Nonlinear Analysis 234 (2023) 113320

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Symmetry of solutions to semilinear PDEs on Riemannian domains

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ARTICLE INFO

Article history:
Received 26 August 2022
Accepted 15 May 2023
Communicated by Francesco Maggi

Keywords:
Symmetry
Stable solutions
Partial differential equations
Riemannian domains
Isoparametric domains
Weighted manifolds

ABSTRACT

This paper deals with symmetry phenomena for solutions to the Dirichlet problem involving semilinear PDEs on Riemannian domains. We shall present a rather general framework where the symmetry problem can be formulated and provide some evidence that this framework is completely natural by pointing out some results for stable solutions. The case of manifolds with density, and corresponding weighted Laplacians, is inserted in the picture from the very beginning.

1. Introduction

This paper deals with symmetry phenomena for solutions to the Dirichlet problem involving semilinear PDEs on Riemannian domains. We shall present a rather general framework where the symmetry problem can be formulated and provide some evidence that this framework is completely natural by pointing out some results for stable solutions. The case of manifolds with density, and corresponding weighted Laplacians, is inserted in the picture from the very beginning. The investigations of the present paper all arise from the *elementary properties of stable solutions* in Euclidean domains as they are presented by L. Dupaigne in [14, Section 1.3] and show how much geometry was (more or less implicitly) contained there.

1.1. Basic notation

Throughout this paper, (M, g) will always denote a connected Riemannian manifold of dimension $\dim M = m$. The symbols Sect and Ric are reserved to its sectional and Ricci curvatures. We set $\mathrm{dist}(x, y)$ for the intrinsic distance of M. The corresponding open metric ball centred at $o \in M$ and of radius $R > 0$ is $B_R^M(o) = \{x \in M : \mathrm{dist}(x, o) < R\}$. When there is no danger of confusion, the overscript M is omitted in

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https://doi.org/10.1016/j.na.2023.113320

the notation and we simply write $B_R(o)$. Moreover, in the special case where $M = \mathbb{R}^n$ is equipped with its standard flat metric g^E we set $\mathbb{B}_R = B_R(0)$.

A class of Riemannian manifolds of special interest is that of *model manifolds*. Let $\sigma : [0, R) \to \mathbb{R}_{\geq 0}$, $0 < R \leq +\infty$, be a smooth function that is positive in $(0, R)$ and satisfies

- $\sigma^{(2k)}(0) = 0$ for all $k \in \mathbb{N}$;
- $\sigma'(0) = 1$.

Then, in polar coordinates around 0, we can define a smooth Riemannian metric on $(0, R) \times \mathbb{S}^{m-1}$ by setting

$$g = dr \otimes dr + \sigma^2(r) g^{\mathbb{S}^{m-1}},$$

where $g^{\mathbb{S}^{m-1}}$ is the standard metric on the unit sphere $\mathbb{S}^{m-1} \subset \mathbb{R}^m$. The corresponding Riemannian manifold $\mathbb{M}^m(\sigma) = (\mathbb{B}_R, g)$, obtained by identifying all the points of the form $(0, \theta)$ with 0 and extending (smoothly) the metric in 0, will be called an m-dimensional *model manifold with warping function* σ. Clearly, $\mathbb{M}(\sigma)$ is complete if and only if $R = +\infty$ and, in any case, the r-coordinate represents the distance from the *pole* $o = 0 \in \mathbb{R}^m$. Thus, $B_T^{\mathbb{M}(\sigma)}(o) = \{x \in \mathbb{B}_R : r(x) < T\}$. For more details on the construction of warped product manifolds and model manifolds we suggest [31].

Example 1.1. The standard spaceforms $\mathbb{R}^m$, $\mathbb{S}^m \setminus \{pt.\}$ and $\mathbb{H}^m$ are model manifolds with the choice, respectively, $\sigma(r) = r$, $\sigma(r) = \sin(r)$, $\sigma(r) = \sinh(r)$.

Now, let the Riemannian manifold (M, g) be endowed with the absolutely continuous measure $\mathrm{dv}_\Psi = e^{-\Psi}\mathrm{dv}$ where dv is the Riemannian measure and $\Psi : M \to \mathbb{R}$ is a smooth function. Usually, the triple

$$M_\Psi = (M, g, \mathrm{dv}_\Psi)$$

is called a *weighted manifold* or a *manifold with density* or a *smooth metric measure space*.

On the weighted manifold M_Ψ we have a natural linear elliptic differential operator. It is the *weighted Laplacian*, also called *Ψ-Laplacian*, which is defined by the formula

$$\Delta_\Psi u = e^\Psi \operatorname{div}(e^{-\Psi}\nabla u) = \Delta u - g(\nabla\Psi, \nabla u).$$

Here,

$$\Delta u = \operatorname{trace}\operatorname{Hess}(u) = \operatorname{div}(\nabla u)$$

stands for the *Laplace–Beltrami* operator of (M, g). We stress that we are using the sign convention according to which, in case $M = \mathbb{R}$, $\Delta = +d^2/dx^2$. In other terms, Δ is a negative definite operator in the spectral sense. Note also that when $\Psi \equiv const$ then $\Delta_\Psi = \Delta$.

Very often, one sets

$$\operatorname{div}_\Psi X = e^\Psi \operatorname{div}(e^{-\Psi}X)$$

so that the Ψ-Laplacian takes the suggestive form

$$\Delta_\Psi u = \operatorname{div}_\Psi(\nabla u).$$

Clearly, we have the validity of the Ψ-divergence theorem on M_Ψ: given a compact domain Ω with smooth boundary and a vector field X, it holds

$$\int_\Omega \operatorname{div}_\Psi X \ \mathrm{dv}_\Psi = \int_{\partial\Omega} g(X, \vec{\nu}) \ \mathrm{da}_\Psi$$

where $\vec{\nu}$ is the exterior unit normal to $\partial\Omega$, $\mathrm{da}_\Psi = e^{-\Psi}\mathrm{da}$ and da is the $(m-1)$-dimensional Hausdorff measure of $\partial\Omega$. As a simple consequence, the operator Δ_Ψ is symmetric on $L^2(M, \mathrm{dv}_\Psi)$.

The geometry of the weighted manifold M_Ψ can be controlled by imposing bounds on its family of Bakry–Emery Ricci tensors. In view of our purposes we limit ourselves to introduce the ∞-Ricci Tensor

$$\mathrm{Ric}_\Psi = \mathrm{Ric} + \mathrm{Hess}(\Psi).$$

Example 1.2. The Gaussian space

$$\mathbb{G}^m = \left(\mathbb{R}^m, g^{\mathbb{R}^m}, e^{-\frac{|x|^2}{2}} dx\right)$$

is an example of great interest in metric and differential geometry, probability, harmonic and geometric analysis. Its weighted Laplacian $\Delta_\Psi u = \Delta u - \langle \nabla u, x\rangle$ is the *Ornstein–Uhlenbeck operator*. Obviously the Gaussian space is a weighted model manifold

$$\mathbb{G}^m = \mathbb{M}^m(\sigma)_\Psi$$

with warping function $\sigma(r) = r$ and symmetric weight $\Psi(x) = r^2(x)/2$. A direct computation shows that $\mathrm{Ric}_\Psi \equiv 1$.

1.2. Symmetry under stability

We are going to address the following classical

Problem 1. Let Ω a (possibly non-compact) domain in the weighted Riemannian manifold M_Ψ and assume that Ω has smooth boundary components $\partial\Omega = (\partial\Omega)_1 \cup \cdots \cup (\partial\Omega)_n$. Let us given a (smooth enough) solution to the semilinear boundary value problem

$$\begin{cases} \Delta_\Psi u = f(u) & \text{in } \Omega \\ u = \phi_j & \text{on } (\partial\Omega)_j \end{cases} \tag{1.1}$$

for some sufficiently regular nonlinearity $f(t)$. Assume that the domain, the differential operator and the boundary data display a certain (and same) symmetry. To what extent the solution inherits this symmetry?

We stress that our solutions will always be assumed to be sufficiently regular (say, at least(!) C^2 in the interior and C^1 up to the boundary). The case of weakly regular solutions introduces other nontrivial difficulties and requires further assumptions, as one can see from the very recent [16] by Dupaigne and Farina where they address the (regularity and) symmetry problem in the Euclidean space. We note in passing that, for the Euclidean Poisson equation, sharp conditions on the nonlinearity ensuring that the solutions have a $C^{1,1}$ interior regularity have been obtained in [25, Theorem 1.1].

In the Euclidean space $M = \mathbb{R}^n$, the celebrated theorem by B. Gidas, W.M. Ni and L. Nirenberg, [20], later extended to spherical and hyperbolic spaceforms in [27], states that if $\Omega = \mathbb{B}$ is the (unit) ball of $\mathbb{R}^n$, $\Delta_\Psi = \Delta$ is the Euclidean Laplacian and $\phi \equiv 0$, then any solution $u > 0$ to (1.1) is rotationally symmetric (and decreasing). The proof makes use of the *moving plane method* and, therefore, requires a lot of homogeneity of the underlying space in order to perform reflections in every direction. It is well known that the positivity of the solution is vital as shown by the (non-symmetric) eigenfunctions relative to higher Dirichlet eigenvalues of the ball. Moreover, the ball itself cannot, in general, be replaced by a non-convex domain, like an annulus, as the seminal example by H. Brezis and L. Nirenberg shows, [8, p. 453].

However, as we are going to see in a quite general geometric setting and as it is proved by N.D. Alikakos and P.W. Bates, [1], in the Euclidean space, both these assumptions become redundant as soon as it is assumed that the solution u is “stable”.

In fact, in this paper we shall only focus the case to *stable solutions* of (1.1), where the nonlinearity $f(t)$ is at least C^1. Stability is a second order condition defined in terms of the first Dirichlet eigenvalue of the linearized (Schrödinger) operator and it is always satisfied if the solution is energy minimizer. More precisely, assume for simplicity that Ω is compact. Let $F(t)$ be a primitive of the C^1 function $f(t)$ and consider the energy functional

$$\mathcal{E}[v] = \int_\Omega \left(\frac{1}{2}|\nabla v|^2 + F(v)\right) \mathrm{dv}_\Psi$$

on the space

$$\mathscr{S} = \{v \in C^2(\overline{\Omega}) : v|_{(\partial\Omega)_j} = \phi_j\}.$$

For any $\varphi \in C_c^\infty(\Omega)$ and $t \in \mathbb{R}$ it holds $u_t = u + t\varphi \in \mathscr{S}$. If u is a classical solution to the problem, then (integrating by parts) u is a weak solution to the PDE and, therefore

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{E}[u_t] = \int_\Omega g(\nabla u, \nabla\varphi)\,\mathrm{dv}_\Psi + \int_\Omega f(u)\varphi\,\mathrm{dv}_\Psi = 0.$$

Definition 1.3 (*Stable and Strongly Stable Solutions*)**.** Say that the solution u is stable if

$$0 \le \left.\frac{d^2}{dt^2}\right|_{t=0} \mathcal{E}[u_t] = \int_\Omega \left(|\nabla\varphi|^2 + f'(u)\varphi^2\right) \mathrm{dv}_\Psi$$

i.e. the stability operator $\mathcal{L} = \Delta_\Psi - f'(u)$ has nonnegative Dirichlet spectrum:

$$\lambda_1^{-\mathcal{L}}(\Omega) := \inf_{\varphi\in C_c^\infty(\Omega),\, \varphi\not\equiv 0} \frac{\int_\Omega (|\nabla\varphi|^2 + f'(u)\varphi^2)\mathrm{dv}_\Psi}{\int_\Omega \varphi^2 \mathrm{dv}_\Psi} \ge 0.$$

The solution u is said to be *strongly stable* if $\lambda_1^{-\mathcal{L}}(\Omega) > 0$.

We observe that the stability plays a central role also in the setting of noncompact domains. To highlight this fact, we mention the article [5] by H. Berestycki, L. A. Caffarelli and L. Nirenberg, where it is proved that any bounded solution $u > 0$ to $\Delta u = f(u)$ in Euclidean half-space $\mathbb{H}^n_+ = \{x_n \ge 0\}$, with homogeneous Dirichlet boundary conditions depends only on the x_n-variable, provided that $f(\sup u) \le 0$. As a by-product, they obtain that the solution u is increasing in x_n, and hence stable. In subsequent works the viewpoint in some sense is reversed: under suitable conditions on the nonlinearity $f(u)$ (and possibly on the dimension of the space), it is used in a crucial way that x_n-monotonic solutions are stable to prove that they in fact depend only on the x_n-variable. In this direction, we mention the very recent [15] by L. Dupaigne and A. Farina.

As it will be clearer later, if one thinks of the half-space $\mathbb{H}^n_+$ as a (unbounded) domain foliated by hyperplanes parallel to $\{x_n = 0\}$, the monodimensionality of u proved in [5,15] coincides with the notion of *symmetry* we will adopt in the present work.

1.3. Organization of the paper

Clearly, in order to carry out an investigation around Problem 1, we need first to clarify what "symmetric" means for a Riemannian domain and, hence, for a solution to (1.1) on it. We choose to define the symmetry of a domain in terms of the existence of a foliation by special hypersurfaces and the corresponding symmetry of functions as the condition that the function is constant on each leaf of the foliation. Equivalently, the function agrees with its averages on the (compact) leaves of the foliation. This is explained in Sections 2 and 3.

