On Braess' paradox and average quality of service in transportation network cooperative games

Mauro Passacantando, Giorgio Gnecco, Yuval Hadas, Marcello Sanguineti

Abstract In the theory of congestion games, the Braess' paradox shows that adding one resource to a network may sometimes worsen, rather than improve, the overall network performance. Here the paradox is investigated under a cooperative game-theoretic setting, in contrast to the non-cooperative one typically adopted in the literature. A family of cooperative games on networks is considered, whose utility function, defined in terms of a traffic assignment problem and the associated Wardrop equilibrium, expresses the average quality of service perceived by the network users.

Key words: Transportation networks; transferable utility games; Braess' paradox; traffic assignment; user equilibrium; quality of service

1 Introduction

In the theory of congestion games [12], Braess' paradox [1] highlights why adding one resource to a network may in some cases worsen, rather than improve, the overall network performance. This phenomenon is typically explained through non-

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Mauro Passacantando
Department of Computer Science, University of Pisa, Largo B. Pontecorvo 3, 56127 Pisa, Italy, e-mail: mauro.passacantando@unipi.it

Giorgio Gnecco
AXES Research Unit, IMT - School for Advanced Studies, Piazza S. Francesco 19, 55110 Lucca, Italy, e-mail: giorgio.gnecco@imtlucca.it

Yuval Hadas
Department of Management, Bar-Ilan University, Max ve-Anna Webb street, Ramat Gan 5290002, Israel, e-mail: yuval.hadas@biu.ac.il

Marcello Sanguineti (corresponding author)
Department of Bioengineering, Robotics, Informatics, and Systems Engineering (DIBRIS), University of Genova, Via Opera Pia 13, 16145 Genova, Italy, e-mail: marcello.sanguineti@unige.it

cooperative game theory, and is related to the concept of price of anarchy [5]: the players (in this context, for example, the users of a road network), being driven by the pursuit of maximizing their own individual interests, tend to reduce social welfare (and sometimes, as an undesired consequence, they even fail to maximize their own individual interests, when compared to the case in which they behave in a more collaborative way). In case several resources are added to the network, however, such a non-cooperative approach, which does not take into account every potential interaction among the resources in all the possible contingent situations, does not allow one to quantify the average marginal contribution (be it positive or negative) of each resource to the overall network performance. This suggests the investigation of a cooperative version of Braess' paradox.

This work, which is in the same research direction as [16], studies Braess' paradox in the context of cooperative games with transferable utility on a graph [20], which can model, for example, transportation networks [6, 8]. The players can be either nodes or arcs of the graph (in this paper, they are arcs). The utility function of each such game is defined in terms of a suitable congestion measure over subgraphs associated with subsets of these nodes/arcs. Such a measure is computed by solving an instance of the classical user equilibrium problem via any traffic assignment algorithm (see, e.g., [15]). Then, the Shapley value of a node (or arc) of the network is used as a measure of its importance, in line with [8, 11]. Differently from the latter works, the goal here is to identify situations for which the Shapley value of a node/arc is negative (as a consequence of the specific choice of the utility function). In this case, the insertion of such an element to the network has a negative average marginal value. This indicates a degradation of the average network performance following its insertion, therefore the inopportunity of such an addition.

The work complements the analysis of [16] in several directions. A different choice of the utility function associated with the transportation network cooperative game is considered, which is proportional to the average quality of service perceived by its users. Moreover, a variation of the example in [16] is adopted for illustration purposes. An additional "fictitious" arc is included in the directed graph modeling the transportation network, in order to make the origin and destination nodes connected, in all its subgraphs derived from all possible coalitions of arcs. A novel numerical example is presented and additional computational issues are discussed.

The paper is structured as follows. Section 2 provides a background on cooperative games with transferable utility, transportation networks cooperative games, and Wardrop first principle, which is used to model the behavior of vehicles in the network. Section 3 details the case of a utility function based on a Wardrop equilibrium, which is proportional to the average quality of service perceived by the network users. Section 4 provides an application to a toy example. Finally, Section 5 is a short discussion.

2 Background

2.1 Cooperative games with transferable utility

A *cooperative game with transferable utility* (*TU game*, see, e.g., [20]) is a pair (N, v), where:

- N is a set of players, and any subset $S \subseteq N$ is called a coalition;
- $v : 2^N \to \mathbb{R}$ is the utility (characteristic) function, with $v(\emptyset) = 0$; $v(S)$ represents the utility that can be achieved jointly by all the players in S, without any contribution from the players in $N \setminus S$.

