Higher-dimensional gauge theories from string theory.

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We review some recent developments regarding supersymmetric field theories in six and five dimensions. In particular, we will describe the classification of supersymmetric six-dimensional theories with a holographic IIA dual; they are "linear quivers" consisting of chains of many SU (or SO/Sp) gauge groups connected by hypermultiplets and tensor multiplets. We will also describe the wider classification of supersymmetric six-dimensional theories that can be engineered in F-theory; these are also chains, but they include exceptional gauge groups and copies of a more exotic "E-string" theory with a single tensor and E_8 flavor symmetry. Finally we discuss some properties of these theories under compactification to lower dimensions.

1 Introduction

There are many reasons to be interested in field theories in higher dimensions ($d > 4$). Defining interacting models is an interesting theoretical challenge: many which are sensible in four dimensions (and fewer) become sick in higher dimensions. For example, the Yang–Mills Lagrangian has dimension four, and thus $g_{\rm YM}$ in $d > 4$ becomes strong at high energies. This is a bit similar to the notorious problem of the Einstein–Hilbert term in $d > 2$.

String theory gives several ways to construct theories in higher dimensions. A famous example in six dimensions is the theory living on a stack of M5-branes, which has $\mathcal{N} = (2,0)$ supersymmetry [1–3]. Other old examples of string-engineered higher-dimensional QFTs include [4–13]. The M5 theory is also useful in understanding physics in four dimensions: when compactified on a Riemann surface Σ, it results in interesting $\mathcal{N} = 2$ theories in four dimensions whose Seiberg–Witten curve is Σ [14].

In the past few years, several developments have led to renewed interest in field theories in six and five dimensions. First, the recent breakthrough in understanding the M2 action [15–18] was achieved by enlarging one's perspective to M2's at singularities. This achieved weak coupling by introducing a new parameter k, at the (temporary) cost of reducing supersymmetry. It is natural to wonder whether we might similarly achieve progress about the M5 theory by introducing more parameters and reducing to $(1,0)$ supersymmetry. Second, the physics of M5 compactifications on Riemann surfaces has turned out to be even much richer than previously thought: among the resulting $\mathcal{N} = 2$ theories, many are non-Lagrangian and have interesting duality properties [19–21]. Understanding compactifications of $\mathcal{N} = (1,0)$ theories might similarly give rise to interesting results in $\mathcal{N} = 1$ theories in four dimensions. Finally (although we will not review this here) localization techniques have progressed to the point where they can compute for example the index of the $(2,0)$ theory [22, 23].

In this talk, we will review some recent progress. We will begin in section 2 with some basics about six-dimensional field theories, showing how one is led naturally to a certain "linear quiver" structure. In section 3 we will see that these linear quiver theories in fact have holographic duals in type IIA. In section 4 we will review a classification of six-dimensional theories that can be engineered in F-theory. Finally, in section 5, we will review some old and new results about theories in five dimensions, and about compactifications from six and five to four dimensions.

2 6d supersymmetric field theories

Let us then start by considering supersymmetric theories in six dimensions.

First let us consider so-called $\mathcal{N} = (1,0)$ supersymmetry. This superalgebra has 8 (chiral) supercharges, and an Sp(1)≅SU(2) R-symmetry. Its multiplets are [24]:[1]

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[1] Indices from the beginning of the Greek alphabet such as α are fermionic, with upper (lower) ones having positive (nega-

- A vector multiplet, consisting of a gaugino $\lambda^{\alpha a}$ and of a gauge vector A^μ.
- A hypermultiplet, consisting of four scalars q_{ab} and a hyperino $\psi_{\alpha b}$.
- A tensor multiplet, consisting of a single real scalar ϕ, a tensorino $\chi_{\alpha a}$, and of a tensor potential $b_{\mu\nu}$ whose field strength $h_{\mu\nu\rho} \equiv 3\partial_{[\mu} b_{\nu\rho]}$ is self-dual.

The peculiar structure of the tensor multiplet looks a little more natural if one notes that a self-dual three-form h can also be written as a tensor with two symmetric spinor indices $h_{\alpha\beta}$; it is the representation of SU(4) with Dynkin indices $(2,0,0)$ and dimension 10. There is also of course a gravity multiplet which includes an anti-self-dual tensor, but we are interested in field theories and we will not consider it. Notice that the gaugino has chirality opposite to the hyperino and tensorino. Notice also that each of the fermions has a symplectic-Majorana reality condition that halves its degrees of freedom.

For $\mathcal{N} = (2,0)$ supersymmetry, the superalgebra has 16 chiral supercharges, and an sp(2)≅so(5) R-symmetry. It also has a tensor multiplet, which famously has 5 scalars transforming in the fundamental of SO(5). This multiplet describes the degrees of freedom of a single M5; the five scalars parameterize the five transverse directions. This $(2,0)$ tensor decomposes under the $(1,0)$ superalgebra as a $(1,0)$ tensor multiplet plus a hypermultiplet. The theory describing a stack of N M5-branes is more mysterious; it is thought to have N^3 degrees of freedom, but a Lagrangian is not known. The same theory is also realized, via M-theory/IIB duality, on $\mathbb{R}^4/\mathbb{Z}_k$ singularities in IIB [1]. One can consider more generally a singularity $\mathbb{R}^4/\Gamma$, where Γ is a discrete subgroup of SU(2). Such subgroups are associated to the Lie groups of the A, D and E series, and one also uses these labels for the corresponding $\mathcal{N} = (2,0)$ theories.

Let us see now how we can construct $\mathcal{N} = (1,0)$ theories. As we mentioned in the introduction, a higher-dimensional gauge theory becomes strongly coupled at high energies; it is non-renormalizable. Even if we choose to ignore this problem, a supersymmetric gauge theory is plagued by a gauge anomaly, since the gaugino $\lambda^{\alpha a}$ is chiral. We can try to cancel this anomaly by introducing a hypermultiplet, which has a fermion $\psi_{\alpha a}$ of the opposite chirality. The most straightforward possibility is to take this hypermultiplet in the adjoint of the gauge group. This basically leads to the $\mathcal{N} = (1,1)$ vector multiplet. A more imaginative possibility is to take the hypermultiplet in a different representation. For example, if the gauge group is $\mathrm{U}(n_\mathrm{c})$, one can consider taking n_f hypermultiplets in the fundamental. As usual, the anomaly I_6^1 can be obtained by a form I_8 in eight dimensions (a polynomial in the gauge field strength two-form F) via the descent formulas $I_8 = dI_7$, $\delta I_7 = dI_6^1$. In our case it reads

$$I_8 = \mathrm{Tr} F^4 - n_\mathrm{f}\, \mathrm{tr} F^4 \ , \tag{1}$$

where Tr and tr denote the traces in the adjoint and fundamental representation respectively. The generators of the two are related by $T_{\mathrm{ad}\,k\bar{l}}^{\,i\bar{j}} = T^i_{\mathrm{f}\,k}\delta^{\bar{j}}_{\bar{l}} + \delta^i_k T^{\bar{j}}_{\mathrm{f}\,\bar{l}}$; so we get $\mathrm{Tr}F^4 = 2n_\mathrm{c}\mathrm{tr}F^4 + 6(\mathrm{tr}F^2)^2$. We can cancel the term $\mathrm{tr}F^4$ by choosing

