SCUOLA DI DOTTORATO
UNIVERSITÀ DEGLI STUDI DI MILANO-BICOCCA

Department of Mathematics and its Applications

PhD program in Mathematics   Cycle XXXV

Central limit theorems for polynomial chaos and fluctuations for 2d directed polymers

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ACADEMIC YEAR 2021/2022

Università degli Studi di Milano-Bicocca

Department of Mathematics and its Applications

Joint PhD Program in Mathematics
Milano Bicocca - Pavia - INdAM
XXXV cycle

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Academic year 2021 – 2022

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Acknowledgments

At the end of this intense journey, and after all these years of dealing with mathematics, I am increasingly convinced that those who believe that this subject is “cold” and has little to do with human relationships are strongly mistaken.

Human connections are vital to mathematics, mathematics itself is made up of relationships. I fully realized this after the Covid pandemic broke out with impeccable timing, just a couple of months into my doctoral program. It was only when I left my makeshift bedroom office and resumed the human connections that I noticed how incredibly my research and work quality had improved. This was not only due to the possibility of discussing mathematics with others in person, but also and above all because it was possible to fully immerse oneself in academic life again.

During the past years, I had the opportunity to travel across Europe and visit many mathematics departments. I met several people from all around the world, with whom I built professional and often even friendly relationships. I listened to different stories and experiences, frequently learning from them alternative approaches to tackle mathematics and, more generally, the daily challenges that come with being a researcher. I exchanged my perspectives and ideas as well, learning that while sharing can be daunting, it always leads to enrichment for oneself and others.

Each one of these relationships contributed to my PhD experience and, going beyond that, to my personal story, too. If I had not cultivated all these relationships over the last three years, I would not be who I am now and I would not be where I am now. For this reason, it feels only natural that this doctoral thesis should open with the acknowledgments I wish to dedicate to some of the people with whom I shared this path.

First and foremost, I wish to express my deepest gratitude to my supervisor, Professor Francesco Caravenna. Over these three years, he has been an invaluable guide and a constant source of support. I am grateful for his impeccable expertise and for all the knowledge he has imparted to me. I would like to thank him for all the attention he has shown me, for his patience when he had to wait for me, for always finding the most delicate ways to put me at ease, both in the research field and from a human perspective, and for being available to offer his point of view on anything from mathematical calculations to life choices. Without him, I am certain that this doctoral path would have been very different, and I would not be as happy for the work accomplished as I am now. For this reason, I will always be grateful to him.

Furthermore, I am truly thankful to the referees of my PhD thesis, Professor Giovanni Peccati and Professor Nikos Zygouras, for accepting to read and review my work, providing me with valuable comments and feedback to enhance its quality.

I also would like to acknowledge Professor Jean–Dominique Deuschel for the insightful discussions we had during my research period at the Technische Universität in Berlin. Thanks to his helpful input, I was able to deepen my understanding and approach my research from new perspectives.

I owe a debt of gratitude to my previous Bachelor's and Master's thesis supervisors, Professor Alfredo Marzocchi, Professor Stefano Bonaccorsi, and Professor Delio Mugnolo, for their constant guidance and support throughout the years. Despite not talking about mathematics every day anymore (at least, for the moment), I know I can always count on their availability and they remain important reference points for me in the academic sphere.

Moreover, I would like to thank the Probability research group at the University of Milano-Bicocca. In particular, I would like to extend a special thank you to two dear friends and colleagues of mine, Professor Maurizia Rossi and Doctor Nicola Turchi, for their reliable support and advice whenever I needed it. Their friendship, sincere words of encouragement, and belief in me and in my abilities have made a significant difference in my work and have given me an extra boost to face challenges and make important decisions.

I would like to dedicate a warm thanks to all the fellow PhD students and postdocs on the second floor of U5 building, who shared with me the daily life inside and outside the department, from organizing and attending our "Insalate di Matematica" seminars, to enjoying all the shared lunches and tea breaks with endless cookies and sweets. All the people I met in that corridor on the second floor and the wonderful and deep friendships grown over the years have been an integral part of my doctoral experience and contributed to making it much better. Doing mathematics is of course fun, but doing it with a group of supportive friends who celebrate each other's successes is definitely far better!

