi^{SU(2)}_{[2]}(\omega)\chi^{SU(2)}_{[2]}(\mu)$ at order x^2 for $N=2$.

The operators with R-charge 1 are the Higgs and Coulomb branch moment maps of $T(U(N))$:

$$
(\mu_H)^i_j\ , \qquad (\mu_C)^{i'}_{j'}\ . \tag{5.3.2}
$$

They are subject to the nilpotent conditions (see [85, below (3.6)] and (2.2.76)):

$$
\mu_H^N=\mu_C^N=0\ . \tag{5.3.3}
$$

These imply that all eigenvalues of μ_H and μ_C are zero and so

$$
\mathrm{tr}(\mu_H^p)=\mathrm{tr}(\mu_C^p)=0\ , \qquad 1\leq p\leq N\ . \tag{5.3.4}
$$

There are two types of marginal operators, namely the pure Higgs or Coulomb branch operators and the mixed branch operators. The pure Higgs or Coulomb branch marginal operators transform in a subrepresentation of

$$
\begin{aligned}
\mathrm{Sym}^2[1,0,\ldots,0,1]=&\,[2,0,\ldots,0,2]+[1,0,\ldots,0,1] \\
&+[0,1,0,\ldots,0,1,0]+[0,\ldots,0]
\end{aligned}
\tag{5.3.5}
$$

of each $SU(N)$. Such operators are

$$
(\mu_H)^i_j(\mu_H)^k_l\ , \qquad (\mu_C)^{i'}_{j'}(\mu_C)^{k'}_{l'} \tag{5.3.6}
$$

Since $\mathrm{tr}(\mu_H^2)=\mathrm{tr}(\mu_C^2)=0$, the singlet $[0,\ldots,0]$ in (5.3.5) vanishes. Thus, each of these operators transform under the representation $[2,0,\ldots,0,2]+[1,0,\ldots,0,1]+[0,1,0,\ldots,0,1,0]$ of each $SU(N)$ for $N\geq 3$[6]. For $N=2$, we have stronger conditions,

[6]For $N=3$, such a representation reduces to $[2,2]+[1,1]$.

namely $\mu_H^2 = \mu_C^2 = 0$, and so each operator in (5.3.6) transforms under [4] of each $SU(2)$.

Next, we consider the marginal mixed branch operators. In the case of $N = 2$, we have

$$(\mu_H)^i_j (\mu_C)^{i'}_{j'} = 0 \ , \qquad \text{for } N = 2 \tag{5.3.7}$$

for the following reason. The F-terms with respect to the chiral multiplets Q and $\widetilde{Q}$ give $Q^i\varphi = 0$ and $\widetilde{Q}_i\varphi = 0$, and so $(\mu_H)^i_j\varphi = Q^i\widetilde{Q}_j\varphi = 0$. Since (V_+, φ, V_-) transform in a triplet of an unbroken $SU(2)$ global symmetry, we have $(\mu_H)^i_j V_\pm = 0$ and so $(\mu_H)^i_j(\mu_C)^{i'}_{j'} = 0$. This explains the absence of the term $\chi^{SU(2)}_{[2]}(\omega)\chi^{SU(2)}_{[2]}(\mu)$ at order x^2 in the index (5.3.1) for $N = 2$. Note, however, that for $N \geq 3$ the operators

$$(\mu_H)^i_j (\mu_C)^{i'}_{j'} \tag{5.3.8}$$

do not vanish.

$U(1)_{k-2}$ gauge theory

In this section, we briefly review S-fold theories with the $T(U(1))$ building block. Although it turns out that these theories are simply ordinary 3d $\mathcal{N} = 3$ Chern–Simons matter theories[7], they are useful for comparing and contrasting with those constructed using the $T(U(N))$ theory with $N > 1$.

The $T(U(1))$ theory is an almost trivial theory with a recipe for coupling external abelian vector multiplets containing gauge fields A_1 and A_2 [85]. Such a coupling is the supersymmetric completion of the following Chern–Simons term:

$$-\frac{1}{2\pi}\int A_1 \wedge dA_2 \ . \tag{5.3.9}$$

In building an S-fold theory starting from $T(U(1))$, when commonly gauging the $U(1) \times U(1)$ symmetry into $U(1)_k$, the term (5.3.9) automatically gives rise to a Chern–Simons level -2. After combining with the Chern–Simons level k, we see that the S-fold theory in question is nothing but the $U(1)_{k-2}$ gauge theory.

From the perspective of the index, the mixed Chern–Simons term in $T(U(N))$ contributes $\omega_N^{n_1+\cdots+n_N}(\mu_1\cdots\mu_N)^{\nu_N}$, where, as in Section (5.1.1), $\{\boldsymbol{\omega}, \boldsymbol{\nu}\}$ are the $U(N)$ topological fugacities and the associated background fluxes and $\{\boldsymbol{\mu}, \boldsymbol{n}\}$ are the $U(N)$ flavour fugacities and the associated background fluxes. When both $U(N)$ are commonly gauged, we set $\mu_i = z_i$, $\omega_i = z_i^{-1}$, $n_i = m_i$, $\nu_i = -m_i$, for $i = 1, \ldots, N$, where z_i are the gauge fugacities and m_i are the corresponding gauge fluxes. This results in $(z_1\cdots z_N)^{-m_N} z_N^{-m_1-\ldots-m_N}$. In the case of $N = 1$, this is simply $z_1^{-2m_1}$, which is the contribution of the $U(1)$ gauge group with Chern–Simons level -2. Together with the term $z_1^{km_1}$ due to Chern–Simons level k of the $U(1)$ gauge group, we have $z_1^{(k-2)m_1}$, which is the contribution of the $U(1)$ gauge group with Chern–Simons level $k-2$, as expected.

The superpotential for the 3d $\mathcal{N} = 3$ $U(1)_{k-2}$ pure gauge theory is

$$W = -\frac{k-2}{4\pi}\varphi^2 \ . \tag{5.3.10}$$

[7] In fact, the pure S-fold theories (*i.e.* those without hypermultiplet matter) of this type were considered in [93, 94]. These are simply pure abelian Chern–Simons theories with several $U(1)$ gauge groups, with mixed Chern–Simons couplings between them.

For $k \neq 2$, φ can be integrated out, and we are left with a topological field theory. For $k = 2$, we have the theory of a free $\mathcal{N} = 4$ abelian vector multiplet.

We can also couple n flavours of hypermultiplets to this theory and obtain the 3d $\mathcal{N} = 3$ $U(1)_{k-2}$ gauge theory with n flavours, whose superpotential is

$$W = -\frac{k-2}{4\pi}\varphi^2 + \widetilde{Q}^i \varphi Q_i \ , \tag{5.3.11}$$

with $i = 1, \ldots, n$. Note that, for $k = 2$, this is in fact the 3d $\mathcal{N} = 4$ $U(1)$ gauge theory with n flavours.

The case of $n \geq 3$ flavours

Let us focus on the case of $n \geq 3$ flavours for the moment. The index of this theory, for $n \geq 3$, is

$$\begin{aligned}
k = 2: \quad & 1 + x\Big(1 + \chi^{SU(n)}_{[1,0,\ldots,0,1]}(\boldsymbol{\mu})\Big) + x^2\Big[\chi^{SU(n)}_{[2,0,\ldots,0,2]}(\boldsymbol{\mu}) - \Big(1 + \chi^{SU(n)}_{[1,0,\ldots,0,1]}(\boldsymbol{\mu})\Big)\Big] \\
& \quad + \ldots + (\omega + \omega^{-1})x^{\frac{n}{2}} + \ldots \\
k \neq 2: \quad & 1 + x\Big(1 + \chi^{SU(n)}_{[1,0,\ldots,0,1]}(\boldsymbol{\mu})\Big) + x^2\Big[\chi^{SU(n)}_{[2,0,\ldots,0,2]}(\boldsymbol{\mu}) - \Big(1 + \chi^{SU(n)}_{[1,0,\ldots,0,1]}(\boldsymbol{\mu})\Big)\Big] \\
& \quad + \ldots
\end{aligned} \tag{5.3.12}$$

We remark that the crucial difference between the cases of $k = 2$ and $k \neq 2$ are the terms $(\omega + \omega^{-1})x^{\frac{n}{2}}$ due to the presence of the gauge invariant monopole operators $X_\pm$ with R-charge $\frac{n}{2}$. For $n = 3, 4$, these monopole operators contribute with the terms at order $x^{\frac{3}{2}}$ and x^2 respectively. For $n \geq 5$, the index up to order x^2 of these cases are equal. Despite this equality, we emphasise that the operators in the cases of $k = 2$ and $k \neq 2$ are different. We will shortly describe these in detail.

For $k = 2$, the term $\mathrm{tr}(\varphi^2)$ in (5.3.11) is absent and the F-terms are

$$\widetilde{Q}^i \varphi = 0 \ , \qquad \varphi Q_i = 0 \ , \qquad \widetilde{Q}^i Q_i = 0 \ . \tag{5.3.13}$$

Due to the last equality, the mesons $M^i_j = \widetilde{Q}^i Q_j$ satisfy

$$M^i_i = 0 \ , \qquad (M^2)^i_j = M^i_k M^k_j = 0 \ . \tag{5.3.14}$$

Moreover, we have

$$\varphi M^i_j = 0 \ . \tag{5.3.15}$$

The operators with R-charge 1 are

$$\varphi \ , \qquad M^i_j \tag{5.3.16}$$

contributing $1+\chi^{SU(n)}_{[1,0,\ldots,0,1]}(\boldsymbol{\mu})$ at order x. The operators at order x^2 that contribute $\chi^{SU(n)}_{[2,0,\ldots,0,2]}(\boldsymbol{\mu})$ are

$$M^i_j M^k_l \tag{5.3.17}$$

satisfying (5.3.14). There is, however, another marginal operator, namely

$$\varphi^2 \ . \tag{5.3.18}$$

The order x^2 of the index in the first line of (5.3.12) should be rewritten as

$$\ldots + x^2\Big[1+\chi^{SU(n)}_{[2,0,\ldots,0,2]}(\boldsymbol{\mu}) - \Big(1+\chi^{SU(n)}_{[1,0,\ldots,0,1]}(\boldsymbol{\mu})\Big) - 1\Big] + \ldots \tag{5.3.19}$$

where the contribution from the $\mathcal{N}=3$ extra SUSY-current is highlighted in brown[8]. Due to the presence of this current, the corresponding IR SCFT has $\mathcal{N}=4$ supersymmetry, as expected.

Let us now assume that $k\neq 2$. The F-terms are

$$\varphi Q_i = 0 \ , \qquad \widetilde{Q}^i\varphi = 0 \ , \qquad \varphi = \frac{2\pi}{k-2}\widetilde{Q}^i Q_i \ . \tag{5.3.20}$$

The meson matrix $M^i_j = \widetilde{Q}^i Q_j$ thus satisfies the conditions

$$\varphi M^i_j = 0 \ , \qquad \varphi = \frac{2\pi}{k-2} M^i_i \ . \tag{5.3.21}$$

Note that φ can be integrated out using the last equality, after which the effective superpotential is

$$W_{\text{eff}} = \frac{\pi}{k-2}(\widetilde{Q}^i Q_i)^2 = \frac{\pi}{k-2}(M^i_i)^2 \ . \tag{5.3.22}$$

Multiplying M^j_k to both sides of the second equation of (5.3.21) and using the first equation of (5.3.21), we obtain

$$(M^i_i)M^j_k = 0 \ . \tag{5.3.23}$$

Contracting the indices j and k, we see that M^i_i is nilpotent:

$$(M^i_i)^2 = 0 \ . \tag{5.3.24}$$

The operators with R-charge 1 are

[8] From the perspective of the $\mathcal{N}=2$ index, this -1 can be viewed as the contribution of the axial symmetry, denoted by $U(1)_d$ in the main text, under which φ carries charge $+2$ and each of $Q_i, \widetilde{Q}^j$ carries charge -1. Note that this symmetry is broken when $k\neq 2$.

$$M_i^i \ , \qquad \widehat{M}_j^i := M_j^i - \frac{1}{n}(M_k^k)\delta_j^i \ . \tag{5.3.25}$$

Using the identity

$$(\widehat{M}^2)_j^i = (M^2)_j^i - \frac{2}{n}(M_k^k)M_j^i + \frac{1}{n^2}(M_k^k)^2\delta_j^i \ , \tag{5.3.26}$$

and the conditions (5.3.23) and (5.3.24), we obtain

$$(\widehat{M}^2)_j^i = (M^2)_j^i = \widetilde{Q}^i Q_k \widetilde{Q}^k Q_j = (M_k^k)M_j^i \overset{(5.3.23)}{=} 0 \ . \tag{5.3.27}$$

Thus, the marginal operators are

$$\widehat{M}_j^i \, \widehat{M}_l^k \tag{5.3.28}$$

satisfying (5.3.27). These contribute the term $\chi^{SU(n)}_{[2,0,\ldots,0,2]}(\boldsymbol{\mu})$ at order x^2 in the index. In this case, we do not see the presence of an extra SUSY-current. The corresponding IR SCFT thus has $\mathcal{N}=3$ supersymmetry.

The case of $n=2$ flavours

The case of $k=2$ is simply the 3d $\mathcal{N}=4$ $U(1)$ gauge theory with 2 flavours or the $T(SU(2))$ theory, whose index is

$$\begin{aligned} &1 + x\left(\chi^{SU(2)}_{[2]}(\omega) + \chi^{SU(2)}_{[2]}(\mu)\right) + x^2\Big[\Big(\chi^{SU(2)}_{[4]}(\omega) + \chi^{SU(2)}_{[4]}(\mu) \\ &\quad - \left(\chi^{SU(2)}_{[2]}(\omega) + \chi^{SU(2)}_{[2]}(\mu)\right) - 1\Big] + \ldots \ , \end{aligned} \tag{5.3.29}$$

where we redefined the topological fugacity ω to be ω^2. The operators with R-charge 1 are M_j^i, satisfying (5.3.14), together with

$$C = \begin{pmatrix} \varphi & X_+ \\ X_- & -\varphi \end{pmatrix} \ , \tag{5.3.30}$$

satisfying $(C^2)^{i'}_{j'} = C^{i'}_{k'} C^{k'}_{j'} = 0$. Due to (5.3.15), we also have

$$C^{i'}_{j'} M^i_j = 0 \ . \tag{5.3.31}$$

The marginal operators are

$$C^{i'}_{j'} C^{k'}_{l'} \ , \qquad M^i_j M^k_l \ . \tag{5.3.32}$$

The contribution of the $\mathcal{N}=3$ extra SUSY-current is highlighted above in brown. The index for the case of $k \neq 2$ is simply (5.3.12) with $n=2$:

$$1 + x\left(1 + \chi_{[2]}^{SU(2)}(\boldsymbol{\mu})\right) + x^2\left[\chi_{[4]}^{SU(2)}(\boldsymbol{\mu}) - \left(1 + \chi_{[2]}^{SU(2)}(\boldsymbol{\mu})\right)\right] . \tag{5.3.33}$$

The operators with R-charges up to 2 are as described previously.

The case of $n = 1$ flavour

For $k = 2$, we have the 3d $\mathcal{N} = 4$ $U(1)$ gauge theory with 1 flavour, which flows to the theory of a free hypermultiplet.

For $k \neq 2$, the operator with R-charge 1 is M, satisfying $M^2 = 0$ due to (5.3.24). There is no marginal operator in this case. The indices are

$$\begin{aligned} k \neq 1,\, 2,\, 3: &\quad 1 + 1x - 1x^2 + 2x^3 + \dots \\ k = 1: &\quad 1 + 1x + (-1 - \omega q^{-1} - \omega^{-1} q)x^2 + (2 + \omega q^{-1} + \omega^{-1} q)x^3 + \dots \\ k = 3: &\quad 1 + 1x + (-1 - \omega q - \omega^{-1} q^{-1})x^2 + (2 + \omega q + \omega^{-1} q^{-1})x^3 + \dots \end{aligned} \tag{5.3.34}$$

where we redefined the μ_1 flavour fugacity as q to highlight its $U(1)$ nature.

For $k \neq 1,\, 2,\, 3$, we don't see the presence of an extra SUSY-current, and so we conclude that the theory has $\mathcal{N} = 3$ supersymmetry. On the other hand, for $k = 1, 3$, where the theory is simply the $U(1)_{\pm 1}$ gauge theory with 1 flavours, we found two $\mathcal{N} = 3$ extra SUSY-currents, and so we conclude that the theory has enhanced $\mathcal{N} = 5$ supersymmetry, as proposed in [97]. From the perspective of the $\mathcal{N} = 2$ index, the negative terms at order x^2 correspond to the conserved current, which indicates that the theory has an $SU(2) \cong Spin(3)$ global symmetry. This is a commutant of the $Spin(2)$ R-symmetry of $\mathcal{N} = 2$ supersymmetry in the $Spin(5)$ R-symmetry of $\mathcal{N} = 5$ supersymmetry.

$U(N)_k$ gauge group and zero flavour

In the following discussion in this paragraph, we assume that $N \geq 2$ and $k \neq 0$. The superpotential is (see also [86] and [88, (31)])

$$W = -\frac{k}{4\pi}\,\mathrm{tr}(\varphi^2) + \mathrm{tr}\left((\mu_C + \mu_H)\varphi\right) \ , \tag{5.3.35}$$

where μ_H and μ_C are the Higgs and Coulomb branch moment maps of $T(U(N))$. For $k \neq 0$, we can integrate out φ using the F-terms with respect to φ:

$$\varphi^a_b = \frac{2\pi}{k}(\mu_H + \mu_C)^a_b \ . \tag{5.3.36}$$

Using (5.3.4), we obtain the effective superpotential

$$W_{\text{eff}} = \frac{2\pi}{k}\,\mathrm{tr}(\mu_C \mu_H) \ . \tag{5.3.37}$$

Since μ_C and μ_H carry the axial $U(1)_d$ charges $+2$ and -2 respectively, the effective superpotential preserves the axial symmetry $U(1)_d$ in this case. This observation was actually pointed out in [88]. In fact, from the perspective of $\mathcal{N} = 3$ supersymmetry, the $U(1)_d$ symmetry plays a role as the extra SUSY-current. Indeed, $U(1)_d$ commutes with the $\mathcal{N} = 3$ R-symmetry $Spin(3)$; the former combines with the latter to become

$Spin(4)$ R-symmetry of the enhanced $\mathcal{N}=4$ supersymmetry. We shall also see this from the perspective of the index, which is given by (5.2.7).

Let us consider the case of $|k| \geq 3$. The indices for $N=2$ are as follows:

$$\begin{aligned} N=2,\ |k|=3: &\qquad 1+0x-2x^2+2(d^2+d^{-2})x^3+\dots\ . \\ N=2,\ |k|\geq 4: &\qquad 1+0x-x^2+(d^2+d^{-2})x^3+\dots\ . \end{aligned} \tag{5.3.38}$$

The case of $|k|=3$ was studied in [88], where it was pointed out that the theory in the IR is a product to two copies of the $\mathcal{N}=4$ SCFTs described by 3d $\mathcal{N}=2$ $U(1)$ gauge theory with CS level $-3/2$ and one chiral multiplet with charge $+1$, whose supersymmetry gets enhanced to $\mathcal{N}=4$ in the IR. The indices for the cases of $|k|\geq 4$ were studied in [97], where it was pointed out that supersymmetry gets enhanced to $\mathcal{N}=4$ in the IR. For $N=3$, the indices for $|k|\geq 3$ read

$$N=3,\ |k|\geq 3: \qquad 1+0x+0x^2-2x^3+\dots\ . \tag{5.3.39}$$

The operators up to R-charge 2 are as follows. Since $\operatorname{tr}\mu_H=\operatorname{tr}\mu_C=0$, there is no operator with R-charge 1. The $\mathcal{N}=3$ flavour symmetry of this theory therefore is empty. Let us now discuss about the marginal operators. From (5.3.4), we have

$$\operatorname{tr}(\mu_H^2)=\operatorname{tr}(\mu_C^2)=(\operatorname{tr}\mu_H)^2=(\operatorname{tr}\mu_C)^2=0\ . \tag{5.3.40}$$

Furthermore, for $N=2$, we also have $\operatorname{tr}(\mu_H\mu_C)=0$ due to the relation (5.3.7); thus the theory with $N=2$ and $|k|\geq 3$ has no marginal operator. In this case, we are able to see clearly the contribution of the extra SUSY current at order x^2 of the indices (5.3.38), since there is no cancellation between the contribution of the conserved currents and that of the marginal operators. For $N=2$ and $|k|\geq 4$, from the perspective of the $\mathcal{N}=2$ index $-x^2$ is the contribution of the $U(1)_d$ symmetry, whereas from the perspective of the $\mathcal{N}=3$ index this is the contribution of the extra SUSY-current. Indeed, we conclude that $\mathcal{N}=3$ supersymmetry gets enhanced to $\mathcal{N}=4$ for $N=2$ and $|k|\geq 4$ [97]. For $N=2$ and $|k|=3$, there are two extra SUSY conserved currents and this is due to the fact that the theory flows to a product of two $\mathcal{N}=4$ SCFTs[9] [88].

For $N=3$, on the other hand, there is precisely one marginal operator, namely $\operatorname{tr}(\mu_H\mu_C)$, which cancels the contribution of the $U(1)_d$ symmetry in the index; this explains the term $(1-1)x^2=0x^2$ in (5.3.39). Again, we identify the $U(1)_d$ conserved current with the $\mathcal{N}=3$ extra SUSY conserved current. We thus conclude that supersymmetry also gets enhanced to $\mathcal{N}=4$ for all $|k|\geq 3$. Although we demonstrated

[9] For $N=2$ and $|k|\geq 3$, there is no relevant, no marginal and no operator with R-charge 3, since $\operatorname{tr}(\mu_{H,C}^3)=\operatorname{tr}(\mu_H^2\mu_C)=\operatorname{tr}(\mu_H\mu_C^2)=0$, etc. It is thus simple to consider the contribution of the conserved currents at order x^3 of index (5.3.38) with $d=1$. From Table (4.5.107) and the remark below, we see that each of the $\mathcal{N}=3$ extra SUSY-current multiplet $A_2[0]_1^{(0)}$ and the $\mathcal{N}=3$ stress tensor multiplet $A_1[1]_{3/2}^{(0)}$ contributes $+x^3$ to $(1-x^2)(\text{index}-1)$. For $N=2$ and $|k|=3$, we have $(1-x^2)(\text{index}-1)=-2x^2+4x^3+\dots$; the term $+4x^3$ is indeed in agreement with the claim that there are two $\mathcal{N}=3$ extra SUSY-currents and two $\mathcal{N}=3$ stress tensors, since the theory is the product of two $\mathcal{N}=4$ SCFTs. For $N=2$ and $|k|\geq 4$, we have $(1-x^2)(\text{index}-1)=-x^2+2x^3+\dots$; the term $+2x^3$ is indeed in agreement with the claim that there are one $\mathcal{N}=3$ extra SUSY-current and one $\mathcal{N}=3$ stress tensor. Unfortunately, when there are relevant and marginal operators in the theory, the analysis of the index at order x^p, with $p\geq 3$, becomes very complicated. In the rest of the section, we focus only on the operators with R-charges up to 2.

this explicitly for $N = 2$ and $N = 3$, we conjecture that this statement holds for all $N \geq 2$.

For $|k| = 2$, we find that the index of the theory diverges and the theory is 'bad' in the sense of [85]. In fact, as we shall discuss in more detail in the next subsection, when n flavours of fundamental hypermultiplets are coupled to the theory with $k = 2$, there are gauge invariant monopole operators with R-charge $n/2$. In the special case of $n = 0$, these monopole operators with R-charge 0 render the theory 'bad'.

For $|k| = 1$ and $k = 0$, we find that the index is equal to unity, and it is expected that the theory flows to a topological theory or an empty theory.

$U(N)_k$ gauge group with $k \neq 0$ and $n \geq 1$ flavours

We propose that the superpotential for theory (5.2.10) is

$$\begin{aligned} W &= -\frac{k}{4\pi}\,\mathrm{tr}(\varphi^2) + \mathrm{tr}\left((\mu_C + \mu_H)\varphi\right) + \widetilde{Q}^i_b \varphi^b_a Q^a_i \\ &= -\frac{k}{4\pi}\,\mathrm{tr}(\varphi^2) + \mathrm{tr}\left((\mu_C + \mu_H + \mu_Q)\varphi\right) \ , \end{aligned} \tag{5.3.41}$$

where we define

$$M^i_j := \widetilde{Q}^i_a Q^a_i \ , \qquad (\mu_Q)^a_b = \widetilde{Q}^i_b Q^a_i \tag{5.3.42}$$

so that we have

$$M^i_i = \mathrm{tr}\,\mu_Q \ . \tag{5.3.43}$$

The following relations follow respectively from the F-terms with respect to $\widetilde{Q}^b_i$, Q^i_a and φ:

$$\varphi^a_b Q^i_a = 0 \ , \qquad \varphi^a_b \widetilde{Q}^b_i = 0 \ , \qquad \varphi^a_b = \frac{2\pi}{k}(\mu_H + \mu_C + \mu_Q)^a_b \ , \tag{5.3.44}$$

We discuss the consequences of these F-term on gauge invariant quantities in Appendix (C).

Using the last equality, we can integrate out φ and obtain the effective superpotential

$$W_{\text{eff}} = \frac{\pi}{k}\,\mathrm{tr}\left(\mu_C + \mu_H + \mu_Q\right)^2 \ . \tag{5.3.45}$$

From this effective superpotential, the F-terms with respect to $\widetilde{Q}^b_i$, Q^i_a are

$$(\mathrm{tr}\,\mu_Q) Q^i_b = 0 \ , \qquad (\mathrm{tr}\,\mu_Q)\widetilde{Q}^a_i = 0 \ . \tag{5.3.46}$$

These imply that

$$(\mathrm{tr}\,\mu_Q)\mu_Q = 0 \ , \qquad (\mathrm{tr}\,\mu_Q)^2 = 0 \ . \tag{5.3.47}$$

Let us define

$$\widehat{M}^i_j = M^i_j - \frac{1}{n}(M^k_k)\delta^i_j = M^i_j - \frac{1}{n}(\mathrm{tr}\,\mu_Q)\delta^i_j \ . \tag{5.3.48}$$

From (C.0.10) and (5.3.47), we obtain

$$
\begin{aligned}
(\widehat{M}^2)^i_j &= -(\mu_H+\mu_C)^b_a \widetilde{Q}^i_b Q^a_j - \frac{2}{n}\widehat{M}^i_j(\operatorname{tr}\mu_Q) \\
(\widehat{M}^2)^i_i &= -\operatorname{tr}\left[(\mu_H+\mu_C)\mu_Q\right] \ .
\end{aligned}
\tag{5.3.49}
$$

Apart from the gauge invariant quantities discussed above, there could possibly be gauge invariant monopole operators for some special values of k. Subsequently, we perform case by case analyses, with the aid of the index.

The case of $|k| \geq 3$, with $n \geq 1$ flavours

For $|k| \geq 3$, with $n \geq 1$, the indices can be computed from (5.2.11) and the results are as follows.

$$
\begin{aligned}
n \geq 3: &\quad 1 + x\left(1+\chi^{SU(n)}_{[1,0,\ldots,0,1]}(\boldsymbol{\mu})\right) + x^2\Big[\chi^{SU(n)}_{[2,0,\ldots,0,2]}(\boldsymbol{\mu}) + \chi^{SU(n)}_{[0,1,0,\ldots,0,1,0]}(\boldsymbol{\mu}) + \\
&\qquad + 3\chi^{SU(n)}_{[1,0,\ldots,0,1]}(\boldsymbol{\mu}) + s - \left(1+\chi^{SU(n)}_{[1,0,\ldots,0,1]}(\boldsymbol{\mu})\right)\Big] + \ldots \ , \\
n = 2: &\quad 1 + x\left(1+\chi^{SU(2)}_{[2]}(\mu)\right) + x^2\Big[\chi^{SU(2)}_{[4]}(\mu) + 2\chi^{SU(2)}_{[2]}(\mu) + s \\
&\qquad - \left(1+\chi^{SU(2)}_{[2]}(\mu)\right)\Big] + \ldots \ , \\
n = 1: &\quad 1 + 1x + (s'-1)x^2 + \ldots
\end{aligned}
\tag{5.3.50}
$$

where

$$
s = \begin{cases} 2 & N=2 \\ 3 & N=3 \end{cases} \qquad s' = \begin{cases} 1 & N=2 \\ 2 & N=3 \end{cases}
\tag{5.3.51}
$$

Let us now analyse the operators with R-charge up to 2 for $n \geq 2$. The operators with R-charge 1 are

$$
M^k_k = \operatorname{tr}\mu_Q \ , \qquad \widehat{M}^i_j
\tag{5.3.52}
$$

and so the flavour symmetry of the theory is $U(1) \times SU(n)$.

The marginal operators are as follows. For $n \geq 3$, the marginal operators contributing $3\chi^{SU(n)}_{[1,0,\ldots,0,1]}(\boldsymbol{\mu})$ to the index (5.3.50) are

$$
\widehat{M}^i_j(\operatorname{tr}\mu_Q) = \widehat{M}^i_j(M^k_k) \ , \qquad (\mathcal{A}_H)^i_j \ , \qquad (\mathcal{A}_C)^i_j \ ,
\tag{5.3.53}
$$

where we define $(\mathcal{A}_H)^i_j$ and $(\mathcal{A}_C)^i_j$ as in (C.0.13):

$$
\begin{aligned}
(\mathcal{A}_H)^i_j &:= (\mu_H)^a_b \widetilde{Q}^i_a Q^b_j - \frac{1}{n}\operatorname{tr}(\mu_H\mu_Q)\delta^i_j \ , \\
(\mathcal{A}_C)^i_j &:= (\mu_C)^a_b \widetilde{Q}^i_a Q^b_j - \frac{1}{n}\operatorname{tr}(\mu_C\mu_Q)\delta^i_j \ .
\end{aligned}
\tag{5.3.54}
$$

However, for $n=2$, we have an extra relation, namely (C.0.15):

$$
(\mathcal{A}_H)^i_j + (\mathcal{A}_C)^i_j = -\widehat{M}^i_j(\operatorname{tr}\mu_Q) = -\widehat{M}^i_j(M^k_k) \ , \quad \text{for } n=2 \ .
\tag{5.3.55}
$$

and so there are only two independent quantities of this type. The marginal operators that contribute to the term $\chi^{SU(n)}_{[0,1,0,\ldots,0,1,0]}(\boldsymbol{\mu})$ are

$$\epsilon^{i_1 i_2 \ldots i_n}\epsilon_{j_1 j_2 \ldots j_n}\widehat{M}^{j_1}_{i_1}\widehat{M}^{j_2}_{i_2} \ . \tag{5.3.56}$$

Those that contribute to the term $\chi^{SU(n)}_{[2,0,\ldots,0,2]}(\boldsymbol{\mu})$ are

$$R^{ik}_{jl} \tag{5.3.57}$$

which is a linear combination $\widehat{M}^i_j\widehat{M}^k_l$ and other quantities such that any contraction between an upper index and a lower index yields zero; for example, for $n=2$, where $\widehat{M}^2$ satisfies (C.0.9), the marginal operators in the representation $[4]_{\boldsymbol{\mu}}$ are

$$R^{ik}_{jl} := \widehat{M}^i_j\widehat{M}^k_l + \frac{1}{6}(\widehat{M}^2)^p_p\delta^i_j\delta^k_l - \frac{1}{3}(\widehat{M}^2)^p_p\delta^i_l\delta^k_j \ , \qquad \text{for } n=2 \ . \tag{5.3.58}$$

The marginal operators in the singlet of $SU(n)$ are

$$\mathrm{tr}(\mu_Q\mu_H) = (\mu_H)^a_b\widetilde{Q}^i_a Q^b_i \ , \quad \mathrm{tr}(\mu_Q\mu_C) = (\mu_C)^a_b\widetilde{Q}^i_a Q^b_i \ , \quad \mathrm{tr}(\mu_H\mu_C) \ . \tag{5.3.59}$$

Thus, there are 3 independent quantities of this type for $N \geq 3$, but for $N=2$ we have $\mathrm{tr}(\mu_H\mu_C)=0$ due to (5.3.7) and so we have only 2 independent quantities of this type. Explicitly, the order x^2 of the indices in (5.3.50) for $n \geq 2$ can be written as

$$\begin{aligned} N=2: &\quad \ldots + x^2\Big[\ldots + 2 - \Big(1+\chi^{SU(n)}_{[1,0,\ldots,0,1]}(\boldsymbol{\mu})\Big)\Big] + \ldots \\ N=3: &\quad \ldots + x^2\Big[\ldots + 3 - \Big(1+\chi^{SU(n)}_{[1,0,\ldots,0,1]}(\boldsymbol{\mu})\Big)\Big] + \ldots \end{aligned} \tag{5.3.60}$$

We do not see the presence of an extra SUSY-current. We thus conclude that, for $n \geq 2$, the theory has $\mathcal{N}=3$ supersymmetry. Although we have shown this explicitly for the cases of $N=2$ and $N=3$, we conjecture that this statement holds for any $N \geq 2$. We point out that, in the above analysis, there is also a symmetry that exchange the quantities with subscripts H and C. We shall shortly see that this symmetry is not present, for example, in the case of $k=2$ and $n=2$.

The above analysis also applies for $n=1$ with the following extra conditions:

$$\widehat{M} = \mathcal{A}_H = \mathcal{A}_C = 0 \ . \tag{5.3.61}$$

Moreover, due to (5.3.47), M is a nilpotent operator satisfying

$$M^2 = 0 \ . \tag{5.3.62}$$

It then follows from (5.3.49) that

$$\mathrm{tr}(\mu_H\mu_Q) = -\,\mathrm{tr}(\mu_C\mu_Q) \tag{5.3.63}$$

The operator with R-charge 1 is

$$M = \mathrm{tr}\, \mu_Q \ . \tag{5.3.64}$$

The $\mathcal{N}=3$ flavour symmetry of the theory is therefore $U(1)$. For $N=2$, there is one marginal operator, given by (5.3.63), contributing $+1x^2$ to the index. For $N=3$, in addition to (5.3.63), there is another marginal operator $\mathrm{tr}(\mu_H \mu_C)$; these two marginal operators contribute $+2x^2$ to the index. We do not see the presence of an extra SUSY-current for both $N=2$ and $N=3$. Thus, we conclude that the theory has $\mathcal{N}=3$ supersymmetry.

The case of $k=2$ and $n \geq 2$ flavours

For $k=2$, there are gauge invariant monopole operators with fluxes $(\pm 1, 0, \ldots, 0)$, denoted by $X_\pm := X_{(\pm 1, 0, \ldots, 0)}$, carrying R-charge $n/2$ and topological fugacity $\omega^{\pm 1}$. These operators contribute contribute with the terms $(\omega + \omega^{-1}) x^{\frac{n}{2}}$ to the index. The presence of these operators is analogous to the $T(U(1))$ case presented in Section (5.3.1), where the mixed CS term of $T(U(1))$ after self-gluing cancels with the bare CS level $k=2$.

For $n \geq 5$, the index up to order x^2 is the same as the case of $|k| \geq 3$ and $n \geq 3$ in (5.3.50), and so we expect that the operators up to R-charge 2 are as described in (5.3.52)–(5.3.59). For $n=4$, there are additional terms $(\omega + \omega^{-1}) x^2$ to the first two lines of (5.3.50), and so the monopole operators $X_\pm$ contribute as the additional marginal operators to those described above. For $n=3$, there are additional terms $(\omega + \omega^{-1}) x^{\frac{3}{2}}$ to the first two lines of (5.3.50), and so $X_\pm$ contribute as the addition operators with R-charge $3/2$ to those describe above.

The case of $k=2$ and $n=2$ flavours

Let us now analyse in detail the case of $k=2$ and $n=2$. From (5.2.11), the indices for $N=2$ and $N=3$ read

$$\begin{aligned} &1 + x \left(\chi^{SU(2)}_{[2]}(\omega) + \chi^{SU(2)}_{[2]}(\mu) \right) + x^2 \Big[\Big(2\chi^{SU(2)}_{[4]}(\omega) + \chi^{SU(2)}_{[4]}(\mu) \\ &\quad + \chi^{SU(2)}_{[2]}(\omega) \chi^{SU(2)}_{[2]}(\mu) + \chi^{SU(2)}_{[2]}(\mu) + s' \Big) - \left(\chi^{SU(2)}_{[2]}(\omega) + \chi^{SU(2)}_{[2]}(\mu) \right) \Big] \\ &\quad + \ldots \ , \end{aligned} \tag{5.3.65}$$

where we redefined the topological fugacity ω as ω^2, we highlighted the contribution of the $\mathcal{N}=3$ flavour symmetry in blue and

$$s' = \begin{cases} 1 & N=2 \ , \\ 2 & N=3 \ . \end{cases} \tag{5.3.66}$$

Note that the index for $N=2$ was computed in (4.25) of [97]. Let us discuss about the operators with R-charge up to 2. The operators with R-charge 1 are

$$\begin{array}{llll} [2]_\omega : & X_+ \ , & M^k_k = \mathrm{tr}\, \mu_Q \ , & X_- \\ [2]_f : & \widehat{M}^i_j & & \end{array} \tag{5.3.67}$$

and so the $\mathcal{N}=3$ flavour symmetry is $SU(2) \times SU(2)$.

Let us now discuss the marginal operators, corresponding to order x^2 in the index. The character $2\chi^{SU(2)}_{[4]}(\omega)$ contains the terms $2\omega^{\pm 4}$. These imply that there are two pairs of marginal operators such that each pair carries topological charges ± 2. One of such pairs is $X_\pm^2$ and we *propose* that the other pair consists of the monopole operators with fluxes $\pm(1,1,0,\ldots,0)$, denoted by $X_{++} := X_{(1,1,0,\ldots,0)}$ and $X_{--} := X_{(-1,-1,0,\ldots,0)}$, each carrying R-charge 2. This proposal is analogous to (B.1.10) of the 3d $\mathcal{N}=4$ $U(2)$ gauge theory with one adjoint and one fundamental hypermultiplet. Moreover, the character $2\chi^{SU(2)}_{[4]}(\omega)$ at order x^2 in the index contains the terms $2\omega^{\pm 2}$. These imply the existence of two pairs of marginal operators such that each pair carries topological charges ± 1. One pair can be immediately identified with $X_\pm(M^k_k)$ and we *propose* that the other pair corresponds to the 'dressed monopole operators' $X_{\pm;(0,1)}$, defined in a similar way to (B.2.5) (see [64]):

$$X_{(\pm 1,0);(r,s)} = (\pm 1,0) m_1^r m_2^s + (0,\pm 1) m_2^r m_1^s \ , \tag{5.3.68}$$

where μ_Q is diagonalised as $\mathrm{diag}(m_1,m_2)$[10]. This proposal is analogous to (B.2.4) of the 3d $\mathcal{N}=4$ $U(2)$ gauge theory with 4 flavours. In summary, the marginal operators that correspond to the terms $2\chi^{SU(2)}_{[4]}(\omega) + \chi^{SU(2)}_{[2]}(\omega)\chi^{SU(2)}_{[2]}(\mu)$ are

$$\begin{array}{llllll}
[4]_\omega : & X_+^2 \ , & X_+(M^k_k) \ , & X_+X_- \ , & X_-(M^k_k) \ , & X_-^2 \\
[4]_\omega : & X_{++} \ , & X_{+;(0,1)} \ , & (\widehat{M}^2)^i_i & X_{-;(0,1)} \ , & X_{--} \\
[2]_\omega[2]_f : & X_+\widehat{M}^i_j & \widehat{M}^i_j(M^k_k) & X_-\widehat{M}^i_j & &
\end{array} \tag{5.3.69}$$

The marginal operators that correspond to $\chi^{SU(2)}_{[4]}(\mu)$ are given in (5.3.58). Noting the relation (C.0.15), we see that the marginal operators corresponding to the term $\chi^{SU(2)}_{[2]}(\mu)$ can be taken to be either $(\mathcal{A}_H)^i_j$ or $(\mathcal{A}_C)^i_j$. Picking any of these choices necessarily breaks the symmetry that exchanges H and C.

