On e-monotonicity and maximality of operators in Banach spaces

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Abstract

In this paper we extend some well known properties of monotone and maximal monotone operators to the wider class of e-monotone and maximal e-monotone operators. The main results concern local boundedness of maximal e-monotone operators, maximal $2e$-monotonicity of the Clarke-Rockafellar subdifferential $\partial^{CR} f$ for an e-convex function f, and the characterization of e-monotonicity of an operator T via the behaviour of its e-Fitzpatrick function outside the graph of T.

Key words: e-monotonicity, maximality, generalized subdifferential; Fitzpatrick function

MSC: 47H05 (49J53, 47H04)

1 Introduction and Preliminaries

In this paper X is a real Banach space, with topological dual space X^*, and $\langle \cdot, \cdot \rangle$ denotes the duality pair.

Given an operator $T : X \to 2^{X^*}$, its domain is $D(T) = \{x \in X : T(x) \neq \emptyset\}$, and its graph $\mathrm{gr}(T) = \{(x, x^*) \in X \times X^* : x^* \in T(x)\}$.

The operator T is said to be monotone if for every $x, y \in D(T)$, $x^* \in T(x)$ and $y^* \in T(y)$,

$$\langle x^* - y^*,\ x - y \rangle \geq 0 \quad \forall (x, x^*), (y, y^*) \in \mathrm{gr}(T), \tag{1}$$

and T is said to be maximal monotone if its graph is not properly included in the graph of any other monotone operator.

In literature monotone and maximal monotone operators have been intensively studied due to their important properties and applications (see, for instance, [16] and the reference therein).

Subsequently, many authors introduced generalized monotone operators with the aim to extend to a larger class of operators some of the properties of the

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monotone ones. In the sequel we will focus on the class of e-monotone operators for which the inequality in (1) is weaker, requiring only that

$$\langle x^* - y^*,\ x - y\rangle \geq -e(x,y),$$

where the *error bifunction* $e : X \times X \to \mathbb{R}$ is nonnegative and symmetric, i.e. $e(x,y) = e(y,x)$.

The primary aim of this work is essentially theoretical and seeks to address a broader class of generalized monotone operators by examining the minimal properties required of the function and to rediscover known properties of monotone operators.

Examples of error bifunctions are a nonnegative constant, $e(x,y) = \|x-y\|$, $e(x,y) = \|x-y\|^2$, $e(x,y) = \min\{\sigma(x),\sigma(y)\}\|x-y\|$, where $\sigma : X \to \mathbb{R}$ is a nonnegative function, to name a few. With any of these particular choices, many different classes of generalized monotone operators studied in the literature can be recovered.

In particular, recently some authors investigated one of these classes, namely the premonotone operators which corresponds to the particular choice $e(x,y) = \min\{\sigma(x),\sigma(y)\}\|x-y\|$ (see, for instance, [2, 12]).

Some of the results of this paper have a counterpart for the premonotone case; when the proofs differ only slightly, we skip them giving more details in case specific properties of the error bifunction e are involved.

The paper is organized as follows: in Section 2 we give the notion of e-monotonicity and maximal e-monotonicity of an operator T, together with some of the properties enjoyed. In Section 3 the notion of (maximal) e-monotonicity is extended to bifunctions, and a relationship between the monotonicity properties of T and the associated bifunction G_T is established. Furthermore, we extend to maximal e-monotone operators the classical result of local boundedness of maximal monotone operators. In Section 4 the connection between e-convexity of a function f and $2e$-monotonicity of its Clarke-Rockafellar subdifferential is explored. The main result of this section is Theorem 18 where it is proved that, under suitable assumptions on the error bifunction e, $\partial^e f = \partial^{CR} f$, and $\partial^{CR} f$ is maximal $2e$-monotone. The last section is devoted to an extension of the Fitzpatrick function for e-monotone operators; in particular, the maximal e-monotonicity of an e-monotone operator T is characterized via the behaviour of its e-Fitzpatrick function outside the graph of T.

2 e-monotone operators and maximality

The definition of e-monotone operator generalizes in a standard way the notion of monotone operator which is recovered assuming $e = 0$:

Definition 1 *Given an operator $T : X \to 2^{X^*}$ and an error bifunction $e : X \times X \to \mathbb{R}_+$, we say that T is* e-monotone *if for every $x, y \in D(T)$, $x^* \in T(x)$ and $y^* \in T(y)$,*

$$\langle x^* - y^*, y - x\rangle \leq e(x,y)\,. \tag{2}$$

Note that if T is an e-monotone operator and e' is an error bifunction such that $e'(x,y) \geq e(x,y)$ for all $x, y \in X$, then T is also e'-monotone.

Given an e-monotone operator T, one may wonder which is the smallest error bifunction e_T with respect to which the operator T is e_T-monotone. To answer this question let us consider for any operator T the bifunction $e_T : X \times X \to \mathbb{R} \cup \{+\infty\}$ defined as follows:

$$e_T(x,y) = \inf\{a \in \mathbb{R}_+ : \langle x^* - y^*, y - x\rangle \leq a, \quad \forall x^* \in T(x),\ \forall y^* \in T(y)\}$$

if $(x,y) \in D(T) \times D(T)$, and $e_T(x,y) = 0$ otherwise or, equivalently:

$$e_T(x,y) = \left(\sup\left\{\langle x^* - y^*, y - x\rangle : \, x, y \in X,\, x^* \in T(x),\, y^* \in T(y)\right\}\right)^+ \tag{3}$$

where, for a function f, we set $(f)^+ = \max\{f, 0\}$.

It is easy to verify by the definition that

$$e_T(x,y) \leq e(x,y) \text{ on } D(T) \times D(T) \tag{4}$$

if T is e-monotone, and that T is e_T-monotone, too. We remark that T is e-monotone for some e if and only if e_T is real-valued.

To introduce the notion of maximality for e-monotone operators, we draw inspiration by an approach used in the monotone and in other generalized monotone cases (see, for instance, [14]) by introducing the reflexive and symmetric binary relation $\widetilde{e}$ on $X \times X^*$ defined as follows:

$$(x, x^*)\widetilde{e}(y, y^*) \iff \langle x^* - y^*, y - x\rangle \leq e\,(x,y)\,. \tag{5}$$

In case (5) holds, we say that (x, x^*) and (y, y^*) are *e-monotonically related.*

Then we define the e-monotone polar $T^{\tilde{e}} : X \to 2^{X^*}$ by setting, for every $x \in X$,

$$T^{\tilde{e}}(x) = \{x^* \in X^* : (x, x^*)\tilde{e}(y, y^*), \quad \forall (y, y^*) \in \mathrm{gr}(T)\}.$$

It is evident that if T is e-monotone, then $T(x) \subseteq T^{\tilde{e}}(x)$ for every $x \in D(T)$. In addition, if $u^* \in T^{\tilde{e}}(u) \setminus T(u)$, the operator $T' : X \to 2^{X^*}$ such that $\mathrm{gr}(T') = \mathrm{gr}(T) \cup (u, u^*)$ is e-monotone. This remark leads to the following definition of e-maximality:

Definition 2 *Given an e-monotone operator $T : X \to 2^{X^*}$, we say that T is maximal e-monotone if* $\mathrm{gr}(T) = \mathrm{gr}(T^{\tilde{e}})$.