I also want to sincerely thank all my friends, mathematicians and non mathematicians alike, who are spread across Italy and Europe. Although we lead different lives and careers, it is always wonderful to rely on each other and share the important moments of our lives.

I want to give a heartfelt thanks to my entire family, in particular to my parents, my brother Alessandro, and my nonna Marisa, who have always been my greatest supporters and have driven me throughout Europe every time I had to move to a new University.

Finally, I want to express a sincere thank you, perhaps the most important one, to Luigi, whom I met right at the beginning of this experience and with whom I shared the joys and challenges of the PhD and beyond. I thank him for his constant support, for showing me a different way of approaching mathematics, for allowing me to express myself sincerely, and for so much more.

Francesca

Declarations

I declare that the work presented in this Ph.D. thesis is enirely my own, except where states otherwise by reference, and have been accomplished under the supervision of Prof. Francesco Caravenna, for the Degree of Doctor of Philosophy in Mathematics. The contents discussed in Chapters 2, 3, 4 and 5 are based on the following articles, published or soon–to–be–published in collaboration with Prof. Francesco Caravenna and Prof. Maurizia Rossi:

[**CC22**] F. Caravenna and F. Cottini. Gaussian limits for subcritical chaos. *Electron. J. Probab.,* 27:81, pp. 1–35, 2022.

[**CCR22+**] F. Caravenna, F. Cottini and M. Rossi. Quasi–critical fluctuations for $2d$ directed polymers. *In preparation.*

Introduction

The core of this work is focused on an important model in statistical physics and probability theory named *directed polymer in random environment*. In chemistry, a *polymer* is a molecule consisting of several smaller units, called *monomers*, which are linked together to form a chain. It is interesting to analyze which configurations a polymer may assume, depending on its features and on its interaction with the *random environment*, also known as *disorder*. From a mathematical perspective, the configurations of a polymer are described by the trajectories of stochastic processes defined on the lattice $\mathbb{N} \times \mathbb{Z}^d$, while the disorder is modeled by a realization of independent random variables indexed by the points in $\mathbb{N} \times \mathbb{Z}^d$.

Concerning our case, the directed polymer in random environment is defined as a disorder perturbation of the *simple symmetric random walk on* $\mathbb{Z}^d$. The growing interest that the directed polymer has attracted in recent years is also due to its connection to singular stochastic partial differential equations (PDEs). In fact, through its partition function the directed polymer provides a *discretization* of the solution of the Stochastic Heat Equation (SHE) with multiplicative noise and of the Kardar–Parisi–Zhang (KPZ) Equation. A robust solution theory for the latter equations has only been developed in the spatial dimension $d = 1$ very recently and is still missing for $d \geq 2$.

In view of the link with the continuum framework of stochastic PDEs, a natural way of investigating the directed polymer is to study suitable scaling limits for its partition function: as the polymer length grows to infinity, the disorder needs to be properly rescaled. Since the polymer partition function solves a discretized version of the Stochastic Heat Equation, which still remains poorly understood in high dimensions, proving the existence of such scaling limits can provide the first step for a candidate solution. Even though many results have been achieved in this direction, there are still many open problems giving rise to intense and challenging research.

This Ph.D. thesis focuses on the special case when the spatial dimension is $d = 2$, which is *critical* for our model and involves subtle and interesting phenomena, as we will discuss. Our work fits into the research carried out in recent years by Caravenna, Sun and Zygouras [**CSZ17b, CSZ19a, CSZ19b, CSZ20, CSZ21+**], who have obtained several convergence results related to the 2d polymer partition functions, the Stochastic Heat Equation and the KPZ Equation.

Our first main result is a novel and more elementary approach to prove limit theorems for random variables with special structures, namely *polynomial and Wiener chaos*, which include the polymer partition functions. Not only are we able to recover previous results with simpler proofs: our new criterion also allows us to obtain new limits and to investigate less known regimes, with more effective tools that are better suited for extensions. The general convergence result for polynomial and Wiener chaos is stated in Chapter 2 (see Theorems 2.2 and 2.9), while the applications to directed polymers in the so-called *subcritical regime* are presented in Chapter 3 (see Theorems 3.2, 3.7, 3.13 and 3.15) and in Chapter 4 (Theorems 4.1 and 4.2).