$$n_\mathrm{f} = 2n_\mathrm{c} \ . \tag{2}$$

(For $\mathcal{N} = 2$ theories in four dimensions this condition is imposed to get conformal invariance rather than for anomaly cancellation.) However, we are still left with a non-zero $I_8 = 6(\mathrm{tr}F^2)^2$. Fortunately, this is of the form $I_8 = I_4^2$ ($I_4 = \mathrm{tr}F^2$), so we can use the Green–Schwarz–West–Sagnotti (GSWS) [25, 26] mechanism. Defining this time $I_4 = dI_3$, $\delta I_3 = I_2^1 (= \mathrm{tr}(\lambda \wedge dA))$, we see now that $I_7 = I_3 \wedge I_4$, $\delta I_7 = \delta I_3 \wedge I_4 = dI_2^1 \wedge I_4$, so that $I_6^1 = I_2^1 \wedge I_4$. Hence the residual anomaly can be canceled by adding to the action a term

$$\int b_2 \wedge I_4 \ , \tag{3}$$

where under a gauge transformation of parameter λ, $\delta b_2 = I_2^1 = \mathrm{tr}(\lambda \wedge dA)$. The gauge-invariant field-strength $h = db_2 - I_2^1$ now obeys the modified Bianchi identity

$$dh = -I_4 = -\mathrm{tr}F^2 \ . \tag{4}$$

So we can cancel the gauge anomaly, at the cost of introducing a two-form potential b_2 and a contribution (3) to the action. Luckily the tensor multiplet indeed includes such a field; so we can hope to cancel the anomaly while still preserving $\mathcal{N} = (1,0)$ supersymmetry. However, such a b_2 is chiral, $db_2 = h = *h$; so the usual action $\int h \wedge *h$ in fact vanishes. The most common reaction is to consider it as a "pseudo-action", in the sense that the constraint $h = *h$ is added by hand after deriving all the equations of motion; another possibility is to introduce a variant of the PST mechanism [27]. If we choose the first possibility, the term (3) modifies the equations of motion, so that they now read $d*h = -I_4 = -\mathrm{tr}F^2$. Fortunately this is consistent with (4) above; this would not have worked had I_8 been simply factorized as $I_4 \wedge \tilde{I}_4$ (with two different four-forms) rather than being a perfect square.

tive) chirality; from the middle, such as μ, are bosonic. Indices from the beginning of the Latin alphabet, such as a, refer to the fundamental of $\mathrm{Sp}(1)_\mathrm{R}$.

The next point would be to write a Lagrangian for this theory. One problem we already mentioned is the self-duality of h. If we bypass it by accepting to write a pseudo-action, we can specialize the Lagrangians of [28], setting to zero their Stückelberg terms and keeping their "embedding tensor" connecting tensors and vectors. For the theory we just discussed, the bosonic Lagrangian is schematically

$$\begin{aligned}\mathscr{L} \sim \phi\,\mathrm{Tr}(|F|^2 - \mathbf{D}^2) + \partial\phi^2 + |h|^2 \\ + *(b \wedge \mathrm{Tr}(F \wedge F)) + |Dq|^2 + q^\dagger \sigma \cdot \mathbf{D} q\end{aligned} \tag{5}$$

where Dq is the appropriate covariant derivative of the hypermultiplet scalars q_{ab}, and $\mathbf{D}$ is a D-term triplet similar to the one of an $\mathcal{N} = 2$ vector multiplet in four dimensions. $|\ |^2$ denotes the norm of a form by index contraction. It is customary to summarize this Lagrangian by a so-called quiver[2] diagram like the one in figure 1(a); the round node denotes the gauge field, the link denotes the tensor multiplet and the hypermultiplets, and the square node denotes the $\mathrm{U}(n_\mathrm{f})$ flavor symmetry mixing the hypermultiplets.

Eq. (5) overlooks a crucial subtlety. In (1) we have ignored terms containing $\mathrm{tr}F$; such terms make the $\mathrm{U}(1)\subset \mathrm{U}(k)$ subgroup still anomalous, even after all the procedure we just explained. There is again a GSWS-like mechanism that can cancel this anomaly [29]; it involves this time introducing a term $b_0 \mathrm{tr}F^3$, with b_0 a scalar charged under gauge transformations. This requires a term $(\partial b_0 + A)^2$ which makes the $\mathrm{U}(1)\subset \mathrm{U}(k)$ massive. This b_0 can be taken to be a combination of the q_{ab} [6], but the full modification to (5) induced by this has not been worked out to my knowledge.

It is easy to see how to generalize (5). We can imagine for example gauging a subgroup of the flavor group $\mathrm{U}(n_\mathrm{f}) = \mathrm{U}(2n_\mathrm{c})$. For example, we can gauge $\mathrm{U}(n_\mathrm{f}) \subset \mathrm{U}(2n_\mathrm{f})$ by introducing a second gauge field. This leads to the quiver in figure 1(b). If one repeats this process, one obtains a chain of the type in figure 1(c).

One could also have chosen to gauge a different fraction of the flavor symmetry at each step. One arrives in this fashion at chains of gauge groups which are called "linear quivers" (see figure 1(d)), where each horizontal link is again associated to a bifundamental hypermultiplet. The condition (2) should be obeyed for all gauge groups, so that all terms $\mathrm{tr}F_i^4$ in the anomaly polynomial cancel. In some case this is achieved by adding some extra hypermultiplets, the vertical links in figure 1(d). (In the example in that figure, the first gauge group U(4) has 1+7 flavors, the second gauge group U(7) has 4+2+8 flavors, and so on.) Let us call the gauge groups $\mathrm{U}(r_1)$, $\mathrm{U}(r_2)\ldots$, and the vertical extra flavors f_1, $f_2\ldots$. Then (2) reads

$$2r_i - r_{i+1} - r_{i-1} = f_i\ . \tag{6}$$

So in a sense the "discrete second derivative" of the r_i gives the extra flavors f_i. Since these numbers are positive by definition, the r_i describe a convex function of the position i. Thus the r_i will rise, then reach a maximum value k (equal to 10 in figure 1(d)), perhaps plateau there for a while, and then go down again. All these data are sometimes summarized visually by two Young diagrams Y_L and Y_R; see figure 2. (For a review of this combinatorics, see [30].) Thus we will label these theories as

$$\mathcal{T}^N_{Y_\mathrm{L},Y_\mathrm{R}}\ . \tag{7}$$

By definition Y_L and Y_R will have the same total number k of boxes; N will be larger than the sum of their two tallest columns.

Once (6) is satisfied, the leftover gauge anomaly is of the form $C_{ij}\mathrm{tr}F_i^2\mathrm{tr}F_j^2$, where C_{ij} is the Cartan matrix of $\mathrm{SU}(N+1)$, with $N =$ the number of gauge groups; it can again be cancelled by a GSWS mechanism, involving N tensor multiplets b_i. (As we will see later, in the string theory realization it is a little more natural to add an extra decoupled "center of mass" tensor multiplet, and to associate the resulting $N+1$ tensors to the horizontal links in figure 1(d).) The resulting theory is very similar to (5) and we will not write it here.

These linear quiver theories make sense classically, but we now have to worry about their quantum properties. Due to the presence of the gauge fields, one might think the theories are simply non-renormalizable: $\mathrm{Tr}|F|^2 = \frac{1}{2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu}$ has dimension four, which is less than the dimension of the spacetime (much like the Einstein-Hilbert R has dimension two, which is less than the dimension of the spacetime for $d > 2$). However, notice that in (5) the gauge kinetic operator is actually $\phi\mathrm{Tr}|F|^2$, so that $1/g^2_{\mathrm{YM}}$ has been promoted to a scalar. Actually ϕ has dimension 2, so this new kinetic operator has dimension 6, which at least *sounds* more promising. In the general linear quiver theories, this phenomenon also appears, with kinetic terms $\phi_i\mathrm{Tr}|F_i|^2$; alternatively, if one chooses to associate tensor multiplets Φ_i to the quiver links as we mentioned earlier, this reads $(\Phi_i - \Phi_{i+1})\mathrm{Tr}F_i^2$.

