Rendiconti del Circolo Matematico di Palermo Series 2 (2024) 73:2513–2526
https://doi.org/10.1007/s12215-024-01053-z

Remarks on asymptotic isometric embeddings of conic transforms for torus actions

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Received: 19 June 2023 / Accepted: 2 May 2024 / Published online: 17 May 2024

Abstract

Consider a Hodge manifold and assume that a torus acts on it in a Hamiltonian and holomorphic manner and that this action linearizes on a given quantizing line bundle. Inside the dual of the line bundle one can define the circle bundle, which is a strictly pseudoconvex CR manifold. Then, there is an associated unitary representation on the Hardy space of the circle bundle. Under suitable assumptions on the moment map, we consider certain loci in unit circle bundle, naturally associated to a ray through an irreducible weight. Their quotients are called conic transforms. We introduce maps which are asymptotic embeddings of conic transforms making use of the corresponding equivariant Szegő projector.

Keywords Symplectic manifold · Group action · Embedding theorem · Conic transform · Torus action

Mathematics Subject Classification 53D20 · 32Q40

Contents

1 Introduction

Let $(M,\ \omega)$ be a compact connected Hodge manifold of complex dimension d with complex structure J and quantum line bundle (L, h) whose curvature form of the unique compatible and holomorphic connection ∇ is $-2i\,\omega$; we shall denote by g the Riemannian structure $\omega(\cdot,\ J\cdot)$. The volume form $\omega^{\wedge d}/d!$ is denoted by dV_M. Let $X \subset L^\vee$ the unit circle bundle,

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with projection $\pi : X \to M$ and connection contact form α such that $\mathrm{d}\alpha = 2\pi^*(\omega)$. Then (X, α) is a CR manifold and α is the contact form, see [22]. There is a natural volume form on X given by $\mathrm{dV}_X = (2\pi)^{-1}\alpha \wedge \pi^*(\mathrm{dV}_M)$. The Hardy space $H(X)$ represents the quantum space and under some suitable hypothesis classical symmetries on M of a Lie group G give rise to a quantum representation of G on $H(X)$. Fix a coprime weight $\boldsymbol{\nu}$ labeling a unitary irreducible representation in $\mathfrak{g}^\vee$. The ladder $\mathcal{L}$ is the set of unitary irreducible representations labeled by integer elements of the ray $i\mathbb{R}_+ \cdot \boldsymbol{\nu}$. In the setting of ladder representations, see [13], one is led to consider spaces $H_{k\boldsymbol{\nu}}(X)$ for a fixed unitary irreducible representation $\boldsymbol{\nu}$ as $k \to +\infty$ and the corresponding projectors on it. The aim of this paper is to study geometric properties of these projectors.

More precisely, we shall consider the Abelian case: given a torus T of dimension n and a holomorphic and Hamiltonian action $\mu : T \times M \to M$, denote the moment map by $\Phi : M \to \mathfrak{t}^\vee$ (where $\mathfrak{t}$ and $\mathfrak{t}^\vee$ is the Lie algebra and co-algebra, respectively and we shall equivariantly identify $\mathfrak{t} \cong \mathfrak{t}^\vee \cong i\,\mathbb{R}^n$). The Hamiltonian action μ naturally induces an infinitesimal contact action of $\mathfrak{t}$ on X (see [14]); explicitly, if $\boldsymbol{\xi} \in \mathfrak{t}$ and

$$\boldsymbol{\xi}_M(m) := \mathrm{d}_{\mathrm{Id}}\mu_m(\boldsymbol{\xi})$$

is the corresponding Hamiltonian vector field on M (here $\mu_m : T \to M$ is given by fixing $m \in M$ and its differential at the identity $\mathrm{Id} \in T$ is denoted by $\mathrm{d}_{\mathrm{Id}}\mu_m : \mathfrak{t} \to T_m M$) then its contact lift $\boldsymbol{\xi}_X$ is

$$\boldsymbol{\xi}_X := \mathrm{d}_{\mathrm{Id}}\tilde{\mu}_{\cdot}(\boldsymbol{\xi}) = \boldsymbol{\xi}_M^\sharp - \langle \Phi \circ \pi, \boldsymbol{\xi} \rangle\, \partial_\theta, \tag{1}$$

where $\boldsymbol{\xi}_M^\sharp$ denotes the horizontal lift on X of a vector field $\boldsymbol{\xi}_M$ on M, and ∂_θ is the generator of the structure circle action on X (similarly as before $\mathrm{d}_{\mathrm{Id}}\tilde{\mu}_x : \mathfrak{t} \to T_x X$ denotes the differential of the map $\tilde{\mu}_x : T \to X$ at the identity).

Furthermore, suppose that the infinitesimal action (1) can be integrated to an action

$$\widetilde{\mu} : T \times X \to X$$

acting via contact and CR automorphism on X. We recall that, since X is the boundary of the strictly pseudoconvex domain

$$D = \{x \in L^\vee : |x|_h < 1\} \subset L^\vee$$

it has a Szegő kernel $\Pi(x, y)$ which projects $L^2(X)$ to the Hardy space $H(X)$ of square summable functions which are boundary values of holomorphic functions in D. Under these assumptions, there is a naturally induced unitary representation of T on the Hardy space $H(X) \subset L^2(X)$ given by

$$\hat{\mu}_t(s)(x) := s(\tilde{\mu}_{t^{-1}(x)}), \qquad s \in H(X),\ x \in X \text{ and } t \in T.$$

Hence $H(X)$ can be equivariantly decomposed over the irreducible representations of T:

$$H(X) = \bigoplus_{\boldsymbol{\nu} \in \widehat{T}} H(X)_{\boldsymbol{\nu}},$$

where $\widehat{T}$ is the collection of all irreducible representations of T which can be identified with $\mathbb{Z}^n$. If $\Phi(m) \neq 0$ for every $m \in M$, then each isotypical component $H(X)_{\boldsymbol{\nu}}$ is finite dimensional, see [16]. For each $\boldsymbol{\nu} \in \widehat{T}$ we denote by

$$\chi_{\boldsymbol{\nu}}(t) := t^{\boldsymbol{\nu}} = e^{i\,\langle \boldsymbol{\nu}, \boldsymbol{\theta} \rangle}, \qquad t \in T,\ \boldsymbol{\theta} \in \mathbb{R}^n \text{ and } \boldsymbol{\nu} \in \mathbb{Z}^n$$

the corresponding character; $\langle \cdot, \cdot \rangle$ denotes the standard scalar product on $i\,\mathbb{R}^n \cong \mathfrak{t}$. Here, we write t the matrix exponential $e^{i\,\boldsymbol{\theta}}$, where $\boldsymbol{\theta} \in \mathbb{R}^n$.