Sometimes, and these are the lucky cases, symmetry properties of generic solutions boil down to uniqueness issues for the relevant class of PDEs. In Section 4 we review (slightly extended versions of) both the classical maximum principle for Schrödinger operators and the uniqueness property of stable solutions. As a consequence of the maximum principle and the fact that the average operator commutes with the differential operator, we observe how, in this general geometric framework, symmetry over compact domains occurs for affine $f(t)$.

In Section 5 we point out that symmetry of stable solutions appears as soon as the domain supports enough Killing vector fields tangential to the leaves of its foliation. This translates the fact that the domain is *homogeneous* in the precise sense of *co-homogeneity one actions of Lie subgroups of isometries*. This simple result encloses in a single view a lot of concrete cases that, at first glance, may appear of different nature, such as balls in model manifolds, annuli in warped products of a real interval with a homoegeneous manifold, tubes around Clifford tori in the n-sphere and many others.

In Section 6, in order to test how much the existence of infinitesimal symmetries influence the problem, we consider the case of a possibly non-compact warped product that, in general, supports no Killing fields at all. Using potential theoretic tools, we are still able to prove a quite general symmetry result for (strongly stable) solutions provided the nonlinearity is concave and somewhat compatible with the geometry. The general result applies e.g. to slabs (the region enclosed between two parallel hyperplanes) in the Gaussian space.

2. Symmetric domains

As we have already mentioned in the Introduction, the first aspect we need to clarify is what does "symmetric" mean in the setting of Riemannian manifolds. At first glance, "radial symmetry" could appear the most natural notion. However, the recent and very active area of research on the geometry of overdetermined problems of various nature, strongly suggests that the appropriate notion is that of an *isoparametric domain*; see especially the seminal paper [39] by V. Shklover, the papers [37,38] by A. Savo and the very recent [35] by L. Provenzano and A. Savo.

Isoparametric hypersurfaces in space-forms have a long history that goes back to the first half of the nineteen century and the modern viewpoint on this theory can be attributed to E. Cartan, [9]. For a gentle introduction on the subject, with plenty of examples and special emphasis on the classification problem in different ambient spaces, we refer the reader to the lecture notes [12] by M. Dominguez-Vazquez and the references therein.

2.1. Isoparametric domains and tubes

We recall that a *singular Riemannian foliation* of the complete Riemannian manifold (M, g) is a foliation $M = \cup_t \Sigma_t$ by smooth, embedded submanifolds such that:

- every geodesic which is perpendicular to one leaf remains perpendicular to every leaf it intersects;
- there exists an integrable distribution $\mathscr{D}$ pointwise tangent to the leaves of the foliation and which is locally generated (actually globally according to [13]) by a finite family of smooth vector fields.

Definition 2.1 (*Isoparametric Domain*)**.** An isoparametric domain $\bar{\Omega} \subseteq M$ is a domain of M endowed with a singular Riemannian foliation $\bar{\Omega} = \cup_t \Sigma_t$ whose regular leaves (i.e. those of maximal dimension) are connected parallel complete hypersurfaces (without boundary) with constant mean curvature and with at most two singular (i.e. of codimension greater than one) leaves.

Here, as usual, we call Σ_1, Σ_2 parallel if, for every $x_1 \in \Sigma_1$ and $x_2 \in \Sigma_2$,

$$\operatorname{dist}(x_1, \Sigma_2) = \operatorname{dist}(\Sigma_1, x_2),$$

in other words, if the distance function to Σ_2 is constant along Σ_1.

Isoparametric domains arise from *isoparametric functions*, i.e. smooth functions f whose norm of the gradient and whose Laplacian can be expressed in terms of the function itself. More precisely, there exist a smooth function α and a continuous function β on the range of f such that

$$|\nabla f|^2 = \alpha(f) \quad \text{and} \quad \Delta f = \beta(f).$$

These two properties imply, respectively, that level sets foliating the domain are parallel and with constant mean curvature. In particular, an isoparametric function f for an isoparametric domain $\overline{\Omega}$ can be provided either by the smooth, signed distance function from a regular leaf or by the smooth absolute distance function from a singular leaf (isoparametric tube). In both cases we call such a leaf the *soul of the domain.*

If $\overline{\Omega}$ is an isoparametric domain arising from a global isoparametric function $f : M \to \mathbb{R}$, the *focal varieties of f* are defined as the sets

$$V^- := \{x \in M \ : \ f(x) = \min_M f\} \quad \text{and} \quad V^+ := \{x \in M \ : \ f(x) = \max_M f\}.$$

From a classical result by Q. M. Wang, [41], later completed by R. Miyaoka in [29], we have that a singular leaf (if any) of the isoparametric domain described by f is a focal variety and it is a minimal submanifold.

2.2. Homogeneous domains

The isoparametric condition provides a very handy model of symmetric domains. However, as we shall see, sometimes the needed notion of symmetry is much stronger.

Definition 2.2. A homogeneous domain $\bar{\Omega} \subseteq M$ of a complete Riemannian manifold (M, g) is an isoparametric domain whose regular leaves are orbits of the action of a closed subgroup $G \subset \mathrm{Iso}_0(M)$, the identity component of the group $\mathrm{Iso}(M)$ of all isometries of M.

Thus, a domain is homogeneous if the regular leaves of the singular Riemannian foliation are homogeneous hypersurfaces with respect to the same group G of isometries of the ambient space.

A straightforward consequence of the fact that G acts transitively on each leaf is that the principal curvatures of the leaves are constant. Moreover, note explicitly that if $\dim M = m$, since each regular leaf is homogeneous and can be written as $\Sigma_t = G/H_p$ for $H_p \subset G$ isotropy subgroup of G at $p \in \Sigma_t$, then $\dim G = k \geq m - 1$.

From the perspective of the present paper, the most important property enjoyed by homogenenous domains is that the leaves display a lot of (and in fact same) isometric symmetries. These symmetries are encoded in the notion of a Killing vector field that we are going to recall.

A smooth vector field X on M is said to be *Killing* if, for every vector fields Y, Z,

$$(L_X g)(Y, Z) = g(\nabla_Y X, Z) + g(\nabla_Z X, Y) = 0.$$

Equivalently, the flow $\phi(x, t)$ of X is a local 1-parameter group of isometries:

$$\phi_t^* g = g.$$

Note that, by the very definition, any Killing vector field X satisfies

$$\operatorname{div} X = 0.$$

Note also that if X is a Killing vector field on (M, g), which is pointwise tangential to an embedded submanifold P, then $X|_P$ is a Killing vector field of P.

Now, let $\bar{\Omega}$ be a homogeneous domain with group G and whose regular leaves are homogeneous hypersurfaces Σ_t and recall has at most two singular leaves P_1 and P_2. Consider the Riemannian submersion given by the projection

$$\begin{array}{rcl}\pi : \bar{\Omega} \setminus (P_1 \cup P_2) & \longrightarrow & \mathbb{R} \\ \Sigma_t & \longmapsto & \Sigma_t / G = point\end{array}$$

and note that

$$\mathcal{V}_p = T_p \Sigma_t \quad \forall p \in \Sigma_t \tag{2.1}$$

where $\mathcal{V}_p = \mathrm{Ker}(d_p \pi)$ is the vertical space at p. For any $p \in \Sigma_t$ the space $\mathcal{V}_p$ is spanned by the set $\mathfrak{K}(\bar{\Omega})$ of all Killing vector fields of $\bar{\Omega}$ evaluated at p. These, in turn, identify with the elements of the Lie algebra $\mathfrak{g}$ of G via the map

$$\begin{array}{rl}\mathfrak{g} & \longrightarrow \mathfrak{K}(\bar{\Omega}) \\ \mathfrak{X} & \longmapsto X\end{array}$$

where

$$X : p \mapsto \left.\frac{d}{dt}\right|_{t=0} \Big(\exp(t\mathfrak{X})(p) \Big).$$

Thus, letting $m-1 \le k = \dim G \le m(m-1)/2$, we can select a distribution of linearly independent Killing vector fields

$$\mathscr{D} = \{X_1, \dots, X_k\} \subseteq \mathfrak{K}(\bar{\Omega})$$

whose integral manifolds are the hypersurfaces Σ_t. For further information on the topic we suggest [31].

2.3. Examples

It is time to present a brief list of concrete examples of isoparametric and homogenenous domains.

Example 2.3 (*Balls in Model Manifolds*)**.** Let $\mathbb{M}^n_\sigma = [0, R) \times_\sigma \mathbb{S}^{n-1}$ be a model manifold, where $R \in (0, +\infty]$. Then, geodesic balls centred at the pole are homogeneous domains with the homogeneous foliation provided by the geodesic spheres concentric to the pole. The corresponding group is $G = \mathbf{SO}(n)$.

Example 2.4 (*Annuli in Warped Products*)**.** Take a warped product manifold $M = I \times_\sigma N$ where (N, g^N) is an $(m-1)$-dimensional Riemannian manifold without boundary, $I \subset \mathbb{R}$ is a real open interval and $\sigma(t) > 0$ is a smooth function on I. Explicitly, the Riemannian metric g of M is given by

$$g = dt \otimes dt + \sigma^2(t) g^N.$$

Take a domain either of the form $\bar{\Omega} = [a, b] \times N$ or $\bar{\Omega} = [a, +\infty) \times N$. Since the (translated) t-coordinate $r(t, \xi) = t - a$ is precisely the (absolute) distance function from the hypersurface $\Sigma_a = \{a\} \times N \hookrightarrow M$ we have that

$$|\nabla r| = 1$$

and the level sets

$$\Sigma_t = r^{-1}(t-a) = \{t\} \times N,$$

with $a \le t \le b$, are parallel hypersurfaces. Moreover, the second fundamental form and the mean curvature of Σ_t with respect to Gauss map $\vec{\nu} = \nabla r$ are given, respectively, by

$$\mathrm{II}_{\Sigma_t} = \mathrm{Hess}(r)|_{\Sigma_t} = \sigma'(t)\sigma(t) g^N$$

and

$$H_{\Sigma_t} = \Delta r = (m-1)\frac{\sigma'}{\sigma}(t) = (m-1)\frac{\sigma'}{\sigma}(r+a).$$

It follows that r is an isoparametric function turning $\bar{\Omega}$ into an isoparametric domain. We note explicitly that each leaf Σ_t is totally umbilical (namely, the traceless second fundamental form vanishes identically).

In case (N, g^N) is a compact Lie group endowed with a left-invariant Riemannian metric, then the domain $\bar{\Omega} = [a, b] \times N$ inside $I \times_\sigma N$ is homogeneous with group N. Actually the same holds if $N = G/H$ is a homogeneous manifold.

Example 2.5 (*Annuli in Harmonic Spaces*)**.** Another interesting class of examples is given by the *harmonic manifolds* introduced by A. Lichnerowicz in [28]. This class includes symmetric spaces and Damek–Ricci spaces. We are grateful to the referee for pointing this out to us.

A Riemannian manifold (M, g) is *locally harmonic* if, for every $p \in M$ there exist a radius $\epsilon(p) > 0$ and a function $\omega_p : [0, \epsilon(p)) \to \mathbb{R}$ such that, in exponential polar coordinates (r, ξ) around p, the volume density takes the form $A(r, \xi) = \omega_p(r)$, for every $r \in (0, \epsilon(p))$. Actually, a-posteriori, the function ω_p is independent of the reference point and defined on the maximal interval $[0, \max_{p\in M} \epsilon(p)) \subseteq [0, +\infty)$. The locally harmonic manifold (M, g) is called *globally harmonic* if it is geodesically complete and $\epsilon \equiv +\infty$. Let us assume that (M, g) is globally harmonic. From the rotational symmetry of the volume density one immediately deduces that:

(a) The conjugate locus of a point $p \in M$ is nonempty only if M is compact.
(b) Since, within the cut-locus, the Laplacian of the distance function $r(x) = \mathrm{dist}(x, p)$ satisfies $\Delta r = \partial_r \log A$, then $\Delta r = (\omega'/\omega)(r)$ is rotationally symmetric.

It follows from the Hadamard–Cartan theorem that a complete, non-compact, simply connected, globally harmonic manifold is diffeomorphic to $\mathbb{R}^n$ and the smooth distance function r from a fixed origin p is a (global) isoparametric function. In particular any annulus inside M is an isoparametric domain. Needless to say, small enough annuli in locally harmonic spaces enjoy the same property.

Simply connected, complete, non-compact, globally harmonic spaces are also *asymptotically harmonic*. This means that the isoparametric property of the distance function is inherited by the Busemann function with respect to any given geodesic line. More precisely, the Busemann function has unit gradient and constant Laplacian. In particular, any *horo-annulus* is an isoparametric domain whose leaves are complete, non-compact, hypersurfaces with the same constant mean curvature. For more information concerning harmonic and asymptotically harmonic manifolds we refer the reader to [6,26,36] and references therein.

Example 2.6 (*Euclidean Homogenenous Domains with Non-Compact Leaves*)**.** Taking the Euclidean space $\mathbb{R}^n$ we easily obtain two different types of isoparametric domains with non-compact leaves:

- *Cylindrical annuli:* consider the tube whose equidistants are the right cylinders $\{\Sigma_t\}_{t\in(a,b)}$ with axis given by a straight line a through the origin $o \in \mathbb{R}^n$. Thanks to the isotropy of the Euclidean space, we can suppose that $a = \mathbb{R}\vec{e}_n = \mathbb{R}(0, \dots, 0, 1)$. Then, each leaf takes the form
$$\Sigma_t = \{(x', x_n) \in \mathbb{R}^n \mid x' \in \mathbb{S}^{n-2}_t,\ x_n \in \mathbb{R}\}$$
for $\mathbb{S}^{n-2}_t$ the $(n-2)$-sphere of radius t, centred at the origin.
In this way we obtain an isoparametric foliation of the domain $\bar{\Omega} = \cup_{t\in[a,b]} \Sigma_t$ with leaves that have constant mean curvature equal to $H(\Sigma_t) = \frac{n-2}{t}$. A possible isoparametric function is
$$f(x_1, \dots, x_n) = \sqrt{x_1^2 + \cdots + x_{n-1}^2} = |x'|$$
- *Slabs:* consider the tube whose equidistants are the hyperplanes $\{\Sigma_t\}_{t\in(a,b)}$ parallel to
$$\Sigma_0 = \{x \in \mathbb{R}^n \mid x \cdot \vec{\nu}_0 = 0\}$$
for a fixed vector $\vec{\nu}_0 \in \mathbb{S}^{n-1}$.

