



Poisson quasi-Nijenhuis deformations of the canonical PN structure

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ABSTRACT

We present a result which allows us to deform a Poisson-Nijenhuis manifold into a Poisson quasi-Nijenhuis manifold by means of a closed 2-form. Under an additional assumption, the deformed structure is also Poisson-Nijenhuis. We apply this result to show that the canonical Poisson-Nijenhuis structure on \mathbb{R}^{2n} gives rise to both the Poisson-Nijenhuis structure of the open (or non periodic) n -particle Toda lattice, introduced by Das and Okubo [7], and the Poisson quasi-Nijenhuis structure of the closed (or periodic) n -particle Toda lattice, described in our recent work [8].

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1. Introduction

The notion of Poisson-Nijenhuis (PN) manifold [16,12] was introduced in connection with the theory of integrable systems. Such a manifold is endowed with a Poisson tensor π and with a tensor field N of type $(1, 1)$, sometimes called “recursion operator”, which is torsionless and compatible (see Section 2) with π . It is a bi-Hamiltonian manifold, and the traces H_k of the powers of N are in involution with respect to both Poisson brackets induced by the Poisson tensors. A Poisson-Nijenhuis structure for the open (or non periodic) n -particle Toda lattice (see [20] and references therein) was presented in [7] (see also [6,17,19]).

A generalization of PN manifolds was introduced in [21], where a Poisson quasi-Nijenhuis (PqN) manifold was defined to be a Poisson manifold with a compatible tensor field \hat{N} of type $(1, 1)$, whose torsion need not vanish but is controlled by a suitable 3-form ϕ . Since, in general, the traces \hat{H}_k of the powers of \hat{N} are not in involution, no application of PqN manifolds to finite-dimensional integrable systems was found until the paper [8], where we obtained a rather stringent sufficient condition to transform, by means of a 2-form Ω , a PN manifold (\mathcal{M}, π, N) into a PqN manifold $(\mathcal{M}, \pi, \hat{N}, \phi)$, and we found suitable compatibility conditions between π , N and Ω entailing that the functions \hat{H}_k are in involution. Then we applied these results to interpret the well known integrability of the closed Toda lattice in the PqN framework, showing that its integrals of motion are the traces of the powers of a suitable tensor field \hat{N} of type $(1, 1)$, which is a deformation of the Das-Okubo recursion operator N of the open Toda lattice.

In this paper we give a stronger result than the above mentioned deformation scheme, in the sense that the only condition on the 2-form Ω , to obtain a PqN manifold, is $d\Omega = 0$. Under an additional assumption, the deformed structure

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is a PN manifold. Then we apply this result to show that the “canonical” PN structure on \mathbb{R}^{2n} (corresponding to the free-particle case) gives rise to both the PN structure [7] of the open Toda lattice and the PqN structure [8] of the closed Toda lattice.

The organization of this paper is the following. In Section 2 we recall the definitions of PN and PqN manifold. Section 3 is devoted to the (improved version of the) above mentioned deformation theorem, with further considerations on the case where π is non degenerate. This result is applied in Section 4 to a wide class of mechanical systems, including the open and closed n -particle Toda systems, whose PN and PqN structures are interpreted as deformations of the canonical PN structure on \mathbb{R}^{2n} .

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2. Poisson quasi-Nijenhuis manifolds

Let $N : T\mathcal{M} \rightarrow T\mathcal{M}$ be a $(1, 1)$ tensor field on a manifold \mathcal{M} . It is well known that its *Nijenhuis torsion* is defined as

$$T_N(X, Y) = [NX, NY] - N([NX, Y] + [X, NY] - N[X, Y]). \tag{1}$$

We also recall that, given a p -form α , with $p \geq 1$, one can construct another p -form $i_N\alpha$ as

$$i_N\alpha(X_1, \dots, X_p) = \sum_{i=1}^p \alpha(X_1, \dots, NX_i, \dots, X_p), \tag{2}$$

and that i_N is a derivation of degree zero (if $i_N f = 0$ for all functions f).

If π is a Poisson bivector on \mathcal{M} and $\pi^\sharp : T^*\mathcal{M} \rightarrow T\mathcal{M}$ is defined by $\langle \beta, \pi^\sharp\alpha \rangle = \pi(\alpha, \beta)$, then π and N are said to be *compatible* [16] if

$$\begin{aligned} N\pi^\sharp &= \pi^\sharp N^*, \quad \text{where } N^* : T^*\mathcal{M} \rightarrow T^*\mathcal{M} \text{ is the transpose of } N; \\ L_{\pi^\sharp\alpha}(N)X - \pi^\sharp L_X(N^*\alpha) + \pi^\sharp L_{NX}\alpha &= 0, \quad \text{for all 1-forms } \alpha \text{ and vector fields } X. \end{aligned} \tag{3}$$

In [21] a *Poisson quasi-Nijenhuis (PqN) manifold* was defined as a quadruple $(\mathcal{M}, \pi, N, \phi)$ such that:

- the Poisson bivector π and the $(1, 1)$ tensor field N are compatible;
- the 3-forms ϕ and $i_N\phi$ are closed;
- $T_N(X, Y) = \pi^\sharp(i_{X \wedge Y}\phi)$ for all vector fields X and Y , where $i_{X \wedge Y}\phi$ is the 1-form defined as $\langle i_{X \wedge Y}\phi, Z \rangle = \phi(X, Y, Z)$.

A slightly more general definition of PqN manifold was recently proposed in [2].