The previous definition means that, for a maximal e-monotone operator T, an e-monotone operator T' such that $\mathrm{gr}(T) \subset \mathrm{gr}(T')$ does not exist. Therefore, following the line of the proof of Theorem 20.21 in [7], we can apply Zorn's Lemma to the set

$$\mathcal{M} = \{T' : X \to 2^{X^*} : T' \text{ is } e - \text{monotone},\ \mathrm{gr}(T) \subseteq \mathrm{gr}(T')\}$$

to show that every e-monotone operator T admits a maximal e-monotone extension.

Remark 3 *In order to prove that an e-monotone operator is maximal, it is enough to prove that every pair* (x, x^*) *e-monotonically related to every* $(y, y^*) \in \mathrm{gr}(T)$ *belongs to* $\mathrm{gr}(T)$ *or, equivalently, for every* $x \in X$ *and* $x^* \notin T(x)$*, there exists* $z \in D(T)$ *and* $z^* \in T(z)$ *such that* (x, x^*) *is not e-monotonically related to* (z, z^*)*.*

Proposition 4 *Let* $T : X \to 2^{X^*}$ *be an e-monotone operator. If* T *is maximal e-monotone, then* T *is maximal* e_T*-monotone.*

Proof. We know that T is e_T-monotone. Since T is maximal e-monotone for every $x \in X$ and $x^* \notin T(x)$, there exists $z \in D(T)$ and $z^* \in T(z)$ such that

$$\langle x^* - z^*, z - x \rangle > e(x, z) \geq e_T(x, z),$$

i.e. T is maximal e_T-monotone. ■

Note that the converse is false:

Example 5 *Consider* $T : \mathbb{R} \to 2^{\mathbb{R}}$ *defined as follows:*

$$T(x) = \begin{cases} x & x \in (-1, 1) \\ (-\infty, -1] & x = -1 \\ [1, +\infty) & x = 1 \end{cases}$$

The operator T *is not only monotone, but also maximal monotone; in particular, it is e-monotone, for every error bifunction* e*, and* $e_T = 0$*. Take the error bifunction* e *defined as follows:*

$$e(x, y) = \begin{cases} 4 - (x - y)^2 & (x, y) \in [-1, 1]^2 \\ 0 & \text{elsewhere} \end{cases}$$

We will show that $(0, 2)\tilde{e}(x, x^*)$ *for every* $(x, x^*) \in \mathrm{gr}(T)$*. In fact, if* $x \in (-1, 1)$*, we have that*

$$\langle x - 0, 2 - x \rangle = 2x - x^2 \leq e(0, x) = 4 - x^2.$$

Let $x = -1$ *: for every* $\alpha \in (-\infty, -1]$*,*

$$\langle -1 - 0, 2 - \alpha \rangle = -2 + \alpha \leq -3 \leq e(-1, 0) = 3;$$

if $x = 1$*, then for every* $\alpha \in [1, +\infty)$ *:*

$$\langle 1 - 0, 2 - \alpha \rangle = 2 - \alpha \leq 1 \leq e(1, 0) = 3.$$

Since $(0, 2) \notin \mathrm{gr}(T)$*, we conclude that* T *is not maximal e-monotone.*

The following result is similar to Proposition 2.7 in [2] and can be proved with the same techniques. Let us recall that given $x_0 \in X$, T is called sequentially norm×weak*-closed at x_0 if for every sequence $(x_n, x_n^*) \in \mathrm{gr}(T)$ such that $x_n \to x_0$ and $x_n^* \xrightarrow{w^*} x_0^*$ one has $x_0^* \in T(x_0)$.

Proposition 6 *Every maximal e-monotone operator T is convex-valued and weak*-closed valued. Moreover, if $x_0 \in \overline{D(T)}$, and $e(\cdot, y)$ is upper semicontinuous at x_0, then T is sequentially norm×weak*-closed at x_0.*

These last results hold in a Hilbert space setting (see [4]).

Proposition 7 *Let $\mathcal{H}$ be a Hilbert space, and $T : \mathcal{H} \to 2^{\mathcal{H}}$ be e-monotone. Suppose that $I + T$ is surjective. If $(y, y^*) \in \mathcal{H} \times \mathcal{H}$ is e-monotonically related to $\mathrm{gr}(T)$, then there exists $(x, x^*) \in \mathrm{gr}(T)$ such that*

$$||x^* - y^*|| = ||x - y|| \le \sqrt{e(x, y)}. \tag{6}$$

Proof. By assumption $R(I + T) = \mathcal{H}$. This implies that there exists an element $x \in \mathcal{H}$ such that $y + y^* \in (I + T)(x)$. This implies that $y + y^* = x + x^*$ for a suitable $x^* \in T(x)$. From this equality, we obtain that

$$y - x = x^* - y^*. \tag{7}$$

Besides, $(y, y^*) \in \mathcal{H} \times \mathcal{H}$ is e-monotonically related to $\mathrm{gr}(T)$. Thus

$$\langle x^* - y^*, y - x \rangle \le e(x, y). \tag{8}$$

From (7) and (8) we infer that

$$\langle y - x, y - x \rangle \le e(x, y).$$

Therefore $||x - y|| \le \sqrt{e(x, y)}$. ■

From the result above one can easily get the following (see Theorem 21.1 in [7])

Corollary 8 *Assume that $T : \mathcal{H} \to 2^{\mathcal{H}}$ is monotone. If $R(I + T) = \mathcal{H}$, then T is maximal monotone.*

3 e-monotone bifunctions and properties of e-monotone operators

As for the case of monotonicity, the notion of e-monotonicity is somehow extended also to bifunction.

Definition 9 *Let C be a nonempty subset of X and $e : X \times X \to \mathbb{R}_+$ be an error bifunction. A bifunction $F : C \times C \to \mathbb{R}$ is called* e-monotone *if*

$$F(x, y) + F(y, x) \le e(x, y), \quad \forall x, y \in C. \tag{9}$$

Note that, taking into account the symmetry of e, the definition is equivalent to saying that $F - \frac{e}{2}$ is a monotone bifunction.

Given any $F : C \times C \to \mathbb{R}$, the operator $A^F : X \to 2^{X^*}$ is defined by

$$A^F(x) = \begin{cases} \{x^* \in X^* : F(x, y) \ge \langle x^*, y - x \rangle, \quad \forall y \in C\} & \text{if } x \in C, \\ \emptyset & \text{if } x \notin C. \end{cases}$$

It is easy to show that for an e-monotone bifunction F, A^F is an e-monotone operator. Moreover, following [10], an e-monotone bifunction F is said to be *maximal e-monotone* if A^F is maximal e-monotone.

An important bifunction intrinsically linked to an operator $T : X \to 2^{X^*}$, is given by $G_T : D(T) \times D(T) \to \mathbb{R} \cup \{+\infty\}$ defined as (see, for instance, [4, 10])

$$G_T(x, y) = \sup_{x^* \in T(x)} \langle x^*, y - x\rangle.$$

For each $x \in D(T)$, $G_T(x, \cdot)$ is lower semicontinuous and convex, and $G_T(x, x) = 0$. The following result shows that G_T is actually real-valued whenever T is e-monotone, and establishes some relations between e-monotonicity of G_T and T as in Proposition 3.3. in [2].

