A finite convergence algorithm for solving linear-quadratic network games with strategic complements and bounded strategies

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ABSTRACT
We propose a new algorithm for solving a class of linear-quadratic network games with strategic complements and bounded strategies. The algorithm is based on the sequential solution of linear systems of equations and we prove that it finds the exact Nash equilibrium of the game after a finite number of iterations. The new algorithm is then applied to a social network model of juvenile delinquency which has been investigated recently where we also consider random perturbations of some data. Experimental results show the efficiency of the algorithm in solving large scale problems.

KEYWORDS
Network games; Nash equilibrium; key player; social network

1. Introduction

Network games are a class of games put forward to model social and economic interactions among various agents whose connections are described with the help of graph-theoretical notions (see, e.g., [2, 3, 18]).

In our work we follow the approach by Ballester et al. [1], where agents (players) are identified with the nodes of a graph and can interact only with their neighbors, i.e., with the players directly connected through an arc. A characteristic of this kind of games is that the adjacency matrix of the underlying graph plays a central role in the utility functions of players. Moreover, its spectral properties are crucial in the analysis of the model. The authors of [1] find the Nash equilibrium of the game in the case of an unbounded strategy space and, when solutions are strictly positive, express the equilibrium using the so called Katz-Bonacich centrality measure [9]. In the last fifteen years this framework has been applied to describe a great variety of social and economic models and the interested readers can refer, for instance, to the nice survey [22], to get an idea of the scope of this approach. It is worth noticing that most of the papers on network games deal with models where the solution is interior to the feasible region

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and can be expressed in closed form. Moreover, although the connection between Nash equilibrium and variational inequalities is a well known fact and dates to the paper [19], the variational inequality theory has been applied to network games only recently. In this respect, a relevant paper is [33], where the authors make a thorough investigation of uniqueness and stability of the Nash equilibrium, with particular emphasis on its connection with the spectral properties of the adjacency matrix of the graph. Another extension of the original model has been proposed in [34], where the authors consider a generalized Nash equilibrium problem on a graph and extend the Katz-Bonacich formula to the new problem. The variational inequality approach has been further used in [35] to derive a Katz-Bonacich formula in the case of a bounded strategy space and to study the price of anarchy related to a class of network games.

In this paper, we consider a class of network games with a bounded strategy space, where the generic player is influenced by her/his neighbors through a term which positively influences her/his action (which is called a strategic complement in the related literature). We specify that our model is based on a slight modification of the linear-quadratic utility function considered in [1], but the introduction of upper bounds in the strategy space of players invalidates the possibility of obtaining closed-form solutions thus requiring a numerical approach. Within this framework, several methods to find Nash equilibria exist in the literature, e.g., the potential-based approach [28], best-response methods [39, 40], variational inequality-based methods [13], and methods based on projected dynamical systems [37]. In this paper we propose a new *ad-hoc* algorithm for finding Nash equilibria, which is based on the sequential solution of linear systems of equations. We will show that this algorithm has both good convergence properties (it allows to find the unique Nash equilibrium in a finite number of iterations) and good computational performance in solving large scale problems.

We then consider an application to a social network model of juvenile delinquency which was originally proposed in [10] and has been further investigated in the very recent paper [25], where a statistical analysis of experimental data has been performed to validate the model. Given that not all of the data are accessible to the econometrician, we also introduce a random perturbation term to take into account this uncertainty. This application is thus studied within the framework of stochastic variational inequalities, a theory that has been developed in the last fifteen years and has been successfully applied to a variety of equilibrium problems (see, e.g., [16, 23, 36, 41, 42]). We also perform a key player analysis under random perturbation, to identify the players who contribute the most to the equilibrium aggregate.

Finally, we perform a series of numerical experiments to show the usefulness and the advantages of using our solution approach to find stochastic Nash equilibria for the considered network game model of delinquency. Specifically, we compare our approach with both the potential-based approach and best-response methods. Numerical results show that our approach outperforms the other methods especially for solving large scale problems.

The rest of the paper is organized as follows. The following Section 2 consists of three subsections: In Section 2.1 we outline the approach in the seminal paper by Rosen [37] and provide his proof of uniqueness which is not based on the variational inequality formulation. We also remark that the results by Rosen have been reformulated only in recent times within the framework of variational inequalities, and point out the advantages of the new formulation. In Section 2.2, we recall the variational inequality problem and its connection with Nash equilibrium problems. We also provide some useful concepts of monotonicity: the strict monotonicity, which is equivalent to Rosen's strict diagonal concavity, and the strong monotonicity which is useful in the case of

unbounded action sets. Section 2.3 is devoted to illustrate the specific model and also to provide some basic definitions and properties of potential games, because our problem falls in this class.

Section 3 is devoted to the description and the convergence proof of our algorithm. In Section 4, we first recall the model investigated in [25] and modify it by introducing upper bounds on the strategy space and a random perturbation term, then we provide its stochastic variational inequality formulation. Section 5 is devoted to numerical experiments. First, we show the convergence of the approximated mean values of the Nash equilibrium and equilibrium aggregate to find the key player on a small size network. Next, we show the efficiency and scalability of the numerical approximation procedure coupled with our algorithm to approximate the stochastic Nash equilibrium and the stochastic key player on a set of randomly generated instances with medium-large size.

2. Network games

2.1. Basic notations and game formulation

In network games players are represented by the nodes of a graph (V, E), where V is the set of nodes and E is the set of arcs formed by pairs of nodes (v, w). Here, we consider undirected simple graphs, where arcs (v, w) and (w, v) are the same, and there are neither multiple arcs connecting the same pair of nodes, nor loops. Two nodes v and w are said to be adjacent if they are connected by an arc, i.e., if (v, w) is an arc. The information about the adjacency of nodes can be stored in the adjacency matrix G whose elements g_{ij} are equal to 1 if (v_i, v_j) is an arc, 0 otherwise. G is thus a symmetric and zero-diagonal matrix. Given a node v, the nodes connected to v with an arc are called the *neighbors* of v. In this context, a *walk* in the graph is a finite sequence of successive nodes, of the form $v_{i_0}, v_{i_1}, \ldots, v_{j_k}$, The *length* of a walk is the number of its arcs. Let us remark that it is allowed to visit a node or go through an arc more than once. The indirect connections between any two nodes in the graph are described by means of the powers of the adjacency matrix G. Indeed, it can be proved that the element $g_{ij}^{[k]}$ of G^k gives the number of walks of length k between v_i and v_j (see, e.g., [4, Theorem 4.1]).

In the sequel, the set of players will be denoted by $\{1, 2, \ldots, n\}$ instead of $\{v_1, v_2, \ldots, v_n\}$. We denote with $A_i \subset \mathbb{R}^{m_i}$ the action space of player i, while $A = A_1 \times \cdots \times A_n$ is called the space of action profiles. For each $a = (a_1, \ldots, a_n)$, $a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$ and the notation $a = (a_i, a_{-i})$ will be used when we want to distinguish the action of player i from the action of all the other players. Each player i is endowed with a payoff function $u_i : A \to \mathbb{R}$ that she/he wishes to maximize. The notation $u_i(a, G)$ is often utilized when one wants to emphasize that the utility of player i also depends on the actions taken by her/his neighbors in the graph.

The solution concept that we consider here is the Nash equilibrium of the game, that is, we seek an element $a^* \in A$ such that for each $i \in \{1, \ldots, n\}$:

$$u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*), \qquad \forall\ a_i \in A_i. \tag{1}$$

A standard set of hypotheses which guarantee the existence of a Nash equilibrium (and which apply to our specific problem) are:

- $u_i : A \to \mathbb{R}$ is continuous, for any $i \in \{1, \ldots, n\}$;

- $u_i(\cdot, a_{-i})$ is concave in the variable a_i, for each $a_{-i} \in A_{-i}$ and $i \in \{1, \ldots, n\}$;
- A_i is compact and convex, for each $i \in \{1, \ldots, n\}$.