Now let us consider the marginal operators that transform as singlets under $SU(n)$. Taking into account of (5.3.49), we can take two of out of three of $(\widehat{M}^2)^i_i$, $\mathrm{tr}(\mu_Q\mu_H)$ and $\mathrm{tr}(\mu_Q\mu_C)$ to be independent operators, but since $(\widehat{M}^2)^i_i$ has already been listed above, we are left with either $\mathrm{tr}(\mu_Q\mu_H)$ or $\mathrm{tr}(\mu_Q\mu_C)$. Hence, for $N\geq 3$, we see that the marginal operators in the singlet of $SU(n)$ are similar to (5.3.59), namely

$$\text{either } \mathrm{tr}(\mu_Q\mu_H) \text{ or } \mathrm{tr}(\mu_Q\mu_C) \ , \qquad \mathrm{tr}(\mu_H\mu_C) \ , \tag{5.3.70}$$

and so there are **two** operators of this type in this case. For $N=2$, $\mathrm{tr}(\mu_H\mu_C)=0$ due to (5.3.7) and so we have **one** operators of this type, namely $\mathrm{tr}(\mu_Q\mu_H)$ or $\mathrm{tr}(\mu_Q\mu_C)$. We can rewrite the indices for $N=2$ and $N=3$ as

$$\begin{aligned}
N=2: &\quad \ldots + x^2\Big[\ldots + 1 - \Big(\chi^{SU(2)}_{[2]}(\omega) + \chi^{SU(2)}_{[2]}(\mu)\Big)\Big] + \ldots \\
N=3: &\quad \ldots + x^2\Big[\ldots + 2 - \Big(\chi^{SU(2)}_{[2]}(\omega) + \chi^{SU(2)}_{[2]}(\mu)\Big)\Big] + \ldots
\end{aligned} \tag{5.3.71}$$

Again we do not see the presence of the extra SUSY-current. We thus conclude that the theory has $\mathcal{N}=3$ supersymmetry. Although we have shown this explicitly for the cases of $N=2$ and $N=3$, we conjecture that this statement holds for any $N\geq 2$.

[10] We dress the bare monopole operators with the components of μ_Q instead of those of φ, because for $k\neq 0$ we have integrated out φ but μ_Q remains massless.

The case of $k = 2$ and $n = 1$ flavour

Here we focus only on the case of $N = 2$ and postpone the discussion of $N = 3$ to future work. This is due to the complication of the computation of the index in the latter case. For $k = 2$ and $n = 1$, the index for $N = 2$ can be computed from (5.2.11) and the result is (see also [97, (4.20)]):

$$\begin{aligned}
&1 + x^{\frac{1}{2}}\left(\omega + \frac{1}{\omega}\right) + x\left(2\omega^2 + \frac{2}{\omega^2} + 2\right) \\
&\quad + x^{\frac{3}{2}}\left(2\omega^3 + \frac{2}{\omega^3} + 2\omega + \frac{2}{\omega}\right) + x^2\left(3\omega^4 + \frac{3}{\omega^4} + 2\omega^2 + \frac{2}{\omega^2} + 1\right) + \dots \\
&= 1 + x^{\frac{1}{2}}\chi^{SU(2)}_{[1]}(\omega) + 2x\chi^{SU(2)}_{[2]}(\omega) + x^{\frac{3}{2}}\left[2\chi^{SU(2)}_{[3]}(\omega) + \chi^{SU(2)}_{[1]}(\omega) - \chi^{SU(2)}_{[1]}(\omega)\right] \\
&\quad + x^2\left[3\chi^{SU(2)}_{[4]}(\omega) + \chi^{SU(2)}_{[2]}(\omega) - 2\chi^{SU(2)}_{[2]}(\omega) - 1\right] + \dots \\
&= \mathcal{I}_{\text{free}}(x;\omega) \times \left[1 + x\chi^{SU(2)}_{[2]}(\omega) + x^2\left(\chi^{SU(2)}_{[4]}(\omega) - \chi^{SU(2)}_{[2]}(\omega) - 1\right) + \dots\right]
\end{aligned} \tag{5.3.72}$$

where the monopole operators $X_\pm$ with fluxes $(\pm 1, 0, \dots, 0)$ have R-charge $1/2$ and decouple as a free hypermultiplet, which contribute to the index as

$$\begin{aligned}
\mathcal{I}_{\text{free}}(x;\omega) &= \frac{(x^{2-\frac{1}{2}}\omega; x^2)_\infty}{(x^{\frac{1}{2}}\omega^{-1}; x^2)_\infty} \frac{(x^{2-\frac{1}{2}}\omega^{-1}; x^2)_\infty}{(x^{\frac{1}{2}}\omega; x^2)_\infty} \\
&= 1 + \chi^{SU(2)}_{[1]}(\omega)x^{\frac{1}{2}} + \chi^{SU(2)}_{[2]}(\omega)x + \left[\chi^{SU(2)}_{[3]}(\omega)\right. \\
&\qquad \left. - \chi^{SU(2)}_{[1]}(\omega)\right]x^{\frac{3}{2}} + \left[\chi^{SU(2)}_{[4]}(\omega) - \chi^{SU(2)}_{[2]}(\omega) - 1\right]x^2 + \dots \,.
\end{aligned} \tag{5.3.73}$$

Note also that $\widehat{M} = 0$ in this case.

We now analyse the operators up to R-charge 2. The operators with R-charge $1/2$ are

$$X_+ \,, \qquad X_- \,, \tag{5.3.74}$$

where $X_\pm$ denote monopole with fluxes $\pm(1, 0, \dots, 0)$.

The operators with R-charge 1 are

$$\begin{array}{llll}
[2]_\omega : & X_{++} \,, & M = \operatorname{tr}\mu_Q \,, & X_{--} \,, \\
[2]_\omega : & X_+^2 \,, & X_+X_- \,, & X_-^2 \,,
\end{array} \tag{5.3.75}$$

where X_{++} and X_{--} denote monopole with fluxes $\pm(1, 1, 0, \dots, 0)$. Upon decoupling the free hypermultiplet containing $X_\pm$, we are left with only the first line, and indeed we see that the $\mathcal{N} = 3$ flavour symmetry of the interacting SCFT is $SU(2)$.

For $N = 2$, the operators with R-charge $3/2$ are

$$\begin{array}{lllll}
[3]_\omega : & X_+^3 \,, & X_+^2X_- \,, & X_+X_-^2 \,, & X_-^3 \\
[3]_\omega : & X_{++}X_+ \,, & X_{++}X_- \,, & X_{--}X_+ \,, & X_{--}X_- \\
[1]_\omega : & X_+M \,, & X_-M & &
\end{array} \tag{5.3.76}$$

where, in the index (5.3.72), the contribution of the operators in the representation $[1]_\omega$ gets cancelled by the same terms with an opposite sign due to the contribution of the free hypermultiplets; see the first term in the last line of (5.3.73). It is worth pointing out the similarity between (5.3.76) and (B.1.11). Note that, upon decoupling the free hypermultiplet, we no longer have an operator at order $x^{\frac{3}{2}}$.

For $N = 2$, the marginal operators are similar to those presented in (B.1.12). It should be noted again that, due to (5.3.47) and (5.3.49), we have

$$M^2 = 0 \ , \qquad \mathrm{tr}(\mu_H \mu_Q) = -\,\mathrm{tr}(\mu_C \mu_Q) \ . \tag{5.3.77}$$

Here is the list of the marginal operators:

$$\begin{array}{llll} [4]_\omega : & X_{++}^2 \ , & X_{++}M \ , & X_{++}X_{--} = \mathrm{tr}(\mu_H\mu_Q) = -\,\mathrm{tr}(\mu_C\mu_Q) \ , \\ & & X_{--}M \ , & X_{--}^2 \\ [4]_\omega : & X_+^4 \ , & X_+^3 X_- \ , & X_+^2 X_-^2 \ , \\ & & X_+ X_-^3 \ , & X_-^4 \\ [4]_\omega : & X_{++}X_+^2 \ , & X_{++}(X_+X_-) \ , & X_{++}X_-^2 = X_+^2 X_{--} \\ & & X_{--}(X_+X_-) \ , & X_{--}X_-^2 \\ [2]_\omega : & X_+^2 M \ , & X_+X_-M \ , & X_-^2 M \end{array} \tag{5.3.78}$$

where the relation

$$X_{++}X_{--} = \mathrm{tr}(\mu_H\mu_Q) = -\,\mathrm{tr}(\mu_C\mu_Q) \tag{5.3.79}$$

is analogous to (B.1.13), where the quantities on the left and right hand sides both have magnetic flux $(0, 0)$. Note that, upon decoupling the free hypermultiplet, we are left with only the operators in the first two lines of (5.3.78). Due to the quantities as listed in (5.3.78), we write the index as in (5.3.72), with the contribution of the extra SUSY conserved current indicated in brown. This leads us to conclude that supersymmetry gets enhanced from $\mathcal{N} = 3$ to $\mathcal{N} = 4$. This conclusion has in fact been already discussed in [97].

Comments on the case of $k = -2$

From (5.2.11) with $\boldsymbol{n} = 0$, we see that the index up to order x^2 for $k = -2$ and $n \geq 3$ is equal to that described in (5.3.50), and the index for $k = -2$ and $n = 2$ reads

$$\begin{aligned} 1 &+ x\left(1 + \chi^{SU(2)}_{[2]}(\mu)\right) + x^2\Big[\left(\chi^{SU(2)}_{[4]}(\mu) + 2\chi^{SU(2)}_{[2]}(\mu) + s' + 2\right) \\ &- \left(1 + \chi^{SU(2)}_{[2]}(\mu)\right)\Big] + \ldots \ , \end{aligned} \tag{5.3.80}$$

where

$$s' = \begin{cases} 1 & N = 2 \ , \\ 2 & N = 3 \ . \end{cases} \tag{5.3.81}$$

Let us interpret this result. The monopole operators (X_+, X_-) and (X_{++}, X_{--}), discussed in the case of $k = 2$, are no longer gauge invariant. However, the above index suggests that quantities like X_+X_- and $X_{++}X_{--}$ are gauge invariant. This can be seen from the observation that the index of the case of $k = -2$ can be obtained from that of $k = 2$ by removing the terms involving ω^p with $p \neq 0$; one can compare (5.3.80) for $k = -2$, $n = 2$ with (5.3.65) for $k = 2$, $n = 2$.

We focus on the case of $k = -2$ and $n = 2$. The operators with R-charge 1 are (5.3.52). The $\mathcal{N} = 3$ flavour symmetry is $SU(2) \times U(1)$. The marginal operators are as follows. Those in $[4]_\mu$ are (5.3.145). Those in $2[2]_\mu$ are either $(\mathcal{A}_H)^i_j$ or $(\mathcal{A}_C)^i_j$, and $\widehat{M}^i_j(M^k_k)$. Those contribute s' are either $\mathrm{tr}(\mu_Q\mu_H)$ or $\mathrm{tr}(\mu_Q\mu_C)$, and $\mathrm{tr}(\mu_H\mu_C)$, which is present for $N = 3$ and absent for $N = 2$. Finally, those contribute $+2$ are X_+X_- and $(\widehat{M^2})^i_i$. We do not see the presence of the extra SUSY-current.

Let us now turn to the case of $k = -2$ and $n = 1$. From (5.2.11), the index is

$$\begin{aligned} N = 2: &\qquad 1 + 2x + x^2\,(4{-}2{-}1) + \ldots \\ N = 3: &\qquad 1 + 2x + x^2\,(5{-}2{-}1) + \ldots \end{aligned} \tag{5.3.82}$$

This, again, can be obtain from (5.3.72) with ω^p ($p \neq 0$) removed. We propose that the operators with R-charge 1 are $M = \operatorname{tr}\mu_Q$ and X_+X_-. The $\mathcal{N} = 3$ flavour symmetry is therefore $U(1)^2$. The four marginal operators of the case of $N = 2$ are as follows: $X_{++}X_{--} = \operatorname{tr}(\mu_H\mu_Q) = -\operatorname{tr}(\mu_C\mu_Q)$, $X_+^2X_-^2$, $X_{++}X_-^2 = X_+^2X_{--}$ and X_+X_-M. For $N = 3$, there is an additional marginal operator $\operatorname{tr}(\mu_H\mu_C)$. There is one extra SUSY-current, indicated in brown. Hence supersymmetry gets enhanced to $\mathcal{N} = 4$.

The case of $k = 1$ and $n = 1$ flavour

Here we focus only on the case of $N = 2$ and postpone the discussion of $N = 3$ to future work, due to the technicality of the index in the latter case. From (5.2.11), the index is

$$N = 2: \quad 1 + 1x + (1{-}1{-}\omega q^{-1} - \omega^{-1}q)x^2 - (\omega q^{-1} + \omega^{-1}q)x^3 + \ldots \tag{5.3.83}$$

where we again redefined μ_1 as q to highlight its $U(1)$ nature.

This case was in fact studied in [97, Section 4.3]. In the following we discuss the operators with R-charge up to 2. In this case, the operator with R-charge 1 corresponds to

$$M = \operatorname{tr}\mu_Q\ . \tag{5.3.84}$$

The $\mathcal{N} = 3$ flavour symmetry is therefore $U(1)$. We indicate the contribution of the flavour current to the index (5.3.83) in blue. Due to (5.3.47), M is a nilpotent operator satisfying $M^2 = 0$. From the relation (5.3.49), namely $M^2 = -\operatorname{tr}(\mu_H\mu_Q) - \operatorname{tr}(\mu_C\mu_Q)$, we have

$$\operatorname{tr}(\mu_H\mu_Q) = -\operatorname{tr}(\mu_C\mu_Q)\ . \tag{5.3.85}$$

This is precisely the marginal operator that contributes to the positive term $+1$ at order x^2 in (5.3.83).

As can be seen from the brown terms in (5.3.83), there are two extra SUSY conserved currents. This leads to the conclusion that supersymmetry gets enhanced from $\mathcal{N} = 3$ to $\mathcal{N} = 5$ in the IR [97]. Note that (5.3.83) also satisfies all of the necessary conditions for the enhanced $\mathcal{N} = 5$ supersymmetry discussed in [74], including that the coefficient of x must be 1.

In fact, if we view (5.3.83) as an $\mathcal{N} = 2$ index, we see that the negative terms at order x^2 indicate that the theory has an $SU(2) \cong Spin(3)$ global symmetry, whose character of the adjoint representation is $1 + \omega q^{-1} + \omega^{-1}q$. This $Spin(3)$ symmetry is indeed the commutant of the $\mathcal{N} = 2$ R-symmetry $U(1) \cong Spin(2)$ in the $\mathcal{N} = 5$ R-symmetry $Spin(5)$.

$U(N)_0$ gauge group and n flavour

This is also known as the S-flip theory [14].

For $n \geq 3$, from (5.2.11), the indices for $N=2$ and $N=3$ read

$$\begin{aligned} n \geq 3: \quad & 1 + x\left(1 + \chi^{SU(n)}_{[1,0,\dots,0,1]}(\boldsymbol{\mu})\right) + x^2\Big[\chi^{SU(n)}_{[2,0,\dots,0,2]}(\boldsymbol{\mu}) + \chi^{SU(n)}_{[0,1,0\dots,0,1,0]}(\boldsymbol{\mu}) \\ & + 3\chi^{SU(n)}_{[1,0,\dots,0,1]}(\boldsymbol{\mu}) + s - \left(1 + \chi^{SU(n)}_{[1,0,\dots,0,1]}(\boldsymbol{\mu})\right)\Big] \\ & + \left(\omega\chi^{SU(n)}_{[0,\dots,2]}(\boldsymbol{\mu}) + \omega^{-1}\chi^{SU(n)}_{[2,0,\dots,0]}(\boldsymbol{\mu})\right) x^{1+\frac{n}{2}} + \dots \, . \end{aligned} \tag{5.3.86}$$

where we highlight the contribution of the $\mathcal{N}=3$ flavour currents in blue and s is defined as

$$s = \begin{cases} 2 & N=2 \\ 3 & N=3 \, . \end{cases} \tag{5.3.87}$$

On the other hand, for $n=2$, the indices are

$$\begin{aligned} (N=2, n=2): \quad & 1 + x\left(1 + \chi^{SU(2)}_{[2]}(\mu)\right) + x^2\Big[\chi^{SU(2)}_{[4]}(\mu) + (2 + \omega + \omega^{-1})\chi^{SU(2)}_{[2]}(\mu) \\ & + 2 - \left(1 + \chi^{SU(2)}_{[2]}(\mu)\right)\Big] + \dots \\ (N=3, n=2): \quad & 1 + x\left(1 + \chi^{SU(2)}_{[2]}(\mu)\right) + x^2\Big[\chi^{SU(2)}_{[4]}(\mu) + (2 + \omega + \omega^{-1})\chi^{SU(2)}_{[2]}(\mu) \\ & + 3 + (\omega + \omega^{-1}) - \left(1 + \chi^{SU(2)}_{[2]}(\mu)\right)\Big] + \dots \, . \end{aligned} \tag{5.3.88}$$

Note that indices (5.3.86) have the same expressions up to order x^2 as the cases of $n \geq 3$ of (5.3.50), except that there are additional terms $\omega\chi^{SU(n)}_{[0,\dots,0,2]}(\mu) + \omega^{-1}\chi^{SU(n)}_{[2,0,\dots,0]}(\mu)$ at order $x^{1+\frac{n}{2}}$. The latter indicate the presence of the gauge invariant dressed monopole operators with R-charge $1 + \frac{n}{2}$. Note that they become marginal for $n=2$.

For $n \geq 2$, the operators up to R-charge 2 are therefore as described in (5.3.52)–(5.3.59)[11], together with the aforementioned monopole operators in the case of $n=2$. We do not see the presence of the extra SUSY-current. We thus conclude that the theory has $\mathcal{N}=3$ supersymmetry.

The special case of $n = 1$

Let us write down explicitly the indices for $N=2$ and $N=3$, which can be computed from (5.2.11):

$$1 + 1x + (\omega q^{-2} + \omega^{-1} q^2) x^{\frac{3}{2}} + (s' - 1)x^2 + \dots \tag{5.3.89}$$

where we again redefined μ_1 as q to highlight its $U(1)$ nature and

[11] Curiously, for $(N=3, n=2)$, the index seems to indicate the presence of extra marginal gauge invariant monopole operators with topological fugacities $\omega^{\pm 1}$. These should be identified with the monopole operators $X_{(\pm 1,0,0)}$ with fluxes $(\pm 1, 0, 0)$. For $n \geq 3$, these operators (if exist) should carry R-charge greater than 2 and is beyond the scope of our analysis. It would be nice to understand these operators better in the future.

$$s' = \begin{cases} 1 & N=2 \\ 2 & N=3 \end{cases} \tag{5.3.90}$$

Note that this is similar to the case of $n = 1$ in (5.3.50), but with additional terms $(\omega q^{-2} + \omega^{-1} q^2)$ at order $x^{\frac{3}{2}}$. Thus, the $\mathcal{N} = 3$ flavour symmetry in each case is $U(1)$. The operators up to R-charge 2 are therefore as described in (5.3.64) and below, together with the aforementioned dressed monopole operators. We do not see the presence of the extra SUSY-current. We thus conclude that the theory has $\mathcal{N} = 3$ supersymmetry, in agreement with the findings in [14, Section 3.1].

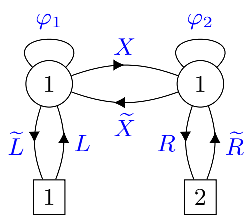
5.3.2 S-fold theories with the $T^{[2,1^2]}_{[2,1^2]}(SU(4))$ building block: Preliminary results

The purpose of this section is to generalise the previous results on the S-fold theories with the $T(U(N))$ building block to those with the $T^{\boldsymbol{\rho}}_{\boldsymbol{\rho}}(SU(N))$ building block. The $T^{\boldsymbol{\sigma}}_{\boldsymbol{\rho}}(SU(N))$ theories were introduced in [85]. They form a large class of 3d $\mathcal{N} = 4$ SCFTs that admits Lagrangian descriptions in terms of linear quivers. They can also be realised using Type IIB brane configurations, involving D3, D5 and NS5 branes [107]. When $\boldsymbol{\sigma} = \boldsymbol{\rho}$ the theory is self-mirror. We therefore can construct S-fold theories by commonly gauging the Higgs and Coulomb branch symmetries of $T^{\boldsymbol{\rho}}_{\boldsymbol{\rho}}(SU(N))$ in the same way as we did for $T(U(N))$. Due to the technicality of the index computation, we shall restrict ourselves to the $T^{[2,1^2]}_{[2,1^2]}(SU(4))$ theory.

We briefly review important details of the $T^{[2,1^2]}_{[2,1^2]}(SU(4))$ theory in Sections (5.3.2) and (5.1.2). We then construct S-fold theories in the subsequent subsections. As we shall see in Sections (5.3.2) and (5.3.2), for some values of CS levels, the theory contains gauge invariant monopole operators in the spectrum. Although we try to study the chiral ring of such operators using the index and other known theories as a guide, we do not have a full understanding of such a chiral ring. The results for the S-fold theories of this section should therefore be taken as preliminary and we shall not study all possible cases as for the $T(U(N))$ case. We hope to revisit this problem in the future.

The $T^{[2,1^2]}_{[2,1^2]}(SU(4))$ theory

Let us redraw the quiver description [85] of this theory in a $\mathcal{N} = 2$ fashion:



(5.3.91)

$$W = L\varphi_1 \widetilde{L} + \widetilde{X}\varphi_1 X - X\varphi_2 \widetilde{X} + \widetilde{R}\varphi_2 R$$

The Higgs and Coulomb branch moment maps

The Higgs branch moment map can be written in terms of the chiral fields in (5.3.91) as

$$(\mu_H)^i_j = \widetilde{R}^i R_j \ , \tag{5.3.92}$$

where $i, j = 1, 2$ are the indices of $U(2)_f$. The F-terms with respect to $\varphi_{1,2}$ imply

$$\mathrm{tr}(\mu_H) = \widetilde{R}^i R_i = X\widetilde{X} = -L\widetilde{L} \ . \tag{5.3.93}$$

As a result, μ_H satisfies the following conditions

$$\mathrm{rank}(\mu_H) \leq 1 \ , \qquad (\mu_H^2)^i_j = (\mu_H)^i_j \,\mathrm{tr}(\mu_H) \ , \qquad \mathrm{tr}(\mu_H^2) = (\mathrm{tr}\,\mu_H)^2 \ . \tag{5.3.94}$$

The Coulomb branch moment map can be written as

$$\mu_C = \begin{pmatrix} \varphi_1 & V_{(1;0)} \\ V_{(-1;0)} & \varphi_2 \end{pmatrix} \ . \tag{5.3.95}$$

where $V_{(m;n)}$ denotes the monopole operator carrying flux m under the left $U(1)$ gauge group in (5.3.91) and flux n under the right $U(1)$ gauge group in (5.3.91). Since $T^{[2,1^2]}_{[2,1^2]}(SU(4))$ is self-mirror, the Coulomb branch moment map also satisfies the same conditions as (5.3.94) with H replaced by C:

$$\mathrm{rank}(\mu_C) \leq 1 \ , \qquad (\mu_C^2)^{i'}_{j'} = (\mu_C)^{i'}_{j'} \,\mathrm{tr}(\mu_C) \ , \qquad \mathrm{tr}(\mu_C^2) = (\mathrm{tr}\,\mu_C)^2 \ . \tag{5.3.96}$$

where $i', j' = 1, 2$ are the $U(2)_w$ indices. It then follows that

$$V_{(1;0)}V_{(-1;0)} = \varphi_1\varphi_2 \ . \tag{5.3.97}$$

Moreover, from the superpotential (5.3.91), the F-terms with respect to $\widetilde{L}$, L, $\widetilde{R}$ and R give

$$L\varphi_1 = 0 \ , \qquad \widetilde{L}\varphi_1 = 0 \ , \qquad R_i\varphi_2 = 0 \ , \qquad \widetilde{R}^i\varphi_2 = 0 \ . \tag{5.3.98}$$

It then follows that

$$0 = \widetilde{R}^i R_j \varphi_2 = (\mu_H)^i_j\varphi_2 \ , \qquad 0 = -(L\widetilde{L})\varphi_1 \overset{(5.3.93)}{=} (\mathrm{tr}\,\mu_H)\varphi_1 \ . \tag{5.3.99}$$

We can rewrite the Coulomb branch symmetry algebra as $SU(2)\times U(1)$, where the $SU(2)$ factor corresponds to the (enhanced) topological symmetry of the left gauge group in (5.3.91) and the $U(1)$ factor corresponds to that of the right one. Indeed, the superpartners of the $SU(2)$ current are the triplet $(V_{(1;0)}, \varphi_1, V_{(-1;0)})$, each of which can be constructed from the fields in the vector multiplet of the left gauge group in (5.3.91) in the UV. On the other hand, the field φ_2 is the superpartner of the aforementioned $U(1)$ symmetry current. Since $V_{(1;0)}$, $V_{(-1;0)}$ and φ_1 transform in the adjoint representation of an unbroken $SU(2)$ symmetry, it follows that the second equality of (5.3.99) has to hold also for $V_{(\pm 1;0)}$, namely:

$$(\mathrm{tr}\,\mu_H)V_{(1,0)} = 0 \ , \qquad (\mathrm{tr}\,\mu_H)V_{(-1,0)} = 0 \ . \tag{5.3.100}$$

We will see that these quantum relations are also consistent with the index.

Contracting the indices i and j in the first equation of (5.3.99), we have $(\mathrm{tr}\,\mu_H)\varphi_2 = 0$. Combining this result with (5.3.100), we obtain

$$(\mathrm{tr}\, \mu_H)(\mu_C)^{i'}_{j'} = 0 \ . \tag{5.3.101}$$

Using mirror symmetry and the fact that the theory is self-mirror, we also have

$$(\mathrm{tr}\, \mu_C)(\mu_H)^{i}_{j} = 0 \ . \tag{5.3.102}$$

Contracting the indices i and j we obtain[12]

$$(\mathrm{tr}\, \mu_H)(\mathrm{tr}\, \mu_C) = 0 \ . \tag{5.3.103}$$

The relevant and marginal operators

The index of the $T^{[2,1^2]}_{[2,1^2]}(SU(4))$ theory can be written as (see Section (5.1.2) for more details)

$$\begin{aligned}
1 &+ x \left[d^2 \left(1 + \chi^{SU(2)}_{[2]}(u)\right) + d^{-2} \left(1 + \chi^{SU(2)}_{[2]}(h)\right)\right] \\
&+ x^{\frac{3}{2}} \left[d^3 (b + b^{-1}) \chi^{SU(2)}_{[1]}(u) + d^{-3} (q + q^{-1}) \chi^{SU(2)}_{[1]}(h)\right] \\
&+ x^2 \Big[d^4 \left(1 + \chi^{SU(2)}_{[2]}(u) + \chi^{SU(2)}_{[4]}(u)\right) + d^{-4} \left(1 + \chi^{SU(2)}_{[2]}(h) + \chi^{SU(2)}_{[4]}(h)\right) \\
&\qquad + \chi^{SU(2)}_{[2]}(u) \chi^{SU(2)}_{[2]}(h) - \left(\chi^{SU(2)}_{[2]}(h) + 1\right) - \left(\chi^{SU(2)}_{[2]}(u) + 1\right) - 1\Big] \\
&+ \ldots \ .
\end{aligned} \tag{5.3.104}$$

Let us analyse the operators that contribute to the index up to order x^2. It is convenient to split the Higgs and Coulomb branch moment maps into the trace and the traceless part, where the latter is denoted by

$$(\widehat{\mu}_{H,C})^i_j := (\mu_{H,C})^i_j - \frac{1}{2} (\mathrm{tr}\, \mu_{H,C}) \delta^i_j \ . \tag{5.3.105}$$

Since the rank of $\mu_{H,C}$ is at most one, we have

$$\mathrm{tr}(\widehat{\mu}^2_{H,C}) = \frac{1}{2} (\mathrm{tr}\, \mu_{H,C})^2 \ . \tag{5.3.106}$$

The coefficient of order x of the index corresponds to the following operators:

$$\begin{aligned}
d^2 \left(1 + \chi^{SU(2)}_{[2]}(u)\right) &: \qquad \mathrm{tr}(\mu_C) \ , \quad (\widehat{\mu}_C)^{i'}_{j'} \\
d^{-2} \left(1 + \chi^{SU(2)}_{[2]}(h)\right) &: \qquad \mathrm{tr}(\mu_H) \ , \quad (\widehat{\mu}_H)^{i}_{j}
\end{aligned} \tag{5.3.107}$$

The coefficient of order $x^{\frac{3}{2}}$ of the index corresponds to the following operators:

[12] This result can also be obtained by contracting the indices i and j in the first equation of (5.3.99) and then summing it with the second equation in (5.3.99), where we have used the fact that $\mathrm{tr}\, \mu_C = \varphi_1 + \varphi_2$.

$$
\begin{aligned}
d^3 b \chi^{SU(2)}_{[1]}(u): &\qquad \mathcal{U}^{i'} := (V_{(1,1)},\, V_{(0,1)})^{i'} \\
d^3 b^{-1} \chi^{SU(2)}_{[1]}(u): &\qquad \widetilde{\mathcal{U}}_{i'} := (V_{(-1,-1)},\, V_{(0,-1)})_{i'} \\
d^{-3} q \chi^{SU(2)}_{[1]}(h): &\qquad \mathcal{H}^{i} := \widetilde{R}^i \widetilde{X} \widetilde{L} \\
d^{-3} q^{-1} \chi^{SU(2)}_{[1]}(h): &\qquad \widetilde{\mathcal{H}}_{i} := L X R_i \;.
\end{aligned}
\tag{5.3.108}
$$

The terms at order x^2 with positive sign correspond to the following marginal operators:

$$
\begin{aligned}
d^4,\, d^{-4}: &\qquad \mathrm{tr}(\widehat{\mu}_C^2) = \frac{1}{2}(\mathrm{tr}\,\mu_C)^2 \;,\; \mathrm{tr}(\widehat{\mu}_H^2) = \frac{1}{2}(\mathrm{tr}\,\mu_H)^2 \\
d^4 \chi^{SU(2)}_{[2]}(u),\, d^{-4}\chi^{SU(2)}_{[2]}(f): &\qquad (\widehat{\mu}_C)^{i'}_{j'}(\mathrm{tr}\,\mu_C) \;,\; (\widehat{\mu}_H)^{i}_{j}(\mathrm{tr}\,\mu_H) \\
d^4 \chi^{SU(2)}_{[4]}(u),\, d^{-4}\chi^{SU(2)}_{[4]}(f): &\qquad (\widehat{\mu}_C)^{i'}_{j'}(\widehat{\mu}_C)^{k'}_{l'} \;,\; (\widehat{\mu}_H)^{i}_{j}(\widehat{\mu}_H)^{k}_{l} \\
\chi^{SU(2)}_{[2]}(u)\chi^{SU(2)}_{[2]}(h): &\qquad (\widehat{\mu}_C)^{i'}_{j'}(\widehat{\mu}_H)^{i}_{j}
\end{aligned}
\tag{5.3.109}
$$

The terms with minus sign confirms that the theory indeed has a $U(1)_b \times SU(2)_u \times U(1)_q \times SU(2)_h \times U(1)_d$ global symmetry, as expected. Note that the terms $+d^0\chi^{SU(2)}_{[2]}(u)$, $+d^0\chi^{SU(2)}_{[2]}(h)$ and $+d^0\chi^{SU(2)}_{[0]}(u)\chi^{SU(2)}_{[0]}(h)$ do not appear at order x^2. The absence of such terms confirms the relations (5.3.101), (5.3.102), (5.3.103), and thus also (5.3.100).

$U(2)_k$ gauge group with zero flavour

We now consider the S-fold building block theory (5.2.12).

The superpotential for (5.2.12) can be written as [86, 88]

$$
W = -\frac{k}{4\pi}\,\mathrm{tr}(\varphi^2) + \mathrm{tr}\left((\mu_C + \mu_H)\varphi\right) \tag{5.3.110}
$$

where φ is a complex scalar in the vector multiplet of the $U(2)$ gauge group, and μ_C and μ_H are the Coulomb branch and Higgs branch moment maps of the $T^{[2,1^2]}_{[2,1^2]}(SU(4))$ SCFT.

Let us assume in the following analysis that $k \neq 0$. We can integrate out φ. The F-terms with respect to φ give

$$
\varphi = \frac{2\pi}{k}(\mu_C + \mu_H) \;. \tag{5.3.111}
$$

Substituting this back to (5.3.110), we obtain the effective superpotential after integrating out φ to be

$$
\begin{aligned}
W_{\text{eff}} &= \frac{\pi}{k}\,\mathrm{tr}(\mu_C + \mu_H)^2 \\
&= \frac{\pi}{k}\left[\mathrm{tr}(\mu_C^2) + \mathrm{tr}(\mu_H^2) + 2\,\mathrm{tr}(\mu_C \mu_H)\right] \\
&= \frac{\pi}{k}\left[(\mathrm{tr}\,\mu_C)^2 + (\mathrm{tr}\,\mu_H)^2 + 2\,\mathrm{tr}(\mu_C \mu_H)\right] \;.
\end{aligned}
\tag{5.3.112}
$$

where in the last line we have used (5.3.94) and (5.3.96). It should be noted that, on the contrary to the effective superpotential (5.3.37) of the S-fold theory with the $T(U(N))$ building block, the $U(1)_d$ axial symmetry is broken in this case[13]. The index of this theory is given by (5.2.13).

The case of $|k| \geq 3$

Evaluating (5.2.13), we obtain the indices for $|k| \geq 3$:

$$\mathcal{I}_{(5.2.12)}(|k| \geq 3; \{\omega, \nu = 0\}) = 1 + 2x + 0x^2 + 0x^3 + \dots \ . \tag{5.3.113}$$

where, for each k such that $|k| \geq 3$, the indices differ at order of x greater than 3. For example,

$$\begin{aligned} k = 3: \qquad & 1 + 2x - 2(\omega + \omega^{-1})x^{\frac{7}{2}} + 5x^4 + \dots \\ k \leq -3,\, k \geq 4: \qquad & 1 + 2x + 5x^4 + \dots \ . \end{aligned} \tag{5.3.114}$$

The coefficient of x indicates that the theory has a $U(1) \times U(1)$ global symmetry. Due to (5.3.111), we can write φ in terms of μ_H and μ_C. As a result, there are only two independent operators with R-charge 1, namely

$$\mathrm{tr}(\mu_H) \ , \qquad \mathrm{tr}(\mu_C) \ , \tag{5.3.115}$$

corresponding to the term $2x$ in the index.

Let us now consider the marginal operators. Taking into account of (5.3.111), (5.3.94) and (5.3.96), we can rewrite any marginal operators in terms of a linear combination of the following quantities: $(\mathrm{tr}\, \mu_H)^2$, $(\mathrm{tr}\, \mu_C)^2$, $(\mathrm{tr}\, \mu_H)(\mathrm{tr}\, \mu_C)$ and $\mathrm{tr}(\mu_H \mu_C)$. However, this set of quantities can be reduced further. Due to (5.3.103), we have $(\mathrm{tr}\, \mu_H)(\mathrm{tr}\, \mu_C) = 0$. Hence, there are three independent marginal operators, which can be taken as

$$(\mathrm{tr}\, \mu_H)^2 \ , \qquad (\mathrm{tr}\, \mu_C)^2 \ , \qquad \mathrm{tr}(\mu_H \mu_C) \ . \tag{5.3.116}$$

Since the coefficient of x^2 in the index is equal to the number of marginal operators minus conserved currents and we have $0x^2$ in (5.3.113), it follows that there are three conserved currents that precisely cancel the contribution of the three marginal operators in (5.3.116). Two of the conserved currents are identified with the $U(1)^2$ flavour currents, as can be seen from order x of the index, and the other one is the extra SUSY current. We thus conclude that $\mathcal{N} = 3$ supersymmetry of theory (5.2.12), with $k \geq 3$, is enhanced to $\mathcal{N} = 4$ in the IR.

Finally, let us point out that there is a symmetry that exchanges μ_H and μ_C for $|k| \geq 3$. As we shall discuss shortly, this symmetry is absent for $k = 2$ and $k = 1$.

The case of $k = 2$

Evaluating (5.2.13), we obtain the index for $k = 2$:

[13] Recall that under the $U(1)_d$ symmetry, μ_C carries charge $+2$ and μ_H carries charge -2.

$$
\begin{aligned}
&\mathcal{I}_{(5.2.12)}(k=2;\{\omega,\nu=0\}) \\
&= 1 + x\left(2+\omega+\frac{1}{\omega}\right) + x^2\left(\omega^2+\frac{1}{\omega^2}\right) + \dots \\
&\overset{\omega\to\omega^2}{=} 1 + x\left[1+\chi^{SU(2)}_{[2]}(\omega)\right] + x^2\left[\left(1+\chi^{SU(2)}_{[4]}(\omega)\right) - \left(1+\chi^{SU(2)}_{[2]}(\omega)\right)\right] + \dots
\end{aligned}
\tag{5.3.117}
$$

where the second equality holds if we redefine ω as ω^2; we also highlighted the contribution of the flavour currents in blue.

From the coefficient of x we see that, in addition to the operators listed in (5.3.115), there are two gauge invariant monopole operators with R-charge 1 that carry topological fugacities $\omega^{\pm 2}$, denoted by $X_\pm$. Hence the operators with R-charge 1 are

$$
1,\, \omega^2,\, 1,\, \omega^{-2}: \qquad \mathrm{tr}(\mu_H)\,, \qquad X_+\,, \qquad \mathrm{tr}(\mu_C)\,, \qquad X_-\,, \tag{5.3.118}
$$

The $\mathcal{N}=3$ flavour symmetry of the SCFT is therefore $SU(2)\times U(1)$. Note that this is larger than that of the case of $|k|\geq 3$, due to the presence of the monopole operators $X_\pm$ with R-charge 1. Here we have to make a choice whether to take $(X_+, (\mathrm{tr}\,\mu_H), X_-)$ or $(X_+, (\mathrm{tr}\,\mu_C), X_-)$ to be a moment map of $SU(2)$. Whatever choice we make will break the symmetry that exchanges μ_H and μ_C. This is a crucial difference between this case and the previously discussed case of $|k|\geq 3$. For definiteness, let us take the triplet $(X_+, (\mathrm{tr}\,\mu_C), X_-)$ to be the moment map of $SU(2)$ and $(\mathrm{tr}\,\mu_H)$ to be that of $U(1)$.[14]

Let us consider the marginal operators. These contribute to order x^2 in the index. We first examine those in the representation [4] of $SU(2)$, whose character is $\chi^{SU(2)}_{[4]}(\omega) = \omega^4+\omega^2+1+\omega^{-2}+\omega^{-4}$. The terms $\omega^{\pm 4}$ should correspond to the operators $X_\pm^2$. In contrast to (5.3.69), there is no gauge invariant monopole operator X_{++} or X_{--} with fluxes $(1,1)$ or $(-1,-1)$. It is also interesting to contrast to the 3d $\mathcal{N}=4$ $U(2)$ gauge theory with four flavours of fundamental hypermultiplets (B.2.4) that there are no operators in the representation [2] of $SU(2)$ in this case. The candidates for the operators that carry fugacities $\omega^{\pm 2}$ are $X_\pm(\mathrm{tr}\,\mu_H)$ and $X_\pm(\mathrm{tr}\,\mu_C)$. However, we argue that the former vanishes for the following reason. Since from (5.3.103) we have $(\mathrm{tr}\,\mu_H)(\mathrm{tr}\,\mu_C)=0$, we must also have

$$
(\mathrm{tr}\,\mu_H)X_\pm = 0\,, \tag{5.3.119}
$$

due to the fact that $(X_+, (\mathrm{tr}\,\mu_C), X_-)$ transform in the adjoint representation of an unbroken $SU(2)$ flavour symmetry. We thus conclude that the marginal operators carrying fugacities $\omega^{\pm 2}$ are $X_\pm(\mathrm{tr}\,\mu_C)$. At this point, it is also worth comment that, in contrast to (B.2.4) and to (5.3.69), there is no dressed monopole operators, like $X_{(\pm1,0);(0,1)}$, in this case. Finally, let us discuss the marginal operators that carry zero charge under the topological symmetry, *i.e.* those with ω^0. The candidates for these are as follows:

$$
(\mathrm{tr}\,\mu_H)^2\,, \quad (\mathrm{tr}\,\mu_C)^2\,, \quad \mathrm{tr}(\mu_H\mu_C)\,, \quad X_+X_-\,. \tag{5.3.120}
$$

From order x^2 in the index, there are the following possibilities:

[14] Of course, we may as well take $(X_+, (\mathrm{tr}\,\mu_H), X_-)$ to be the moment map of $SU(2)$ and $(\mathrm{tr}\,\mu_C)$ to be that of $U(1)$. The arguments below still hold with H interchanged with C.

1. Among (5.3.120), there are only two independent operators. There is no $\mathcal{N}=3$ extra SUSY-current.

2. Among (5.3.120), there are three independent operators. There is one $\mathcal{N}=3$ extra SUSY-current.

3. All of the four operators in (5.3.120) are independent from each other. There are two $\mathcal{N}=3$ extra SUSY-currents.

Let us discuss each of these possibilities in more detail.

Possibility 1 is the most unlikely. This is because we do not have two relations that reduce four quantities in (5.3.120) to two independent quantities.