Proposition 10 *Let T be an operator and $e : X \times X \to \mathbb{R}$ be an error bifunction. Then the following statements are true:*

(i) T is e-monotone if and only if G_T is e-monotone; in particular, if T is an e-monotone operator, then G_T is real-valued on $D(T) \times D(T)$.

(ii) If T is maximal e-monotone, then $A^{G_T} = T$ and G_T is a maximal e-monotone bifunction.

(iii) Suppose that T is an e-monotone operator with w^*-closed convex values and $D(T) = X$. If G_T is maximal e-monotone, then T is maximal e-monotone.

Proof. (i) Let $T : X \to 2^{X^*}$ be e-monotone. For any $x, y \in D(T)$ we have

$$\langle y^*, x - y\rangle + \langle x^*, y - x\rangle \le e(x, y), \quad \forall x^* \in T(x), y^* \in T(y),$$

i.e.

$$\sup_{y^* \in T(y)} \langle y^*, x - y\rangle + \sup_{x^* \in T(x)} \langle x^*, y - x\rangle \le e(x, y),$$

or, equivalently,

$$G_T(x, y) + G_T(y, x) \le e(x, y). \tag{10}$$

This means that G_T is real valued and e-monotone. The converse holds trivially starting from (10).

(ii) For every $x, y \in D(T)$ we have

$$G_T(x, y) = \sup_{x^* \in T(x)} \langle x^*, y - x\rangle \ge \langle z^*, y - x\rangle \quad \forall z^* \in T(x).$$

This means that $T(x) \subseteq A^{G_T}(x)$. Since T is maximal e-monotone, and A^{G_T} is e-monotone, we conclude that $T = A^{G_T}$.

(iii) Since G_T is maximal e-monotone, according to the definition, A^{G_T} is maximal e-monotone. Let $x \in X$ and $z^* \in A^{G_T}(x)$. Then

$$G_T(x, y) = \sup_{x^* \in T(x)} \langle x^*, y - x\rangle \ge \langle z^*, y - x\rangle.$$

Now, according to the separation theorem, it follows that $z^* \in T(x)$. Consequently, $T = A^{G_T}$ and T is maximal e-monotone. ■

In the following result, we extend the well-known fact that every set-valued monotone operator T from X to X^* is locally bounded within the interior of its domain. We will denote by $B(x,r)$ the open ball with centre $x \in X$ and radius r.

We first recall the following definition from [2]:

Definition 11 *A bifunction $F : C \times C \to \mathbb{R}$ is said to be locally bounded at $x_0 \in C$ if there exist $R > 0$ and $M \in \mathbb{R}$ such that $F(x,y) \leq M$ for all $x, y \in C \cap B(x_0, R)$. We call F locally bounded on a set C if it is locally bounded at every $x \in C$.*

Proposition 12 *Let X be a Banach space, and $T : X \to 2^{X^*}$ be an e-monotone operator, where the error bifunction e is locally bounded at every point within* $\operatorname{int} D(T)$. *Then T is locally bounded at every point of* $\operatorname{int} D(T)$.

Proof. First, let us establish that the function G_T is locally bounded on the interior of $D(T)$. Take any point $x_0 \in \operatorname{int} D(T)$, and let $\epsilon > 0$ be such that $B(x_0, \epsilon) \subset \operatorname{int} D(T)$ and e is bounded on $B(x_0,\epsilon) \times B(x_0,\epsilon)$ by a constant M_{x_0}. Define the function $g : B(x_0,\epsilon) \to \mathbb{R} \cup \{+\infty\}$ as follows:

$$g(y) = \sup\{G_T(x,y),\, x \in B(x_0,\epsilon)\}.$$

Note that g is real-valued; indeed, since G_T is e-monotone by Proposition 10 (i), we have that

$$G_T(x,y) \leq e(x,y) - G_T(y,x) \leq M_{x_0} + \|y_0^*\|(\epsilon + \|y - x_0\|),$$

for some $y_0^* \in T(y)$, and for every $(x,y) \in B(x_0,\epsilon) \times B(x_0,\epsilon)$. Therefore, $g(y)$ is real-valued on $B(x_0,\epsilon)$. Since g is convex and lower semicontinuous, and $x_0 \in \operatorname{int}\operatorname{dom}(g)$, there exists $0 < \delta < \epsilon$ and $M > 0$ such that $g(y) \leq M$, for every $y \in B(x_0,\delta)$ (see, for instance, Theorem 2.2.8 in [13]). This implies that $G_T(x,y) \leq M$ for every $x, y \in B(x_0,\delta)$, i.e. G_T is locally bounded at x_0. By Remark 6 (b) in [3], T is locally bounded at x_0. ■

Under the assumption of T being maximal e-monotone, the aforementioned result has a converse that generalizes a property of maximal monotone operators established by Vesely (see [16]).

Let us denote by $\operatorname{co}(D(T))$ the convex hull of $D(T)$. Recall that in case $D(T)$ is a convex set, then $\operatorname{int} D(T) = \operatorname{int} \overline{D(T)}$. In addition, the normal cone to $D(T)$ at a point $x_0 \in D(T)$ is given by

$$N_{D(T)}(x_0) = \{x^* \in X^* : \langle x^*, y - x_0 \rangle \leq 0,\ \forall y \in D(T)\}.$$

We now prove the following lemma:

Lemma 13 *Let X be a Banach space, and $T : X \to 2^{X^*}$ be a maximal e-monotone operator. If* $\operatorname{int}\operatorname{co}(D(T)) \neq \emptyset$, *then $T(x_0)$ is unbounded, for every point $x_0 \in D(T) \setminus \operatorname{int}\operatorname{co}(D(T))$.*

Proof. Let $x_0 \in D(T) \setminus \text{int co}(D(T))$; this implies that x_0 belongs to the boundary of the closed and convex set $\text{co}(D(T))$. Since, by assumption, we have that $\text{int co}(D(T)) \neq \emptyset$, there exists a supporting hyperplane to $\text{co}(D(T))$ at x_0; this means that there exists $0 \neq w^* \in X^*$ such that $\langle w^*, x_0 \rangle \geq \langle w^*, x \rangle$ for all $x \in D(T)$. This implies that $w^* \in N_{D(T)}(x_0)$.

Take any $x^* \in T(x_0)$ and $w^* \in N_{D(T)}(x_0)$. For each $(y, y^*) \in \text{gr}(T)$ and every $\lambda \geq 0$ we would have

$$\langle x^* + \lambda w^* - y^*, y - x_0 \rangle = \langle x^* - y^*, y - x_0 \rangle + \lambda \langle w^*, y - x_0 \rangle \leq e(y, x_0),$$

which implies that $(x_0, x^* + \lambda w^*)$ is e-monotonically related with all $(y, y^*) \in \text{gr}(T)$. By maximal e-monotonicity of T we obtain

$$x^* + \lambda w^* \in T(x_0), \quad \forall \lambda \geq 0. \tag{11}$$

From (11) we infer that the set $T(x_0)$ is not bounded and so T is not locally bounded at x_0. ■

Theorem 14 *Suppose that T is maximal e-monotone with a convex domain $D(T)$, $\text{int}\, D(T) \neq \emptyset$, and $e(\cdot, y)$ is upper semicontinuous for each $y \in D(T)$. If $x_0 \in \overline{D(T)}$ and T is locally bounded at x_0, then $x_0 \in \text{int}\, D(T)$.*

Proof. The first part of the proof follows the line of Theorem 2.14 in [16].