[2] A quiver diagram should actually contain dots and *arrows*, as the name indicates; such diagrams are useful for $\mathcal{N} = 1$ theories in four dimensions. For $\mathcal{N} = 2$ theories in four dimensions and $\mathcal{N} = (1,0)$ theories in six dimensions, the direction of the arrows is not important and can be dropped, but the name "quiver" is usually retained.

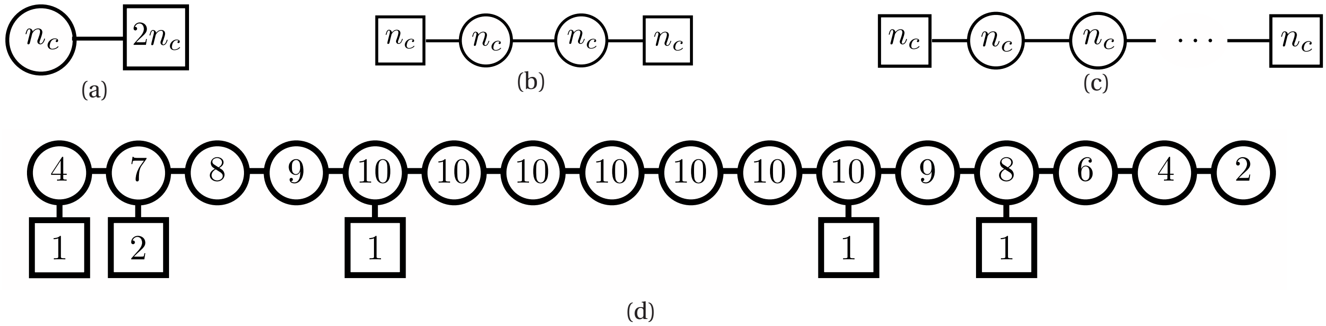


Figure 1 In (a), the quiver summarizing the Lagrangian (5). Gauging an $\mathrm{U}(n_c)$ subgroup of the flavor group $\mathrm{U}(2n_c)$ leads to the theory in (b). Iterating the process leads to the chain in (c). In (d) we see a more general linear quiver; the condition (2) is met at every node.

Figure 2 The Young diagrams Y_L and Y_R associated to the theory in figure 1(d). Their columns are given by $s_i = |r_i - r_{i-1}|$ for both "tails" left and right of the central plateau.

When quantizing (5), one has to choose a vacuum expectation value $\langle\phi\rangle \geq 0$; the space of such possibilities is called "tensor" or sometimes "Coulomb" branch, since it reduces to an ordinary Coulomb branch upon compactification to four dimensions. For $\langle\phi\rangle > 0$, the gauge field will now effectively have an ordinary gauge coupling $\langle\phi\rangle\mathrm{Tr}|F|^2$, and the theory will be non-renormalizable again. If we choose $\langle\phi\rangle = 0$, we cannot conclude this; but it is not clear how to set up perturbation theory around this vacuum. Indeed, if one sees $\langle\phi\rangle \sim 1/g^2_{\mathrm{YM}}$, it is clear that $\langle\phi\rangle = 0$ corresponds to a strong coupling limit: we cannot really use (5) reliably any more. The discussion for the linear quivers is similar: the strong coupling point is now the origin $\langle\phi_i\rangle = 0$ of the tensor branch.

Fortunately, as we will see in the next section, the linear quiver theories can be engineered in string theory; that engineering suggests that in fact $\langle\phi_i\rangle = 0$ corresponds to a CFT. Presumably this CFT has a tensor branch in its moduli space of vacua; choosing a point in it breaks conformal invariance and triggers an RG flow to the linear quiver Lagrangian with $\langle\phi_i\rangle > 0$.

Even though we cannot directly give a Lagrangian description of these CFTs, we can estimate its number of degrees of freedom. The Weyl anomaly reads $\langle T^\mu_\mu\rangle \propto aE + \sum_i c_i I_i$; E is the Euler density and I_i are combinations of the Weyl tensor, of which there are three in six dimensions. a is expected to be monotonic under RG flows, just like in two [31] and four [32] dimensions. Since Weyl transformations are in the same superalgebra as R- and Poincaré symmetries, a is (linearly) related to the R-symmetry and diffeomorphism anomaly [33] (similar formulas can be obtained for the c_i [34]). These in turn can be computed reliably in the Lagrangian of the type (5), since along the tensor branch flow described earlier neither R-symmetry nor diffeomorphisms are broken. This was done in some particular case in [35, 36], and for general linear quivers in [30]. An important contribution is given by the GSWS term (3); in the limit where the number N of gauge groups is large, it gives the dominant term:

$$a \sim \frac{192}{7} C^{-1}_{ij} r_i r_j \,, \tag{8}$$

which goes roughly like N^3. As we will see, this is related to the well-known scaling $a \sim \frac{16}{7}N^3$ for the $\mathcal{N} = (2,0)$ theory on multiple M5s.

Before we turn to the string embedding of these theories, it is natural to wonder if our construction can be generalized further. For example at some point we might have tried to introduces "loops" or "branches" in the quiver, or to use gauge groups other than $\mathrm{SU}(k)$. As we have seen, supersymmetry and anomaly cancellation impose tight constraints even at the classical level. A classification of the possibilities was given recently in [37]; loops are impossible, and branching structures are severely constrained. Gauge groups other than $\mathrm{SU}(k)$ are possible; indeed some of these also have a string theory embedding by adding orientifold actions to the constructions of the next section. The next step would be to try to analyze the actions in [37] quantum-mechanically. As we saw, this is not so easy, and at present we have to rely on the presence of a string theory embedding. We will review a classification of F-theory-engineered theories in section 4.

3 Holographic duals

The linear quivers in the previous section can be realized on a brane system [6, 7]. The configuration involves $N+1$ NS5-branes extended along directions $0\dots5$, a varying number of D6-branes extended along $0\dots6$, and D8-branes extended along all directions except 6. The combinatorial rules to extract the spectrum from such a configuration are similar to the ones in three [38] and four [14] dimensions. For example, we give in figure 3 a brane configuration which engineers the linear quiver in figure 1(d). The stack of r_i D6-branes on the segment between the i-th and $(i+1)$-th NS5s gives rise to the $\mathrm{SU}(r_i)$ gauge group. The i-th NS5 gives rise to a bifundamental hypermultiplet associated to the i-th horizontal link in the quiver, and to the tensor multiplet Φ_i. The scalar in the latter is identified with the position of the NS5 in direction 6; so the conformal point $\Phi_i = \Phi_{i+1}$ on the tensor branch corresponds to the point where the NS5 coincide (figure 4(a)).[3] Finally, the D8's give rise to the f_i fundamental hypermultiplets associated to the vertical links (when they are present).