As before, we can suppose $\vec{\nu}_0 = \vec{e}_n$. Then, the leaves are

$$\Sigma_t = \Sigma_0 + t\vec{\nu}_0 = \{(x', t) \mid x' \in \mathbb{R}^{n-1} \equiv \Sigma_0\}$$

These hyperplanes give the domain $\bar{\Omega} = \cup_{t\in[a,b]}\Sigma_t$ an isoparametric structure, whose leaves have vanishing mean curvature. A possible isoparametric function is

$$f(x_1, \dots, x_n) = x_n$$

In both cases, the domain $\bar{\Omega}$ is homogeneous with groups, respectively, $G = \mathbf{SO}(n)$ and $G = \mathbb{R}^{n-1}$.

Example 2.7 (*Generalized Hopf-Fibration*)**.** Let $M = \mathbb{S}^3$ and $F(x) = x_1^2 + x_2^2 - x_3^2 - x_4^2$ be the Cartan–Munzner polynomial that gives rise to Clifford tori $T(r) = \mathbb{S}^1(r) \times \mathbb{S}^1(\sqrt{1-r^2})$ with $0 < r < 1$. Then $F^{-1}([t_1, t_2])$ is a homogeneous domain by the action of $G = \mathbf{SO}(2) \times \mathbf{SO}(2)$. Similar examples can be constructed in the higher dimensional spheres $\mathbb{S}^n$, using the isoparametric functions $F(x) = l(x_1^2 + \cdots + x_k^2) - k(x_{k+1}^2 + \cdots + x_n^2)$ for $k+l = n+1$. Note that the leaves of these isoparametric domains are not totally umbilical (and, in particular, $F^{-1}([t_1, t_2])$ does not have a warped product structure of the form $I \times_\sigma N$).

Example 2.8 (*Cartan Homogenenous Domains*)**.** Tubes around tori are just one of the possible families of examples of homogenenous domains in the sphere $\mathbb{S}^m$. For different choices of the Cartan–Munzner polynomial, corresponding to different choices of the Lie subgroup $G \subset \mathbf{SO}(m+1)$, we refer to [39]. An account of more examples, in different ambient spaces, can be found in [12].

2.4. Weighted symmetric domains

When formulated in the context of a weighted Riemannian manifold M_Ψ, the notion of isoparametric domain can be naturally generalized as follows.

Recall that, given a smooth hypersurface Σ oriented by $\vec{\nu}$ inside the weighted manifold M_Ψ, its *weighted mean curvature* (in the sense of Gromov) $\vec{H}_\Psi = H_\Psi\vec{\nu}$ is given by

$$H_\Psi = H - g(\nabla\Psi, \vec{\nu})$$

where $\vec{H} = H\vec{\nu}$ is the usual mean curvature vector field, i.e., the (unnormalized) trace of the second fundamental form.

Definition 2.9 (Ψ*-Isoparametric Domain*)**.** Let M_Ψ be a weighted Riemannian manifold. A Ψ-isoparametric domain $\bar{\Omega} \subseteq M_\Psi$ is a domain of M_Ψ endowed with a singular Riemannian foliation $\bar{\Omega} = \cup_t\Sigma_t$ whose regular leaves (i.e. those of maximal dimension) are connected parallel complete hypersurfaces (without boundary) with constant mean curvature and with at most two singular (i.e. of codimension greater than one) leaves.

Similarly to the unweighted case, Ψ-isoparametric domains arise as domains foliated by the level sets of Ψ*-isoparametric functions*, that are smooth functions f whose norm of the gradient and whose weighted Laplacian can be expressed in terms of f itself

$$|\nabla f|^2 = \alpha(f) \quad \text{and} \quad \Delta_\Psi f = \beta(f),$$

for α smooth and β continuous in the range of f.

The notion of a homogeneous domain can be extended to the weighted setting using a similar spirit. In this case, however, it is not a-priori clear how to incorporate the weighted structure into the homogeneity condition. We choose to adopt the following

Definition 2.10 (*Ψ-Homogenenous Domain*)**.** Let M_Ψ be a weighted Riemannian manifold. Say that $\bar{\Omega}$ is a Ψ-homogeneous domain if it is a Ψ-isoparametric domain and a homogeneous domain simultaneously.

Equivalently, $\bar{\Omega}$ is Ψ-homogeneous if it is a homogeneous domain satisfying the "weight compatibility condition"

$$g(\nabla\Psi, \vec{\nu}) = const \text{ on each leaf } \Sigma_t \tag{2.2}$$

The equivalence of these two conditions come from the very definition of weighted mean curvature and the fact that a homogenenous domain has constant (ordinary) mean curvature.

Remark 2.11 (*From Homogenenous to Ψ-Homogenenous*)**.** It is worth noting that, if P is the soul of $\bar{\Omega}$ and $d(x) = \operatorname{dist}(x, P)$, the natural choice $\Psi(x) = \hat{\Psi}(d(x))$ turns any(!) homogeneous domain into a Ψ-homogeneous domain. However, as we shall see, there are interesting Ψ-homogeneous domains that do not fall in this category. See Example 2.13.

Example 2.12. By definition of Ψ-symmetry and according to Remark 2.11, Examples 2.3 and 2.4 trivially generalize, respectively, to the case of weighted model manifolds and annuli in weighted warped product manifolds, up to assuming that the weight has the form $\Psi(x) = \hat{\Psi}(d(x, o))$ and $\Psi(x) = \hat{\Psi}(\operatorname{dist}(x, \Sigma_a))$.

Example 2.13 (*Gaussian Isoparametric Domains with Non-Compact Leaves*)**.** Take the Gaussian space $\mathbb{G}^n$. The weighted mean curvature of a $\vec{\nu}$-oriented smooth hypersurface $\Sigma \subset \mathbb{G}^n$ is

$$H_\Psi = H - g(-x, \vec{\nu}) = H + g(x, \vec{\nu})$$

Using this fact, we can easily generalize the two examples obtained in (2.6):

- *Weighted cylindrical annuli:* As done in the non-weighted case, we consider

$$\Sigma_t = \{(x', x_n) \in \mathbb{R}^n \mid x' \in \mathbb{S}^{n-2}_t,\ x_n \in \mathbb{R}\}$$

  for $\mathbb{S}^{n-2}_t$ the $(n-2)$-sphere of radius t, centred at the origin.
  It follows that the normal vector field to the leaf Σ_t is

$$\vec{\nu}_t(x) = \vec{\nu}_t\Big((x', x_n)\Big) = \frac{x'}{|x'|} \quad \forall x \in \Sigma_t$$

  where we are identifying x' with $(x', 0)$. So

$$g(x, \vec{\nu}_t(x)) = \frac{|x'|^2}{|x'|} = |x'| = t$$

  is constant on each Σ_t. Using this equality and the fact that the mean curvature of Σ_t is $H(\Sigma_t) = \frac{n-2}{t}$, we obtain that

$$H_\Psi(\Sigma_t) = \frac{n-2}{t} + t$$

  is constant on each Σ_t.
- *Weighted slabs:* As before, let $\vec{\nu}_0 = \vec{e}_n$ and consider

$$\Sigma_t = \Sigma_0 + t\vec{\nu}_0 = \{(x', t) \mid x' \in \mathbb{R}^{n-1} \equiv \Sigma_0\}$$

  with normal vector field to Σ_t given by

$$\vec{\nu}_t(x) = \vec{\nu}_t\Big((x', x_n)\Big) = \frac{(0, x_n)}{|x_n|} = \frac{t}{|t|}\vec{e}_n$$

So

$$g(x, \vec{\nu}_t(x)) = \frac{|x_n|^2}{|x_n|} = |x_n| = t$$

and thus

$$H_\Psi(\Sigma_t) = H(\Sigma_t) + t = t$$

is constant on each Σ_t.

In particular, both weighted cylindrical annuli and weighted slabs are Ψ-homogeneous domains whose weight Ψ is not symmetric.

Example 2.14 (*Gaussian-Like Weighted Spaces*)**.** Consider the weighted space $\mathbb{R}^n_\Psi = \left(\mathbb{R}^n, g^{\mathbb{R}^n}, e^{-\Psi}dx\right)$ for a symmetric weight $\Psi(x) = A|x|^2 + B$ and $A, B \in \mathbb{R}$, $A \neq 0$. Then, the previous examples with non-compact leaves (parallel hyperplanes and coaxial cylinders) and the spherical tube continue to be Ψ-homogeneous domains.

Indeed, the gradient of the weight is

$$\nabla\Psi(x) = 2Ax$$

and following the previous calculations, we obtain that the weighted mean curvature of each equidistant of the above mentioned domains is constant.

3. Symmetric functions

Laid the foundations of the theory of isoparametric domains, we must specify what we mean by *symmetry* when we talk about functions defined on them. Accordingly, one introduces the *average operator*

$$\mathcal{A}_\Psi(u)(x) = \frac{1}{\operatorname{area}_\Psi \Sigma_{t(x)}} \int_{\Sigma_{t(x)}} u(y) \mathrm{da}_\Psi \tag{3.1}$$

and put the following

Definition 3.1. Let $\bar{\Omega}$ be a compact weighted isoparametric domain inside the weighted manifold M_Ψ. Say that the function u on $\bar{\Omega}$ is symmetric if

$$u(x) = \mathcal{A}_\Psi(u)(x).$$

Remark 3.2 (*Symmetry Condition Using Distance Function*)**.** If $\bar{\Omega}$ is a compact Ψ-isoparametric domain with soul P and $d(x) = \operatorname{dist}(x, P)$, then the following are equivalent:

(a) $u = \mathcal{A}_\Psi(u)$.
(b) $u(x) = \hat{u}(d(x))$.

The advantage of characterization (b) over (a) is that it makes sense even if P is non-compact and u is not necessarily integrable on the leaves of the foliation.

One of the main features of weighted isoparametric domains is that the corresponding average operator, that preserves the smoothness of functions, commutes with the weighted Laplacian. This property is formalized in the following Lemma that extends [38, Proposition 13] to the weighted setting.

Lemma 3.3 (*Savo*)**.** *Let Ω be a smooth, compact, weighted isoparametric domain with soul P inside the weighted manifold M_Ψ. Let $\mathcal{A}_\Psi$ be the average operator defined on $L^1(\Omega, \mathrm{dv}_\Psi)$ by (3.1). Then the following hold:*

(a) *If* $u \in C^{k+2}(\Omega)$, *then* $\mathcal{A}_\Psi(u) \in C^k(\Omega)$.
(b) *Given* $u \in C^4(\Omega)$, $\mathcal{A}_\Psi(\Delta_\Psi u) = \Delta_\Psi \mathcal{A}_\Psi(u)$.

Notation 3.4. For the sake of brevity, we shall write condition (b) as the commutation rule

$$[\mathcal{A}_\Psi, \Delta_\Psi] = 0.$$

A similar convention will be adopted during the paper for other operators.

The proof is a minor variation of the original one in the Riemannian setting.

3.1. Local vs. global symmetry

The notion of symmetry defined in the previous subsection can be formulated equivalently in terms of a first order condition.

Let $\bar{\Omega}$ be an isoparametric domain with compact soul P inside the weighted Riemannian manifold M_Ψ. We set, as usual, $d(x) = \mathrm{dist}(x, P)$ so that $\bar{\Omega} = \cup_{r\in[r_1,r_2]}\Sigma_r$ is foliated by the smooth, embedded, parallel hypersurface $\Sigma_r = \{x \in M : d(x) = r\}$ in the same isotopy class.

Definition 3.5 (*Local Symmetry*)**.** Say that $u \in C^1(\bar{\Omega})$ is symmetric at $x_0 \in \bar{\Omega}$ if, for any smooth vector field X on $\bar{\Omega}$ satisfying

$$(i)\ X|_{x_0} \neq 0, \quad (ii)\ g(X|_{x_0}, \nabla d(x_0)) = 0,$$

it holds

$$X(u)(x_0) = g(X|_{x_0}, \nabla u(x_0)) = 0.$$

In case u is symmetric at every point $x \in \bar{\Omega}$ we say that u is locally symmetric on $\bar{\Omega}$.

Remark 3.6. Clearly, the local symmetry at x_0 can be formulated in either of the following equivalent ways.