By assumption T is locally bounded at x_0, so there is an open neighborhood U of x_0 so that $T(U)$ is a bounded set. Let $\{x_n\} \subset D(T) \cap U$ be a sequence so that $x_n \to x_0$ and choose $(x_n, x_n^*) \in \text{gr}(T)$. According to Alaoglou's theorem there exist a subnet (x_α, x_α^*) of (x_n, x_n^*) and $x_0^* \in X^*$ such that $x_\alpha^* \xrightarrow{w^*} x_0^*$. Since the net $\{x_\alpha^*\}$ is in the bounded set $T(U)$, we infer that $\langle x_\alpha^*, x_\alpha \rangle \to \langle x_0^*, x_0 \rangle$. Consequently, for all $(y, y^*) \in \text{gr}(T)$

$$\begin{aligned} \langle x_0^* - y^*, y - x_0 \rangle &= \lim_\alpha \langle x_\alpha^* - y^*, y - x_\alpha \rangle \\ &\leq \limsup_\alpha e(x_\alpha, y) \leq e(x_0, y). \end{aligned}$$

Hence (x_0, x_0^*) is e-monotonically related to all $(y, y^*) \in \text{gr}(T)$. By the assumptions, we get $x_0 \in D(T)$. Now by applying Lemma 13, we conclude that $x_0 \in \text{int}\, D(T)$. ■

4 Some results on generalized subdifferential and maximality

In this section we will focus both on the Clarke-Rockafellar subdifferentials and the e-subdifferentials. The main result shows the maximal $2e$-monotonicity of these operators. Let us first recall some definitions from [1].

Definition 15 *Let a function* $f : X \to \mathbb{R} \cup \{+\infty\}$ *and an error bifunction* e *be given. Then* f *is called* e-convex *if*

$$f\left(tx+(1-t)\,y\right) \le tf\left(x\right)+(1-t)\,f\left(y\right)+t\left(1-t\right)e\left(x,y\right) \tag{12}$$

for all $x, y \in X$, *and* $t \in]0,1[$.

Note that the domain of an e-convex function is necessarily convex.

Suppose that $f : X \to \mathbb{R} \cup \{+\infty\}$ is a proper function. The e-subdifferential of f is the multivalued operator $\partial^e f : X \to 2^{X^*}$ defined as

$$\partial^e f\left(x\right) = \{x^* \in X^* : \langle x^*, y-x\rangle \le f\left(y\right) - f\left(x\right) + e(x,y), \quad \forall y \in X\}$$

if $x \in \mathrm{dom}(f)$; otherwise it is empty. It is easy to show that the e-subdifferential of any function is a $2e$-monotone operator.

Moreover, we recall that the Clarke-Rockafellar subdifferential of a proper, lower semicontinuous function f at the point $x \in \mathrm{dom}(f)$ is the set

$$\partial^{CR} f(x) = \{p \in X^* : \; f^0(x;v) \ge \langle p, v\rangle, \quad \forall v \in X\},$$

where

$$f^0(x;v) = \lim_{\epsilon\downarrow 0}\; \limsup_{y\to_f x,\, t\downarrow 0}\; \inf_{w\in v+\epsilon B} \frac{f(y+tw)-f(y)}{t},$$

$B = B(0,1)$, and $y \to_f x$ means that $(y, f(y))$ tends to $(x, f(x))$ in $X \times \mathbb{R}$. If $x \notin \mathrm{dom}(f)$, then $\partial^{CR} f(x) = \emptyset$.

The next results generalize properties well known for generalized monotone functions. In Example 22.3 in [7] the authors provide similar results in case of other classes of generalized monotone functions. Note that, unlike the monotone case which corresponds to $e = 0$, we have no equivalence between e-convexity of the function and e-monotonicity of the subdifferential.

Proposition 16 *Let* $f : X \to \mathbb{R} \cup \{+\infty\}$ *be a proper, lower semicontinuous function, and* $e : X \times X \to \mathbb{R}$ *be an error bifunction such that* $e(\cdot, y)$ *is upper semicontinuous on* $\mathrm{dom}(f)$, *for every* $y \in \mathrm{dom}(f)$. *Consider the following statements:*

(i) f *is e-convex;*

(ii) $f(y) - f(x) + e(x,y) \ge \langle x^*, y-x\rangle$, *for every* $x \in \mathrm{dom}(f)$, $y \in X$, $x^* \in \partial^{CR} f(x)$;

(iii) $\partial^{CR} f$ *is* $2e$*-monotone.*

We have that (i) $\Longrightarrow$ *(ii)* $\Longrightarrow$ *(iii).*

Proof. (i) implies (ii): Under the assumptions, from Theorem 3.5 in [1] the inclusion $\partial^{CR} f(x) \subseteq \partial^e f(x)$ holds for every $x \in \mathrm{dom}(f)$. Thus, (ii) follows.

(ii) implies (iii): it trivially follows by adding the l.h.s. and r.h.s. of the inequalities (ii), where we exchange the role of x and y, and from the symmetry of the error bifunction e. ■

Proposition 17 *Let f be a proper, lower semicontinuous function with convex domain, and $e : X \times X \to \mathbb{R}$ be an error bifunction such that $e(x,x) = 0$ for all $x \in \mathrm{dom}(f)$, $e(\cdot, y)$ is upper semicontinuous and convex on $\mathrm{dom}(f)$, for every $y \in \mathrm{dom}(f)$. In addition, suppose that $D(\partial^{CR} f) = \mathrm{dom}(f)$. Consider the following statements:*

(i) f is $2e$-convex;

(ii) $f(y) - f(x) + e(x,y) \geq \langle x^*, y - x\rangle$, for every $x \in \mathrm{dom}(f)$, $y \in X$, and $x^* \in \partial^{CR} f(x)$;

(iii) $\partial^{CR} f$ is e-monotone.

We have that (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i).

Proof. (iii) implies (ii): Let $x \in \mathrm{dom}(f)$ and $x^* \in \partial^{CR} f(x)$; by Zagrodny Mean Value Theorem (see, for instance, [17]), for every $y \in \mathrm{dom}(f)$ there exists $c \in [y, x)$ and sequences $x_n \to c$ and $x_n^* \in \partial^{CR} f(x_n)$ such that

a. $\dfrac{\|x-y\|}{\|x-c\|} \liminf\limits_{n\to+\infty} \langle x_n^*, x - x_n\rangle \geq f(x) - f(y)$;

b. $\liminf_{n\to+\infty} \langle x_n^*, x - y\rangle \geq f(x) - f(y)$.