The traditional way of finding a holographic dual to a brane-engineered theory consists in writing down a gravity solution for the brane system, and performing a near-horizon limit. In this case, this is challenging, because of the notorious problems in finding localized gravity solutions associated to intersecting branes. In the case without D8s, one can consider the known solution for a stack of M5 branes on a $\mathbb{Z}_k$ singularity and reduce to IIA along a Hopf- (or Taub–NUT-) like isometry; the resulting solution describes an NS5-D6 system accurately in the near-D6 region. In M-theory, the near-horizon limit is $\mathrm{AdS}_7 \times S^4/\mathbb{Z}_k$; in IIA, the near-horizon limit of the NS5-D6 system is $\mathrm{AdS}_7 \times S^3$, with two D6 stacks at the poles of the S^3 [41, Sec. 5.1]. This shows that it is possible for an NS5-D6 intersection to have an AdS_7 near-horizon limit; it is dual to the theory in 1(c), which in the language of (7) has both $Y_{\mathrm{L}} = Y_{\mathrm{R}} = [N]$ (a single vertical column). However, it is not clear how to generalize the procedure in this paragraph to include D8-branes.

Fortunately, there is a shortcut: AdS_7 solutions were classified in [41]. In IIB, it was found that there are no solutions; but in IIA infinitely many were obtained, whose analytic form was later uncovered in [42].[4] Even if we cannot follow the near-horizon process that takes us from the brane configuration to the AdS_7 solution, the data of the solutions suggest a correspondence [45] that we will now describe.

First we need a description of the AdS_7 solutions themselves. The internal space M_3 is topologically an S^3; the metric consists of a round S^2, whose volume $v(y)$ varies as a function of an interval $I \ni y$, such that v goes to zero at the two endpoints of the interval I. The simplest example consists of the solution we mentioned earlier, obtained by reduction from eleven dimensions. A more interesting, and almost as simple, example is

$$ds^2 = \frac{n_{\mathrm{D6}}}{F_0}\sqrt{y+2}\left(\frac{4}{3}ds^2_{\mathrm{AdS}_7} + \frac{dy^2}{4(1-y)(y+2)} + \frac{1}{3}\frac{(1-y)(y+2)}{8-4y-y^2}ds^2_{S^2}\right); \tag{9}$$

in this case $y \in [-2, 1] = I$. The active fluxes are F_0 (the so-called "Romans mass"), $F_2 \propto \mathrm{vol}_{S^2}$, $H \propto \mathrm{vol}_{M_3}$. With appropriate coordinate changes one can show that $y = 1$ is a regular point, while at $y = -2$ a stack of n_{D6} D6-branes is present; these D6s are extended along AdS_7. In figure 5 we see a sketch of M_3 in this case.

More general solutions have D8-branes as well as D6-branes. These D8s are again extended along AdS_7, and wrap the S^2 at various particular values y_i of y. Along the S^2 it is also possible to have a topologically non-trivial worldsheet gauge bundle. A bundle on S^2 is always direct sum of bundles of rank 1; their first Chern class $c_1 \equiv \mu$ can also be interpreted as the D6-charge of the D8. In other words, each D8 is in fact a D8-D6 bound state. Supersymmetry fixes the positions y_i of the D8-branes in terms of their D6-charges; so D8s with the same D6 charge μ will be on top of each other. The condition dictates that the radius of the S^2 wrapped by the D8, divided by the string coupling, must be proportional to μ; this is basically a Myers effect [46]. In figure 4(b) we see a typical solution with several D8-branes. The "creases" in the figure represent the D8s, where the metric has an angular point (just like for a D8 in flat space); they give M_3 a "crescent

[3] Irrespectively of our argument in section 2, the presence of a CFT at this point in moduli space is also suggested by the appearance of tensionless strings given by D2-branes suspended between the NS5s. It is also possible to count the states of such strings [39, 40].

[4] For some earlier literature on AdS_7 solutions with branes, see for example [43, 44].

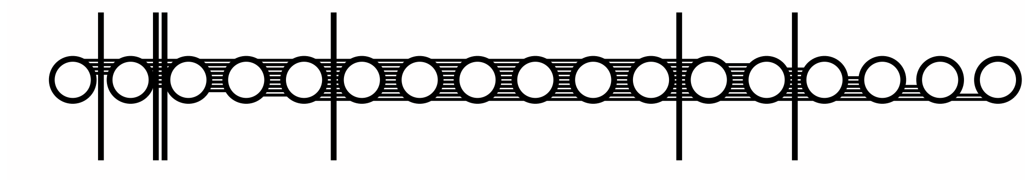
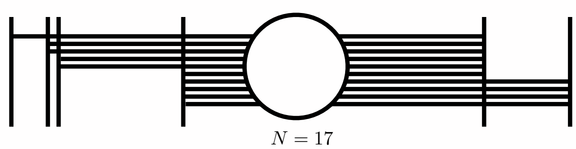
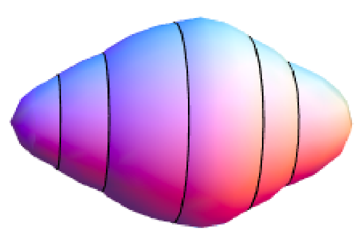


Figure 3 A brane configuration that engineers the linear quiver in figure 1(d). The dots represent NS5-branes, the horizontal lines D6s, the vertical lines D8s.

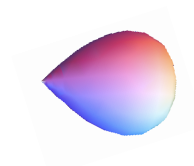


(a)

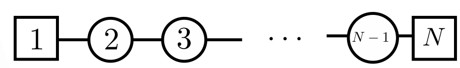


(b)

Figure 4 In (a), the brane configuration in 3 after one puts the NS5s on top of each other. In (b), a sketch of the internal space M_3 for the corresponding AdS_7 solution. The solution is in IIA: there is no lift to eleven dimensions. The D8-branes are still visible as loci where the metric has an angular point, while the D6s have become magnetization on the D8 worldvolume and the NS5 have dissolved into flux. Notice that the Young diagrams Y_L and Y_R in figure 2 summarize visually the numbers of D6s ending on the D8s in (a), the D6-charge of the D8s in (b), and the positions of the D8s in figure 3.



(a)



(b)

Figure 5 In (a), a sketch of M_3 in (9); in (b), the dual quiver. The Young diagrams are $[1^N]$ (single horizontal row) and $[N]$ (single vertical column).

roll"-like shape. (One can also include D6-branes, but in the limit where the size of the M_3 is large these can be approximated by D8s with a very small μ.)

In fact one finds that the data characterizing these theories are the same N, Y_L, Y_R that label the field theories in (7), with the same restrictions mentioned there. This suggests that the set of theories $\mathcal{T}^N_{Y_L,Y_R}$ should be holographically dual to each other [45].

Here is a rough picture of why such a correspondence should be true. One has performed a near-horizon limit on the brane configuration in figure 4(a); this limit gives rise to an M_3 such as the one depicted in figure 4(b). The $N+1$ NS5s dissolve into flux: in other words, in the AdS_7 solution one has a flux integer $\frac{1}{4\pi^2}\int_{M_3} H = N+1$. Each D8 with μ D6's ending on them becomes a D8-D6 bound state with D6 charge equal to μ.[5]

As a cross-check of the proposed holographic duality, notice that the presence of "vertical" fundamental flavors f_i corresponds to the presence of D8-brane stacks in the AdS_7 solution. For example, in figure 1(d) we see 5 non-zero f_i, which correspond to the five D8 stacks depicted in figure 4(b). The value of F_0 jumps across the D8s, as it should; between two D8s related to two non-zero f_i, it is given by $F_0 = \frac{r_i - r_{i-1}}{2\pi}$. In particular, the central plateau in the quiver of figure 1(d) corresponds in figure 4(b) to a central region where $F_0 = 0$. Such a central region is generically present, simply because setting to zero the length of the central plateau is in a sense a fine tuning. For a more detailed review of these aspects see [30].

In any case, a strong check on the proposed correspondence is provided by the holographic computation of the

[5] In a sense, already in the brane picture it is better to understand the D6–D8 system as a "fuzzy funnel", described in gauge theory terms by a "Nahm pole" [47].