(i) Let $(\nabla u(x_0))^\top$ denote the orthogonal projection of $\nabla u(x_0)$ on the tangent space $T_{x_0}\Sigma_{d(x_0)}$. Then

$$(\nabla u(x_0))^\top = 0.$$

(ii) The gradient of u at x_0 is parallel to $\nabla d(x_0)$:

$$\nabla u(x_0) \in \mathrm{span}\nabla d(x_0) = (T_x\Sigma_{d(x_0)})^\perp.$$

Lemma 3.7. *Keeping the above notation, the function u is locally symmetric on $\bar{\Omega}$ if and only if u is symmetric in the global sense, i.e., $u(x) = \hat{u}(d(x))$.*

Proof. Assume that u is locally symmetric and suppose by contradiction that there exist $r \geq 0$ and $x, y \in \Sigma_r$ such that $u(x) > u(y)$. Each leaf Σ_r is connected, therefore we can consider a smooth immersed[1] curve $\gamma : [0,1] \to \Sigma_r$ joining $\gamma(0) = x$ to $\gamma(1) = y$. Since $u \circ \gamma$ is a C^1 function satisfying $u\circ\gamma(0) > u\circ\gamma(1)$, there exists $\bar{t} \in [0,1]$ such that

$$g((\nabla u)(\gamma(\bar{t})), \dot{\gamma}(\bar{t})) = \frac{d}{dt}(u\circ\gamma)(\bar{t}) < 0.$$

This contradicts the local symmetry because $0 \neq \dot{\gamma}(\bar{t}) \in T_{\gamma(\bar{t})}\Sigma_r$. $\square$

[1] A connected smooth manifold N can be always endow with a complete Riemannian metric h. Therefore, any two given points $x, y \in N$ are connected by a minimizing h-geodesic, which is a smooth immersed curve of N.

4. Maximum principles, uniqueness and symmetry

Maximum principles for Schrödinger operators and uniqueness issues for solutions to semilinear PDEs permeate the whole theory of symmetry problems and the whole paper. Therefore, we devote this preliminary section to review briefly these topics both in the compact and in the non-compact settings.

4.1. Compact maximum principle

In their book [34, Section 5, Theorem 10], Protter–Weinberger introduced a form of the Maximum Principle valid for elliptic operators in the presence of zeroth order terms. Their celebrated result states as follows.

Proposition 4.1 (*Compact Maximum Principle*)**.** *Let $M_\Psi = (M, g, \mathrm{dv}_\Psi)$ be a compact weighted Riemannian manifold with boundary $\partial M \neq \emptyset$ and suppose we are given on M_Ψ the Schrödinger operator $\mathcal{L} = \Delta_\Psi - q$, where $q \in C^0(M)$. Assume that there exists a function $\varphi \in C^0(M) \cap C^2(\mathrm{int}(M))$ solution to the problem*

$$\begin{cases} \mathcal{L}\varphi \leq 0 & \mathrm{int}M \\ \varphi > 0 & M \end{cases} \tag{4.1}$$

Then, any solution $u \in C^0(M) \cap W^{1,2}_{\mathrm{loc}}(\mathrm{int}M)$ of

$$\begin{cases} \mathcal{L}u \geq 0 & \mathrm{int}M \\ u \leq 0 & \partial M \end{cases}$$

satisfies $u \leq 0$ in M.

Proof. Consider the positive part of the function u

$$u_+ = \max\{u, 0\}$$

Then u_+ satisfies

$$\begin{cases} \mathcal{L}u_+ \geq 0 & \mathrm{int}M \\ u_+ = 0 & \partial M; \end{cases}$$

see e.g. [32, Lemma 6.1] for a proof that works in the nonlinear setting. Defining the function $0 \leq \omega = \frac{u_+}{\varphi}$ on the weighted manifold M_Φ, where $\Phi = \log(\varphi^{-2}) + \Psi$, we get

$$\begin{cases} \Delta_\Phi \omega \geq 0 & \mathrm{int}M \\ \omega = 0 & \partial M, \end{cases}$$

By the usual maximum principle we obtain $\omega \leq 0$ in M that implies $\omega = 0$ in M, i.e. $u_+ = 0$ in M, as claimed. □

Observe that for a compact Riemannian manifold with boundary M there is no loss of generality in assuming that M is a smooth bounded domain inside a closed Riemannian manifold (N, g^N); [33, Theorem A]. Thus, the existence of a function φ satisfying (4.1) is guaranteed under the assumption that $\lambda_1^{-\mathcal{L}}(M) > 0$. Indeed, in this case, once q and Ψ are extended with the same regularity to N, we can slightly enlarge M to some smooth domain $\Omega \Subset N$ with $\lambda_1^{-\mathcal{L}}(\Omega) > 0$ and take as φ the restriction to M of the first eigenfunction on Ω. The existence of such a domain Ω could be seen as a trivial consequence of a deep continuity property of the Dirichlet eigenvalues with respect to the (Gromov-)Hausdorff convergence. See e.g. the paper [10] by Chenais for the case of Hausdorff converging uniformly Lipschitz domains of the Euclidean space. However, one can obtain the existence of Ω using much more elementary considerations. We are going to provide the arguments for the sake of completeness.

Lemma 4.2. *Let $N_\Psi = (N, g^N, \mathrm{dv}_\Psi)$ be a complete weighted Riemannian manifold (without boundary) and $\mathcal{L} = \Delta_\Psi - q$ with $q \in C^0(N)$. Let $D \Subset N$ be a smooth domain such that $\lambda_1^{-\mathcal{L}}(D) > 0$. Then there exists a smooth domain $D \Subset \Omega \Subset N$ satisfying $\lambda_1^{-\mathcal{L}}(\Omega) > 0$.*

Proof. Consider a sequence of nested smooth domains $N \Supset \Omega_1 \Supset \Omega_2 \Supset \dots \Omega_n \Supset \Omega_{n+1} \dots \Supset D$ satisfying $\bigcap_n \Omega_n = \bar{D}$ and let Q_n and Q be the quadratic forms associated to the Rayleigh quotient on Ω_n and on D respectively

$$Q_n(u) := \int_{\Omega_n} \left(|\nabla u|^2 + qu^2\right) \mathrm{dv}_\Psi, \qquad u \in W_0^{1,2}(\Omega_n, \mathrm{dv}_\Psi)$$
$$Q(u) := \int_{D} \left(|\nabla u|^2 + qu^2\right) \mathrm{dv}_\Psi, \qquad u \in W_0^{1,2}(D, \mathrm{dv}_\Psi).$$

By the domain monotonicity of the first Dirichlet eigenvalue we have

$$\lambda_1^{-\mathcal{L}}(D) \geq \lambda_1^{-\mathcal{L}}(\Omega_n), \; \forall n \in \mathbb{N}.$$

Therefore, if $\{u_n\}_n \subset C^\infty(\bar{\Omega}_n)$ is the sequence of first Dirichlet eigenfunctions corresponding to $\lambda_1^{-\mathcal{L}}(\Omega_n)$, normalized so to have

$$\begin{cases} u_n \geq 0 & \text{in } \Omega_n \\ \|u_n\|_{L^2(\Omega_n, \mathrm{dv}_\Psi)} = 1, \end{cases}$$

then, by extending each u_n to 0 in $\Omega_1 \setminus \Omega_n$ so that $u_n \in W_0^{1,2}(\Omega_1)$, we get

$$\begin{cases} \|\nabla u_n\|^2_{L^2(\Omega_1, \mathrm{dv}_\Psi)} = \|\nabla u_n\|^2_{L^2(\Omega_n, \mathrm{dv}_\Psi)} \\ \|u_n\|_{L^2(\Omega_1, \mathrm{dv}_\Psi)} = \|u_n\|_{L^2(\Omega_n, \mathrm{dv}_\Psi)} = 1 \\ Q_1(u_n) = Q_n(u_n) = \lambda^{-\mathcal{L}}(\Omega_n) \leq \lambda^{-\mathcal{L}}(D). \end{cases}$$

In particular

$$\begin{aligned} \|\nabla u_n\|^2_{L^2(\Omega_1, \mathrm{dv}_\Psi)} &= \lambda_1^{-\mathcal{L}}(\Omega_n) - \int_{\Omega_n} q u_n^2 \, \mathrm{dv}_\Psi \\ &\leq \lambda_1^{-\mathcal{L}}(D) + \|q\|_{L^\infty(\Omega_1, \mathrm{dv}_\Psi)}. \end{aligned}$$

We have deduced that $\{u_n\}_n$ is a bounded sequence in $W_0^{1,2}(\Omega_1, \mathrm{dv}_\Psi)$. Then there exists a subsequence $\{u_{n_k}\}_k$ converging weakly in $W_0^{1,2}(\Omega_1, \mathrm{dv}_\Psi)$ and strongly in $L^2(\Omega_1, \mathrm{dv}_\Psi)$ to some function $v \in W_0^{1,2}(\Omega_1, \mathrm{dv}_\Psi)$. Clearly,

$$\|v\|_{L^2(\Omega_1, \mathrm{dv}_\Psi)} = 1.$$

Moreover, since we can always assume that $u_{n_k} \xrightarrow{a.e.} v$ and, by assumption, $\bigcap_n \Omega_n = \bar{D}$, we have $v = 0$ a.e. on $\Omega_1 \setminus \bar{D}$. But, in fact,

$$v = 0 \text{ a.e. on } \Omega_1 \setminus D$$

because the smooth boundary ∂D of D has measure zero. It follows from [4, Proposition 2.11] that

$$v \in W_0^{1,2}(D)$$

and thus

$$\lambda_1^{-\mathcal{L}}(D) \leq Q(v) = Q_1(v).$$

Now, using the lower semicontinuity of the quadratic form Q_1 with respect to the weak $W^{1,2}$-topology, we obtain

$$Q_1(v) \geq \lambda_1^{-\mathcal{L}}(D)$$

$$
\begin{aligned}
&\geq \limsup_k \lambda_1^{-\mathcal{L}}(\Omega_{n_k}) \\
&\geq \liminf_k \lambda_1^{-\mathcal{L}}(\Omega_{n_k}) \\
&= \liminf_k Q_1(u_{n_k}) \\
&\geq Q_1(v),
\end{aligned}
$$

showing that

$$\lim_k \lambda_1^{-\mathcal{L}}(\Omega_{n_k}) = \lambda_1^{-\mathcal{L}}(D) > 0.$$

The desired conclusion now follows by choosing $\Omega = \Omega_{k_0}$ with k_0 large enough. □

As a consequence of Proposition 4.1 and Lemma 4.2, on noting also that if $\lambda_1^{-\mathcal{L}}(\mathrm{int}M) = 0$ then the corresponding first Dirichlet eigenfunction $u \geq 0$ violates the maximum principle, we have the validity of the following well known characterization.

Corollary 4.3. *Let $M_\Psi = (M, g, \mathrm{dv}_\Psi)$ be a compact weighted Riemannian manifold with smooth boundary. Then, the compact maximum principle of Proposition 4.1 for the Schrödinger operator $\mathcal{L}$ holds if and only if $\lambda_1^{-\mathcal{L}}(\mathrm{int}M) > 0$.*

When specified to the stability operator, the previous result takes the following form.

Corollary 4.4. *Let $M_\Psi = (M, g, \mathrm{dv}_\Psi)$ be a compact weighted Riemannian manifold with smooth boundary $\partial M \neq \emptyset$. Assume that $u \in C^0(M) \cap C^2(int(M))$ is a strongly stable solution to $\Delta_\Psi u = f(u)$ on M. If $v \in C^0(M) \cap W^{1,2}_{\mathrm{loc}}(int(M))$ satisfies*

$$\begin{cases} \Delta_\Psi v \geq f'(u) v & \mathrm{int}M \\ v \leq 0 & \partial M \end{cases}$$

then $v \leq 0$ on M.

4.2. Non-compact maximum principle: parabolicity

Let M_Ψ be a (connected) weighted manifold with (possibly empty) boundary ∂M and outward pointing unit normal $\vec{\nu}$. Say that M_Ψ is *Neumann-parabolic* ($\mathcal{N}$-parabolic for short) if, for any given $v \in C^0(M) \cap W^{1,2}_{loc}(\mathrm{int}M, \mathrm{dv}_\Psi)$ satisfying

$$\begin{cases} \Delta_\Psi v \geq 0 & \mathrm{int}M \\ \partial_{\vec{\nu}} v \leq 0 & \partial M \\ \sup_M v < +\infty \end{cases}$$

it holds

$$v \equiv const.$$

Obviously, in case $\partial M = \emptyset$, the normal derivative condition is void.

In order to give an alternative (and equivalent) definition of the $\mathcal{N}$-parabolicity, we first recall that the *capacity* of a compact set $K \subset M_\Psi$ is defined as

$$\mathrm{cap}_\Psi\, K := \inf \left\{ \int_M |\nabla u|^2 \, \mathrm{dv}_\Psi \ : \ u \in C^\infty_c(M),\ u \geq 1 \text{ on } K \right\}.$$

We have the following characterization (see e.g. [24, Theorem 1.5]).

Theorem 4.5. *Let M_Ψ be an oriented, connected, weighted Riemannian manifold with nonempty boundary. Then, the following are equivalent*

(1) *cap_Ψ $K = 0$ for every compact set $K \subset M_\Psi$;*
(2) *M_Ψ is $\mathcal{N}$-parabolic.*

As the definition shows, parabolicity is a kind of compactness from the viewpoint of the (weighted) Laplacian. This is also visible in the next theorem. Further instances will be presented in Section 6.2.