If $\phi = 0$, then the torsion of N vanishes and \mathcal{M} becomes a *Poisson-Nijenhuis manifold* (see [12] and references therein). In this case, the bivector field π_N defined by $\pi_N^\sharp = N\pi^\sharp$ is a Poisson tensor compatible with π , so that \mathcal{M} is a bi-Hamiltonian manifold. Moreover, the functions

$$H_k = \frac{1}{2k} \text{Tr}(N^k), \quad k = 1, 2, \dots, \tag{4}$$

satisfy $dH_{k+1} = N^*dH_k$, entailing their involutivity with respect to both Poisson brackets induced by π and π_N .

In [8] we called *involutive* a PqN manifold such that the traces (4) of the powers of N are in involution (with respect to the unique Poisson bracket defined on \mathcal{M} , i.e., the one associated with π) and we observed that there are non involutive PqN manifolds. Moreover, we found some hypotheses (see Remark 3) to be added to obtain a class of involutive PqN manifolds, and we gave an application to the closed Toda lattice, whose PqN structure has been defined as a suitable deformation (see next section) of the PN structure [7] of the open Toda lattice.

3. Deformations of PN manifolds

In this section we first present a few basic facts about the theory of PN and PqN manifolds from the view-point of differential graded Lie algebras. Then we prove a result which gives a sufficient condition to deform a PN structure into a PqN one.

First of all, we recall that, given a tensor field $N : T\mathcal{M} \rightarrow T\mathcal{M}$, the usual Cartan differential can be modified as follows,

$$\begin{aligned}
 (d_N\alpha)(X_0, \dots, X_q) &= \sum_{j=0}^q (-1)^j L_{NX_j}(\alpha(X_0, \dots, \widehat{X}_j, \dots, X_q)) \\
 &+ \sum_{i < j} (-1)^{i+j} \alpha([X_i, X_j]_N, X_0, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_q),
 \end{aligned}
 \tag{5}$$

where α is a q -form, the X_i are vector fields, and $[X, Y]_N = [NX, Y] + [X, NY] - N[X, Y]$. Note that $d_N f = N^*df$ for all $f \in C^\infty(\mathcal{M})$. Moreover,

$$d_N = i_N \circ d - d \circ i_N, \tag{6}$$

where i_N is given by (2), and consequently $d \circ d_N + d_N \circ d = 0$. Finally, $d_N^2 = 0$ if and only if the torsion of N vanishes.

We also remind that one can define a Lie bracket between 1-forms on a Poisson manifold (\mathcal{M}, π) as

$$[\alpha, \beta]_\pi = L_{\pi^\sharp\alpha}\beta - L_{\pi^\sharp\beta}\alpha - d\langle\beta, \pi^\sharp\alpha\rangle, \tag{7}$$

and that this Lie bracket can be uniquely extended to all forms on \mathcal{M} in such a way that, if η is a q -form and η' is a q' -form, then $[\eta, \eta']_\pi$ is a $(q + q' - 1)$ -form and

$$(K1) \quad [\eta, \eta']_\pi = -(-1)^{(q-1)(q'-1)}[\eta', \eta]_\pi;$$

$$(K2) \quad [\alpha, f]_\pi = i_{\pi^\sharp\alpha}df = \langle df, \pi^\sharp\alpha \rangle \text{ for all } f \in C^\infty(\mathcal{M}) \text{ and for all 1-forms } \alpha;$$

$$(K3) \quad [\eta, \cdot]_\pi \text{ is a derivation of degree } q - 1 \text{ of the wedge product, that is, for any differential form } \eta'',$$

$$[\eta, \eta' \wedge \eta'']_\pi = [\eta, \eta']_\pi \wedge \eta'' + (-1)^{(q-1)q'} \eta' \wedge [\eta, \eta'']_\pi. \tag{8}$$

This extension is a *graded* Lie bracket, in the sense that (besides (K1)) the graded Jacobi identity holds:

$$(-1)^{(q_1-1)(q_3-1)}[\eta_1, [\eta_2, \eta_3]_\pi]_\pi + (-1)^{(q_2-1)(q_1-1)}[\eta_2, [\eta_3, \eta_1]_\pi]_\pi + (-1)^{(q_3-1)(q_2-1)}[\eta_3, [\eta_1, \eta_2]_\pi]_\pi = 0 \tag{9}$$

where q_i is the degree of η_i . It is sometimes called the Koszul bracket – see, e.g., [10] and references therein. We warn the reader that the Koszul bracket we used in [8] is the opposite of the one used here, since a minus sign in (K2) was inserted. For future reference, we remark that (K2) holds for any differential form η and for all $f \in C^\infty(\mathcal{M})$. More precisely,

$$[f, \eta]_\pi = i_{\pi^\sharp df} \eta. \tag{10}$$

Indeed, $[f, \cdot]_\pi$ and $i_{\pi^\sharp df}$ coincide on 0-forms (they both vanish) and on 1-forms (thanks to (K1) and (K2)), and they are derivations of degree -1 with respect to the wedge product.

It was proved in [11] that the compatibility conditions (3) between a Poisson tensor π and a tensor field $N : T\mathcal{M} \rightarrow T\mathcal{M}$ hold if and only if d_N is a derivation of $[\cdot, \cdot]_\pi$, that is,

$$d_N[\eta, \eta']_\pi = [d_N\eta, \eta']_\pi + (-1)^{(q-1)}[\eta, d_N\eta']_\pi \tag{11}$$

if η is a q -form and η' is any differential form. In particular, taking $N = \text{Id}$, one has that the Cartan differential d is always a derivation of $[\cdot, \cdot]_\pi$.