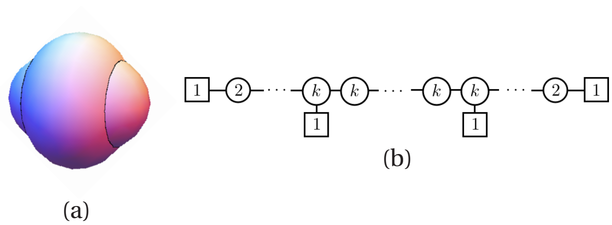


Figure 6 A solution with two D8s of opposite D6 charge, and the corresponding quiver. The Young diagrams are both $[1^N]$ in this case.

a Weyl anomaly [30]. This can be done by adapting the older computation for the $\mathcal{N} = (2,0)$ theory [48], and it reduces to computing $\int_{M_3} e^{5A-2\phi}\mathrm{vol}_3$. (This is also proportional to the coefficient dictating the behavior of the free energy with the temperature, which is another possible measure of the number of degrees of freedom). The result of this integral can then be compared with the field theory computation described around (8). One needs to identify a holographic limit, namely a limit in which the gravity solution is under control. For our linear quiver theories, one has to take to infinity the number of gauge groups N, rather than the size of the individual gauge groups $\mathrm{SU}(r_i)$. This is less strange than it sounds, once one recalls that the case without D8s lifts in eleven dimensions to the theory of $N+1$ M5-branes, where N is indeed the number of gauge groups. Moreover, in order to keep the D8 from shrinking in this limit, one also needs to rescale the positions of all the non-zero f_i.

Consider for example the theory in figure 6(a). In this case, the holographic limit is achieved by taking both N and k to infinity, such that N/k remains fixed. The field theoretic result in this case gives

$$a = \frac{16}{7}k^2\left(N^3 - 4Nk^2 + \frac{16}{5}k^3\right); \tag{10}$$

since N and k are both large, all three terms are of the same order. (Stringy corrections will give terms $O(N)$ or $O(k)$.) This result precisely matches the one obtained holographically as explained above. (The example (10) was already computed in [42]).

One can prove that the agreement works in general [30]. A heuristic argument comes from (8): the Cartan matrix C can be seen as a discrete second derivative, and its inverse as a double primitive. One can then show that the holographic integral for a can be written as $\int \ddot{\alpha}\alpha$, where $\ddot{\alpha}$ is a piecewise linear function that appears in the metric which happens to interpolate between the r_i.

There are several possible extensions. One could for example take an O8 orientifold acting on y. This would introduce an O8-plane at the equator of M_3; on the field theory side it would correspond to having an SO or Sp gauge group at the end of the quiver. Or one could take an O6 orientifold acting on the S^2; this would include an O6-plane at the poles, and would turn the SU quiver into a chain of alternating SO and Sp gauge groups.

Instead of pursuing these generalizations, we will now describe a much more general construction of field theories from F-theory.

4 F-theory engineering

We have seen that the SU linear quivers have an engineering in IIA in terms of NS5s on top of D6s. For the case without D8s, we also mentioned that this can be lifted in M-theory to a configuration of M5s on top of a line of $\mathbb{Z}_k$ singularities; see figure 7. We can read off the quiver in this duality frame as well: M-theory on a $\mathbb{R}^4/\mathbb{Z}_k$ singularity realizes seven-dimensional $\mathrm{SU}(k)$ super-Yang–Mills, and the M5s give boundary conditions that put this 7d theory on a segment; just like in the Hanany–Witten construction, we end up with a copy of six-dimensional super-Yang–Mills for each segment. Instead of going to eleven dimensions, from IIA we can T-dualize to IIB, say along direction 7 (which was transverse to both D6s and NS5s). In this case the D6s turn into D7-branes, but the NS5s turn into geometry; each segment of D6s is now better thought of as a D7 stack on a non-trivial two-cycle (see again figure 7).

This picture works very similarly also for dihedral D_k singularities; one now needs to introduce an O6 on the IIA side. For $G = E_k$, however, we don't have the IIA picture. We still have the M-theory picture: the M5s are now placed on a line of singularities given by quotienting $\mathbb{R}^4$ by Γ_{E_k}, the discrete group associated to E_k. The IIB picture is also available as an F-theory configuration, where this time we have exceptional seven-branes wrapping the non-trivial two-cycles Σ_i. The two pictures are related by M/F-theory duality; see figure 8.

However, while in the A_k case we know that the links in the quiver represent a hypermultiplet plus a tensor multiplet, in the D_k and the E_k case it is not clear what they should represent.

This can be clarified on the F-theory side; we will illustrate it in the case of a link joining to E_8 nodes. The geometry consists of $\mathbb{R}^6 \times M_4$, where M_4 contains a chain of two-cycles Σ_i (as in the bottom of figure 8), on each of which an E_8 seven-brane is present (which is also ex-

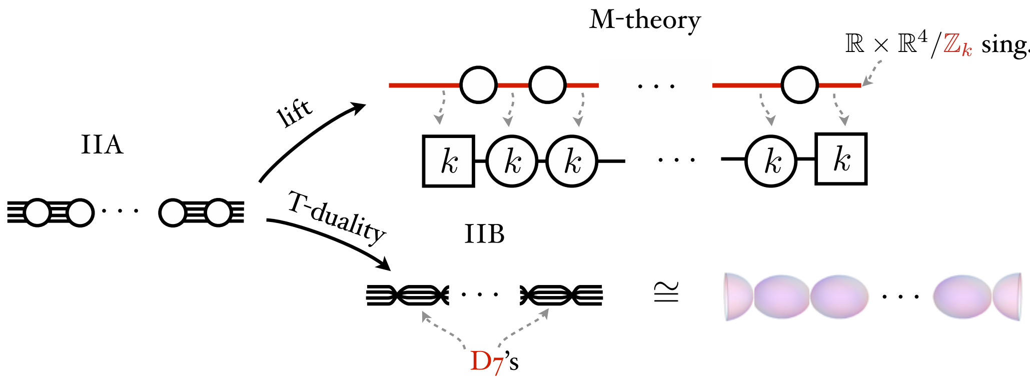


Figure 7 Various ways of realizing the theory in figure 1(c).

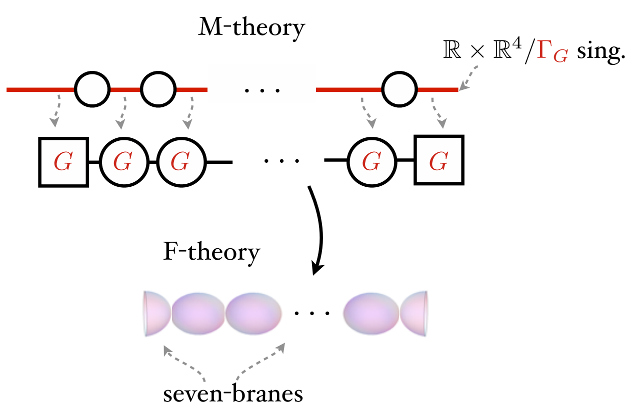


Figure 8 A generalization of the theory in figure 1(c) with exceptional gauge groups.

tended along $\mathbb{R}^6$). As is usual in F-theory, this means that there is a certain non-trivial configuration for the axio-dilaton $\tau = C_0 + ie^{-\phi}$, which is summarized by a torus fibration M_6 over M_4, in the sense that the modular parameter for the torus fibre is equal to τ. The presence of a seven-brane on the Σ_i means that the fiber degenerates in a certain way; the gauge group can be read off from this degeneration. One often writes M_6 as the locus $\{y^2 = x^3 + fx + g\}$, where f and g are functions on M_4, and y, x are local coordinates on a patch of $\mathbb{CP}^2$ or a weighted projective space, where the torus is embedded. The gauge algebra can then be read off by looking at the local behavior of f and g; see for example [49, Table 1] (or [50] for more details).