Fix $\boldsymbol{\nu} \in \widehat{T}$, let $\Pi_{\boldsymbol{\nu}} : L^2(X) \to H(X)_{\boldsymbol{\nu}}$ be the corresponding equivariant Szegő projection; we also denote by $\Pi_{\boldsymbol{\nu}} \in \mathcal{C}^\infty(X \times X)$ its distributional kernel:

$$\Pi_{\boldsymbol{\nu}}(x, y) := \sum_{j=1}^{d_{\boldsymbol{\nu}}} s_j^{\boldsymbol{\nu}}(x) \cdot \overline{s_j^{\boldsymbol{\nu}}(y)}$$

where $\{s_j^{\boldsymbol{\nu}}\}_{j=1}^{d_{\boldsymbol{\nu}}}$ is an orthonormal basis of $H(X)_{\boldsymbol{\nu}}$. It is given explicitly by the formula:

$$\Pi_{\boldsymbol{\nu}}(x, y) = \int_T \overline{\chi_{\boldsymbol{\nu}}(t)}\, \Pi(\tilde{\mu}_{t^{-1}}(x), y)\, \mathrm{dV}_T(t)$$

where Π is the Szegő kernel, dV_T is the Haar measure and $\chi_{\boldsymbol{\nu}}$ is the character of the representation $\boldsymbol{\nu}$. In [16], the dimensions of isotypes $H(X)_{k\boldsymbol{\nu}}$ and the behavior of $\Pi_{k\boldsymbol{\nu}}(x, x)$ are studied as $k \to +\infty$. In particular if $\mathbf{0} \notin \Phi(M)$ and Φ is transversal to the ray $i\mathbb{R}_+ \cdot \boldsymbol{\nu}$ the asymptotic of $\Pi_{k\boldsymbol{\nu}}(x, x)$ is rapidly decreasing whenever $x \notin X_{\boldsymbol{\nu}} := \pi^{-1}(M_{\boldsymbol{\nu}})$, where $M_{\boldsymbol{\nu}} := \Phi^{-1}(i\,\mathbb{R}_+ \cdot \boldsymbol{\nu})$, as k goes to infinity. In [16, Theorem 2] point-wise expansions and scaling asymptotics are considered at $x \in X_{\boldsymbol{\nu}}$.

If $\mathbf{0} \notin \Phi(M)$ then the dimension of isotype $H(X)_{k\boldsymbol{\nu}}$ is finite. There is a canonical map

$$\widetilde{\varphi_{k\boldsymbol{\nu}}} : X \to H_{k\boldsymbol{\nu}}(X)^\vee$$

given by

$$x \mapsto \widetilde{\varphi_{k\boldsymbol{\nu}}}(x) = \Pi_{k\boldsymbol{\nu}}(x, \cdot).$$

Let us consider an orthonormal basis of $H_{k\boldsymbol{\nu}}(X)$ given by $s_1^{k\boldsymbol{\nu}}, \dots, s_{d_{k\boldsymbol{\nu}}}^{k\boldsymbol{\nu}}$ with respect to the standard L^2 product. Now, as a consequence of the scaling asymptotics of the equivariant Szegő kernel we can investigate geometric properties of the map

$$\widetilde{\varphi_{k\boldsymbol{\nu}}} : X_{\boldsymbol{\nu}} \to H_{k\boldsymbol{\nu}}(X)^\vee, \quad x \mapsto \Pi_{k\boldsymbol{\nu}}(x, \cdot) = \left(s_1^{k\boldsymbol{\nu}}(x), \dots, s_{d_{k\boldsymbol{\nu}}}^{k\boldsymbol{\nu}}(x)\right),$$

as k goes to infinity, where we write the image $\widetilde{\varphi_{k\boldsymbol{\nu}}}(x)$ in components with respect to the dual basis.

An equivariant map $R \to S$ between two T-spaces descends to a map $R/T \to S/T$. Thus, $\widetilde{\varphi_{k\boldsymbol{\nu}}}$ is T-equivariant and it descends to a map $\varphi_{k\boldsymbol{\nu}}$ between the corresponding quotients:

$$\varphi_{k\boldsymbol{\nu}} : N_{\boldsymbol{\nu}} \to \mathbb{CP}^{d_{k\boldsymbol{\nu}}-1}_{\chi_{k\boldsymbol{\nu}}},$$

here we denote by $N_{\boldsymbol{\nu}}$ the conic transform $X_{\boldsymbol{\nu}}/T$, see [19]. Furthermore we shall recall in Sect. 2.2 from [19] that it has a Kähler structure $\eta_{\boldsymbol{\nu}}$. The target space is given by taking the quotient of $H_{k\boldsymbol{\nu}}(X)^\vee \setminus \{\mathbf{0}\}$ with respect to the action induced by the representation

$$s\left(\tilde{\mu}_{t^{-1}}(x)\right) = \chi_{k\boldsymbol{\nu}}(t)\, s(x)$$

where $\chi_{k\boldsymbol{\nu}} : T \to S^1$ is the character. Consider the symplectic form on $\mathbb{C}^{d_k} \setminus \{\mathbf{0}\} \cong H_{k\boldsymbol{\nu}}(X)^\vee \setminus \{\mathbf{0}\}$:

$$\tilde{\omega}_{d_k} := \frac{i}{2}\, \partial\overline{\partial} \log|\zeta|^2.$$

The restriction of the action of T on the unit sphere in $(\mathbb{C}^{d_k} \setminus \{\mathbf{0}\}, \tilde{\omega}_{d_k})$ has the same orbit of the standard S^1 action on S^{d_k-1}, $\tilde{\omega}_{d_k}$ descends to the Fubini Study form $\omega_{FS,k}$ on $\mathbb{CP}^{d_{k\boldsymbol{\nu}}-1}_{\chi_{k\boldsymbol{\nu}}}$.

Let us denote by T_x the stabilizer of a point $x \in X_{\boldsymbol{\nu}}$. The cardinality $|T_x|$ need not be constant on $X_{\boldsymbol{\nu}}$, but it does attain a generic minimal value $|T^{\boldsymbol{\nu}}_{\min}|$ on some dense open subset $X'_{\boldsymbol{\nu}}$, where $T^{\boldsymbol{\nu}}_{\min} \subseteq T$ is the stabilizer of each point in $X'_{\boldsymbol{\nu}}$ (Corollary B.47 of [12]). Then $T^{\boldsymbol{\nu}}_{\min}$ stabilizes every $x \in X_{\boldsymbol{\nu}}$. Suppose that $x \in X_{\boldsymbol{\nu}} \setminus X'_{\boldsymbol{\nu}}$ and consider the projection $p : X_{\boldsymbol{\nu}} \to N_{\boldsymbol{\nu}}$, then $p(x)$ is a singular point in $(N_{\boldsymbol{\nu}})_{\text{sing}} \subseteq N_{\boldsymbol{\nu}}$.

If k is sufficiently large, we shall prove that $\varphi_{k\boldsymbol{\nu}}$ is a asymptotically rescaled Kähler embedding, in a sense specified below. By making use of a variant of [16, Theorem 4] we can prove the following theorem, which is an analogue of embedding theorems in Kähler geometry, see [22]. For terminology concerning orbifolds, see Sect. 2.1.

Theorem 1.1 *Assume that $\boldsymbol{\nu} \neq \mathbf{0}$ is coprime, $\mathbf{0} \notin \Phi(M)$, $T^{\boldsymbol{\nu}}_{\min}$ is trivial and Φ is transversal to $i\,\mathbb{R}_+ \cdot \boldsymbol{\nu}$. Then there exists a subsequence k_n such that the maps $\varphi_{k_n\boldsymbol{\nu}}$ form an asymptotic sequence of Kähler embeddings of the Kähler orbifold $(N_{\boldsymbol{\nu}}, \eta_{\boldsymbol{\nu}})$ into $(\mathbb{CP}^{d_{k\boldsymbol{\nu}}-1}_{\chi_{k\boldsymbol{\nu}}}, \omega_{FS,k})$ in the following sense:*

(1) *$\varphi_{k_n\boldsymbol{\nu}}$ is an orbifold embedding if n is sufficiently large.*
(2) *Uniformly on compact subsets of the smooth locus $N_{\boldsymbol{\nu}} \setminus (N_{\boldsymbol{\nu}})_{\text{sing}}$, there exists $\epsilon > 0$ such that $\varphi_{k_n\boldsymbol{\nu}}$ satisfies:*

$$\varphi_{k\boldsymbol{\nu}}{}^*(\omega_{FS,k}) = \eta_{\boldsymbol{\nu}} + O(k^{-\epsilon}).$$

Theorem 2.1 is based on micro-local techniques that work in the almost complex symplectic setting, see [2] and [21]. Following similar idea as in [21], we study off-locus asymptotic of the projector $\Pi_{k\boldsymbol{\nu}}$ for arbitrary displacement as k goes to infinity, we refer to [16] and Sect. 2.3.