Possibility 2 is possible *if* we postulate a relation like

$$X_+X_- = (\mathrm{tr}\,\mu_C)^2\ . \tag{5.3.121}$$

We will shortly comment on the validity of this assumption. As a result, the marginal operators transforming under the representation [4] of $SU(2)$ are

$$X_+^2\ ,\quad X_+(\mathrm{tr}\,\mu_C)\ ,\quad X_+X_- = (\mathrm{tr}\,\mu_C)^2\ ,\quad X_-(\mathrm{tr}\,\mu_C)\ ,\quad X_-^2\ , \tag{5.3.122}$$

whereas those transforming as singlets are

$$(\mathrm{tr}\,\mu_H)^2\ ,\qquad \mathrm{tr}(\mu_H\mu_C)\ . \tag{5.3.123}$$

In this possibility, the terms at order x^2 should be rewritten as

$$x^2\left[\left(2+\chi^{SU(2)}_{[4]}(\omega)\right)-\left(1+\chi^{SU(2)}_{[2]}(\omega)\right)-1\right] \tag{5.3.124}$$

where the term -1, highlighted in purple, indicates the presence of an extra SUSY-current. If this were true, we would conclude that the theory flows to an SCFT with enhanced $\mathcal{N}=4$ supersymmetry. We emphasise again that this conclusion relies heavily on assumption (5.3.121). It may be argued that this cannot be true because if $X_\pm$ correspond to the monopole operators with fluxes $(\pm1,0)$, then X_+X_- carries flux $(1,-1)$[15] and not $(0,0)$; hence it should not be equated to $(\mathrm{tr}\,\mu_C)^2$. Indeed, the relation of type (5.3.121) does not hold for the 3d $\mathcal{N}=4$ $U(2)$ gauge theory with 4 flavours; see (B.2.4). It would hold if we had an abelian gauge group, like 3d $\mathcal{N}=4$ $U(1)$ gauge theory with 2 flavours.

Possibility 3 is the most likely. In this possibility, the marginal operators transforming under the representation [4] of $SU(2)$ are

$$X_+^2\ ,\quad X_+(\mathrm{tr}\,\mu_C)\ ,\quad X_+X_-\ ,\quad X_-(\mathrm{tr}\,\mu_C)\ ,\quad X_-^2\ , \tag{5.3.125}$$

whereas those transforming as singlets are

$$(\mathrm{tr}\,\mu_H)^2\ ,\qquad (\mathrm{tr}\,\mu_C)^2\ ,\qquad \mathrm{tr}(\mu_H\mu_C)\ . \tag{5.3.126}$$

The terms at order x^2 should then be rewritten as

[15] After applying the Weyl symmetry, the flux (m,n) of the monopole operator $X_{(m,n)}$ should be written such that $m\geq n>-\infty$. The flux of of X_- should thus be written as $(0,-1)$. Since X_+ has flux $(+1,0)$, it follows that X_+X_- has flux $(1,-1)$.

$$x^2\left[\left(3+\chi^{SU(2)}_{[4]}(\omega)\right)-\left(1+\chi^{SU(2)}_{[2]}(\omega)\right)-2\right] \tag{5.3.127}$$

where the term -2, highlighted in purple, indicates the two $\mathcal{N}=3$ extra SUSY-currents. Note that supersymmetry cannot get enhanced to $\mathcal{N}=5$, since this would violate a necessary condition for $\mathcal{N}=5$ supersymmetry which states that the coefficient of x has to be 1 [74]. We are obliged to conclude that the theory flows to a product of two SCFTs, each with $\mathcal{N}=4$ supersymmetry. This situation is similar to that studied in [88]. It would be interesting to verify this conclusion using other methods and, if it were true, it would be also nice to identify such $\mathcal{N}=4$ SCFTs. We leave this for future work.

The case of $k=1$

Evaluating (5.2.13), we obtain the index for $k=1$ as

$$\begin{aligned}
&\mathcal{I}_{(5.2.12)}(k=1;\{\omega,\nu=0\})\\
&=1+x\left(2+\omega^2+\frac{1}{\omega^2}\right)+x^2\left(-1+\omega^4+\frac{1}{\omega^4}\right)+x^{\frac{5}{2}}\left(-2\omega-\frac{2}{\omega}\right)+\dots\\
&=1+x\left[1+\chi^{SU(2)}_{[2]}(\omega)\right]+x^2\left[\chi^{SU(2)}_{[4]}(\omega)-\left(1+\chi^{SU(2)}_{[2]}(\omega)\right)\right]\\
&\quad-2x^{\frac{5}{2}}\chi^{SU(2)}_{[1]}(\omega)+\dots\,.
\end{aligned} \tag{5.3.128}$$

We *propose* that the gauge invariant operators with R-charge 1 that carry fugacities $w^{\pm 2}$ are the monopole operators with fluxes $\pm(1,1)$, denoted by $X_{++}:=X_{(1,1)}$ and $X_{--}:=X_{(-1,-1)}$. It is interesting to point out that there is no gauge invariant monopole operator with fluxes $\pm(1,0)$ in this theory, since there are no terms $\omega^{\pm1}$ at order x. The operators with R-charge 1 are

$$1,\,\omega^2,\,1,\,\omega^{-2}: \qquad \mathrm{tr}(\mu_H)\,, \qquad X_{++}\,, \qquad \mathrm{tr}(\mu_C)\,, \qquad X_{--}\,, \tag{5.3.129}$$

corresponding to the coefficient of x. The $\mathcal{N}=3$ flavour symmetry of the SCFT is therefore $SU(2)\times U(1)$. Similarly to the case of $k=2$, we have to make a choice whether to take $(X_{++},(\mathrm{tr}\,\mu_C),X_{--})$ or $(X_{++},(\mathrm{tr}\,\mu_H),X_{--})$ to be a moment map of $SU(2)$. Picking any of these choices amounts to breaking the symmetry that exchanges μ_H and μ_C. For definiteness, we take the triplet $(X_{++},(\mathrm{tr}\,\mu_C),X_{--})$ to be the moment map of $SU(2)$ and $(\mathrm{tr}\,\mu_H)$ to be that of $U(1)$.[16]

Let us now examine the marginal operators of this theory. It is convenient to start from those in the representation [4] of $SU(2)$. Those carrying fugacities $\omega^{\pm4}$ are X_{++}^2 and X_{--}^2. Those carrying fugacities $\omega^{\pm2}$ are $X_{++}(\mathrm{tr}\,\mu_C)$ and $X_{--}(\mathrm{tr}\,\mu_C)$. It should be noted that $X_{++}(\mathrm{tr}\,\mu_H)$ and $X_{--}(\mathrm{tr}\,\mu_H)$ vanish due to the following argument (very similar to that of the case of $k=2$). Since $(\mathrm{tr}\,\mu_C)(\mathrm{tr}\,\mu_H)=0$ due to (5.3.103) and $(X_{++},(\mathrm{tr}\,\mu_C),X_{--})$ transforms as a triplet under an unbroken $SU(2)$ flavour symmetry, we have

$$X_{++}(\mathrm{tr}\,\mu_H)=X_{--}(\mathrm{tr}\,\mu_H)=0\,. \tag{5.3.130}$$

The marginal operators carrying fugacity ω^0 are

[16] Similarly to footnote (14), we may as well take $(X_{++},(\mathrm{tr}\,\mu_H),X_{--})$ to be the moment map of $SU(2)$ and $(\mathrm{tr}\,\mu_C)$ to be that of $U(1)$. The arguments below still hold with H interchanged with C.

$$(\mathrm{tr}\,\mu_H)^2\ , \qquad (\mathrm{tr}\,\mu_C)^2\ , \qquad \mathrm{tr}(\mu_H\mu_C)\ , \qquad X_{++}X_{--}\ . \tag{5.3.131}$$

Analogously to (B.1.13) of the $U(2)$ gauge theory with one adjoint and one fundamental hypermultiplet, we propose that $X_{++}X_{--}$ satisfies a quantum relation:

$$X_{++}X_{--} = (\mathrm{tr}\,\mu_C)^2\ . \tag{5.3.132}$$

Note that both left and right hand sides of this equation have magnetic flux $(0,0)$. In summary, the marginal operators in the representation [4] of $SU(2)$ are

$$X_{++}^2\ , \quad X_{++}(\mathrm{tr}\,\mu_C)\ , \quad X_{++}X_{--} = (\mathrm{tr}\,\mu_C)^2\ , \quad X_{--}(\mathrm{tr}\,\mu_C)\ , \quad X_{--}^2\ , \tag{5.3.133}$$

and those transforming as singlets under $SU(2)$ are

$$(\mathrm{tr}\,\mu_H)^2\ , \quad \mathrm{tr}(\mu_H\mu_C)\ . \tag{5.3.134}$$

These operators contribute to the terms $\left(2+\chi^{SU(2)}_{[4]}(\omega)\right)$ at order x^2 in the index. As a result, the x^2 term in (5.3.128) should be rewritten as

$$x^2\left[\left(2+\chi^{SU(2)}_{[4]}(\omega)\right)-\left(1+\chi^{SU(2)}_{[2]}(\omega)\right)-2\right]\ . \tag{5.3.135}$$

The extra -2, highlighted in purple, indicates the presence of two extra SUSY-currents. The same remark for the case of $k=2$ applies here. Supersymmetry cannot get enhanced to $\mathcal{N}=5$, since it would violate a necessary condition for $\mathcal{N}=5$ supersymmetry which states that the coefficient of x has to be 1 [74]. We are again obliged to conclude that the theory flows to a product of two SCFTs, each with $\mathcal{N}=4$ supersymmetry, similarly to the situation encountered in [88]. It would be interesting to verify this conclusion using other methods and, if it were true, it would be also nice to identify such $\mathcal{N}=4$ SCFTs. We leave this for future work.

$U(2)_k$ gauge group with n flavour

Let us now couple to theory (5.2.12) n flavours of hypermultiplets in the fundamental representation of $U(2)$ and obtain theory (5.2.14).

We propose that the superpotential for this theory is the same as (5.3.41), namely

$$\begin{aligned} W &= -\frac{k}{4\pi}\,\mathrm{tr}(\varphi^2)+\mathrm{tr}\left((\mu_C+\mu_H)\varphi\right)+\widetilde{Q}^i_b\varphi^b_a Q^a_i \\ &= -\frac{k}{4\pi}\,\mathrm{tr}(\varphi^2)+\mathrm{tr}\left((\mu_C+\mu_H+\mu_Q)\varphi\right)\ , \end{aligned} \tag{5.3.136}$$

The F-terms are the same as (5.3.44) and the consequences of them are as analysed in Appendix (C). The index of this theory is discussed in Section (5.2.2).

The case of $n \geq 2$ flavours

We focus on the cases of ($n \geq 3$, $|k| \geq 1$) and ($n = 2$, $|k| \geq 3$). Evaluating (5.2.15) with the background fluxes for the flavour symmetry being set to zero, $\boldsymbol{n} = 0$, we obtain the indices, up to order x^2, as follows:

$$\begin{aligned}(n \geq 3,\, |k| \geq 1) : 1 + x \left[3 + \chi^{SU(n)}_{[1,0,\dots,0,1]}(\boldsymbol{h})\right] + x^2 \Big[2q\chi^{SU(n)}_{[1,0,\dots,0]}(\boldsymbol{h}) + 2q^{-1}\chi^{SU(n)}_{[0,\dots,0,1]}(\boldsymbol{h}) \\ + \chi^{SU(n)}_{[2,0,\dots,0,2]}(\boldsymbol{h}) + 5\chi^{SU(n)}_{[1,0,\dots,0,1]}(\boldsymbol{h}) + \chi^{SU(n)}_{[0,1,0,\dots,0,1,0]}(\boldsymbol{h}) + 7 \\ - \left(3 + \chi^{SU(3)}_{[1,0,\dots,0,1]}(\boldsymbol{h})\right)\Big] + \dots\end{aligned} \tag{5.3.137}$$

$$\begin{aligned}(n = 2,\, |k| \geq 3) : 1 + x \left[3 + \chi^{SU(2)}_{[2]}(\boldsymbol{h})\right] + x^2 \Big[2q\chi^{SU(2)}_{[1]}(\boldsymbol{h}) + 2q^{-1}\chi^{SU(2)}_{[1]}(\boldsymbol{h}) \\ + \chi^{SU(2)}_{[4]}(\boldsymbol{h}) + 4\chi^{SU(2)}_{[2]}(\boldsymbol{h}) + 7 \\ - \left(3 + \chi^{SU(2)}_{[2]}(\boldsymbol{h})\right)\Big] + \dots\end{aligned} \tag{5.3.138}$$

where we used the fugacity map (5.2.16) and highlighted the contribution of the $U(1)^3 \times SU(n)$ flavour symmetry current in blue. Let us now analyse the operators with R-charges 1 and 2.

The operators with R-charge 1 are

$$\mathrm{tr}\,\mu_H\ , \qquad \mathrm{tr}\,\mu_C\ , \qquad M^k_k = \mathrm{tr}\,\mu_Q\ , \qquad \widehat{M}^i_j \tag{5.3.139}$$

where we remark that $\widehat{M}^i_j$ transforms in the adjoint representation $[1, 0, \dots, 0, 1]$ of $SU(n)$, and that we can always rewrite φ in terms of μ_H, μ_C and μ_Q due to (5.3.44).

Let us now discuss about the marginal operators. These contribute to positive terms at order x^2 of the index. The terms $2q\chi^{SU(n)}_{[1,0,\dots,0]}(\boldsymbol{h})$ and $2q^{-1}\chi^{SU(n)}_{[0,\dots,0,1]}(\boldsymbol{h})$ correspond to the gauge invariant combinations constructed by "dressing" Q or $\widetilde{Q}$ to the operators in (5.3.108):

$$\begin{aligned} 2q\chi^{SU(n)}_{[1,0,\dots,0]}(\boldsymbol{h}) : \qquad & Q^a_i\widetilde{\mathcal{H}}_a\ , \qquad Q^a_i\widetilde{\mathcal{U}}_a\ , \\ 2q^{-1}\chi^{SU(n)}_{[0,\dots,0,1]}(\boldsymbol{h}) : \qquad & \widetilde{Q}^i_a\mathcal{H}^a\ , \qquad \widetilde{Q}^i_a\mathcal{U}^a\ . \end{aligned} \tag{5.3.140}$$

The term $5\chi^{SU(n)}_{[1,0,\dots,0,1]}(\boldsymbol{h})$ corresponds to

$$\begin{aligned} &\widehat{M}^i_j(\mathrm{tr}\,\mu_H)\ , \qquad \widehat{M}^i_j(\mathrm{tr}\,\mu_C)\ , \qquad \widehat{M}^i_j(\mathrm{tr}\,\mu_Q) = \widehat{M}^i_j(M^k_k)\ , \\ &(\mathcal{A}_H)^i_j\ , \qquad (\mathcal{A}_C)^i_j\ , \end{aligned} \tag{5.3.141}$$

where we have defined $\underline{\widehat{M}^2}$ in (C.0.11) and $\mathcal{A}_{H,C}$ in (C.0.13). It should be noted that, from (C.0.12), the quantity $(\underline{\widehat{M}^2})^i_j$ can be written in terms of a linear combination of $(\mathcal{A}_H)^i_j$, $(\mathcal{A}_C)^i_j$ and $\widehat{M}^i_j(M^k_k) = \widehat{M}^i_j(\mathrm{tr}\,\mu_Q)$. For the special case of $n = 2$, we have an extra relation (C.0.15):

$$(\mathcal{A}_H)^i_j + (\mathcal{A}_C)^i_j = -\widehat{M}^i_j(\mathrm{tr}\,\mu_Q) = -\widehat{M}^i_j(M^k_k) \qquad (\text{for } n=2) \tag{5.3.142}$$

and so we have only four independent quantities, which correspond to the term $4\chi^{SU(2)}_{[2]}(\boldsymbol{h})$ in the index. The term $\chi^{SU(n)}_{[0,1,0,\ldots,0,1,0]}(\boldsymbol{h})$ corresponds to

$$\epsilon^{i_1 i_2 \ldots i_n}\epsilon_{j_1 j_2 \ldots j_n}\widehat{M}^{j_1}_{i_1}\widehat{M}^{j_2}_{i_2} \ . \tag{5.3.143}$$

The term $\chi^{SU(n)}_{[2,0,\ldots,0,2]}(\boldsymbol{h})$ corresponds to the quantity

$$R^{ik}_{jl} \tag{5.3.144}$$

which is a linear combination $\widehat{M}^i_j\widehat{M}^k_l$ and other quantities such that any contraction between an upper index and a lower index yields zero; for example, for $n=2$, where $\widehat{M}^2$ satisfies (C.0.9), the marginal operators in [4] are

$$R^{ik}_{jl} := \widehat{M}^i_j\widehat{M}^k_l + \frac{1}{6}(\widehat{M}^2)^p_p\delta^i_j\delta^k_l - \frac{1}{3}(\widehat{M}^2)^p_p\delta^i_l\delta^k_j \ , \qquad \text{for } n=2 \ . \tag{5.3.145}$$

Finally the candidates for the marginal operators that do not carry q and $\boldsymbol{h}$ fugacities are

$$\begin{array}{ll} \mathrm{tr}(\mu_H^2) = (\mathrm{tr}\,\mu_H)^2 \ , & \mathrm{tr}(\mu_C^2) = (\mathrm{tr}\,\mu_C)^2 \ , \\ \mathrm{tr}(\mu_Q\mu_H) = (\mu_H)^a_b\widetilde{Q}^i_a Q^b_i \ , & (\mathrm{tr}\,\mu_Q)(\mathrm{tr}\,\mu_H) \ , \\ \mathrm{tr}(\mu_Q\mu_C) = (\mu_C)^a_b\widetilde{Q}^i_a Q^b_i \ , & (\mathrm{tr}\,\mu_Q)(\mathrm{tr}\,\mu_C) \ , \\ (\widehat{M}^2)^i_i = \widehat{M}^i_j\widehat{M}^j_i \ , & (\mathrm{tr}\,\mu_Q)^2 = (M^k_k)^2 \\ \mathrm{tr}(\mu_H\mu_C) \ , & (\mathrm{tr}\,\mu_H)(\mathrm{tr}\,\mu_C) \overset{(5.3.103)}{=} 0 \ . \end{array} \tag{5.3.146}$$

where we recall from (C.0.6) that $\mathrm{tr}(\mu_Q^2)$ is not independent from the above quantities, since it can be written as

$$\mathrm{tr}(\mu_Q^2) = M^i_jM^j_i = \widehat{M}^i_j\widehat{M}^j_i + \frac{1}{n}(\mathrm{tr}\,\mu_Q)^2 = -\,\mathrm{tr}(\mu_Q\mu_H) - \mathrm{tr}(\mu_Q\mu_C) \ . \tag{5.3.147}$$

However, the quantities in (5.3.146) are not all independent from each other. Let us try to reduce them into a smaller set as follows. From (5.3.103), we see that $(\mathrm{tr}\,\mu_H)(\mathrm{tr}\,\mu_C)$ vanishes. From (C.0.10), we see that $(\mathrm{tr}\,\mu_Q)^2$ is a linear combination of $\mathrm{tr}(\mu_Q\mu_H)$ and $\mathrm{tr}(\mu_Q\mu_C)$ and $(\widehat{M}^2)^i_i$. In summary, we have eight of such marginal operators:

$$\begin{array}{ll} \mathrm{tr}(\mu_H^2) = (\mathrm{tr}\,\mu_H)^2 \ , & \mathrm{tr}(\mu_C^2) = (\mathrm{tr}\,\mu_C)^2 \ , \\ \mathrm{tr}(\mu_Q\mu_H) = (\mu_H)^a_b\widetilde{Q}^i_a Q^b_i \ , & (\mathrm{tr}\,\mu_Q)(\mathrm{tr}\,\mu_H) \ , \\ \mathrm{tr}(\mu_Q\mu_C) = (\mu_C)^a_b\widetilde{Q}^i_a Q^b_i \ , & (\mathrm{tr}\,\mu_Q)(\mathrm{tr}\,\mu_C) \ , \\ (\widehat{M}^2)^i_i = \widehat{M}^i_j\widehat{M}^j_i \ , & \mathrm{tr}(\mu_H\mu_C) \end{array} \tag{5.3.148}$$

As a result, the x^2 term in (5.3.137) and (5.3.138) should be rewritten as

$$x^2\left[\ldots + 8 - \left(3 + \chi^{SU(3)}_{[1,0,\ldots,0,1]}(\boldsymbol{h})\right) - 1\right] \ . \tag{5.3.149}$$

where the term -1, highlighted in brown, indicates the presence of an extra SUSY-current. We conclude that supersymmetry gets enhanced to $\mathcal{N} = 4$.

We also observe that, for $k = 2$, the coefficient of $x^{\frac{n}{2}+1}$ in the index contains the terms $\omega + \omega^{-1}$. Similarly, for $k = 1$, the coefficient of x^{n+1} in the index contains the terms $\omega^2 + \omega^{-2}$. These indicate that

- for $k = 2$, there are gauge invariant monopole operators $X_\pm$ with topological charges ± 1 with R-charge $\frac{n}{2} + 1$; and
- for $k = 1$, there are gauge invariant monopole operators X_{++} and X_{--} with topological charges ± 2 with R-charge $n + 1$.

In fact, we have encountered such monopole operators for the case of zero flavour ($n = 0$) in sections (5.3.2) and (5.3.2). The above statements generalise the previous results to any n. In particular, for ($n = 2$, $k = 2$), the gauge invariant monopole operator $X_\pm$ are marginal operators. This can be seen from the index that can be computed from (5.2.15) with $\boldsymbol{n} = 0$:

$$\begin{aligned}(n = 2,\, k = 2): \quad 1 + x \left[3 + \chi^{SU(2)}_{[2]}(\boldsymbol{h})\right] &+ x^2 \Big[2q\chi^{SU(2)}_{[1]}(\boldsymbol{h}) + 2q^{-1}\chi^{SU(2)}_{[1]}(\boldsymbol{h}) \\ &+ \chi^{SU(2)}_{[4]}(\boldsymbol{h}) + 4\chi^{SU(2)}_{[2]}(\boldsymbol{h}) + \omega + \omega^{-1} + 7 \\ &- \left(3 + \chi^{SU(2)}_{[3]}(\boldsymbol{h})\right)\Big] + \dots \,,\end{aligned} \tag{5.3.150}$$

where there are extra terms $\omega + \omega^{-1}$ at order x^2 in comparison to (5.3.138).

The case of $n = 1$ flavour

In this subsection, we discuss the special case of $n = 1$. The operators are as discussed in the previous subsection, but with the flavour indices $i, j, k = 1$, and so they can be dropped. As a result, we have

$$\widehat{M} = 0 \,, \qquad \mathcal{A}_H = 0 \,, \qquad \mathcal{A}_C = 0 \,. \tag{5.3.151}$$

The cases of $|k| \geq 3$

For $|k| \geq 3$, the index can be computed from (5.2.15) with $n = 1$ and $n_1 = 0$:

$$\begin{aligned}&1 + 3x + \left(3 + 2q + 2q^{-1}\right) x^2 - x^3 + \dots \\ &= 1 + 3x + \left(6 + 2q + 2q^{-1} - 3\right) x^2 - x^3 + \dots \,,\end{aligned} \tag{5.3.152}$$

where we highlight the contribution of the flavour currents in blue and rewrite the fugacity μ_1 as q to emphasise its $U(1)$ nature.

From (5.3.139) and (5.3.151), we see that the three independent operators with R-charge 1 are

$$\operatorname{tr} \mu_H \,, \qquad \operatorname{tr} \mu_C \,, \qquad M = \operatorname{tr} \mu_Q \,. \tag{5.3.153}$$

The flavour symmetry of this theory is therefore $U(1)^3$.

Let us now discuss the marginal operators. The terms $2q + 2q^{-1}$ in (5.3.152) correspond to the operators in (5.3.140), namely

$$
\begin{array}{lll}
2q: & Q^a\widetilde{\mathcal{H}}_a \ , & Q^a\widetilde{\mathcal{U}}_a \ , \\
2q^{-1}: & \widetilde{Q}_a\mathcal{H}^a \ , & \widetilde{Q}_a\mathcal{U}^a \ .
\end{array}
\tag{5.3.154}
$$

Note that all of the operators in (5.3.141) vanish identically for $n = 1$, due to (5.3.151) and the fact that the flavour indices can be dropped. The marginal operators that do not carry fugacity q are as listed in (5.3.148); since $\widehat{M} = 0$, there are 7 independent quantities:

$$
\begin{array}{ll}
\mathrm{tr}(\mu_H^2) = (\mathrm{tr}\,\mu_H)^2 \ , & \mathrm{tr}(\mu_C^2) = (\mathrm{tr}\,\mu_C)^2 \ , \\
\mathrm{tr}(\mu_Q\mu_H) \ , & (\mathrm{tr}\,\mu_Q)(\mathrm{tr}\,\mu_H) \ , \\
\mathrm{tr}(\mu_Q\mu_C) \ , & (\mathrm{tr}\,\mu_Q)(\mathrm{tr}\,\mu_C) \ , \\
\mathrm{tr}(\mu_H\mu_C) &
\end{array}
\tag{5.3.155}
$$

These operators, together with (5.3.154), contribute $7 + 2q + 2q^{-1}$ to order x^2 in the index. The x^2 term of the index should then be rewritten as $(7+2q+2q^{-1})-3-1$, where the term -1 indicates the presence of the extra SUSY-current. Hence we conclude that supersymmetry gets enhanced to $\mathcal{N} = 4$.

The cases of $k = 2$

The index in this case can be computed from (5.2.15) with $k = 2$, $n = 1$ and $n_1 = 0$:

$$
\begin{split}
&1 + 3x + (\omega + \omega^{-1})x^{\frac{3}{2}} + \left(7 + 2q + 2q^{-1} - 3 - 1\right) x^2 \\
&\quad + (\omega + \omega^{-1})x^{\frac{5}{2}} + (-1 + \omega^2 + \omega^{-2})x^3 \ldots \ .
\end{split}
\tag{5.3.156}
$$

where we rewrite the fugacity μ_1 as q to highlight its $U(1)$ nature.

As can be seen from order x, the $\mathcal{N} = 3$ flavour symmetry of the theory is $U(1)^3$. The operators with R-charge 1 are (5.3.153). In this case, there are also gauge invariant monopole operators $X_\pm$, carrying topological fugacities $\omega^{\pm 1}$, with R-charge 3/2. (This is consistent with the observation that the theory with $k = 2$ and n flavours, there are gauge invariant monopole operators with R-charge $\frac{1}{2}n + 1$; see section (5.3.2)). The marginal operators are listed in (5.3.154) and (5.3.155). Again, the term -1 at order x^2 of the index indicates the presence of the extra SUSY-current, and we conclude that supersymmetry gets enhanced to $\mathcal{N} = 4$.

The cases of $k = 1$

The index can be computed from (5.2.15) with $k = 1$, $n = 1$ and $n_{f_1} = 0$:

$$
\begin{split}
&1 + 3x + x^2\left(7 + 2q + 2q^{-1} + \omega^2 + \omega^{-2} - 3 - 1\right) \\
&\quad - x^3\left[2\left(q + q^{-1}\right)\left(\omega + \omega^{-1}\right) + 4\left(\omega + \omega^{-1}\right) + 2\right] + \ldots \ .
\end{split}
\tag{5.3.157}
$$

with the fugacity μ_1 being rewritten as q to highlight its $U(1)$ nature.

The $\mathcal{N} = 3$ flavour symmetry of this theory is $U(1)^3$, and the operators with R-charge 1 are (5.3.153). The marginal operators are (5.3.154) and (5.3.155), together with the gauge invariant monopole operators X_{++} and X_{--}, carrying topological fugacities $\omega^{\pm 2}$. (This is consistent with the observation that in the theory with $k = 1$ and n flavours there are gauge invariant monopole operators with topological charges

± 2 and R-charge $n+1$; see section (5.3.2)). The term -1 at order x^2 of the index indicates the presence of the extra SUSY-current, and we conclude that supersymmetry gets enhanced to $\mathcal{N}=4$.

Chapter 6

Zero-form and one-form symmetries of the ABJ and related theories

After a brief review of generalised global symmetries, in this chapter we will examine in detail the zero-form and one-form global symmetries of the Aharony-Bergman-Jafferis (ABJ) and related theories, with at least $\mathcal{N} = 6$ supersymmetry in three dimensions.

The Aharony-Bergman-Jafferis-Maldacena (ABJM) $U(N)_k \times U(N)_{-k}$ theories [6] and the Aharony-Bergman-Jafferis (ABJ) $U(N+x)_k \times U(N)_{-k}$ theories [1] constitute a large class of three-dimensional superconformal field theories (SCFTs) with $\mathcal{N} = 6$ and in some special cases $\mathcal{N} = 8$ supersymmetry.

As observed in [1] and further studied in [54], some of these theories are dual to the ABJ theories with orthogonal and symplectic gauge groups:

$$\begin{aligned} O(2N)_2 \times USp(2N)_{-1} &\longleftrightarrow U(N)_4 \times U(N)_{-4} \\ O(2N+2)_2 \times USp(2N)_{-1} &\longleftrightarrow U(N+2)_4 \times U(N)_{-4} \\ O(2N+1)_2 \times USp(2N)_{-1} &\longleftrightarrow U(N+1)_4 \times U(N)_{-4} \end{aligned} \tag{6.0.1}$$

Starting from these well-known dualities, we gauge their one-form symmetries or their subgroups and obtain new dualities. As pointed out in [2, 87], the study of higher-form symmetries and extended operators leads to new insight on several structures of the theory, especially distinctions between theories with the same gauge algebra but with different global structures of the gauge group.

We thus study the refined superconformal indices of such theories and map the symmetries across the dualities, with particular attention to their discrete part.

As a generalisation, we also find a new duality between a circular quiver with a discrete quotient of alternating special orthogonal and symplectic gauge groups and a three-dimensional $\mathcal{N} = 4$ circular (Kronheimer-Nakajima) quiver with unitary gauge groups, whose Higgs or Coulomb branch describes an instanton on a singular orbifold.

6.1 Generalised global symmetries

It is well known that standard symmetry transformations form a group G that can have both a continuous or a discrete nature. If the group is continuous, for every continuous generator, there is an associated conserved Noether current J which is a 1-form.

The conserved charge Q is constructed starting from J as the integral of the Hodge dual $\star J$ over a co-dimension one submanifold $\mathcal{M}$ of the spacetime X; namely

$$Q(\mathcal{M}) = \int_{\mathcal{M}} \star J \tag{6.1.1}$$

where, from the conservation law of J, it follows that $\star J$ is conserved too and thus

$$dJ = d \star J = 0 \tag{6.1.2}$$

Typically the submanifold $\mathcal{M}$ is a closed $d-1$-dimensional space separating spacetime into two regions and it can be non-compact.

To introduce the concept of higher-form symmetries we will closely follow [87]. Thus, first of all, we shall recast the symmetry transformation as the action of an operator $U_g(\mathcal{M})$ with g a group element of the global symmetry G. The fact that $U_g(\mathcal{M})$ is now associated with a symmetry means that its dependence on the submanifold $\mathcal{M}$ is topological, *i.e.* it is unchanged when $\mathcal{M}$ is slightly deformed.

In fact, considering a smooth deformation $\widetilde{\mathcal{M}}$ of the original manifold $\mathcal{M}$, we can always find a d-dimensional space $X^{(d)}$ interpolating between the two where $d \star J$ is conserved as in (6.1.2), and thus

$$\int_{\mathcal{M}} \star J - \int_{\widetilde{\mathcal{M}}} \star J = \int_{X^{(d)}} d \star J = 0 \tag{6.1.3}$$

The quantity (6.1.3) can be non-zero only when the deformation of $\mathcal{M}$ crosses an operator $\mathcal{O}(x)$ charged under the symmetry, so that the conservation fails due to the presence of a source term; namely

$$d \star J = Q(\mathcal{M})\delta^{(d)}(x) \tag{6.1.4}$$

In the continuous case, the symmetry transformation operator $U_g(\mathcal{M})$ can be easily obtained by exponentiating $Q(\mathcal{M})$. By considering a sphere S^{d-1} surrounding a spacetime point $x \in X$, $U_g(S^{d-1})$ can then act on charged operators $\mathcal{O}^I(x)$, as

$$U_g(S^{d-1})\mathcal{O}^I(x) = g^{Q(S^{d-1})}\mathcal{O}^I(x) = R^I_J(g)\mathcal{O}^J(x) \tag{6.1.5}$$

where R^I_J are the generators of the group in the representation $\mathcal{R}$ carried by $\mathcal{O}^I(x)$.

The symmetry transformation operators satisfy the group law

$$U_g(\mathcal{M})U_{g'}(\mathcal{M}) = U_{g''}(\mathcal{M}) \tag{6.1.6}$$

where $g'' = g \cdot g' \in G$ with $\cdot$ the group product.

In the discrete case, there is clearly no J current but, nonetheless, the generator can be still associated to a co-dimension one manifold even without being an integral of a local quantity. Thus, even in the discrete case, we can define a symmetry transformation operator $U_g(\mathcal{M})$ that automatically inherits all the properties of the continuous one.

We will thus refer to both continuous and discrete symmetries of this type as *zero-form symmetries*. In fact, all the previous concepts can be easily generalised to the so-called higher-form symmetries by changing the dimensions of the conserved Noether current J along with the submanifold $\mathcal{M}$.

If the symmetry is continuous, we can thus define a *q-form symmetry* when:

- The Noether current J is a $(q+1)$-form and the generator can again be written as an integral similarly to (6.1.1);
- The associated submanifold $\mathcal{M} \subset X$ is a co-dimension $q+1$ manifold, *i.e.* it is $(d-q-1)$-dimensional;

- The charged operators $\mathcal{O}^I$ have dimension q, *i.e.* they are defined on q-dimensional manifolds.

The symmetry transformation operator $U_g(\mathcal{M})$ is then defined on a $(d-q-1)$-dimensional sphere S^{d-q-1} similarly to (6.1.5). If the symmetry is discrete one must then rely on the $U_g(\mathcal{M})$ operator only.

In general, a higher form symmetry can thus be detected by the existence of topological operators $U_g(\mathcal{M})$ associated with co-dimension $q+1$ manifolds $\mathcal{M}$ that glue according to the group law (6.1.6). Then we get that

- If $q=0$, the manifolds $\mathcal{M}$ are co-dimension one and we can make sense of the group product (6.1.6) by time ordering such manifolds. Hence operators $U_g(\mathcal{M})$ at different times might not commute and G can be non-Abelian.

- If $q>0$, there cannot be such ordering; the manifold $\mathcal{M}$ at time $t+\epsilon$ can be continuously deformed to the one at time $t-\epsilon$. This means that the operators must all commute with each other and hence G must be Abelian.

For the rest of this work, we will be interested in $3d$ one-form symmetries only and thus we will only treat the case $q=1$.

6.1.1 One-form symmetries in $3d$ Yang-Mills theories

We will now consider one-form symmetries in three dimensional Yang-Mills theories.

Let us start with pure Yang-Mills theory with gauge group G. These theories possess a one-form electric symmetry whose 2-form Noether current is just the field strength $F_{\mu\nu}$ with conservation law (2.1.10).

The charged operators will then be the Wilson line operators

$$W[\mathcal{C}] := \mathrm{Tr}\left(\mathcal{P}e^{i\int_{\mathcal{C}} A}\right) \tag{6.1.7}$$

where $\mathcal{P}$ stands for the path ordering prescription and $\mathcal{C}$ for the path itself. Moreover, A is the one-form associated to the gauge boson, namely $A = A^a_\mu T^{\mathcal{R}}_a dx^\mu$, where $T^{\mathcal{R}}_a$ are the generators of G in a given representation $\mathcal{R}$. So, in principle, these operators can be labelled by any representation $\mathcal{R}$ and associated to any weight vector in the weight lattice $\Lambda_w(\mathfrak{g})$ [2].

Because the gauge bosons A^a_μ are in the adjoint representation, they are blind to any transformation which sits in the centre $Z(G)$ of G. In fact, from the algebraic point of view, the centre is defined as

$$Z(\mathfrak{g}) = \{A \in \mathfrak{g} \mid [A,\ \cdot] = 0\} \tag{6.1.8}$$

The one-form electric symmetry thus acts by shifting the gauge field by a flat $Z(G)$-valued gauge connection $\mathscr{A}$ such that the field strength $F_{\mu\nu}$ is invariant thanks to the fact that

$$d\mathcal{A} = 0\ , \qquad\qquad [\mathcal{A}, \cdot\ ] = 0 \tag{6.1.9}$$

The action of this symmetry on the Wilson line operators can be easily obtained by considering the symmetry transformation operators $U_g(\mathcal{M})$.

Let us now see some examples for the unitary groups.

Example: $G = U(1)$

The simplest pure Yang-Mills theory is the $3d$ Maxwell theory with $G = U(1)$. This theory possesses a global $U(1)_e^{[1]}$ one-form electric symmetry with associated Noether 2-form current

$$J^e = \frac{1}{2\pi} F \tag{6.1.10}$$

The symmetry transformation operators $U_g(\mathcal{M})$ are obtained by exponentiating the conserved charges (6.1.1), namely

$$U^e_{g\in U(1)}(\mathcal{C}) = e^{i\alpha \frac{1}{2\pi}\int_{\mathcal{C}} \star F} \tag{6.1.11}$$

where the group element g is just a phase and $e^{i\alpha}$.

The one-form electric symmetry acts on Wilson lines W as

$$W[\mathcal{C}] \to U^e_{e^{i\alpha}}(S^1) W[\mathcal{C}] = e^{i\alpha \frac{1}{2\pi}\oint \star F} W[\mathcal{C}] \tag{6.1.12}$$

where the charge of the Wilson line $\frac{1}{2\pi}\oint \star F$ is nothing but the magnetic flux through the S^1 loop.

Example: $G = SU(N)$

Let us now consider a pure $3d$ Yang-Mills theory with $G = SU(N)$. Since $Z(SU(N)) = \mathbb{Z}_N$, this theory has a global discrete $(\mathbb{Z}_N^{[1]})_e$ one-form electric symmetry that acts on Wilson lines as

$$W[\mathcal{C}] \to U^e_{g\in\mathbb{Z}_N}(S^1) W[\mathcal{C}] = e^{i\frac{2\pi k}{N}\mathfrak{n}} W[\mathcal{C}] \tag{6.1.13}$$

where the group element $g = e^{i\frac{2\pi k}{N}}$ and the charge of the Wilson line $\mathfrak{n}$ is the so-called N-ality and corresponds to the number of boxes (modulo N) in the Young tableau defining the representation $\mathcal{R}$ of $W[\mathcal{C}]$.

Since this one-form symmetry is discrete we cannot define the conserved Noether current as in the previous example.

As for these unitary theories, the starting point to understand the one-form symmetries for the orthosymplectic gauge groups is via their centre. Table (6.1.14) contains all the centre subgroups of these groups.

Group	N (Mod4)			
	0	1	2	3
$SO(N)$	$\mathbb{Z}_2$	1	$\mathbb{Z}_2$	1
$O(N)$	$\mathbb{Z}_2$			
$USp(2N)$	$\mathbb{Z}_2$			

(6.1.14)

Naively one could think that these centres are automatically identified with the one-form symmetries of the respective theories. In a given three dimensional theory, however, whenever a $\mathbb{Z}_r^{[0]}$ zero-form symmetry is gauged, this results in an emergent $\mathbb{Z}_r^{[1]}$ one-form global symmetry for the new theory (and *vice versa*) [87]. Thus, since the $O(N)$ gauge group is obtained by gauging the $(\mathbb{Z}_2^{[0]})_{\mathcal{C}}$ charge conjugation symmetry of $SO(N)$ (see Fig. (4.6.6)), we get an additional quantum $(\mathbb{Z}_2^{[1]})_{\widehat{\mathcal{C}}}$ one-form global symmetry which combines with the pre-existent one of Tab. (6.1.14) [60, 112]. Moreover, the exact form of the resulting one-form global symmetry group depends on N. Since we are interested in the even case only, we get that the final form for the one-form global symmetry of $O(2N)$ is $(\mathbb{Z}_2^{[1]})_{\widehat{\mathcal{C}}} \times (\mathbb{Z}_2^{[1]})_{\text{centre}}$. In this sense, when talking about the one-form global symmetries, Table (6.1.14) must be modified as following

Group	One-form symmetry
$SO(2N)$	$\mathbb{Z}_2^{[1]}$
$O(2N)$	$(\mathbb{Z}_2^{[1]})_{\widehat{\mathcal{C}}} \times (\mathbb{Z}_2^{[1]})_{\text{centre}}$
$USp(2N)$	$\mathbb{Z}_2^{[1]}$

(6.1.15)

where, for the sake of simplicity, we considered the even cases only.

In the following, however, we will work with $3d$ Chern–Simons theories of the form G_k and not with pure Yang-Mills theories. So, we must see what happens when introducing a Chern–Simons interaction of the form (2.1.7) in the previous theories. Thanks to this new term, monopole operators V_m with gauge charge $m = \sum_a^{\text{Rank}\, G} m_a$ acquire an electric charge proportional to km. The presence of such new charged matter screens some of the pre-existing Wilson lines of the theory and the original one-form symmetry group is reduced.

In the unitary cases we get:

- For $N = 1$ the naive $U(1)^{[1]}$ one-form symmetry is reduced to $\mathbb{Z}_k^{[1]}$.

  This is because the most general Wilson line has charge m (see (6.1.12)) and a collection of k such Wilson lines can be screened by the monopole operator V_m.