For the uniqueness of a Nash equilibrium we report here a theorem due to Rosen [37]. In his seminal paper Rosen also considered the more general case where the constraints of the different players are coupled, which goes beyond the framework of our work. However, for the sake of generality, and for possible further extensions, we keep the general notation of [37] for the individual constraints of players. Thus, let $A_i \subset \mathbb{R}^{m_i}$, $m_1 + \ldots + m_n = m$. Specifically, each set A_i is described by a vector function $h_i : \mathbb{R}^{m_i} \to \mathbb{R}^{k_i}$ such that its components $h_{ij} : \mathbb{R}^{m_i} \to \mathbb{R}$, $j = 1, \ldots, k_i$, are concave functions of a_i. We thus have:

$$A_i = \{a_i \in \mathbb{R}^{m_i} : h_{ij}(a_i) \geq 0, \quad j = 1, \ldots, k_i\} \qquad i = 1, \ldots, n.$$

In order to apply the KKT conditions to the maximum problems defined in (1), we will further assume that, for each $i = 1, \ldots, n$, $h_i \in C^1(\mathbb{R}^{m_i})$ and u_i has continuous partial derivatives with respect to the components of a_i. Moreover, for any scalar function $\varphi : \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_n} \to \mathbb{R}$, the symbol $\nabla_i \varphi(a)$ will denote the partial gradient of φ with respect to $a_i \in \mathbb{R}^{m_i}$. Moreover, we will assume that some constraint qualification holds (see Theorem 2.2 below).

Let us now define, for each $r \in \mathbb{R}^n_+$, the weighted sum of the utility functions:

$$\sigma(a, r) = \sum_{i=1}^{n} r_i u_i(a),$$

and consider, for r fixed, the map $g(a, r) : \mathbb{R}^m \to \mathbb{R}^m$ given by:

$$g(a, r) = [r_1 \nabla_1 u_1(a)^\top, r_2 \nabla_2 u_2(a)^\top, \ldots, r_n \nabla_n u_n(a)^\top]^\top, \tag{2}$$

which is called the *pseudogradient* of σ (or also the pseudogradient of the game).

Definition 2.1. For a fixed $r \geq 0$, the function $\sigma(a, r)$ is called *diagonally strictly concave* on A if, for each $a^0, a^1 \in A$ with $a^0 \neq a^1$, we have:

$$(a^1 - a^0)^\top g(a^0, r) + (a^0 - a^1)^\top g(a^1, r) > 0.$$

Remark 1. The above inequality can also be written as

$$(a^1 - a^0)^\top [-g(a^1, r) - (-g(a^0, r))] > 0,$$

which in the language of monotone operator theory expresses the strict monotonicity of $-g$ (see Section 2.2 on variational inequalities below). Moreover, it is well known that the strict monotonicity of $-g$ holds if the matrix $J_a g(a, r) + J_a g(a, r)^\top$ is negative definite in A, where $J_a g(a, r)$ is the Jacobian matrix of g with respect to a (see, e.g., [30]).

We now provide the uniqueness theorem of the Nash equilibrium (Theorem 2 of [37]). We point out that this classical result, along with the other results in Rosen's paper, have been reformulated in recent times in the framework of variational inequality theory (see, e.g., [14, 17, 29]) and monotone operator theory. This new approach

allows for a more concise and elegant formulation and, above all, gives access to a large number of numerical algorithms.

Theorem 2.2. *Let u_i and h_i satisfy the differentiability and concavity assumptions above and let $\sigma(r,a)$ be diagonally strictly concave for some $\bar{r} > 0$. Moreover, assume that the following constraint qualification holds: for each $i \in \{1,\dots,n\}$ it exists $\bar{x}_i \in \mathbb{R}^{m_i}$ such that $h_{ij}(\bar{x}_i) > 0$ for any $j = 1,\dots,k_i$. Then, there is a unique Nash equilibrium which solves* (1).

2.2. Variational inequality formulation

Given a closed and convex set $K \subset \mathbb{R}^n$ and a map $F : K \to \mathbb{R}^n$, the variational inequality problem $VI(F,K)$ consists of looking for an element $x^* \in K$ such that

$$F(x^*)^\top (x - x^*) \geq 0, \qquad \forall\, x \in K.$$

The inequality above can also be written as:

$$F(x^*)^\top x^* \leq F(x^*)^\top x, \qquad \forall\, x \in K,$$

which can be thought of as a minimum problem for the linear function $F(x^*)^\top x$. Although this minimum problem assumes the *a priori* knowledge of x^*, when the set K is described by inequality and/or equality constraints, it allows to associate with the variational inequality suitable KKT conditions. Specifically, if

$$K = \{x \in \mathbb{R}^n :\ h(x) \leq 0\},$$

where $h : \mathbb{R}^n \to \mathbb{R}^l$ and each h_j, $j = 1,\dots,l$, is continuously differentiable and convex and some constraint qualification holds, the variational inequality is equivalent to the following KKT conditions (see Proposition 1.3.4 in [13] for more details):

$$h(x^*) \leq 0, \tag{3}$$

$$\forall\, j = 1,\dots,l \quad \exists\, \mu_j \geq 0 : \quad \mu_j h_j(x^*) = 0, \tag{4}$$

$$F(x^*) + \sum_{j=1}^{l} \mu_j \nabla h_j(x^*) = 0. \tag{5}$$

For the subsequent development it is important to recall that if the u_i are continuously differentiable functions on A, and each $u(\cdot, a_{-i})$ is concave in the variable a_i for each a_{-i}, the Nash equilibrium problem is equivalent to the variational inequality $VI(F,A)$: find $a^* \in A$ such that

$$F(a^*)^\top (a - a^*) \geq 0, \qquad \forall\, a \in A, \tag{6}$$

where

$$[F(a)]^\top := -\left[\nabla_1 u_1(a), \dots, \nabla_n u_n(a)\right] \tag{7}$$

is the opposite of the pseudogradient of the game, obtained for $r = (1,\dots,1)$. For an account of variational inequalities the interested reader can refer to [13, 24, 30]. We

recall here some useful monotonicity properties.

Definition 2.3. A map $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be monotone on A iff:

$$[T(x) - T(y)]^\top (x - y) \geq 0, \qquad \forall\, x, y \in A.$$

T is said to be strictly monotone on A if the equality holds only when $x = y$.
T is said to be β-strongly monotone on A iff there exists $\beta > 0$ such that

$$[T(x) - T(y)]^\top (x - y) \geq \beta \|x - y\|^2, \qquad \forall\, x, y \in A.$$

For affine operators on $\mathbb{R}^n$ the two concepts of strict and strong monotonicity coincide and are equivalent to the positive definiteness of the corresponding jacobian matrix.

Conditions that ensure the unique solvability of a variational inequality problem are given by the following theorem (see, e.g., [30]).

Theorem 2.4. *If $K \subset \mathbb{R}^n$ is a compact convex set and $T : \mathbb{R}^n \to \mathbb{R}^n$ is continuous on K, then the variational inequality problem $VI(T, K)$ admits at least one solution. In the case that K is unbounded, the existence of a solution may be established under the following coercivity condition:*

$$\lim_{\|x\| \to +\infty} \frac{[T(x) - T(x_0)]^\top (x - x_0)}{\|x - x_0\|} = +\infty, \tag{8}$$

for $x \in K$ and some $x_0 \in K$. Furthermore, if T is strictly monotone on K the solution is unique.

Remark 2. A sufficient condition for the coercivity condition (8) is the strong monotonicity of T. Hence, in the case of an unbounded set and of a nonlinear operator the continuity and strong monotonicity of T ensure solvability of $VI(T, K)$.

In the following subsection, we describe in detail a specific class of games to which our algorithm applies.