From (iii), $\langle x_n^* - x^*, x_n - x\rangle \geq -e(x_n, x)$, thus, from a.,

$$\begin{aligned} f(x) - f(y) &\leq \frac{\|x-y\|}{\|x-c\|} \liminf_{n\to+\infty} (\langle x_n^* - x^*, x - x_n\rangle + \langle x^*, x - x_n\rangle) \\ &\leq \frac{\|x-y\|}{\|x-c\|} (\liminf_{n\to+\infty} e(x, x_n) + \langle x^*, x - c\rangle) \\ &\leq \frac{\|x-y\|}{\|x-c\|} (\limsup_{n\to+\infty} e(x, x_n) + \langle x^*, x - c\rangle) \\ &\leq \frac{\|x-y\|}{\|x-c\|} (e(x, c) + \langle x^*, x - c\rangle). \end{aligned}$$

From the assumption of convexity of $e(\cdot, y)$, simple computations give

$$f(y) - f(x) \geq \langle x^*, y - x\rangle - e(x, y),$$

thereby proving (ii) for $y \in \mathrm{dom}(f)$. If $y \notin \mathrm{dom}(f)$, then the inequality in (ii) trivially holds.
(ii) implies (i): Take any $x, y \in \mathrm{dom}(f)$, $t \in (0,1)$, and set $x_t = (1-t)x + ty$. For every $x_t^* \in \partial^{CR} f(x_t)$ we have that

$$\begin{aligned} f(x) - f(x_t) &\geq \langle x_t^*, x - x_t\rangle - e(x, x_t) \\ f(y) - f(x_t) &\geq \langle x_t^*, y - x_t\rangle - e(y, x_t). \end{aligned}$$

Multiplying the first inequality by $(1-t)$, the second one by t, adding up l.h.s. and r.h.s. and taking into account the convexity of $e(\cdot, y)$ we get the assertion. ■

The next result shows that, under suitable conditions, $\partial^{CR} f = \partial^e f$, and the operator $\partial^e f$ is maximal $2e$-monotone. The proof is partially inspired by Lemma 4.2 and Theorem 4.3 [11]. With the notation ∂h we will denote the classical subdifferential of convex analysis.

Theorem 18 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous and e-convex function. Suppose that, for every $x \in \mathrm{dom}(f)$, $e(x, x) = 0$, $e(x, \cdot)$ is Fréchet differentiable at x with derivative 0, $e(x, \cdot)$ is convex, and it is Lipschitz on X with a same constant for every $x \in X$ (i.e. there exists $\alpha > 0$ such that $|e(x, z) - e(x, y)| \leq \alpha \|y - z\|$ for every $y, z \in X$). Then $\partial^{CR} f = \partial^e f$, and it is maximal $2e$-monotone.*

Proof. First of all, let us show that, under the assumptions, $\partial^e f = \partial^{CR} f$. From Theorem 3.5 in [1] we have that $\partial^{CR} f(x) \subseteq \partial^e f(x)$. Let us prove the opposite inclusion. Take any $x^* \in \partial^e f(x)$, i.e.,

$$f(x + v) - f(x) \geq \langle x^*, v\rangle - e(x, x + v), \qquad \forall v \in X.$$

We will show that $x^* \in \partial^{CR} f(x)$. Let $v \in X \setminus \{0\}$. For all $x \in \mathrm{dom}(f)$ we have that

$$\begin{aligned}
f^0(x; v) &= \lim_{\epsilon\downarrow 0} \limsup_{y\to_f x,\, t\downarrow 0} \inf_{w\in v+\epsilon B} \frac{f(y + tw) - f(y)}{t} \\
&\geq \lim_{\epsilon\downarrow 0} \limsup_{t\downarrow 0} \inf_{w\in v+\epsilon B} \frac{f(x + tw) - f(x)}{t} \\
&\geq \lim_{\epsilon\downarrow 0} \limsup_{t\downarrow 0} \inf_{w\in v+\epsilon B} \left(\langle x^*, w\rangle - \frac{e(x, x + tw)}{t}\right) \\
&= \lim_{\epsilon\downarrow 0} \limsup_{t\downarrow 0} \inf_{w'\in B} \left(\langle x^*, v\rangle + \epsilon\langle x^*, w'\rangle - \frac{e(x, x + t(v + \epsilon w'))}{t}\right).
\end{aligned}$$

Taking into account that $e(x, x) = 0$ and $De(x, \cdot)|_x = 0$ for every $x \in \mathrm{dom}(f)$, if $t\|v + \epsilon w'\| \to 0$ we have that

$$\begin{aligned}
e(x, x + t(v + \epsilon w')) &= e(x, x) + \langle De(x, \cdot)|_x, t(v + \epsilon w')\rangle + o(t\|v + \epsilon w'\|) \\
&= o(t\|v + \epsilon w'\|).
\end{aligned}$$

Let us assume, without loss of generality, that $\epsilon < \|v\|$. In particular for every $\eta > 0$ there exists $\delta > 0$ such that

$$0 \leq \frac{e(x, x + t(v + \epsilon w'))}{t\|v + \epsilon w'\|} < \eta$$

if $0 < t\|v + \epsilon w'\| < \delta$. This implies that if $0 < t < \frac{\delta}{\|v\|+\epsilon}$ we have that

$$0 \leq \frac{e(x, x + t(v + \epsilon w'))}{t} < \eta\|v + \epsilon w'\| \leq \eta(\|v\| + \epsilon).$$

Thus,

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\limsup_{t\downarrow 0}\inf_{w'\in B}&\left(\langle x^*,v\rangle+\epsilon\langle x^*,w'\rangle-\frac{e(x,x+t(v+\epsilon w'))}{t}\right)\\ &\geq \lim_{\epsilon\downarrow 0}\left(\langle x^*,v\rangle-\epsilon\|x^*\|-\eta(\|v\|+\epsilon)\right)\\ &=\langle x^*,v\rangle-\eta\|v\|.\end{aligned}$$

Thereby from the arbitrariness of η we get that $x^*\in\partial^{CR}f(x)$.

Since Proposition 16 entails that $\partial^{CR}f$ is $2e$-monotone, to prove maximality we need to show that for every $y\in X$ and $y^*\notin\partial^{CR}f(y)$, there exists $z\in\operatorname{dom}(f)$, $z^*\in\partial^{CR}f(z)$ such that

$$\langle y^*-z^*,y-z\rangle<-2e(y,z).$$

Let $g(x)=f(x)-\langle y^*,x\rangle$ and $\phi_y(x)=g(x)+2e(x,y)$. Since $y^*\notin\partial^{CR}f(y)$, it is easy to verify that the function ϕ_y does not attain its minimum at y. Indeed, otherwise, we would have that $0\in\partial^{CR}\phi_y(y)$, i.e., $0\in\partial^{CR}f(y)-y^*$, a contradiction. Therefore there exist $r\in\mathbb{R}$ and $x_1\in\operatorname{dom}(f)$ such that

$$\inf_{x\in X}\phi_y(x)\leq\phi_y(x_1)<r<\phi_y(y)=g(y).$$

Set $K=\sup_{x\in X,x\neq y}\dfrac{r-\phi_y(x)}{\|y-x\|}$; we will show that $0<K<+\infty$.