- For $N > 1$ we can write the gauge group $U(N)_k$ as

$$U(N)_k = SU(N)_k \times U(1)_{Nk}/\mathbb{Z}_N \tag{6.1.16}$$

  The $SU(N)_k$ theory in (6.1.16) possesses a $\mathbb{Z}_N^{[1]}$ one-form symmetry since its monopoles cannot be electrically charged, *i.e.* $\sum_a m_a = 0$. On the other hand, the $U(1)_{Nk}$ theory in (6.1.16) possesses a $\mathbb{Z}_{Nk}^{[1]}$ one-form symmetry. Modding out a combined $\mathbb{Z}_N$ from the two, we are left with a $\mathbb{Z}_k^{[1]}$ one-form symmetry.

  Here a Wilson line in the fundamental representation $(\boldsymbol{N})$ has charge 1 (see (6.1.13)), and a collection of k such Wilson lines can be screened by the unit monopole operator V_1.

In the orthosymplectic case, only the one-form symmetry for the orthogonal group $O(N)_k$ can be modified by the presence of the CS level and, thus, its exact form will depend on both N and k. In the even case, for example, we get

$O(2N)_k$	k (Mod4)	
	0	2
One-form symmetry	$(\mathbb{Z}_2^{[1]})_{\widehat{\mathcal{C}}} \times (\mathbb{Z}_2^{[1]})_{\text{centre}}$	$\mathbb{Z}_4^{[1]}$

(6.1.17)

where, for $k = 2$ Mod4, the $(\mathbb{Z}_2^{[1]})_{\widehat{\mathcal{C}}}$ and $(\mathbb{Z}_2^{[1]})_{\text{centre}}$ one-form symmetries combines into an overall $\mathbb{Z}_4^{[1]}$.

As explained in [60, Sections 2.3 and 2.4] and [112, Section 6.2], the $\mathbb{Z}_4^{[1]}$ one-form symmetry of the $O(N)_k$ theory arises from a "non-trivial extension" between the two original one form symmetries. Whenever such a non-trivial extension between two symmetries exists[1], the resulting overall symmetry gets enhanced. To keep track of such a peculiar behaviour, one can define the so-called "short exact sequence" between the two symmetries; in this case it reads:

$$0 \;\to\; (\mathbb{Z}_2^{[1]})_{\widehat{\mathcal{C}}} \;\to\; \mathbb{Z}_4^{[1]} \;\to\; (\mathbb{Z}_2^{[1]})_{\text{centre}} \;\to\; 0 \tag{6.1.18}$$

In the following we will see lots of examples of short exact sequences and non-trivial extensions.

To conclude this section, let us stress that the screening process happens whenever we insert generic charged matter into a theory. The representation $\mathcal{R}$ in which the matter sits establishes which Wilson lines are screened and, thus, how the naive one-form symmetry of the centre is reduced. Then, if this one-form symmetry shares a non-trivial extension with some other symmetry, it can enhance to a new one.

6.1.2 One-form symmetry gauging

As any other global symmetry, the one-form symmetry of a given theory can be gauged too [87]. If the one-form symmetry is continuous this can be done in the usual way by coupling the one-form electric current J^e to a generic background 2-form connection B. If, on the other hand, the one-form symmetry is discrete the gauging procedure can be performed by summing summing over all possible insertions of the symmetry transformations operators $U_g(\mathcal{M})$.

In both cases, when we gauge a subgroup C of the global one-form symmetry, we obtain a gauge theory with gauge group G/C. This can be seen by analysing the Wilson lines before and after the gauging is performed.

As we already saw, in the absence of charged matter, for simply-connected gauge group G, one allows Wilson lines of any possible representation.

Suppose, however, we start with a gauge group G/C instead of G. Since the representations of G/C are a subset of those of G, any representation that transforms non-trivially under C is prohibited for the Wilson lines. This limits the allowed Wilson lines of the theory to be a subgroup of those we would have had considering G as a gauge group.

When for example $G = SU(N)$ and $C \equiv Z(G) = \mathbb{Z}_N$, in the theory with gauge group $SU(N)/\mathbb{Z}_N$ only the Wilson lines carrying tensor products of the adjoint representation remain. However these lines are screened by the gluons which sit in the adjoint representation too. So in the $SU(N)/\mathbb{Z}_N$ theory there are no unscreened Wilson lines and, therefore, no remaining one-form symmetry.

Since the complete screening of Wilson lines corresponds to the gauging of the entire one-form symmetry $\mathbb{Z}_N^{[1]}$ of the $SU(N)$ theory, this means that the gauged theory must then be $SU(N)/\mathbb{Z}_N$.

We should remark again that every time we gauge a discrete q-form symmetry we gain a new quantum $(d-q-2)$-form symmetry. In this sense, as already anticipated, in our three dimensional case, whenever a one-form electric symmetry $C^{[1]}$ is gauged, a new zero-form magnetic global symmetry $C^{[0]}$ emerges in the gauged theory (and

[1] As we will see in the following sections, the involved symmetries can also be of different nature such a p-form symmetry and a q-form one.

vice versa). This feature can be detected by looking at the monopole operators of the gauged theory; in fact some of these monopoles are absent in the original theory and are exactly the operators charged under the new zero-form magnetic symmetry.

Suppose in fact that $G/C = U(N)_k/\mathbb{Z}_k$. This theory admits additional monopole operators with respect to the $U(N)_k$ theory, corresponding to fractional magnetic fluxes [31]

$$m_i = j \,\mathrm{diag}\left(\frac{1}{k}, \cdots, \frac{1}{k}\right) \quad \text{where } j \in \mathbb{Z}_k \tag{6.1.19}$$

Indeed, after gauging the $\mathbb{Z}_k^{[1]}$ one-form electric symmetry, we get a new $\mathbb{Z}_k^{[0]}$ zero-form magnetic symmetry which acts on these newly added monopole operators.

This new zero-form magnetic symmetry can then combine with the pre-existing topological symmetry (if existing at all) and form a new global symmetry group. We will present this phenomenon in detail later for the ABJM and ABJ theories we are interested in.

The introduction of this fractional magnetic fluxes in the G/C theory has some consequences on the computation of the 3d superconformal index (4.5.100). Indeed, when summing over all the possible values of the magnetic fluxes $\{m_a\}$, one must take into account the fact that now they arrange into different sectors $\mathcal{S}_p$ which must be summed up separately.

Suppose again that $G/C = U(N)_k/\mathbb{Z}_k$. Then the different sectors are defined as follows

$$\mathcal{S}_p = \left\{ m_a \in \mathbb{Z} + \frac{p}{k} \mid \mathrm{Mod}(k m_a, k) = p,\ \forall a = 1, \ldots, N \right\} \tag{6.1.20}$$

then the summation over the magnetic fluxes in (4.5.100) becomes

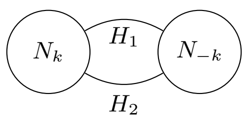
$$\sum_{\{m_a\}} \quad \xrightarrow{\text{gauging } \mathbb{Z}_k} \quad \sum_{p=0}^{k-1} g^p \sum_{\{m_a \in \mathcal{S}_p\}} \tag{6.1.21}$$

where we introduced a new fugacity g encoding the belonging of the magnetic fluxes to a specific sector $\mathcal{S}_p$. This fugacity will represent at the index level the new $\mathbb{Z}_k^{[0]}$ zero-form magnetic symmetry of the gauged theory (see, for example, (6.4.2)).

6.2 The ABJM theories

We want to apply all the concepts of the previous section to the ABJ theories with both unitary and orthosymplectic gauge groups. These are generalizations of the aforementioned ABJM theories which constitute the most basic cases. Thus, we will now analyse the latter and come back later to the first.

The ABJM $U(N)_k \times U(N)_{-k}$ theories [6] are a family of three-dimensional SCFTs with at least $\mathcal{N} = 3$ supersymmetry whose quiver diagram is depicted in Fig. (6.2.1).



(6.2.1)

They contain two hypermultiplets $\{H_1, H_2\}$ in the bi-fundamental representation of the two gauge groups $U(N)_k \times U(N)_{-k}$. These two hypermultiplets can be written in a $\mathcal{N} = 2$ formalism as four chiral fields

$$C_I = (A_1, A_2, B_{1'}, B_{2'}) \tag{6.2.2}$$

where A_α and $B_{\alpha'}$ transform in the $(\mathbf{2}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{2})$ representations respectively of the R-symmetry group $SU(2)_L \times SU(2)_R \subset SU(4)_R$.

The supersymmetry of such theories gets enhanced to $\mathcal{N} = 6$. To see this, let us first write down the superpotential, which takes the form

$$\mathcal{W} = \frac{k}{8\pi} \operatorname{Tr}\left(\Phi_{(2)}^2 - \Phi_{(1)}^2\right) + \sum_{\alpha=\alpha'} \left(\operatorname{Tr}\left(B_{\alpha'}\Phi_{(1)}A_\alpha\right) + \operatorname{Tr}\left(A_\alpha \Phi_{(2)} B_{\alpha'}\right)\right) \tag{6.2.3}$$

Once the scalar fields $\Phi_{(i)}$ are integrated out, the resulting superpotential can then be rewritten in a more compact way as follows

$$\mathcal{W} = \frac{2\pi}{k} \varepsilon^{\alpha\beta} \varepsilon^{\alpha'\beta'} \operatorname{Tr}\left(A_\alpha B_{\alpha'} A_\beta B_{\beta'}\right) \tag{6.2.4}$$

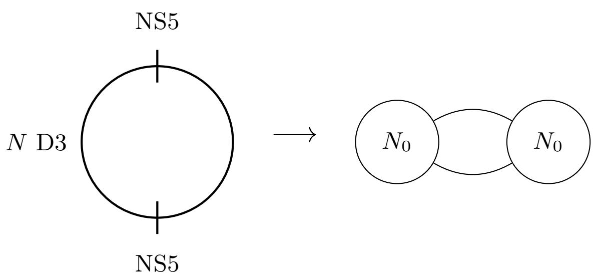
which explicitly exhibits an $SU(2)_A \times SU(2)_B$ flavour symmetry.

This flavour symmetry does not commute with the aforementioned R-symmetry, thus combining the two together the resulting global symmetry is $SU(4)_C$ which acts on the four chiral fields C_I collectively. Since the supercharges cannot be singlets under this combination of flavour and R- symmetries, the supersymmetry gets enhanced to at least $\mathcal{N} = 6$ so that the R-symmetry becomes $SO(6) \simeq SU(4)$.

6.2.1 Hanany-Witten brane construction

Let us now focus on the Hanany-Witten brane construction of the ABJM theories [6, Section 3].

The Hanany-Witten brane setup for $k = 0$ is depicted in Fig. (6.2.5), where the direction x^6 of Table (3.1.1) has been compactified to a circle.



(6.2.5)

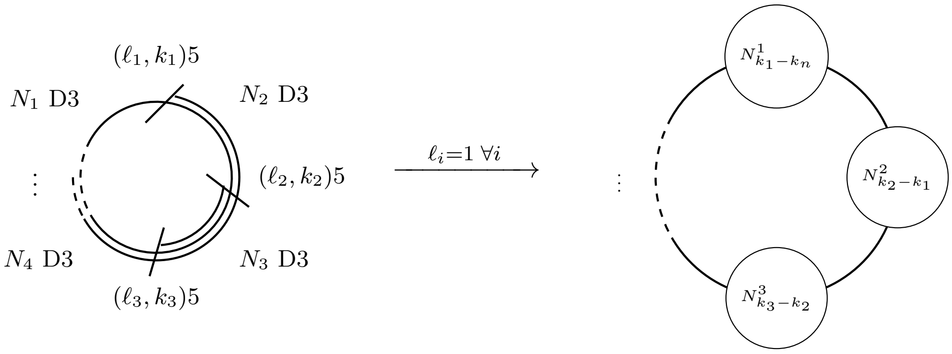
Chern–Simons interaction terms can be obtained within the Hanany-Witten brane construction by a particular mass deformation of the usual configuration of Table (3.1.1). This deformation combines k D5 branes and a NS5 brane into a new type of five brane which we call $(1, k)$ fivebrane by rotating them in the $(5, 9)$, $(3, 7)$ and $(4, 8)$ planes by the same discrete θ angle, such that $\tan\theta = k$.

In the end, the new brane configuration becomes

Type	0	1	2	3	4	5	6	7	8	9
NS5	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$				
D3	$\times$	$\times$	$\times$				$\times$			
$(1,k)5$	$\times$	$\times$	$\times$	$(3,7)_\theta$	$(4,8)_\theta$	$(5,9)_\theta$		$(3,7)_\theta$	$(4,8)_\theta$	$(5,9)_\theta$

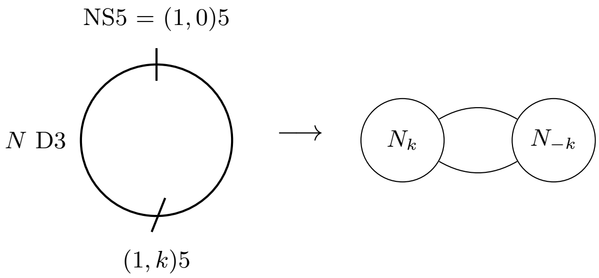
(6.2.6)

According to this new five brane classification, the standard NS5 brane becomes a $(1,0)$ fivebrane while the standard D5 brane becomes a $(0,1)$ fivebrane. Thus in general we could have (ℓ, k) fivebrane according to how many D5 branes and NS5 brane we respectively mix in the new five brane. A generic Hanany-Witten brane configuration of this type is depicted in Fig. (6.2.7) (see for example [109]). When all the ℓ_i are equal to one, the worldvolume theory becomes a circular quiver with the gauge group $U(N_1)_{k_1-k_n} \times U(N_2)_{k_2-k_1} \times \ldots \times U(N_n)_{k_n-k_{n-1}}$.



(6.2.7)

Thus, considering, as in Fig. (6.2.8), one NS5 brane and one $(1,k)$ fivebrane on top of N D3 branes, we get a $U(N)_k \times U(N)_{-k}$ Yang-Mills Chern–Simons theory, which is exactly a standard ABJM theory.



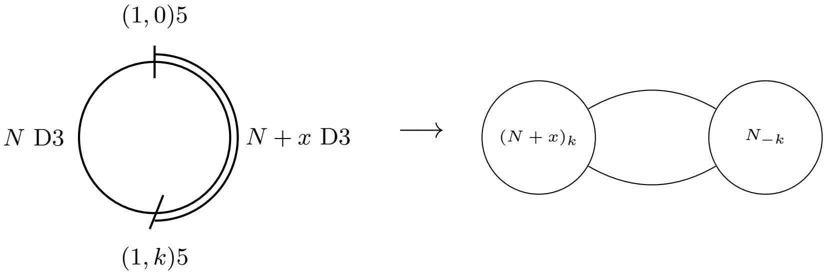
(6.2.8)

One can then lift the Hanany-Witten brane configuration to M-theory. Here, taking the IR limit of the new configuration, the system becomes the near-horizon limit of M2 branes probing a $\mathbb{C}^4/\mathbb{Z}_k$ singularity. This interesting fact will become very useful in the following sections.

Now that we know the brane setup for the ABJM theories, it will be very easy to generalise this construction and to build the ABJ and related theories.

6.3 The ABJ theories

The first generalization of the ABJM theories consists in the ABJ theories of the form $U(N+x)_k \times U(N)_{-k}$ [1]. These are obtained by slightly modifying the brane setup (6.2.8) as follows

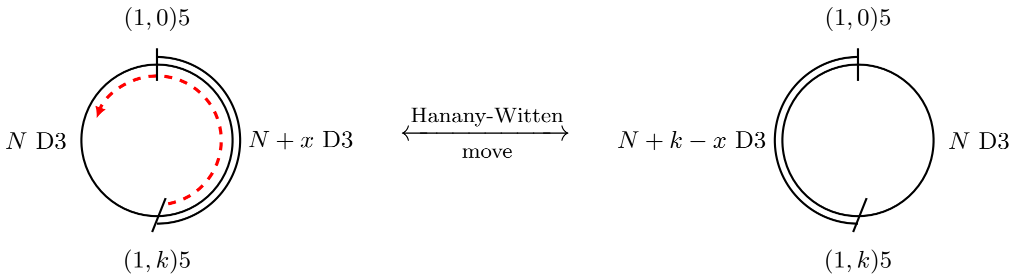


(6.3.1)

where we only added x D3 branes on the right part of the brane setup. According to the rules of the previous section, this transform the $U(N)_k$ gauge node into a $U(N+x)_k$ one by leaving untouched the hypermultiplets (their number in fact corresponds to the number of D5 branes inserted in the setup, which, in this case, are not modified). Clearly we could also add the x D3 branes on the left part of the brane setup (6.2.8), leading to the $U(N)_k \times U(N+x)_{-k}$ ABJ theory.

These additional x suspended D3 branes, however, are constrained by the properties of the Hanany-Witten brane setup. First of all, since they are stretched between two different types of five branes, they are locked into position and have fixed coordinates along the $(3, 4, 5)$ and $(7, 8, 9)$ directions (see Sec. (3.1)). Moreover, due to the s-rule, which prevents more than one D3 brane to be stretched between a NS5- and a D5 brane pair, we must have $k \geq x$. In other words, the number of suspended D3 branes must be less or equal to the number of D5 branes mixed in the $(1, k)$ fivebrane.

Furthermore, we can also perform the Hanany-Witten move of Fig. (3.1.9) on the additional x suspended D3 branes. Moving the $(1, k)$ fivebrane through the NS5 brane, as in Fig. (6.3.2), and taking into account the s-rule constraints, the pre-existing x D3 branes are annihilated while $k - x$ branes are created. The latter come from the $k - x$ D5 branes mixed in the $(1, k)$ fivebrane that did not possess any stretched D3 brane prior to the move.



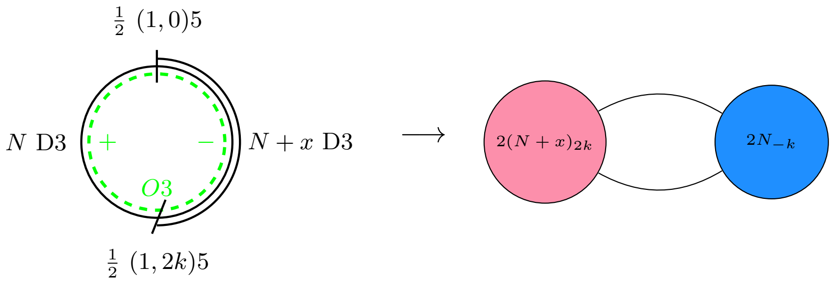
(6.3.2)

After the Hanany-Witten move, the resulting theory on the right side of Fig. (6.3.2) is the $U(N)_k \times U(N + k - x)_{-k}$ ABJ theory. Since the two brane setups of Fig. (6.3.2) are completely equivalent, the corresponding theories are dual and thus

$$U(N + x)_k \times U(N)_{-k} \longleftrightarrow U(N)_k \times U(N + k - x)_{-k} \tag{6.3.3}$$

Another possible variation of this setup consists in including an orientifold $O3$-plane wrapped on the circle. As we already saw in Sec. (3.4), the presence of such $O3$-plane makes the five branes become half branes and changes the gauge groups related to the D3 branes according to Table (3.4.4). We can see an example in Fig.

(6.3.4), where we depicted the brane setup for the $O(2(N+x))_{2k} \times USp(2N)_{-k}$ theory[2].



(6.3.4)

This allows us to obtain an entire class of new $\mathcal{N} = 5$ SCFTs with gauge groups $O(M) \times USp(2N)$ which we call "ABJ-like theories" and which exact form depend on the type of the inserted $O3$-plane and on the number of D3 branes. The possible theories are the following:

$$O(2N+2x)_{2k} \times USp(2N)_{-k} \text{ with } k \geq x-1 \tag{6.3.5}$$
$$USp(2N+2x)_{k} \times O(2N)_{-2k} \text{ with } k \geq x+1 \tag{6.3.6}$$
$$O(2N+2x+1)_{2k} \times USp(2N)_{-k} \text{ with } k \geq x \tag{6.3.7}$$
$$USp(2N+2x)_{k} \times O(2N+1)_{-2k} \text{ with } k \geq x \tag{6.3.8}$$

and other four obtained by sending $k \to -k$. The restrictions on x, as in the unitary case, are obtained by taking into account the s-rule constraints (see Sec. (3.4.1)). However, when an $O3$-plane is introduced in the brane setup, the s-rule gets more complicated, since the creation or annihilation of the D3 branes depends on the charge q of the orientifold plane and, thus, one has to pay more attention.

These theories possess dual relations of the form (6.3.3) when performing the Hanany-Witten move of Fig. (3.4.5) on the additional x suspended D3 branes. These are

$$\begin{aligned} O(2N+2x)_{2k} \times USp(2N)_{-k} &\longleftrightarrow O(2N+2(k-x+1))_{-2k} \times USp(2N)_{k} \\ USp(2N+2x)_{k} \times O(2N)_{-2k} &\longleftrightarrow USp(2N+2(k-x-1))_{-k} \times O(2N)_{2k} \\ O(2N+2x+1)_{2k} \times USp(2N)_{-k} &\longleftrightarrow O(2N+2(k-x)+1)_{-2k} \times USp(2N)_{k} \\ USp(2N+2x)_{k} \times O(2N+1)_{-2k} &\longleftrightarrow USp(2N+2(k-x))_{-k} \times O(2N+1)_{2k} \end{aligned} \tag{6.3.9}$$

Moreover, when considering ABJ-like theories of the form $O(2N)_{2k} \times USp(2N)_{-k}$, the brane configuration can again be lifted to M-theory. Here, in the IR, it is found to be equal to the near-horizon limit of M2 branes probing a $\mathbb{C}^4/\widehat{D}_k$ singularity, where $\widehat{D}_k$ is the binary dihedral group with $4k$ elements. Thanks to this, recalling what we said in the previous section, the duality in the first line of (6.0.1) is evident from the simple fact that $\widehat{D}_1 = \mathbb{Z}_4$ as groups and thus the singularities probed by the M2 branes in the near-horizon limit are exactly the same. This duality was indeed first conjectured in [1] on this basis and then studied in more detail in [54].

In the following, we will consider also theories of the form $SO(M) \times USp(2N)$. Since these theories differ from the ones in (6.3.8) just for the ungaged charge conjugation discrete $\mathbb{Z}_2^{\mathcal{C}}$ symmetry of the $SO(M)$ gauge node (see Fig. (4.6.6)), we will

[2]We remark again that each red node with a label N denotes an $O(N)$ or $SO(N)$ group and each blue node with an even label $2N$ denotes a $USp(2N)$ group.

treat them as ABJ theories too.

The matter content of the ABJ-like theories of the form $(S)O(2N)_{2k} \times USp(2N)_{-k}$ is exactly the same as in the unitary case (as depicted, for example, in the quiver of Fig. (6.3.4)). However, because of the presence of orthosymplectic gauge groups, the four $\mathcal{N}=2$ chiral fields C_I (6.2.2) are now subject to the reality conditions

$$A_\alpha = B_{\alpha'} J \tag{6.3.10}$$

where where J is the invariant antisymmetric matrix of the symplectic group. In this sense the two $\mathcal{N}=4$ hypermultiplets $\{H_1, H_2\}$ becomes two real "half hypermultiplets" preserving a subgroup $USp(4)_R \subset SU(4)_R$ of the original R-symmetry.

Moreover, due to the identifications (6.3.10), the original $SU(2)_A \times SU(2)_B$ flavour symmetry is reduced to $SU(2)_A$ only.

6.3.1 Dualities and one-form symmetry gauging

One of the main objectives of this chapter is to take the first duality of (6.0.1), namely

$$I: \quad O(2N)_2 \times USp(2N)_{-1} \;\longleftrightarrow\; U(N)_4 \times U(N)_{-4} \tag{6.3.11}$$

and see what happens if we gauge the global one-form symmetries of the two theories involved in the duality. To do so, however, we need to study each theory and their global symmetries in detail. For convenience, in the following we respectively use (L) and (R) to denote the left and right descriptions of each duality we will examine.

1. The I(R) description.

   In examining the global symmetries of this theory we will closely follow [31, Section 2.1] and make use of their notation for monopole operators. Thus, from now on, $T_{\{\boldsymbol{m}_L;\boldsymbol{m}_R\}}$ will denote a monopole operator with fluxes $\boldsymbol{m}_L$ and $\boldsymbol{m}_R$ in the left and right gauge nodes respectively. Its electric gauge charge will then be $(km_L, -km_R)$ with

   $$m_L = \sum_a^N (\boldsymbol{m}_L)_a, \qquad m_R = \sum_a^N (\boldsymbol{m}_R)_a \tag{6.3.12}$$

   At first glance, recalling what said in Sec. (6.1.1), the one-form symmetry is naively $\mathbb{Z}_4^{[1]} \times \mathbb{Z}_4^{[1]}$, with one $\mathbb{Z}_4^{[1]}$ factor for each gauge node. To understand the real form of this symmetry, we need however to see which Wilson line are present in the theory and which ones are screened.

   First of all, the bifundamental matter fields in the $(\boldsymbol{N}, \overline{\boldsymbol{N}})$ representation screen the Wilson lines in the $(\boldsymbol{N}, \overline{\boldsymbol{N}})$ representation charged under the anti-diagonal combination of $\mathbb{Z}_4^{[1]} \times \mathbb{Z}_4^{[1]}$. Both have in fact electric gauge charges $(1,1)$ under the two $U(1)$ factors of the gauge groups.

   Then, since there is no other charged matter field in the theory apart from monopole operators $T_{\{\boldsymbol{m}_1;\boldsymbol{m}_2\}}$ with electric gauge charge $(4m_1, -4m_2)$, Wilson lines in the $(\boldsymbol{N}, \boldsymbol{N})$ representation with electric gauge charge $(1,-1)$ cannot be screened by monopoles unless if taken in groups of four.

   The resulting one-form symmetry is thus the diagonal $\mathbb{Z}_4^{[1]}$ subgroup only.

   Focusing now on zero-form symmetries, theory I(R) also possesses a $U(1)^{[0]}_{\text{top}}$ topological symmetry which is the diagonal combination of the two $U(1)$ factors coming from the two gauge nodes[3]. This symmetry possesses a mixed anomaly with the $\mathbb{Z}_4^{[1]}$ one-form symmetry characterised by the following short exact sequence:

$$\mathrm{I(R)}: \quad 0 \to \mathbb{Z}_4^{[1]} \to \mathbb{Z}_4^{[1]} \times U(1)^{[0]}_{\mathrm{top}} \to U(1)^{[0]}_{\mathrm{top}} \to 0 \tag{6.3.13}$$

which shows that there is no non-trivial extension between these two symmetries.

Finally, as seen in Sec (6.2), theory I(R) also possesses the standard $SU(2)_A \times SU(2)_B$ flavour symmetry of the ABJM theories.

2. The I(L) description.

   As already said in Sec. (6.1.1), both $O(2N)$ and $USp(2N)$ gauge groups have a $\mathbb{Z}_2$ centre thus the one-form symmetry is naively $\mathbb{Z}_2^{[1]} \times \mathbb{Z}_2^{[1]}$. However, similarly to the I(R) theory, the half-hypermultiplets in the bifundamental representation screen the diagonal combination of $\mathbb{Z}_2^{[1]} \times \mathbb{Z}_2^{[1]}$, and so we are left with the anti-diagonal $\mathbb{Z}_2^{[1]}$ subgroup only, which we denote by $(\mathbb{Z}_2^{[1]})_{\mathrm{centre}}$. Then, since in our case the CS level is $k = 2$, the anti-diagonal $(\mathbb{Z}_2^{[1]})_{\mathrm{centre}}$ combines with the quantum $(\mathbb{Z}_2^{[1]})_{\widehat{\mathcal{C}}}$ into an overall $\mathbb{Z}_4^{[1]}$ one-form symmetry [60, 112]. This symmetry is characterised by the short exact sequence (6.1.18).

   Coming to the global zero-form symmetries, the $U(1)^{[0]}_{\mathrm{top}}$ topological symmetry of theory I(R) is not manifest in the I(L) description but we got a discrete $(\mathbb{Z}_2^{[0]})_{\mathcal{M}}$ zero-form magnetic symmetry[4] coming from the centre of the gauge nodes. Moreover, the $SU(2)_A \times SU(2)_B$ flavour symmetry of the ABJM theories is reduced to $SU(2)_A$ by the identifications (6.3.10).

Thanks to the superconformal index, in Subsection (6.4.3) we study in detail the zero-form symmetries of the dual theories I(L) and I(R) of (6.3.11) and their matching, which can be performed even if the symmetries are manifestly different by mapping the index fugacities across the duality. The result of the analysis is summarized in the first line of Table (6.3.20) along with all the zero-form symmetries and their respective fugacities.

Now, let us see what happens to the duality (6.3.11) if we gauge the one-form symmetry in each theory. Even if both theories I(L) and I(R) possess a $\mathbb{Z}_4^{[1]}$ one-form symmetry, the correct way of doing this is by step, gauging only $\mathbb{Z}_2$ subgroups of it. In such a way, we can understand in a clear way what happens to the theories and to their global symmetries. Thus, in the theory I(L), this is obtained by gauging $(\mathbb{Z}_2^{[1]})_{\widehat{\mathcal{C}}}$, whereas in the theory I(R), this corresponds to gauging a generic $\mathbb{Z}_2^{[1]}$ subgroup of the $\mathbb{Z}_4^{[1]}$ one-form symmetry. As a result, we should obtain an emergent new $\mathbb{Z}_2^{[0]}$ zero-form symmetry in each gauged theory.

As we will see in details in the following sections, the superconformal index shows that the resulting theories are indeed again dual to eachother and, thus, that we are left with the duality

$$II: \quad SO(2N)_2 \times USp(2N)_{-1} \longleftrightarrow U(N)_4 \times U(N)_{-4}/\mathbb{Z}_2 \tag{6.3.14}$$

[3]The anti-diagonal $U(1)$ acts non-trivially on the bifundamental matter fields and therefore defines a $U(1))^{[0]}_B$ zero-form baryonic symmetry.

[4]Notice also that in the special case of $N = 1$ this symmetry is actually continuous and becomes $U(1)^{[0]}_{\mathcal{M}}$.

1. The II(R) description.

   This new emergent $\mathbb{Z}_2^{[0]}$ zero-form symmetry is identified with the discrete topological symmetry which, naively, we call $(\mathbb{Z}_2^{[0]})_g$. However, as pointed out in [31, Section 2.5], the theory II(R) of the new duality (6.3.14) actually has a $U(1)^{[0]} \times \mathbb{Z}^{[0]}_{\mathrm{GCD}(N,2)}$ zero-form symmetry. This can be seen by considering the action of $(\mathbb{Z}_2^{[0]})_g$ on the new monopole operators introduced in theory II(R) with fractional magnetic flux of the form (6.1.19). This action can in fact be reproduced by the $U(1)^{[0]}_{\mathrm{top}}$ topological symmetry except for the elements in the $\mathbb{Z}^{[0]}_{\mathrm{GCD}(N,2)}$ subgroup of $(\mathbb{Z}_2^{[0]})_g$. Thus, the global symmetry actually is $U(1)^{[0]} \times \mathbb{Z}^{[0]}_{\mathrm{GCD}(N,2)}$.

   Similarly to [31, (2.18)], this phenomenon can also be explained as the existence of a non-trivial extension of the $U(1)^{[0]}_{\mathrm{top}}$ topological symmetry and the zero-form discrete topological symmetry $(\mathbb{Z}_2^{[0]})_g$, characterised by the short exact sequence:

$$\mathrm{II(R)}: \quad 0 \;\to\; (\mathbb{Z}_2^{[0]})_g \;\to\; U(1)^{[0]} \times \mathbb{Z}^{[0]}_{\mathrm{GCD}(N,2)} \;\to\; U(1)^{[0]}_{\mathrm{top}} \;\to\; 0\,. \tag{6.3.15}$$

   Clearly, the other global symmetries and the flavour symmetries are left untouched by the one-form symmetry gauging.

2. The II(L) description.

   The new emergent $\mathbb{Z}_2^{[0]}$ zero-form symmetry is identified with the charge conjugation symmetry $(\mathbb{Z}_2^{[0]})_{\mathcal{C}}$, making the orthogonal gauge node change from $O(2N)_2$ to $SO(2N)_2$. This adds up to the already existing $(\mathbb{Z}_2^{[0]})_{\mathcal{M}}$ zero-form magnetic symmetry.

   Again,the flavour symmetries are unchanged by the gauging.

The mapping of the discrete zero-form symmetries across duality (6.3.14) then qualitatively changes depending on the value of $\mathrm{GCD}(N,2)$ and will be illustrated in details in the following sections, when studying the superconformal index for some specific values of N.

After this first $\mathbb{Z}_2$ gauging, each description of the duality (6.3.14) also has a remnant $\mathbb{Z}_4^{[1]}/\mathbb{Z}_2^{[1]} = \mathbb{Z}_2^{[1]}$ one-form symmetry.

Following the discussion in [151], if we start from the (6.1.18) exact sequence of theory II(L) and gauge the $(\mathbb{Z}_2^{[1]})_{\widehat{\mathcal{C}}}$ one-form symmetry, we obtain the new short exact sequence:

$$\mathrm{II(L)}: \quad 0 \;\to\; (\mathbb{Z}_2^{[0]})_{\mathcal{C}} \;\to\; (\mathbb{Z}_2^{[0]})_{\mathcal{C}} \times (\mathbb{Z}_2^{[1]})_{\mathrm{centre}} \;\to\; (\mathbb{Z}_2^{[1]})_{\mathrm{centre}} \;\to\; 0 \tag{6.3.16}$$

with a mixed anomaly between the zero-form charge conjugation symmetry $(\mathbb{Z}_2^{[0]})_{\mathcal{C}}$ and the centre $\mathbb{Z}^{[1]}_{2\,\mathrm{centre}}$ one-form symmetry (as discussed in [60, Section 2.4]).

Since there is no mixed anomaly between $(\mathbb{Z}_2^{[1]})_{\widehat{\mathcal{C}}}$ and $(\mathbb{Z}_2^{[1]})_{\mathrm{centre}}$ in (6.1.18), there is no non-trivial extension between $(\mathbb{Z}_2^{[0]})_{\mathcal{C}}$ and $(\mathbb{Z}_2^{[1]})_{\mathrm{centre}}$ (and so the exact sequence (6.3.16) is split).

Given the identification of the global symmetries between the theories II(L) and II(R) that we will see later, the same statements hold on the side of theory II(R) between $\mathbb{Z}^{[0]}_{\mathrm{GCD}(N,2)}$ and the remnant $\mathbb{Z}^{[1]}_2$. This is consistent with the fact that the description II(R) has indeed a $\mathbb{Z}^{[1]}_2$ one-form symmetry [31], without any extension with a zero-form symmetry. We will describe how to see the relation between the two global symmetries of the theories, namely $(\mathbb{Z}^{[0]}_2)_{\mathbb{C}} \times (\mathbb{Z}^{[0]}_2)_{\mathrm{top}}$ and $U(1)^{[0]}_{\mathrm{top}} \times \mathbb{Z}^{[0]}_{\mathrm{GCD}(N,2)}$, around (6.4.36) in terms of the superconformal index. See also the second line of Table (6.3.20) for a summary of the zero-form symmetries of theories II(L) and II(R).

Finally, we gauge the remnant $\mathbb{Z}^{[1]}_2$ one-form symmetry in each description of the duality (6.3.14). As a result, we obtain the following new duality between the gauged theories

$$III: \quad SO(2N)_2 \times USp(2N)_{-1}/\mathbb{Z}_2 \;\longleftrightarrow\; U(N)_4 \times U(N)_{-4}/\mathbb{Z}_4 \tag{6.3.17}$$

We also gain a $\mathbb{Z}^{[0]}_2$ zero-form symmetry in both descriptions of the duality (6.3.17). For convenience, let us denote it by $(\mathbb{Z}^{[0]}_2)_{\mathfrak{g}}$.

1. The III(R) description.

   There is a non-trivial extension $\mathbb{Z}^{[0]}_4$ zero-form symmetry that arises from the mixed anomaly (6.3.16) between the discrete topological symmetry $\mathbb{Z}^{[0]}_{\mathrm{GCD}(N,4)}$ of the theory II(R) and the remnant one-form symmetry $(\mathbb{Z}^{[1]}_2)_{\mathrm{centre}}$, which we gauged. As pointed out in [31, (2.17)], for the same reasons of theory II(R), the description III(R) has now a $U(1)^{[0]} \times \mathbb{Z}^{[0]}_{\mathrm{GCD}(N,4)}$ zero-form symmetry, which is a further non-trivial extension between the aforementioned $\mathbb{Z}^{[0]}_4$ zero-form symmetry and the topological symmetry $U(1)^{[0]}_{\mathrm{top}}$:[5]

$$\text{III(R)}: \quad 0 \;\to\; \mathbb{Z}^{[0]}_4 \;\to\; U(1)^{[0]} \times \mathbb{Z}^{[0]}_{\mathrm{GCD}(N,4)} \;\to\; U(1)^{[0]}_{\mathrm{top}} \;\to\; 0 \tag{6.3.18}$$

2. The III(L) description.

   On the other hand, by the same reasoning as in [60, Section 2.4 and Footnote 20], the (abelian) zero-form symmetry is either $(\mathbb{Z}^{[0]}_2)_{\mathfrak{g}} \times (\mathbb{Z}^{[0]}_2)_{\mathcal{M}}$ if $2N$ in the $SO(2N)$ gauge factor is not equal to $2 \mod 4$, or the non-trivial extension $\mathbb{Z}^{[0]}_4$ of the former if $2N$ in the $SO(2N)$ gauge factor is equal to $2 \mod 4$.[6]

Note that the $U(1)^{[0]}_{\mathrm{top}}$ topological symmetry is again not manifest in theory III(L). As a result, not every generator of the $U(1)^{[0]} \times \mathbb{Z}^{[0]}_{\mathrm{GCD}(N,4)}$ symmetry is manifest in theory III(L). For example, in (6.4.32), we show that in the case of $N = 2$, the $\mathbb{Z}^{[0]}_2$ subgroup of the $U(1)^{[0]}$ symmetry of theory III(R) is identified with the $\mathbb{Z}^{[0]}_2$ subgroup of the $(\mathbb{Z}^{[0]}_2)_{\mathfrak{g}} \times (\mathbb{Z}^{[0]}_2)_{\mathcal{M}}$ symmetry or of the $\mathbb{Z}^{[0]}_4$ symmetry of theory III(L). See also

[5] For $N = 1$ the non-trivial extension implies that the symmetry is only $U(1)^{[0]}$. This is compatible with what happens on the side of theory III(L) as discussed in Footnote (6). We show that $U(1)^{[0]}$ on the side III(R) is identified with $U(1)^{[0]}_{\mathcal{M}}$ on the side III(L) at the level of the index in (6.4.12).

[6] The above statements hold only for $N > 1$ since [60, (2.18)] for the anomaly applies only to discrete symmetries, while the case $N = 1$ where the magnetic symmetry is $U(1)^{[0]}$ should be treated separately. In (6.4.10) we show at the level of the index that $(\mathbb{Z}^{[0]}_2)_{\mathfrak{g}}$ can be absorbed into $U(1)^{[0]}$, indicating that the non-trivial extension between the two symmetries occurs also for $N = 1$.

the third line of Table (6.3.20) for a summary of the zero-form symmetries of theories III(L) and III(R).

A recap is thus essential. In the previous paragraphs we analyzed the following dualities

$$
\begin{array}{lccc}
\text{I:} \quad O(2N)_2 \times USp(2N)_{-1} & \longleftrightarrow & U(N)_4 \times U(N)_{-4} \\
& \mathbb{Z}_2^{[1]} \downarrow \quad \uparrow \mathbb{Z}_2^{[0]} & \\
\text{II:} \quad SO(2N)_2 \times USp(2N)_{-1} & \longleftrightarrow & [U(N)_4 \times U(N)_{-4}]/\mathbb{Z}_2 \\
& \mathbb{Z}_2^{[1]} \downarrow \quad \uparrow \mathbb{Z}_2^{[0]} & \\
\text{III:} \ [SO(2N)_2 \times USp(2N)_{-1}]/\mathbb{Z}_2 & \longleftrightarrow & [U(N)_4 \times U(N)_{-4}]/\mathbb{Z}_4
\end{array}
\tag{6.3.19}
$$

where a downwards arrow with the label $\mathbb{Z}_2^{[1]}$ denotes the gauging of the $\mathbb{Z}_2^{[1]}$ one-form symmetry and an upwards arrow with the label $\mathbb{Z}_2^{[0]}$ denotes the gauging of the $\mathbb{Z}_2^{[0]}$ zero-form symmetry.