Theorem 4.6 (*Ahlfors Maximum Principle, [23,24]*). *If M_Ψ is a $\mathcal{N}$-parabolic weighted manifold with $\partial M \neq \emptyset$, then for any $v \in C^0(M) \cap W^{1,2}_{loc}(\mathrm{int}M)$ satisfying*

$$\begin{cases} \Delta_\Psi v \geq 0 & \mathrm{int}M \\ \sup_M v < +\infty \end{cases}$$

it holds

$$\sup_M v = \sup_{\partial M} v.$$

Using Theorem 4.6, the proof of Proposition 4.1 extends to the context of non-compact parabolic Riemannian manifolds: in addition, we only have to require suitable bounds on the functions u and φ:

Proposition 4.7 (*Non-Compact Maximum Principle*). *Let $M_\Psi = (M, g, \mathrm{dv}_\Psi)$ be a $\mathcal{N}$-parabolic weighted Riemannian manifold with boundary $\partial M \neq \emptyset$ and set $\mathcal{L} = \Delta_\Psi - q$ with $q \in C^0(M)$. Assume that there exists $\varphi \in C^2(M)$ satisfying*

$$\begin{cases} \mathcal{L}\varphi \leq 0 & \mathrm{int}M \\ \varphi \leq C & M \end{cases} \tag{4.2}$$

for some constant $C \geq 1$. Then, any solution $u \in C^0(M) \cap W^{1,2}_{\mathrm{loc}}(\mathrm{int}M)$ of

$$\begin{cases} \mathcal{L}u \geq 0 & \mathrm{int}M \\ u \leq 0 & \partial M \\ \sup_M u < +\infty \end{cases}$$

satisfies $u \leq 0$ in M.

Proof. Note that, thanks to the bounds on φ, defining $\Phi = \log(\varphi^{-2}) + \Psi$ as in the compact case, the weighted manifold M_Φ inherits the $\mathcal{N}$-parabolicity of M_Ψ. For instance, this can be seen by using the capacitary characterization of parabolicity as explained in Theorem 4.5. Therefore, the proof of Proposition 4.1 can be carried out verbatim up to replacing the classical maximum principle for the operator Δ_Φ with the corresponding Ahlfors Maximum Principle of Theorem 4.6. □

4.3. Uniqueness

It is well known that, for convex or concave nonlinearities, stable solutions to the corresponding semilinear equations on compact domains are (essentially) unique. More precisely, we recall the following result from [14, Proposition 1.3.1].

Theorem 4.8. *Let $M_\Psi = (M, g, \mathrm{dv}_\Psi)$ be a compact weighted Riemannian manifold with boundary components $(\partial M)_j \neq \emptyset$, $j = 1, 2$. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^2 function satisfying either $f''(t) \leq 0$ or $f''(t) \geq 0$. Then, the boundary value problem*

$$\begin{cases} \Delta_\Psi u = f(u) & \mathrm{int}M \\ u = c_j \in \mathbb{R} & (\partial M)_j \end{cases} \tag{4.3}$$

has at most one $C^2(M)$-stable solution unless $f(t) = -\lambda_1 t + c$, with $\lambda_1 = \lambda_1^{-\Delta_\Psi}(M) > 0$ the first Dirichlet eigenvalue. In this case, if u_1 and u_2 are two solutions, then $u_1 - u_2 = \alpha\varphi_1$, where $\alpha \in \mathbb{R}$ and φ_1 is a first Dirichlet eigenfunction of $-\Delta_\Psi$ on M.

We are going to show how the proof of this uniqueness property extends to complete manifolds under a global Sobolev regularity condition. To this end, we first adapt to complete manifolds with boundary the classical global Stokes theorem by Gaffney, [19].

Theorem 4.9 (*Gaffney with Boundary*)**.** *Let $M_\Psi = (M, g, \mathrm{dv}_\Psi)$ be a complete weighted Riemannian manifold with (possibly empty) boundary ∂M. Let X be a vector field on M such that:*

$$(i)\,|X| \in L^1(M, \mathrm{dv}_\Psi),\ (ii)\ \mathrm{div}_\Psi(X) \in L^1(M, \mathrm{dv}_\Psi),\ (iii)\, g(X, \vec{\nu}) \in L^1(\partial M, \mathrm{dv}_\Psi),$$

where $\vec{\nu}$ is the outward-pointing unit normal to ∂M. Then

$$\int_M div_\Psi(X)\ \mathrm{dv}_\Psi = \int_{\partial M} g(X, \vec{\nu})\ da_\Psi.$$

Proof. It is a consequence of the Riemannian extension property of complete manifolds that, even for manifolds with boundary, the completeness of M implies the existence of a sequence of cutoff functions $\{\rho_k\}_k \subset C_c^\infty(M)$ satisfying

$$\begin{cases} 0 \le \rho_k \le 1 \\ \|\nabla \rho_k\|_{L^\infty(M,\mathrm{dv})} \to 0 \\ \rho_k \nearrow 1. \end{cases} \tag{4.4}$$

See [33, Page 16]. Since the vector field $\rho_k X$ is compactly supported, by the classical (weak) divergence theorem we have

$$\int_M \mathrm{div}_\Psi(\rho_k X)\ \mathrm{dv}_\Psi = \int_{\partial M} g(\rho_k X, \vec{\nu})\ \mathrm{da}_\Psi.$$

On the other hand,

$$\int_M \mathrm{div}_\Psi(\rho_k X)\ \mathrm{dv}_\Psi = \int_M g(\nabla \rho_k, X)\ \mathrm{dv}_\Psi + \int_M \rho_k\, \mathrm{div}_\Psi(X)\ \mathrm{dv}_\Psi.$$

Whence, we obtain

$$\int_{\partial M} g(\rho_k X, \vec{\nu})\ \mathrm{da}_\Psi = \int_M g(\nabla \rho_k, X)\ \mathrm{dv}_\Psi + \int_M \rho_k\, \mathrm{div}_\Psi(X)\ \mathrm{dv}_\Psi. \tag{4.5}$$

To conclude the validity of (4.5) we take the limit as $k \to +\infty$ once we have noted that, by dominated convergence,

$$\int_M \rho_k\, \mathrm{div}_\Psi(X)\ \mathrm{dv}_\Psi \to \int_M \mathrm{div}_\Psi(X)\ \mathrm{dv}_\Psi$$

and

$$\int_{\partial M} g(\rho_k X, \vec{\nu})\ \mathrm{da}_\Psi \to \int_{\partial M} g(X, \vec{\nu})\ \mathrm{da}_\Psi$$

while

$$\left| \int_M g(\nabla \rho_k, X)\ \mathrm{dv}_\Psi \right| \le \|\nabla \rho_k\|_{L^\infty(M,\mathrm{dv})} \|X\|_{L^1(M,\mathrm{dv}_\Psi)} \to 0. \quad \square$$

Using this global divergence theorem, we can now extend to complete manifolds the uniqueness result of Theorem 4.8.

Theorem 4.10. *Let $M_\Psi = (M, g, \mathrm{dv}_\Psi)$ be a complete weighted Riemannian manifold with boundary $\partial M \neq \emptyset$, and $u_1, u_2 \in C^0(M) \cap W^{1,2}(\mathrm{int} M, \mathrm{dv}_\Psi) \cap L^\infty(M)$ be stable solutions to (4.3) with $f \in C^1$ concave (or convex). Then $u_1 = u_2$ unless $f(t) = At + B$ for some $A, B \in \mathbb{R}$.*

Proof. Observe that $\omega = u_2 - u_1$ solves

$$\begin{cases} \Delta_\Psi \omega = f(u_2) - f(u_1) & \text{in int}M \\ \omega = 0 & \text{on } \partial M, \end{cases} \tag{4.6}$$

Let $\omega_+ = \max(\omega, 0) \in W^{1,2}(\text{int}M) \cap C^0(M)$. Using a standard approximation argument that relies on the completeness of M, we easily see that

$$\omega_+ \in W_0^{1,2}(\text{int}M).$$

Indeed, let $\{\rho_k\}_k \subset C_c^\infty(M)$ be the sequence of cutoff functions introduced in Theorem 4.9 and consider the corresponding sequence $\{\varphi_k = \rho_k \omega_+\}_k \subset W_0^{1,2}(\text{int}M)$. Since, by dominated convergence, $\varphi_k \xrightarrow{L^2} \omega_+$ and, moreover,

$$\int_M |\nabla(\varphi_k - \omega_+)|^2 \, \mathrm{dv}_\Psi \\ \leq \underbrace{2\int_M |\omega_+|^2 |\nabla \rho_k|^2 \mathrm{dv}_\Psi}_{\xrightarrow{DCT} 0} + \underbrace{2\int_M (1-\rho_k)^2 |\nabla \omega_+|^2 \, \mathrm{dv}_\Psi}_{\xrightarrow{MCT} 0} \longrightarrow 0$$

we have $\varphi_k \xrightarrow{W^{1,2}} \omega_+$. The claimed property thus follows form the fact that $W_0^{1,2}(\text{int}M)$ is a closed subspace of $W^{1,2}(\text{int}M)$.

Now consider the vector field $X = \omega_+ \nabla \omega_+$. By the very definition, X and $\text{div}_\Psi(X)$ are L^1-functions and X vanishes on the boundary ∂M. Thus, we can apply Theorem 4.9 obtaining

$$\int_M |\nabla \omega_+|^2 \, \mathrm{dv}_\Psi = -\int_M \Big(f(u_2) - f(u_1)\Big)\omega_+ \, \mathrm{dv}_\Psi. \tag{4.7}$$

On the other hand, since u_2 is a stable solution, using $\varphi_k = \rho_k \omega_+ \in W_0^{1,2}(M, \mathrm{dv}_\Psi)$ as test functions in the stability condition, we obtain

$$\int_M |\nabla \varphi_k|^2 \, \mathrm{dv}_\Psi \geq -\int_M f'(u_2)\varphi_k^2 \, \mathrm{dv}_\Psi$$

where

$$\int_M |\nabla \varphi_k|^2 \, \mathrm{dv}_\Psi = \underbrace{\int_M \rho_k^2 |\nabla \omega_+|^2 \, \mathrm{dv}_\Psi}_{\xrightarrow{MCT} \int_M |\nabla \omega_+|^2 \, \mathrm{dv}_\Psi} + \underbrace{\int_M \omega_+^2 |\nabla \rho_k|^2 \, \mathrm{dv}_\Psi}_{\xrightarrow{DCT} 0} \\ + \underbrace{2\int_M \rho_k \omega_+ g(\nabla \rho_k, \nabla \omega_+) \, \mathrm{dv}_\Psi}_{=c_k}$$

and

$$|c_k| \leq 2 \underbrace{\left(\int_M \rho_k^2 |\nabla \omega_+|^2 |\nabla \rho_k|^2 \, \mathrm{dv}_\Psi\right)^{\frac{1}{2}}}_{\xrightarrow{DCT} 0} \left(\int_M \omega_+^2 \, \mathrm{dv}_\Psi\right)^{\frac{1}{2}}.$$

Thus

$$\int_M |\nabla \varphi_k|^2 \, \mathrm{dv}_\Psi \to \int_M |\nabla \omega_+|^2 \, \mathrm{dv}_\Psi.$$

Moreover

$$-\int_M f'(u_2)\varphi_k^2 \, \mathrm{dv}_\Psi = -\int_M f'(u_2)\rho_k^2 \omega_+^2 \, \mathrm{dv}_\Psi \xrightarrow{DCT} -\int_M f'(u_2)\omega_+^2 \, \mathrm{dv}_\Psi.$$

It follows that

$$\int_M |\nabla \omega_+|^2 \, \mathrm{dv}_\Psi \geq -\int_M f'(u_2)\omega_+^2 \, \mathrm{dv}_\Psi$$

and this latter, together with (4.7), implies

$$-\int_M f'(u_2)\omega_+^2 \, \mathrm{dv}_\Psi \leq -\int_M \Big(f(u_2) - f(u_1)\Big)\omega_+ \, \mathrm{dv}_\Psi$$

i.e.

$$\int_M \Big(f(u_2) - f(u_1) - f'(u_2)\omega_+\Big)\omega_+ \, \mathrm{dv}_\Psi \leq 0.$$

Since, by concavity, the above integrand is non-negative we deduce that

$$\Big(f(u_2) - f(u_1) - f'(u_2)\omega_+\Big)\omega_+ = 0$$

and two possibilities can occur: either $f(t)$ is strictly concave and, hence, $w_+ \equiv 0$, or $f(t)$ is affine. Clearly, in the first case, $u_2 \leq u_1$ and by reversing the role of u_1 and u_2 we conclude $u_1 = u_2$ as desired. □

4.4. Symmetry via average

As a warm-up for the investigations of the paper we observe that, clearly, if the boundary value problem at hand

$$\begin{cases} \Delta_\Psi u = f(u) & \text{in } \Omega \\ u = c_j \in \mathbb{R} & \text{on } (\partial\Omega)_j \end{cases} \tag{1.1}$$

has a unique solution, and we are able to construct at least one symmetric solution, then we are done. This happens e.g. in the affine setting $f(t) = At + B$. Indeed, the equation is clearly preserved by the average procedure, hence a symmetric solution exists. In order for the maximum principle to hold, we just need to assume that either $A \geq 0$ or, more generally, that Ω is small enough in the spectral sense, i.e. $\lambda_1^{-\Delta_\Psi + A}(\Omega) > 0$. Thus, any solution to the corresponding Dirichlet problem (1.1) is automatically strictly stable. This is the simplest situation that can occur.

Proposition 4.11. *Let M_Ψ be a weighted manifold and let $\bar{\Omega}$ be a smooth, compact, Ψ-isoparametric domain. The connected components of its boundary are denoted by $(\partial\Omega)_j$, $j = 1, 2$.*

Let $u \in C^\infty(\Omega) \cap C^0(\bar{\Omega})$ be a strictly stable solution to the problem

$$\begin{cases} \Delta_\Psi u = Au + B & \textit{in } \Omega \\ u = c_j & \textit{on } (\partial\Omega)_j \end{cases} \tag{4.8}$$

where $B, c_j \in \mathbb{R}$. Then, u is symmetric.