Near the intersection of two E_8 seven-branes, the fibration reads $y^2 = x^3 + u^5v^5$, where u, v are local coordinates on M_4 and the seven-branes are located at $\{u = 0\}$ and $\{v = 0\}$. Over this point, M_6 has a singularity that goes beyond the cases listed in [49, Table 1]; this means that M_6 cannot be resolved while staying a Calabi–Yau, which in turn means that the equations of motion of F-theory are not satisfied. In order to cure this problem, one has to blow up this singularity; this generates a new two-cycle, on which f and g again degenerate, this time in a way corresponding to an F_4 gauge group (see [49] for a detailed description). Near the intersection of the E_8 and F_4 branes, after a change of coordinates, M_6 now looks like $y^2 = x^3 + u^5v^4$; this again needs to be blown up, generating a new two-cycle with a G_2 gauge group. One has to perform this process several more times, until one gets a model where M_6 is a Calabi–Yau; it contains a total of eleven new two-cycles, most (but not all) of which have a gauge group. One can finally read off the theory associated to this brane configuration: it is illustrated in figure 9. This result appeared in several contexts in F-theory, beginning with [49, 51].

As one can see, there are some "empty nodes", where there is no gauge group. When such a node is marked by a 2, it indeed just means the absence of a gauge group; the tensor multiplets that would be associated to the links are still there, and the node is a placeholder, so to speak. An empty node marked by a 1, however, has a different meaning; it describes a certain peculiar $\mathcal{N} = (1,0)$ CFT$_6$ that we have not discussed so far, called "E-string". This is a non-Lagrangian theory without gauge groups, with an E_8 flavor symmetry, and with a single tensor multiplet. It appears for example when describing M5-branes on an E_8 Hořava–Witten wall [4, 11, 52]. (Recall that here, however, the origin of the E_8 is the Γ_{E_8} singularity; there is no E_8 wall.) Notice that such a 1 node also appears inside the diagram in figure 9, where no E_8 is visible: in this example,

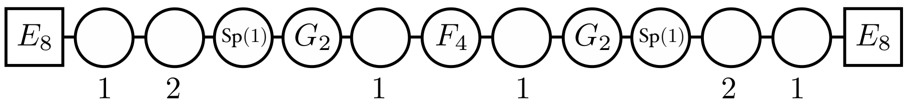


Figure 9 The "conformal matter" theory obtained by repeatedly resolving the contact point between two E_8 seven-branes in F-theory; it represents one of the links in the quiver in figure 8, for $G = E_8$.

it is connected to an F_4 and a G_2 gauge groups. This means that we have gauged a $F_4 \times G_2$ subgroup of the E_8 flavor symmetry of the E-string theory.

Thus, in the quiver with gauge groups $G = E_8$ of figure 8, each link is actually to be understood as one of the chains in figure 9. The results for $G = E_6, E_7, D_k$ can be computed similarly; the resulting chains have 5, 3, 1 inner nodes (rather than 11 as figure 9) respectively and can be found in [49, 51, 53]. These chains are thus "building blocks" that make it possible to construct linear quivers with gauge groups more general than the SU(k)'s of the previous section. This is illustrated by the simple quiver in figure 8, but these building blocks keep appearing prominently in the classification we will soon review. For this reason, it is natural to call these chains "conformal matter" theories. According to this terminology, the theory in figure 9 is thus (E_8, E_8) conformal matter.

Before we go on to more elaborate theories, it is interesting to try to interpret the resolution process in M-theory. Since so far "links" were associated to M5s or NS5s, it is natural to think [53] that the chain in 9 corresponds to a single M5 splitting in 12 "fractional M5s"; see figure 10. Similarly for E_7, E_6, D_k we would have 6, 4, 2 fractions; for A_k no fractionation occurs. (The D_k case is similar to a phenomenon where NS5s split in two half NS5s along an orientifold [54].) While at this point this idea might look speculative, it can be checked in several ways. For example [55, Sec. 3] if one compactifies three directions of the M5s on a T^3, making it very small, one ends up in M-theory again with a dual large T^3; the M5 on the singularity line becomes an M2 realized as a gauge theory instanton on $\mathbb{R} \times T^3$. The fractional M2s can now be thought of as domain walls connecting different values of the Chern–Simons invariant on the T^3. (Such values are described by "triples" [56].) An alternative understanding was also given in [57, 58].

One can also wonder what happens if in the IIA configuration of figure 7 one were to include D8s. The M-theory understanding is a bit mysterious, because of the notorious problems in incorporating the Romans mass F_0 in M-theory — see for example [59]. (On the other hand, in a simpler configuration, D8s can be realized as Wilson lines on the M-theory E_8 walls [60].) One can try to go directly to IIB, however, again T-dualizing along direction 7; now

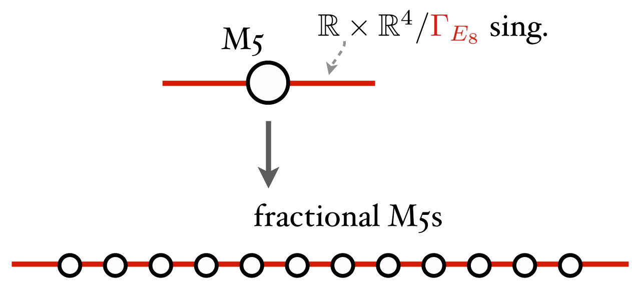


Figure 10 The M-theory interpretation of the repeated blow-up process in F-theory: a single M5 has broken up in 12 "fractional M5s".

both the D6s and the D8s become D7s. However, just as one can argue that the D6–D8 system is in fact a single "Nahm pole" (see footnote 5 and figure 2), the two types of D7 after this T-duality are more likely to fuse into a single nontrivial D7. This was conjectured in [53] to be described by a "Hitchin pole"; in F-theory this is known as a T-brane [61]. Recall that the theories in section 2 were labeled by two Young diagrams (see (7)). The two "ramps" in r_i left and right of this plateau can be thought of as two T-branes attached to the central region with $F_0 = 0$, labeled by the Young diagrams $Y_{\rm L}$ and $Y_{\rm R}$.

Even in the more general case where the IIA picture is not available, it is possible to decorate the quiver in figure 8 with a T-brane on each side; now the T-brane data are no longer labeled by a Young diagram, but by a nilpotent element in G. It would be interesting to identify these theories explicitly.

We have seen so far that F-theory can generalize the linear quiver theories we saw in section 2 and 3. In fact, a classification was recently obtained [62, 63] of six-dimensional theories that can be engineered in F-theory.

The strategy of the classification is the following. Recall from section 3 that the tensor multiplet scalars ϕ_i, which parameterize what we called the tensor branch of the theory, represent in the string theory realization the distances between the NS5s in the direction where the D6s are extended (the horizontal direction in figure 3). The conformal point was realized at the origin $\{\phi_i = 0\}$, where all the NS5s are on top of each other. In the IIB dual the NS5s become geometry: they become the points where the non-trivial two-cycles Σ_i of figure 7 are touching. Thus, the tensor branch is now parameterized by the sizes of the two-cycles, and the conformal point is where all the two-cycles are shrunk to zero.