Our second main result is an investigation of directed polymers in a new *quasi-critical regime*, which interpolates between the subcritical and the critical regimes: in Chapter 5 we present new Gaussian fluctuations results in this setting, see Theorem 5.1. A fundamental tool to face the novel and more subtle framework is given by fine estimates for the high moments of the polymer partition function, see Proposition 5.4.

The techniques that we exploit are mostly probabilistic, including *second moment estimates for polynomial and Wiener chaos* (Chapters 2, 3, 4, 5), *Lindeberg principles* and the *hypercontractivity* for polynomial chaos (Chapter 2), *coarse–graining* procedures (Chapters 3, 4, 5), *renewal theory* (Chapter 5) and *operator bounds based on functional inequalities* (Chapter 5).

The thesis is organized as follows. In Chapter 1, we define the d-dimensional directed polymer in random environment and recall its main properties, then we introduce the Stochastic Heat Equation and the KPZ Equation by emphasizing their connection with this discrete system. We discuss motivations for the key problems and we present an overview of the related literature. Our novel contributions are presented and motivated in Subsections 1.4.1, and 1.4.2.

Chapters 2, 3, 4 and 5 contain the original results obtained in this Ph.D. thesis, as well as some refinements of known results.

CHAPTER 1

Directed polymers and stochastic PDEs

The *d-dimensional directed polymer in random environment*, d-DPRE for short, is a probabilistic model which describes the shape of a *polymer* (namely a chain of smaller units called *monomers*) affected by any impurities it might encounter in its environment. The d-DPRE belongs to the family of so-called *disordered systems* studied in Statistical Mechanics (see, for instance, [**Bov06**]) in order to analyze the interaction between a *pure/homogeneous system* (which depicts the polymer configurations, in this case) and a *disorder* (the impurities) which can affect the model.

First introduced by Huse and Henley in the physics literature [**HH85**] to examine interfaces of the Ising model with random impurities and then mathematically reformulated by Imbrie and Spencer [**IS88**], the d-DPRE has significantly attracted interest in recent years, also because of its close connection to some stochastic partial differential equations (SPDEs), as we will discuss more in detail further in this chapter. We refer to [**Com17**] for an extensive and recent review of the d-DPRE model.

Throughout this thesis, we will mainly deal with the discrete d-DPRE. The *pure system* is the simple random walk $S = (S_n)_{n \geq m}$ on $\mathbb{Z}^d$ starting from $z \in \mathbb{Z}^d$ at initial time $m \in \mathbb{N}_0 := \{0, 1, 2, 3, \ldots\}$, defined on the probability space $\Omega := (\mathbb{Z}^d)^{\mathbb{N}_0}$ equipped with the cylindric σ-algebra $\mathcal{F}$ and probability measure $\mathrm{P}_{m,z}$. In the sequel, we denote by $\mathrm{E}_{m,z}$ the expectaction with respect to $\mathrm{P}_{m,z}$ and we write $\mathrm{P} := \mathrm{P}_{0,0}$, $\mathrm{E} := \mathrm{E}_{0,0}$. Intuitively, trajectories of the random walk S represent polymer configurations.

On the other hand, the *environment*, or *disorder*, is described by a family of independent and identically distributed random variables $\omega = (\omega(n,x))_{n \in \mathbb{N}, x \in \mathbb{Z}^d}$, independent of S, defined on a probability space $(\widetilde{\Omega}, \mathcal{G}, \mathbb{P})$ such that

$$\mathbb{E}[\omega(n,x)] = 0\,, \qquad \mathbb{E}[\omega(n,x)^2] = 1\,, \qquad \lambda(\beta) := \log \mathbb{E}[e^{\beta\omega(n,x)}] < \infty \quad \forall\, \beta > 0\,,$$

where we denoted by $\mathbb{E}$ the expectation with respect to the law $\mathbb{P}$. Here, we follow the convention such that $\mathbb{N} := \{1, 2, 3, \ldots\}$.