Proof. Using the commutation rule $[\mathcal{A}_\Psi, \Delta_\Psi] = 0$ we see that the smooth function

$$w = u - \mathcal{A}_\Psi(u)$$

solves the problem

$$\begin{cases} \Delta_\Psi w = Aw & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

The maximum principle yields $w = 0$ which means

$$u = \mathcal{A}_\Psi(u) \quad \text{on } \Omega$$

as desired. □

5. Symmetry of solutions on Ψ-homogeneous domains

The main result of the section is a geometric interpretation of the arguments in [14, Proposition 1.3.4]. The original symmetry result, for rotationally symmetric domains in the Euclidean spaces, is proved in [1, Lemma 1.1].

Theorem 5.1. *Let $\bar{\Omega}$ be a compact Ψ-homogeneous domain with soul P inside the weighted manifold M_Ψ. If $\mathcal{D} = \{X_1, \ldots, X_k\}$ is an integrable distribution of Killing vector fields associated to the foliation of $\bar{\Omega}$, suppose that Ψ satisfies the compatibility condition*

$$g(X_i, \nabla \Psi) \equiv const \quad on\ \Omega, \tag{5.1}$$

for every $i = 1, \ldots, k$.

Then, a stable solution $u \in C^3(\Omega) \cap C^1(\bar{\Omega})$ of

$$\begin{cases} \Delta_\Psi u = f(u) & \Omega \\ u = c_j & (\partial\Omega)_j \end{cases} \tag{5.2}$$

is symmetric if and only if at least one of the following conditions hold:

(a) *$g(\nabla\Psi, X_i) \equiv 0$ for every $i = 1, \ldots, k$, i.e. $\Psi(x) = \widehat{\Psi}(dist(x, P))$ is symmetric;*
(b) *the mean value of u over $\bar{\Omega}$ is zero.*

Remark 5.2. For a Killing vector field X, condition (5.1) can be seen as a Ψ-compatibility property. Indeed, since $\operatorname{div}(X) = 0$,

$$g(X, \nabla\Psi) \equiv const$$
$$\Updownarrow$$
$$\operatorname{div}_\Psi(X) = \operatorname{div}(X) - g(X, \nabla\Psi) \equiv const.$$

Thus, in condition (5.1), we are requiring that the divergence-free property of the Killing field X is (in a certain sense) inherited by the weighted manifold.

The proof of Theorem 5.1 relies on the fact that (Ψ-)Killing vector fields well behave with respect to the (weighted) Laplace–Beltrami operator. We first recall the following known characterization.

Lemma 5.3. *Let (M, g) be a Riemannian manifold. Then, the vector field X is Killing if and only if the commutation rule $[\Delta, X] = 0$ holds. This means that, for any smooth function u, $\Delta X(u) = X(\Delta u)$.*

Proof. See [17] for a computational proof that involves generic vector fields. On the other hand, following V. Matveev, the commutation rule can be also deduced directly from the fact that the flow of a Killing vector field is an infinitesimal isometry. Conversely, if the commutation rule holds then the flow of X preserves the Laplacian and the Laplacian determines uniquely the Riemannian metric. □

In the special case of a Killing vector field tangential to the leaves of a weighted isoparametric domain, the commutation extends to the weighted Laplacian. This is a special case of the following

Lemma 5.4. *Let M_Ψ be a weighted manifold. If X is a Killing vector field satisfying condition (5.1), then*

$$[\Delta_\Psi, X] = 0, \quad on\ \Omega$$

in the sense that, for any smooth function u on Ω,

$$\Delta_\Psi X(u) = X(\Delta_\Psi u).$$

Proof. Recall that

$$\Delta_{\Psi} u = \Delta u - g(\nabla \Psi, \nabla u)$$

and that, since X is Killing,

$$[\Delta, X] = 0.$$

Therefore, we are reduced to verify that

$$g(\nabla \Psi, \nabla X(u)) = D_X g(\nabla \Psi, \nabla u). \tag{5.3}$$

To this end, let us start by computing

$$\begin{aligned} g(\nabla \Psi, \nabla X(u)) &= g(\nabla \Psi, \nabla g(X, \nabla u)) \\ &= D_{\nabla \Psi} g(X, \nabla u) \\ &= g(D_{\nabla \Psi} X, \nabla u) + g(X, D_{\nabla \Psi} \nabla u) \\ &= -g(D_{\nabla u} X, \nabla \Psi) + \mathrm{Hess}(u)(X, \nabla \Psi), \end{aligned}$$

where in the last equality we have used that X is Killing and the definition of the Hessian tensor. Now

$$\begin{aligned} g(X, \nabla \Psi) = const &\Longrightarrow D_{\nabla u}\, g(X, \nabla \Psi) = 0 \\ &\Longrightarrow g(D_{\nabla u} X, \nabla \Psi) + g(X, D_{\nabla u} \nabla \Psi) = 0 \\ &\Longrightarrow -g(D_{\nabla u} X, \nabla \Psi) = \mathrm{Hess}(\Psi)(X, \nabla u). \end{aligned}$$

Inserting into the above gives

$$g(\nabla \Psi, \nabla X(u)) = \mathrm{Hess}(u)(X, \nabla \Psi) + \mathrm{Hess}(\Psi)(X, \nabla u). \tag{5.4}$$

On the other hand,

$$\begin{aligned} D_X g(\nabla \Psi, \nabla u) &= g(D_X \nabla \Psi, \nabla u) + g(\nabla \Psi, D_X \nabla u) \\ &= \mathrm{Hess}(\Psi)(X, \nabla u) + \mathrm{Hess}(u)(X, \nabla \Psi). \end{aligned} \tag{5.5}$$

Putting together (5.4) and (5.5) we conclude the validity of (5.3) as desired. □

We are now in the position to give the

Proof of Theorem 5.1. Consider a distribution $\mathcal{D} = \{X_1, \ldots, X_k\}$ of Killing vector fields tangential to the leaves of the foliation and satisfying $g(\nabla \Psi, X_i) = const$ for every $i = 1, \ldots, k$. Let $X = X_j$ and define

$$v = X(u) = g(\nabla u, X).$$

Since u is locally constant on $\partial\Omega$ and $X|_{\partial\Omega}$ is tangential to $\partial\Omega$, we have

$$v = 0 \quad \text{on } \partial\Omega.$$

On the other hand, by Lemma 5.4 we deduce that

$$\Delta_{\Psi} v = X(\Delta_{\Psi} u) = X(f(u)) = f'(u) X(u) = f'(u) v.$$

It follows that $v \in C^2(\Omega)$ is a solution to the problem

$$\begin{cases} \Delta_{\Psi} v = f'(u)\, v & \Omega \\ v = 0 & \partial\Omega. \end{cases}$$

In particular, since by stability $\lambda_1^{-\Delta_\Psi + f'(u)}(\Omega) = 0 \geq 0$, it follows that $\lambda_1^{-\Delta_\Psi + f'(u)}(\Omega) = 0$ and v is a first eigenfunction corresponding to this Dirichlet eigenvalue. By the nodal domain theorem,

$$v \geq 0.$$

We are going to prove that the validity of at least one of the conditions (a) or (b) is equivalent to

$$\int_\Omega v \, \mathrm{dv}_\Psi = 0 \tag{5.6}$$

and, hence to

$$v \equiv 0.$$

To this end, we use the Ψ-divergence theorem with the vector field $Z = uX$. Since $\operatorname{div} X = 0$ and X_x is tangential to $\Sigma_{d(x)}$, on the one hand we have

$$\begin{aligned}\int_\Omega \operatorname{div}_\Psi Z \, \mathrm{dv}_\Psi &= \int_\Omega g(\nabla u, X) \, \mathrm{dv}_\Psi + \int_\Omega u \operatorname{div}_\Psi X \, \mathrm{dv}_\psi \\ &= \int_\Omega v \, \mathrm{dv}_\Psi + \int_\Omega u \operatorname{div} X \, \mathrm{dv}_\Psi - \int_\Omega u \, g(\nabla \Psi, X) \mathrm{dv}_\Psi \\ &= \int_\Omega v \, \mathrm{dv}_\Psi - g(\nabla \Psi, X) \int_\Omega u \, \mathrm{dv}_\Psi .\end{aligned}$$

On the other hand,

$$\int_\Omega \operatorname{div}_\Psi Z \, \mathrm{dv}_\Psi = \int_{\partial\Omega} g(Z, \vec{\nu}) \mathrm{da}_\Psi = \int_{\partial\Omega} u \, g(X, \pm\nabla d) \mathrm{da}_\Psi = 0,$$

where $d(x) = \operatorname{dist}(x, P)$. By putting together these two expressions we obtain

$$\int_\Omega v \, \mathrm{dv}_\Psi = g(\nabla \Psi, X) \int_\Omega u \, \mathrm{dv}_\Psi$$

That is, (5.6) holds if and only if either $g(\nabla \Psi, X) \equiv 0$ or u has vanishing integral.

We have thus proved that if at least one of the conditions (a) and (b) is satisfied, then

$$X_j(u)(x_0) = 0, \quad \forall j = 1, \ldots, k, \quad \forall x_0 \in \bar{\Omega}.$$

Thanks to the fact that $\{X_1|_{x_0}, \ldots, X_k|_{x_0}\}$ generates $T_{x_0}\Sigma_{d(x_0)}$, this implies that u is locally symmetric, and hence symmetric, on $\bar{\Omega}$. The proof of Theorem 5.1 is completed. □

6. Symmetry of solutions in a non-homogeneous case

In this section we discuss a case where we cannot apply Theorem 5.1 due to the absence of enough (if any) Killing vector fields tangential to the leaves of the tube. In fact, recall that, in nonpositive curvature, Killing fields tangential to the (concave) boundary of a domain are trivial as the following classical theorem shows; see [42].

Theorem 6.1 (*Weighted Yano–Bochner*)**.** *Let $M_\Psi = (M, g, \mathrm{dv}_\Psi)$ be a compact weighted Riemannian manifold with (possibly empty) concave boundary ∂M. This means that, if $\vec{\nu}$ denote the outer unit normal to ∂M, then $\mathrm{II}(Z, Z) = g(D_Z(-\vec{\nu}), Z) \geq 0$ for every $Z \in T\partial M$. Assume also that $\operatorname{Ric}_\Psi = \operatorname{Ric} + \operatorname{Hess} \Psi \leq 0$.*

Then, every Killing vector field X on M such that $X|_{\partial M} \in T\partial M$ and satisfying $\operatorname{div}_\Psi(X) \equiv const$ must be parallel. In particular, $|X| \equiv const$. Moreover, if $\operatorname{Ric}_\Psi < 0$ at some point, then $X = 0$.

Proof. The weighted version of Bochner formula for Killing vector fields satisfying $\operatorname{div}_\Psi(X) \equiv const$ states that

$$\frac{1}{2}\Delta_\Psi |X|^2 = |DX|^2 - \operatorname{Ric}_\Psi(X,X).$$

Therefore, using the curvature assumption,

$$\Delta_\Psi |X|^2 \geq 0.$$

By the Killing condition and the fact that $X|_{\partial\Omega}$ is tangential to $\partial\Omega$ we get

$$\partial_{\vec{\nu}}|X|^2 = -2\mathrm{II}(X,X), \quad \text{on } \partial\Omega.$$

It follows that $v = |X|^2$ is a solution to the problem

$$\begin{cases} \Delta_\Psi v \geq 0 & \Omega \\ \partial_{\vec{\nu}} v = -2\mathrm{II}(X,X) \leq 0 & \partial\Omega. \end{cases}$$

By the Hopf Lemma, $v \equiv const$. Using this information into the Bochner formula gives that $|DX| = 0$, i.e. X is parallel, and $\operatorname{Ric}_\Psi(X,X) = 0$. □

Remark 6.2. For a general Killing vector field, without any request on the Ψ-divergence, the weighted Bochner formula states that

$$\frac{1}{2}\Delta_\Psi |X|^2 = |DX|^2 - \operatorname{Ric}_\Psi(X,X) + Xg(X,\nabla\Psi)$$

or, equivalently,

$$\frac{1}{2}\Delta_\Psi |X|^2 = |DX|^2 - \operatorname{Ric}_\Psi(X,X) + g(X,\nabla \operatorname{div}_\Psi(X))$$

Thus, the previous Theorem can be slightly generalized to Killing vector fields tangent to the boundary of the manifold and satisfying

$$g(X,\nabla \operatorname{div}_\Psi(X)) \geq 0$$

Remark 6.3. Formally, the conclusion of Theorem 6.1 can be extended to Killing fields of bounded length on a complete Riemannian manifold with boundary and with quadratic volume growth. See Sections 4.2 and 6.2.

Example 6.4. Take the annulus $A(-1,+1) = [-1,+1] \times N$ inside the Riemannian warped cylinder $M = \mathbb{R} \times_\sigma N$ where:

(i) (N, g^N) is compact, $\partial N = \emptyset$, and $\mathrm{Sect}^N \equiv -k^2 < 0$;
(ii) $\sigma'(-1) \leq 0$, $\sigma'(+1) \geq 0$;
(iii) $\sigma''(r) \geq 0$ in $[-1,1]$.

We have already observe in Example 2.4 that $A(-1,1)$ is an isoparametric domain with totally umbilical leaves $\Sigma_t = \{t\} \times N$, $-1 \leq t \leq 1$. In particular,

$$\mathrm{II}_{\Sigma_\pm} = \pm\sigma'(\pm 1)\sigma(\pm 1) g^N.$$

It follows from (ii) that

(a) $\partial A(-1,1) = \Sigma_{\pm 1}$ is concave.