2.3. The linear-quadratic model with strategic complements and bounded strategies

In the following, we consider a network game where the action space $A_i = [0, L_i]$ for any $i \in \{1, \ldots, n\}$, hence $A = [0, L_1] \times \cdots \times [0, L_n]$, and the payoff function of player i is given by:

$$u_i(a, G) = -\frac{1}{2} a_i^2 + \alpha_i a_i + \phi \sum_{j=1}^{n} g_{ij} a_i a_j, \qquad \alpha_i, \phi > 0. \tag{9}$$

We now remark that with the utility functions defined above, our game has the property of strategic complements, according to the following definition.

Definition 2.5. The network game has the property of strategic complements if

$$\frac{\partial^2 u_i}{\partial a_j \partial a_i}(a) > 0, \qquad \forall (i,j) : \; g_{ij} = 1, \; \forall \; a \in A.$$

Let us observe that when any two players i and j are directly connected by an arc (i.e., $g_{ij} = 1$), we get $\frac{\partial^2 u_i}{\partial a_j \partial a_i}(a) = \phi > 0$, hence our game has the property of strategic complements according to Definition 2.5. In mathematical economics this property has been formalized in the paper [6] in the framework of an oligopolistic market, and further developed in [27] and [43]. The linear-quadratic model above specifically takes into account the network structure of relationships. Thus, the marginal utility $\frac{\partial u_i}{\partial a_i}$ of player i increases when the actions of her neighbors increases. In network games the linear-quadratic model with strategic complements, after the seminal paper [1], has been applied by different authors to describe various socio-economics relationships. For instance, in [11] the influence of peers on educational networks was studied extensively, while in [21] the linear-quadratic model was used to investigate the interaction between the social space (i.e., the network) and the geographic space (i.e., the city). The linear-quadratic model has been further investigated in [7], with a particular focus on the relationship between the centrality of players in the networks and the intensity of their actions at equilibrium.

Moreover, we remark that this class of linear-quadratic network games is a specific case of the 2-groups partitionable Cournot oligopoly models defined and analyzed in [39, 40].

The opposite of the pseudogradient of this game is easily computed and give

$$F_i(a) = a_i - \alpha_i - \phi \sum_{j=1}^{n} g_{ij} a_j, \qquad i = 1, \ldots, n,$$

which can be written in compact form as

$$F(a) = (I - \phi G)a - \alpha,$$

where $\alpha = \; (\alpha_1, \ldots, \alpha_n)^\top \in \mathbb{R}^n_+$. We will seek Nash equilibrium points by solving the variational inequality

$$F(a^*)^\top (a - a^*) \geq 0, \qquad \forall \; a \in A. \tag{10}$$

To ensure uniqueness of the equilibrium we require F to be strictly monotone, a condition that is ensured by the following lemma.

Lemma 2.6 (see, e.g., [22])**.** *The matrix $I - \phi G$ is positive definite iff $\phi\rho(G) < 1$, where $\rho(G)$ is the spectral radius of G.*

The following result guarantees the uniqueness of the Nash equilibrium.

Theorem 2.7. *If $\phi\rho(G) < 1$, then a unique Nash equilibrium exists.*

Proof. Lemma 2.6 guarantees that the matrix $(I - \phi G)$ is positive definite, thus the mapping $F(a) = (I - \phi G)a - \alpha$ is strongly monotone and there exists a unique solution

of the variational inequality (10) (see, e.g., [13]), which is the unique Nash equilibrium of the game. □

We remark that the class of linear-quadratic games with strategic complements is a particular kind of potential games. The use of potential functions for games in strategic form was introduced by Rosenthal [38] and developed by Monderer and Shapley in their seminal paper [28]. Further generalizations can be found in [12, 15, 32]. We recall here some basic facts about potential games (see, e.g., [28]).

Definition 2.8. A game in strategic form is called a potential game if it exists a function $P : A \to \mathbb{R}$ such that, for each $i \in \{1, \dots, n\}$, for all $a_{-i} \in A_{-i}$ and for all $z \in A_i$:

$$u_i(a_i, a_{-i}) - u_i(z_i, a_{-i}) = P(a_i, a_{-i}) - P(z_i, a_{-i}).$$

Lemma 2.9. *Consider a potential game in strategic form. If the utility functions are continuous and the strategy sets are compact, then there exists a Nash equilibrium of the game.*

Since we deal with differentiable games (and the strategy sets are intervals of real numbers), the following characterization of a potential is very useful.

Lemma 2.10. *Assume that $u_i(\cdot, a_{-i})$ are continuously differentiable, for each $a_{-i} \in A_{-i}$, and let $P : A \to \mathbb{R}$. Then P is a potential if and only if P is continuously differentiable and*

$$\frac{\partial u_i}{\partial a_i} = \frac{\partial P}{\partial a_i}, \qquad \forall\ i \in \{1, \dots, n\}.$$

A very useful theorem is the following.

Theorem 2.11. *Consider a game where the strategy sets are intervals of real numbers and assume that the utility functions are twice continuously differentiable. Then the game is a potential game if and only if:*

$$\frac{\partial^2 u_i}{\partial a_i \partial a_j} = \frac{\partial^2 u_j}{\partial a_i \partial a_j}, \qquad \forall\ i, j \in \{1, \dots, n\}.$$

In this case, if z is an arbitrarily fixed strategy profile in A, then a potential is given by:

$$P(a) = \sum_{i=1}^{n} \int_0^1 \frac{\partial u_i}{\partial a_i}(x(t))\,(x^i)'(t)dt$$

where $x : [0, 1] \to A$ is a piecewise continuously differentiable path in A that connects z to a (i.e., $x(0) = z$, $x(1) = a$).

In our game, a potential function is given by

$$P(a, G, \phi) = \sum_{i=1}^{n} u_i(a, G) - \frac{\phi}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij} a_i a_j,$$

thus the pseudogradient of the game coincides with the gradient of the potential. Monderer and Shapley have proved that, in general, the solutions of the optimization problem

$$\max_{a \in A} \; P(a, G, \phi) \tag{11}$$

form a subset of the Nash equilibria. Because under the condition $\phi\rho(G) < 1$ both problems have a unique solution, it follows that the two problems share the same solution.

3. A new solution algorithm with finite convergence

In the class of network games described in Section 2.3, the presence of upper bounds in the strategy space of players implies that a closed-form Nash equilibrium can be obtained only in the case of an interior solution. In general, Nash equilibria can have some boundary components which cannot be analytically derived as in [1]. On the other hand, there are several approaches in the literature to find Nash equilibria of this class of games. Since we are dealing with potential games, a number of optimization algorithms can be applied to solve the maximization problem (11). Moreover, since the considered class of games is a specific case of the 2-groups partitionable Cournot oligopoly models analyzed in [39, 40], the best-response method proposed in [40] is guaranteed to be convergent in a finite number of iterations to an ε-approximate Nash equilibrium. Then, if we exploit the variational inequality reformulation of the game, then a number of algorithms can be used to solve the variational inequality (10) (see, e.g., [13]). Finally, solution approaches based on projected dynamical systems can be applied as well. In particular, the strict monotonicity assumption on the pseudogradient of the game guarantees that the trajectories of suitable projected dynamical systems converge to the Nash equilibrium (see, e.g., [31, 37]).

In this section, we propose a new *ad-hoc* algorithm to find the Nash equilibrium of the considered network game that is different from the solution approaches mentioned above. The algorithm, inspired by the approach for general VIs proposed in [26], is based on the sequential solution of linear systems of equations. We will show in the following that this algorithm has both good theoretical properties of convergence and very good numerical performance. In fact, it allows to find the exact Nash equilibrium after a finite number of iterations (see Theorem 3.1), it needs to solve a generally very small number of linear systems and it is computationally efficient to solve large scale problems, as the numerical results in Section 5.2 show.