We also studied in detail the zero-form and one-form symmetries of each pair of dual theories (L) and (R) and their matching. The result of such analysis is summarized in Table (6.3.20)

	Theory L	Theory R	Map
I	$SU(2)_A \times (\mathbb{Z}_2^{[0]})_{\mathcal{M}}$	$SU(2)_A \times SU(2)_B \times U(1)^{[0]}_{\text{top}}$	$(6.4.65)_{N=1}$
	f, ζ (ζ' when $N=1$)	u, v, w'	$(6.4.76)_{N=2}$
II	$SU(2)_A \times (\mathbb{Z}_2^{[0]})_{\mathbb{C}} \times (\mathbb{Z}_2^{[0]})_{\mathcal{M}}$	$SU(2)_A \times SU(2)_B \times U(1)^{[0]} \times \mathbb{Z}^{[0]}_{\text{GCD}(N,2)}$	$(6.4.46)_{N=1}$
	f, χ, ζ (ζ' when $N=1$)	u, v, w', g''	$(6.4.58)_{N=2}$
III	$SU(2)_A \times (\mathbb{Z}_2^{[0]})_{\mathfrak{g}} \times (\mathbb{Z}_2^{[0]})_{\mathcal{M}}$	$SU(2)_A \times SU(2)_B \times U(1)^{[0]} \times \mathbb{Z}^{[0]}_{\text{GCD}(N,4)}$	$(6.4.12)_{N=1}$
	f, $\mathfrak{g}$, ζ (ω when $N=1$)	u, v, w, g'	$(6.4.32)_{N=2}$

(6.3.20)

Let us stress that the **manifest** zero-form symmetries in the left (L) and right (R) frames of each of the three dualities I, II and III do not straightforwardly match between the dual theories, since part of the full IR symmetry can be emergent in one or both of the dual frames, rendering the mapping of symmetries across the duality non-trivial. In the last column of Table (6.3.20) we refer to the equations in the main text where the such mapping is described for the $N=1$ and $N=2$ cases at the level of the indices refined with fugacities for all the manifest symmetries. Note that for the special case of $N=1$ the zero-form $\mathbb{Z}_2$ magnetic symmetry in the first column should be replaced by the zero-form $U(1)$ topological symmetry.

Special cases

There are many interesting special cases that can be considered.

1. For $N=1$, duality II becomes

$$SO(2)_2 \times USp(2)_{-1} \;\leftrightarrow\; [U(1)_4 \times U(1)_{-4}]/\mathbb{Z}_2 \;\leftrightarrow\; U(1)_2 \times U(1)_{-2}\ . \tag{6.3.21}$$

We discuss the indices of these theories in Section (6.4.2). They are different descriptions of the worldvolume theory of a single M2 brane on $\mathbb{C}^4/\mathbb{Z}_2$ singularity, and so they all have $\mathcal{N} = 8$ supersymmetry. The second arrow is, in fact, a special case of the following duality for abelian theories:

$$[U(1)_{kp} \times U(1)_{-kp}]/\mathbb{Z}_p \quad \longleftrightarrow \quad U(1)_k \times U(1)_{-k} \; . \tag{6.3.22}$$

2. The theories involved in duality III are also related to others as follows.

$$\begin{aligned} [SO(2N)_2 \times USp(2N)_{-1}]/\mathbb{Z}_2 \quad &\longleftrightarrow \quad [U(N)_4 \times U(N)_{-4}]/\mathbb{Z}_4 \\ &\overset{[31,\,108,\,153]}{\longleftrightarrow} \quad [SU(N)_4 \times SU(N)_{-4}]/\mathbb{Z}_N \end{aligned} \tag{6.3.23}$$

   (a) For $N = 1$, we have the theory of two free hypermultiplets:

$$\begin{aligned} [SO(2)_2 \times USp(2)_{-1}]/\mathbb{Z}_2 \quad &\leftrightarrow \quad [U(1)_4 \times U(1)_{-4}]/\mathbb{Z}_4 \\ &\leftrightarrow \quad 2 \text{ free hypermultiplets} \\ &\overset{(6.3.22)}{\leftrightarrow} \quad U(1)_1 \times U(1)_{-1} \end{aligned} \tag{6.3.24}$$

   (b) For $N = 2$, we have

$$\begin{aligned} [SO(4)_2 \times USp(4)_{-1}]/\mathbb{Z}_2 \quad &\leftrightarrow \quad [U(2)_4 \times U(2)_{-4}]/\mathbb{Z}_4 \\ &\overset{[31,\,108,\,153]}{\leftrightarrow} \quad [SU(2)_4 \times SU(2)_{-4}]/\mathbb{Z}_2 \\ &\overset{[153]}{\leftrightarrow} \quad U(3)_2 \times U(2)_{-2} \\ &\overset{[153]}{\leftrightarrow} \quad Spin(5)/\mathbb{Z}_2 \text{ or } USp(4)/\mathbb{Z}_2 \text{ SYM} \end{aligned} \tag{6.3.25}$$

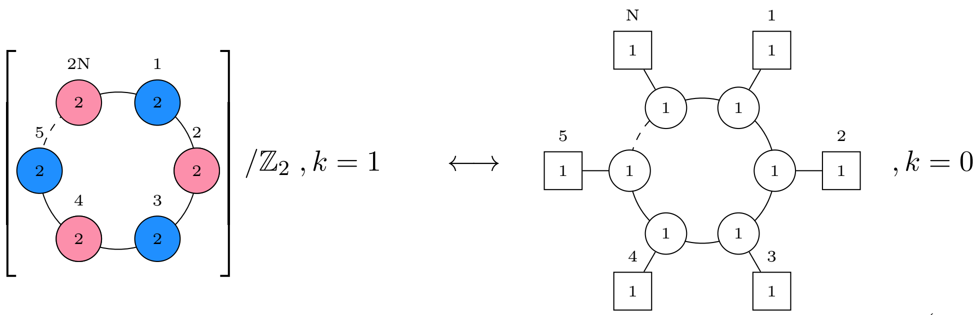
   These theories have $\mathcal{N} = 8$ supersymmetry. On the other hand, we find that the theory $SO(4)_2 \times USp(4)_{-1}$, which is dual to $[U(2)_4 \times U(2)_{-4}]/\mathbb{Z}_2$, has $\mathcal{N} = 6$ supersymmetry (see Section (6.4.2)). The $\mathbb{Z}_2$ discrete quotient, indeed, brings about extra operators carrying a non-trivial charge under the new $\mathbb{Z}_2$ zero-form topological symmetry. The conserved currents associated to these operators lead to $\mathcal{N} = 8$ supersymmetry.

   (c) On the other hand, for the case of $N = 3$ in (6.3.23), from the index computation, we see that $[U(3)_4 \times U(3)_{-4}]/\mathbb{Z}_4$ possesses $\mathcal{N} = 6$ supersymmetry (see below (6.4.29)).

Generalisations

The above results can be generalised in many ways. First, we consider the $U(3)_4 \times U(1)_{-4}$ and its dual $O(4)_2 \times USp(2)_{-1}$. According to the discussion around [153, (3.19)], such theories have a non-anomalous $\mathbb{Z}_2$ one-form symmetry. Upon gauging this symmetry, we obtain a duality pair: $[U(3)_4 \times U(1)_{-4}]/\mathbb{Z}_2 \leftrightarrow SO(4)_2 \times USp(2)_{-1}$. We discuss the symmetries of these theories in Sections (6.4.4) and (6.4.5).

We then generalise (6.3.24) to a circular quiver with $2N$ alternating $SO(2)_2$ and $USp(2)_{-1}$ gauge groups, with a discrete $\mathbb{Z}_2$ quotient. It turns out that the theories in this class are dual to 3d $\mathcal{N}=4$ gauge theories described by a circular quiver with a collection of N $U(1)$ gauge groups and with a hypermultiplet with charge 1 under each gauge group (see Fig. (6.3.26)). More details are provided in Section (6.4.6).



(6.3.26)

Finally, we study the dual pair $O(2N+1)_2 \times USp(2N)_{-1} \leftrightarrow U(N+1)_4 \times U(N)_{-4}$, as well as the dual pair $SO(2N+1)_2 \times USp(2N)_{-1} \leftrightarrow [U(N+1)_4 \times U(N)_{-4}]/\mathbb{Z}_2$. As a surprise, it turns out that these four theories have the same superconformal indices, even refined with fugacities for their 0-form discrete symmetries (see Section (6.4.7)). In particular, the zero-form charge conjugation symmetry in the $SO(2N+1)_2 \times USp(2N)_{-1}$ theory acts trivially and is unfaithful, so as the $\mathbb{Z}_2$ zero-form symmetry arising from the $\mathbb{Z}_2$ discrete gauging in the $[U(N+1)_4 \times U(N)_{-4}]/\mathbb{Z}_2$ theory. We conjecture that the $\mathbb{Z}_2$ one-form symmetry of the first two theories acts trivially on the spectrum of the line operators.

6.4 Dualities and superconformal indices

6.4.1 $[SO(2N)_2 \times USp(2N)_{-1}]/\mathbb{Z}_2 \leftrightarrow [U(N)_4 \times U(N)_{-4}]/\mathbb{Z}_4$

In this subsection, we consider the duality between these two theories:

$$\text{III(L): } [SO(2N)_2 \times USp(2N)_{-1}]/\mathbb{Z}_2 \quad \leftrightarrow \quad \text{III(R): } [U(N)_4 \times U(N)_{-4}]/\mathbb{Z}_4 \tag{6.4.1}$$

The case of $N=1$

For $N=1$, the theory III(R): $[U(1)_4 \times U(1)_{-4}]/\mathbb{Z}_4$ is dual to $SU(1)_4 \times SU(1)_{-4}$ [31, 153]. We expect the latter to be identical to the theory of two free hypermultiplets, which is also dual to the $U(1)_1 \times U(1)_{-1}$ theory. Subsequently, we study these theories in detail with the aid of the superconformal index.

The index of theory III(R) is given by (we summarized our conventions for the index in Section (4.6), see in particular (4.6.12) for the contribution Z_{chir} of the chiral multiplet)

$$
\begin{aligned}
\mathcal{I}^{N=1}_{\mathrm{III(R)}}(u,v,w) &= \sum_{p=0}^{3} g^p \sum_{(m_1;m_2)\in\left(\mathbb{Z}+\frac{p}{4}\right)^2} \oint \frac{dz_1}{2\pi i z_1} \oint \frac{dz_2}{2\pi i z_2} z_1^{4m_1} z_2^{-4m_2} w_1^{m_1} w_2^{m_2} \times \\
&\qquad \prod_{s=\pm1} Z_{\mathrm{chir}}(u^s z_1 z_2^{-1}; m_1-m_2; 1/2) Z_{\mathrm{chir}}(v^s z_2 z_1^{-1}; m_2-m_1; 1/2) \\
&= \prod_{s=\pm1} Z_{\mathrm{chir}}\left(g^3 u^s \widetilde{w}^{-1/2}; 0; 1/2\right) Z_{\mathrm{chir}}\left(g v^s \widetilde{w}^{1/2}; 0; 1/2\right) \\
&= \prod_{s=\pm1} Z_{\mathrm{chir}}\left(w^{-1} u^s; 0; 1/2\right) Z_{\mathrm{chir}}\left(w v^s; 0; 1/2\right)
\end{aligned}
\tag{6.4.2}
$$

where u and v are the fugacities for the $SU(2)_A \times SU(2)_B$ flavour symmetry and w is the fugacity for the $U(1)^{[0]}$ zero-form topological symmetry. In the above, g is the $\mathbb{Z}_4$ discrete topological fugacity satisfying $g^4 = 1$, but it can be absorbed into a $U(1)$ global symmetry by a redefinition (see Table (6.3.20)). In particular, we have defined

$$
\widetilde{w} = (w_1 w_2)^{1/2} \ , \qquad w = g\widetilde{w}^{1/2} = g(w_1 w_2)^{1/4} \ . \tag{6.4.3}
$$

We remark that if one makes a change of variables $s_1 = z_1 z_2$ and $s_2 = z_1 z_2^{-1}$, the contribution of the matter fields Z_{chir} is independent of s_1 and so the integration over s_1 leads to a delta-function that sets (see also [153, (4.15)])

$$
m_1 = m_2 \ . \tag{6.4.4}
$$

This is in agreement with the discussion of [65, Section 4.1.4]. As a result, only the combination $w_1 w_2$, but not w_1/w_2, appears in the index.

The last line of (6.4.2) is indeed the index of the theory with two free hypermultiplets. These are identified as the gauge invariant dressed di-baryons, discussed in [31, (2.11)] (with $N = 1$ and $k = 4$ in their notation):

$$
\mathcal{B}_\alpha = T_{\{-\frac{1}{4};-\frac{1}{4}\}} A_\alpha \ , \quad \mathcal{B}'_{\alpha'} = T_{\{\frac{1}{4};\frac{1}{4}\}} B_{\alpha'} \ ; \tag{6.4.5}
$$

where we denote by $\alpha, \beta, \ldots = 1,2$ the indices for the $SU(2)_A$ flavour symmetry and by $\alpha', \beta', \ldots = 1,2$ the indices for the $SU(2)_B$ flavour symmetry. Note that the gauge invariant dressed monopole operators

$$
\begin{aligned}
(\mathcal{M}_{-1})_{\alpha_1\cdots\alpha_4} &= T_{\{-1;-1\}} A_{\alpha_1} \cdots A_{\alpha_4} \ , \\
(\mathcal{M}_{+1})_{\alpha'_1\cdots\alpha'_4} &= T_{\{+1;+1\}} B_{\alpha'_1} \cdots B_{\alpha'_4} \ ,
\end{aligned}
\tag{6.4.6}
$$

are related to the di-baryons by the relations

$$
(\mathcal{M}_{-1})_{\alpha_1\cdots\alpha_4} = \prod_{j=1}^{4} \mathcal{B}_{\alpha_j} \ , \quad (\mathcal{M}_{+1})_{\alpha'_1\cdots\alpha'_4} = \prod_{j=1}^{4} \mathcal{B}'_{\alpha'_j} \ . \tag{6.4.7}
$$

In order to obtain the index for $U(1)_1 \times U(1)_{-1}$, we proceed as follows. We rewrite the above index using the variables $\widetilde{m}_1 = 4m_1$ and $\widetilde{m}_2 = 4m_2$. The contribution from the Chern–Simons levels is therefore $z_1^{\widetilde{m}_1} z_2^{\widetilde{m}_2}$. The summation of $(m_1, m_2) \in (\mathbb{Z} + p/4)^2$ is then equivalent to the summation of $(\widetilde{m}_1, \widetilde{m}_2) \in (4\mathbb{Z} + p)^2$, where p is summed from 0 to 3. The factors corresponding to the topological fugacities are $w_1^{m_1} w_2^{m_2} = w_1^{\frac{1}{4}\widetilde{m}_1} w_2^{\frac{1}{4}\widetilde{m}_2}$. We can now shift $\widetilde{m}_{1,2} \to \widetilde{m}_{1,2} + p$ and so, together with g^p, we have

$$(g(w_1 w_2)^{\frac{1}{4}})^p w_1^{\frac{1}{4}\widetilde{m}_1} w_2^{\frac{1}{4}\widetilde{m}_2} = w^p w_1^{\frac{1}{4}\widetilde{m}_1} w_2^{\frac{1}{4}\widetilde{m}_2} \tag{6.4.8}$$

where $w = g(w_1 w_2)^{\frac{1}{4}}$ as stated in (6.4.3). Using (6.4.4), namely $\widetilde{m}_1 = \widetilde{m}_2 \equiv \widetilde{m}$, and writing $w_1 = ws$ and $w_2 = ws^{-1}$, (6.4.8) becomes $w^{p+\frac{1}{2}\widetilde{m}}$. Upon shifting $\widetilde{m} \to \widetilde{m} - 2p$, we are left with $w^{\frac{1}{2}\widetilde{m}} = w^{\frac{1}{4}\widetilde{m}_1} w^{\frac{1}{4}\widetilde{m}_2}$. Observe that the discrete fugacity g as well as the factor w^p disappear from the index. At this point, the summation of $p \in \{0, 1, 2, 3\}$ together with the summation of $(\widetilde{m}_1; \widetilde{m}_2) \in (4\mathbb{Z} + p)^2$ can be replaced by the summation of $(\widetilde{m}_1; \widetilde{m}_2) \in \mathbb{Z}^2$. Moreover, the argument in Z_{chir} depends only on $m_1 - m_2$, which is an integer, and so we can replace it by $\widetilde{m}_1 - \widetilde{m}_2$. Overall, we obtain the index for $U(1)_1 \times U(1)_{-1}$, as required. This procedure can be easily generalised to show duality (6.3.22), namely $[U(1)_{kp} \times U(1)_{-kp}]/\mathbb{Z}_p \leftrightarrow U(1)_k \times U(1)_{-k}$.

The index of theory III(L): $[SO(2)_2 \times USp(2)_{-1}]/\mathbb{Z}_2$ is given by

$$\begin{aligned} &\mathcal{I}^{N=1}_{\text{III(L)}}(f, \omega) \\ &= \sum_{p=0}^{1} \mathfrak{g}^p \sum_{(m_1, m_2) \in \left(\mathbb{Z} + \frac{p}{2}\right)^2} \oint \frac{dz_1}{2\pi i z_1} \oint \frac{dz_2}{2\pi i z_2} z_1^{2m_1} \zeta^{m_1} z_2^{-2m_2} Z_V^{USp(2)}(z_2, m_2) \times \\ &\quad \prod_{s, s_1, s_2 = \pm 1} Z_{\text{chir}}(f^s z_1^{s_1} z_2^{s_2}; s_1 m_1 + s_2 m_2; 1/2) \\ &= \prod_{s=\pm 1} Z_{\text{chir}}\left(\mathfrak{g}\, \zeta^{\frac{1}{2}} f^s; 0; 1/2\right) Z_{\text{chir}}\left(\mathfrak{g}\, \zeta^{-\frac{1}{2}} f^s; 0; 1/2\right) \\ &= \prod_{s=\pm 1} Z_{\text{chir}}\left(\omega f^s; 0; 1/2\right) Z_{\text{chir}}\left(\omega^{-1} f^s; 0; 1/2\right) \end{aligned} \tag{6.4.9}$$

where ζ is the fugacity of the $U(1)^{[0]}_{\mathcal{M}}$ magnetic (topological) symmetry of $SO(2)$, f is the fugacity of the $SU(2)_A$ flavour symmetry, $\mathfrak{g}$ is the fugacity associated with the $\mathbb{Z}_2$ topological symmetry (see Table (6.3.20)), the fugacity ω is defined as

$$\omega = \mathfrak{g}\, \zeta^{\frac{1}{2}} \ , \tag{6.4.10}$$

and the contribution $Z_V^{USp(2)}$ from the $USp(2)$ vector multiplet is as defined in (4.6.4).

Observe that the $\mathbb{Z}_2$ zero-form symmetry, associated with the fugacity $\mathfrak{g}$, can be absorbed into the magnetic symmetry. This results in a $U(1)^{[0]}$ zero-form global symmetry with fugacity ω. The last line of (6.4.30) is indeed the index of the theory with two free hypermultiplets. These are identified with the gauge invariants dressed di-baryons:

$$\mathcal{B}^+_\alpha = T_{\left\{\frac{1}{2};\frac{1}{2}\right\}} A_\alpha \ , \qquad \mathcal{B}^-_\alpha = T_{\left\{-\frac{1}{2};-\frac{1}{2}\right\}} A_\alpha \ . \tag{6.4.11}$$

where in this case $\alpha = 1, 2$ is an index for the $SU(2)_A$ flavour symmetry.

Comparing (6.4.2) with (6.4.30), we see that

$$\mathcal{I}^{N=1}_{\text{III(L)}}(f, \omega) = \mathcal{I}^{N=1}_{\text{III(R)}}(u = f, v = f, w = \omega) \ . \tag{6.4.12}$$

The $U(1)^{[0]}$ zero-form topological symmetry of theory III(R) with fugacity w is mapped to the $U(1)^{[0]}$ zero-form symmetry of theory III(L) with fugacity ω, whereas the $SU(2)_A$ flavour symmetry of theory III(L) with fugacity f is identified with the diagonal subgroup of the $SU(2)_A \times SU(2)_B$ flavour symmetry of theory III(R) with fugacities u and v respectively.

$[U(N)_4 \times U(N)_{-4}]/\mathbb{Z}_4$ **with** $N \geq 2$

The index for theory III(R): $[U(N)_4 \times U(N)_{-4}]/\mathbb{Z}_4$ can be written as

$$\begin{aligned}
&\mathcal{I}_{\text{III(R)}}(u, v, w_1, w_2, g) \\
&= \sum_{p=0}^{3} g^p \sum_{\mathcal{S}_p} \left(\prod_{j=1}^{N} \oint \frac{dz_1^{(j)}}{2\pi i z_1^{(j)}} \oint \frac{dz_2^{(j)}}{2\pi i z_2^{(j)}} (z_1^{(j)})^{4m_1^{(j)}} (z_2^{(j)})^{-4m_2^{(j)}} \right) \times \\
&\qquad w_1^{\sum_{j=1}^N m_1^{(j)}} w_2^{\sum_{j=1}^N m_2^{(j)}} \prod_{\ell=1}^{2} Z_V^{U(N)}(z_\ell^{(1)}, \ldots, z_\ell^{(N)}; m_\ell^{(1)}, \ldots, m_\ell^{(N)}) \times \\
&\qquad \prod_{i,j=1}^{N} \prod_{s=\pm 1} Z_{\text{chir}}(u^s z_1^{(i)}/z_2^{(j)}; m_1^{(i)} - m_2^{(j)}; 1/2) Z_{\text{chir}}(v^s z_2^{(i)}/z_1^{(j)}; m_2^{(i)} - m_1^{(j)}; 1/2)
\end{aligned} \tag{6.4.13}$$

where $g^4 = 1$, the notation $\mathcal{S}_p$ stands for the summation over

$$(m_1^{(1)}, \ldots, m_1^{(N)}; m_2^{(1)}, \ldots, m_2^{(N)}) \in \left(\mathbb{Z} + \frac{p}{4}\right)^{2N} \tag{6.4.14}$$

and the contribution $Z_V^{U(N)}$ of the $U(N)$ vector multiplet is as defined in (4.6.2).

Note that the integration over the diagonal gauge $U(1)$ leads to a delta-function imposing the constraint[7]

[7] As explained in [65, Section 4.1.4], due to the D-term equations, the gauge invariant quantities can be formed provided that the magnetic fluxes of the two gauge groups are paired:

$$m_1^{(j)} = m_2^{(j)} \equiv m^{(j)} \ , \quad j = 1, \ldots, N \ , \tag{6.4.15}$$

provided that we use the Weyl symmetry to order fluxes so that $m_1^{(1)} \geq m_1^{(2)} \geq \cdots \geq m_1^{(N)}$ and $m_2^{(1)} \geq m_2^{(2)} \geq \cdots \geq m_2^{(N)}$. We point out that (6.4.15) only holds at the level of the moduli space of vacua. In fact, it can be shown that there are non-trivial contributions to the index (at higher orders than x^3) from the magnetic fluxes that satisfy (6.4.16) but not (6.4.15). We thank Luca Viscardi for pointing this out to us.

$$\sum_{j=1}^{N} m_1^{(j)} = \sum_{j=1}^{N} m_2^{(j)} \ . \tag{6.4.16}$$

According to [153, Section 2.3], the apparent $\mathbb{Z}_4$ zero-form symmetry, associated with the fugacity g, is actually $\mathbb{Z}_{\mathrm{GCD}(N,4)}$. More generally, in the $[U(N)_k \times U(N)_{-k}]/\mathbb{Z}_k$ theory part of the $\mathbb{Z}_k$ zero-form symmetry can be absorbed into the $U(1)_{\mathrm{top}}$ topological symmetry and only $\mathbb{Z}_{\mathrm{GCD}(N,k)}$ remains. By explicitly computing the index, one can indeed check that it can be rewritten solely in terms of two new fugacities[8]

$$w = g(w_1 w_2)^{N/k} \ , \quad g' = g^{k/\mathrm{GCD}(N,k)} \ , \tag{6.4.17}$$

where w is the fugacity for the "new" topological symmetry $U(1)^{[0]}$ while g' is the $\mathbb{Z}_{\mathrm{GCD}(N,k)}$-valued fugacity for the $\mathbb{Z}^{[0]}_{\mathrm{GCD}(N,k)}$ discrete symmetry (see Table (6.3.20)).

This means that the actual zero-form global symmetry of the theory is $SU(2)_A \times SU(2)_B \times U(1)^{[0]} \times \mathbb{Z}^{[0]}_{\mathrm{GCD}(N,k)}$.

The special case of $N = 2$

After redefining the fugacities as in (6.4.17)

$$w = g(w_1 w_2)^{1/2} \ , \quad g' = g^2 \ , \tag{6.4.18}$$

the index (6.4.13) for the case of $N = 2$ can be written as

$$\begin{aligned}
&\mathcal{I}^{N=2}_{\mathrm{III(R)}}(u, v, w, g') \\
&= 1 + x\left[\chi^{SU(2)}_{[1]}(u)\chi^{SU(2)}_{[1]}(v) + w^{-1}\chi^{SU(2)}_{[2]}(u) + w\chi^{SU(2)}_{[2]}(v)\right] \\
&\quad + x^2\Big[(g'+1)w^{-2}\chi^{SU(2)}_{[4]}(u) + (g'+1)w^{-1}\chi^{SU(2)}_{[3]}(u)\chi^{SU(2)}_{[1]}(v) \\
&\qquad + \left(w \leftrightarrow w^{-1}, u \leftrightarrow v\right) + (g'+2)\chi^{SU(2)}_{[2]}(u)\chi^{SU(2)}_{[2]}(v) + w^2 + w^{-2} \\
&\qquad - \chi^{SU(2)}_{[2]}(u) - \chi^{SU(2)}_{[2]}(v)\Big] + \ldots \ .
\end{aligned} \tag{6.4.19}$$

The unrefined index is (cf. [90, (4.5)])

$$\mathcal{I}^{N=2}_{\mathrm{III(R)}}(u=1, v=1, w=1, g'=1) = 1 + 10x + 75x^2 + 230x^3 + 449x^4 + \ldots \ . \tag{6.4.20}$$

[8] We normalise the power of w such that the di-baryon operators, which involve the monopole operators $T_{\pm\{\underbrace{1/k,\cdots,1/k}_{N};\underbrace{1/k,\cdots,1/k}_{N}\}}$, carry $U(1)^{[0]}$ topological charge ±1; see (6.4.21). This explains the power N/k of $w_1 w_2$ in (6.4.17).

Note that the operators with R-charge 1 are[9]

$$\begin{aligned} \chi^{SU(2)}_{[1]}(u)\chi^{SU(2)}_{[1]}(v) : &\qquad M_{\alpha\alpha'} = (A_\alpha)^a_i (B_{\alpha'})^i_a \\ w^{-1}\chi^{SU(2)}_{[2]}(u) : &\qquad \mathcal{B}_{\alpha\beta} = T_{\{-\frac14,-\frac14;-\frac14,-\frac14\}} (A_\alpha)^a_i (A_\beta)^b_j \epsilon^{ij}\epsilon_{ab} \\ w\chi^{SU(2)}_{[2]}(v) : &\qquad \mathcal{B}'_{\alpha'\beta'} = T_{\{+\frac14,+\frac14;+\frac14,+\frac14\}} (B_{\alpha'})^i_a (B_{\beta'})^j_b \epsilon_{ij}\epsilon^{ab} \end{aligned} \tag{6.4.21}$$

where the last two are the gauge invariant dressed di-baryons. Here $a, b, \ldots = 1, 2$ and $i, j, \ldots = 1, 2$ are the gauge indices for each $U(2)$ gauge group, $\alpha, \beta, \ldots = 1, 2$ are the indices for the $SU(2)_A$ flavour symmetry, and $\alpha', \beta', \ldots = 1, 2$ are the indices for the $SU(2)_B$ flavour symmetry. Let us discuss some examples of marginal operators, contributing at order x^2 of the index. The combinations $\mathcal{B}\mathcal{B}$ transform in the representation $\mathrm{Sym}^2[2; 0] = [4; 0] + [0; 0]$ of $SU(2)_A \times SU(2)_B$ and similarly for $\mathcal{B}'\mathcal{B}'$:

$$\begin{aligned} w^{-2}\chi^{SU(2)}_{[4]}(u) + w^{-2} : &\qquad \mathcal{B}_{\alpha\beta}\mathcal{B}_{\gamma\delta} \\ w^{2}\chi^{SU(2)}_{[4]}(v) + w^{2} : &\qquad \mathcal{B}'_{\alpha'\beta'}\mathcal{B}'_{\gamma'\delta'} \ . \end{aligned} \tag{6.4.22}$$

Note that the index (6.4.19) can be rewritten in terms of characters of $SU(4)$ representations as follows:

$$\begin{aligned} (6.4.13)_{N=2} = 1 &+ x\chi^{SU(4)}_{[0,0,2]}(\boldsymbol{s}) \\ &+ x^2\left[(g'+1)\chi^{SU(4)}_{[0,0,4]}(\boldsymbol{s}) + \chi^{SU(4)}_{[0,2,0]}(\boldsymbol{s}) - \chi^{SU(4)}_{[1,0,1]}(\boldsymbol{s})\right] + \ldots \ , \end{aligned} \tag{6.4.23}$$

where we have taken

$$w = q^2 \tag{6.4.24}$$

and have used the fugacity map[10]

$$s_1 = qu \ , \quad s_2 = q^2 \ , \quad s_3 = qv \ . \tag{6.4.25}$$

Since this theory is known to be dual to the Bagger-Lambert-Gustavson [16, 103] theory $[SU(2)_4 \times SU(2)_{-4}]/\mathbb{Z}_2$ and the $USp(4)/\mathbb{Z}_2$ super-Yang-Mills (see [153, Table 1]), it has $\mathcal{N} = 8$ supersymmetry. This can be seen from the index (6.4.13) as follows. (The argument given below is the same as that of [28, Appendix C.1][11]). We rewrite

[9] As pointed out in [22, 24, 31, 125, 128], for a $U(N)_k$ gauge group, the monopole operators with the magnetic fluxes $(m_1, m_2, \cdots, m_N)$, with $m_1 \geq m_2 \geq \cdots \geq m_N$, transform under the representation of the $SU(N)$ gauge factor with the Dynkin label $[k(m_1 - m_2), k(m_2 - m_3), \cdots, k(m_{N-1} - m_N)]$ and carry $U(1)$ gauge charge $k\sum_{i=1}^N m_i$. Consequently, the monopoles $T_{\{\pm\frac14,\pm\frac14;\pm\frac14,\pm\frac14\}}$ have charge $(\pm2, \mp2)$ under the $U(1)$ subgroups of the two $U(1) \cong SU(2) \times U(1)$ gauge groups. Moreover, in (6.4.21) the ϵ-tensors are with respect to the $SU(2)$ parts. Hence, the object $(A_\alpha)^a_i (A_\beta)^b_j \epsilon^{ij}\epsilon_{ab}$ is invariant under the $SU(2)$ parts, while it has charge $(\pm2, \mp2)$ under the $U(1)$ parts, making the operator $\mathcal{B}_{\alpha\beta}$ gauge invariant and similarly for $\mathcal{B}'_{\alpha'\beta'}$.

[10] In this convention, the character of the fundamental representation $[1, 0, 0]$ of $SU(4)$ is written as $\chi^{SU(4)}_{[1,0,0]}(\boldsymbol{s}) = s_1 + s_2 s_1^{-1} + s_3 s_2^{-1} + s_3^{-1}$.

[11] See also [88, 92, 97] for other examples of supersymmetry enhancements in 3d detected with the index.

(6.4.19), or equivalently (6.4.23), as an $\mathcal{N}=3$ index.[12] This can be achieved by setting $u=v=f$ and $w=q^2$ in (6.4.19), or by using branching rules of representations of $SU(4)$ to those of a maximal subgroup of $USp(4)$ in (6.4.23). Either way, we obtain the $\mathcal{N}=3$ index in terms of characters of representations of $USp(4)$ as

$$\begin{aligned}(6.4.13)_{N=2} = 1 + x\chi^{USp(4)}_{[2,0]}(\boldsymbol{h}) + x^2\Big[&(g'+1)\chi^{USp(4)}_{[4,0]}(\boldsymbol{h}) + \chi^{USp(4)}_{[0,2]}(\boldsymbol{h}) + 1 \\ &+ \chi^{USp(4)}_{[0,1]}(\boldsymbol{h}) - \chi^{USp(4)}_{[2,0]}(\boldsymbol{h}) - \chi^{USp(4)}_{[0,1]}(\boldsymbol{h})\Big] + \ldots\end{aligned} \tag{6.4.26}$$

where we have used the fugacity map[13]

$$\begin{aligned} &h_1 = qf\ , \quad h_2 = q^{-1}f \\ &s_1 = h_1\ , \quad s_2 = h_1h_2^{-1}\ , \quad s_3 = h_1\ . \end{aligned} \tag{6.4.27}$$

Note that (6.4.26) satisfies all of the necessary conditions for the enhanced $\mathcal{N}=8$ supersymmetry [74]. The blue term in (6.4.26) is the contribution of 5 marginal operators in the representation $[0,1]$ of $USp(4)$, whereas the red term in (6.4.26) is the contribution of the extra supersymmetry currents. These two contributions precisely cancel with each other. Since we have 5 extra supersymmetry currents, supersymmetry gets enhanced from $\mathcal{N}=3$ to $\mathcal{N}=3+5=8$, as expected. Let us discuss the marginal operators corresponding to the blue term in (6.4.26) in detail. First of all, since $[0,1]$ is a subrepresentation of $\mathrm{Sym}^2[2,0] = [4,0]\oplus[0,2]\oplus[0,1]\oplus[0,0]$, we expect that such marginal operators can be constructed by appropriately multiplying those in (6.4.21). Secondly, since the representation $[0,1]$ of $USp(4)$ decomposes into those of $SU(2)_f\times U(1)_q$ as $[2]_0\oplus[0]_{-2}\oplus[0]_2$, we propose that the corresponding operators are respectively[14]

$$(\mathrm{Tr}\ M)\widehat{M}_{\alpha\beta}\ , \qquad \widehat{M}_{\alpha\beta}\mathcal{B}_{\gamma\delta}\epsilon^{\beta\gamma}\epsilon^{\alpha\delta}\ , \qquad \widehat{M}_{\alpha\beta}\mathcal{B}'_{\gamma\delta}\epsilon^{\beta\gamma}\epsilon^{\alpha\delta}\ , \tag{6.4.28}$$

where $\alpha,\beta,\gamma,\delta=1,2$ are the indices for $SU(2)_f$, which is a diagonal subgroup of $SU(2)_A\times SU(2)_B$ of theory III(R), and we have defined

$$\mathrm{Tr}\ M = M_{\alpha\beta}\epsilon^{\alpha\beta}\ , \qquad \widehat{M}_{\alpha\beta} = M_{\alpha\beta} - \frac{1}{2}(\mathrm{Tr}\ M)\epsilon_{\alpha\beta}\ . \tag{6.4.29}$$

[12] As a requirement of an $\mathcal{N}=3$ index, the order x receives a contribution solely from the $\mathcal{N}=3$ flavour current, and so the coefficient of x must be an adjoint representation of the flavour symmetry of the corresponding $\mathcal{N}=3$ theory. The index (6.4.19), or equivalently (6.4.23), is an $\mathcal{N}=2$ index, not an $\mathcal{N}=3$ index, since the coefficient of order x is not an adjoint representation of $SU(4)$. As can be seen below, one can rewrite this as an $\mathcal{N}=3$ index by tuning some fugacities to be equal and reexpressing the index in terms of characters of representations of $USp(4)$.

[13] In this convention, the character of the fundamental representation $[1,0]$ of $USp(4)$ is written as $\chi^{USp(4)}_{[1,0]}(\boldsymbol{h}) = h_1 + h_1^{-1} + h_2 + h_2^{-1}$.

[14] There are other two operators in the representations $[0]_{\mp 2}$ of $SU(2)_f\times U(1)_q$ that are contained in the branching rule of the representation $[4,0]$ of $USp(4)$, namely

$$\begin{aligned}&\epsilon^{\alpha\beta}\epsilon^{\gamma\delta}(A_\alpha)^a_i(A_\beta)^b_j(B_\gamma)^i_b(A_\delta)^c_k\epsilon^{jk}\epsilon_{ac}T_{\{-\frac14,-\frac14;-\frac14,-\frac14\}}\ , \\ &\epsilon^{\alpha\beta}\epsilon^{\gamma\delta}(B_\alpha)^i_a(B_\beta)^j_b(A_\gamma)^b_i(B_\delta)^k_c\epsilon_{jk}\epsilon^{ac}T_{\{+\frac14,+\frac14;+\frac14,+\frac14\}}\ .\end{aligned}$$

Finally, let us remark that in the case of $N = 2$ the di-baryon operators $\mathcal{B}$ and $\mathcal{B}'$ have R-charge 1 and so they contribute at order x of the index, which makes it fulfil the condition for having $\mathcal{N} = 8$ supersymmetry. For a general N, the di-baryons have R-charge $N/2$, and so for $N \geq 3$ they contribute at a higher order of the index. In the latter case, the only contribution at order x comes from the operators M, which have 4 components. We thus expect the theory with $N \geq 3$ to have $\mathcal{N} = 6$ supersymmetry [74].

$[SO(2N)_2 \times USp(2N)_{-1}]/\mathbb{Z}_2$ **with** $N \geq 2$

The index for the theory III(L): $[SO(2N)_2 \times USp(2N)_{-1}]/\mathbb{Z}_2$ is

$$
\begin{aligned}
&\mathcal{I}_{\text{III(L)}}(f, \mathfrak{g}, \zeta) \\
&= \sum_{p=0}^{1} \mathfrak{g}^p \sum_{\mathcal{S}'_p} \left(\prod_{j=1}^{N} \oint \frac{dz_1^{(j)}}{2\pi i z_1^{(j)}} \oint \frac{dz_2^{(j)}}{2\pi i z_2^{(j)}} (z_1^{(j)})^{2m_1^{(j)}} (z_2^{(j)})^{-2m_1^{(j)}} \right) \zeta^{\sum_{j=1}^{N} m_1^{(j)}} \times \\
&\quad Z_V^{SO(2N)}(z_1^{(1)}, \ldots, z_1^{(N)}; m_1^{(1)}, \ldots, m_1^{(N)}) Z_V^{USp(2N)}(z_2^{(1)}, \ldots, z_2^{(N)}; m_2^{(1)}, \ldots, m_2^{(N)}) \times \\
&\quad \prod_{i,j=1}^{N} \prod_{s, s_1, s_2 = \pm 1} Z_{\text{chir}} \left(f^s (z_1^{(i)})^{s_1} (z_2^{(j)})^{s_2}; s_1 m_1^{(i)} + s_2 m_2^{(j)}; 1/2 \right)
\end{aligned}
\tag{6.4.30}
$$

where $\mathcal{S}'_p$ stands for the summation over $(m_1^{(1)}, \ldots, m_1^{(N)}, m_2^{(1)}, \ldots, m_2^{(N)}) \in \left(\mathbb{Z} + \frac{p}{2}\right)^{2N}$, f is the fugacity associated with the $SU(2)_A$ flavour symmetry, ζ is the fugacity associated with the $(\mathbb{Z}_2^{[0]})_{\mathcal{M}}$ magnetic symmetry of the $SO(2N)$ gauge group satisfying $\zeta^2 = 1$, and $\mathfrak{g}$ is the fugacity for the topological $(\mathbb{Z}_2^{[0]})_{\mathfrak{g}}$ symmetry satisfying $\mathfrak{g}^2 = 1$ (see Table (6.3.20)).

Let us provide an explicit expression for $N = 2$ up to order x^2:

$$
\begin{aligned}
&\mathcal{I}_{\text{III(L)}}^{N=2}(f, \mathfrak{g}, \zeta) \\
&= 1 + x \left[1 + (\mathfrak{g} + \zeta + \mathfrak{g}\zeta) \chi_{[2]}^{SU(2)}(f) \right] + x^2 \Big[5\chi_{[4]}^{SU(2)}(f) + 5 \\
&\qquad + (\mathfrak{g} + \zeta + \mathfrak{g}\zeta) \left(2\chi_{[4]}^{SU(2)}(f) + 2\chi_{[2]}^{SU(2)}(f) \right) - \chi_{[2]}^{SU(2)}(f) \Big] + \ldots
\end{aligned}
\tag{6.4.31}
$$

with the unrefined index $\mathcal{I}_{\text{III(L)}}^{N=2}(f = 1, \mathfrak{g} = 1, \zeta = 1)$ given by (6.4.20). Note that it is not possible to absorb $\mathfrak{g}$ with a redefinition of ζ as in (6.4.17). The manifest zero-form global symmetry of this theory is therefore $SU(2)_A \times (\mathbb{Z}_2^{[0]})_{\mathfrak{g}} \times (\mathbb{Z}_2^{[0]})_{\mathcal{M}}$. We can match the indices (6.4.19) and (6.4.31) as follows:

$$
\mathcal{I}_{\text{III(L)}}^{N=2}(f, \mathfrak{g} = \xi, \zeta = \xi) = \mathcal{I}_{\text{III(R)}}^{N=2}(u = f, v = f, w = \xi, g' = 1)\Big|_{\xi^2 = 1} . \tag{6.4.32}
$$

In the theory III(L) only the diagonal subgroup $SU(2)_A$ of the flavour symmetry $SU(2)_A \times SU(2)_B$ of the theory III(R) is manifest. Moreover, the $U(1)^{[0]}$ zero-form topological symmetry of the theory III(R) is not manifest in the theory III(L), but its $\mathbb{Z}_2$ subgroup is identified with the diagonal $\mathbb{Z}_2$ symmetry of $(\mathbb{Z}_2^{[0]})_{\mathfrak{g}} \times (\mathbb{Z}_2^{[0]})_{\mathcal{M}}$ in III(L). Furthermore, the $\mathbb{Z}_{\text{GCD}(N,k)}^{[0]} = \mathbb{Z}_2^{[0]}$ symmetry with fugacity g' of the theory III(R) is not manifest in the theory III(L). Since we claim that the theories III(L) and III(R) are dual to each other, the theory III(L) is expected to have an emergent zero-form symmetry $SU(2)_A \times SU(2)_B \times U(1)^{[0]} \times \mathbb{Z}_2^{[0]}$.