2 Motivations and preliminaries

Given a compact Hodge manifold (M, ω) the sections of the quantizing line bundle L can be used to define an embedding $\phi_k : M \to \mathbb{CP}^{d_k-1}$ into the complex projective space $(\mathbb{CP}^{d_k-1}, \omega_{FS})$ for k large, where $\omega_{FS,k}$ is the Fubini-Study form and d_k is the dimension of the space of holomorphic sections of the k-th tensor power of the line bundle L. These spaces of sections can be naturally identified with spaces of functions on X lying in the k-th Fourier component of the Hardy space. The map ϕ_k can be studied by using the Szegő kernel, in fact its k-th Fourier component defines, for k sufficiently large, a map

$$\widetilde{\phi}_k : X \to H_k(X)^* \setminus \{\mathbf{0}\}, \quad x \mapsto \Pi_k(x, \cdot),$$

which in turn define a map

$$\phi_k : M \to \mathbb{P}(H_k(X)^*).$$

It is natural to ask how the pull-back of ω_{FS} on M is related to ω. S. Zelditch and D. Catlin strengthen a previous result of Tian by proving that $(1/k)\,\phi_k{}^*(\omega_{FS})$ converges to ω in the smooth topology, see [3], [6] and [22]. Furthermore, B. Shiffman and S. Zelditch generalize this picture in the symplectic setting in [21]. In this setting, small displacements from a fixed $x \in X$ are conveniently expressed in Heisenberg local coordinates on X centered at x. A choice of Heisenberg local coordinates at x gives a meaning to the expression $x + \mathbf{v}$, where $\mathbf{v} \in T_{\pi(x)}M$ has sufficiently small norm. By using re-scaling asymptotic expansion

$$k^{-d}\,\Pi_k\left(x + \frac{\mathbf{v}}{\sqrt{k}},\, x + \frac{\mathbf{w}}{\sqrt{k}}\right) \sim \frac{1}{\pi^d} e^{\mathbf{v}\cdot\mathbf{w} - \frac{1}{2}(\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2)} \left[1 + \sum_{j \geq 1} k^{-j/2} a_j(x,\, \mathbf{v},\, \mathbf{w})\right]$$

where $a_j(x, \mathbf{v}, \mathbf{w})$ are polynomial functions in $\mathbf{v}$ and $\mathbf{w}$ and $\mathbf{v}, \mathbf{w} \in T_{m_x}M$, they prove the convergence of the rescaled pull-backs of the Fubini Study forms. Here, $\sim$ means "as the same asymptotic as". Furthermore in [9], we obtain an analogue Kodaira embedding theorem and Tian's almost-isometry theorem for compact quantizable pseudo-Kähler manifold making use of the asymptotic expansion of the Bergman kernels for $L^{\otimes k}$-valued $(0, q)$-forms.

In this paper we shall focus on Szegő kernel asymptotics along rays in weight space and we aim to obtain analogue of results we have discussed in this section, we shall now briefly review the literature. In [16] the author focuses on asymptotic expansions locally reflecting the equivariant decomposition of $H(X)$ over the irreducible representations of a compact torus. In [10] and [11] the focus was on respectively the case $G = U(2)$ and $G = SU(2)$ making use of the Weyl integration formula; in [18] local scaling asymptotics were generalized to the case of a connected compact Lie group G. Furthermore, we refer to [17] where asymptotics of the associated Szegő and Toeplitz operators were studied in detail for the case of Hamiltonian circle action lifting to the quantizing line bundle, see [7] for some related results concerning the torus action case. It is important to notice that, in general, the spaces $H(X)_{k\boldsymbol{\nu}}$ are not contained in any space of global sections $H^0(M, A^{\otimes k})$.

The analog of "quantization commutes with reduction" for torus action in this framework was first studied in [19]. In [8] we generalize the main theorem in [19] to compact "quantizable" pseudo-Kähler manifolds.

2.1 Definitions concerning orbifolds

We recall basic definitions we need about orbifolds, for a more precise discussion see [20], [1], [15] and references therein. We define the category $\mathcal{M}_s$ in the following way. The objects are pairs (U, G), where U is a connected smooth manifold and G is a finite group acting smoothly on U. A morphism $\Phi : (U, G) \to (U', G')$ is family of open embeddings $\varphi : U \to U'$ such that:

i) For each $\varphi \in \Phi$ there exists an injective group homomorphism $\lambda_\varphi : G \to G'$ satisfying

$$\varphi(g \cdot x) = \lambda_\varphi(g) \cdot \varphi(x), \qquad x \in U,\ g \in G.$$

ii) For $g \in G'$, $\varphi \in \Phi$, if $(g\,\varphi)(U) \cap \varphi(U) \neq \emptyset$, then $g \in \lambda_\varphi(G)$.
iii) For $\varphi \in \Phi$, we have $\Phi = \{g\,\varphi,\ g \in G'\}$.

Let X be a paracompact Hausdorff space and let $\mathcal{U}$ be a covering of X consisting of connected open subsets. We assume $\mathcal{U}$ satisfies the condition: for any $x \in U \cap V$, $U, V \in \mathcal{U}$, there is $W \in \mathcal{U}$ such that $x \in W \subset U \cap V$. Then we say that the topological space X has an orbifold structure if

i) for each $U \in \mathcal{U}$ there exists $(\widetilde{U}, G_U) \in \mathcal{M}_s$ and a ramified covering $(\widetilde{U}, G_U) \to U$ such that U is homeomorphic to $\widetilde{U}/G_U$, where $\widetilde{U}/G_U$ is endowed with the quotient topology;
ii) for any $U, V \in \mathcal{U}$, $U \subset V$, there exists a morphism $\varphi_{V,U} : (\widetilde{U}, G_U) \to (\widetilde{V}, G_V)$ which covers the inclusion $U \subset V$ and satisfies $\varphi_{WU} = \varphi_{WV} \circ \varphi_{VU}$ for any $U, V, W \in \mathcal{U}$ such that $U \subset V \subset W$.

Let us denote by $\pi_U : (\widetilde{U}, G_U) \to U$ the ramified covering. We recall that for each $p \in X$ one can always find local coordinates $x = (x_1, \dots, x_n)$ on $(\widetilde{U}, G_U)$, $\widetilde{U} \subseteq \mathbb{R}^n$, and $\tilde{p}$ with $\pi_U(\tilde{p}) = p$ such that $x(\tilde{p}) \equiv 0 \in \mathbb{R}^n$ is a fixed point of G_U and G_U acts linearly on $\mathbb{R}^n$. We call these coordinates *standard coordinates* at p. If $|G_U| > 1$ then we say that p is a singular point, otherwise we say that p is regular.

A *vector orbibundle* (E, p) on an orbifold X is an orbifold E together with a continuous projection $p : E \to X$ such that for $U \in \mathcal{U}$ there exists a G_U-equivariant lift $\widetilde{p} : \widetilde{E}_U \to \widetilde{U}$ defining a trivial vector bundle such that the diagram

$$\begin{array}{ccc} (\widetilde{E}_U, G_U^E) & \xrightarrow{\widetilde{p}} & (\widetilde{U}, G_U) \\ \downarrow & & \downarrow \\ E_U & \xrightarrow{p} & U \end{array}$$

commutes. As before, when we say that $\widetilde{p} : \widetilde{E}_U \to \widetilde{U}$ is G_U-equivariant we mean that there exists an injective group homomorphism $\lambda_p : G_U^E \to G_U$ such that

$$\widetilde{p}(g \cdot e) = \lambda_p(g) \cdot \widetilde{p}(e) \qquad e \in \widetilde{E}_U,\ g \in G_U^E;$$

if λ_p is a group isomorphism, we say that the vector orbibundle is *proper*.