The left inequality is true since $\phi_y(x_1)<r$. To show the right inequality it is sufficient to consider the points $x\in L_y=\{x\in\operatorname{dom}(f):\ \phi_y(x)\leq r\}$. Note that L_y is non-empty and closed, and $y\notin L_y$. Therefore, $0<\operatorname{dist}(y,L_y)\leq\|y-x\|$ for all $x\in L_y$.

Since the function g is, in particular, also $2e$-convex, then by Theorem 3.5 in [1], for every $x\in\operatorname{dom}(f)$, we have $\partial^{CR}g(x)\subseteq\partial^{2e}g(x)$.

Since the domain of $\partial^{CR}f$ is dense in $\operatorname{dom}(f)$ from Corollary 3.2 in [6] we can find $u\in\operatorname{dom}(f)$ and $u^*\in\partial^{2e}g(u)$. Then for every $x\in X$ we have

$$\begin{aligned}g(x)&\geq g(u)+\langle u^*,x-u\rangle-2e(x,u)\\ &\geq g(u)+\langle u^*,y-u\rangle-\|u^*\|\|x-y\|-2e(x,u)\end{aligned}$$

Thus, for every $x\in L_y$,

$$\frac{r-g(x)-2e(x,y)}{\|y-x\|}\leq\frac{(r-g(u)-\langle u^*,y-u\rangle+2e(x,u)-2e(x,y))^+}{\operatorname{dist}(y,L_y)}+\|u^*\|.$$

From the inequality above, taking into account the Lipschitz property of $e(x,\cdot)$, we get $K<+\infty$.

Let now $H_y:X\to\mathbb{R}\cup\{+\infty\}$ given by $H_y(x)=K\|y-x\|+\phi_y(x)$. The function H_y is lower semicontinuous and $H_y(x)\geq r$ for every $x\in X$. Given $0<\epsilon<K$, by the definition of K there exists x_0 such that $H_y(x_0)<r+\epsilon\|y-x_0\|\leq\inf_X H_y(x)\epsilon\|y-x_0\|$. Therefore, by applying the Ekeland variational principle (see, for instance Theorem 4.2.5 in [13]), there exists a point $z\in X$ satisfying the following conditions:

(i) $\|z - x_0\| \leq \|y - x_0\|$

(ii) $H_y(z) \leq H_y(x_0) - \epsilon\|z - x_0\|$

(iii) $H_y(z) < H_y(x) + \epsilon\|z - x\|$ for all $x \neq z$.

From (iii) the point z is a global minimum of $H_y(\cdot) + \epsilon\|z - \cdot\|$, therefore $0 \in \partial^{CR}(H_y + \epsilon\|z - \cdot\|)(z)$. Since ∂^{CR} is an absubdifferential (see Definition 2.7 in [11]), and the functions $e(\cdot, y), \|y - \cdot\|, \|z - \cdot\|$ are convex with domain X, then from Theorem 3.4.2 in [13] we get that

$$0 \in \partial K\|y - \cdot\|(z) + \partial^{CR} g(z) + \partial(2e)(\cdot, y)(z) + \partial\epsilon\|z - \cdot\|(z).$$

We can then find $q^* \in \partial K\|y - \cdot\|(z)$, $u^* \in \partial^{CR} g(z)$, and $p^* \in \partial(2e)(\cdot, y)(z)$ such that $\|w^*\| = \|q^* + p^* + u^*\| \leq \epsilon$. Then,

$$\begin{aligned}
\langle u^*, y - z\rangle &= \langle p^*, z - y\rangle + \langle q^*, z - y\rangle + \langle w^*, y - z\rangle \\
&\geq 2e(z, y) + K\|y - z\| - \|w^*\|\|y - z\| \\
&\geq 2e(z, y) + (K - \|w^*\|)\|y - z\| \\
&> 2e(z, y).
\end{aligned}$$

To conclude the proof, it is enough to note that $u^* = z^* - y^*$, where $z^* \in \partial^{CR} f(z)$. ■

In the following example we exhibit an error bifunction satisfying all the assumptions of the theorem above and an e-convex function for which $\partial^e f$ is maximal $2e$-monotone.

Example 19 *Let $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ given by*

$$f(x) = \begin{cases} -x^2 & \text{if } x \in [-1, 1] \\ +\infty & \text{otherwise} \end{cases}$$

and set $e(x, y) = g(y - x)$ where

$$g(t) = \begin{cases} -4t - 4 & \text{if } t \leq -2 \\ t^2 & \text{if } -2 < t < 2 \\ 4t - 4 & \text{if } t \geq 2 \end{cases}$$

The error bifunction e satisfies all the assumptions in Theorem 18. Taking into account Example 3 in [5], we can verify that f is e-convex. Moreover, easy computations show that

$$\partial^e f(x) = \begin{cases} -2x & \text{if } x \in (-1, 1) \\ [-2, +\infty) & \text{if } x = 1 \\ (-\infty, 2] & \text{if } x = -1 \\ \emptyset & \text{otherwise} \end{cases}$$

We now verify that $\partial^e f$ *is maximal* $2e$*-monotone.*

Note that $(u,u^*) \in \mathrm{gr}\left((\partial^e f)^{\widetilde{2e}}\right)$ *if and only if, for every* $(x,x^*) \in \mathrm{gr}(\partial^e f)$,

$$(u^* - x^*)(u - x) \geq -2e(x,u).$$

In the case where $u > 1$*, this inequality is false for* $x = 1$*, while for* $u < -1$*, it is false for* $x = -1$*.*

For $-1 \leq u \leq 1$*, the inequality for* $x \in (-1,1)$ *yields* $(u^* + 2x)(u - x) \geq 0$*. It is easy to check that the only possible choice for* u^* *is to have* $u^* \in \partial^e f(u)$*. Therefore,* $\partial^e f(u) = (\partial^e f)^{\widetilde{2e}}(u)$*.*

5 The Fitzpatrick function of an e-monotone operator

The Fitzpatrick function of a monotone operator was introduced by Fitzpatrick in [9] and it makes a bridge between convex functions and maximal monotone operators (see, for instance, [7, 8, 15] and the references therein).

For a monotone operator T, let us recall that its Fitzpatrick function $\mathcal{F}_T : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ is given by

$$\mathcal{F}_T(x,x^*) = \sup_{(y,y^*)\in \mathrm{gr}(T)} \left(\langle x^*, y\rangle + \langle y^*, x - y\rangle\right), \tag{13}$$

or, equivalently,

$$\mathcal{F}_T(x,x^*) = \langle x^*, x\rangle - \inf_{(y,y^*)\in \mathrm{gr}(T)} \langle y^* - x^*, y - x\rangle \tag{14}$$

Since $\langle y^* - x^*, y - x\rangle \geq 0$ on $\mathrm{gr}(T)$, on this set $\mathcal{F}_T(x,x^*) = \langle x^*, x\rangle$, and thus it is proper. In addition, $\mathcal{F}_T$ is a lower semicontinuous and convex function.

In case of an e-monotone operator, we slightly change the definition of the Fitzpatrick function involving also the error bifunction e.