6.4.2 $SO(2N)_2 \times USp(2N)_{-1} \leftrightarrow [U(N)_4 \times U(N)_{-4}]/\mathbb{Z}_2$

In this subsection, we consider the duality between the following two theories:

$$\text{II(L): } SO(2N)_2 \times USp(2N)_{-1} \quad \leftrightarrow \quad \text{II(R): } [U(N)_4 \times U(N)_{-4}]/\mathbb{Z}_2 \ . \tag{6.4.33}$$

The theory II(L) can be obtained by gauging the $(\mathbb{Z}_2^{[0]})_{\mathfrak{g}}$ zero-form symmetry of the theory III(L), where at the level of the index this corresponds to summing over $\mathfrak{g} \in \{\pm 1\}$ in (6.4.30). Note also that in the description II(L) there is also a $(\mathbb{Z}_2^{[0]})_{\mathcal{C}}$ zero-form charge conjugation symmetry, whose fugacity will be denoted by χ.

On the other hand, we can obtain the theory II(R) from the theory III(R) by gauging a $\mathbb{Z}_2$ subgroup of the $U(1)^{[0]} \times \mathbb{Z}^{[0]}_{\text{GCD}(N,4)}$ zero-form symmetry. In general, we can only gauge a $\mathbb{Z}_{\mathfrak{m}'}$ subgroup of this symmetry such that $k = \mathfrak{m}'\mathfrak{m} = 4$, with $\mathfrak{m}', \mathfrak{m} \in \mathbb{Z}$. In our case $\mathfrak{m}' = \mathfrak{m} = 2$. From the perspective of the index, this discrete gauging can be done as follows.

First, we rewrite the index (6.4.13) using the variables w and g' as indicated in (6.4.17); these are indeed the correct fugacities for the $U(1)^{[0]} \times \mathbb{Z}^{[0]}_{\text{GCD}(N,4)}$ symmetry to explicitly appear in the index.

Then, we define a new fugacity

$$\widetilde{g} \in \mathbb{Z}_{\mathfrak{m}'} = \{\exp(2\pi i\, j/\mathfrak{m}') \,|\, j = 0, 1, \ldots, \mathfrak{m}' - 1\} \tag{6.4.34}$$

to substitute the g fugacity in the index (6.4.13). Thus, taking

$$w = \widetilde{g}(w_1 w_2)^{N/k} \ , \quad g' = \widetilde{g}^{k/\text{GCD}(N,k)} \ , \tag{6.4.35}$$

and summing over $\widetilde{g} \in \mathbb{Z}_{\mathfrak{m}'}$, we are left with the fugacities $(w_1 w_2)^{N/k}$ and g' such that $(g')^{\text{GCD}(N,k)} = 1$.

By computing the index one can check that we can further redefine[15]

$$w' = g'(w_1 w_2)^{N/\mathfrak{m}'} = g'(w_1 w_2)^{\mathfrak{m}N/k} \ , \qquad g'' = (g')^{\frac{\text{GCD}(N,k)}{\text{GCD}(N,\mathfrak{m}')}} \ , \tag{6.4.36}$$

where

$$(g'')^{\text{GCD}(N,\mathfrak{m}')} = (g')^{\text{GCD}(N,k)} = 1 \ . \tag{6.4.37}$$

The zero-form symmetry of the theory II(R) is therefore $SU(2)_A \times SU(2)_B \times U(1)^{[0]} \times \mathbb{Z}^{[0]}_{\text{GCD}(N,\mathfrak{m}')}$ with fugacities u, v, w' and g'' respectively. This is in agreement with the discussion in [31, Section 2.5] and, in fact, it works for any k, $\mathfrak{m}$ and $\mathfrak{m}'$.

[15] Similarly to Footnote (8), we normalise the power of w' such that the di-baryon operators, which involve the monopole operators $T_{\pm\{\underbrace{1/\mathfrak{m}', \cdots, 1/\mathfrak{m}'}_{N};\, \underbrace{1/\mathfrak{m}', \cdots, 1/\mathfrak{m}'}_{N}\}}$, carry $U(1)^{[0]}$ charge ± 1; see (6.4.52). This explains the power $N/\mathfrak{m}' = \mathfrak{m}N/k$ of $w_1 w_2$ in (6.4.36).

As a result of gauging a $\mathbb{Z}_2^{[0]}$ zero-form symmetry, both of II(L) and II(R) have a $\mathbb{Z}_2^{[1]}$ one-form symmetry.

The case of $N = 1$

Let us examine the theory II(R): $[U(1)_4 \times U(1)_{-4}]/\mathbb{Z}_2$. The index of this theory is almost the same as (6.4.2), with two exceptions: the summation over (m_1, m_2) is in $(\mathbb{Z}+p/2)^2$, and the summation over p is from $p = 0$ to 1. The index, up to order x^2, can be written as

$$\begin{aligned}\mathcal{I}^{N=1}_{\text{II(R)}} &= 1 + x\left[\chi^{SU(2)}_{[1]}(u)\chi^{SU(2)}_{[1]}(v) + w^{-1}\chi^{SU(2)}_{[2]}(u) + w\chi^{SU(2)}_{[2]}(v)\right] \\ &\quad + x^2\Big[w^{-2}\chi^{SU(2)}_{[4]}(u) + w^{-1}\chi^{SU(2)}_{[3]}(u)\chi^{SU(2)}_{[1]}(v) + \big(w \leftrightarrow w^{-1}, u \leftrightarrow v\big) \\ &\quad + \chi^{SU(2)}_{[2]}(u)\chi^{SU(2)}_{[2]}(v) - (w + w^{-1})\chi^{SU(2)}_{[1]}(u)\chi^{SU(2)}_{[1]}(v) \\ &\quad - \chi^{SU(2)}_{[2]}(u) - \chi^{SU(2)}_{[2]}(v) - 2\Big] + \dots\ .\end{aligned} \tag{6.4.38}$$

It turns out that the theory II(R) coincides with the ABJM theory $U(1)_2\times U(1)_{-2}$. To see this equivalence, we make the following change of variables: $m_1' = 2m_1$ and $m_2' = 2m_2$. The contribution from the Chern–Simons levels is therefore $z_1^{2m_1'} z_2^{2m_2'}$. The summation of $(m_1, m_2) \in (\mathbb{Z} + p/2)^2$ is then equivalent to the summation of $(m_1', m_2') \in (2\mathbb{Z} + p)^2$, where p is summed over $\{0, 1\}$. At this point, we can just set $g = 1$ since $\text{GCD}(N, \mathfrak{m}) = \text{GCD}(1, 2) = 1$ and take the summation of (m_1', m_2') to be over $\mathbb{Z}^2$. Since Z_{chir} only depends on $m_1 - m_2$, which is an integer, we can replace the latter by $m_1' - m_2'$. Overall, we obtain the index of $U(1)_2 \times U(1)_{-2}$. Due to this equivalence, we conclude that the theory $[U(1)_4 \times U(1)_4]/\mathbb{Z}_2$ also has $\mathcal{N} = 8$ supersymmetry.

Let us comments on the operators that contribute to the index (6.4.38). From the perspective of the $[U(1)_4 \times U(1)_4]/\mathbb{Z}_2$ theory, there are monopole operators

$$T_{\pm\frac{m}{2}} \equiv T_{\pm m\left\{\frac{1}{2};\frac{1}{2}\right\}}\ , \quad m \in \mathbb{Z}\ , \tag{6.4.39}$$

which carry gauge charges $\pm m(2, -2)$ under the gauge group $U(1)_4 \times U(1)_{-4}$, due to the discrete $\mathbb{Z}_2$ quotient. On the other hand, the monopole operators $T_{\pm\frac{1}{2}}$ do not exist in the $U(1)_2 \times U(1)_{-2}$ theory, but there are instead the monopole operators

$$V_{\pm m} \equiv V_{\pm m\{1;1\}}\ , \quad m \in \mathbb{Z}\ , \tag{6.4.40}$$

which carry gauge charges $\pm m(2, -2)$ under the gauge group $U(1)_2 \times U(1)_{-2}$. The operators that contribute at order x are

$$\begin{aligned}\chi^{SU(2)}_{[1]}(u)\chi^{SU(2)}_{[1]}(v) :&\qquad M_{\alpha\alpha'} = A_\alpha B_{\alpha'} \\ w^{-1}\chi^{SU(2)}_{[2]}(u) :&\qquad T_{-\frac{1}{2}}A_\alpha A_\beta \;\leftrightarrow\; V_{-1}A_\alpha A_\beta \\ w\chi^{SU(2)}_{[2]}(v) :&\qquad T_{+\frac{1}{2}}B_{\alpha'}B_{\beta'} \;\leftrightarrow\; V_{+1}B_{\alpha'}B_{\beta'}\end{aligned} \tag{6.4.41}$$

Observe that the di-baryon operators in $[U(1)_4 \times U(1)_4]/\mathbb{Z}_2$ get mapped to the dressed monopole operators in $U(1)_2 \times U(1)_{-2}$. The marginal operators, contributing to order x^2, are

$$\begin{aligned} w^{-2}\chi^{SU(2)}_{[4]}(u) : &\qquad V_{-2}\, A_{\alpha_1}A_{\alpha_2}A_{\alpha_3}A_{\alpha_4} \\ w^{-1}\chi^{SU(2)}_{[3]}(u)\chi^{SU(2)}_{[1]}(v) : &\qquad V_{-1}\, A_{\alpha_1}A_{\alpha_2}A_{\alpha_3}B_{\alpha'_1} \\ \chi^{SU(2)}_{[2]}(u)\chi^{SU(2)}_{[2]}(v) : &\qquad A_{\alpha_1}A_{\alpha_2}B_{\alpha'_1}B_{\alpha'_2}\ , \end{aligned} \tag{6.4.42}$$

where in the first two lines we can obtain those correspond to the terms $w^2\chi^{SU(2)}_{[4]}(v)$ and $w\chi^{SU(2)}_{[3]}(v)\chi^{SU(2)}_{[1]}(u)$ by simply simultaneously exchanging $V_{-m} \leftrightarrow V_{+m}$ and $A \leftrightarrow B$. These gauge invariant combinations are written from the perspective of the $U(1)_2\times U(1)_{-2}$ theory. In the $[U(1)_4 \times U(1)_4]/\mathbb{Z}_2$ duality frame, one simply needs to replace $V_{\pm m}$ by $T_{\pm m/2}$ in the above expressions.

Similarly to (6.4.23) and (6.4.26), the index of the theory $[U(1)_4 \times U(1)_4]/\mathbb{Z}_2 \cong U(1)_2 \times U(1)_{-2}$ can be written in terms of $SU(4)$ characters and $USp(4)$ characters as follows:

$$\begin{aligned} \mathcal{I}^{N=1}_{\mathrm{II(R)}} &= 1 + x\chi^{SU(4)}_{[0,0,2]}(\boldsymbol{s}) + x^2\left[\chi^{SU(4)}_{[0,0,4]}(\boldsymbol{s}) - \chi^{SU(4)}_{[1,0,1]}(\boldsymbol{s}) - 1\right] + \dots \\ &= 1 + x\chi^{USp(4)}_{[2,0]}(\boldsymbol{h}) + x^2\left[\chi^{USp(4)}_{[4,0]}(\boldsymbol{h}) - \chi^{USp(4)}_{[2,0]}(\boldsymbol{h}) - \chi^{USp(4)}_{[0,1]}(\boldsymbol{h}) - 1\right] \end{aligned} \tag{6.4.43}$$

where we use the fugacity maps (6.4.25) and (6.4.27). The first line should be regarded as an $\mathcal{N} = 2$ index, whereas the second line should be regarded as an $\mathcal{N} = 3$ index, since *e.g.* the coefficient of x is an adjoint representation of the flavour symmetry of the $\mathcal{N} = 3$ theory. The red term is the contribution of the $\mathcal{N} = 3$ extra supersymmetry currents. Since there are 5 of them in the representation $[0, 1]$ of $USp(4)$, supersymmetry gets enhanced from $\mathcal{N} = 3$ to $\mathcal{N} = 3 + 5 = 8$. The term -1 at order x^2 worths some explanations. This corresponds to a conserved current associated with the $U(1)$ global symmetry that gives charge 1 to all of the chiral multiplets A_α and $B_{\alpha'}$. For convenience, we shall denote this symmetry by $U(1)_D$. Note that this symmetry is specific to the abelian ABJM theory, since the superpotential vanishes. In the non-abelian case, the superpotential (6.2.4) does not vanish and so the $U(1)_D$ symmetry is explicitly broken. Moreover, the current of the $U(1)_D$ symmetry does not belong to the $\mathcal{N} = 3$ flavour current multiplet and so does not contribute at order x of the index. This is because the $\mathcal{N} = 3$ superpotential (6.2.3) of the abelian ABJM theory does not allow such a charge assignment.

Let us now analyse the index of the $SO(2)_2 \times USp(2)_{-1}$ theory. This can be computed using (6.4.30) with two modifications: the summation of (m_1, m_2) is over $\mathbb{Z}^2$, and the part $\sum_{p=0}^{1} \mathfrak{g}^p$ is removed. The result is the same as (6.4.38), with $u = v = f$ and $w = \zeta$. In other words, the $SU(2)_A$ flavour symmetry of theory II(L) is identified with the diagonal subgroup of $SU(2)_A \times SU(2)_B$ of theory II(R), and the $U(1)_\mathcal{M}$ magnetic symmetry of theory II(L) is identified with the topological symmetry of theory II(R). In fact, we can also turn on the fugacity χ for the $(\mathbb{Z}_2^{[0]})_\mathcal{C}$ zero-form charge conjugation symmetry [4]. For $\chi = 1$, the index is the same as that of $SO(2)_2 \times USp(2)_{-1}$, *i.e.* as discussed before. For $\chi = -1$, the index is

$$\begin{aligned}&\sum_{m_2\in\mathbb{Z}}\oint\frac{dz_2}{2\pi i z_2}z_2^{-2m_2}Z_V^{USp(2)}(z_2,m_2)\prod_{s,s_1,s_2=\pm1}Z_{\text{chir}}(s_1f^sz_2^{s_2};s_2m_2;1/2)\\&=1+\left(-f^2-\frac{1}{f^2}\right)x+\left(f^4+\frac{1}{f^4}+1\right)x^2+\left(-f^6-\frac{1}{f^6}\right)x^3+\dots\ .\end{aligned} \tag{6.4.44}$$

Let ζ' be the fugacity for a $\mathbb{Z}_2$ subgroup of the $U(1)_{\mathcal{M}}$ magnetic symmetry. The index can be written in terms of the fugacities f, ζ' and χ as

$$\begin{aligned}1&+x\left[1+(\zeta'+\chi+\zeta'\chi)\chi^{SU(2)}_{[2]}(f)\right]\\&+x^2\left[(\zeta'+\chi+\zeta'\chi+2)\chi^{SU(2)}_{[4]}(f)-\chi^{SU(2)}_{[2]}(f)-(\zeta'+\chi+\zeta'\chi)\right]+\dots\end{aligned} \tag{6.4.45}$$

where $\zeta'^2=\chi^2=1$. Note that ζ' and χ appear on an equal footing and they can be interchanged. In this notation, we can match the indices (6.4.38) and (6.4.45) as

$$[(6.4.38)](u=f,v=f,w=\chi)\Big|_{\chi^2=1}=[(6.4.45)](f,\zeta'=\chi,\chi)\ . \tag{6.4.46}$$

The unrefined indices of theories II(L) and II(R) with $N=1$ are given by [54, (4.2)]:

$$1+10x+19x^2+26x^3+49x^4+26x^5+\dots\ . \tag{6.4.47}$$

$[U(2)_4\times U(2)_4]/\mathbb{Z}_2$

The index of theory II(R): $[U(2)_4\times U(2)_4]/\mathbb{Z}_2$ can be written as

$$\begin{aligned}1&+x\left[\chi^{SU(2)}_{[1]}(u)\chi^{SU(2)}_{[1]}(v)\right]+x^2\Big[(g''+1)w'^{-1}\chi^{SU(2)}_{[4]}(u)+(g''+1)w'\chi^{SU(2)}_{[4]}(v)\\&+g''(w'+w'^{-1})+(g''+2)\chi^{SU(2)}_{[2]}(u)\chi^{SU(2)}_{[2]}(v)+1\\&-\chi^{SU(2)}_{[2]}(u)-\chi^{SU(2)}_{[2]}(v)-1\Big]+\dots\end{aligned} \tag{6.4.48}$$

where $(g'')^2=1$ and we have used the notation as discussed around (6.4.36). The corresponding unrefined index is given by

$$1+4x+43x^2+108x^3+241x^4+\dots\ . \tag{6.4.49}$$

The operators that contribute at order x are

$$\chi^{SU(2)}_{[1]}(u)\chi^{SU(2)}_{[1]}(v):\qquad M_{\alpha\alpha'}=(A_\alpha)^a_i(B_{\alpha'})^i_a\ . \tag{6.4.50}$$

Due to the $\mathbb{Z}_2$ quotient, the elementary monopole operators are

$$T_{\pm\frac{1}{2}} \equiv T_{\{\pm\frac{1}{2},\pm\frac{1}{2};\pm\frac{1}{2},\pm\frac{1}{2}\}} \ . \tag{6.4.51}$$

Arising from the discrete gauging, they transform non-trivially under the $\mathbb{Z}^{[0]}_{\mathrm{GCD}(4,2)} = \mathbb{Z}^{[0]}_2$ zero-form symmetry. They carry charges ± 1 under the $U(1)^{[0]}$ zero-form topological symmetry. Moreover, $T_{-\frac{1}{2}}$ carries gauge charges $4\left(-\frac{1}{2}-\frac{1}{2},\frac{1}{2}+\frac{1}{2}\right) = (-4,+4)$ under the $U(1)\times U(1)$ gauge subgroup of the $U(2)_4 \times U(2)_{-4}$ gauge group. Similarly, $T_{+\frac{1}{2}}$ carries such gauge charges $(+4,-4)$.

Now let us discuss the marginal operators, contributing the positive terms at order x^2. The di-baryon gauge invariant operators are[16]

$$\begin{aligned} g''w'^{-1}\left(\chi^{SU(2)}_{[4]}(u)+1\right): \quad & \mathcal{B}_{\alpha_1\ldots\alpha_4} = T_{-\frac{1}{2}}(A_{\alpha_1})^{a_1}_{i_1}(A_{\alpha_2})^{a_2}_{i_2}(A_{\alpha_3})^{a_3}_{i_3}(A_{\alpha_4})^{a_4}_{i_4}\epsilon^{i_1i_2}\epsilon_{a_1a_2}\epsilon^{i_3i_4}\epsilon_{a_3a_4} \\ g''w'\left(\chi^{SU(2)}_{[4]}(v)+1\right): \quad & \mathcal{B}'_{\alpha'_1\ldots\alpha'_4} = T_{+\frac{1}{2}}(B_{\alpha'_1})^{i_1}_{a_1}(B_{\alpha'_2})^{i_2}_{a_2}(B_{\alpha'_3})^{i_3}_{a_3}(B_{\alpha'_4})^{i_4}_{a_4}\epsilon_{i_1i_2}\epsilon^{a_1a_2}\epsilon_{i_3i_4}\epsilon^{a_3a_4} \end{aligned} \tag{6.4.52}$$

where the representations $[4]\oplus[0]$ come from the decomposition of $\mathrm{Sym}^2[2]$ of $SU(2)_A$ or $SU(2)_B$. There are also the following marginal operators:

$$\begin{aligned} & g''\chi^{SU(2)}_{[2]}(u)\chi^{SU(2)}_{[2]}(v): \\ & \mathcal{G}_{(\alpha\beta)(\alpha'\beta')} = (T_{\{\frac{1}{2},-\frac{1}{2};\frac{1}{2},-\frac{1}{2}\}})^{(i_1i_2i'_3i'_4)}_{(a_1a_2a'_3a'_4)}(A_\alpha)^{a_1}_{i_1}(A_\beta)^{a_2}_{i_2}(B_{\alpha'})^{i_3}_{a_3}(B_{\beta'})^{i_4}_{a_4}\times \\ & \qquad \epsilon^{i'_3i_3}\epsilon^{i'_4i_4}\epsilon_{a'_3a_3}\epsilon_{a'_4a_4} \ . \end{aligned} \tag{6.4.53}$$

where we remark that the monopole operator $T_{\{\frac{1}{2},-\frac{1}{2};\frac{1}{2},-\frac{1}{2}\}}$ transforms in the representation $[4_0;4_0]$ of the $U(2)\times U(2)$ gauge group. There are gauge invariant dressed monopole operators, contributing $w'^{-1}\chi^{SU(2)}_{[4]}(u)$ and $w'\chi^{SU(2)}_{[4]}(v)$ at order x^2,

$$\begin{aligned} (\mathcal{M}_{-1})_{\alpha_1\ldots\alpha_4} &= (T_{\{-1,0;-1,0\}})^{(i_1\cdots i_4)}_{(a_1\cdots a_4)}(A_{\alpha_1})^{a_1}_{i_1}(A_{\alpha_2})^{a_2}_{i_2}(A_{\alpha_3})^{a_3}_{i_3}(A_{\alpha_4})^{a_4}_{i_4} \\ (\mathcal{M}_{+1})_{\alpha'_1\ldots\alpha'_4} &= (T_{\{+1,0;+1,0\}})^{(a_1\cdots a_4)}_{(i_1\cdots i_4)}(B_{\alpha'_1})^{i_1}_{a_1}(B_{\alpha'_2})^{i_2}_{a_2}(B_{\alpha'_3})^{i_3}_{a_3}(B_{\alpha'_4})^{i_4}_{a_4} \end{aligned} \tag{6.4.54}$$

where, as for the ABJM theory, $T_{\pm\{1,0;1,0\}}$ transform in the representation $[4_{\pm 4};4_{\mp 4}]$ of the $U(2)_4\times U(2)_{-4}$ gauge group.[17] Finally, there are also the following marginal operators:

$$\begin{aligned} \chi^{SU(2)}_{[2]}(u)\chi^{SU(2)}_{[2]}(v)+1: \quad & M_{\alpha\alpha'}M_{\beta\beta'} \ , \\ \chi^{SU(2)}_{[2]}(u)\chi^{SU(2)}_{[2]}(v): \quad & \mathcal{Q}_{\alpha\beta\alpha'\beta'} = (A_\alpha)^a_i(B_{\alpha'})^i_b(A_\beta)^b_j(B_{\beta'})^j_a \end{aligned} \tag{6.4.55}$$

where the latter are subject to the relations (6.4.71) coming from the F-terms. We will discuss these two operators in more detail around (6.4.70).

[16] The dressing of the monopole operators works similarly to to what was explained in Footnote (9).

[17] Here $[k_q]$ stands for a $U(2)\cong SU(2)\times U(1)$ representation consisting of the spin $k/2$ representation of the $SU(2)$ part and having charge q under the $U(1)$ part.

Theory II(R): $[U(2)_4 \times U(2)_4]/\mathbb{Z}_2$, in fact, has $\mathcal{N}=6$ supersymmetry. This can be seen from the index as follows. It is convenient to rewrite (6.4.48) in terms of an $\mathcal{N}=3$ index simply by setting $u=v=f$ and using the fact that $[2]\otimes[2]=[4]\oplus[2]\oplus[0]$:

$$\begin{aligned} 1 &+ x\left[1+\chi^{SU(2)}_{[2]}(f)\right] + x^2\Big[(g''+1)(w'+w'^{-1})\chi^{SU(2)}_{[4]}(f) \\ &+ g''(w'+w'^{-1}) + (g''+2)(\chi^{SU(2)}_{[4]}(f)+\chi^{SU(2)}_{[2]}(f)+1)+1 \\ &- \chi^{SU(2)}_{[2]}(f) - (\chi^{SU(2)}_{[2]}(f)+1)\Big] + \ldots \end{aligned} \tag{6.4.56}$$

where the contribution of the $\mathcal{N}=3$ flavour currents is denoted in blue and the the contribution of the $\mathcal{N}=3$ extra supersymmetry current is written in red. Since there are 3 of the latter, we conclude that supersymmetry gets enhanced from $\mathcal{N}=3$ to $\mathcal{N}=3+3=6$.

$SO(4)_2 \times USp(4)_{-1}$

The index of theory II(L): $SO(4)_2 \times USp(4)_{-1}$ can be written as

$$\begin{aligned} 1 &+ x\left[1+\zeta\chi^{SU(2)}_{[2]}(f)\right] + x^2\Big[(1+2(1+\chi)+(1+\chi)\zeta)\,\chi^{SU(2)}_{[4]}(f) \\ &+ (1+\chi)\zeta\chi^{SU(2)}_{[2]}(f) + (1+2(1+\chi)) - (1-\chi)\zeta - \chi^{SU(2)}_{[2]}(f)\Big] + \ldots \,. \end{aligned} \tag{6.4.57}$$

with the unrefined index given by (6.4.49). The indices (6.4.48) and (6.4.57) can be matched as follows:

$$[(6.4.48)](u=v=f,\, w'=1,\, g''=\chi) = [(6.4.57)](f,\chi,\zeta=1)\,. \tag{6.4.58}$$

In other words, the $SU(2)_A$ flavour symmetry of theory II(L) is identified with the diagonal subgroup of the flavour symmetry $SU(2)_A \times SU(2)_B$ of theory II(R). The $Z_2^{[0]}$ zero-form symmetry of theory II(R) with fugacity g'' is identified with the zero-form charge conjugation symmetry $(\mathbb{Z}_2^{[0]})_{\mathcal{C}}$ of theory II(L) with fugacity χ. However, the $U(1)^{[0]}$ zero-form topological symmetry of theory II(R) with fugacity w' is not manifest in theory II(L), whereas the magnetic symmetry $(\mathbb{Z}_2^{[0]})_{\mathcal{M}}$ of theory II(L) with fugacity ζ is not manifest in theory II(R) (see Table (6.3.20)).

6.4.3 $O(2N)_2 \times USp(2N)_{-1} \;\leftrightarrow\; U(N)_4 \times U(N)_{-4}$

In this subsection, we consider the well-known duality between the following two theories:

$$\text{I(L): } O(2N)_2 \times USp(2N)_{-1} \quad \leftrightarrow \quad \text{I(R): } U(N)_4 \times U(N)_{-4}\,. \tag{6.4.59}$$

Theory I(L) can be obtained from theory II(L) by gauging the zero-form charge conjugation symmetry of the latter. At the level of the index, this can be done by summing over $\chi \in \{-1,+1\}$. As a result, we are left with the fugacity f for the $SU(2)_A$ flavour symmetry and the fugacity ζ for the $(\mathbb{Z}_2^{[0]})_{\mathcal{M}}$ zero-form magnetic symmetry.

On the other hand, theory I(R) can be obtained from theory II(R) by gauging the zero-form symmetry $\mathbb{Z}^{[0]}_{\mathrm{GCD}(N,2)}$ of the latter with fugacity g''. In particular, given the index of theory II(R) written in terms of u, v, w' and g'', where $(g'')^{\mathrm{GCD}(N,2)} = 1$, we are summing over $g'' \in \{e^{2\pi i j/\mathrm{GCD}(N,2)} \,|\, j = 0, 1, \ldots, \mathrm{GCD}(N,2) - 1\}$. As a result, we are left with the fugacities u and v for the $SU(2)_A \times SU(2)_B$ flavour symmetry and the fugacity w' for the $U(1)^{[0]}_{\mathrm{top}}$ topological symmetry.

The case of $N = 1$

The $\mathcal{N} = 2$ index for theory I(R): $U(1)_4 \times U(1)_{-4}$ is

$$\begin{aligned} 1 &+ x\left[\chi^{SU(2)}_{[1]}(u)\chi^{SU(2)}_{[1]}(v)\right] + x^2\Big[w'\chi^{SU(2)}_{[4]}(v) + w'^{-1}\chi^{SU(2)}_{[4]}(u) \\ &+ \chi^{SU(2)}_{[2]}(u)\chi^{SU(2)}_{[2]}(v) - \chi^{SU(2)}_{[2]}(u) - \chi^{SU(2)}_{[2]}(v) - 2\Big] + \ldots \end{aligned} \tag{6.4.60}$$

In order to write this in terms of the $\mathcal{N} = 3$ index, we set $u = v = f$ and use the tensor product decomposition $[2] \otimes [2] = [4] \oplus [2] \oplus [0]$:

$$\begin{aligned} 1 &+ x\left[1 + \chi^{SU(2)}_{[2]}(f)\right] + x^2\Big[(w' + w'^{-1})\chi^{SU(2)}_{[4]}(f) \\ &+ \chi^{SU(2)}_{[4]}(f) + \chi^{SU(2)}_{[2]}(f) + 1 - (\chi^{SU(2)}_{[2]}(f) + 1) - \chi^{SU(2)}_{[2]}(f) - 1\Big] + \ldots \,, \end{aligned} \tag{6.4.61}$$

where the blue terms denote the contribution of the $\mathcal{N} = 3$ flavour currents in $SU(2)_f \times U(1)^{[0]}_{\mathrm{top}}$, and the red terms denote the contribution of the $\mathcal{N} = 3$ extra supersymmetry current. Since there are three of the latter, we conclude that supersymmetry gets enhanced from $\mathcal{N} = 3$ to $\mathcal{N} = 6$, as expected. The last -1 term at order x^2 is the contribution of the current of the $U(1)_D$ symmetry, discussed below (6.4.43).

The operators contributing at order x of (6.4.60) correspond to

$$M_{\alpha\alpha'} = A_\alpha B_{\alpha'} \ . \tag{6.4.62}$$

Those contributing to the positive terms at order x^2 (*i.e.* $\mathcal{N} = 2$ preserving marginal operators) are gauge invariant dressed monopole operators and the square of M:

$$\begin{aligned} w'^{-1}\chi^{SU(2)}_{[4]}(u) &: \quad (\mathcal{M}_{-1})_{\alpha_1\cdots\alpha_4} = T_{\{-1;-1\}} A_{\alpha_1}A_{\alpha_2}A_{\alpha_3}A_{\alpha_4} \ , \\ w'\chi^{SU(2)}_{[4]}(v) &: \quad (\mathcal{M}_{+1})_{\alpha'_1\cdots\alpha'_4} = T_{\{+1;+1\}} B_{\alpha'_1}B_{\alpha'_2}B_{\alpha'_3}B_{\alpha'_4} \ , \\ \chi^{SU(2)}_{[2]}(u)\chi^{SU(2)}_{[2]}(v) &: \quad \mathcal{Q}_{\alpha\beta\alpha'\beta'} = (A_\alpha A_\beta)(B_{\alpha'}B_{\beta'}) = M_{\alpha\alpha'}M_{\beta\beta'} \ . \end{aligned} \tag{6.4.63}$$

On the other hand, the index of theory I(L): $O(2)_2 \times USp(2)_{-1}$ can be obtained by summing over $\chi \in \{-1, 1\}$ in (6.4.45)

$$
\begin{aligned}
&1 + x\left[1 + \zeta'\,\chi^{SU(2)}_{[2]}(f)\right] + x^2\Big[(\zeta'+1)\chi^{SU(2)}_{[4]}(f) \\
&\quad + \chi^{SU(2)}_{[4]}(f) + \chi^{SU(2)}_{[2]}(f) + 1 - (\chi^{SU(2)}_{[2]}(f)+1) - \chi^{SU(2)}_{[2]}(f) - \zeta'\Big] + \dots
\end{aligned}
\tag{6.4.64}
$$

where $(\zeta')^2 = 1$.[18] The indices (6.4.60) and (6.4.64) can be matched as follows:

$$
[(6.4.60)](u=f, v=f, w'=1) = [(6.4.64)](f, \zeta'=1)\ . \tag{6.4.65}
$$

In other words, the $U(1)^{[0]}_{\text{top}}$ topological symmetry of theory I(R) is not manifest in theory I(L), whereas the $(\mathbb{Z}^{[0]}_2)_{\mathcal{M}}$ magnetic symmetry of theory I(L) is not manifest in theory I(R). As usual, the $SU(2)_A$ flavour symmetry of theory I(L) is identified with the diagonal subgroup of the $SU(2)_A \times SU(2)_B$ flavour symmetry of theory I(R) (see Table (6.3.20)).

The unrefined indices of theories I(L) and I(R) with $N = 1$ are, of course, equal and are given by [54, Table 1]:

$$
1 + 4x + 11x^2 + 12x^3 + 25x^4 + 12x^5 + \dots\ . \tag{6.4.66}
$$

The case of $N = 2$

The $\mathcal{N} = 2$ index for theory I(R): $U(2)_4 \times U(2)_{-4}$ is

$$
\begin{aligned}
&1 + x\left[\chi^{SU(2)}_{[1]}(u)\chi^{SU(2)}_{[1]}(v)\right] + x^2\Big[w'\chi^{SU(2)}_{[4]}(v) + w'^{-1}\chi^{SU(2)}_{[4]}(u) \\
&\quad + 2\chi^{SU(2)}_{[2]}(u)\chi^{SU(2)}_{[2]}(v) + 1 - \chi^{SU(2)}_{[2]}(u) - \chi^{SU(2)}_{[2]}(v) - 1\Big] + \dots
\end{aligned}
\tag{6.4.67}
$$

As before, the $\mathcal{N} = 3$ index can be obtain by setting $u = v = f$:

$$
\begin{aligned}
&1 + x\left[1 + \chi^{SU(2)}_{[2]}(f)\right] + x^2\Big[(w' + w'^{-1})\chi^{SU(2)}_{[4]}(f) \\
&\quad + 2\chi^{SU(2)}_{[4]}(f) + 2\chi^{SU(2)}_{[2]}(f) + 2 + 1 - (\chi^{SU(2)}_{[2]}(f)+1) - \chi^{SU(2)}_{[2]}(f)\Big] + \dots
\end{aligned}
\tag{6.4.68}
$$

The operators contributing at order x is

$$
\chi^{SU(2)}_{[1]}(u)\chi^{SU(2)}_{[1]}(v): \qquad M_{\alpha\alpha'} = (A_\alpha)^a_i (B_{\alpha'})^i_a\ . \tag{6.4.69}
$$

[18] The magnetic symmetry of $O(2)_2$ is not $U(1)^{[0]}_{\mathcal{M}}$, but rather $(\mathbb{Z}^{[0]}_2)_{\mathcal{M}}$, see for example [60, Appendix H]. It is interesting to point out that the index is sensitive to this. Indeed, if we compute the index of the $O(2)_2 \times USp(2)_{-1}$ model treating the magnetic fugacity ζ' as a $U(1)^{[0]}_{\mathcal{M}}$ fugacity, that is without imposing $(\zeta')^2 = 1$ as in (6.4.64), we would get fractional coefficients such as $\frac{1}{2}(\zeta' + 1/\zeta')$, which is clearly incosistent. This signals that the actual symmetry is $(\mathbb{Z}^{[0]}_2)_{\mathcal{M}}$ and by accordingly setting $\zeta' = 1/\zeta'$ we get the sensible result (6.4.64).

The marginal operators, which contribute to the positive terms at order x^2, are[19]

$$\begin{aligned}
w'^{-1}\chi^{SU(2)}_{[4]}(u): &\quad (\mathcal{M}_{-1})_{\alpha_1\cdots\alpha_4} = (T_{\{-1,0;-1,0\}})^{(i_1\cdots i_4)}_{(a_1\cdots a_4)}(A_{\alpha_1})^{a_1}_{i_1}\cdots(A_{\alpha_4})^{a_4}_{i_4}\ , \\
w'\chi^{SU(2)}_{[4]}(v): &\quad (\mathcal{M}_{+1})_{\alpha'_1\cdots\alpha'_4} = (T_{\{+1,0;+1,0\}})^{(a_1\cdots a_4)}_{(i_1\cdots i_4)}(B_{\alpha'_1})^{i_1}_{a_1}\cdots(B_{\alpha'_4})^{i_4}_{a_4}\ , \\
\chi^{SU(2)}_{[2]}(u)\chi^{SU(2)}_{[2]}(v)+1: &\quad M_{\alpha\alpha'}M_{\beta\beta'}\ , \\
\chi^{SU(2)}_{[2]}(u)\chi^{SU(2)}_{[2]}(v): &\quad \mathcal{Q}_{\alpha\beta\alpha'\beta'} = (A_\alpha)^a_i(B_{\alpha'})^i_b(A_\beta)^b_j(B_{\beta'})^j_a
\end{aligned} \tag{6.4.70}$$

where we comment on the above operators as follows:

- The monopole operators $T_{\{+1,0;+1,0\}}$ and $T_{\{-1,0;-1,0\}}$ transform in the representations $[4_{+4};4_{-4}]$ and $[4_{-4};4_{+4}]$ of the gauge group $U(2)\times U(2)$, respectively.
- The gauge invariant combinations MM in the third line transform in the representation $\mathrm{Sym}^2[1;1] = [2;2]+[0;0]$ of $SU(2)_A\times SU(2)_B$.
- The gauge invariant combinations $\mathcal{Q}_{\alpha\beta\alpha'\beta'}$ are subject to the F-terms coming from the superpotential of the ABJM theory (6.2.4) and so

$$\epsilon^{\alpha\beta}\mathcal{Q}_{\alpha\beta\alpha'\beta'} = 0\ , \qquad \epsilon^{\alpha'\beta'}\mathcal{Q}_{\alpha\beta\alpha'\beta'} = 0\ . \tag{6.4.71}$$

  Thus, $\mathcal{Q}_{\alpha\beta\alpha'\beta'}$ transform under the representation $[2;2]$ of $SU(2)_A\times SU(2)_B$.
- Note that one could also consider the following gauge invariant combinations:

$$\begin{aligned}
&(A_\alpha)^{a_1}_{i_1}(A_\beta)^{a_2}_{i_2}(B_{\alpha'})^{j_1}_{b_1}(B_{\beta'})^{j_2}_{b_2}\epsilon_{a_1a_2}\epsilon^{i_1i_2}\epsilon^{b_1b_2}\epsilon_{j_1j_2} \\
&= (A_\alpha)^{a_1}_{i_1}(A_\beta)^{a_2}_{i_2}(B_{\alpha'})^{j_1}_{b_1}(B_{\beta'})^{j_2}_{b_2}\,\delta^{[b_1}_{a_1}\delta^{b_2]}_{a_2}\,\delta^{[i_1}_{j_1}\delta^{i_2]}_{j_2} \\
&= (A_\alpha)^{a_1}_{i_1}(B_{\alpha'})^{i_1}_{a_1}(A_\beta)^{a_2}_{i_2}(B_{\beta'})^{i_2}_{a_2} - (\alpha'\leftrightarrow\beta') \\
&= M_{\alpha\alpha'}M_{\beta\beta'} - (\alpha'\leftrightarrow\beta')
\end{aligned} \tag{6.4.72}$$

  and so they are not independent from those in (6.4.70).

The index of theory I(L): $O(4)_2\times USp(4)_{-1}$ is

$$\begin{aligned}
&1 + x\left[1+\zeta'\chi^{SU(2)}_{[2]}(f)\right] + x^2\Big[(\zeta'+1)\chi^{SU(2)}_{[4]}(f) + 2\chi^{SU(2)}_{[4]}(f) \\
&\quad + (1+\zeta')\chi^{SU(2)}_{[2]}(f) + 2 + 1 - (\chi^{SU(2)}_{[2]}(f)+1) - \chi^{SU(2)}_{[2]}(f) + 1 - \zeta'\Big] + \ldots
\end{aligned} \tag{6.4.73}$$

Let us discuss the operators, contributing to order x, in this theory. In the following, $i,j = 1,...,4$ are the $O(4)$ gauge indices; $a,b = 1,...,4$ are the $USp(4)$ gauge indices; and $\alpha,\beta = 1,2$ are the $SU(2)_A$ flavour indices. They are

[19] Notice that the monopoles $T_{\{\pm1,0;\pm1,0\}}$ are in a representation of the $U(2)\times U(2)$ gauge group that have charges $(\pm1/2,\mp1/2)$ under the $U(1)\times U(1)$ part and that are in the symmetric representation of each of the two $SU(2)$ part.

$$\begin{aligned} 1: \qquad & \mathfrak{m}_{[\alpha\beta]} = (A_\alpha)^{i_1}_{a_1}(A_\beta)^{i_2}_{b_2}\delta_{i_1 i_2}J^{a_1 a_2} \ , \\ \zeta'\,\chi^{SU(2)}_{[2]}(f): \qquad & \mathfrak{M}_{(\alpha\beta)} = (T_{\{1,0;1,0\}})^{(a_1 a_2)}_{(i_1 i_2)}(A_\alpha)^{i_1}_{a_1}(A_\beta)^{i_2}_{a_2} \end{aligned} \tag{6.4.74}$$

where $\mathfrak{m}$ transforms as a singlet under $SU(2)_A$, due to the total antisymmetrisation of the indices α and β, and $\mathfrak{M}$ transforms as a triplet under $SU(2)_A$, due to the total symmetrisation of the gauge indices in the elementary monopole operator $T_{\{1,0;1,0\}}$. These operators are mapped to the mesons (6.4.69) of the unitary theory I(R). Hence, from the perspective of the $\mathcal{N}=3$ theory, these are the moment map operators of the $U(1)\times SU(2)_A$ symmetry, whose contribution of the currents is denoted in blue. The contributions in red are instead identified as the $\mathcal{N}=3$ extra supersymmetry-currents that make $\mathcal{N}=3$ supersymmetry become $\mathcal{N}=3+3=6$ supersymmetry[20 21].