A smooth section of a vector orbibundle (E, p) over X is a smooth map $s : X \to E$ such that $p \circ s = 1_X$, we denote the space of smooth sections as $\mathcal{C}^\infty(X, E)$. We recall that a *smooth map* $f : X \to Y$ between orbifolds is a continuous map between the underlying topological spaces such that for each $U \in \mathcal{U}$ there exists a local lift $(\widetilde{f}_U, \overline{f}_U)$ such that $\widetilde{f}_U : \widetilde{U} \to \widetilde{V}$ is smooth, $\overline{f}_U : G_U \to G_V$ is a group homomorphism, the diagram

$$\begin{array}{ccc} (\widetilde{U}, G_U) & \xrightarrow{\widetilde{f}} & (\widetilde{V}, G_V) \\ \downarrow & & \downarrow \\ U & \xrightarrow{f} & V \end{array}$$

commutes and $\widetilde{f}_U$ is $\overline{f}_U$-equivariant:

$$\widetilde{f}_U(g \cdot x) = \overline{f}_U(g)\widetilde{f}_U(x), \qquad g \in G_U,\ x \in \widetilde{U}.$$

The tangent orbibundle TX is defined by gluing together the bundles defined over the charts

$$(T\widetilde{U},\ G_U) \to (\widetilde{U},\ G_U),$$

where the action of G_U on $T_{\widetilde{x}}\widetilde{U}$ is induced by differentiating the action on $\widetilde{U}$, $\mathrm{d}g$. It has a natural structure of proper vector orbibundle. The smooth sections of it are called vector fields.

We can then define the *differential* of a smooth map $f : X \to Y$ to be the smooth map $\mathrm{d}f : TX \to TY$ such that locally for a given $x \in U$, $f(x) \in V$, and *standard coordinates* at x we have a commutative diagram

$$\begin{array}{ccc} (T_{\widetilde{x}}\widetilde{U}, G_U) & \xrightarrow{\mathrm{d}_{\widetilde{x}}\widetilde{f}} & (T_{\widetilde{f}(\widetilde{x})}\widetilde{V}, G_V) \\ \downarrow & & \downarrow \\ T_x U & \xrightarrow{\mathrm{d}_x f} & T_{f(x)} V \end{array}$$

where $\mathrm{d}_{\widetilde{x}}\widetilde{f}$ is $\overline{f}_U$-equivariant. We say that f is an *immersion* at $x \in X$ if $\mathrm{d}f$ is injective, so this means that there is a local lift $\widetilde{f}_U : \widetilde{U} \to \widetilde{V}$ such that both $\mathrm{d}_{\widetilde{x}}\widetilde{f}$ and $\overline{f}_U$ are injective.

2.2 Conic transforms for torus actions

We work with the same assumptions as in [19, Basic Assumption 1.1]. Let us recall that $M_{\boldsymbol{\nu}} = \Phi^{-1}(i\mathbb{R}_+ \cdot \boldsymbol{\nu})$ and for each $m \in M_{\boldsymbol{\nu}}$ we have

$$\Phi(m) = \varsigma(m)\, i\, \boldsymbol{\nu},$$

where $\varsigma : M \to \mathbb{R}$ is a suitable smooth positive function. Here, we always assume that the action of T on $X_{\boldsymbol{\nu}}$ is locally free. By [19], transversality is tantamount to T acting locally freely on $X_{\boldsymbol{\nu}}$ so that $N_{\boldsymbol{\nu}}$ is an orbifold.

In [19, Section 5.3] a Kähler orbifold structure for $N_{\boldsymbol{\nu}}$ is defined in the following way. Let us denote by $i\,\boldsymbol{\nu}^0$ the annihilator of $i\,\boldsymbol{\nu}$ in $\mathfrak{t}$. Consider the quotient $Y_{\boldsymbol{\nu}} := X_{\boldsymbol{\nu}}/\exp_T(i\,\boldsymbol{\nu}^0)$, then $Y_{\boldsymbol{\nu}}$ is as principal S^1 orbibundle over $N_{\boldsymbol{\nu}}$, let $\pi^{\boldsymbol{\nu}}$ be the projection. Adopting the same notation as in [19, Section 5.3], $\Phi \circ \pi$ descends to a smooth function $\overline{\Phi} : Y_{\boldsymbol{\nu}} \to \mathfrak{t}^\vee$; hence $\Phi^{\boldsymbol{\nu}} = \langle \Phi,\, i\,\boldsymbol{\nu}\rangle$ descends to a smooth function $\overline{\Phi}^{\boldsymbol{\nu}} : Y_{\boldsymbol{\nu}} \to \mathbb{R}$.

Let $\alpha^{X_{\boldsymbol{\nu}}} := j_{\boldsymbol{\nu}}^*(\alpha)$, where $j_{\boldsymbol{\nu}} : X_{\boldsymbol{\nu}} \to X$ is the inclusion. Then, $\alpha^{X_{\boldsymbol{\nu}}}$ is T-invariant, and by definition of $X_{\boldsymbol{\nu}}$ for any $\boldsymbol{\xi} \in \boldsymbol{\nu}^0$ we have $\iota((i\,\boldsymbol{\xi})_{X_{\boldsymbol{\nu}}})\alpha^{X_{\boldsymbol{\nu}}} = 0$, see [19]. Hence $\alpha^{X_{\boldsymbol{\nu}}}$ is the pullback of an orbifold 1-form $\alpha^{Y_{\boldsymbol{\nu}}}$ on $Y_{\boldsymbol{\nu}}$. Let us define

$$\beta_{\boldsymbol{\nu}} := \frac{\|\boldsymbol{\nu}\|^2}{\overline{\Phi}^{\boldsymbol{\nu}}}\, \alpha^{Y_{\boldsymbol{\nu}}}.$$

By [19, Equation (46)], we see that

$$\mathrm{d}\beta_{\boldsymbol{\nu}} = \|\boldsymbol{\nu}\|^2 \left[\frac{1}{\overline{\Phi}^{\boldsymbol{\nu}}}\, \mathrm{d}\alpha^{Y_{\boldsymbol{\nu}}} - \frac{1}{(\overline{\Phi}^{\boldsymbol{\nu}})^2}\, \mathrm{d}\overline{\Phi}^{\boldsymbol{\nu}} \wedge \alpha^{Y_{\boldsymbol{\nu}}} \right].$$

By [19, Corollary 16] and [19, Lemma 11]:

Proposition 2.1 *There exists a 2-form $\eta_{\boldsymbol{\nu}}$ on $N_{\boldsymbol{\nu}}$ such that $\mathrm{d}\beta_{\boldsymbol{\nu}} = 2\pi_{\boldsymbol{\nu}}^*(\eta_{\boldsymbol{\nu}})$ and a complex structure $J^{N_{\boldsymbol{\nu}}}$ so that $(N_{\boldsymbol{\nu}},\, J^{\boldsymbol{\nu}},\, \eta_{\boldsymbol{\nu}})$ is a Kähler orbifold.*

First, we introduce a decomposition of $T_m M$ when $m \in M_{\boldsymbol{\nu}}$, as in [16]. We adopt the following notation. Let $\mathfrak{l} \leq \mathfrak{t}$ be a Lie subalgebra of $\mathfrak{t}$, we denote by $\mathfrak{l}_M(m)$ the space of infinitesimal vector fields $\boldsymbol{\xi}_M(m)$ on M at m, $\boldsymbol{\xi} \in \mathfrak{l}$. Explicitly,

$$\mathfrak{l}_M(m) = \{\boldsymbol{\xi}_M(m) \in T_m M : \boldsymbol{\xi} \in \mathfrak{l}\}.$$

Similarly we denote by $\mathfrak{l}_X(x)$ the space of infinitesimal vector fields $\boldsymbol{\xi}_X(x)$ on X at x, $\boldsymbol{\xi} \in \mathfrak{l}$. Under the transversality assumption, for each $m \in M_{\boldsymbol{\nu}}$ we have

$$T_m M = T_m^{\mathrm{t}} M \oplus T_m^{\mathrm{v}} M \oplus T_m^{\mathrm{hor}} M, \tag{2}$$

where

$$T_m^{\mathrm{t}} M := J_m\,(i\,\boldsymbol{\nu}^0)_M(m), \qquad T_m^{\mathrm{v}} M := (i\,\boldsymbol{\nu}^0)_M(m)$$

and

$$T_m^{\mathrm{hor}} M := \left(T_m^{\mathrm{v}} M \oplus T_m^{\mathrm{t}} M\right)^{\perp_g}.$$

By the proof of [19, Lemma 11], see [19, Equation (48)], we have the following lemma.