Moreover, recalling that

$$\mathrm{Sect}_M(X \wedge Y) = \begin{cases} 0 & X, Y = \nabla r \\ -\frac{\sigma''(r)}{\sigma(r)} & X = \nabla r,\ Y \in TN \\ \frac{-k^2 - \sigma'(r)^2}{\sigma(r)^2} & X, Y \in TN \end{cases}$$

by (iii) we have

(b) $\mathrm{Sect}_M < 0$.

An application of Theorem 6.1 gives that any Killing vector field X of $\bar{A}(-1,1)$ tangential to $\partial A(-1,1)$ must vanish identically.

As we are going to show, in the situation of Example 6.4 we are still able to deduce a symmetry result. But there is a price to pay: besides the assumption that the solution to the boundary value problem is (strictly) stable, the nonlinearity $f(t)$ has to be concave. In particular, when the fibre N is compact, we are in the regime of uniqueness of the solution; see Theorem 4.8. Despite of this drawback, on the one hand, it is not clear how to produce a-priori a symmetric solution (clearly, average does not work) and, on the other hand, the method we use works in a more general setting where, apparently, the non-compact uniqueness result of Theorem 4.10 is not applicable. See Remark 6.7.

6.1. A non-compact symmetry result: statement and comments

Let $M_\Psi = (M, g^M, \mathrm{dv}_\Psi)$ be the m-dimensional weighted Riemannian manifold given as the warped product

$$M = I \times_\sigma N$$

where (N, g^N) is a possibly non-compact $(m-1)$-dimensional Riemannian manifold with $\partial N = \emptyset$, $I \subseteq \mathbb{R}$ is an interval, $\sigma : I \to \mathbb{R}_{>0}$ is a smooth function and

$$\Psi(r, \xi) = \Phi(r) + \Gamma(\xi) \tag{6.1}$$

splits into the sum of two smooth functions depending respectively on the I-variable and on the N-variable. Consider the annulus $\bar{A}(r_1, r_2) = [r_1, r_2] \times N$. By the coarea formula, the volume of $\bar{A}(r_1, r_2)$ has the expression

$$\mathrm{vol}_\Psi(\bar{A}(r_1, r_2)) = \mathrm{vol}_\Gamma(N) \int_{r_1}^{r_2} e^{-\Phi(r)}\ \sigma^{m-1}(r)\ \mathrm{d}r.$$

Moreover, we note explicitly that

$$\Delta^M u = \partial_r^2 u + (m-1)\frac{\sigma'}{\sigma}\partial_r u + \frac{1}{\sigma^2}\Delta^N u$$

and thus

$$\begin{aligned} \Delta_\Psi^M u &= \partial_r^2 u + (m-1)\frac{\sigma'}{\sigma}\partial_r u + \frac{1}{\sigma^2}\Delta^N u - g(\nabla^M u, \nabla^M \Psi) \\ &= \partial_r^2 u + \left((m-1)\frac{\sigma'}{\sigma} - \Phi'\right)\partial_r u + \frac{1}{\sigma^2}\Delta^N u - \sigma^2 g^N\left(\frac{\nabla^N u}{\sigma^2}, \frac{\nabla^N \Gamma}{\sigma^2}\right) \\ &= \partial_r^2 u + \left((m-1)\frac{\sigma'}{\sigma} - \Phi'\right)\partial_r u + \frac{1}{\sigma^2}\Delta^N u - \frac{1}{\sigma^2} g^N(\nabla^N u, \nabla^N \Gamma) \\ &= \partial_r^2 u + \left((m-1)\frac{\sigma'}{\sigma} - \Phi'\right)\partial_r u + \frac{1}{\sigma^2}\Delta_\Gamma^N u \end{aligned}$$

In particular, $\bar{A}(r_1, r_2)$ is Ψ-isoparametric and we have the validity of the commutation rule

$$[\Delta_{\Psi}^{M}, \Delta_{\Gamma}^{N}] = 0. \tag{6.2}$$

We are now ready to state our non-compact symmetry result. Since the underlying manifold is always M_Ψ and there is no danger of confusion, from now on we shall omit the overscript M in the corresponding quantities and operators.

Theorem 6.5. *Let $M_\Psi = (I \times_\sigma N)_\Psi$ where (N, g^N) is a complete (possibly non-compact), connected, $(m-1)$-dimensional Riemannian manifold with finite Γ-volume $\mathrm{vol}_\Gamma(N) < +\infty$.*

Let $u \in C^4(\bar{A}(r_1, r_2))$ be a solution to the Dirichlet problem

$$\begin{cases} \Delta_\Psi u = f(u) & \text{in } A(r_1, r_2) \\ u \equiv c_1 & \text{on } \{r_1\} \times N \\ u \equiv c_2 & \text{on } \{r_2\} \times N. \end{cases} \tag{6.3}$$

where $c_j \in \mathbb{R}$ are given constants and the function $f(t)$ is of class C^2 and satisfies $f''(t) \le 0$. If

$$\|u\|_{C^2_{rad}} := \sup_{A(r_1,r_2)} |u| + \sup_{A(r_1,r_2)} |\partial_r u| + \sup_{A(r_1,r_2)} |\partial_r^2 u| < +\infty, \tag{6.4}$$

and $f'(u) \ge -B$, for some constant $B \ge 0$ satisfying

$$0 \le B < \left(\int_{r_1}^{r_2} \frac{\int_{r_1}^{s} e^{-\Phi(z)} \sigma^{m-1}(z)\, dz}{e^{-\Phi(s)} \sigma^{m-1}(s)}\, \mathrm{d}s \right)^{-1} \tag{6.5}$$

then $u(r, \xi) = \hat{u}(r)$ is symmetric.

Remark 6.6. Under the additional assumption $[\Delta_\Psi, \Delta_\Gamma^N](u) \le 0$, this symmetry result can be easily generalized to every smooth weight $\Psi(r, \xi)$ satisfying the condition $\partial_r \Psi \in L^\infty(A(r_1, r_2))$. This is needed to ensure the existence of the function φ claimed in Lemma 6.13. Clearly, in this case condition (6.5) need to be slightly modified.

Remark 6.7. Some observations on the statement of Theorem 6.5 are in order.

(a) Obviously, if N is compact, assumption (6.4) is automatically satisfied. In this case, if there exists at least one symmetric solution u of (6.7), then each solution must coincide with the symmetric one, thanks to the uniqueness result contained in Theorem 4.10. In the opposite direction, the symmetry result could be useful in establishing whether a symmetric solution actually exists. In fact, the concave non-linearity $f(t)$ is so general that neither standard conditions for the coerciveness of the energy functional are automatically satisfied nor min–max and sub/super-solution methods can be applied directly to construct a symmetric, say one-dimensional, solution. See for instance [2,40].

(b) In the non-compact case, the boundedness assumption (6.4) of Theorem 6.5 is skew with the $W^{1,2}$ global regularity needed in Theorem 4.10. Thus, we do not know whether or not there is some global uniqueness of the (stable) solution.

(c) Condition (6.5) is clearly satisfied if $f'(u) \ge -B = 0$. As a matter of fact, it will be clear from Lemma 6.13 that there is a (strong) stability condition hidden in (6.5). Indeed, the validity of (6.5) implies the existence of a smooth solution $\varphi > 0$ of $\mathcal{L}\varphi \le 0$ on $\mathrm{int} M$, where $\mathcal{L} = \Delta_\Psi - f'(u)$ is the stability operator. According to a classical result independently due to Fischer-Colbrie and Schoen, [18], and to Moss and Piepenbrink, [30] (see also [11]), we have that $\lambda_1^{-\mathcal{L}}(A(r_1, r_2)) \ge 0$. But in fact more is true because we can even obtain that $C^{-1} \le \varphi \le C$ on the whole $\bar{A}(r_1, r_2)$.

(d) In an upcoming work by the first author, [7], condition (6.5) will be replaced by the strong stability assumption on the function u. This is possible thanks to the validity of a maximum principle for unbounded domains, which is based on an Alexandroff–Bakelman–Pucci estimate on manifolds.
(e) It could be interesting to note that condition (6.5) can be written as

$$0 \le \int_{r_1}^{r_2} \frac{\mathrm{vol}_\Psi A(r_1, s)}{\mathrm{area}_\Psi \Sigma_s} \, \mathrm{d}s < \frac{1}{B}$$

where the integrand is the inverse of the Cheeger isoperimetric quotient.
(f) From a different perspective, symmetry on Riemannian (warped) products have been previously investigated in [17] by A. Farina, L. Mari and E. Valdinoci. Their viewpoint is that of the De Giorgi conjecture where, a-priori, it is not known along which direction the stable solution to the Allen-Cahn type equation is symmetric. Thus, their result takes the form of a geometric splitting of the underlying space. See also [3] by M. Batista and I.J. Santos for the case of weighted manifolds and negative Ricci lower bounds.

As a concrete example where to set Theorem 6.5 in, we can consider the weighted slabs of Example 2.13, thus obtaining the following

Corollary 6.8. *Let $\bar{A}(r_1, r_2) = [r_1, r_2] \times \mathbb{R}^{n-1} \subset \mathbb{G}^n = \mathbb{R}^n_\Psi$ be a slab in the Gaussian space, whose weight writes as $\Psi(r, \xi) = \frac{r^2}{2} + \frac{|\xi|^2}{2}$.*
Let $u \in C^4(\bar{A}(r_1, r_2))$ be a solution to the Dirichlet problem

$$\begin{cases} \Delta_\Psi u = f(u) & \text{in } A(r_1, r_2) \\ u \equiv c_1 & \text{on } \{r_1\} \times N \\ u \equiv c_2 & \text{on } \{r_2\} \times N. \end{cases}$$

where $c_j \in \mathbb{R}$ are given constants and the function $f(t)$ is of class C^2 and satisfies $f''(t) \le 0$. If

$$\|u\|_{C^2_{rad}} < +\infty$$

and $f'(u) \ge -B$, for some constant $B \ge 0$ satisfying

$$0 \le B < \left(\int_{r_1}^{r_2} \frac{\int_{r_1}^{s} e^{-z^2/2} \, dz}{e^{-s^2/2}} \, \mathrm{d}s \right)^{-1}$$

then $u(r, \xi) = \hat{u}(r)$ is symmetric.

Proof. Thanks to the presence of the Gaussian weight, the leaves of the foliation have finite volume. Thus we can apply Theorem 6.5, obtaining the claim. □

Observe that this is not true for the same domains in Euclidean space: this fact points out how the presence of a weight that deforms the Riemannian measure may strongly influence the structure of solutions to the equation $\Delta u = f(u)$.

A second important consequence of Theorem 6.5 concerns weights with vanishing tangential component.

Corollary 6.9. *Let $M_\Psi = (I \times_\sigma N)_\Psi$ where $\Psi(r, \xi) = \hat{\Psi}(r)$ is a symmetric smooth function and (N, g^N) is a complete (possibly non-compact), connected, $(m-1)$-dimensional Riemannian manifold with finite volume* $\mathrm{vol}(N) < +\infty$.

Let $u \in C^4(\bar{A}(r_1, r_2))$ be a solution to the Dirichlet problem

$$\begin{cases} \Delta_\Psi u = f(u) & \text{in } A(r_1, r_2) \\ u \equiv c_1 & \text{on } \{r_1\} \times N \\ u \equiv c_2 & \text{on } \{r_2\} \times N. \end{cases}$$

where $c_j \in \mathbb{R}$ are given constants and the function $f(t)$ is of class C^2 and satisfies $f''(t) \leq 0$. If

$$\|u\|_{C^2_{rad}} < +\infty$$

and $f'(u) \geq -B$, for some constant $B \geq 0$ satisfying

$$0 \leq B < \left(\int_{r_1}^{r_2} \frac{\int_{r_1}^{s} e^{-\Psi(z)} \sigma^{m-1}(z) \, \mathrm{d}z}{e^{-\Psi(s)} \sigma^{m-1}(s)} \, \mathrm{d}s \right)^{-1}$$

then $u(r, \xi) = \hat{u}(r)$ is symmetric.

6.2. Some preliminary lemmas

We have already mentioned that the notion of $\mathcal{N}$-parabolicity, introduced in Section 4.2, is a kind of compactness from many viewpoints. The following result contains further instances.

Theorem 6.10. *Let M_Ψ be a weighted Riemannian manifold with (possibly empty) boundary ∂M.*

(a) *(Stokes theorem: general vector fields, [24]) If M_Ψ is $\mathcal{N}$-parabolic then, given a vector field X satisfying $|X| \in L^2(M, \mathrm{dv}_\Psi)$, $g(X, \vec{\nu}) \in L^1(\partial M, da_\Psi)$, $\mathrm{div}_\Psi(X) \in L^1(M, \mathrm{dv}_\Psi)$, it holds*

$$\int_M \mathrm{div}_\Psi(X) \, \mathrm{dv}_\Psi = \int_{\partial M} g(X, \vec{\nu}) \, \mathrm{da}_\Psi.$$

(b) *(Stokes theorem: gradient vector fields and no boundary, [22, Prop. 3.1]) If M_Ψ is parabolic and $\partial M = \emptyset$ then, given $u \in W^{1,2}_{loc}(M, \mathrm{dv}_\Psi)$ satisfying $u \in L^\infty(M, \mathrm{dv}_\Psi)$ and $\Delta_\Psi u \in L^1(M, \mathrm{dv}_\Psi)$, it holds*

$$\int_M \Delta_\Psi u \, \mathrm{dv}_\Psi = 0.$$

(c) *(Volume growth, [21]) Assume that M_Ψ is complete(!) and that $\frac{R}{\mathrm{vol}_\Psi B_R(o)} \notin L^1(+\infty)$ for some (any) $o \in \mathrm{int} M$. Then M_Ψ is $\mathcal{N}$-parabolic.*

Keeping the notation and the assumptions of Theorem 6.5, the above potential theoretic tools enable us to deduce some useful preliminary properties of the Ψ-isoparametric domain $\bar{A}(r_1, r_2)$ and of the solution u.