We now describe the proposed method in detail (see Algorithm 1). At the first iteration, the linear system providing the Nash equilibrium $\bar{a}$ in the case of unbounded strategy sets is solved (line 1). If the solution $\bar{a}$ satisfies all the upper bound constraints on the strategies, then the algorithm stops since $\bar{a}$ is the Nash equilibrium of the game (lines 2–3). Otherwise, the set of players is partitioned into two subsets: $C_0 = \{i : \; \bar{a}_i > L_i\}$ and $U_0 = \{i : \; \bar{a}_i \leq L_i\}$ (line 5). Then, a new linear system is solved, where the variables a_i with $i \in U_0$ are free, while the variables a_i with $i \in C_0$ are set to $a_i = L_i$ (lines 8–9). Next, the sign of a suitable vector μ (that is part of the KKT multipliers vector associated to the Nash equilibrium) is checked (line 10): if it is nonnegative, the algorithm stops (line 12), otherwise the variables a_i with $i \in C_0$ corresponding to negative components of μ becomes free in the next iteration (lines 14–17), and so

on. Under the uniqueness condition of Lemma 2.6, we will prove that the algorithm converges to the exact Nash equilibrium after a finite number of iterations.

In the following, we denote $L = (L_1, \dots, L_n)^\top$. Moreover, given any vector $v \in \mathbb{R}^n$, any matrix $M \in \mathbb{R}^{n\times n}$ and two sets of indices $S, T \subseteq \{1, \dots, n\}$, we denote the subvector $v_S = (v_i)_{i\in S}$ and the submatrix $M_{ST} = (M_{ij})_{i\in S,\, j\in T}$.

Algorithm 1: Sequential resolution of linear systems

1 Solve the linear system $(I - \phi G)\,\bar{a} = \alpha$
2 **if** $\bar{a}_i \le L_i$ *for any* $i = 1, \dots, n$ **then**
3 | STOP ($\bar{a}$ is the Nash equilibrium)
4 **else**
5 | set $C_0 := \{i :\ \bar{a}_i > L_i\}$, $U_0 := \{i :\ \bar{a}_i \le L_i\}$ and $k := 0$
6 **end**
7 **while** *true* **do**
8 | Solve the linear system $(I_{U_kU_k} - \phi G_{U_kU_k})\, z^k = \alpha_{U_k} + \phi G_{U_kC_k} L_{C_k}$
9 | Set $a_i^k := \begin{cases} L_i & \text{if } i \in C_k \\ z_i^k & \text{if } i \in U_k \end{cases}$ for any $i = 1, \dots, n$
10 | Set $\mu^k := \alpha_{C_k} - (I_{C_kC_k} - \phi G_{C_kC_k}) L_{C_k} + \phi G_{C_kU_k} z^k$
11 | **if** $\mu^k \ge 0$ **then**
12 | | STOP (a^k is the Nash equilibrium)
13 | **else**
14 | | set $N_k := \{i \in C_k :\ \mu_i^k < 0\}$
15 | | set $C_{k+1} := C_k \setminus N_k$
16 | | set $U_{k+1} := N_k \cup U_k$
17 | | set $k := k + 1$
18 | **end**
19 **end**

Theorem 3.1. *If $\phi\rho(G) < 1$, then Algorithm 1 finds the Nash equilibrium after a finite number of iterations.*

Proof. The assumption guarantees that there exists a unique Nash equilibrium a^*, which is also the unique solution of the following KKT system:

$$(I - \phi G)a^* - \alpha + \lambda - \nu = 0, \tag{12}$$

$$\lambda_i \ge 0, \qquad a_i^* \le L_i, \qquad \lambda_i(a_i^* - L_i) = 0, \qquad i = 1, \dots, n, \tag{13}$$

$$\nu_i \ge 0, \qquad a_i^* \ge 0, \qquad \nu_i a_i^* = 0, \qquad i = 1, \dots, n, \tag{14}$$

where λ and ν are the KKT multipliers related to a^*.

Since $(I - \phi G)$ is a positive definite matrix, the linear system at line 1 admits a unique solution $\bar{a}$.

Notice that the inverse matrix of $(I - \phi G)$ is nonnegative since $\phi > 0$, $G \ge 0$ and

$$(I - \phi G)^{-1} = \sum_{p=0}^{\infty} \phi^p G^p \ge 0,$$

hence $\bar{a} = (I - \phi G)^{-1}\alpha \geq 0$. Hence, if $\bar{a} \leq L$ holds, then $\bar{a}$ satisfies the KKT system (12)–(14) with multipliers $\lambda = \nu = 0$, that is $\bar{a}$ is the Nash equilibrium. Therefore, the stopping criterion at line 2 is correct.

Since G is nonnegative and $G_{U_kU_k}$ is a principal submatrix of G, we have $\rho(G_{U_kU_k}) \leq \rho(G)$ holds for any $k \geq 0$ (see, e.g., [8, Corollary 2.1.6]). Hence, the assumption $\phi\rho(G) < 1$ also implies that the matrix $(I_{U_kU_k} - \phi G_{U_kU_k})$ is nonsingular, and its inverse matrix is nonnegative since

$$(I_{U_kU_k} - \phi G_{U_kU_k})^{-1} = \sum_{p=0}^{\infty} \phi^p (G_{U_kU_k})^p \geq 0.$$

Therefore, the linear system at line 8 admits a unique solution.

We now prove by induction that the sequence $\{a^k\}$ generated by Algorithm 1 is feasible.

Base case. Since the vector $\alpha_{U_k} + \phi G_{U_kC_k} L_{C_k}$ and the matrix $(I_{U_kU_k} - \phi G_{U_kU_k})^{-1}$ are nonnegative, we get

$$z^k = (I_{U_kU_k} - \phi G_{U_kU_k})^{-1}(\alpha_{U_k} + \phi G_{U_kC_k} L_{C_k}) \geq 0,$$

thus $a^k \geq 0$ holds for any $k \geq 0$. Moreover, the linear system $(I - \phi G)\bar{a} = \alpha$ can be written as follows:

$$\begin{pmatrix} I_{C_0C_0} - \phi G_{C_0C_0} & -\phi G_{C_0U_0} \\ -\phi G_{U_0C_0} & I_{U_0U_0} - \phi G_{U_0U_0} \end{pmatrix} \begin{pmatrix} \bar{a}_{C_0} \\ \bar{a}_{U_0} \end{pmatrix} = \begin{pmatrix} \alpha_{C_0} \\ \alpha_{U_0} \end{pmatrix},$$

and $L_{C_0} < \bar{a}_{C_0}$ by definition, thus we have the following chain of equalities and inequalities:

$$\begin{aligned} a^0_{U_0} = z^0 &= (I_{U_0U_0} - \phi G_{U_0U_0})^{-1}(\alpha_{U_0} + \phi G_{U_0C_0} L_{C_0}) \\ &\leq (I_{U_0U_0} - \phi G_{U_0U_0})^{-1}(\alpha_{U_0} + \phi G_{U_0C_0} \bar{a}_{C_0}) \\ &= \bar{a}_{U_0} \\ &\leq L_{U_0}. \end{aligned}$$

Since $a^0_{C_0} = L_{C_0}$ by definition, we get $a^0 \leq L$, thus a^0 is feasible.