The indices (6.4.67) and (6.4.73) can be matched as follows:

$$(6.4.67)[u=f, v=f, w=1] = (6.4.73)[f, \zeta'=1] \tag{6.4.76}$$

The fugacity ζ' for the $(\mathbb{Z}_2^{[0]})_{\mathcal{M}}$ magnetic symmetry of theory I(L) cannot be mapped to any fugacity in theory I(R), and so it is not manifest in theory I(R) and should be considered as emergent in theory I(R). Similarly, the $U(1)^{[0]}_{\text{top}}$ zero-form topological symmetry of theory I(R) with fugacity w' should be considered as emergent in theory I(L). As usual, the $SU(2)_A$ flavour symmetry of theory I(L) is identified as a diagonal subgroup of the $SU(2)_A\times SU(2)_B$ flavour symmetry of theory I(R) (see Table (6.3.20)).

[20]It is also interesting to analyse this theory from the perspective of $\mathcal{N}=5$ theory. For an SCFT with $\mathcal{N}=5$ (and not higher) supersymmetry, it is necessary that the coefficient of x in the index must be 1 [74], whose contribution comes from the $\mathcal{N}=5$ stress-tensor multiplet decomposed into one $\mathcal{N}=2$ multiplet $L\overline{B}_1[0]^{(1)}_1$ (in the notation of [59]). However, for an SCFT with $\mathcal{N}=6$ (and not higher) supersymmetry, the coefficient of x must be 4 [74], which is the case for (6.4.73). Observe that the singlet operator $\mathfrak{m}$ is present in any $O(2N)_{2k}\times USp(2N)_{-k}$ theory, where for $k\geq 2$ the theory has $\mathcal{N}=5$ supersymmetry [1]. We thus conclude that the operator $\mathfrak{m}$ resides in the $\mathcal{N}=5$ stress-tensor multiplet $B_1[0]^{(1,0)}_1$, whereas the triplet operators $\mathfrak{M}$ reside in the $\mathcal{N}=5$ extra supersymmetry-current multiplet $B_1[0]^{(0,2)}_1$. The singlet in the tensor product decomposition of $[0,2]\otimes[0,2]$, where each $[0,2]$ is the representation of the latter multiplet under $\mathfrak{so}(5)_R$ symmetry, corresponds to the $U(1)$ global symmetry, which must be present in any $\mathcal{N}=6$ SCFT [17]. This is mapped to the $U(1)^{[0]}_{\text{top}}$ symmetry of the unitary theory I(R). We thank Oren Bergman for explaning and pointing this out to us.

[21]To elucidate further Footnote (20), we provide, as a reference, the index of the $O(4)_4\times USp(4)_{-2}$ theory, which has $\mathcal{N}=5$ supersymmetry:

$$1+1x+x^2\left[2+(1+\zeta)\chi^{SU(2)}_{[4]}(f)-\chi^{SU(2)}_{[2]}(f)\right]+\ldots \tag{6.4.75}$$

From $\mathcal{N}=2$ perspective, the negative term at order x^2 indicates the contribution of the $\mathcal{N}=2$ $SU(2)_A$ flavour currents in the multiplet $A_2\overline{A}_2[0]^{(0)}_1$. From the $\mathcal{N}=3$ perspective, the three components of $SU(2)_A$ currents split into two parts. Suppose that we write the character of the adjoint representation of $SU(2)_A$ as f^2+1+f^{-2}. Two components (f^2, f^{-2}) of this $SU(2)_A$ symmetry currents are identified as the $\mathcal{N}=3$ extra SUSY-currents in the multiplet $A_2[0]^{(0)}_1$; this makes $\mathcal{N}=3$ supersymmetry become $\mathcal{N}=3+2=5$ supersymmetry. The remaining component (corresponding to 1) of this $SU(2)_A$ symmetry currents resides in the $\mathcal{N}=3$ $U(1)$ flavour current multiplet $B_1[0]^{(2)}_1$. The corresponding moment map operator is the singlet operator $\mathfrak{m}$, contributing $+1x$ to the index. Thus, this $U(1)$ symmetry from the $\mathcal{N}=3$ perspective is identified as the Cartan subalgebra of $SU(2)_A$.

The unrefined indices for theories I(L) and I(R) for $N = 2$ are, of course, equal and are given by [54, Table 1]:

$$1 + 4x + 22x^2 + 56x^3 + 131x^4 + 252x^5 + \dots \; . \tag{6.4.77}$$

6.4.4 $SO(4)_2 \times USp(2)_{-1} \;\leftrightarrow\; [U(3)_4 \times U(1)_{-4}]/\mathbb{Z}_2$

As pointed out in [153, Section 3.3], the consistency conditions of the quotient $[U(N+x)_k \times U(N)_k]/\mathbb{Z}_p$ are

$$p \text{ divides } k \qquad \text{and} \qquad \frac{kx}{p^2} \in \mathbb{Z} \; . \tag{6.4.78}$$

In this section, we take $N = 1$, $x = 2$, $k = 4$ and $p = 2$. Indeed, the theory in question can be obtained by gauging the $\mathbb{Z}_2^{[1]}$ one-form symmetry of the $U(3)_4 \times U(1)_{-4}$ theory. The index for $[U(3)_4 \times U(1)_{-4}]/\mathbb{Z}_2$ reads (here we define $w = (w_1 w_2)^{\frac{1}{4}}$)

$$\begin{aligned} 1 &+ x\left[\chi^{SU(2)}_{[1]}(u)\chi^{SU(2)}_{[1]}(v)\right] + x^2\Big[w^{-1}\chi^{SU(2)}_{[4]}(u) + w\chi^{SU(2)}_{[4]}(v) \\ &+ \chi^{SU(2)}_{[2]}(u)\chi^{SU(2)}_{[2]}(v) + g(w^{1/2} + w^{-1/2}) - \chi^{SU(2)}_{[2]}(u) - \chi^{SU(2)}_{[2]}(v) - 1\Big] + \dots \; . \end{aligned} \tag{6.4.79}$$

As before, we can obtain the $\mathcal{N} = 3$ index by setting $u = v = f$ and compute the relevant tensor product decompositions:

$$\begin{aligned} 1 &+ x\left[1 + \chi^{SU(2)}_{[2]}(f)\right] + x^2\Big[(w + w^{-1})\chi^{SU(2)}_{[4]}(f) + \chi^{SU(2)}_{[4]}(f) + \chi^{SU(2)}_{[2]}(f) + 1 \\ &+ g(w^{1/2} + w^{-1/2}) - (\chi^{SU(2)}_{[2]}(f) + 1) - \chi^{SU(2)}_{[2]}(f)\Big] + \dots \end{aligned} \tag{6.4.80}$$

where the blue terms are the contribution of the $\mathcal{N} = 3$ flavour currents and the red term is the contribution of the $\mathcal{N} = 3$ extra supersymmetry current. Therefore, $\mathcal{N} = 3$ supersymmetry gets enhanced to $\mathcal{N} = 6$.

As usual, the operators contributing at order x of (6.4.79) are the mesons,

$$M_{\alpha\alpha'} = (A_\alpha)^a (B_{\alpha'})_a \tag{6.4.81}$$

where $a, b, c = 1, 2, 3$ are the $U(3)$ gauge indices. As usual, the monopole operators $T_- \equiv T_{\{-1,0,0;-1\}}$ and $T_+ \equiv T_{\{+1,0,0;+1\}}$ transform in the representations $[[0,4]_{-4}; +4]$ and $[[4,0]_{+4}; -4]$ of the gauge symmetry $U(3) \times U(1)$ respectively, where $[0,4]_{-4}$ and $[4,0]_{+4}$ are from the 4th symmetric power of the antifundamental and fundamental representation of $U(3)$ respectively.

We can write down the marginal operators, contributing to the positive terms at order x^2, as follows:

$$\begin{aligned} w^{-1}\chi^{SU(2)}_{[4]}(u): &\qquad (\mathcal{M}_{-1})_{\alpha_1\cdots\alpha_4} = (T_-)_{(a_1a_2a_3a_4)}(A_{\alpha_1})^{a_1}\cdots(A_{\alpha_4})^{a_4} \\ w\chi^{SU(2)}_{[4]}(v): &\qquad (\mathcal{M}_{+1})_{\alpha'_1\cdots\alpha'_4} = (T_+)^{(a_1a_2a_3a_4)}(B_{\alpha_1})_{a_1}\cdots(B_{\alpha_4})_{a_4} \\ \chi^{SU(2)}_{[2]}(u)\chi^{SU(2)}_{[2]}(v): &\qquad \mathcal{Q}_{\alpha\beta\alpha'\beta'} = M_{\beta\alpha'}M_{\alpha\beta'} \end{aligned} \tag{6.4.82}$$

Moreover, there are marginal operators, associated with the terms $gw^{\pm 1/2}$ at order x^2 in the index, that involve monopole operators $T_{\pm\frac{1}{2}} \equiv T_{\pm\{+\frac{1}{2},+\frac{1}{2},-\frac{1}{2};\frac{1}{2}\}}$, arising from the $\mathbb{Z}_2$ discrete quotient. Here $\mathcal{Q}_{\alpha\beta\alpha'\beta'}$ is defined as in (6.4.70) with the absence of the indices i, j, and $\mathcal{M}_\pm$ are the gauge invariant dressed monopole operators.

The index of $SO(4)_2 \times USp(2)_{-1}$ reads

$$\begin{aligned} &1 + x\left[1 + \zeta\chi^{SU(2)}_{[2]}(f)\right] \\ &\quad + x^2\left[(\zeta+2)\chi^{SU(2)}_{[4]}(f) + \zeta\chi + \chi + 1 - \zeta - \chi^{SU(2)}_{[2]}(f)\right] + \ldots\ . \end{aligned} \tag{6.4.83}$$

The indices (6.4.79) and (6.4.83) can be matched as follows:

$$[(6.4.79)](u = f, v = f, w = 1, g = \chi) = [(6.4.83)](f, \zeta = 1, \chi)\ . \tag{6.4.84}$$

The flavour symmetry $SU(2)_A$ of the orthosymplectic theory can be identified with the diagonal subgroup of the flavour symmetry $SU(2)_A \times SU(2)_B$ of the unitary theory. The $\mathbb{Z}_2^{[0]}$ zero-form symmetry of the unitary theory with fugacity g is identified with the $(\mathbb{Z}_2^{[0]})_\mathcal{C}$ zero-form charge conjugation symmetry of the orthosymplectic theory. The $U(1)^{[0]}$ zero-form topological symmetry of the unitary theory is not manifest in the orthosymplectic theory, whereas the $(\mathbb{Z}_2^{[0]})_\mathcal{M}$ zero-form magnetic symmetry of the orthosymplectic theory is not manifest in the unitary theory.

The unrefined indices for both theories are equal to

$$\begin{aligned} [(6.4.79)](u = 1, v = 1, w = 1, g = 1) &= [(6.4.83)](f = 1, \zeta = 1, \chi = 1) \\ &= 1 + 4x + 14x^2 + 35x^4 + \ldots\ . \end{aligned} \tag{6.4.85}$$

6.4.5 $O(4)_2 \times USp(2)_{-1} \ \leftrightarrow\ U(3)_4 \times U(1)_{-4}$

The $O(4)_2 \times USp(2)_{-1}$ theory can be obtained from the $SO(4)_2 \times USp(2)_{-1}$ theory by gauging the $(\mathbb{Z}_2^{[0]})_\mathcal{C}$ charge conjugation symmetry of the latter. Correspondingly, the $U(3)_4 \times U(1)_{-4}$ can be obtained from the $[U(3)_4 \times U(1)_{-4}]/\mathbb{Z}_2$ theory by gauging the $\mathbb{Z}_2^{[0]}$ zero-form symmetry with fugacity g of the latter.

Thus, summing over $g \in \{\pm 1\}$ in (6.4.79), we obtain the index for the $U(3)_4 \times U(1)_{-4}$ theory as

$$\begin{aligned} &1 + x\left[\chi^{SU(2)}_{[1]}(u)\chi^{SU(2)}_{[1]}(v)\right] + x^2\left[w^{-1}\chi^{SU(2)}_{[4]}(u) + w\chi^{SU(2)}_{[4]}(v)\right. \\ &\quad \left. + \chi^{SU(2)}_{[2]}(u)\chi^{SU(2)}_{[2]}(v) - \chi^{SU(2)}_{[2]}(u) - \chi^{SU(2)}_{[2]}(v) - 1\right] + \ldots\ . \end{aligned} \tag{6.4.86}$$

The operators are as listed in (6.4.81) and (6.4.82), except that there are no monopole operators $T_{\pm\frac{1}{2}}$ due to the absence of the discrete $\mathbb{Z}_2$ quotient. By the same argument as in the precedent subsection, the index indicates that the theory has $\mathcal{N} = 6$ supersymmetry, in agreement with [1].

Similarly, summing over $\chi \in \{\pm 1\}$ in (6.4.83) gives the index of the $O(4)_2 \times USp(2)_{-1}$ theory:

$$1 + x\left[1 + \zeta \chi^{SU(2)}_{[2]}(f)\right] + x^2\left[(\zeta + 2)\chi^{SU(2)}_{[4]}(f) + 1 - \zeta - \chi^{SU(2)}_{[2]}(f)\right] + \dots \; . \tag{6.4.87}$$

The indices (6.4.86) and (6.4.87) can be matched as follows:

$$[(6.4.86)](u = f, v = f, w = 1) = [(6.4.87)](f, \zeta = 1) \; . \tag{6.4.88}$$

The correspondence between the global symmetries of the $U(3)_4 \times U(1)_{-4}$ theory and the $O(4)_2 \times USp(2)_{-1}$ are as discussed below (6.4.83). The $SU(2)_A$ flavour symmetry of the orthosymplectic theory is identified with the diagonal subgroup of the $SU(2)_A \times SU(2)_B$ of the unitary theory. The $U(1)^{[0]}_{\text{top}}$ zero-form topological symmetry of the unitary theory is not manifest in the orthosymplectic theory, whereas the $(\mathbb{Z}_2^{[0]})_{\mathcal{M}}$ zero-form magnetic symmetry of the orthosymplectic theory is not manifest in the unitary theory. The unrefined indices for both theories are equal

$$\begin{aligned}[(6.4.86)](u = 1, v = 1, w = 1) &= [(6.4.87)](f = 1, \zeta = 1) \\ &= 1 + 4x + 12x^2 + 8x^3 + 27x^4 + 36x^5 + \dots \; ,\end{aligned} \tag{6.4.89}$$

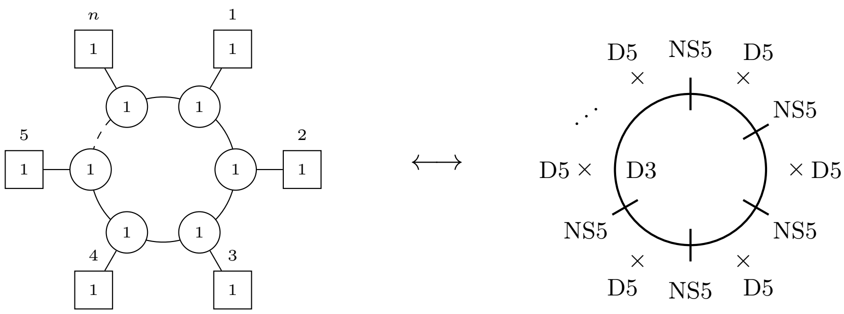
as computed in [54, (2.8)].

6.4.6 Circular quivers

In this subsection, we examine the following duality for $n \geq 3$:

$$\begin{aligned} &\underbrace{[SO(2)_2 \times USp(2)_{-1} \times \cdots \times SO(2)_2 \times USp(2)_{-1}]}_{2n \text{ gauge groups}}/\mathbb{Z}_2 \\ &\longleftrightarrow \quad \text{circular quiver (6.4.91)} \\ &\longleftrightarrow \quad \text{circular quiver } \underbrace{U(1)_1 \times U(1)_{-1} \times \cdots \times U(1)_1 \times U(1)_{-1}}_{2n \text{ gauge groups}} \end{aligned} \tag{6.4.90}$$

where the theory on the second line, also known as a Kronheimer–Nakajima quiver [131], is described by



(6.4.91)

This theory is self-mirror, and its Higgs/Coulomb branch describes one $PSU(n) \cong U(n)/U(1)$ instanton on $\mathbb{C}^2/\mathbb{Z}_n$ with the holonomy of the gauge field at infinity that brakes $PSU(n)$ into $U(1)^n/U(1)$.

The duality between theories in the second and third lines of (6.4.90) is well-known and can be seen from the brane system (see *e.g.* [13, 135]) by applying the T^t $SL(2,\mathbb{Z})$ transformation, which reads

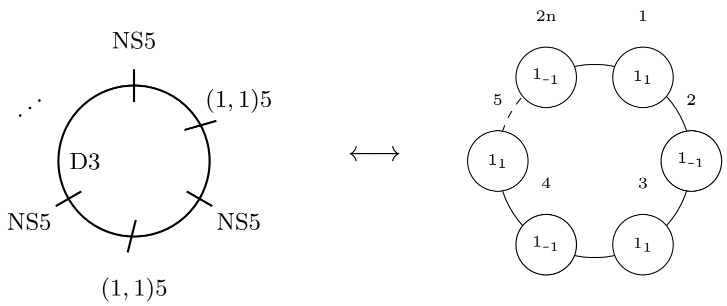
$$T^t = -TST = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{6.4.92}$$

where T and S are the generators (3.2.2) such that $S^2 = -1$ and $(ST)^3 = 1$.

Under such transformation the NS5 branes remain invariant but each D5 brane turns into a $(1,1)$ brane, as follows

$$\text{NS5} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{D5} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{6.4.93}$$

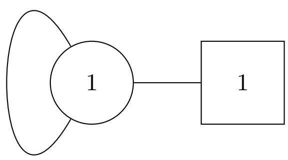
Upon applying the T^t transformation to the brane system of Fig. (6.4.91), we obtain the configuration of Fig. (6.4.94) giving rise to the theory in the third line of (6.4.90).



(6.4.94)

The case of $n = 1$

In the case of $n = 1$, we have seen in Section (6.4.1) that the $[SO(2)_2 \times USp(2)_{-1}]/\mathbb{Z}_2$ theory flows to a theory of two free hypermultiplets. This is indeed dual to the special case of (6.4.91) with $n = 1$, namely one $U(1)$ instanton on $\mathbb{C}^2$ [15] with quiver

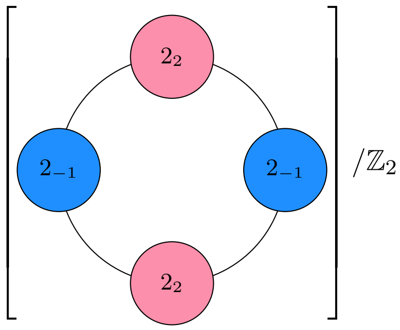


(6.4.95)

where the two free hypermultiplets come from the adjoint hypermultiplet of the $U(1)$ gauge group and the elementary monopole operators $T_{\pm 1}$. By the above argument and the discussion in Section (6.4.1), this is also dual to the ABJM theory $U(1)_1 \times U(1)_{-1}$ and $[U(1)_k \times U(1)_{-k}]/\mathbb{Z}_k$.

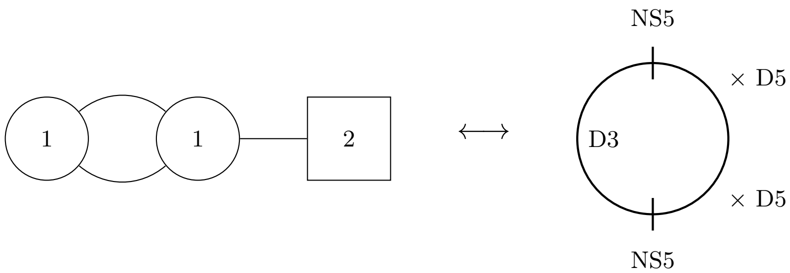
The case of $n = 2$

The case of $n = 2$ requires a separate discussion. We find that the $[SO(2)_2 \times USp(2)_{-1} \times SO(2)_2 \times USp(2)_{-1}]/\mathbb{Z}_2$ theory with quiver



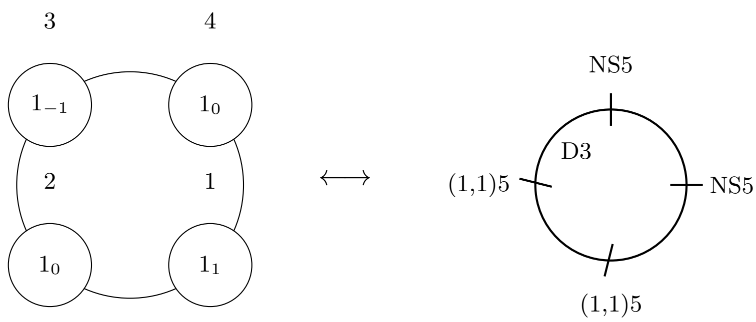
(6.4.96)

is dual to



(6.4.97)

which is the Kronheimer-Nakajima quiver whose Higgs/Coulomb branch describes one $PSU(2) \cong U(2)/U(1)$ instanton on $\mathbb{C}^2/\mathbb{Z}_2$ with the monodromy that preserves the $PSU(2)$ symmetry. This is also dual to the following circular quiver:



(6.4.98)

The duality between (6.4.97) and (6.4.98) can be realised by applying the action T^t (6.4.92) on the brane system as discussed above. The $\mathcal{N} = 2$ indices for (6.4.97) and (6.4.98) can be written in terms of the fugacity d of the $U(1)_d$ symmetry and the fugacities p_1, p_2, p_3, p_4 of the $SU(2)^4$ global symmetry as follows:

$$\begin{aligned}
1 + x & \left[d^{-2} \sum_{i=1}^{2} \chi_{[2]}^{SU(2)}(p_i) + d^{2} \sum_{i=3}^{4} \chi_{[2]}^{SU(2)}(p_i) \right] \\
& + x^2 \left[d^{-4} \sum_{i=1}^{2} \chi_{[4]}^{SU(2)}(p_i) + d^{4} \sum_{i=3}^{4} \chi_{[4]}^{SU(2)}(p_i) \right. \\
& \quad + d^{-4} \chi_{[2]}^{SU(2)}(p_1) \chi_{[2]}^{SU(2)}(p_2) + \chi_{[2]}^{SU(2)}(p_1) \chi_{[2]}^{SU(2)}(p_4) \\
& \quad + d^{4} \chi_{[2]}^{SU(2)}(p_3) \chi_{[2]}^{SU(2)}(p_4) + \chi_{[2]}^{SU(2)}(p_3) \chi_{[2]}^{SU(2)}(p_2) \\
& \quad \left. - \left(\sum_{i=1}^{4} \chi_{[2]}^{SU(2)}(p_i) \right) - 2 \right] + \ldots
\end{aligned} \tag{6.4.99}$$

where the origin of the each $U(1)_d \times SU(2)_{p_i}$ in each theory is as follows.

For (6.4.97), $U(1)_d$ is identified with the axial symmetry that assigns charges -1 to each chiral multiplet and $+2$ to the scalar fields in the vector multiplet, $SU(2)_{p_1}$ can be identified with the flavour symmetry that exchanges the two bifundametal hypermultiplets, $SU(2)_{p_2}$ can be identified with the flavour symmetry of the two fundamental hypermultiplets denoted by the square node, $SU(2)_{p_3}$ can be identified with the enhanced $U(1)$ topological symmetry of the left gauge node, and $SU(2)_{p_4}$ can be identified with the enhanced $U(1)$ diagonal subgroup of the $U(1) \times U(1)$ symmetry of the left and right gauge nodes.[22]. Since the theory is self-mirror, the index is invariant under the simultaneous exchange of $d \leftrightarrow d^{-1}$ and $(p_1, p_2) \leftrightarrow (p_3, p_4)$

For (6.4.98), let $w_1, \ldots, w_4$ be fugacities for topological symmetries of node 1 to 4 and let c_i, c_i^{-1} to be the fugacities for the $U(1)$ symmetry that gives charge $+1$ and -1 to the chiral multiplets $Q_i, \widetilde{Q}_i$ carrying gauge charges $(1,-1), (-1,1)$ between the i-th and the $(i+1)$-th nodes. Then, we have the following fugacity maps:

$$p_1^2 = w_4, \quad p_2^2 = \frac{c_1 c_2}{w_1 w_2 w_3 w_4} \ , \quad p_3^2 = c_3 c_4 (w_1 w_2 w_3 w_4) \ , \quad p_4^2 = w_2 \ . \tag{6.4.100}$$

In other words, the $U(1)$ topological symmetries of the two nodes with zero CS levels get enhanced to $SU(2)$. The operators associated with the currents of the $SU(2)_{p_2}$ and $SU(2)_{p_3}$ flavour symmetries are, respectively, the dressed monopole operators:

$$\begin{aligned}
& T_{\{-1;-1;-1;-1\}} Q_1 Q_2 \ , \qquad T_{\{+1;+1;+1;+1\}} \widetilde{Q}_1 \widetilde{Q}_2 \ , \\
& T_{\{+1;+1;+1;+1\}} Q_3 Q_4 \ , \qquad T_{\{-1;-1;-1;-1\}} \widetilde{Q}_3 \widetilde{Q}_4 \ .
\end{aligned} \tag{6.4.101}$$

For (6.4.98), the $U(1)_d$ symmetry assigns the charges -1, -1, $+1$, $+1$ to the $(Q_1, \widetilde{Q}_1)$, $(Q_2, \widetilde{Q}_2)$, $(Q_3, \widetilde{Q}_3)$, $(Q_4, \widetilde{Q}_4)$, respectively. Matching of the unrefined indices of theories (6.4.97) and (6.4.98) is demonstrated in [90, (5.3)]:

$$1 + 12x + 42x^2 + 48x^3 + 115x^4 + \ldots \ . \tag{6.4.102}$$

The $\mathcal{N} = 3$ indices of (6.4.97) and (6.4.98) can be obtained from (6.4.99) by setting $d = 1$.

[22]Note that the monopole operators $T_{\{1;0\}}$ and $T_{\{1;1\}}$ have R-charge 1. The former corresponds to $SU(2)_{p_3}$ symmetry and the latter corresponds to the $SU(2)_{p_4}$ symmetry.

Let us now discuss the theory (6.4.96). As usual, not all symmetries of the unitary quivers (6.4.97) and (6.4.98) are manifest in the orthosymplectic quiver (6.4.96). The index of the theory (6.4.96) can be obtained from (6.4.99) by setting $p_3 = p_2$ and $p_4 = p_1$, where the origin of $U(1)_d$, $SU(2)_{p_1}$ and $SU(2)_{p_2}$ can be explained as follows. Let ζ_1 and ζ_2 be the fugacities for the $U(1)$ magnetic symmetries for the first and the third $SO(2)$ gauge group respectively. Let g be a $\mathbb{Z}_2$ zero-form symmetry arising from the $\mathbb{Z}_2$ discrete gauging, so that $g^2 = 1$. If we denote the half-hypermultiplets in the bifundametal representations of the gauge groups in (6.4.96), from left to right, by A_1, A_2, A_3 and A_4, the $U(1)_d$ symmetry assigns the charges $+1$, -1, $+1$ and -1 to them, respectively. In this notation, the index can be written as

$$1 + x\left[2d^{-2} + 2d^2 + g \sum_{s_1,s_2,s_3=\pm 1} d^{2s_1} \zeta_1^{\frac{1}{2}s_2} \zeta_2^{\frac{1}{2}s_3}\right] + \dots \,. \tag{6.4.103}$$

However, the index does not really depend on g, since it can be absorbed into a fugacity for the magnetic symmetry. In particular, $\zeta_{1,2}$ and g are related to the fugacities $p_{1,2}$ as follows:

$$\zeta_1^{1/2} = g p_1 p_2 \ , \qquad \zeta_2^{1/2} = p_1^{-1} p_2 \ . \tag{6.4.104}$$

Using this fugacity map, we obtain (6.4.99) with $p_3 = p_2$ and $p_4 = p_1$, as required.

The case of $n = 3$

The $\mathcal{N} = 2$ index of the unitary theories in the second and third lines of (6.4.90) can be written as

$$\begin{aligned}
&1 + x(3d^2 + 3d^{-2}) + x^{\frac{3}{2}}\Big[d^{-3}\Big(p_1p_2p_3 + p_1^{-1}p_2^{-1}p_3^{-1} + \sum_{i=1}^{3}(p_i + p_i^{-1})\Big) \\
&+ d^3\Big(w_1w_2w_3 + w_1^{-1}w_2^{-1}w_3^{-1} + \sum_{i=1}^{3}(w_i + w_i^{-1})\Big)\Big] \\
&+ x^2\Big[-3 + (6d^4 + 6d^{-4}) + d^{-4}\sum_{1\le i<j\le 3}(p_ip_j + p_i^{-1}p_j^{-1}) \\
&+ d^4 \sum_{1\le i<j\le 3}(w_iw_j + w_i^{-1}w_j^{-1})\Big] + \dots
\end{aligned} \tag{6.4.105}$$

where the theory has a $U(1)^6 \times U(1)_d$ global symmetry, where the fugacities for $U(1)^6$ are denoted by $p_{1,2,3}$ and $w_{1,2,3}$. The $\mathcal{N} = 3$ index can be obtained by setting $d = 1$.

For the theory (6.4.91) with $n = 3$, the $U(1)_d$ symmetry corresponds to the axial symmetry that gives that assigns charges -1 to each chiral multiplet and $+2$ to the scalar fields in the vector multiplet; the fugacities $w_{1,2,3}$ correspond to the $U(1)^3$ topological symmetry; and the fugacities $p_{1,2,3}$ correspond to the $U(1)^3$ flavour symmetry. Since the theory is self-mirror, the index is invariant under the simultaneous exchange of $d \leftrightarrow d^{-1}$ and $(p_1, p_2, p_3) \leftrightarrow (w_1, w_2, w_3)$. To specify our parametrisation of p_1, p_2, p_3, let us first define c_i, c_i^{-1} to be the fugacities for the $U(1)$ symmetry that gives charge $+1$ and -1 to the chiral multiplets $Q_i, \widetilde{Q}_i$ carrying gauge charges

$(1,-1),(-1,1)$ between the i-th and the $(i+1)$-th gauge nodes, and let f_i, f_i^{-1} be the flavour charges of the fundamental chiral multiplets carrying gauge charge -1 and $+1$ under the i-th gauge node. Then, $p_{1,2,3}$ are related to these fugacities as

$$p_1 = f_1\, c_1\, f_2^{-1}, \qquad p_2 = f_2\, c_2\, f_3^{-1}, \qquad p_3 = f_3\, c_3\, f_1^{-1} \; . \tag{6.4.106}$$

For the theory on the third line of (6.4.90), namely the circular unitary quiver with alternating CS levels, we label the nodes as $1,\ldots,6$ from left to right. The $U(1)_d$ assigns alternating charges $(-1)^{i+1}$ to the chiral multiplets $(Q_i, \widetilde{Q}_i)$ in the bifundamental representation of the i-th and the $(i+1)$-th gauge nodes. Let us define c_i (with $i = 1,2,\ldots,6$) as above. Then, $p_{1,2,3}$ are related to these fugacities as

$$\begin{array}{lll} p_1 = \frac{c_1}{w_1 w_2} \;, & p_2 = \frac{c_3}{w_3 w_4} \;, & p_3 = \frac{c_5}{w_5 w_6} \\ w_1 = c_2 w_2 w_3 \;, & w_2 = c_4 w_4 w_5 \;, & w_3 = c_6 w_6 w_1 \;. \end{array} \tag{6.4.107}$$

where w_i (with $i = 1,2,\ldots,6$) the topological symmetry associated with the i-th node.

As usual, not all symmetries of these unitary quivers are manifest in the orthosymplectic quiver in the first line of (6.4.90). In fact, the index of the latter can be obtained from (6.4.105) by setting $w_i = p_i$, with $i = 1,2,3$. Indeed, if we denote by ζ_1,ζ_2,ζ_3 the $U(1)^3$ magnetic symmetry associated with each $SO(2)$ gauge group from left to right, we then have the fugacity map

$$\zeta_1^{1/2} = g p_2^{1/2} p_3^{1/2} \;, \qquad \zeta_2^{1/2} = p_1^{1/2} p_2^{1/2} \;, \qquad \zeta_3^{1/2} = p_1^{1/2} p_3^{1/2} \;, \tag{6.4.108}$$

where g is the fugacity associated with a $\mathbb{Z}_2$ zero-form symmetry associated with the $\mathbb{Z}_2$ discrete quotient in the first line of (6.4.90) such that $g^2 = 1$. We emphasise that g can be absorbed in a redefinition of a fugacity of the magnetic symmetry and so the index does not really depend on g. The $U(1)_d$ symmetry assigns the charges $(-1)^{i+1}$ to the half-hypermultiplets A_i, with $i = 1,\ldots,6$, in the bifundamental representation of $SO(2)\times USp(2)$ from left to right in the circular quiver in the first line of (6.4.90).

This discussion can be generalised in a straightforward manner to the cases of $n > 3$.

6.4.7 $(S)O(2N+1)_2 \times USp(2N)_{-1}$ and $[U(N+1)_4 \times U(N)_{-4}](/\mathbb{Z}_2)$

In this subsection, we demonstrate that the indices of the following four theories are equal:

$$\begin{array}{ll} O(2N+1)_2 \times USp(2N)_{-1} & U(N+1)_4 \times U(N)_{-4} \\ SO(2N+1)_2 \times USp(2N)_{-1} & [U(N+1)_4 \times U(N)_{-4}]/\mathbb{Z}_2 \end{array} \tag{6.4.109}$$

The duality of the theories in the first line were pointed out in [1]. The one-form symmetry of each theory in the first line is $\mathbb{Z}_2$, which can be realised as follows. For the orthosymplectic quiver in the first line, the $O(2N+1)$ and $USp(2N)$ gauge groups both have a $\mathbb{Z}_2$ centre and the bifundamental matter screens a diagonal combination, so we are left with one $\mathbb{Z}_2$ centre symmetry. For the unitary quiver in the first line, namely $U(N+1)_4 \times U(N)_{-4}$, the presence of the $\mathbb{Z}_2$ one-form symmetry was pointed out in [153, Section 3.3].

The theories in the second line arise from gauging the $\mathbb{Z}_2$ one-form symmetries of the theories on the first line. This is consistent because the conditions (6.4.78) are satisfied. We thus expect that the theories in the second line are also dual to each other. However, what is surprising is that all of the four theories have the same indices. Let us demonstrate this point as follows.

We first provide an argument to show that the indices of the $U(N+1)_4 \times U(N)_{-4}$ theory and the $[U(N+1)_4 \times U(N)_{-4}]/\mathbb{Z}_2$ theory are equal. We emphasise that, in each of these theories, there is an overall $U(1)$ that does not act on matter fields. Upon integrating over such a $U(1)$ fugacity in the index, we obtain a delta-function which imposes the following condition that the magnetic fluxes of the $U(N+1)$ gauge group, $m_L^{(i)}$, with $i = 1, \dots, N+1$, and those of the $U(N)$ gauge group, $m_R^{(j)}$, with $j = 1, \dots, N$:

$$\sum_{i=1}^{N+1} m_L^{(i)} = \sum_{j=1}^{N} m_R^{(j)} \tag{6.4.110}$$

In the $[U(N+1)_4 \times U(N)_{-4}]/\mathbb{Z}_2$ theory, we have to sum over the fluxes

$$(m_L^{(1)}, \dots, m_L^{(N+1)}; m_R^{(1)}, \dots, m_R^{(N)}) \in (\mathbb{Z} + p/2)^{2N+1} \tag{6.4.111}$$

and sum over $p \in \{0, 1\}$, whereas in $U(N+1)_4 \times U(N)_{-4}$ theory there is a contribution only from the $p = 0$ sector. Observe that, for $p = 1$, if one of the two sides of (6.4.110) is half-integral the other is integral.[23] This means that there is no contribution from the $p = 1$ sector to the index, since it is forbidden by (6.4.110). As a result, the index of the $[U(N+1)_4 \times U(N)_{-4}]/\mathbb{Z}_2$ is the same as that of the $U(N+1)_4 \times U(N)_{-4}$ theory. We see that the $\mathbb{Z}_2$ zero-form symmetry arises from the $\mathbb{Z}_2$ discrete gauging of the former theory acts trivially on the theory and hence it is an unfaithful symmetry. This leads us to conclude that the $\mathbb{Z}_2$ one-form symmetry of the $U(N+1)_4 \times U(N)_{-4}$ theory also acts trivially on the line operators.

Similarly, we can provide an argument to show that the zero-form charge conjugation symmetry of the $SO(2N+1)_2 \times USp(2N)_{-1}$ theory acts trivially on the theory and hence it is unfaithful. The index of this theory can be written as

$$\begin{aligned}
&\mathcal{I}_{SO(2N+1)_{2k} \times USp(2N')_{k'}} \\
&= \frac{1}{2^N N!} \sum_{\boldsymbol{m} \in \mathbb{Z}^N} \oint \prod_{a=1}^{N} \frac{\mathrm{d}z_a}{2\pi i z_a} \prod_{a=1}^{N} x^{-|m_a|} (1 - \chi(-1)^{m_a} x^{|m_a|} z_a^{\pm 1}) \\
&\times \prod_{a<b}^{N} x^{-|\pm m_a + m_b|} (1 - (-1)^{\pm m_a \pm m_b} x^{|\pm m_a \pm m_b|} z_a^{\pm 1} z_b^{\pm 1}) \prod_{a=1}^{N} z_a^{2k m_a} \zeta^{m_a} \\
&\times \frac{1}{2^{N'} N'!} \sum_{\boldsymbol{n} \in \mathbb{Z}^{N'}} \oint \prod_{i=1}^{N'} \frac{\mathrm{d}u_i}{2\pi i u_i} \prod_{i=1}^{N'} x^{-|n_i|} (1 - (-1)^{n_i} x^{|n_i|} u_i^{\pm 2})
\end{aligned}$$

[23] This argument can be generalised to any theory of the form $[U(N+x)_{2k} \times U(N)_{-2k}]/\mathbb{Z}_2$ with x odd. In such a theory, the $\mathbb{Z}_2$ zero-form symmetry arising from the discrete gauging acts trivially.

$$
\begin{aligned}
&\times \prod_{i<j}^{N'} x^{-|\pm n_i + n_j|}(1-(-1)^{\pm n_i \pm n_j} x^{|\pm n_i \pm n_j|} u_i^{\pm 1} u_j^{\pm 1}) \prod_{i=1}^{N'} u_i^{2k' n_i} \\
&\times \prod_{i=1}^{N'} x^{\frac{-|u_i|}{2}} \frac{\left((-1)^{n_i} \chi\, u^{\mp 1}\, f^{\mp 1}\, x^{\frac{3}{2}+|n_i|}; x^2\right)_\infty}{\left((-1)^{n_i} \chi\, u^{\pm 1}\, f^{\pm 1}\, x^{\frac{1}{2}+|n_i|}; x^2\right)_\infty} \\
&\times \prod_{a=1}^{N} \prod_{i=1}^{N'} x^{\frac{-|\pm z_a + u_i|}{2}} \frac{\left((-1)^{\pm m_a + n_i} z_a^{\mp 1}\, u^{\mp 1}\, f^{\mp 1}\, x^{\frac{3}{2}+|\pm m_a + n_i|}; x^2\right)_\infty}{\left((-1)^{m_i} z^{\pm 1}\, u^{\pm 1}\, f^{\pm 1}\, x^{\frac{1}{2}+|\pm m_a + n_i|}; x^2\right)_\infty} ,
\end{aligned}
\tag{6.4.112}
$$

where f is the fugacity for the $SU(2)_A$ flavour symmetry, ζ is the fugacity for the $(\mathbb{Z}_2^{[0]})_{\mathcal{M}}$ topological symmetry satisfying $\zeta^2 = 1$ and χ is the fugacity for the zero-form charge conjugation symmetry $(\mathbb{Z}_2^{[0]})_{\mathcal{C}}$. In the problem at hand, we take $N' = N$, $k = 1$ and $k' = -1$, but the following argument holds for general N, k and k'. We claim that the charge conjugation symmetry can be re-absorbed with a gauge transformation. This can be seen in the index from the fact that if we simultaneously rescale

$$
z_a \to \chi z_a, \quad a = 1, \cdots, N \qquad u_i \to \chi u_i, \quad i = 1, \cdots, N' \tag{6.4.113}
$$

then the fugacity χ completely disappears from the matrix integral since χ is a square root of unity $\chi = \mathrm{e}^{i\pi n}$ with $n = 0, 1$. What is crucial for this to happen is that the CS level of the $SO(2N+1)_{2k}$ group is even.[24] This observation leads us to conclude that the $\mathbb{Z}_2$ one-form symmetry of the $O(2N+1)_2 \times USp(2N)_{-1}$ theory also acts trivially on the spectrum of line operators. Since we can obtain the theories in the second line of (6.4.109) from those in the first line by gauging the $\mathbb{Z}_2^{[1]}$ one-from symmetry in the latter, from the perspective of the $U(N+1)_4 \times U(N)_{-4}$ theory, such gauging removes from the spectrum Wilson lines in representations that are not multiple of 2 of $(\mathbf{N+1}, \mathbf{N})$. We thus conjecture that there exist only the Wilson lines in the representation $((\mathbf{N+1})^{2\mathfrak{m}}, \mathbf{N}^{2\mathfrak{m}})$, with $\mathfrak{m} \geq 1$, in the spectrum of this theory, and so the action of such a $\mathbb{Z}_2^{[1]}$ one-form symmetry is trivial. We leave the verification of this statement to future work.