To engineer a more general theory in F-theory (rather than simply IIB), we want to put seven-branes on an M_4 with several two-cycles, and shrink all the two-cycles simultaneously to reach a conformal point. As it turns out, this is not always possible. Once one finds a T^2-fibration M_6 over M_4 which has the desired degeneration properties on each two-cycle (as briefly reviewed at the beginning of this section), the moduli space of M_6 may or may not have a point where all the two-cycles Σ_i in M_4 shrink simultaneously. As pointed out in [62], algebraic geometry demands that the intersection matrix $\Sigma_i \cdot \Sigma_j$ of the two-cycles should be negative-definite. In any case, the classification is now a geometrical problem.

To make the task more manageable, originally [62] considered theories that are "minimal". This means that they cannot be Higgsed, and that they have no "unnecessary elements", as we will explain shortly. So far we have only discussed the tensor branches of our theories, but in general a theory might have Higgs branches, or mixed Higgs-tensor branches, similarly to what happens for $\mathcal{N} = 2$ theories in four dimensions. For example we can consider giving expectation values to the hypermultiplets we had in our linear quivers of section 2. In the simple chain of figures 1(c) and 7, this corresponds to taking an NS5 off the D6 stack. On the IIB side, it corresponds to "fusing" two neighboring two-cycles into a single one. This can be achieved by a complex structure deformation. For F-theory seven-branes, more generally Higgsing corresponds to a complex structure deformation that fuses together two seven-branes.

Crucially, [64] obtained a classification of which non-Higgsable quiver theories can be obtained in F-theory. These theories can be "connected" via the E-string CFT we introduced earlier, again by the mechanism of gauging a subgroup of E_8 (as for the $F_4 \times G_2$ in figure 9). The theories classified in [62] are all the possible CFTs obtained in this way, without introducing any more such E-string theories than necessary. Interestingly, for all these theories the base in the singular limit is a quotient

$$\mathbb{C}^2/\Gamma \tag{11}$$

with Γ is a discrete subgroup of $\mathrm{U}(2)$. In type II theories Γ usually acts as a subgroup of $\mathrm{SU}(2)$, such that the quotient $\mathbb{C}^2/\Gamma$ is supersymmetric and the resulting space is a singular Calabi–Yau; we mentioned these singularities at the beginning of section 2. In F-theory, however, M_4 need not be Calabi–Yau, essentially because of the presence of the varying axio-dilaton τ; for this reason Γ can act as a more exotic subgroup of $\mathrm{U}(2)$, to produce a so-called "Hirzebruch–Jung singularity". The precise action is related to the data of the theory under consideration.

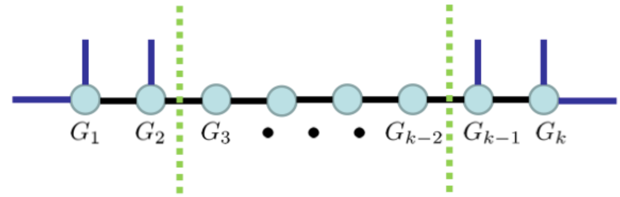


Figure 11 The structure of a general CFT_6 obtained from F-theory, from [63].

From these minimal theories one can in principle produce all the others by making the gauge groups bigger, adding matter fields or more E-string CFTs. This was tackled in [63] in a very detailed way; here we will only point out a couple of interesting features of this classification.

One is that the structure of the theories is basically still "linear", with some optional decoration towards the end (see figure 11). Notice that the gauge groups G_i in that figure are now only either of D or E type; all the other possible gauge groups are included in the links. In fact each link can be a very long quiver itself, such as one of the conformal matter theories we mentioned earlier (e.g. the (E_8, E_8) one in figure 9). The possible links are listed in [63]. Another interesting point is that the G_i satisfy

$$G_1 \subseteq \ldots \subseteq G_m \supseteq \ldots \supseteq G_k \; ; \tag{12}$$

the gauge groups go "up and then down". This is reminiscent of the structure we found for the $\mathrm{SU}(r_i)$ gauge groups in section 2 (see for example figure 1(d)).

5 Five and four dimensions

We will now comment a bit about what happens if we go to lower dimensions.

First let us consider five-dimensional theories. Many ideas are similar to six-dimensional ones, and we will be brief. We will be interested in the minimal amount of supersymmetry, $\mathcal{N} = 1$, which (as in six dimensions) already has $\mathrm{SU}(2)$ R-symmetry. Both the tensor and the vector multiplets in six dimensions reduce to the vector multiplet in five dimensions, which has a gauge field A_μ, a gaugino λ_α and a scalar ϕ. On the other hand, the hypermultiplet in six dimensions reduces to a five-dimensional multiplet also called hypermultiplet, with four scalars and a single spinor.

The description of possible $\mathcal{N} = 1$ vector multiplet Lagrangians in $d = 5$ is very similar to $\mathcal{N} = 2$ in $d = 4$; they can also be summarized by a prepotential $\mathcal{F}(\mathcal{A}_i)$, where $\mathcal{A}_i$ are the vector superfields, but now $\mathcal{F}$ can be at most cubic [12]. With a single vector multiplet and a purely cubic $\mathcal{F}$, the Lagrangian reads schematically

$$\phi \mathrm{Tr}|F|^2 + \phi(\partial\phi)^2 + *\mathrm{Tr}(A \wedge F \wedge F) \,. \tag{13}$$

Notice that the gauge kinetic term is just like in the Lagrangian (5) in six dimensions. Again it makes the dimension of the operator match that of the spacetime: normally a scalar in five dimensions would have dimension 3/2, but in (13) the scalar kinetic term is also unconventional; as a result ϕ has dimension 1, and $\phi\mathrm{Tr}|F|^2$ has dimensions 5. The last term is a Chern–Simons-like coupling.

In any case, once again the promising point is $\langle\phi\rangle = 0$, where (13) is strongly coupled and we might find interesting physics. Once again we can ask whether we can engineer such a situation in string theory, and argue that indeed $\langle\phi\rangle = 0$ (or a generalization thereof, in more complicated theories) corresponds to a CFT.

Fortunately there are many ways to engineer five-dimensional theories in string theory. One is to consider [12] a D4-brane (and its image) near an O8-plane, with $n < 8$ D8-branes on top. This produces an SU(2) gauge theory with n flavors; the single scalar ϕ parameterizes the distance from the O8–D8. The metric for an O8–D8 system depends (as any D-brane or O-plane metric in flat space) on a single harmonic function H of the transverse directions; since there is only one such transverse direction ϕ, this H is linear, $H = H_0 + (n-8)\phi$. The constant H_0 can be tuned to zero; in this situation one recovers the gauge kinetic term $\phi\mathrm{Tr}|F|^2$ in (13). This string-theoretic embedding tells us there is indeed a CFT at $\phi = 0$. In fact a duality chain also allows to show that this theory has enhanced flavor symmetry equal to E_{n+1} flavor symmetry (with a suitable definition for $n < 6$; for example, $E_5 = \mathrm{Spin}(10)$, $E_4 = \mathrm{SU}(5)$).