Remark 1. It is worth noting that (K2) is a consistency requirement. In fact, for $f, g \in C^\infty(\mathcal{M})$ one has that

$$d[dg, f]_\pi \stackrel{(11)}{=} [dg, df]_\pi \stackrel{(7)}{=} d\{g, f\}. \tag{12}$$

On the other hand,

$$d[dg, f]_\pi \stackrel{(K2)}{=} d(i_{\pi^\sharp dg}(df)) = d\{g, f\}. \tag{13}$$

The following result generalizes Theorem 3 in [8].

Theorem 2. Let (\mathcal{M}, π, N) be a PN manifold and let Ω be a closed 2-form. If $\widehat{N} = N + \pi^\sharp\Omega^b$, where $\Omega^b : T\mathcal{M} \rightarrow T^*\mathcal{M}$ is defined as usual by $\Omega^b(X) = i_X\Omega$, and

$$\phi = d_N\Omega + \frac{1}{2}[\Omega, \Omega]_\pi, \tag{14}$$

then $(\mathcal{M}, \pi, \widehat{N}, \phi)$ is a PqN manifold. In particular, if

$$d_N \Omega + \frac{1}{2}[\Omega, \Omega]_\pi = 0, \tag{15}$$

then $(\mathcal{M}, \pi, \widehat{N})$ is a PN manifold.

Proof. First we prove the compatibility between π and \widehat{N} . If $d_{\widehat{N}} = d_N + d_{\pi^\sharp \Omega^b}$ is the differential defined by $\widehat{N} = N + \pi^\sharp \Omega^b$, we observe that

$$d_{\pi^\sharp \Omega^b} = [\Omega, \cdot]_\pi. \tag{16}$$

Indeed, both the left and the right-hand side of the previous formula are graded derivations with respect to the wedge product, anti-commuting with d (since $d\Omega = 0$), and coinciding on 0-forms. To prove the last assertion, we first notice that, for all $f \in C^\infty(\mathcal{M})$,

$$d_{\pi^\sharp \Omega^b} f = (\pi^\sharp \Omega^b)^*(df) = \Omega^b(\pi^\sharp df) = i_{\pi^\sharp} df \Omega = [\Omega, f]_\pi,$$

where we used (10) and (K1) in the last equality. Hence, since d_N and $[\Omega, \cdot]_\pi$ are both derivations of $[\cdot, \cdot]_\pi$, it follows at once that $d_{\widehat{N}}$ is also a derivation of $[\cdot, \cdot]_\pi$, yielding the compatibility between \widehat{N} and π . The closedness of ϕ easily follows from that of Ω , recalling that $d \circ d_N = -d_N \circ d$ and that d is a derivation of $[\cdot, \cdot]_\pi$. Moreover,

$$d_{\widehat{N}} \phi = d_N \phi + [\Omega, \phi]_\pi \stackrel{(14)}{=} d_N \left(d_N \Omega + \frac{1}{2}[\Omega, \Omega]_\pi \right) + \left[\Omega, d_N \Omega + \frac{1}{2}[\Omega, \Omega]_\pi \right]_\pi = \frac{1}{2}[\Omega, [\Omega, \Omega]_\pi]_\pi = 0 \tag{17}$$

thanks to $d_N^2 = 0$, the Leibniz rule (11) for d_N , the commutation rule (K1), and the graded Jacobi identity (9). Then it follows from (6) and $d\phi = 0$ that $i_{\widehat{N}} \phi$ is closed. Finally, with the same argument used to show (16), we can prove that $d_{\widehat{N}}^2 = [\phi, \cdot]_\pi$. Indeed, using again $d_N^2 = 0$, we obtain

$$\begin{aligned} d_{\widehat{N}}^2 f &= (d_N + [\Omega, \cdot]_\pi)^2 f = (d_N + [\Omega, \cdot]_\pi)(d_N f + [\Omega, f]_\pi) \\ &= d_N^2 f + d_N[\Omega, f]_\pi + [\Omega, d_N f] + [\Omega, [\Omega, f]_\pi]_\pi \\ &\stackrel{(11)}{=} [d_N \Omega, f]_\pi - \cancel{[\Omega, d_N f]_\pi} + \cancel{[\Omega, d_N f]_\pi} + [\Omega, [\Omega, f]_\pi]_\pi \\ &\stackrel{(9)}{=} [d_N \Omega, f]_\pi + \frac{1}{2}[[\Omega, \Omega]_\pi, f]_\pi = [\phi, f]_\pi. \end{aligned} \tag{18}$$

To conclude the proof of the first assertion in the theorem, it suffices to use the fact (see [21]) that, for any 3-form ϕ ,

$$d_{\widehat{N}}^2 = [\phi, \cdot]_\pi \quad \text{if and only if} \quad \begin{cases} T_{\widehat{N}}(X, Y) = \pi^\sharp(i_{X \wedge Y} \phi) & \text{for all vector fields } X, Y \\ i_{(\pi^\sharp \alpha) \wedge (\pi^\sharp \beta) \wedge (\pi^\sharp \gamma)}(d\phi) = 0 & \text{for all 1-forms } \alpha, \beta, \gamma \end{cases} \tag{19}$$

The proof of the second assertion simply follows, recalling that a PqN manifold $(\mathcal{M}, \pi, \widehat{N}, \phi)$ whose 3-form ϕ vanishes is a PN manifold. \square

We remark that in a similar way one can prove the following result (see [8]): If $(\mathcal{M}, \pi, N, \phi)$ is a PqN manifold, Ω is a closed 2-form such that

$$-\phi = d_N \Omega + \frac{1}{2}[\Omega, \Omega]_\pi,$$

and $\widehat{N} = N + \pi^\sharp \Omega^b$, then $(\mathcal{M}, \pi, \widehat{N})$ is a PN manifold.