As a final remark, we see that the four theories in (6.4.109) seem to be dual to each other, even though the $\mathbb{Z}_2$ one-form symmetry seems to be present in the theories in the first line of (6.4.109), but not in the theories in second line. One might ask if there exists a topological field theory that provides the $\mathbb{Z}_2$ one-form symmetry in the former. The answer seems to be no. This is in contrast with, for example, the duality appetiser [119], which is a duality between the 3d $\mathcal{N} = 2$ $SU(2)_1$ gauge theory with one adjoint chiral multiplet and a free chiral multiplet together with a topological quantum field theory (TQFT) given by $U(1)_{-2}$. Indeed, the $SU(2)_1$ gauge theory has a $\mathbb{Z}_2$ one-form symmetry (as it can be seen from the centre of the gauge group), whereas the theory of a free chiral multiplet does not have any one-form symmetry; in this case the $\mathbb{Z}_2$ one-form symmetry is provided by the TQFT $U(1)_{-2}$. The latter can be detected by the index by turning on an appropriate background magnetic flux, which is the one associated with the $U(1)$ flavour symmetry, as we demonstrate in Appendix (D). However, upon turning on background magnetic fluxes for the theories on the first line of (6.4.109), we are not able to detect the presence of the TQFT that

[24] Indeed, if we consider the index of $SO(2N+1)_{2k+1} \times USp(2N')_{k'}$, the CS factor $\prod_{a=1}^{N} z_a^{(2k+1)m_a}$ would produce odd powers of χ after the shift (6.4.113), thus leaving a non-trivial χ dependence.

supports the $\mathbb{Z}_2$ one-form symmetry. We thus conclude that such a symmetry acts trivially on the spectrum of the line operators.

Let us report the index of the theories (6.4.109) when $N = 1$. It turns out that, up to order x^2, those of the unitary theories are given by (6.4.86), and those of the orthosymplectic theories are given by (6.4.87). Note, however, that from order x^5 onwards, they are different; see [54, Table 1] for the unrefined indices of these theories:

$$\begin{aligned} (6.4.109)_{N=1}: \qquad & 1 + 4x + 12x^2 + 8x^3 + 27x^4 + 32x^5 + \dots \\ (6.4.89): \qquad & 1 + 4x + 12x^2 + 8x^3 + 27x^4 + 36x^5 + \dots \end{aligned} \tag{6.4.114}$$

As a final remark, we also observe that the circular quivers $SO(3)_2 \times USp(2)_{-1} \times SO(3)_2 \times USp(2)_{-1}$ and $O(3)_2 \times USp(2)_{-1} \times O(3)_2 \times USp(2)_{-1}$ have the same indices; up to order x^2, they are

$$\begin{aligned} 1 + x^2 \Big[&(\zeta_1 + \zeta_2 + \zeta_1\zeta_2 + 2)\chi^{SU(2)}_{[4]}(f) + \zeta_1\zeta_2 + 1 \\ &- (\zeta_1 + \zeta_2 + \zeta_1\zeta_2 + 2)\chi^{SU(2)}_{[2]}(f) \Big] + \dots, \end{aligned} \tag{6.4.115}$$

where $\zeta_{1,2}$ are fugacities for the magnetic symmetry of the (special)orthogonal gauge groups. For the theory with special orthogonal gauge groups, the index does not depend on the fugacity for the charge conjugation symmetry. For reference, we report the unrefined index up to order x^4 as follows:

$$1 + 12x^2 + 4x^4 + \dots \ . \tag{6.4.116}$$

Chapter 7

Conclusions and perspectives

In this final chapter we want to collect the main results of the original parts of the work and point out possible future directions and leftover open problems. The aim of the this thesis was to use the three-dimensional indices, such as the Hilbert series and the superconformal index, to study different properties of a variety of SCFTs from several perspectives.

In Chapter (3) we propose a description of the three-dimensional mirror dual theories of the circle reduction of the four-dimensional twisted $\chi(\mathfrak{a}_{2N})$ theories of class-S. This is an "almost" star-shaped quiver with the central gauge node being $USp(2N)$. In checking such proposals, we compute the Hilbert series for both the Higgs and Coulomb branches of such mirror theories and compare them to those of the four-dimensional ones. We show, in fact, that the Higgs branch of the 4d SCFT matches with the Coulomb branch of the corresponding 3d mirror theory. Furthermore, the quaternionic dimension of the Higgs branch of such mirror theories matches with the rank of the 4d SCFT. In many cases, there are more than one description of the mirror theory, where one is constructed using the proposal of this paper and the other involves only unitary gauge groups. One of the important features of these dualities is that in many cases not all Coulomb branch symmetries of the 3d mirror quiver is manifest in the quiver itself; in other word, one cannot turn on in the Hilbert series all of the fugacities associated with the full global symmetry of the SCFT in the IR.

Let us discuss some of the open questions that arise from these findings. First of all, some of the quiver descriptions that we proposed are "bad" theories in the sense of [85]. Even though we manage to use such a description to compute various quantities, such as the Coulomb branch dimension and the Higgs branch Hilbert series, it would be nice to come up with a "good" description for such theories. Secondly, it would be nice to understand better the dualities between different descriptions of the mirror theory of the S^1 reduction of the same 4d SCFT, such that as how to "derive" one description from the others.

In Chapter (5), thanks to the superconformal index, we investigate the $\mathcal{N}=2$ preserving exactly marginal operators of two different families of 3d S-fold SCFTs. One such family is constructed by gauging the diagonal flavour symmetry of the $T(U(2))$ and $T(U(3))$ theories, and the other one by gauging the diagonal flavour symmetry of the $T^{[2,1^2]}_{[2,1^2]}(SU(4))$ theory. In both cases, it is possible to turn on a Chern–Simons level for each gauge group and to couple to each theory various numbers of hypermultiplets. This detailed analysis, allows us to determine whether supersymmetry gets enhanced in the infrared and to deduce the amount of supersymmetry of the corresponding SCFT.

The results for the $T^{[2,1^2]}_{[2,1^2]}(SU(4))$ theory are just preliminary results since, despite using other known theories as a guide, we do not have a full understanding of the chiral ring of the theory. In this sense, one appealing line of possible future investigation would be to better understand the chiral ring structure of this theory. Furthermore, one could study other specific cases of S-fold theories, changing the $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ partitions along with N.

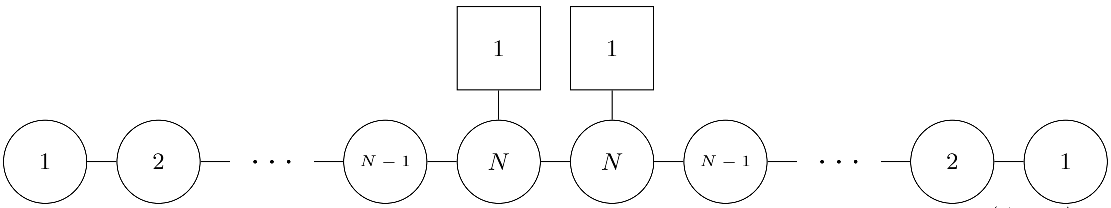
In Chapter (6) we have obtained several new dualities between ABJ and related theories, with at least $\mathcal{N} = 6$ supersymmetry, by gauging zero-form or one-form symmetries. We analysed in details the symmetries of these theories and how they are mapped across each duality, paying particular attention on the discrete symmetries. This result is also generalised to a circular quiver with alternating $SO(2)_2$ and $USp(2)_{-1}$ gauge groups and a discrete $\mathbb{Z}_2$ quotient.

There are several interesting directions for further study. First, it would be interesting to generalise these results to theories with orthosymplectic gauge groups with $\mathcal{N} = 5$ supersymmetry, as well as more general $U(N + x)_k \times U(N)_{-k}$ and $[U(N + x)_k \times U(N)_{-k}]/\mathbb{Z}_p$ theories with $\mathcal{N} = 6$ supersymmetry. Moreover, regarding the duality involving the circular quivers, it would be nice to find the analog for the higher ranks theories, such as those involving $SO(2N)_2$ and $USp(2N)_{-1}$ gauge groups.

Appendix A

Reduction of the $D_2[SU(2N+1)]$ theory on S^1

The 4d $\mathcal{N}=2$ $D_2[SU(2N+1)]$ theory was first studied in [49, 50]. For $N=1$, the $D_2[SU(3)]$ theory is simply the (A_1, D_4) Argyres-Douglas theory. Upon reduction on S^1 to 3d, the mirror theory is described by the following quiver [169, Figure 3]:



(A.0.1)

The quaternionic dimension of the Coulomb branch of (A.0.1) is $2\sum_{j=1}^{N} j = N(N+1)$, in agreement with the dimension of the Higgs branch of the 4d $D_2(SU(N+1))$ theory, which is given by

$$\begin{aligned} \dim_{\mathbb{H}} \mathcal{H}[D_2(SU(2N+1))] &= 24(c-a) \\ &= 24\left[\frac{1}{3}N(N+1) - \frac{7}{24}N(N+1)\right] \\ &= N(N+1) \ , \end{aligned} \tag{A.0.2}$$

where $a = \frac{7}{24}N(N+1)$ and $c = \frac{1}{3}N(N+1)$ are the conformal anomalies [50]. The Higgs branch of (A.0.1) is $2\sum_{j=1}^{N-1} j(j+1) + N^2 + 2N - 2\sum_{j=1}^{N} j^2 = N$ quaternionic dimensional; this is in agreement with the fact that the $D_2[SU(2N+1)]$ theory is a rank N theory.

The Coulomb branch and Higgs branch Hilbert series can be computed as described in the main text. The Coulomb branch symmetry is $SU(2N+1)$, whereas the Higgs branch symmetry is $U(1)$. In this work we focus mainly on the case of $N=1,2$. The case of $N=1$ was discussed in the main text in the context of the (A_1, D_4) theory. For $N=2$, the highest weight generating function of the Coulomb branch Hilbert series of theory (A.0.1) admits the following simple closed form:

$$\begin{aligned} \text{HWG}\left[H_{\mathcal{C}}[(A.0.1)_{N=2}]\right] &= \text{HWG}[H_{\mathcal{H}}[D_2[SU(5)]]] \\ &= \text{PE}\left[t^2\mu_1\mu_4 + t^4\mu_2\mu_3\right] \ . \end{aligned} \tag{A.0.3}$$

If we set the fugacities in the $SU(5)$ characters to unity, we obtain the closed form for the following unrefined Coulomb branch Hilbert series for $N=2$:

$$\frac{1+12t^2+53t^4+88t^6+53t^8+12t^{10}+t^{12}}{(1-t)^{12}(1+t)^{12}} \ . \tag{A.0.4}$$

Observe that the order of the pole at $t=1$ is 12, equal to the complex dimension of the Coulomb branch.

Appendix B

Monopole operators in some 3d $\mathcal{N} = 4$ gauge theories

In this section, we analyse the Coulomb branch operators of two 3d $\mathcal{N} = 4$ gauge theories, namely the $U(N)$ gauge theory (with $N = 2, 3$) with one adjoint and one fundamental hypermultiplets and the $U(2)$ gauge theory with four flavours, using the indices and Coulomb branch Hilbert series. The aim is to write down explicitly the Coulomb branch operators with R-charges up to 2 and their relations. These turn out to be extremely useful in drawing an analogy with operators in the S-fold theories discussed in the main text.

B.1 $U(2)$ and $U(3)$ gauge theories with one adjoint and one fundamental hypermultiplets

Let us first consider the $U(2)$ gauge group. The index of this theory is

$$\begin{aligned} &1 + x^{\frac{1}{2}}(d[1]_w + d^{-1}[1]_c) + x(2d^2[2]_w + 2[1]_w[1]_c + 2d^{-2}[2]_c) \\ &+ x^{\frac{3}{2}}\Big[d^3(2[3]_w + [1]_w) + 3d[2]_w[1]_c + 3d^{-1}[2]_c[1]_w + d^{-3}(2[3]_c + [1]_c)\Big] \\ &+ x^2\Big[d^4(3[4]_w + [2]_w + 1) + 4d^2[3]_w[1]_c + (d \to d^{-1}, w \leftrightarrow c) + 5[2]_w[2]_c \\ &\qquad - [2]_c - [2]_w - 2\Big] + \ldots \end{aligned} \tag{B.1.1}$$

The terms at order $x^{\frac{1}{2}}$ indicate that the theory contains two free hypermultiplets, and so the above expression can be rewritten as

$$\begin{aligned} &\mathcal{I}_{\text{free}}(x; cd^{-1})\,\mathcal{I}_{\text{free}}(x; c^{-1}d^{-1})\,\mathcal{I}_{\text{free}}(x; wd)\,\mathcal{I}_{\text{free}}(x; w^{-1}d) \\ &\times \Big[1 + x\left(d^2[2]_w + [1]_w[1]_c + d^{-2}[2]_c\right) + x^2\Big(d^4[4]_w + d^2[3]_w[1]_c + \\ &\quad + d^{-4}[4]_c + d^{-2}[3]_c[1]_w + [2]_w[2]_c \\ &\quad - d^2[1]_w[1]_c - d^{-2}[1]_w[1]_c - [2]_w - [2]_c - 1\Big) + \ldots\Big] \end{aligned} \tag{B.1.2}$$

where $\mathcal{I}_{\text{free}}(x; \omega)$ is defined in (5.3.73). In fact, this index can be rewritten in terms of characters of $SU(4)$ representations as

$$\begin{aligned} &\mathcal{I}_{\text{free}}(x; cd^{-1})\,\mathcal{I}_{\text{free}}(x; c^{-1}d^{-1})\,\mathcal{I}_{\text{free}}(x; wd)\,\mathcal{I}_{\text{free}}(x; w^{-1}d) \\ &\times \Big[1 + [2,0,0]x + \Big([4,0,0] - [1,0,1]\Big)x^2 + \ldots\Big]\,, \end{aligned} \tag{B.1.3}$$

where we have used the following decompositions of representations of $SU(4)$ into $SU(2)_w \times SU(2)_c \times U(1)_d$:

$$\begin{aligned}
[2,0,0] &\longrightarrow [2;0]_{+2} + [1;1]_0 + [0;2]_{-2} \\
[4,0,0] &\longrightarrow [4;0]_{+4} + [3;1]_{+2} + [2;2]_0 + [1;3]_{-2} + [0;4]_{-4} \\
[1,0,1] &\longrightarrow [1;1]_{+2} + [2;0]_0 + [0;0]_0 + [0;2]_0 + [1;1]_{-2} \ .
\end{aligned} \tag{B.1.4}$$

Let us discuss (B.1.2) from the perspective of the $\mathcal{N}=3$ index, in which case we have to set $d=1$. The index can then be rewritten in terms of characters of $USp(4) \cong Spin(5)$ representations as follows:

$$\begin{aligned}
&\mathcal{I}_{\text{free}}(x;c)\,\mathcal{I}_{\text{free}}(x;c^{-1})\,\mathcal{I}_{\text{free}}(x;w)\,\mathcal{I}_{\text{free}}(x;w^{-1}) \\
&\times \Big[1 + [0,2]x + x^2\left([0,4] - [0,2] - [1,0]\right) + \dots\Big] \ .
\end{aligned} \tag{B.1.5}$$

The $\mathcal{N}=3$ flavour current is in the adjoint representation $[0,2]$ of $Spin(5)$. We indicate its contribution to the index in blue. The brown negative term at order x^2 in (B.1.2) implies that there are five extra SUSY conserved currents in the vector representation $[1,0]$ of $Spin(5)$. We thus conclude that the interacting SCFT part of this theory has $\mathcal{N}=3+5=8$ enhanced supersymmetry, in agreement with [124, Section 5.1]. Indeed, the symmetry $Spin(5)$ is the commutant of the $\mathcal{N}=3$ R-symmetry $Spin(3)$ in the $\mathcal{N}=8$ R-symmetry $Spin(8)$. Another way to see this is to view (B.1.3) as an $\mathcal{N}=2$ index, in which the $SU(4) \cong Spin(6)$ global symmetry is manifest. This is actually the commutant of the $\mathcal{N}=2$ R-symmetry $Spin(2)$ in $Spin(8)$, which is the R-symmetry of an $\mathcal{N}=8$ SCFT.

We remark that, in (B.1.1), we include the contribution from the free hypermultiplets. In particular they contribute negative terms $-(d[1]_w + d^{-1}[1]_c)$ at order $x^{3/2}$ and $-([2]_w + d^2[1]_w[1]_c + d^{-2}[1]_w[1]_c + [2]_c + 2)$ at order x^2; see (5.3.73). These can combine with the contribution of the interacting SCFT part and cancel that of the operators constructed from products with the aforementioned free fields.

We denote the monopole operator with flux (m,n) by $X_{(m,n)}$, which carries topological charge $m+n$ and R-charge $\frac{1}{2}(|m|+|n|)$. Note that one can always use the Weyl symmetry of $U(2)$ to arrange the flux into the form $m \geq n > -\infty$. As in the main text, we use the following shorthand notations below:

$$X_\pm := X_{(\pm 1,0)} \ , \qquad X_{++} := X_{(1,1)} \ , \qquad X_{--} := X_{(-1,-1)} \ . \tag{B.1.6}$$

In the following analysis we focus on the Coulomb branch operators. Up to order x^2, these correspond to the terms with the highest power of d in (B.1.1). Another convenient way is to compute a quantity that counts such operators, known as Coulomb branch Hilbert series, which can be regarded as a limit of the index (see (3.41) of [144]). For the theory in question, the Hilbert series is computed in section 4.1 of [64]:

$$\begin{aligned}
&\sum_{m \geq n > -\infty} x^{\frac{1}{2}(|m|+|n|)} P_{U(2)}(x;m,n) w^{m+n} \\
&= \mathrm{PE}\left[x^{\frac{1}{2}}[1]_w + x[2]_w - x^2\right] \\
&= 1 + x^{\frac{1}{2}}[1]_w + 2x[2]_w + x^{\frac{3}{2}}(2[3]_w + [1]_w) + x^2(3[4]_w + [2]_w + 1) + \dots \ ,
\end{aligned} \tag{B.1.7}$$

with

$$P_{U(2)}(x;m,n) = \begin{cases} (1-x)^{-2} \ , & m \neq n \\ (1-x)^{-1}(1-x^2)^{-1} \ , & m = n \end{cases} \tag{B.1.8}$$

The second line of (B.1.7) indicates that the Coulomb branch is isomorphic to $\mathbb{C}^2 \times (\mathbb{C}^2/\mathbb{Z}_2)$.

The Coulomb branch operators that carry R-charge 1/2 are the monopole operators with fluxes $(\pm 1, 0)$

$$[1]_w : \qquad X_+ \ , \quad X_- \tag{B.1.9}$$

They parametrise the $\mathbb{C}^2$ factor of the Coulomb branch and decouple as a free hypermultiplet. These correspond to the term $x^{\frac{1}{2}}[1]_w$ inside the PE in (B.1.7).

The Coulomb branch operators with R-charge 1 are

$$\begin{array}{llll} [2]_w : & X_{++} \ , & (\mathrm{tr}\,\varphi) \ , & X_{--} \\ [2]_w : & X_+^2 \ , & X_+X_- \ , & X_-^2 \ . \end{array} \tag{B.1.10}$$

It should be noted that $X_+X_- = X_{(1,0)}X_{(-1,0)} = X_{(1,0)}X_{(0,-1)}$ is not subject to any relation and is an independent operator; it can be identified with the monopole operator with flux $(1,-1)$. The quantities in the first line are generators of the Coulomb branch, corresponding to the term $x[2]_w$ inside the PE in (B.1.7).

The Coulomb branch operators with R-charge 3/2 are

$$\begin{array}{lllll} [3]_w : & X_+^3 \ , & X_+^2 X_- \ , & X_+ X_-^2 \ , & X_-^3 \\ [3]_w : & X_{++}X_+ \ , & X_{++}X_- \ , & X_{--}X_+ \ , & X_{--}X_- \\ [1]_w : & X_+(\mathrm{tr}\,\varphi) \ , & X_-(\mathrm{tr}\,\varphi) \ . & & \end{array} \tag{B.1.11}$$

The Coulomb branch operators with R-charge 2 are

$$\begin{array}{llll} [4]_w : & X_+^4 \ , & X_+^3 X_- \ , & X_+^2 X_-^2 \ , \\ & & X_+ X_-^3 \ , & X_-^4 \\ [4]_w : & X_{++}^2 \ , & X_{++}(\mathrm{tr}\,\varphi) \ , & X_{++}X_{--} = (\mathrm{tr}\,\varphi)^2 \ , \\ & & X_{--}(\mathrm{tr}\,\varphi) \ , & X_{--}^2 \\ [4]_w : & X_{++}X_+^2 \ , & X_{++}(X_+X_-) \ , & X_{++}X_-^2 = X_+^2 X_{--} \\ & & X_{--}(X_+X_-) \ , & X_{--}X_-^2 \\ [2]_w : & X_+^2(\mathrm{tr}\,\varphi) \ , & X_+X_-(\mathrm{tr}\,\varphi) \ , & X_-^2(\mathrm{tr}\,\varphi) \\ [0]_w : & \mathrm{tr}(\varphi^2) & & \end{array} \tag{B.1.12}$$

where the relation

$$X_{++}X_{--} = (\mathrm{tr}\,\varphi)^2 \tag{B.1.13}$$

is the defining equation of the factor $\mathbb{C}^2/\mathbb{Z}_2$ of the Coulomb branch. Notice that the left hand side $X_{++}X_{--} = X_{(1,1)}X_{(-1,-1)}$ occupies the point $(0,0)$ on the magnetic lattice and so as the right hand side. This relation corresponds to the term $-x^2$ inside the PE in (B.1.7). Moreover, the relation

$$X_{++}X_-^2 = X_+^2 X_{--} \tag{B.1.14}$$

follows from the fact that the monopole operators on the left and right hand sides of the equation occupy the same point $(1,-1)$ in the magnetic lattice.

In the case of the $U(3)$ gauge group, the Coulomb branch Hilbert series reads

$$\begin{aligned} &\mathrm{PE}\left[x^{\frac{1}{2}}[1]_w + x[2]_w + x^{\frac{3}{2}}[3]_w - x^{\frac{5}{2}}[1]_w - x^3[2]_w + \dots\right] \\ &= 1 + x^{\frac{1}{2}}[1]_w + 2x[2]_w + x^{\frac{3}{2}}(3[3]_w + [1]_w) + x^2(4[4]_w + 2[2]_w + 2) + \dots \ . \end{aligned} \tag{B.1.15}$$

The notations need to be slightly modified as follows:

$$X_{\pm} := X_{(\pm1,0,0)} \ , \qquad X_{\pm\pm} := X_{\pm(1,1,0)} \ , \qquad X_{\pm\pm\pm} := X_{\pm(1,1,1)} \ . \tag{B.1.16}$$

As we can see from the above Hilbert series, the generators of the Coulomb branch are the same as for $N = 2$, except that there are additional ones with R-charge $3/2$ in the representation $[3]_w$:

$$[3]_w : \quad X_{+++} \ , \quad X_{+;(0,1)} \ , \quad X_{-;(0,1)} \ , \quad X_{---} \ . \tag{B.1.17}$$

The dressed monopole operators $X_{\pm;(0,1)}$ are as discussed in (5.4) of [64]:

$$X_{\pm;(r,s)} := X_{(\pm1,0,0);(r,s)} = (\pm1,0,0)\phi_1^r(\phi_2^s + \phi_3^s) + \text{permutations} \ , \tag{B.1.18}$$

where along the Coulomb branch φ can be diagonalised as $\mathrm{diag}(\phi_1, \phi_2, \phi_3)$.

B.2 $U(2)$ gauge theory with four flavours of fundamental hypermultiplets

The index of this theory reads

$$\begin{aligned} 1 + x\left(d^2[2]_w + d^{-2}[1,0,1]_{\boldsymbol{f}}\right) + x^2\Big[&d^4([4]_w + [2]_w + 1) + [2]_w[1,0,1]_{\boldsymbol{f}} \\ &+ d^{-4}([2,0,2]_{\boldsymbol{f}} + [0,2,0]_{\boldsymbol{f}}) - [2]_w - [1,0,1]_{\boldsymbol{f}} - 1\Big] + \dots \ . \end{aligned} \tag{B.2.1}$$

The monopole operator $X_{(m,n)}$ with flux (m,n) carries the topological charge $m+n$ and R-charge $2(|m|+|n|) - |m-n|$. The Coulomb branch operators are captured by the highest powers of d at each order of x in the index. The information about the Coulomb branch chiral ring is contained in the Hilbert series, which was discussed in (5.6) of [64]:

$$\begin{aligned} &\sum_{m \geq n > -\infty} x^{2(|m|+|n|)-|m-n|} P_{U(2)}(x;m,n) w^{2(m+n)} \\ &= \mathrm{PE}\left[x[2]_w + x^2[2]_w - x^3 - x^4\right] \\ &= 1 + x[2]_w + x^2([4]_w + [2]_w + 1) + \dots \ . \end{aligned} \tag{B.2.2}$$

The Coulomb branch operators with R-charge 1 are

$$[2]_w : \qquad X_{(1,0)} \ , \quad (\mathrm{tr}\,\varphi) \ , \quad X_{(-1,0)} \ . \tag{B.2.3}$$

These correspond to the term $x[2]_w$ in the PE in (B.2.2).

The Coulomb branch operators with R-charge 2 are

$$
\begin{aligned}
&[4]_w: \quad X_{(1,0)}^2 \ , \quad X_{(1,0)}(\operatorname{tr}\varphi) \ , \quad X_{(1,0)}X_{(-1,0)} \ , \quad X_{(-1,0)}(\operatorname{tr}\varphi) \ , \quad X_{(-1,0)}^2 \\
&[2]_w: \quad X_{(1,0);(0,1)} \ , \quad \operatorname{tr}(\varphi^2) \ , \qquad X_{(-1,0);(0,1)} \\
&[0]_w: \quad (\operatorname{tr}\varphi)^2
\end{aligned}
\tag{B.2.4}
$$

The second line contains the dressed monopole operators, as discussed in (5.4) of [64]:

$$
X_{(\pm 1,0);(r,s)} = (\pm 1, 0)\phi_1^r \phi_2^s + (0, \pm 1)\phi_2^r \phi_1^s \ , \tag{B.2.5}
$$

where along the Coulomb branch φ can be diagonalised as $\mathrm{diag}(\phi_1, \phi_2)$. The quantities in the second line correspond to the term $x^2[2]_w$ inside the PE in (B.2.2). The quantities in the first and third lines of (B.2.4) correspond to the symmetric product $\mathrm{Sym}^2[2] = [4] + [0]$.

In order to understand the relations at order x^3 and x^4, as indicated by the Hilbert series (B.2.2), it is convenient to define the following traceless matrices, containing the generators of the Coulomb branch:

$$
\mathcal{X}_1 := \begin{pmatrix} \operatorname{tr}\varphi & X_{(1,0)} \\ X_{(-1,0)} & -\operatorname{tr}\varphi \end{pmatrix} \ , \qquad \mathcal{X}_2 := \begin{pmatrix} \operatorname{tr}(\varphi^2) & X_{(1,0);(0,1)} \\ X_{(-1,0);(0,1)} & -\operatorname{tr}(\varphi^2) \end{pmatrix} \ , \tag{B.2.6}
$$

each of which transforms in the adjoint representation of $SU(2)$. Similarly to (4.19) and (4.20) of [105], the relations at order x^3 and x^4 can be written respectively as

$$
\begin{aligned}
x^3: \quad & \operatorname{tr}(\mathcal{X}_1\mathcal{X}_2) = 0 \\
& \Leftrightarrow \quad X_{(1,0)}X_{(-1,0);(0,1)} + X_{(-1,0)}X_{(1,0);(0,1)} + 2(\operatorname{tr}\varphi)\operatorname{tr}(\varphi^2) = 0 \ , \\
x^4: \quad & \operatorname{tr}(\mathcal{X}_2^2) + \alpha(\operatorname{tr}\mathcal{X}_1^2)^2 = 0 \\
& \Leftrightarrow \quad X_{(1,0);(0,1)}X_{(-1,0);(0,1)} + [\operatorname{tr}(\varphi^2)]^2 \\
& \qquad\qquad + 2\alpha[X_{(1,0)}X_{(-1,0)} + (\operatorname{tr}\varphi)^2]^2 = 0 \ ,
\end{aligned}
\tag{B.2.7}
$$

where α is a non-zero constant, which can be absorbed by a redefinition of $\mathcal{X}_1$ or $\mathcal{X}_2$.

Appendix C

Consequences of the F-term equations (5.3.44)

In this appendix, we discuss consequences of the F-term equations (5.3.44) on gauge invariant quantities.

It is convenient to define

$$M^i_j := \widetilde{Q}^i_a Q^a_j \ , \qquad (\mu_Q)^a_b = \widetilde{Q}^i_b Q^a_i \tag{C.0.1}$$

so that we have

$$M^i_i = \mathrm{tr}\,\mu_Q \ . \tag{C.0.2}$$

It then follows that

$$\begin{aligned} Q^b_j M^j_i &\overset{(C.0.1)}{=} Q^a_i (\mu_Q)^b_a \overset{(5.3.44)}{=} Q^a_i \left(\frac{k}{2\pi}\varphi - \mu_C - \mu_H\right)^b_{\ a} \overset{(5.3.44)}{=} -Q^a_i (\mu_C + \mu_H)^b_{\ a} \ , \\ \widetilde{Q}^j_b M^i_j &\overset{(C.0.1)}{=} \widetilde{Q}^i_a (\mu_Q)^a_b \overset{(5.3.44)}{=} \widetilde{Q}^i_a \left(\frac{k}{2\pi}\varphi - \mu_C - \mu_H\right)^a_{\ b} \overset{(5.3.44)}{=} -\widetilde{Q}^i_a (\mu_C + \mu_H)^a_{\ b} \ , \end{aligned} \tag{C.0.3}$$

or, equivalently,

$$Q^a_j \left[(\mu_H + \mu_C)^b_a \delta^j_i + M^j_i \delta^b_a\right] = 0 \ , \qquad \widetilde{Q}^j_a \left[(\mu_H + \mu_C)^a_b \delta^i_j + M^i_j \delta^a_b\right] = 0 \ . \tag{C.0.4}$$

Multiplying $\widetilde{Q}^k_b$ to both sides of the first equation in (C.0.4), we obtain

$$M^k_j M^j_i = (M^2)^k_i = -(\mu_H + \mu_C)^b_a \widetilde{Q}^k_b Q^a_i \ . \tag{C.0.5}$$

Contracting the indices k and i, we obtain

$$(M^2)^l_l = \mathrm{tr}(\mu_Q^2) = -(\mu_H + \mu_C)^a_b \widetilde{Q}^l_a Q^b_l = -\,\mathrm{tr}\left[(\mu_H + \mu_C)\mu_Q\right] \ . \tag{C.0.6}$$

For $n \geq 2$, it is convenient to define

$$\widehat{M^i_j} = M^i_j - \frac{1}{n}(M^k_k)\delta^i_j = M^i_j - \frac{1}{n}(\mathrm{tr}\,\mu_Q)\delta^i_j \ . \tag{C.0.7}$$

It satisfies the following identifies:

$$
\begin{aligned}
(\widehat{M^2})^i_j &= (M^2)^i_j - \frac{2}{n}(M^k_k)M^i_j + \frac{1}{n^2}(M^k_k)^2\delta^i_j \ , \\
(\widehat{M^2})^i_i &= (M^2)^j_j - \frac{1}{n}(M^k_k)^2 \ .
\end{aligned}
\tag{C.0.8}
$$

In the special case of $n = 2$, due to the Hamilton–Cayley theorem[1], we also have

$$
(\widehat{M^2})^i_j = \frac{1}{2}(\widehat{M^2})^k_k\,\delta^i_j \ , \qquad \text{for } n = 2 \ . \tag{C.0.9}
$$

Using (C.0.2), (C.0.5), (C.0.6) and (C.0.8), we obtain

$$
\begin{aligned}
(\widehat{M^2})^i_j &= -(\mu_H + \mu_C)^b_a \widetilde{Q}^i_b Q^a_j - \frac{2}{n}\widehat{M}^i_j(\operatorname{tr}\mu_Q) - \frac{1}{n^2}(\operatorname{tr}\mu_Q)^2\delta^i_j \ , \\
(\widehat{M^2})^i_i &= -\operatorname{tr}\left[(\mu_H + \mu_C)\mu_Q\right] - \frac{1}{n}(\operatorname{tr}\mu_Q)^2 \ .
\end{aligned}
\tag{C.0.10}
$$

It is also convenient to define

$$
(\underline{\widehat{M^2}})^i_j := (\widehat{M^2})^i_j - \frac{1}{n}(\widehat{M^2})^k_k\delta^i_j \ . \tag{C.0.11}
$$

Then, from (C.0.10), we have

$$
\begin{aligned}
(\underline{\widehat{M^2}})^i_j &= -(\mu_H + \mu_C)^a_b \widetilde{Q}^i_a Q^b_j + \frac{1}{n}\operatorname{tr}(\mu_H\mu_Q + \mu_C\mu_Q)\delta^i_j - \frac{2}{n}\widehat{M}^i_j(\operatorname{tr}\mu_Q) \\
&= -(\mathcal{A}_H)^i_j - (\mathcal{A}_C)^i_j - \frac{2}{n}\widehat{M}^i_j(\operatorname{tr}\mu_Q) \ ,
\end{aligned}
\tag{C.0.12}
$$

where we define

$$
\begin{aligned}
(\mathcal{A}_H)^i_j &:= (\mu_H)^a_b \widetilde{Q}^i_a Q^b_j - \frac{1}{n}\operatorname{tr}(\mu_H\mu_Q)\delta^i_j \ , \\
(\mathcal{A}_C)^i_j &:= (\mu_C)^a_b \widetilde{Q}^i_a Q^b_j - \frac{1}{n}\operatorname{tr}(\mu_C\mu_Q)\delta^i_j \ .
\end{aligned}
\tag{C.0.13}
$$

Using (C.0.9), we also have

$$
(\underline{\widehat{M^2}})^i_j = 0 \ , \quad \text{for } n = 2 \ , \tag{C.0.14}
$$

and so it follows from (C.0.12) that

$$
(\mathcal{A}_H)^i_j + (\mathcal{A}_C)^i_j = -\widehat{M}^i_j(\operatorname{tr}\mu_Q) = -\widehat{M}^i_j(M^k_k) \ , \quad \text{for } n = 2 \ . \tag{C.0.15}
$$

[1]For a 2×2 matrix A, it satisfies $A^2 - (\operatorname{tr} A)A + \frac{1}{2}\left[(\operatorname{tr} A)^2 - \operatorname{tr}(A^2)\right]\mathbf{1}_{2\times 2} = 0$.

Appendix D

Topological sectors and indices: the example of the duality appetiser

In Section 6.4.7 we mentioned that the 3d index can be useful for detecting the presence of topological sectors in a theory. In this appendix we give some details of this for the case of the duality appetiser of [119]. This is a duality which relates an $SU(2)_1$ gauge theory with one adjoint chiral to the product theory of a free chiral multiplet plus a topological sector consisting of a $U(1)_{-2}$ TQFT. The topological sector was detected in [119] using the $\mathbb{S}^3$ partition function, where it was observed that the $U(1)_f$ symmetry acting on the adjoint chiral on the $SU(2)_1$ side of the duality is mapped to a combination of the R-symmetry and the topological symmetry of the $U(1)_{-2}$ TQFT on the dual side. The topological sector can actually be detected also in the index by turning on a background magnetic flux m_f for the $U(1)_f$ symmetry. The duality is indeed represented by the following identity of indices:

$$\begin{aligned}&\frac{1}{2}\sum_{m=-\infty}^{+\infty}\oint\frac{\mathrm{d}z}{2\pi i z}z^{2m}x^{-2m}(1-x^{2m}z^{\pm 2})(f^2x^{R_\Phi-1})^{-|m_f|}\frac{(f^{-2}\,x^{2-R_\Phi+|2m_f|};x^2)_\infty}{(f^2\,x^{R_\Phi+|2m_f|};x^2)_\infty}\\&\times(z^{\pm 2}f^2x^{R_\Phi-1})^{-|\pm m+m_f|}\frac{(z^{\mp 2}\,f^{-2}\,x^{2-R_\Phi+|\mp 2m+2m_f|};x^2)_\infty}{(z^{\pm 2}\,f^2\,x^{R_\Phi+|\pm 2m+2m_f|};x^2)_\infty}=\\&=(f^4x^{2R_\Phi-1})^{-|2m_f|}\frac{(f^{-4}\,x^{2-2R_\Phi+|4m_f|};x^2)_\infty}{(f^4\,x^{2R_\Phi+|4m_f|};x^2)_\infty}\sum_{m=-\infty}^{+\infty}\oint\frac{\mathrm{d}z}{2\pi i z}z^{2m+2m_f}\left(x^{R_\Phi+1}f^2\right)^m\,,\end{aligned}\tag{D.0.1}$$

where f, m_f are the fugacity and the flux for the $U(1)_f$ global symmetry and R_Φ is the R-charge of the adjoint chiral field. The identity holds provided that the background flux is quantised as $m_f\in\mathbb{Z}$ and for generic values of f and R_Φ. Notice that the map between the $U(1)_f$ symmetry acting on Φ and the topological symmetry of the TQFT is compatible with the one found in [119] at the level of the $\mathbb{S}^3$ partition function. The prefactor to the integral on the right hand side is the index of the free chiral, while the remaining integral is the index of the $U(1)_{-2}$ TQFT, which evaluates to

$$\sum_{m=-\infty}^{+\infty}\oint\frac{\mathrm{d}z}{2\pi i z}z^{2m+2m_f}\left(x^{R_\Phi+1}f^2\right)^m=\left(x^{R_\Phi+1}f^2\right)^{-m_f}\tag{D.0.2}$$

and, as we anticipated, is trivial if we turn off the background magnetic flux $m_f=0$, while it is non-trivial if we take $m_f\in\mathbb{Z}_{\neq 0}$. In other words, the topological sector is detectable by the index provided that we introduce such background flux.

The identity (D.0.1) can be tested by perturbatively expanding both sides in x. Taking $R_\Phi = \frac{1}{4}$, which is the value corresponding to the superconformal R-symmetry [119], we find that the indices of both of the dual theories are for $m_f = 0$

$$\begin{aligned}\mathcal{I}_{m_f=0} = &\, 1 + f^4 x^{\frac{1}{2}} + \left(f^{12} - f^{-4}\right) x^{\frac{3}{2}} + f^8 x + (f^{16} - 1)x^2 + f^{20} x^{\frac{5}{2}} + f^{24} x^3 + \\ &+ \left(f^{28} - f^{-4}\right) x^{\frac{7}{2}} + (f^{32} - 2)x^4 + \left(f^{36} - f^4\right) x^{\frac{9}{2}} + \left(f^{40} + f^{-8}\right) x^5 + \\ &+ f^{44} x^{\frac{11}{2}} + (f^{48} - 2)x^6 + \mathcal{O}\left(x^{\frac{13}{2}}\right) ,\end{aligned} \tag{D.0.3}$$

which is the same result that was found in [119, (9)]. Notice that -1 at order x^2, which represents the fermionic superpartner of the $U(1)_f$ conserved current. For non-trivial m_f we find that the two indices still match and that the topological sector (D.0.2) is crucial for the matching. For example for $m_f = 1$ we get

$$\mathcal{I}_{m_f=\pm 1} = f^{-10} x^{-\frac{1}{4}} + f^{-6} x^{\frac{17}{4}} - f^{-14} x^{\frac{21}{4}} + \mathcal{O}\left(x^{\frac{25}{4}}\right) , \tag{D.0.4}$$

but we also checked (D.0.1) for several other values of m_f and R_Φ.

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