In view of the next Lemma, recall that N_Γ is complete weighted manifold with $\partial N = \emptyset$ and $\mathrm{vol}_\Gamma(N) < +\infty$.

Lemma 6.11. *The following hold.*

(i) *N_Γ is parabolic;*

(ii) *The closed annulus $\bar{A}(r_1, r_2)_\Psi$ endowed with the weight and the warped product metric inherited from M_Ψ is a weighted $\mathcal{N}$-parabolic manifold with $\partial \bar{A}(r_1, r_2) \neq \emptyset$.*

Proof. (i) is a direct consequence of Theorem 6.10.c. Concerning (ii), let $\alpha = \min_{[r_1,r_2]} \sigma(r) > 0$ and $\beta = \max_{[r_1,r_2]} \sigma(r) < +\infty$ so that, on $\bar{A}(r_1, r_2)$,

$$dr \otimes dr + \alpha \cdot g^N \leq g \leq dr \otimes dr + \beta \cdot g^N$$

in the sense of quadratic forms. Since the LHS metric is complete and the RHS metric has finite Ψ-volume the conclusion follows again from Theorem 6.10.c. □

For the next Lemma recall also that $\|u\|_{C^2_{rad}} < +\infty$.

Lemma 6.12. *We have*

$$\Delta^N_\Gamma u \in L^\infty(A(r_1, r_2)).$$

Moreover, for every fixed $\bar{r} \in [r_1, r_2]$,

$$\Delta^N_\Gamma u\,(\bar{r}, \cdot) \in L^1(N, dv_\Gamma)$$

and

$$\int_N \Delta^N_\Gamma u(\bar{r}, \xi)\, \mathrm{dv}_\Gamma = 0.$$

Proof. Using the fact that $\Delta_\Psi u = f(u)$ we can write

$$\Delta^N_\Gamma u = \sigma^2 f(u) - \sigma^2 \partial^2_r u - \Big((m-1)\sigma\sigma' - \Phi'\sigma^2\Big)\partial_r u.$$

From this expression, since $\sup_{[r_1,r_2]}(\sigma + |\sigma'| + |\Phi'|) < +\infty$, $\|u\|_{C^2_{rad}} < +\infty$ and, hence, $\sup_{A(r_1,r_2)} |f(u)| < +\infty$, we get

$$\Delta^N_\Gamma u \in L^\infty(A(r_1, r_2)).$$

In particular, for every $\bar{r} \in [r_1, r_2]$,

$$\Delta^N_\Gamma u(\bar{r}, \cdot) \in L^\infty(N).$$

Recalling that $\mathrm{vol}_\Gamma(N) < +\infty$ it follows that $\Delta^N_\Gamma u(\bar{r}, \cdot) \in L^1(N, \mathrm{dv}_\Gamma)$. Since $u(\bar{r}, \cdot) \in L^\infty(N)$ and N_Γ is parabolic without boundary, by Theorem 6.10.b we conclude that $\int_N \Delta^N_\Gamma u(\bar{r}, \xi)\, \mathrm{dv}_\Gamma(\xi) = 0$, as required. □

The previous Lemmas, stemming from potential theoretic considerations, will play a fundamental role in the proof of Theorem 6.5. Besides them, we shall also need the validity of the non-compact maximum principle from Proposition 4.7. This follows from the next

Lemma 6.13. *There exists a function* $\varphi \in C^2(A(r_1, r_2)) \cap C^0(\bar{A}(r_1, r_2))$ *satisfying condition (4.2) of Proposition 4.7, namely,*

$$\begin{cases} \mathcal{L}\varphi \leq 0 & A(r_1, r_2) \\ \frac{1}{C} \leq \varphi \leq C & \bar{A}(r_1, r_2), \end{cases}$$

where, as usual, $\mathcal{L} = \Delta_\Psi - f'(u)$ *is the stability operator.*

Proof. Let us start by considering the differential inequality $\big(\Delta_\Psi - f'(u)\big)\varphi \leq 0$ when applied to a symmetric function $\varphi(r, \xi) = \varphi(r)$, that is,

$$\varphi'' + \Big((m-1)\frac{\sigma'}{\sigma} - \Phi'\Big)\varphi' - f'(u) \leq 0 \quad \text{in } I = (r_1, r_2).$$

Since f' is continuous and u is bounded, then there exists $B \geq 0$ such that

$$-f'(u) \leq B$$

obtaining

$$\varphi'' + \Big((m-1)\frac{\sigma'}{\sigma} - \Phi'\Big)\varphi' - f'(u) \leq \varphi'' + \Big((m-1)\frac{\sigma'}{\sigma} - \Phi'\Big)\varphi' + B.$$

Under condition (6.5), a function φ solving the above can be obtained by considering the solution to

$$\begin{cases} \varphi'' + \Big((m-1)\frac{\sigma'}{\sigma} - \Phi'\Big)\varphi' + B = 0 & \text{in } I \\ \varphi(r_1) = 1 \\ \varphi'(r_1) = b < 0 \end{cases} \tag{6.6}$$

for a suitable choice of $b \in \mathbb{R}$. Indeed, letting

$$B(t) = B\int_{r_1}^{t} e^{\Phi(s)}\sigma^{1-m}(s)\int_{r_1}^{s} e^{-\Phi(z)}\sigma^{m-1}(z)\ \mathrm{d}z\ \mathrm{d}s \geq 0$$
$$A(t) = b\ e^{-\Phi(r_1)}\ \sigma^{m-1}(r_1)\int_{r_1}^{t} e^{\Phi(s)}\sigma^{1-m}(s)\ \mathrm{d}s \leq 0,$$

if (6.5) is satisfied, then it is possible to choose $b < 0$ such that

$$-1 < A(r_2) - B(r_2) < 0.$$

It follows that the function

$$\varphi(t) = 1 + A(t) - B(t)$$

is a positive and decreasing solution to (6.6). In particular, φ is bounded above by $\varphi(r_1) = 1$, so it clearly solves the differential inequality

$$\varphi'' + \Big((m-1)\frac{\sigma'}{\sigma} - \Phi'\Big)\varphi' - f'(u)\varphi \leq \varphi'' + \Big((m-1)\frac{\sigma'}{\sigma} - \Phi'\Big)\varphi' + B = 0.$$

The proof of the Lemma is completed. □

6.3. *Proof of Theorem 6.5*

Let us define

$$v(r,\xi) = \Delta_{\Gamma}^{N} u(r,\xi).$$

It is enough to show that, for every $\bar{r} \in [r_1, r_2]$,

$$\xi \mapsto v(\bar{r},\xi) \text{ is constant on } N.$$

Indeed, if this is the case, then $u(\bar{r},\cdot)$ is a bounded (sub/super) harmonic function on the parabolic weighted manifold N_Γ, therefore it must be constant on N. This is precisely what we have to prove.

Now, since u is (locally) constant on the boundary $\partial A(r_1, r_2)$ then

$$v = 0 \quad \text{on } \partial A(r_1, r_2).$$

On the other hand, using the commutation rule (6.2), the fact that $\Delta_\Psi u = f(u)$ and the properties of f we see that

$$\begin{aligned} \Delta_\Psi v &= \Delta_\Gamma^N f(u) \\ &= \Delta^N f(u) - g^N(\nabla^N f(u), \nabla^N \Gamma) \\ &= \mathrm{div}^N(\nabla^N f(u)) - f'(u)\ g^N(\nabla^N u, \nabla^N \Gamma) \\ &= \mathrm{div}^N(f'(u)\nabla^N u) - f'(u)\ g^N(\nabla^N u, \nabla^N \Gamma) \end{aligned}$$

$$\begin{aligned}&= f''(u)|\nabla^N u|_N^2 + f'(u)\Delta^N u - f'(u)\, g^N(\nabla^N u, \nabla^N \Gamma)\\&\leq f'(u)\Delta^N u - f'(u)\, g^N(\nabla^N u, \nabla^N \Gamma)\\&= f'(u)v.\end{aligned}$$

Summarizing, the C^2 function v solves

$$\begin{cases}\Delta_\Psi(-v) \geq f'(u)(-v) & \text{in } A(r_1, r_2)\\(-v) = 0 & \text{on } \partial A(r_1, r_2).\end{cases}$$

By Lemma 6.13 we can apply the non-compact Protter–Weinberger maximum principle of Proposition 4.7, and we get

$$v \geq 0 \text{ in } A(r_1, r_2).$$

On the other hand,

$$\begin{aligned}\int_{A(r_1,r_2)} v \, \mathrm{dv}_\Psi &= \int_{r_1}^{r_2} \left(\int_{\{t\}\times N} v(t,\xi)\, \mathrm{dv}_\Gamma(\xi) \right) e^{-\Phi(t)}\sigma^{m-1}(t)\mathrm{d}t\\&= \int_{r_1}^{r_2} \left(\int_N \Delta_\Gamma^N u(t,\xi)\, \mathrm{dv}_\Gamma(\xi) \right) e^{-\Phi(t)}\sigma^{m-1}(t)\mathrm{d}t\\&= 0\end{aligned}$$

where, for the last equality, we have used Lemma 6.12. As a consequence,

$$v \equiv 0 \text{ on } A(r_1, r_2),$$

as required. The proof of the theorem is completed.

6.4. Infinite annuli

Theorem 6.5 can be easily generalized to the case of infinite annuli, under suitable assumptions that are trivially satisfied in the case of finite annuli.

To this end, consider $A(r_0, +\infty) = (r_0, +\infty) \times_\sigma N$ with $r_0 \in \mathbb{R}_{>0}$ and suppose that $\bar{A}(r_0, +\infty)$ is $\mathcal{N}$-parabolic. If the warping function σ is a bounded function with bounded derivative, then Lemma 6.12 extends trivially to this setting. Moreover, if the function

$$\theta : s \mapsto \frac{\int_{r_0}^s e^{-\Phi(z)}\sigma^{m-1}(z)\, \mathrm{d}z}{e^{-\Phi(s)}\sigma^{m-1}(s)}$$

is integrable over $(r_0, +\infty)$, then the proof of Lemma 6.13 can be readapted, ensuring the existence of the function φ and allowing the non-compact Maximum Principle of Proposition 4.7 to hold.

In this way, the whole proof of Theorem 6.5 can be retraced step by step also in the context of infinite annuli, obtaining the next

Theorem 6.14. *Let $M_\Psi = (\mathbb{R}_{\geq 0} \times_\sigma N)_\Psi$ where (N, g^N) is a complete (possibly non-compact), connected, $(m-1)$-dimensional Riemannian manifold with finite Γ-volume $\mathrm{vol}_\Gamma(N) < +\infty$ and $\sigma \in L^\infty(\mathbb{R}_{\geq 0})$ satisfies $\sigma' \in L^\infty(\mathbb{R}_{\geq 0})$. Suppose also that $\bar{A}(r_0, +\infty)$ is a $\mathcal{N}$-parabolic manifold.*

Let $u \in C^4(\bar{A}(r_0, +\infty))$ be a solution to the Dirichlet problem

$$\begin{cases}\Delta_\Psi u = f(u) & \text{in } A(r_0, +\infty)\\u \equiv c_0 & \text{on } \{r_0\} \times N.\end{cases} \tag{6.7}$$

where $c_0 \in \mathbb{R}$ *is a given constant and the function* $f(t)$ *is of class* C^2 *and satisfies* $f''(t) \leq 0$. *If*

$$\|u\|_{C^2_{rad}} < +\infty \tag{6.8}$$

$$\Delta^N_\Gamma u \in L^1(N, dv_\Gamma) \tag{6.9}$$

$$\theta(s) = \frac{\int_{r_0}^{s} e^{-\Phi(z)} \sigma^{m-1}(z)\, \mathrm{d}z}{e^{-\Phi(s)} \sigma^{m-1}(s)} \in L^1(r_0, +\infty) \tag{6.10}$$

and $f'(u) \geq -B$, *for some constant* $B \geq 0$ *satisfying*

$$0 \leq B < \left(\int_{r_0}^{+\infty} \theta(s)\, \mathrm{d}s \right)^{-1} \tag{6.11}$$

then $u(r, \xi) = \hat{u}(r)$ *is symmetric.*

Remark 6.15. Note that, when specified to a model manifold, $A(r_0, +\infty)$ is the exterior domain $\mathbb{M}(\sigma) \setminus B_r(o)$.

Theorem 6.14 paves the way for further interesting studies about infinite annuli, such as a deeper understanding of the link between the warping function σ and the weight function Ψ. Indeed, it is only in the context of annuli with infinite radius that we can really understand how the behaviour of σ at infinity plays a role when combined with that of Ψ.

Lastly, it could also be interesting to better understand the $\mathcal{N}$-parabolicity and its compatibility with the conditions just required for infinite annuli.

Acknowledgements

The authors would like to thank Giona Veronelli for his suggestions related to the proof of Lemma 4.2 and Alberto Farina for explanations about the content of [17] and for some interesting discussions concerning maximum principles. Finally, the authors are indebted to the anonymous referee for a careful reading of the paper and for several remarks and suggestions.

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