We also notice that, in the case where $N = \text{Id}$, equation (15) was studied in [14], in the framework of the theory of Manin triples for Lie algebroids. Starting from a Poisson manifold (\mathcal{M}, π) , it was shown that every solution of

$$d\Omega + \frac{1}{2}[\Omega, \Omega]_\pi = 0 \tag{20}$$

defines a Dirac subbundle $\Gamma_\Omega \subset T^* \mathcal{M} \oplus T \mathcal{M}$ transversal to $T^* \mathcal{M}$, and that every solution of

$$d\Omega = 0 \quad \text{and} \quad [\Omega, \Omega]_\pi = 0 \tag{21}$$

defines a PN structure on \mathcal{M} . As we will see in Corollary 8, the latter result becomes an equivalence under the further hypothesis that π is non degenerate.

Remark 3. The involutivity problem of a PqN structure, that is, the possible involutivity of the functions $\widehat{H}_k = \frac{1}{2k} \text{Tr}(\widehat{N}^k)$, where \widehat{N} is obtained as in Theorem 2, is, in our opinion, an open question that deserves further attention. A preliminary result was obtained in [8], where it was shown that this happens if the following conditions hold:

1. $[\Omega, \Omega]_{\pi} = 0$;
2. $d_N \Omega = d\widehat{H}_1 \wedge \Omega$;
3. $i_{Y_k} \Omega = 0$, where $Y_k = \widehat{N}^{k-1} X_1 - X_k$ and $X_k = \pi^{\sharp} d\widehat{H}_k$;
4. $\{\widehat{H}_1, \widehat{H}_k\} = 0$ for all $k \geq 2$.

Then this result was applied to the PqN structure of the closed Toda lattice (see Example 12), seen as a deformation of the PN structure of the open Toda lattice.

3.1. Deformations of symplectic-Nijenhuis manifolds

In this subsection we consider the important, though particular, case of a PN manifold (\mathcal{M}, π, N) whose Poisson tensor is non degenerate, i.e., such that π^{\sharp} is invertible, so that it defines a symplectic form ω via the identity

$$\pi^{\sharp} \omega^b = -\text{Id}. \tag{22}$$

The relevance of this case stems from the theory of classical integrable systems, where \mathcal{M} is a cotangent bundle endowed with its canonical symplectic structure.

Remark 4. The defining relation (22) is the same chosen in [23], but we alert the reader that it is not universally adopted. For example, the authors of [21] chose $\pi^{\sharp} \omega^b = +\text{Id}$.

Following [21,23], we introduce the notion of a *symplectic-Nijenhuis manifold*, which is a Poisson-Nijenhuis manifold (\mathcal{M}, π, N) whose Poisson tensor π is non degenerate. Hereafter a symplectic-Nijenhuis manifold will be denoted by a triple (\mathcal{M}, ω, N) where ω is the symplectic form defined in (22). In this more specialized framework, the properties of the PN structure find a covariant analogue. In particular, the bilinear form $\omega_N : T\mathcal{M} \times T\mathcal{M} \rightarrow \mathbb{R}$ defined by

$$\omega_N(X, Y) = \omega(NX, Y),$$

where X, Y are vector fields on \mathcal{M} , is skew-symmetric; hence it defines a 2-form on M , called the *associated 2-form* of (\mathcal{M}, ω, N) . The following result provides a characterization of the associated 2-forms, see [23, Theorem 2.1].

Proposition 5. *The associated 2-form of a symplectic-Nijenhuis manifold (\mathcal{M}, ω, N) satisfies the conditions*

$$d\omega_N = 0 \quad \text{and} \quad [\omega_N, \omega_N]_{\pi} = 0, \tag{23}$$

where π is the Poisson tensor corresponding to ω . On the other hand, if a 2-form ω_N on a symplectic manifold (\mathcal{M}, ω) satisfies (23), then ω_N is the associated 2-form of the symplectic-Nijenhuis manifold (M, ω, N) , where $N = (\omega^b)^{-1} \omega_N^b = -\pi^{\sharp} \omega_N^b$.

Remark 6. For future reference, we notice that (16) entails

$$[\omega, \cdot]_{\pi} = d_{\pi^{\sharp} \omega^b} = d_{-\text{Id}} = -d, \quad [\omega_N, \cdot]_{\pi} = d_{\pi^{\sharp} \omega_N^b} = -d_N. \tag{24}$$

Now we show that the sufficient condition contained in the PN part of Theorem 2 is also necessary when π is non degenerate.