Lemma 2.1 *Let $Q_{\boldsymbol{\nu}} : X_{\boldsymbol{\nu}} \to Y_{\boldsymbol{\nu}}$ be the projection. For each $x \in X_{\boldsymbol{\nu}}$, with $\pi(x) = m$, and for each $\mathbf{v}, \mathbf{w} \in T_m^{\mathrm{hor}} M$, we have*

$$Q_{\boldsymbol{\nu}}^*(\mathrm{d}\beta_{\boldsymbol{\nu}})_x(\mathbf{v}^\sharp,\, \mathbf{w}^\sharp) = \frac{2}{\varsigma(m)}\, \pi^*\omega_m(\mathbf{v}^\sharp,\, \mathbf{w}^\sharp).$$

2.3 Equivariant Szego kernels

Let $\Pi : L^2(X) \to H(X)$ be the Szegő projector, $\Pi(\cdot,\cdot)$ its kernel. By [4], Π is a Fourier integral operator with complex phase:

$$\Pi(x, y) = \int_0^{+\infty} e^{iu\psi(x,y)}\, s(x, y, u)\, \mathrm{d}u,$$

where the imaginary part of the phase satisfies $\Im(\psi) \geq 0$ and

$$s(x, y, u) \sim \sum_{j\geq 0} u^{d-j}\, s_j(x, y).$$

We shall also make use of the description of the phase ψ in Heisenberg local coordinates (see §3 of [21]).

The Heisenberg coordinates for the unit circle bundle X at a given point x, with $\pi(x) = m$, are defined in Section §1.2 in [21]. The Heisenberg coordinates play a crucial role in the description of the scaling asymptotics of the Szegő kernel in [21]. Here, we will only recall that to define them one needs to choose *preferred local coordinates* $z = (z_1, \ldots, z_d)$ centered at a point $m \in M$ and choose a *preferred local frame* for the line bundle L. Recall that the coordinates $(z_1, \ldots, z_d)$ are preferred at $m \in M$ if and only if

$$\sum_{j=1}^{d} \mathrm{d}z_j \otimes \mathrm{d}\overline{z}_j = (g - i\,\omega)_{|m},$$

where g is the Riemannian metric $\omega(\cdot,\ J\cdot)$. Furthermore, a preferred frame for $L \to M$ at a point $m \in M$ is a local frame e_L in a neighborhood of m such that

$$(\|e_L\|_{h_L})_{|m} = 1, \quad \nabla(e_L)_{|m} = 0,$$

and

$$\nabla^2(e_L)_{|m} = -(g + i\,\omega) \otimes (e_L)_{|m}.$$

The preferred frame and preferred coordinates together give us the Heisenberg local coordinates on the circle bundle X: a Heisenberg coordinate chart at a point x ($m = \pi(x)$) is a coordinate chart $\rho : U \approx V$ with $0 \in U \subset \mathbb{C}^d \times \mathbb{R}$ (denote by $p_{\mathbb{C}^d} : \mathbb{C}^d \times \mathbb{R} \to \mathbb{C}^d$ the projection) and $\rho(0) = x \in V \subset X$ of the form

$$\rho(\theta, z_1, \ldots, z_d) = e^{i\theta}\, a(z)^{-\frac{1}{2}}\, e_L^*(z),$$

where a is a real-valued smooth function on $p_{\mathbb{C}^d}(U)$, e_L is a preferred local frame at m and $(z_1, \ldots, z_d)$ are preferred coordinates centered at m.

For ease of notation, on a Heisenberg local chart, we write $x + (\theta,\ \mathbf{v})$; here $\theta \in (-\pi,\ \pi)$ and $\mathbf{v} \in B_{2d}(\mathbf{0},\ \delta)$, the open ball of center the origin and radius $\delta > 0$ in $\mathbb{C}^d \cong \mathbb{R}^{2d}$. If $\mathbf{v} \in T_m M$ and $\|\mathbf{v}\| < \epsilon$, we write $x + (0,\ \mathbf{v}) = x + \mathbf{v}$. Furthermore, accordingly with the decomposition (2), we write $\mathbf{v} = \mathbf{v}_{\mathrm{t}} + \mathbf{v}_{\mathrm{v}} + \mathbf{v}_{\mathrm{hor}}$.

Now, we need to state a variant, Theorem 2.1 below, of the main result concerning off-diagonal asymptotics of the equivariant Szegő kernel obtained in [16] and [5]. In fact, [16, Theorem 4] concerns off-locus asymptotics expansions along directions in the normal bundle $T^{\mathrm{t}}M$ but one can generalize them for arbitrary displacements as in [5]. More precisely, in [5], the author considers that a compact and connected Lie group G and a compact torus T act on M in a holomorphic and Hamiltonian manner, that the actions commute, and linearize to

L. Given a nonzero integral weight ν for T, the author considers the isotypical components associated to the multiples $k\,\nu_T$, $k \to +\infty$, and focuses on how their structure as G-modules is reflected by certain local scaling asymptotics on X. Thus, [5, Theorem 1.6] reduces to Theorem 2.1 below for trivial G.

Choose $\boldsymbol{\xi}_1 \in \mathrm{span}(\boldsymbol{\nu})$ so that $\|\boldsymbol{\xi}_1\| = 1$ and $\langle \boldsymbol{\xi}_1,\, \boldsymbol{\nu}\rangle = \|\boldsymbol{\nu}\|$. Hence, if $v_j = (\theta_j,\, \mathbf{v}_j) \in T_xX$, set

$$\mathbf{A}(v_1,\, v_2) := \frac{\theta_2 - \theta_1}{\|\Phi(m)\|}\, \boldsymbol{\xi}_{1,\,M}(m).$$

Define $E : T_xX_{\boldsymbol{\nu}} \times T_xX_{\boldsymbol{\nu}} \to \mathbb{C}$ by setting

$$E(v_1,\, v_2) := i\,\omega_m(\mathbf{A}(v_1,\, v_2),\, \mathbf{v}_{1\,h}) + \psi_2(\mathbf{v}_{1\mathrm{h}} - \mathbf{A}(v_1,\, v_2),\, \mathbf{v}_{2\,h})$$

where

$$\psi_2\,(\mathbf{v}_1,\, \mathbf{v}_2) = -i\,\omega_m(\mathbf{v}_1,\, \mathbf{v}_2) - \frac{1}{2}\|\mathbf{v}_1 - \mathbf{v}_2\|^2.$$

Given $v = (\theta,\, \mathbf{v}) \in T_xX$, for every $t \in T_x$, we write

$$v^{(t)} = \mathrm{d}_x\tilde{\mu}_t(v) = (\theta,\, \mathrm{d}_{m_x}\mu_t(\mathbf{v})).$$

Now, we can state the theorem concerning off diagonal scaling asymptotics for directions tangent to $X_{\boldsymbol{\nu}}$.