Induction step. Given any $k \geq 0$, let us assume that a^k is feasible and prove that a^{k+1} is feasible as well. By definition of N_k we have $\mu^k_{N_k} < 0$. On the other hand, by definition of μ^k, we have

$$\mu^k_{N_k} = \alpha_{N_k} - (I_{N_kC_k} - \phi G_{N_kC_k}) L_{C_k} + \phi G_{N_kU_k} z^k.$$

Since the set C_k is partitioned into the two subsets N_k and C_{k+1}, we can decompose the vector L_{C_k} as follows:

$$L_{C_k} = \begin{pmatrix} L_{N_k} \\ L_{C_{k+1}} \end{pmatrix}.$$

Since N_k and C_{k+1} are disjoint subsets, the submatrix $I_{N_kC_{k+1}}$ is null, thus

$$\begin{aligned}(I_{N_kC_k} - \phi G_{N_kC_k})L_{C_k} &= (I_{N_kN_k} - \phi G_{N_kN_k})L_{N_k} + (I_{N_kC_{k+1}} - \phi G_{N_kC_{k+1}})L_{C_{k+1}} \\ &= (I_{N_kN_k} - \phi G_{N_kN_k})L_{N_k} - \phi G_{N_kC_{k+1}}L_{C_{k+1}}.\end{aligned}$$

Hence, we get

$$\begin{aligned}0 &> \mu^k_{N_k} \\ &= \alpha_{N_k} - (I_{N_kC_k} - \phi G_{N_kC_k})L_{C_k} + \phi G_{N_kU_k}z^k \\ &= \alpha_{N_k} - (I_{N_kN_k} - \phi G_{N_kN_k})L_{N_k} + \phi G_{N_kC_{k+1}}L_{C_{k+1}} + \phi G_{N_kU_k}z^k.\end{aligned}$$

Since the set U_{k+1} is partitioned into the two subsets N_k and U_k, we have

$$\begin{aligned}(I_{U_{k+1}U_{k+1}} - \phi G_{U_{k+1}U_{k+1}})\begin{pmatrix} L_{N_k} \\ z^k \end{pmatrix} &= \begin{pmatrix} I_{N_kN_k} - \phi G_{N_kN_k} & -\phi G_{N_kU_k} \\ -\phi G_{U_kN_k} & I_{U_kU_k} - \phi G_{U_kU_k} \end{pmatrix}\begin{pmatrix} L_{N_k} \\ z^k \end{pmatrix} \\ &= \begin{pmatrix} \alpha_{N_k} \\ \alpha_{U_k} \end{pmatrix} + \begin{pmatrix} \phi G_{N_kC_{k+1}} \\ \phi G_{U_kC_{k+1}} \end{pmatrix} L_{C_{k+1}} - \begin{pmatrix} \mu^k_{N_k} \\ 0 \end{pmatrix} \\ &\geq \begin{pmatrix} \alpha_{N_k} \\ \alpha_{U_k} \end{pmatrix} + \begin{pmatrix} \phi G_{N_kC_{k+1}} \\ \phi G_{U_kC_{k+1}} \end{pmatrix} L_{C_{k+1}} \\ &= \alpha_{U_{k+1}} + \phi G_{U_{k+1}C_{k+1}}L_{C_{k+1}}.\end{aligned}$$

Since the matrix $(I_{U_{k+1}U_{k+1}} - \phi G_{U_{k+1}U_{k+1}})^{-1}$ is nonnegative, we get

$$\begin{pmatrix} L_{N_k} \\ z^k \end{pmatrix} \geq (I_{U_{k+1}U_{k+1}} - \phi G_{U_{k+1}U_{k+1}})^{-1}\left[\alpha_{U_{k+1}} + \phi G_{U_{k+1}C_{k+1}}L_{C_{k+1}}\right] = z^{k+1}.$$

Since a^k is feasible by induction hypothesis, we have

$$a^{k+1}_{U_{k+1}} = z^{k+1} \leq \begin{pmatrix} L_{N_k} \\ z^k \end{pmatrix} = \begin{pmatrix} L_{N_k} \\ a^k_{U_k} \end{pmatrix} \leq \begin{pmatrix} L_{N_k} \\ L_{U_k} \end{pmatrix} = L_{U_{k+1}},$$

hence $a^{k+1} \leq L$, i.e., a^{k+1} is feasible. Since the cardinality of the set C_k is decreasing at each iteration, the condition $\mu^k \geq 0$ at line 11 has to be satisfied after a finite number of iterations, before the set C_k becomes empty. In fact, if $C_{k+1} = \emptyset$, then $U_{k+1} = \{1, \ldots, n\}$ and the feasible solution z^{k+1} should coincide with the unfeasible vector $\bar{a}$, which is impossible.

When $\mu^k \geq 0$ holds, the vector a^k is the Nash equilibrium since it solves the KKT system (12)–(14) with multipliers λ and ν defined as follows:

$$\lambda_i = \begin{cases} \mu^k_i & \text{if } i \in C_k, \\ 0 & \text{if } i \in U_k, \end{cases} \qquad \nu = 0.$$

In fact, we have:

$$(I-\phi G)a^k = \begin{pmatrix} I_{C_kC_k} - \phi G_{C_kC_k} & -\phi G_{C_kU_k} \\ -\phi G_{U_kC_k} & I_{U_kU_k} - \phi G_{U_kU_k} \end{pmatrix} \begin{pmatrix} L_{C_k} \\ z^k \end{pmatrix}$$
$$= \begin{pmatrix} \alpha_{C_k} - \mu^k \\ \alpha_{U_k} \end{pmatrix} = \alpha - \lambda + \nu,$$

hence equation (12) holds. Moreover, a^k is feasible, $\lambda \geq 0$ and $\lambda^\top(a^k - L) = 0$ hold by definition, thus conditions (13)–(14) hold as well. This concludes the proof. □

4. A network-game model of delinquency

We now present a model which has been used recently to describe a network of juvenile delinquency in the United States. This model is inspired to the quantitative economic approach to crime introduced in [5] and, along the same lines as in [10], introduces network effects in the utility function assigned to each player (which represents a young person involved in criminal activities). It has been used in the very recent paper [25] as the microeconomic starting point for a statistical analysis of survey collected data on the social environment of students in grades 7-12 from 130 schools in the United States. The form of the utility functions in [25] belongs to the class (9) considered in Section 2.3 for which our algorithm is designed and we also introduce an upper bound on the actions of the players, to be more realistic. We remark that the introduction of upper bounds on the actions makes it impossible to find the Nash equilibrium in closed form. It is thus of paramount importance to use an efficient algorithm to numerically solve the problem under consideration. This is particularly true in our model because we also consider random perturbations which, due to discretization of the random variables involved, will yield to a large number of subproblems.

Moreover, the authors of [25] also perform a key player analysis aimed at identifying the criminal that mostly contribute to raise the aggregate level of delinquency in the network. More precisely, the key player of a network game where each action space $A_i \subset \mathbb{R}$ is defined as follows.

Definition 4.1. Let $a^*(G \setminus \{i\})$ be the Nash equilibrium of the game where player i is removed from the network and $S_i := \sum_{j \neq i} a_j^*(G \setminus \{i\})$ the corresponding equilibrium aggregate. A key player of the game is a player k such that

$$S_k = \min_{i=1,\dots,n} S_i,$$

that is, after its removal from the network, the new equilibrium aggregate is the minimum possible with respect to all possible removals of one player.

Some comments are in order. In general, a key player is not necessarily one whose action at equilibrium is maximum with respect to the other players' actions. Indeed, the contribution of a given player to the aggregate can also be indirect so that a "weaker" player can contribute more than others to increase the equilibrium actions of her/his neighbors (see Example 5.1 below). Moreover, the use of the centrality measure based on the Katz-Bonacich vector pioneered in [1] can be exploited to find the key player only in the case of an interior solution. As for the network game model of juvenile delinquents, targeting and isolating a key player corresponds to the best

possible reduction of the overall criminal activity of the network. As already pointed out in [10], this is a policy which will be effective until the network has reorganized, i.e., until players have established new connections among themselves.