Proposition 7. *Let (\mathcal{M}, ω, N) be a symplectic-Nijenhuis manifold, \widehat{N} a (1,1) tensor field, and Ω the (0,2) tensor field defined by*

$$\widehat{N} = N + \pi^{\sharp} \Omega^b, \tag{25}$$

where π is given by (22). Then \widehat{N} is torsionless and compatible with π if and only if Ω is a closed 2-form such that

$$d_N \Omega + \frac{1}{2} [\Omega, \Omega]_{\pi} = 0. \tag{26}$$

Proof. If the above mentioned conditions on Ω hold, then Theorem 2 implies that $(\mathcal{M}, \pi, \widehat{N})$ is a PN manifold. Viceversa, multiplying (25) on the left by the inverse of π^{\sharp} and considering the associated 2-forms ω_N and $\omega_{\widehat{N}}$, one obtains

$$\Omega = \omega_N - \omega_{\widehat{N}}, \tag{27}$$

showing that Ω is a closed 2-form thanks to Proposition 5. Moreover, from (18) and $d_N^2 = d_{\widehat{N}}^2 = 0$ we have that $\phi = d_N \Omega + \frac{1}{2} [\Omega, \Omega]_{\pi}$ satisfies $[\phi, f]_{\pi} = 0$ for all functions f . Then (10) implies that $i_{\pi^{\sharp} d f} \phi = 0$, so that $\phi = 0$ follows from the fact that π is non degenerate. \square

Putting $N = \text{Id}$, we obtain the following corollary, to be compared with the results of [14], Section 6, and those of [3], Section 4.4.

Corollary 8. Let (\mathcal{M}, ω) be a symplectic manifold and let π be the Poisson tensor defined by ω , see (22). A tensor field \widehat{N} of type (1,1) is torsionless and compatible with π if and only if the (0, 2) tensor field Ω defined by

$$\widehat{N} = \text{Id} + \pi^\sharp \Omega^b \tag{28}$$

is a 2-form satisfying the conditions

$$d\Omega = 0 \quad \text{and} \quad [\Omega, \Omega]_\pi = 0. \tag{29}$$

In the next section we will deform the so called *canonical PN structure*, i.e., the symplectic-Nijenhuis manifold $(\mathbb{R}^{2n}, \pi, N)$, where $\pi = \sum_{i=1}^n \partial_{p_i} \wedge \partial_{q_i}$ is the canonical Poisson tensor and

$$N = \sum_{i=1}^n p_i (\partial_{q_i} \otimes dq_i + \partial_{p_i} \otimes dp_i) \tag{30}$$

is easily seen to be torsionless and compatible with π . It was shown in [22] that this is the local form of any $2n$ -dimensional symplectic-Nijenhuis manifold, provided that the eigenvalues of the recursion operator are n independent functions p_1, \dots, p_n . In this context, $(q_1, \dots, q_n, p_1, \dots, p_n)$ were called Darboux-Nijenhuis coordinates and used as separation variables in [9]. In the following example we show that N itself can be seen as a deformation.

Example 9. The above mentioned symplectic-Nijenhuis manifold $(\mathbb{R}^{2n}, \pi, N)$ can be obtained, using Corollary 8, as a deformation of the symplectic manifold $(\mathbb{R}^{2n}, \omega)$, where $\omega = \sum_{i=1}^n dp_i \wedge dq_i$ is the canonical symplectic form, i.e., the one corresponding to π . Indeed, let us consider the 2-form

$$\Omega = \sum_{i=1}^n dp_i \wedge dq_i - \sum_{i=1}^n p_i dp_i \wedge dq_i. \tag{31}$$

Then $N = \text{Id} + \pi^\sharp \Omega^b$, and Ω can be checked to be a solution of (29). Moreover, observe that, according to (27), $\Omega = \omega - \omega_N$, where $\omega_N = \sum_{i=1}^n p_i dp_i \wedge dq_i$ is the 2-form associated to N .

The final part of this section is devoted to two remarks on Corollary 8.

Remark 10. If (\mathcal{M}, ω, N) is a symplectic-Nijenhuis manifold, then the 2-form Ω defined by $N = \text{Id} + \pi^\sharp \Omega^b$ satisfies (29). One can show that the pair (π, Ω) endows \mathcal{M} with a $P\Omega$ structure (see [16,13]), in the sense that π is a Poisson tensor, Ω is closed, and $\Omega^b \pi^\sharp \Omega^b$ is closed too. Indeed, $(\mathcal{M}, \omega, \pi^\sharp \Omega^b)$ is also a symplectic-Nijenhuis manifold, so that the same is true for $(\mathcal{M}, \omega, (\pi^\sharp \Omega^b)^2)$. But the 2-form associated to this latter is $-\Omega^b \pi^\sharp \Omega^b$, and this is closed by Proposition 5. Since $\Omega^b \pi^\sharp \Omega^b$ is closed, one can easily check that the pair (Ω, N) endows \mathcal{M} with an ΩN structure [15], in the sense that Ω and $\Omega^b N$ are both closed, and $\Omega^b N = N^* \Omega^b$. It follows that (Ω_{N^k}, N) is an ΩN structure too, where $(\Omega_{N^k})^b = \Omega^b N^k$. In particular, all the 2-forms Ω_{N^k} are closed.