Definition 20 *Given an* e*-monotone operator* T*, we define the* e*-Fitzpatrick function* $\mathcal{F}_T^e : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ *as follows:*

$$\mathcal{F}_T^e(x,x^*) = \sup_{(y,y^*)\in \mathrm{gr}(T)} \left(\langle x^*, y\rangle + \langle y^*, x - y\rangle - e(x,y)\right).$$

Equivalently,

$$\mathcal{F}_T^e(x,x^*) = \langle x^*, x\rangle - \inf_{(y,y^*)\in \mathrm{gr}(T)} \left(\langle y^* - x^*, y - x\rangle + e(x,y)\right). \tag{15}$$

Note that, by e-monotonicity of T, if $e(x,x) = 0$ for every $x \in D(T)$, and $(x,x^*) \in \mathrm{gr}(T)$, we have again that

$$\mathcal{F}_T^e(x,x^*) = \langle x^*, x\rangle. \tag{16}$$

Therefore $\mathcal{F}_T^e$ is proper as well. However, in general, it does not possess the property of being lower semicontinuous and convex.

Remark 21 *The (classical) Fitzpatrick function defined in* (13) *applied to e-monotone operators can give rise to a non proper function. Take, for instance, the operator* $T : \mathbb{R} \to 2^{\mathbb{R}}$ *defined by* $T(x) = [0,1]$ *for every* $x \in \mathbb{R}$. *Then* T *is maximal e-monotone with* $e = |x - y|$ *and* $\mathcal{F}_T \equiv +\infty$, *but* $\mathcal{F}_T^e(x, 0^*) = 0$ *and so* $\mathcal{F}_T^e$ *is proper.*

In the next proposition we give some elementary properties of $\mathcal{F}_T^e$; for similar results see Proposition 20.47 in [7].

Proposition 22 *Suppose that* $T : X \to 2^{X^*}$ *is an e-monotone operator for some error bifunction e, with* $e(x,x) = 0$ *for all* $x \in D(T)$. *Then*

(i) $\mathcal{F}_T^e(x, x^*) \leq \langle x^*, x\rangle$ *if and only if* $\mathrm{gr}(T) \cup \{(x, x^*)\}$ *is e-monotone;*

(ii) $(\mathcal{F}_T^e(\cdot,\cdot))^*(x^*, x) \geq \mathcal{F}_T(x, x^*) \geq \mathcal{F}_T^e(x, x^*)$ *for every* $(x, x^*) \in \mathrm{gr}(T)$;

(iii) for any $\alpha > 0$, *if* T *is instead* e/α*-monotone, then* $\mathcal{F}_{(\alpha T)}^e(x, x^*) = \alpha \mathcal{F}_T^{\frac{e}{\alpha}}\left(x, \frac{x^*}{\alpha}\right)$.

Proof. The proof of (i) follows from the following equivalent statements:

$$\begin{aligned}
\langle x^*, x\rangle \geq \mathcal{F}_T^e(x, x^*) &= \langle x^*, x\rangle - \inf_{(y,y^*)\in \mathrm{gr}(T)} \left(\langle y^* - x^*, y - x\rangle + e(x,y)\right) \\
&\Longleftrightarrow \inf_{(y,y^*)\in \mathrm{gr}(T)} \left(\langle y^* - x^*, y - x\rangle + e(x,y)\right) \geq 0 \\
&\Longleftrightarrow \langle y^* - x^*, y - x\rangle + e(x,y) \geq 0 \quad \forall (y, y^*) \in \mathrm{gr}(T) \\
&\Longleftrightarrow \mathrm{gr}(T) \cup \{(x, x^*)\} \text{ is } e\text{-monotone.}
\end{aligned}$$

To prove (ii): let $(x, x^*) \in X \times X^*$. Simple computations show that

$$\begin{aligned}
(\mathcal{F}_T^e(\cdot,\cdot))^*(x, x^*) &= \sup_{(y,y^*)\in X\times X^*} \left(\langle (y, y^*), (x^*, x)\rangle - \mathcal{F}_T^e(y, y^*)\right) \\
&= \sup_{(y,y^*)\in X\times X^*} \left(\langle x^*, y\rangle + \langle y^*, x\rangle - \mathcal{F}_T^e(y, y^*)\right) \\
&\geq \sup_{(y,y^*)\in \mathrm{gr}\, T} \left(\langle x^*, y\rangle + \langle y^*, x\rangle - \mathcal{F}_T^e(y, y^*)\right) \\
&= \sup_{(y,y^*)\in \mathrm{gr}\, T} \left(\langle x^*, y\rangle + \langle y^*, x\rangle - \langle y^*, y\rangle\right) \\
&= \mathcal{F}_T(x, x^*) \geq \mathcal{F}_T^e(x, x^*),
\end{aligned}$$

where we use the fact that $\mathcal{F}_T^e(x, x^*) = \langle x^*, x\rangle$ on the graph of T.

For the the proof of (iii), note that for $\alpha \in \mathbb{R}\setminus\{0\}$, $(x, x^*) \in \mathrm{gr}(T)$ if and only if $(x, \alpha x^*) \in \mathrm{gr}(\alpha T)$. Then the assertion follows from the following equalities:

$$\begin{aligned}
\alpha \mathcal{F}_T^{\frac{e}{\alpha}}\left(x, \frac{x^*}{\alpha}\right) &= \alpha \sup_{(y,y^*)\in \mathrm{gr}(T)} \left(\left\langle \frac{x^*}{\alpha}, y\right\rangle + \langle y^*, x - y\rangle - \frac{e(x,y)}{\alpha}\right) \\
&= \sup_{(y,y^*)\in \mathrm{gr}(T)} \left(\langle x^*, y\rangle + \langle \alpha y^*, x - y\rangle - e(x,y)\right) \\
&= \sup_{(y,\alpha y^*)\in \mathrm{gr}(\alpha T)} \left(\langle x^*, y\rangle + \langle \alpha y^*, x - y\rangle - e(x,y)\right) = \mathcal{F}_{(\alpha T)}^e(x, x^*).
\end{aligned}$$

■

The next result characterizes maximal e-monotone operators by generalizing a well-known result for monotone operators (see [9], Theorem 3.8):

Theorem 23 *Suppose that $T : X \to 2^{X^*}$ is an e-monotone operator for some error bifunction e, with $e(x,x) = 0$ for all $x \in D(T)$. Then, T is maximal e-monotone if and only if*

$$\mathcal{F}_T^e(x,x^*) > \langle x^*, x\rangle \tag{17}$$

whenever $(x,x^*) \notin \mathrm{gr}(T)$.

Proof. For the necessary condition, note that if $\mathcal{F}_T^e(x,x^*) \leq \langle x^*, x\rangle$, by (i) in Proposition 22 we have that $\mathrm{gr}\,(T) \cup \{(x,x^*)\}$ is e-monotone, which contradicts maximality.