Given a network of n players, we assume the action $a_i \in [0, L_i]$ of each player i represents her/his effort in delinquent activities. The utility function of each player which was proposed in [25] is:

$$u_i(a) = \left(\pi_i + \phi \sum_{j=1}^{n} g_{ij} a_j \right) a_i - \left(p a_i + \frac{1}{2} a_i^2 \right). \tag{15}$$

The term π_i represents the specific ability of player i in criminal activities and can be partially estimated with the help of statistical analysis of data. To take into account the contributions which are not observable to the econometrician, we model π_i as the sum of a deterministic term β_i and a random perturbation $\gamma_i r$, where γ_i is a fixed number and r is a random variable following a given distribution:

$$\pi_i = \beta_i + \gamma_i r. \tag{16}$$

The term $\phi \sum_{j=1}^{n} g_{ij} a_j$ in the marginal payoff of the utility function has the following interpretation: an agent can improve her/his utility directly if her/his neighbors are acting with her/him in the same delinquent activity. She/he can also have indirect benefits acquiring knowledge or useful skills from her/his neighbors, even if they are not engaged in the same criminal activity. The cost part of (15) consists of two terms: the cost pa_i of being caught, which is assumed to be increasing with the action level a_i, and the direct cost of criminal activity, given by $\frac{1}{2} a_i^2$. The utility function can then be re-written as:

$$u_i(r, a) = -\frac{1}{2} {a_i}^2 + (\beta_i + \gamma_i r - p) a_i + \phi \sum_{j=1}^{n} g_{ij} a_j a_i. \tag{17}$$

Due to the presence of the random variable r, we now look for a Nash equilibrium random vector $a^* : r \mapsto a^*(r) \in \mathbb{R}^n$ such that, for each $i \in \{1, \ldots, n\}$, $a_i^*(r) \in [0, L_i]$ and

$$u_i(r, a_i^*(r), a_{-i}^*(r)) \geq u_i(r, a_i, a_{-i}^*(r)), \qquad \forall\ a_i \in A_i. \tag{18}$$

In the above formula, it is enough to require the inequality to be satisfied up to a negligible set, with respect to the probability measure P according to which r is distributed, that is P-almost surely, in the language of Probability. As a consequence, the opposite of the pseudogradient of the game is the map $F : (r, a) \mapsto F(r, a) \in \mathbb{R}^n$ given by:

$$F_i(r, a) = a_i - \beta_i - \gamma_i r + p - \phi \sum_{j=1}^{n} g_{ij} a_j. \tag{19}$$

The Nash equilibrium can then be computed by solving the variational inequality corresponding to the operator above and to the set $A = [0, L_1] \times \cdots \times [0, L_n]$: for each

$r \in \mathbb{R}$, find $a^*(r) \in A$ such that for each $a \in A$ we have:

$$\sum_{i=1}^{n}[a_i^*(r) - \phi \sum_{j=1}^{n} g_{ij} a_j^*(r)]\,[a_i - a_i^*(r)] \geq \sum_{i=1}^{n}[\beta_i + \gamma_i r - p]\,[a_i - a_i^*(r)]. \qquad (20)$$

The above variational inequality is known in the literature as a stochastic (or also random) variational inequality. Several methodologies and solution concepts have been proposed in the last fifteen years and the interested reader can refer to the references given in the introduction and to the recent monograph [20] for a comprehensive treatment of the subject. Here, we only mention that to compute the expected value of the Nash equilibrium $a^*(r)$ with respect to the probability measure P, we follow the so-called L^2 approach which consists of considering an integral version of (20). To this end, we introduce the space $L^2(\mathbb{R}, P, \mathbb{R}^n)$ of the vector-valued functions, defined on $\mathbb{R}$, which possess finite first and second moments with respect to the probability measure P. It is thus well-posed the following problem:

Find $a^* \in L^2(\mathbb{R}, P, \mathbb{R}^n)$ such that for all the functions $a \in L^2(\mathbb{R}, P, \mathbb{R}^n)$ with $0 \leq a_i(r) \leq L_i$, P-almost surely:

$$\begin{aligned} \int_{-\infty}^{+\infty} \sum_{i=1}^{n}[a_i^*(r) - \phi \sum_{j=1}^{n} g_{ij} a_j^*(r)]\,[a_i(r) - a_i^*(r)]dP \\ \geq \int_{-\infty}^{+\infty} \sum_{i=1}^{n}[\beta_i + \gamma_i r - p]\,[a_i(r) - a_i^*(r)]dP. \end{aligned} \qquad (21)$$

The quantity we are interested in is the expectation of the unique solution $a^*(r)$:

$$E_P[a^*(r)] = \int_{-\infty}^{+\infty} a^*(r)dP, \qquad (22)$$

for which we construct a converging approximation. The detailed approximation procedure used to solve (21), which also yields a sequence converging to $E_P[a^*(r)]$, can be found in the previously given references. Here, for the reader's convenience, we outline the steps of our procedure. We start with a discretization of the support of the probability measure P (which we assume compact) in N subintervals. We then introduce the L_N^2 space of step functions on the partition thus obtained and solve (21) in L_N^2 to get the step function $a_N^*(r)$. Letting $N \to \infty$ yields a sequence of step functions which is proved to be norm-convergent to $a^*(r)$. From the implementation point of view, we remark that variational inequality (21), when restricted to the space L_N^2, is split in N finite-dimensional variational inequalities on $\mathbb{R}^n$, of the same kind as (10). Moreover, the approximated mean values $E_P[a_N^*(r)]$ form a sequence which converges to the exact mean value $E_P[a^*(r)]$.

5. Numerical experiments

In this section, we report some numerical experiments for the stochastic network game model described in Section 4. The approximation procedure was implemented in MATLAB R2022a and tested on an Apple M1 Max system with 64 GB of RAM running

under macOS 13.1. For any given discretization of the support of the probability measure P, a deterministic network game equivalent to (10) was solved for any subinterval of the discretization. The linear systems within Algorithm 1 were solved by exploiting the MATLAB functions $\backslash$ if $n \leq 100$ or `pcg` if $n > 100$.

In the following, Section 5.1 shows the convergence of the approximated mean values of the Nash equilibrium and equilibrium aggregate to find the key player of the network on a small size example. Moreover, the impact of different probability densities of the random variable r is reported. Section 5.2 shows the computational efficiency of the numerical approximation procedure coupled with Algorithm 1 to solve instances of increasing size up to 10,000 players. In particular, we compare our solution approach with both the potential-based approach (where three different optimization solvers were exploited to maximize the potential function) and two variants of the best-response method introduced in [40]. Numerical results show that our approach is faster than all other methods especially for large scale problems.

5.1. Convergence of approximated mean values of Nash equilibria

Example 5.1. We consider the network shown in Fig. 1 (see also [1]) with 11 nodes (players). The spectral radius of the adjacency matrix is $\rho(G) \simeq 4.4040$. We set parameter $\phi = 0.2$ to guarantee the matrix $I - \phi G$ is positive definite. Moreover, we set parameters $\beta = (10, \ldots, 10)$, $\gamma = (1, \ldots, 1)$ and $p = 1$. The upper bounds on the player strategies are set to $L = (100, \ldots, 100)$. We assume that the random variable r varies in the interval $[-5, 5]$ with either uniform distribution or truncated normal distribution with mean 0 and standard deviation 1. The approximation procedure considers a uniform partition of the interval $[-5, 5]$ into N subintervals and solves a deterministic network game for any subinterval by exploiting Algorithm 1.

Tables 1 and 2 show the convergence of the approximate mean values of the Nash equilibrium for different values of N, when the random variable r varies in the interval $[-5, 5]$ with uniform distribution (Table 1) or with truncated normal distribution with mean 0 and standard deviation 1 (Table 2). We note that, due to the network symmetry, players 2, 6, 7 and 11 have equal approximated values, and the same fact holds for players 3, 4, 5, 8, 9 and 10. Moreover, the mean values of the Nash equilibrium increase by about 3-5% from uniform to truncated normal distribution.