Considering the symplectic-Nijenhuis manifold $(\mathcal{M}, \omega, N^k)$, one can introduce the closed 2-form Ω_k defined by $N^k = \text{Id} + \pi^\sharp \Omega_k^b$ and check that, for all $k \geq 1$,

$$\Omega_k = \sum_{l=0}^{k-1} \Omega_{N^l}.$$

Indeed, a simple induction shows that

$$N^k = \text{Id} + \pi^\sharp \left(\sum_{l=0}^{k-1} \Omega^b N^l \right).$$

Remark 11. It is well known (see, e.g., [15,1]) that the closed 1-forms α on a symplectic-Nijenhuis manifold (\mathcal{M}, ω, N) that fulfills $d_N \alpha = 0$ (or, equivalently, $d(N^* \alpha) = 0$) play a fundamental role in the applications to integrable systems. Such forms coincide with the (closed) ones such that $L_{\pi^\sharp \alpha} \Omega = 0$. Indeed,

$$L_{\pi^\sharp \alpha} \Omega = d(i_{\pi^\sharp \alpha} \Omega) = d(\Omega^b \pi^\sharp \alpha) = d(N^* \alpha).$$

In the case discussed in and before Example 9, it can be easily checked that the closed 1-form such that $L_{\pi^\sharp \alpha} \Omega = 0$ are the differentials of the functions depending only on p_1, \dots, p_n . The algebra of these functions is abelian (with respect to the Poisson bracket) and is generated by the traces of the powers of N .

4. Deformations of the canonical PN structure

In this section we start from the canonical PN structure $(\mathbb{R}^{2n}, \pi, N)$, described before Example 9, and we apply Theorem 2 to recover the PN (respectively PqN) structure of the open (respectively closed) Toda lattice. The traces of the powers of N are simply $H_k = \frac{1}{2k} \text{Tr}(N^k) = \sum_{i=1}^n p_i^k$, so that from the physical point of view the starting point is nothing but the free-particle case. On the contrary, in [8] we used a preliminary version of Theorem 2 to construct the PqN structure of the closed Toda lattice out of the PN structure of the open one.

We consider the 2-form

$$\Omega = \sum_{i < j} (V_{ij}(q_i - q_j) dq_j \wedge dq_i + dp_j \wedge dp_i), \tag{32}$$

where V_{ij} are C^∞ functions of a single variable. One has that $\Omega = d\theta$, where

$$\theta = \sum_{i < j} \left(-\tilde{V}_{ij} dq_i + p_j dp_i \right) \tag{33}$$

and \tilde{V}_{ij} is a primitive function of V_{ij} . The deformed tensor field $\hat{N} = N + \pi^\sharp \Omega^\flat$ turns out to be

$$\hat{N} = \sum_{i=1}^n p_i (\partial_{q_i} \otimes dq_i + \partial_{p_i} \otimes dp_i) + \sum_{i < j} (\partial_{q_i} \otimes dp_j - \partial_{q_j} \otimes dp_i) + \sum_{i < j} V_{ij} (\partial_{p_j} \otimes dq_i - \partial_{p_i} \otimes dq_j). \tag{34}$$

We refer to Subsection 4.1 for the matrix form of \hat{N} in the case $n = 2$, from which the general expression is easily guessed. Theorem 2 entails that $(\mathbb{R}^{2n}, \pi, \hat{N}, \phi)$ is a PqN manifold, where $\phi = d_N \Omega + \frac{1}{2} [\Omega, \Omega]_\pi$. One can show that

$$\begin{aligned} d_N \Omega &= \sum_{i < j} V_{ij} dq_i \wedge dq_j \wedge (dp_i + dp_j), \\ [\Omega, \Omega]_\pi &= 2 \sum_{i < j} V'_{ij} dq_i \wedge dq_j \wedge \sum_{k < l} ((\delta_{il} - \delta_{jl}) dp_k + (\delta_{jk} - \delta_{ik}) dp_l), \end{aligned} \tag{35}$$

where δ_{ij} is the Kronecker delta. Notice that $H_2 = \frac{1}{4} \text{Tr}(\hat{N}^2) = \frac{1}{2} \sum_{i=1}^n p_i^2 + \sum_{i < j} V_{ij}(q_i - q_j)$ is the Hamiltonian of n interacting particles (with unit mass) and that $\{H_1, H_k\} = 0$ for all k , since \hat{N} depends only on the differences $q_i - q_j$. However, in general $\{H_j, H_k\} \neq 0$, i.e., the PqN structure is not involutive (see [8] and the Calogero example below).

Example 12. We consider the case where $V_{i,i+1}(x) = f_i e^x$ for all $i = 1, \dots, n - 1$ and $V_{1n}(x) = f_n e^{-x}$ (where the f_i are some constants), the other V_{ij} vanishing. The deformed tensor field \hat{N} is given by

$$\begin{aligned} \hat{N} &= \sum_{i=1}^n p_i (\partial_{q_i} \otimes dq_i + \partial_{p_i} \otimes dp_i) + \sum_{i < j} (\partial_{q_i} \otimes dp_j - \partial_{q_j} \otimes dp_i) \\ &+ \sum_{i=1}^{n-1} f_i e^{q_i - q_{i+1}} (\partial_{p_{i+1}} \otimes dq_i - \partial_{p_i} \otimes dq_{i+1}) - f_n e^{q_n - q_1} (\partial_{p_1} \otimes dq_n - \partial_{p_n} \otimes dq_1), \end{aligned} \tag{36}$$

while $\phi = 2 f_n e^{q_n - q_1} dq_1 \wedge dq_n \wedge \sum_{i=1}^n dp_i$.

If $f_i = 1$ for all $i = 1, \dots, n$, then we obtain the PqN structure of the closed Toda lattice, see [8]. The H_k can be shown to be in involution since the PqN manifold is involutive. For example,

$$H_1 = \frac{1}{2} \text{Tr}(\hat{N}) = \sum_{i=1}^n p_i, \quad H_2 = \frac{1}{4} \text{Tr}(\hat{N}^2) = \frac{1}{2} \sum_{i=1}^n p_i^2 + \sum_{i=1}^{n-1} e^{q_i - q_{i+1}} + e^{q_n - q_1} \tag{37}$$

are respectively the total momentum and the energy.