Conversely, if T is not maximal e-monotone, there exists $(z,z^*) \in X \times X^*$, $(z,z^*) \notin \mathrm{gr}(T)$, such that $\langle y^* - z^*, y - z\rangle + e(z,y) \geq 0$, for every $(y,y^*) \in \mathrm{gr}(T)$. Thus,

$$\langle z^*, z\rangle \geq \langle z^*, y\rangle + \langle y^*, z-y\rangle - e(z,y) \qquad \forall (y,y^*) \in \mathrm{gr}(T),$$

and therefore $\mathcal{F}_T^e\,(z,z^*) \leq \langle z^*, z\rangle$. ■

By the previous result, we easily get the following

Corollary 24 *Let T be maximal e-monotone for some error bifunction e, with $e(x,x) = 0$ for all $x \in D(T)$. Then $\mathcal{F}_T^e(x,x^*) \geq \langle x^*, x\rangle$ for every $(x,x^*) \in X \times X^*$, and $\mathcal{F}_T^e(x,x^*) = \langle x^*, x\rangle$ if and only if $(x,x^*) \in \mathrm{gr}(T)$.*

In this last part, we focus on the $2e$-Fitzpatrick function of an e-subdifferential. Let us first recall the following definition (see [5]):

Definition 25 *Suppose that $f : X \to \mathbb{R} \cup \{+\infty\}$ is a proper and e-convex function, and $y \in X$ is fixed. Then the function $f^*_{e,y} : X^* \to \mathbb{R} \cup \{+\infty\}$ defined by*

$$f^*_{e,y}\,(x^*) := \sup_{x \in X} \left\{\langle x^*, x\rangle - f\,(x) - e\,(x,y)\right\}, \quad \forall x^* \in X^*$$

is called the (e,y)-conjugate of f. Also, the function $f^{**}_{e,y} : X \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$f^{**}_{e,y}\,(x) := \sup_{x^* \in X^*} \left\{\langle x^*, x\rangle - f^*_{e,y}\,(x^*)\right\}, \quad \forall x \in X$$

is called the (e,y)-biconjugate of f.

Proposition 26 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be an e-convex function, and denote by $\partial^e f : X \to 2^{X^*}$ its e-subdifferential. Then*

$$\mathcal{F}^{2e}_{\partial^e f}(x,x^*) \leq f(x) + f^*_{e,x}(x^*), \qquad \forall (x,x^*) \in X \times X^*.$$

In addition, if $e(x,x) = 0$ for every $x \in \mathrm{dom}(f)$, the equality holds if $x^* \in \partial^e f(x)$.

Proof. Taking into account that, from the definition of $\partial^e f(y)$, $\langle y^*, x-y\rangle \le f(x)-f(y)+e(x,y)$ for every $y^*\in\partial^e f(y)$, simple computations show that

$$\begin{aligned}\mathcal{F}^{2e}_{\partial^e f}(x,x^*) &= \sup_{y\in D(\partial^e f)}\left(\langle x^*,y\rangle - 2e(x,y) + \sup_{y^*\in\partial^e f(y)}\langle y^*, x-y\rangle\right)\\ &\le \sup_{y\in D(\partial^e f)}\left(\langle x^*,y\rangle + f(x)-f(y)-e(x,y)\right)\\ &\le f(x)+\sup_{y\in X}\left(\langle x^*,y\rangle - f(y)-e(x,y)\right)\\ &= f(x)+f^*_{e,x}(x^*).\end{aligned}$$

Take now $x^*\in\partial^e f(x)$; by (16) and Proposition 11 in [5] we can easily get

$$\langle x^*,x\rangle = \mathcal{F}^{2e}_{\partial^e f}(x,x^*) \le f(x)+f^*_{e,x}(x^*) = \langle x^*,x\rangle. \tag{18}$$

■

Remark 27 *Let $f: X\to\mathbb{R}\cup\{+\infty\}$ be an e-convex function, such that $\partial^e f$ is maximal 2e-monotone, and $e(x,x)=0$ for every $x\in\mathrm{dom}(f)$. Then, by combining Corollary 24 and Proposition 26, we obtain that, for every $(x,x^*)\in X\times X^*$,*

$$\langle x^*,x\rangle \le \mathcal{F}^{2e}_{\partial^e f}(x,x^*) \le f(x)+f^*_{e,x}(x^*) \le \langle x^* x\rangle + \iota_{\mathrm{gr}(\partial^e f)}(x,x^*), \tag{19}$$

where ι_A denotes the indicator function of A. Note that (19) provides a kind of refinement of the Fenchel-Young inequality.

Moreover, the second inequality implies that for each $x\in X$, we have $\mathrm{dom}(f)\times\mathrm{dom}(f^*_{e,x})\subset\mathrm{dom}(\mathcal{F}^{2e}_{\partial^e f})$.

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Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

[1] M.H. Alizadeh, *On e-convex functions and e-subdifferentials in locally convex spaces*, Optimization **72** (2023), 1347-1362.

[2] M.H. Alizadeh, N. Hadjisavvas and M. Roohi, *Local boundedness properties for generalized monotone operators*, J. Convex Anal. **19** (2012), 49-61.

[3] M.H. Alizadeh and N. Hadjisavvas, *Local boundedness of monotone bifunctions*, J. Global Optim. **53** (2012), 231-241.

[4] M.H. Alizadeh, *Fitzpatrick function for generalized monotone operators*, to appear in J. Nonlinear Convex Anal.

[5] M.H. Alizadeh, A.Y. Zanjani, *Generalized convex functions and their applications in optimality conditions*, arXiv:2402.15597.

[6] D. Aussel, J.N Corvellec and M. Lassonde, *Subdifferential characterization of quasiconvexity and convexity*, J. Convex Anal. **1** (1994), 195-201.

[7] H.H. Bauschke and P.L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces. Second Edition*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer (2017).

[8] H.H. Bauschke, D.A. McLaren and H.S. Sendov, *Fitzpatrick functions: inequalities, examples and remarks on a problem of S. Fitzpatrick*, J. Convex Anal. **13** (2006), 499-523.

[9] S. Fitzpatrick, *Representing monotone operators by convex functions,* in: Functional Analysis and Optimization, Workshop and Mini-conference, Canberra, Australia 1988, pp. 59-65, Proc. Center Math. Anal. Australian Nat. Univ. **20** (1988).

[10] N. Hadjisavvas and H. Khatibzadeh, *Maximal monotonicity of bifunctions*, Optimization **59** (2010), 147-160.

[11] M. Ivanov and N. Zlateva, *Abstract subdifferential calculus and semi-convex functions,* Serdica Math. J. **23** (1997), 35-58.

[12] A.N. Iusem, G. Kassay and W. Sosa, *An existence result for equilibrium problems with some surjectivity consequences*, J. Convex Anal. **16** (2009), 807-826.

[13] R. Lucchetti, *Convexity and Well-Posed Problems,* CMS Books in Mathematics, Springer (2006).

[14] J.-E. Martinez-Legaz and B.F Svaiter, *Monotone operators representable by l.s.c. convex functions*, Set-Valued Anal. **13** (2005) 21-46.

[15] J.-P. Penot and C. Zalinescu, *Some problems about the representation of monotone operators by convex functions,* ANZIAM J. **47** (2005), 1-20.

[16] R. Phelps, *Lectures on maximal monotone operators.* Extracta Math. **12** (1997), 193-200.

[17] D. Zagrodny, *Approximate mean value theorem for upper subderivatives.* Nonlinear Anal. **12** (1988), 1413-1428.