Tables 3 and 4 show the convergence of the approximate mean values of the equilibrium aggregate for different values of N, when r varies in the interval $[-5, 5]$ with

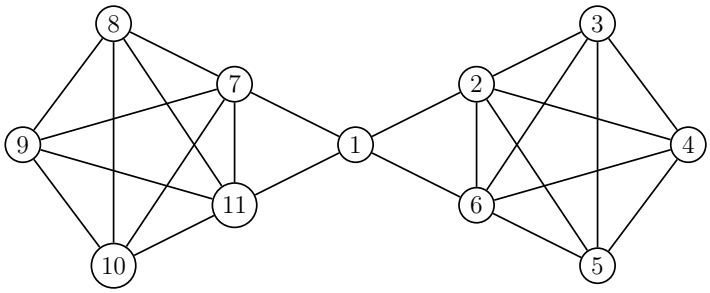

Figure 1. Network topology of Example 5.1.

Table 1. The convergence of the approximated mean values of the Nash equilibrium for $r \sim \mathcal{U}(-5, 5)$.

Nash	N			
equilibrium	100	1,000	10,000	100,000
a_1^*	71.193	71.467	71.494	71.497
a_2^*	77.804	78.090	78.118	78.121
a_3^*	66.786	67.051	67.078	67.081
a_4^*	66.786	67.051	67.078	67.081
a_5^*	66.786	67.051	67.078	67.081
a_6^*	77.804	78.090	78.118	78.121
a_7^*	77.804	78.090	78.118	78.121
a_8^*	66.786	67.051	67.078	67.081
a_9^*	66.786	67.051	67.078	67.081
a_{10}^*	66.786	67.051	67.078	67.081
a_{11}^*	77.804	78.090	78.118	78.121

Table 2. The convergence of the approximated mean values of the Nash equilibrium for $r \sim \mathcal{N}(0, 1)$ on $[-5, 5]$.

Nash	N			
equilibrium	100	1,000	10,000	100,000
a_1^*	74.514	74.880	74.917	74.920
a_2^*	81.955	82.356	82.397	82.401
a_3^*	69.553	69.896	69.930	69.934
a_4^*	69.553	69.896	69.930	69.934
a_5^*	69.553	69.896	69.930	69.934
a_6^*	81.955	82.356	82.397	82.401
a_7^*	81.955	82.356	82.397	82.401
a_8^*	69.553	69.896	69.930	69.934
a_9^*	69.553	69.896	69.930	69.934
a_{10}^*	69.553	69.896	69.930	69.934
a_{11}^*	81.955	82.356	82.397	82.401

uniform distribution and $\phi = 0.1$ (Table 3) or $\phi = 0.2$ (Table 4). The key players with the minimum value of the approximated equilibrium aggregate are highlighted in bold. Notice that for $\phi = 0.1$ the key players are 2, 6, 7 and 11 (i.e., the nodes most connected to the others), while for $\phi = 0.2$ the unique key player is 1 (i.e., the bridge connecting the two complete subgraphs). Moreover, we remark that, when $\phi = 0.2$, the key player's action at equilibrium is not maximum with respect to the other players' actions.

Table 3. The convergence of the approximated mean values of the equilibrium aggregate for $r \sim \mathcal{U}(-5, 5)$ and $\phi = 0.1$.

Equilibrium	N			
aggregate	100	1,000	10,000	100,000
S_1	149.167	149.917	149.992	149.999
S_2	145.929	146.663	146.736	**146.744**
S_3	150.360	151.116	151.192	151.199
S_4	150.360	151.116	151.192	151.199
S_5	150.360	151.116	151.192	151.199
S_6	145.929	146.663	146.736	**146.744**
S_7	145.929	146.663	146.736	**146.744**
S_8	150.360	151.116	151.192	151.199
S_9	150.360	151.116	151.192	151.199
S_{10}	150.360	151.116	151.192	151.199
S_{11}	145.929	146.663	146.736	**146.744**

Table 4. The convergence of the approximated mean values of the equilibrium aggregate for $r \sim \mathcal{U}(-5, 5)$ and $\phi = 0.2$.

Equilibrium	N			
aggregate	100	1,000	10,000	100,000
S_1	447.500	449.750	449.975	**449.998**
S_2	459.267	461.525	461.750	461.773
S_3	523.915	526.304	526.543	526.567
S_4	523.915	526.304	526.543	526.567
S_5	523.915	526.304	526.543	526.567
S_6	459.267	461.525	461.750	461.773
S_7	459.267	461.525	461.750	461.773
S_8	523.915	526.304	526.543	526.567
S_9	523.915	526.304	526.543	526.567
S_{10}	523.915	526.304	526.543	526.567
S_{11}	459.267	461.525	461.750	461.773

5.2. Scalability of Algorithm 1 and comparison with other solution approaches

We now show the performance of the numerical approximation procedure coupled with Algorithm 1 to solve instances of increasing size, where the number of players varies from 10 to 10,000. Instances are produced exploiting the generator of uniformly distributed pseudo-random numbers of MATLAB. The adjacency matrix of any random network is generated according to the following code:

```
G = rand(n);
G = floor((G+G')/2 + δ);
```

`G = G - diag(diag(G));`
so that G is an $n \times n$ zero-diagonal binary symmetric matrix and the parameter $\delta \in (0, 1)$ represents the density of the network, e.g., $\delta = 0$ corresponds to an empty network, while $\delta = 1$ to a complete network. We set parameters $\beta = (4, \ldots, 4)$, $\gamma = (1, \ldots, 1)$ and $p = 1$. We assume that the random variable r varies in the interval $[-1, 1]$ with uniform distribution and the approximation procedure considers a uniform partition of the interval $[-1, 1]$ into $N = 100$ subintervals. The upper bounds on the player strategies are set in such a way the approximated Nash equilibrium computed for any subinterval does not generally belong to the interior of the feasible region. Specifically, first we find the "unconstrained" Nash equilibrium $\bar{a}$ by solving the linear system $(I - \phi G)\bar{a} = \alpha$, where $\alpha_i = \beta_i - p$ for any $i = 1, \ldots, n$. Then, for any $i = 1, \ldots, n$, the upper bound L_i is chosen equal to a uniform pseudo-random number in the interval

$$\left[\frac{1}{2} \min_{1 \le j \le n} \{\bar{a}_j\},\ 2 \max_{1 \le j \le n} \{\bar{a}_j\}\right].$$

Table 5 shows the average number of linear systems solved by Algorithm 1 for different values of parameters n (from 10 to 10,000), ϕ (from $0.1/\rho(G)$ to $0.9/\rho(G)$) and δ (equal to 0.2 or 0.5). The figures reported are the average values obtained on a set of five random instances.

Table 5. The average number of linear systems solved by Algorithm 1 for different values of n, ϕ and δ.

n	$\delta = 0.2$			$\delta = 0.5$		
	$\phi = \frac{0.1}{\rho(G)}$	$\phi = \frac{0.5}{\rho(G)}$	$\phi = \frac{0.9}{\rho(G)}$	$\phi = \frac{0.1}{\rho(G)}$	$\phi = \frac{0.5}{\rho(G)}$	$\phi = \frac{0.9}{\rho(G)}$
10	1.94	2.18	2.38	2.05	2.26	2.54
20	2.05	2.20	2.94	2.12	2.42	2.97
50	2.16	2.73	3.15	2.30	2.78	3.43
100	2.37	2.90	3.77	2.49	2.99	3.83
200	2.59	3.07	4.05	2.64	3.09	4.04
500	2.85	3.21	4.21	2.83	3.30	4.31
1,000	2.94	3.45	4.46	2.94	3.42	4.54
2,000	2.97	3.59	4.64	2.99	3.63	4.68
5,000	3.02	3.77	4.82	3.02	3.78	4.82
10,000	3.03	3.86	4.89	3.03	3.88	4.89

The results suggest that the average number of linear systems solved by Algorithm 1 is very low and quite stable since it varies from 2 to 5 for any choice of parameters. In particular, it seems to be lightly increasing with respect to n and ϕ, while the density parameter δ does not seem to have a major impact.