In TU games, the utilities can be transferred from one player to another without any loss.

The Shapley value [18] is the most important point-solution in cooperative game theory, and corresponds to a suitable way of allocating the total utility in a "fair way" among the players. For each player $i \in N$, it is defined as

$$Sh(i) = \sum_{S \subseteq N} \frac{(|S|-1)!(|N|-|S|)!}{|N|!} \left[v(S) - v(S \setminus \{i\}) \right] .$$

It represents the average marginal contribution of each player across all possible coalitions, according to a suitable probability distribution (i.e., when players, starting from the empty coalition, enter the grand coalition randomly, in such a way that all orders are equally likely). It is worth noting that, due to the interpretation above, the Shapley value can be applied as a measure of players' importance not only in classical contexts in which the players are modeled as rational decision makers, but also in other more general situations in which this does not occur, e.g., when players are features in supervised machine learning problems [2], genes in microarray games [13], or joints in the analysis of motion capture datasets [10].

2.2 Transportation network cooperative games

Consider a graph $G = (V, A)$, where

- V is the set of nodes;
- $A \subseteq V \times V$ is the set of arcs;
- W is the set of Origin-Destination pairs;
- d_w is the traffic demand of the OD pair w, and $d = (d_w)$;
- P_w is the set of paths joining the elements of the OD pair w;
- x_p is the flow on path p, and $x = (x_p)$;
- f_a is the flow on arc a, and $f = (f_a)$;
- $c_a(f)$ is the (non-negative) cost on arc a associated with the flow vector f;

- $C_p(x)$ is the (non-negative) cost on path p, equal to the sum of the costs on the arcs of the path p.

The set N of players is a given subset of arcs such that any OD pair can be served in the subgraph $(V, A \setminus N)$. This is a subset of suitable arcs chosen a priori because they are deemed to be important for the analysis of the specific network under investigation. Specifically, for any OD pair $w \in W$ there exists a path joining the elements of w in the subgraph $(V, A \setminus N)$ obtained by removing all the arcs in N from the arc set A (see Section 4 for an example).

2.3 Wardrop first principle

We consider a transportation network whose arcs model one-way traffic roads and their weights the associated travel costs (e.g., travel times translated to monetary costs, combined with tolls, if present), which are functions of the respective arc flows. The number of vehicles traveling is considered so large that each vehicle contributes with an infinitesimally small amount of flow. According to *Wardrop first principle* [21], a Wardrop equilibrium state is such that no vehicle can unilaterally reduce its travel cost by shifting to another route. So, the resulting equilibrium (called Wardrop equilibrium) models the realistic case in which all the drivers behave in a selfish way. This equilibrium can be interpreted as a Nash equilibrium in the case of an infinite number of infinitesimal players (the vehicles) [9].

For any coalition $S \subseteq N$, the subgraph associated to S is

$$G(S) := (V, (A \setminus N) \cup S)\,.$$

A path flow $x(S)$ in the subgraph $G(S)$ is feasible if for any $w \in W$ the demand d_w is satisfied by using paths belonging to the set $P_w(S)$ of paths joining the OD pair w in $G(S)$.

A *Wardrop equilibrium* (or *user equilibrium*) in $G(S)$ is defined as a feasible path flow $\bar{x}(S)$ such that for any OD pair $w \in W$ and any $p \in P_w(S)$ one has

$$C_p(\bar{x}(S)) \begin{cases} = \lambda_w(S) & \text{if } \bar{x}_p(S) > 0, \\ \geq \lambda_w(S) & \text{if } \bar{x}_p(S) = 0, \end{cases}$$

where $\lambda_w(S)$ is the "equilibrium disutility" for the OD pair w, and $\bar{x}_p(S)$ is the component of $\bar{x}(S)$ which is associated with the path p. It follows from the definition of Wardrop equilibrium that, for any $w \in W$, one incurs the same cost $C_p(\bar{x}(S))$ on all the paths $p \in P_w(S)$ for which $\bar{x}_p(S) > 0$, and such a cost is smaller than or equal to the costs $C_p(\bar{x}(S))$ on all the other paths $p \in P_w(S)$ for which $\bar{x}_p(S) = 0$. The specific value of $\lambda_w(S)$ is obtained a posteriori by imposing all the conditions above.