If $f_n = 0$, then

$$\phi = d_N \Omega + \frac{1}{2} [\Omega, \Omega]_\pi = 0, \tag{38}$$

and we obtain the PN structure introduced in [7] to study the open Toda lattice (with generic values of the constants f_i , for $i = 1, \dots, n - 1$).

Remark 13. Using the notations of Example 9, we note that one can recover the Das-Okubo (1, 1) tensor field

$$\widehat{N} = \sum_{i=1}^n p_i (\partial_{q_i} \otimes dq_i + \partial_{p_i} \otimes dp_i) + \sum_{i < j} (\partial_{q_i} \otimes dp_j - \partial_{q_j} \otimes dp_i) + \sum_{i=1}^{n-1} f_i e^{q_i - q_{i+1}} (\partial_{p_{i+1}} \otimes dq_i - \partial_{p_i} \otimes dq_{i+1}), \tag{39}$$

which is simply (36) with $f_n = 0$, by deforming the identity with the 2-form $\widehat{\Omega} = \Omega + \Omega_1$, where Ω is given by (31) and

$$\Omega_1 = \sum_{i=1}^{n-1} e^{q_i - q_{i+1}} dq_{i+1} \wedge dq_i + \sum_{i < j} dp_j \wedge dp_i, \tag{40}$$

see (32). This means that $\widehat{N} = \text{Id} + \pi^\sharp \widehat{\Omega}^\flat$ and

$$d\widehat{\Omega} + \frac{1}{2}[\widehat{\Omega}, \widehat{\Omega}]_\pi = 0.$$

Note that since $\widehat{\Omega}$ is closed, one should have $[\widehat{\Omega}, \widehat{\Omega}]_\pi = 0$. To double check the latter, we recall that $\Omega = \omega - \omega_N$ is a solution of (29), so that

$$[\widehat{\Omega}, \widehat{\Omega}]_\pi = [\Omega, \Omega]_\pi + 2[\Omega, \Omega_1]_\pi + [\Omega_1, \Omega_1]_\pi = 2[\omega, \Omega_1]_\pi - 2[\omega_N, \Omega_1]_\pi + [\Omega_1, \Omega_1]_\pi, \tag{41}$$

where $[\omega, \Omega_1]_\pi = 0$ since $[\omega, \cdot]_\pi = -d$, providing the required identity thanks to $[\omega_N, \cdot]_\pi = -d_N$ (see Remark 6) and the fact that Ω_1 solves (38).

Example 14. Another interesting particular case of (32) and (34) is $V_{ij}(x) = x^{-2}$ for all i, j . We have that

$$H_2 = \frac{1}{4} \text{Tr}(\widehat{N}^2) = \frac{1}{2} \sum_{i=1}^n p_i^2 + \sum_{i < j} (q_i - q_{i+1})^{-2}$$

is the Hamiltonian of the rational Calogero system. However, one can check that the corresponding PqN manifold is involutive for $n = 3$ but not for $n = 4$. In other words, the powers of the traces of \widehat{N} give the integral of motions of the Calogero system only for $n = 3$. The question whether the general rational Calogero system can be framed within the theory of PqN manifolds is still open.

4.1. The case $n = 2$

In this subsection we present some explicit formulas for the trivial case $n = 2$. We first notice that, given a bivector π and a coordinate system (x_1, \dots, x_n) on a manifold \mathcal{M} , we have that $X = \pi^\sharp \alpha$ if and only if $X^i = \pi^{ij} \alpha_j$. Since we prefer to use column rather than row vectors, whenever we write $\pi^\sharp = A$ and A is a matrix, we mean that the (i, j) entry of A is π^{ji} . Also, if Ω is a 2-form and $\Omega^\flat = A$, then $A_{ij} = \Omega_{ji}$. For the same reason, when N is a (1,1) tensor field and we write $N = A$, we mean that the (i, j) entry of A is N^i_j .

We start with the canonical PN structure on \mathbb{R}^4 , defined by

$$\pi^\sharp = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array} \right), \quad N = \left(\begin{array}{cc|cc} p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ \hline 0 & 0 & p_1 & 0 \\ 0 & 0 & 0 & p_2 \end{array} \right). \tag{42}$$

Then we consider the 2-form $\Omega = V(q_1, q_2) dq_2 \wedge dq_1 + dp_2 \wedge dp_1$, where V is any C^∞ function of two variables. One has that $\Omega = d\theta$, where $\theta = \tilde{V} dq_1 + p_2 dp_1$ and \tilde{V} is any function such that $\partial_{q_2} \tilde{V} = V$. The deformed tensor field turns out to be

$$\widehat{N} = N + \pi^\sharp \Omega^\flat = \left(\begin{array}{cc|cc} p_1 & 0 & 0 & 1 \\ 0 & p_2 & -1 & 0 \\ \hline 0 & -V & p_1 & 0 \\ V & 0 & 0 & p_2 \end{array} \right).$$

One can check that

$$d_N \Omega = V dq_1 \wedge dq_2 \wedge (dp_1 + dp_2), \quad [\Omega, \Omega]_\pi = 2dq_1 \wedge dq_2 \wedge ((\partial_{q_2} V) dp_1 - (\partial_{q_1} V) dp_2), \tag{43}$$

and apply Theorem 2 to show that $(\mathbb{R}^4, \pi, \widehat{N}, \phi)$ is a PqN manifold, where $\phi = d_N \Omega + \frac{1}{2}[\Omega, \Omega]_{\pi}$. Notice that $\phi = 0$ (i.e., the manifold is PN) if and only if $V = f_1 e^{q_1 - q_2}$ for some constant f_1 , corresponding to the 2-particle Toda system.