In order to show the usefulness and the advantages of using our solution approach, we now compare the performance of Algorithm 1 with the performances of the potential-based approach and the best-response method developed in [40]. Specifically, in the potential-based approach we consider three different quadratic programming solvers to solve problem (11): `Gurobi` (with default options), the MATLAB `quadprog` function with the `'interior-point-convex'` algorithm, and the `quadprog` function with the `'trust-region-reflective'` algorithm. As for the best-response method introduced in [40], we consider the Jacobi variant and the Gauss-Seidel variant (in which the order of play is $1, 2, \ldots, n$), both with starting point equal to the upper bound vector $L = (L_1, \ldots, L_n)^\top$.

We consider a set of random instances, generated as described above, where the number n of players varies from 100 to 10,000 and $\phi = 0.8/\rho(G)$.

Tables 6 and 7 report the average CPU times (in seconds) of our approach based on Algorithm 1, of the three variants of the potential-based approach, and of the two variants of the best-response method developed in [40], when $\delta = 0.2$ (Table 6) or $\delta = 0.5$ (Table 7). For problems of larger size (when n varies from 2,000 to 10,000), we compare our approach only with the best variant of the potential-based approach and the best variant of the best-response method. The figures reported are the average values obtained on a set of five random instances.

Table 6. Comparison between the approach based on Algorithm 1, three variants of the potential-based approach, and two variants of the best-response method [40] to find the approximated stochastic Nash equilibrium, when $\delta = 0.2$ (CPU times in seconds).

		Potential-based approach			Best-response method	
n	Algorithm 1	`Gurobi`	`quadprog` interior-point	`quadprog` trust-region	Jacobi	Gauss-Seidel
100	0.025	0.966	0.103	0.529	0.047	0.027
200	0.268	2.707	0.807	0.560	0.154	0.089
300	0.430	4.613	2.223	0.851	0.322	0.183
400	0.602	10.686	4.301	1.060	0.510	0.299
500	0.701	13.383	8.233	1.273	0.766	0.447
600	0.813	17.230	14.838	1.513	1.617	0.948
700	1.406	22.793	30.908	1.778	2.349	1.370
800	1.291	27.865	28.714	2.138	3.880	2.256
900	1.355	34.323	38.835	2.587	3.936	2.341
1,000	1.527	37.187	50.834	2.922	4.989	2.911
2,000	2.916			10.913		13.366
3,000	6.181			23.372		49.398
4,000	9.950			36.831		68.642
5,000	14.256			54.906		368.093
6,000	19.233			74.899		465.769
7,000	25.112			101.163		783.910
8,000	31.950			128.381		959.708
9,000	39.738			161.549		1,075.657
10,000	59.745			199.125		1,102.629

The results show that the performance of the potential-based approach strongly depends on which optimization solver is used to maximize the potential function. The `quadprog` function with the `'trust-region-reflective'` algorithm seems to be the best solver among those tested (especially when $\delta = 0.2$). Moreover, the Gauss-Seidel variant of the best-response method results to be always more efficient than the Jacobi variant. When n varies from 100 to 700, our approach is comparable with the best variant of the potential-based approach and the best-response method. However, for large scale problems, results clearly show that our approach outperforms both the potential-based approach and the best-response method by 4 to 20 times in terms of CPU time.

Finally, in Tables 8 and 9 we compare our approach, the three variants of the potential-based approach, and the two variants of the best-response method to find the approximated stochastic key player, when n varies from 10 to 1,000 and $\delta = 0.2$ (Table 8) or $\delta = 0.5$ (Table 9). Also in this case, for problems of larger size (with n from 400 to 1,000), we compare our approach only with the best variant of the potential-based approach and the best variant of the best-response method. The results confirm that our approach outperforms the other methods also with regard to the search for

Table 7. Comparison between the approach based on Algorithm 1, three variants of the potential-based approach, and two variants of the best-response method [40] to find the approximated stochastic Nash equilibrium, when $\delta = 0.5$ (CPU times in seconds).

		Potential-based approach			Best-response method	
n	Algorithm 1	`Gurobi`	`quadprog` interior-point	`quadprog` trust-region	Jacobi	Gauss-Seidel
100	0.031	1.018	0.100	0.574	0.045	0.027
200	0.162	3.168	0.899	0.879	0.140	0.082
300	0.316	5.609	2.189	1.485	0.290	0.170
400	0.472	11.615	4.235	2.228	0.474	0.281
500	0.561	14.638	9.232	3.102	0.706	0.421
600	0.682	18.530	15.711	4.389	1.552	0.900
700	0.933	23.384	22.033	5.864	2.065	1.195
800	1.102	28.769	30.683	7.767	3.105	1.814
900	1.139	35.450	40.387	9.538	3.398	2.006
1,000	1.319	42.587	47.510	11.332	4.786	2.816
2,000	3.052			39.689		12.917
3,000	6.163			79.893		46.715
4,000	10.069			135.675		68.777
5,000	14.298			218.514		453.096
6,000	19.006			311.270		575.458
7,000	24.688			405.576		758.007
8,000	31.279			534.597		710.733
9,000	38.238			933.030		941.853
10,000	56.980			1,163.213		1,119.356

Table 8. Comparison between the approach based on Algorithm 1, three variants of the potential-based approach, and two variants of the best-response method [40] to find the approximated stochastic key player, when $\delta = 0.2$ (CPU times in seconds).

		Potential-based approach			Best-response method	
n	Algorithm 1	`Gurobi`	`quadprog` interior-point	`quadprog` trust-region	Jacobi	Gauss-Seidel
10	0.018	5.578	0.501	2.920	0.023	0.012
20	0.051	11.559	0.631	7.382	0.114	0.064
30	0.125	17.945	1.035	11.880	0.255	0.144
40	0.231	25.029	1.492	17.483	0.499	0.279
50	0.373	32.653	2.021	21.759	0.809	0.454
60	0.588	41.113	2.700	27.101	1.370	0.767
70	0.863	50.718	3.718	32.936	1.810	1.017
80	1.202	60.931	4.780	38.656	2.639	1.507
90	1.682	72.882	6.133	44.656	3.433	1.971
100	2.303	86.363	7.714	50.396	5.096	2.630
200	41.005	536.232	153.362	124.031	28.949	16.657
300	113.084	1,362.593	562.649	266.226	96.002	55.025
400	199.194			399.094		115.136
500	318.320			605.750		219.864
600	465.053			879.512		476.712
700	733.321			1,221.094		937.606
800	1,027.401			1,670.024		1,403.227
900	1,198.161			2,204.351		1,956.338
1,000	1,423.880			2,804.093		2,354.519

the key player in large scale problems.

Table 9. Comparison between the approach based on Algorithm 1, three variants of the potential-based approach, and two variants of the best-response method [40] to find the approximated stochastic key player, when $\delta = 0.5$ (CPU times in seconds).

		Potential-based approach			Best-response method	
n	Algorithm 1	`Gurobi`	`quadprog` interior-point	`quadprog` trust-region	Jacobi	Gauss-Seidel
10	0.021	5.882	0.307	3.484	0.025	0.013
20	0.062	11.924	0.654	7.611	0.118	0.059
30	0.138	18.634	1.088	12.481	0.256	0.134
40	0.251	25.993	1.580	17.471	0.445	0.241
50	0.399	34.005	2.143	21.991	0.857	0.472
60	0.624	43.515	2.951	27.570	1.233	0.674
70	0.952	54.180	4.121	33.312	1.773	0.956
80	1.432	66.265	5.376	39.578	2.643	1.456
90	2.021	80.751	6.987	46.149	3.459	1.945
100	2.883	97.537	10.277	54.596	4.372	2.426
200	29.644	630.424	169.477	173.197	27.461	15.983
300	91.677	1,637.397	621.744	445.772	91.666	51.738
400	192.747			827.213		111.295
500	257.223			1,478.674		209.505
600	381.021			2,718.983		478.374
700	629.128			3,989.585		888.171
800	736.379			6,032.577		1,397.354
900	960.393			8,565.410		1,796.690
1,000	1,243.579			11,498.981		2,064.835

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