It is known [4, 19] that $\bar{x}(S)$ is a Wardrop equilibrium if and only if it solves the variational inequality

$$\langle C(\bar{x}(S)), x(S) - \bar{x}(S)\rangle \geq 0, \qquad \text{for any feasible path flow } x(S).$$

3 Utility function based on user equilibrium

Consider the following utility function:

$$v^{ue}(S) := \sum_{w\in W}\sum_{p\in P_w(S)} \frac{\bar{x}_p(S)}{C_p(\bar{x}_p(S))} - \sum_{w\in W}\sum_{p\in P_w(\emptyset)} \frac{\bar{x}_p(\emptyset)}{C_p(\bar{x}_p(\emptyset))},$$

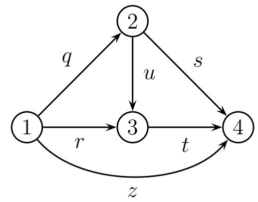
where $\bar{x}(S)$ and $\bar{x}(\emptyset)$ are Wardrop equilibria in $G(S)$ and $G(\emptyset)$, respectively. Notice that, by the definition of the Wardrop equilibrium $\bar{x}(S)$, one has $C_p(\bar{x}_p(S)) = \lambda_w(S)$ for all the paths $p \in P_w(S)$ with $\bar{x}_p(S) > 0$, so the denominators in the inner summations above do not depend on $p \in P_w(S)$. Such a utility function is well-defined when the equilibrium costs of the traveled paths are unique, which occurs when the arc costs are non-decreasing functions of the respective arc flows [3], even under possibly non-unique equilibria (indeed, for all such equilibria, one has $\sum_{p\in P_w(S)} \bar{x}_p(S) = d_w$).

The utility function introduced above is inspired by a measure of network performance versus efficiency for congested networks, which was considered in [14], but not in a cooperative setting therein. Equivalently, the term $\bar{x}_p(S)/C_p(\bar{x}_p(S))$ in the utility function $v^{ue}(S)$ represents the product between the flow $\bar{x}_p(S)$ which is served by path $p \in P_w(S)$ in the Wardrop equilibrium $\bar{x}(S)$, and the "quality of service" $1/C_p(\bar{x}_p(S))$ perceived by its vehicles. In the present work the served demand does not depend on S (indeed, even the empty coalition is able to serve it - possibly inefficiently - by using the arcs belonging to $A \setminus N$). Hence, the utility function $v^{ue}(S)$ is proportional to the improvement in the average quality of service one gets when one moves from the empty coalition to the coalition S, i.e., when the arcs in S are included in the transportation network.

Finally, it is worth remarking that the chosen utility function is not generally monotone (i.e., it is not necessarily true that $v(S) \leq v(T)$ for any $S \subseteq T \subseteq N$). Hence, the Shapley value of some arcs may be negative and the Braess' paradox can occur (see Section 4). This follows from the interpretation of the Shapley value as average marginal contribution of a player to the utility of a randomly generated coalition.

4 An illustrative example

Consider the following network with one OD pair $w = (1, 4)$ with demand d.



There are 4 paths connecting the OD pair:
$p_1 = (q, s)$, $p_2 = (r, t)$, $p_3 = (q, u, t)$, $p_4 = z$.

We assume that the arc cost functions are defined as follows:

$$c_q = 9 f_q, \quad c_r = f_r{+}50, \quad c_s = f_s{+}50, \quad c_t = 9 f_t, \quad c_u = f_u{+}10, \quad c_z = 40d{+}50.$$

Hence, the path cost functions are:

$$C_1(x) = 10x_1 + 9x_3 + 50, \qquad C_2(x) = 10x_2 + 9x_3 + 50,$$
$$C_3(x) = 9x_1 + 9x_2 + 19x_3 + 10, \qquad C_4(x) = 40d + 50.$$

The arc z represents an additional “fictitious” arc, not present in the topology of the original Braess’ network, which has been included here in order to make all the demand served, independently of the specific coalition S. Hence, we consider a TU game where the set of players is $N := A \setminus \{z\}$. In such a way, being the demand always served, negative Shapley values will arise only as a consequence of a deterioration of the average quality of service perceived by the network users.

The user equilibrium $\bar{x}(S)$ and the disutility $\lambda(S)$ for each coalition $S \subseteq N$ have the expressions reported in Table 1.