Another possible string theory realization of five-dimensional theories is obtained by considering (p,q)-fivebranes, sometimes including also (p,q)-sevenbranes (see for example [65,66]). These branes are extended along directions $0\dots4$: the field theory will live here. They are all transverse to directions $7,8,9$: these will realize the $\mathrm{SU}(2)_\mathrm{R}$. Each (p,q)-fivebrane is further extended along a linear combination of directions 5 and 6 dictated by p and q (when the zero-form potential $C_0 = 0$, it is simply $px^5 + qx^6$). The fivebranes can meet and merge, or split, paying attention however that around each such

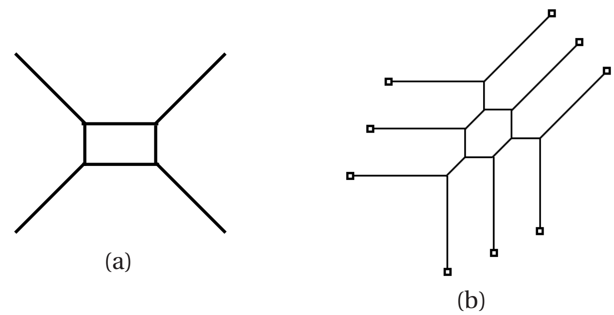


Figure 12 Examples of (p,q)-fivebrane webs. The square dots denote (p,q)-sevenbranes.

intersection charges are conserved (see for example figure 12). Additionally, each (p,q)-fivebrane may end on a (p,q)-sevenbrane (with the same p and q). These are in a sense a bit optional for many configurations: a semi-infinite fivebrane can be replaced with a fivebrane that ends on a sevenbrane, with no change in the engineered five-dimensional field theory.

These webs are in a sense the closest analogue to the six-dimensional theories we considered in section 2 and to the similar theories in three [38] and four [14] dimensions; the procedure to extract the field theory is similar. For example, the configuration in figure 12(a) can be thought of as two horizontal D5s which have been suspended between two vertical NS5s; outside the contact points, the vertical D5s have been "bent" and have become oblique. This is similar to the logarithmic bending pointed out in [14], except that in this case it is linear rather than logarithmic because $1/g_{\mathrm{YM}^2}$ has dimension mass. Thus in the end 12(a) engineers pure $\mathcal{N} = 1$ SU(2).

A third way of engineering five-dimensional theories is by compactifying M-theory on a Calabi–Yau M_6 (see for example [13]). Just like in the six-dimensional case, one expects a CFT in limits where M_6 develops singularities. When M_6 is toric, one can reduce along the torus directions, ending up in IIB. Where the torus directions degenerate, T-duality gives rise to (p,q)-fivebranes; in fact one can see that the toric diagram becomes exactly a fivebrane web [67].

In fact, also the D4–O8–D8 configuration we reviewed earlier can be dualized to a fivebrane web [66, 68] (one uses the fact that an O7 splits into a (1,1) and a (1,−1)-sevenbrane). But with brane webs one can engineer more complicated quivers. In general one expects CFTs when the inner regions (e.g. the rectangle in figure 12(a)) shrink to a point.

One can also wonder about the holographic duals of five-dimensional CFTs. Unfortunately, few AdS_6 solutions are known. In IIA, there is a solution $\mathrm{AdS}_6 \times S^4$ [69], proven to be unique in [70], up to orbifolds. This has been proposed to be dual to some simple quivers [68]. Since fivebrane webs in IIB seem to be a more promising source of CFTs, one may think it more promising to look for AdS_6 solutions in IIB. A general analysis was performed in [71]; the problem was reduced to two PDEs, and the geometry of the internal four-manifold was shown to be an S^2-fibration over a two-dimensional space, somewhat similarly to the solutions in section 3. However, only two solutions are known: they have been obtained by abelian and non-abelian [72–74] T-duality from the IIA solution.

Let us now consider some of the many relations between theories in six, five and four dimensions. These relations come about because of dimensional reduction. They can both clarify the higher-dimensional theories, and define new interesting lower-dimensional ones.

First of all let us consider going from six to four dimensions. This can be achieved by compactifying on a Riemann surface Σ. Let us first consider the $\mathcal{N} = (2,0)$ theory living on the worldvolume of an M5 stack. For $\Sigma = T^2$, we simply get $\mathcal{N} = 4$ super-Yang–Mills. For different Σ, this leads to the so-called "class S" theories [19]. These are $\mathcal{N} = 2$ theories described by "generalized" quivers, which together with the usual nodes and links also have certain special "vertices" representing a certain T_N theory. This is a non-Lagrangian theory (for a recent review see [75]). Each generalized quiver corresponds to a pants decomposition of Σ; the Seiberg–Witten curve is a branched cover of Σ. The T_N theory in particular is associated to compactifying on a sphere with three punctures. On each puncture one has an additional piece of data: a Young diagram (or partition) that summarizes the structure of the branched covering around it. The AdS_5 gravity duals of these theories are obtained from the $\mathrm{AdS}_7 \times S^4$ solution (dual to the M5 stack) by replacing AdS_7 with $\mathrm{AdS}_5 \times \Sigma$, distorting S^4 in a certain way, and fibering it over Σ [20,76]. A more general fibration can produce $\mathcal{N} = 1$ theories [76,77].

There are also class S theories of type D and E, obtained compactifying the D and E $\mathcal{N} = (2,0)$ theories (the ones which arise from IIB on a $\mathbb{R}^4/\Gamma$ singularity). In this case, the punctures are parameterized by nilpotent elements [78,79] in the G Lie group corresponding to Γ (just like for T-branes; see section 4).

The compactifications of the $\mathcal{N} = (1,0)$ theories on Riemann surfaces should also give rise to interesting theories in four dimensions. For $\Sigma = T^2$, we should obtain $\mathcal{N} = 2$ theories. These were studied in [55,80,81]. In [55] it was pointed out that the T^2 compactification of the "conformal matter" theories described in section 4 (for example 9) coincide with the class S theory with three punctures (two of which "maximal", one "minimal"). [81] found that, in absence of Wilson lines, the compactification of the theory $\mathcal{T}^N_{Y_\mathrm{L},Y_\mathrm{R}}$ often contains an infrared-free vector; so it is not a CFT. This does not happen when one of the two Young diagrams is $[N]$ (a single vertical column). On the other hand, [80] include Wilson lines and find plenty of CFT's in four dimensions.

Compactifications of $\mathcal{N} = (1,0)$ theories on different Riemann surfaces Σ should give $\mathcal{N} = 1$ theories in four dimensions; they are less understood. [82] initiated this study for the theory $\mathcal{T}^N_{[N],[N]}$ in figure 1(c), whose corresponding gravity solution has $F_0 = 0$ everywhere. On the other hand, in [83] an analytic AdS_5 solution was found associated to each of the solutions of section 3. The solutions are similar in spirit to the ones described earlier for the $\mathcal{N} = (2,0)$ theory [76]: namely, AdS_7 is replaced by $\mathrm{AdS}_5 \times \Sigma$, and the internal M_3 is fibred over Σ. The latter is however assumed to have genus $g \geq 2$ and to have no punctures. It was later shown that a holographic RG flow connects the AdS_7 and $\mathrm{AdS}_5 \times \Sigma$ solutions, using a consistent truncation approach [84].

Let us now go from six to five dimensions. For the $\mathcal{N} = (2,0)$ theory on an M5 stack, reduction on an S^1 gives $\mathcal{N} = 1$ super-Yang–Mills in five dimensions. It was even conjectured [85–88] that the solitons of this theory are already the KK states needed to go back to six dimensions. Interestingly, more recently other theories have been found which seem to already include KK modes among their solitons [89,90]. On the other hand, reductions of the theories in sections 2 and 4 generically give an extra five-dimensional vector multiplet with finite gauge coupling related to the compactification scale [81].

Finally, going from five to four, it is interesting to note that the T_N theories can be obtained from five dimensions as well [91]. For example, the fivebrane web in figure 12(b) happens to reduce to T_3, which is the E_6 Minahan–Nemeschansky theory [92].

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