It is immediate to check that this PqN manifold is involutive if and only if $V(q_1, q_2)$ depends only on the difference $q_1 - q_2$. However, as seen in Remark 3, one of the hypotheses of the involutivity Theorem 6 in [8] is that $[\Omega, \Omega]_{\pi} = 0$, and one can see from the second of (43) that this is true if and only if V is constant. Similar considerations hold for the other hypotheses. Hence this theorem cannot be applied to the very simple example $V = V(q_1 - q_2)$. In particular, the well known integrability of the Toda lattice associated to the simple Lie algebra B_2 , whose potential is $V(q_1, q_2) = e^{q_1 - q_2} + e^{q_2}$, cannot be interpreted with the help of the PqN manifold $(\mathbb{R}^4, \pi, \widehat{N}, \phi)$. We leave for future investigations how to suitably modify the 2-form Ω , and we close our paper with an application of Proposition 7 to the case of the so called orthogonal Toda systems.

Example 15. A bi-Hamiltonian formulation for the open orthogonal Toda systems was described in [18], see also [5,4]. In particular, the authors of the (first) above cited reference found a Poisson tensor $\widehat{\pi}$ which, together with the canonical one π , forms a PN structure for the Toda systems whose Hamiltonians are

$$H(p, q) = \frac{1}{2} \sum_{i=1}^n p_i^2 + \sum_{i=1}^{n-1} e^{q_i - q_{i+1}} + e^{kq_n}, \tag{44}$$

where $k = 1$ (respectively, $k = 2$) for the Lie algebras B_n (respectively, C_n). The framework of [18] is the same of Proposition 7. More precisely, this result guarantees that the 2-form Ω defined by $\widehat{N} = N + \pi^{\sharp} \Omega^b$, where N is as in (30) and $\widehat{N} = \widehat{\pi}^{\sharp} \pi^{\sharp - 1}$, is closed. Said differently, one can think that the Poisson tensor $\widehat{\pi}$ described in [18] can be obtained as a deformation of the tensor N by a suitable closed 2-form Ω . For the reader convenience, we will compute below the 2-form Ω providing this deformation in the B_2 case.

To this end, first we recall [18, equation (38)] that

$$\widehat{\pi}^{\sharp} = \left(\begin{array}{cc|cc} 0 & 2p_2 & -p_1^2 - 2e^{q_1 - q_2} & e^{q_1 - q_2} - 2e^{q_2} \\ -2p_2 & 0 & -e^{q_1 - q_2} & -p_2^2 - 2e^{q_2} \\ \hline p_1^2 + 2e^{q_1 - q_2} & e^{q_1 - q_2} & 0 & e^{q_1 - q_2}(p_1 + p_2) \\ 2e^{q_2} - e^{q_1 - q_2} & p_2^2 + 2e^{q_2} & -e^{q_1 - q_2}(p_1 + p_2) & 0 \end{array} \right),$$

which entails that

$$\widehat{N} = \left(\begin{array}{cc|cc} -p_1^2 - 2e^{q_1 - q_2} & e^{q_1 - q_2} - 2e^{q_2} & 0 & -2p_2 \\ -e^{q_1 - q_2} & -p_2^2 - 2e^{q_2} & 2p_2 & 0 \\ \hline 0 & e^{q_1 - q_2}(p_1 + p_2) & -p_1^2 - 2e^{q_1 - q_2} & -e^{q_1 - q_2} \\ -e^{q_1 - q_2}(p_1 + p_2) & 0 & e^{q_1 - q_2} - 2e^{q_2} & -p_2^2 - 2e^{q_2} \end{array} \right).$$

Hence we obtain

$$\Omega^b = \left(\begin{array}{cc|cc} 0 & -e^{q_1 - q_2}(p_1 + p_2) & p_1^2 + 2e^{q_1 - q_2} + p_1 & e^{q_1 - q_2} \\ e^{q_1 - q_2}(p_1 + p_2) & 0 & 2e^{q_2} - e^{q_1 - q_2} & p_2^2 + 2e^{q_2} + p_2 \\ \hline -p_1^2 - 2e^{q_1 - q_2} - p_1 & e^{q_1 - q_2} - 2e^{q_2} & 0 & -2p_2 \\ -e^{q_1 - q_2} & -p_2^2 - 2e^{q_2} - p_2 & 2p_2 & 0 \end{array} \right)$$

or, equivalently,

$$\begin{aligned} \Omega &= e^{q_1 - q_2}(p_1 + p_2) dq_1 \wedge dq_2 - (p_1^2 + 2e^{q_1 - q_2} + p_1) dq_1 \wedge dp_1 - e^{q_1 - q_2} dq_1 \wedge dp_2 \\ &\quad + (e^{q_1 - q_2} - 2e^{q_2}) dq_2 \wedge dp_1 - (p_2^2 + 2e^{q_2} + p_2) dq_2 \wedge dp_2 + 2p_2 dp_1 \wedge dp_2, \end{aligned}$$

that a simple computation shows to be closed.

Data availability

No data was used for the research described in the article.

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