From the computational point of view, it can be observed that, if the flow f_a on an arc $a \in S$ is equal to 0 in correspondence of the Wardrop equilibrium $\bar{x}(S)$ for the subgraph $G(S)$, then $\bar{x}(S)$ is a Wardrop equilibrium also for the subgraph $G(S \setminus \{a\})$ obtained by removing the arc a from it. In Table 1 this occurs, e.g., for $S = \{q, r, t\}$: indeed the arc q has 0 flow in $G(S)$, because in that subgraph there is no path that connects the OD pair and uses arc q. It occurs also for $S = \{q, r, t, u\}$ when $d \leq 4$: the arc r is not used in such a case, because only the flow x_3 on path $p_3 = (q, u, t)$ is different from 0 in $\bar{x}(S)$. These arguments could help speeding up the evaluation of the utility function (and, as a consequence, of the Shapley value) for larger networks, and could be combined with empirical approximations of the Shapley value based on a subset of sampled coalitions (as done in [7], for a different and easier to compute utility function), or even based on approximate solutions of the variational inequalities that define the Wardrop equilibrium $\bar{x}(S)$ for different coalitions S.

Since there is a unique OD pair, the utility function v^{ue} has the following form:

$$v^{ue}(S) = \frac{d}{\lambda(S)} - \frac{d}{\lambda(\emptyset)}.$$

Moreover, it follows from Table 1 that the Shapley value of arc u is given by the following explicit formula:

Coalition S	$\bar{x}(S)$	$\lambda(S)$
$\emptyset$	$(0, 0, 0, d)$	$40d + 50$
$\{q\}, \{r\}, \{s\}$ $\{t\}, \{u\}$ $\{q, r\}, \{q, t\}$ $\{q, u\}, \{r, s\}$ $\{r, u\}, \{s, t\}$ $\{s, u\}, \{t, u\}$ $\{q, r, u\}$ $\{r, s, u\}$ $\{s, t, u\}$	$(0, 0, 0, d)$	$40d + 50$
$\{q, s\}$ $\{q, r, s\}$ $\{q, s, u\}$ $\{q, s, t\}$ $\{q, r, s, u\}$	$(d, 0, 0, 0)$	$10d + 50$
$\{r, t\}$ $\{q, r, t\}$ $\{r, s, t\}$ $\{r, t, u\}$ $\{r, s, t, u\}$	$(0, d, 0, 0)$	$10d + 50$
$\{q, t, u\}$	$(0, 0, d, 0)$	$19d + 10$
$\{q, r, s, t\}$	$(d/2, d/2, 0, 0)$	$5d + 50$
$\{q, s, t, u\}$	$\begin{cases} (0, 0, d, 0) & \text{if } d \leq 4 \\ (\frac{10d-40}{11}, 0, \frac{d+40}{11}, 0) & \text{if } d \geq 4 \end{cases}$	$\begin{cases} 19d + 10 & \text{if } d \leq 4 \\ \frac{109d+510}{11} & \text{if } d \geq 4 \end{cases}$
$\{q, r, t, u\}$	$\begin{cases} (0, 0, d, 0) & \text{if } d \leq 4 \\ (0, \frac{10d-40}{11}, \frac{d+40}{11}, 0) & \text{if } d \geq 4 \end{cases}$	$\begin{cases} 19d + 10 & \text{if } d \leq 4 \\ \frac{109d+510}{11} & \text{if } d \geq 4 \end{cases}$
$\{q, r, s, t, u\}$	$\begin{cases} (0, 0, d, 0) & \text{if } d \leq 4 \\ \left(\frac{5d-20}{6}, \frac{5d-20}{6}, \frac{40-4d}{6}, 0\right) & \text{if } d \in [4, 10] \\ (\frac{d}{2}, \frac{d}{2}, 0, 0) & \text{if } d \geq 10 \end{cases}$	$\begin{cases} 19d + 10 & \text{if } d \leq 4 \\ \frac{7d+230}{3} & \text{if } d \in [4, 10] \\ 5d + 50 & \text{if } d \geq 10 \end{cases}$

Table 1 User equilibrium $\bar{x}(S)$ and disutility $\lambda(S)$ for each coalition $S \subseteq N$.

$$Sh(u) = \begin{cases} \frac{d}{3(19d+10)} - \frac{d}{30(40d+50)} - \frac{d}{10(10d+50)} - \frac{d}{5(5d+50)} & \text{if } d \leq 4, \\ \frac{d}{30(19d+10)} + \frac{11d}{10(109d+510)} + \frac{3d}{5(7d+230)} & \\ -\frac{d}{30(40d+50)} - \frac{d}{10(10d+50)} - \frac{d}{5(5d+50)} & \text{if } d \in [4, 10], \\ \frac{d}{30(19d+10)} + \frac{11d}{10(109d+510)} - \frac{d}{30(40d+50)} - \frac{d}{10(10d+50)} & \text{if } d \geq 10. \end{cases}$$

The Shapley value of arc u is negative (i.e., a sort of cooperative version of Braess' paradox occurs) for $d \in (3.57, 7.67)$. This conclusion is similar to the one obtained in [16], where a different utility function - still based on Wardrop equilibria - was considered in the analysis. The Shapley values of the other arcs are positive for any $d > 0$ and, because of the symmetry of the arc cost functions, arcs q and t have the same Shapley value for any demand, and the same fact holds for arcs r and s. Figure 1 shows the Shapley value of each arc as a function of the traffic demand.