Theorem 2.1 *For each $x \in X_{\boldsymbol{\nu}}$, with $\pi(x) = m$, and $v_j = (\theta_j,\, \mathbf{v}_j) \in T_xX_{\boldsymbol{\nu}}$, $j = 1,\, 2$, such that $\mathbf{v}_j \in T_m^{\mathrm{hor}}M$ and $\|\mathbf{v}_i\| \le C\,k^{\epsilon}$, for sufficiently small ϵ, we have*

$$\begin{aligned} &\Pi_{k\boldsymbol{\nu}}\left(x + \frac{v_1}{\sqrt{k}},\, x + \frac{v_2}{\sqrt{k}}\right) \\ &\quad\sim \frac{e^{i\sqrt{k}\,(\theta_1-\theta_2)/\varsigma(m)}}{(\sqrt{2}\,\pi)^{n-1}\,\mathcal{D}(m)}\left(\|\boldsymbol{\nu}\|\cdot\frac{k}{\pi}\right)^{d+(1-n)/2}\cdot\left(\frac{1}{\|\Phi(m)\|}\right)^{d+1+(1-n)/2} \\ &\quad\cdot\left(\sum_{t\in T_x}\chi_{\boldsymbol{\nu}}(t)^k\, e^{\frac{1}{\varsigma(m)}E\left(v_1^{(t)},\, v_2\right)}\right)\cdot\left(1 + \sum_{j\ge 1} R_j(m,\, v_1,\, v_2)\,k^{-j/2}\right), \end{aligned} \tag{3}$$

where $\mathcal{D}$ is a smooth positive function on $M_{\boldsymbol{\nu}}$, R_j is polynomial in the v_i's of degree $\le 3\,j$ and parity j.

The re-scaled asymptotic expansions (3) of Theorem 2.1 will be the main tool for proving Theorem 1.1.

Let us compare the exponent $E\,(v_1,\, v_2)$ with the standard one appearing in [21]. For ease of notation, we set

$$\begin{aligned} F_k^{\mu}&\left(\left(\frac{\theta_1}{\sqrt{k}},\, \frac{\mathbf{v}_1}{\sqrt{k}}\right),\, \left(\frac{\theta_2}{\sqrt{k}},\, \frac{\mathbf{v}_2}{\sqrt{k}}\right)\right) := i\,\sqrt{k}\,(\theta_1 - \theta_2) + E\,(v_1,\, v_2) \\ &= i\,\sqrt{k}\left[\theta_1\left(1 - \omega\left(\frac{\boldsymbol{\xi}_{1,\,M}}{\|\Phi(m)\|},\, \mathbf{v}_{1,h}\right)\right) - \theta_2\left(1 - \omega\left(\frac{\boldsymbol{\xi}_{1,\,M}}{\|\Phi(m)\|},\, \mathbf{v}_{2,\,h}\right)\right)\right] \\ &\quad + \psi_2\left(\mathbf{v}_{1,\,h} + \theta_1\cdot\frac{\boldsymbol{\xi}_{1,\,M}}{\|\Phi(m)\|},\, \mathbf{v}_{2,\,h} + \theta_2\cdot\frac{\boldsymbol{\xi}_{1,\,M}}{\|\Phi(m)\|}\right) \end{aligned}$$

and we note that, when $T = T^1$ acts trivially on M, with constant moment map equal to 1, we recover the exponent of [21]:

$$F_k^{SZ}\left(\left(\frac{\theta_1}{\sqrt{k}},\ \frac{\mathbf{v}_1}{\sqrt{k}}\right),\ \left(\frac{\theta_2}{\sqrt{k}},\ \frac{\mathbf{v}_2}{\sqrt{k}}\right)\right) = i\sqrt{k}\,(\theta_1 - \theta_2) + \psi_2\,(\mathbf{v}_1,\ \mathbf{v}_2)\,.$$

Given θ sufficiently small, $x \in X_{\boldsymbol{\nu}}$ and $\mathbf{v} \in T_x X_{\boldsymbol{\nu}}$, we note that by Corollary 2.2 in [16] we get

$$\tilde{\mu}_{\exp_T\left(-\theta\,\tilde{\boldsymbol{\xi}}_1\right)}\,(x + \mathbf{v}) = x + \left(\theta + \theta\cdot\omega\left(\frac{\boldsymbol{\xi}_{1,\,M}}{\|\Phi(m)\|},\ \mathbf{v}\right),\ \mathbf{v} - \theta\cdot\frac{\boldsymbol{\xi}_{1,\,M}}{\|\Phi(m)\|}\right) + O\left(\|(\theta,\ \mathbf{v})\|^2\right)$$

and eventually, we note that

$$\begin{aligned}&F_k^{\mu}\left(\left(\frac{\theta_1}{\sqrt{k}},\ \frac{\mathbf{v}_1}{\sqrt{k}}\right),\ \left(\frac{\theta_2}{\sqrt{k}},\ \frac{\mathbf{v}_2}{\sqrt{k}}\right)\right)\\ &\quad= F_k^{SZ}\left(\tilde{\mu}_{\exp_T\left(-\frac{\theta_1}{\sqrt{k}}\,\tilde{\boldsymbol{\xi}}_1\right)}\left(x - \frac{\mathbf{v}_1}{\sqrt{k}}\right),\ \tilde{\mu}_{\exp_T\left(-\frac{\theta_2}{\sqrt{k}}\,\tilde{\boldsymbol{\xi}}_1\right)}\left(x - \frac{\mathbf{v}_2}{\sqrt{k}}\right)\right) + O\left(1/\sqrt{k}\right)\end{aligned}$$

where

$$\tilde{\boldsymbol{\xi}}_1 = \frac{\boldsymbol{\nu}}{\|\boldsymbol{\nu}\|\cdot\|\Phi(m)\|}.$$

3 Proof of Theorem 1.1

Let us denote the dimension of $H_{k\boldsymbol{\nu}}(X)$ by $d_{k\boldsymbol{\nu}}$ and consider an orthonormal basis of $H_{k\boldsymbol{\nu}}(X)$ given by $s_1^{k\boldsymbol{\nu}},\ \dots,\ s_{d_{k\boldsymbol{\nu}}}^{k\boldsymbol{\nu}}$ with respect to the standard L^2 product.

We distinguish two cases: the stabilizer of x is trivial or its cardinality is finite and equals $|T_x| > 1$. First, let us consider a neighborhood of a point x with $|T_x| = 1$.

Now, by definition, we can write in coordinates with respect to the dual basis

$$\widetilde{\varphi_{k\boldsymbol{\nu}}}\,:\,X_{\boldsymbol{\nu}} \to H_{k\boldsymbol{\nu}}(X)^{\vee},\quad x \mapsto \left(s_1^{k\boldsymbol{\nu}}(x),\ \dots,\ s_{d_{k\boldsymbol{\nu}}}^{k\boldsymbol{\nu}}(x)\right),$$

and we note that

$$\Pi_{k\boldsymbol{\nu}}(x,\ y) := \widetilde{\varphi_{k\boldsymbol{\nu}}}(x)\cdot\overline{\widetilde{\varphi_{k\boldsymbol{\nu}}}(y)}. \tag{4}$$

We shall follow [21] and we consider the 2-form Ω on a sufficiently small neighborhood of the diagonal $(\mathbb{C}^{d_{k\boldsymbol{\nu}}}\setminus\{0\}) \times (\mathbb{C}^{d_{k\boldsymbol{\nu}}}\setminus\{0\})$ given by

$$\Omega = \frac{i}{2}\,\mathrm{d}^{(1)}\mathrm{d}^{(2)}\log\zeta\cdot\overline{\eta}$$

where ζ and η are the variables respectively on the first and second component of

$$(\mathbb{C}^{d_{k\boldsymbol{\nu}}}\setminus\{0\}) \times (\mathbb{C}^{d_{k\boldsymbol{\nu}}}\setminus\{0\})$$

and the formal differentials $\mathrm{d}^{(1)}$ and $\mathrm{d}^{(2)}$ denote the exterior differentiation on the first and second factors, respectively. On the diagonal of $(\mathbb{C}^{d_k}\setminus\{0\}) \times (\mathbb{C}^{d_k}\setminus\{0\})$, we have

$$j^*(\Omega) = \tilde{\omega}_{d_k} := \frac{i}{2}\,\partial\overline{\partial}\log|\zeta|^2$$

where j is the diagonal embedding.

Furthermore define

$$\Phi_{k\boldsymbol{\nu}} = \widetilde{\varphi_{k\boldsymbol{\nu}}} \times \widetilde{\varphi_{k\boldsymbol{\nu}}} : X_{\boldsymbol{\nu}} \times X_{\boldsymbol{\nu}} \to \mathbb{C}^{d_{k\boldsymbol{\nu}}} \times \mathbb{C}^{d_{k\boldsymbol{\nu}}}, \quad \Phi_{k\boldsymbol{\nu}}(x, y) := (\widetilde{\varphi_{k\boldsymbol{\nu}}}(x), \widetilde{\varphi_{k\boldsymbol{\nu}}}(y)).$$

The proof of the following Lemma is left to the reader.

Lemma 3.1 $\Phi^*_{k\boldsymbol{\nu}}$ *commutes with* $\mathrm{d}^{(1)}$ *and* $\mathrm{d}^{(2)}$.

Thus we can pullback Ω via $\Phi_{k\boldsymbol{\nu}}$, by (4) and Lemma 3.1 we can easily obtain

$$\frac{1}{k}\Phi^*_{k\boldsymbol{\nu}}\Omega = \frac{i}{2k}\,\Phi^*_{k\boldsymbol{\nu}}\mathrm{d}^{(1)}\mathrm{d}^{(2)}\log\zeta\cdot\overline{\eta} = \frac{i}{2k}\,\mathrm{d}^{(1)}\mathrm{d}^{(2)}\Phi^*_{k\boldsymbol{\nu}}\log\zeta\cdot\overline{\eta} = \frac{i}{2k}\,\mathrm{d}^{(1)}\mathrm{d}^{(2)}\log\Pi_{k\boldsymbol{\nu}}.$$

Consider a small open neighborhood $U \subset Y_{\boldsymbol{\nu}} = X_{\boldsymbol{\nu}}/\exp_T(i\,\boldsymbol{\nu}^0)$ of a point y in $Y_{\boldsymbol{\nu}}$ and $Q_{\boldsymbol{\nu}}^{-1}(U) \subset X_{\boldsymbol{\nu}}$. Let $x \in X_{\boldsymbol{\nu}}$ such that $Q_{\boldsymbol{\nu}}(x) = y$. Since $Q_{\boldsymbol{\nu}} : X_{\boldsymbol{\nu}} \to Y_{\boldsymbol{\nu}}$ is a principal $\exp_T(i\,\boldsymbol{\nu}^0)$-bundle over $Y_{\boldsymbol{\nu}}$, we can assume U to be sufficiently small so that there exists a trivializing section $s : U \to Q_{\boldsymbol{\nu}}^{-1}(U)$ with $V := s(U) \subseteq Q_{\boldsymbol{\nu}}^{-1}(U)$. Let us denote by $T^{\mathrm{t}}_m M^\sharp$ the space of vectors in $T_x X$ which are the horizontal lift of vectors in $T^{\mathrm{t}}_m M$. By (2), where $x \in X_{\boldsymbol{\nu}}$ with $\pi(x) = m$ we have

$$T_x X = T^{\mathrm{t}}_m M^\sharp \oplus T_x X_{\boldsymbol{\nu}}$$

since, by formula (1), for every $\boldsymbol{\xi} \in i\,\boldsymbol{\nu}^0$, we have $\boldsymbol{\xi}_X(x) = \boldsymbol{\xi}^\sharp_M(m)$. Since the vectors in $T^{\mathrm{v}}_m M = (i\,\boldsymbol{\nu})^0_M(m)$ are tangent to the fibers of the principal bundle $Q_{\boldsymbol{\nu}} : X_{\boldsymbol{\nu}} \to Y_{\boldsymbol{\nu}}$ in $X_{\boldsymbol{\nu}} \subseteq X$, then s can be chosen so that a basis for $T_x V$ consists of a basis for $T^{\mathrm{hor}}_m M$ and $\boldsymbol{\xi}_{1,\,X}(x)$. Note that that $T^{\mathrm{hor}}_x M^\sharp$ projects isomorphically to $T_n N_{\boldsymbol{\nu}}$ where x is such that $p(x) = n$ and $p : X_{\boldsymbol{\nu}} \to N_{\boldsymbol{\nu}}$ is the projection.

Thus, let us consider $y \in U \subseteq Y_{\boldsymbol{\nu}}$ with $Q_{\boldsymbol{\nu}}(x) = y$. We compose with the diagonal map diag and we make use of Theorem 2.1, using Heisenberg coordinate, for $\mathbf{v}_1,\ \mathbf{v}_2 \in T^{\mathrm{hor}}_m M$ we get

$$\begin{aligned}\frac{1}{k}\widetilde{\varphi_{k\boldsymbol{\nu}}}^*\omega_{d_k}\big|_V &= \frac{i}{2k}\,\mathrm{diag}^*\mathrm{d}^{(1)}\mathrm{d}^{(2)}\log\Pi_{k\boldsymbol{\nu}}\left(x + \left(\frac{\theta_1}{\sqrt{k}}, \frac{\mathbf{v}_1}{\sqrt{k}}\right), x + \left(\frac{\theta_2}{\sqrt{k}}, \frac{\mathbf{v}_2}{\sqrt{k}}\right)\right)\\ &= \frac{i}{2k}\,\mathrm{diag}^*\mathrm{d}^{(1)}\mathrm{d}^{(2)}\left[\frac{1}{\varsigma(m_x)}\,F^\mu_k\left(\left(\frac{\theta_1}{\sqrt{k}}, \frac{\mathbf{v}_1}{\sqrt{k}}\right), \left(\frac{\theta_2}{\sqrt{k}}, \frac{\mathbf{v}_2}{\sqrt{k}}\right)\right)\right] + O\left(1/\sqrt{k}\right).\end{aligned}$$

Thus, by the description of local coordinates on $V \subset X_{\boldsymbol{\nu}}$ for the circle bundle $\pi_{\boldsymbol{\nu}} : Y_{\boldsymbol{\nu}} \to N_{\boldsymbol{\nu}}$ at the end of Sect. 2.3, we obtain

$$\begin{aligned}\frac{1}{k}\widetilde{\varphi_{k\boldsymbol{\nu}}}^*\omega_{d_k}\big|_V &= \frac{i}{2k}\,\mathrm{diag}^*\mathrm{d}^{(1)}\mathrm{d}^{(2)}\left[\frac{1}{\varsigma(m_x)}\,\psi_2\left(\mathbf{v}_1,\ \mathbf{v}_2\right)\right] + O\left(1/\sqrt{k}\right)\\ &= \frac{1}{\varsigma(m_x)}\,\pi^*\omega|_V + O\left(1/\sqrt{k}\right).\end{aligned}$$

Since in Heisenberg local coordinates ω is the standard form on $\mathbb{C}^{d-r+1}$ (where r is the dimension of the torus T), see [21], and by the discussion in Sect. 2.2, see Lemma 2.1, we get (2) in Theorem 1.1. Thus, we also get (1) since, as a consequence of (2) we get that $\mathrm{d}\varphi_{k\boldsymbol{\nu}}$ is injective for k large.