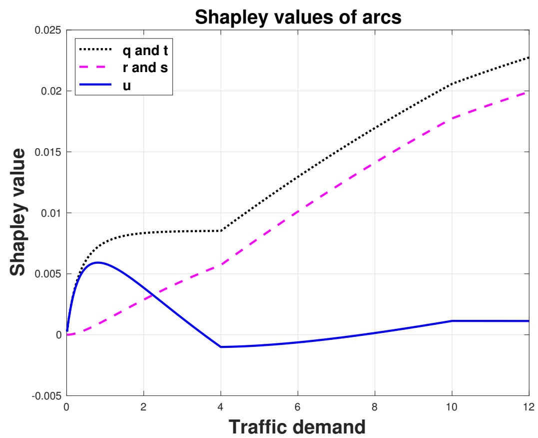


Fig. 1 Shapley value of each arc as a function of the traffic demand.

However, it can be verified (using similar expressions for the user equilibria and disutilities for each coalition as the ones reported in Table 1) that no negative Shapley value occurs if the arc cost function of arc u is modified to $c_u = 10 f_u + 50$ (as a consequence, e.g., of the introduction of a suitable congestion pricing scheme).

It is also worth noting that, likewise in [16], no negative Shapley value is obtained if, in the definition of the utility associated with each coalition S, the Wardrop equilibrium $\bar{x}(S)$ is replaced by a flow vector $\hat{x}(S)$ maximizing the expression $\sum_{w \in W} \sum_{p \in P_w(S)} \frac{x(S)}{C_p(x(S))}$ over all flow vectors $x(S)$ that are feasible in $G(S)$, and $\bar{x}(\emptyset)$ is replaced by a vector $\hat{x}(\emptyset)$ maximizing the expression $\sum_{w \in W} \sum_{p \in P_w(\emptyset)} \frac{x(\emptyset)}{C_p(x(\emptyset))}$ over all flow vectors $x(\emptyset)$ that are feasible in $G(\emptyset)$. In this case, indeed, the resulting *system optimum* utility function

$$v^{so}(S) := \sum_{w \in W} \sum_{p \in P_w(S)} \frac{\hat{x}_p(S)}{C_p(\hat{x}_p(S))} - \sum_{w \in W} \sum_{p \in P_w(\emptyset)} \frac{\hat{x}_p(\emptyset)}{C_p(\hat{x}_p(\emptyset))}$$

is monotone, so no negative Shapley value can occur, being the Shapley value the average marginal utility of a player when it is inserted in a suitably randomly generated coalition.

5 Discussion

A first future research direction of the work is aimed at further investigating the use of congestion pricing as a way to deal with negative Shapley values. For instance, in the case of the occurrence of negative Shapley values, one could be interested in finding the minimal amount of change in the arc cost functions (induced by congestion pricing) able to make all Shapley values non-negative. A second possible extension consists in reducing the computational effort in the evaluation of the Shapley values, making it possible to analyze realistic networks characterized by a large number of arcs and various traffic demands. Indeed, in such cases, closed forms expressions of the Shapley values could be not available, or their exact evaluation could be computationally expensive. However, even a sufficiently accurate approximate evaluation of the Shapley values would be enough to achieve the final goal of detecting arcs with negative such values. A promising approach in this direction appears to be the application of supervised machine learning techniques [17] which, based on a suitable set of supervised training pairs - e.g., depending on the context, input/output pairs of the form (input vector of arc cost functions $c_a(f)$, output vector of Shapley values $Sh(i)$) or (input vector of traffic demands d, output vector of Shapley values $Sh(i)$) - could allow one to predict the output vectors of Shapley values associated with test examples (not used in the training phase), starting from the corresponding input vectors. Moreover, the possibility of guaranteeing a good generalization capability of the resulting trained machines could be investigated via a sensitivity analysis of Wardrop equilibria with respect to a change in the vector of arc cost functions.

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