Now, suppose that the stabilizer $|T_x| > 1$, hence n is a singular point in $N_{\boldsymbol{\nu}}$ and fix x such that $p(x) = n$ where $p : X_{\boldsymbol{\nu}} \to N_{\boldsymbol{\nu}}$ is the projection. Since $N_{\boldsymbol{\nu}}$ is an orbifold there exists a local chart $(T_x,\ \widetilde{U_n})$. By the Slice Theorem, see Appendix B in [12], a neighborhood of the orbit of x is equivariantly diffeomorphic to a neighborhood of the zero section of the associated principal bundle $T \times_{T_x} D$ where D is a sufficiently small neighborhood of the origin in $T^{\mathrm{hor}}_m M_{\boldsymbol{\nu}}$ with $\pi(x) = m$.

Now, $\Pi_{k\mathbf{\nu}}(x,\ x)$ can be zero for some k even when k is large. To solve this problem, we note that there exists a subsequence k_n such that for each $x \in X_{\mathbf{\nu}}$

$$\sum_{t \in T_x} \chi_{k_n \mathbf{\nu}}(t) \neq 0.$$

In fact, we can choose k_n to be divisible by the least common multiple of $|T_x|$, for $x \in X_{\mathbf{\nu}}$, (there are only finitely many possible stabilizers). As a consequence of

$$\widetilde{\varphi_{k\mathbf{\nu}}}(\tilde{\mu}_{t^{-1}}(x)) = \chi_{k\mathbf{\nu}}(t) \sum_{j=1}^{d_{k\mathbf{\nu}}} s_j^{k\mathbf{\nu}}(x)(s_j^{k\mathbf{\nu}}) = \hat{\mu}_t\left(\widetilde{\varphi_{k\mathbf{\nu}}}(x)\right),$$

we note $\varphi_{k\mathbf{\nu}}$ is a T_x-equivariant map from an open neighborhood $\widetilde{U_n} \cong D$ of $\widetilde{n}_x$ to $\mathbb{CP}^{d_{k\mathbf{\nu}}-1}_{\chi_{k\mathbf{\nu}}}$, where in the orbifold local chart $\pi_{U_n} : (\widetilde{U_n},\ G_{U_n}) \to U_n$ we have that $\pi_{U_n}(\widetilde{n}_x) = n_x$.

Eventually, we shall prove that the maps $\varphi_{k\mathbf{\nu}}$ are embeddings for k sufficiently large. In order to prove it, we study long-range and short-range injectivity as in [21]. Thus, given $x,\ y \in X_{\mathbf{\nu}}$ in the range where $\sqrt{k}\,\mathrm{dist}_X(T \cdot x,\ y)$ tends to infinity as $k \to +\infty$, it follows in a similar way as in [21] that $\varphi_{k\mathbf{\nu}}(n_x) \neq \varphi_{k\mathbf{\nu}}(n_y)$ as an easy consequence of [16].

For short-range injectivity we consider $\mathbf{v}_k \in T_m^{\mathrm{hor}} M^\sharp$, where $m = \pi(x)$ and $x \in X_{\mathbf{\nu}}$, such that $0 \neq \|\mathbf{v}_k\| \leq C$ and assume

$$\varphi_{k\mathbf{\nu}}\left(n_{x+\mathbf{v}_k/\sqrt{k}}\right) = \varphi_{k\mathbf{\nu}}(n_x) \tag{5}$$

where recall that $T_x^{\mathrm{hor}} M^\sharp$ projects isomorphically to $T_n N_{\mathbf{\nu}}$ where x is such that $p(x) = n_x$ and $p : X_{\mathbf{\nu}} \to N_{\mathbf{\nu}}$ is the projection. Thus, (5) is equivalent to

$$\frac{\widetilde{\varphi_{k\mathbf{\nu}}}(x)}{\|\widetilde{\varphi_{k\mathbf{\nu}}}(x)\|} \wedge \frac{\widetilde{\varphi_{k\mathbf{\nu}}}(x + \mathbf{v}_k/\sqrt{k})}{\|\widetilde{\varphi_{k\mathbf{\nu}}}(x + \mathbf{v}_k/\sqrt{k})\|} = 0. \tag{6}$$

Thus, by (4), Eq. (6) is equivalent to

$$\left| \frac{\Pi_{k\mathbf{\nu}}(x,\ x + \mathbf{v}_k/\sqrt{k})}{\sqrt{\Pi_{k\mathbf{\nu}}(x,\ x)} \cdot \sqrt{\Pi_{k\mathbf{\nu}}(x + \mathbf{v}_k/\sqrt{k},\ x + \mathbf{v}_k/\sqrt{k})}} \right| = 1. \tag{7}$$

Define, for a sufficiently small ϵ, $f : [-\epsilon,\ 1+\epsilon] \to \mathbb{R}$

$$f_k(\tau) = \frac{|\Pi_{k\mathbf{\nu}}(x,\ x + \tau\, \mathbf{v}_k/\sqrt{k})|^2}{\Pi_{k\mathbf{\nu}}(x,\ x) \cdot \Pi_{k\mathbf{\nu}}(x + \tau\, \mathbf{v}_k/\sqrt{k},\ x + \tau\, \mathbf{v}_k/\sqrt{k})}$$

and note that $f_k(0) = 1$ and, by (7), $f_k(1) = 1$ (the denominator is non-zero provided we pass to a subsequence). Furthermore, by (4) and the Cauchy-Schwartz inequality, $0 \leq f_k \leq 1$. Hence we have $f_k'(0) = 0$ (note that f is defined on $[-\epsilon,\ 1+\epsilon]$). Thus, there exists $\tau_k \in (0,\ 1)$ such that $f_k''(\tau_k) = 0$.

Eventually, starting with (7) and expanding in decreasing half-integer powers of k,

$$f_k(\tau) = \sum_{t \in T_x} \chi_{k\mathbf{\nu}}(t) \cdot e^{-\frac{\tau^2}{2\,\varsigma(m)}\,\|\mathbf{v}_k\|^2}\left[1 + k^{-1/2} R(\tau,\ \mathbf{v}_k)\right]$$

where $R_3(\tau,\ \mathbf{v}_k) = O(1)$. Note that, since $f_k(1) = 1$, we get $\|\mathbf{v}_k\|^2 = O(k^{-1/2})$. Thus, we obtain

$$f_k''(\tau) = (-1 + O(1)) \sum_{t \in T_x} \chi_{k\mathbf{\nu}}(t) \cdot \|\mathbf{v}_k\|^2.$$

By evaluating at τ_k, we get $\mathbf{v}_k = \mathbf{0}$.

Acknowledgements I express my gratitude to the referee for valuable suggestions and corrections, which have contributed to the improvement of this paper. I would like to thank the referee of a previous version of this paper for providing several comments and for pointing out a mistake in a previous version of this paper that led to a change in the statement of Corollary 1.1. I would also like to express my gratitude to Prof. R. Paoletti for suggesting the study of immersion problems. This project was initiated during my postdoctoral fellowship at the National Center for Theoretical Sciences in Taiwan and was completed at the University of Milano-Bicocca. I would like to thank both institutions for their support. Additionally, I extend my thanks to the Mathematics Institute of Universität zu Köln for their hospitality throughout my scholarship: the author is supported by INdAM (Istituto Nazionale di Alta Matematica) foreign scholarship.

Funding Open access funding provided by Università degli Studi di Milano - Bicocca within the CRUI-CARE Agreement.

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