Fixed a scale parameter $N \in \mathbb{N}$, a starting time-space point $(m, z) \in \{0, \ldots, N\} \times \mathbb{Z}^d$ and an inverse temperature $\beta > 0$ which also represents the disorder strength of the model, the law of the d-DPRE is defined as a Gibbs perturbation of $\mathrm{P}_{m,z}$ by

$$\mathrm{dP}^{\beta}_{N,m,z}(S) := \frac{1}{Z^{\beta}_N(m,z)}\, e^{\sum_{n=m+1}^{N} (\beta\omega(n,S_n) - \lambda(\beta))}\, \mathrm{dP}_{m,z}(S)\,, \tag{1.1}$$

where the normalizing constant

$$Z_N^\beta(m,z) := \mathrm{E}_{m,z}\big[\, e^{\sum_{n=m+1}^N (\beta\omega(n,S_n)-\lambda(\beta))}\,\big] = \mathrm{E}\big[\, e^{\sum_{n=m+1}^N (\beta\omega(n,S_n)-\lambda(\beta))}\,\big|\, S_m = z\,\big] \tag{1.2}$$

is called *partition function* of the d-DPRE and it captures some of the essential information about the model. It has such a physical significance since starting from its logarithm, namely the *free energy* of the system, we can derive the main thermodynamic quantities (for more details, see [**Bov06**, Chapter 2]).

The partition function of the d-DPRE is *the main object of interest in our work*: throughout this thesis, we will focus on the case $d = 2$ and we will investigate the asymptotic behaviour of $Z_N^\beta(m,z)$ when the scale parameter N, namely the polymer length, grows to infinity.

Notice that $\big(Z_N^\beta(m,z)\big)_{(m,z)\in\{0,\dots,N\}\times\mathbb{Z}^d}$ is a family of random variables *with respect to the law* $\mathbb{P}$ *of disorder* ω and it is stationary for $z \in \mathbb{Z}^2$ (for fixed $m \in \{0,\dots,N\}$), thanks to the translation invariance of the simple random walk S. To simplify the notation, we denote $Z_N^\beta(z) := Z_N^\beta(0,z)$ and $Z_N^\beta := Z_N^\beta(0) = Z_N^\beta(0,0)$. We stress that the constant $-\lambda(\beta)$ in (1.2) does not significantly affect the randomness of $Z_N^\beta(m,z)$, since it can be factorized to yield the quantity $e^{-(N-m)\lambda(\beta)}$, which is purely deterministic. However, it is convenient because it provides a normalization such that $\mathbb{E}\big[Z_N^\beta(m,z)\big] = 1$ for any $(m,z) \in \{0,\dots,N\}\times\mathbb{Z}^d$. Moreover, it is easy to show that $(Z_N^\beta)_{N\in\mathbb{N}}$ is a martingale with respect to the filtration $(\mathcal{G}_N)_{N\in\mathbb{N}} := \sigma\big(\omega(i,x) \,:\, i \le N\,,\, x \in \mathbb{Z}^d\big)$.

Before presenting the key problem we aim to study, in the following two sections we briefly recall some properties of the d-DPRE and some related models from two different perspectives, which also provide interesting motivations for our research.

1.1. The directed polymer as a disordered system

From a mathematical perspective, the d-DPRE models a perturbation of the simple random walk given by the presence of a random disorder. One should think of each $\omega(n,S_n)$ in the definition (1.1) as reward or penalty (according to its sign and amplified by the disorder strength $\beta > 0$) which the simple random walk collects along its path (see Figure 1). Therefore, the law $\mathrm{P}_{N,m,z}^\beta$ favors the trajectories such that the total "energy" $\sum_{n=m+1}^N(\beta\omega(n,S_n)-\lambda(\beta)) > 0$ and penalizes those with $\sum_{n=m+1}^N(\beta\omega(n,S_n)-\lambda(\beta)) < 0$. Clearly, when $\beta = 0$ the disorder is absent and the law of the d-DPRE coincides with $\mathrm{P}_{m,z}$, thus all trajectories of length $N-m$ have the same probability equal to $(2d)^{-(N-m)}$. In this case, there is an entropic gain due to the fact that the polymer has access to any possible space configuration. On the other hand, when β grows until formally reaching $\beta = \infty$, the law $\mathrm{P}_{N,m,z}^\beta$ degenerates into a measure concentrated in a single trajectory, that is the one which maximizes the energy $\beta\sum_{n=m+1}^N \omega(n,S_n)$ and thus targets high values of the disorder ω. It is then clear that there is a competition

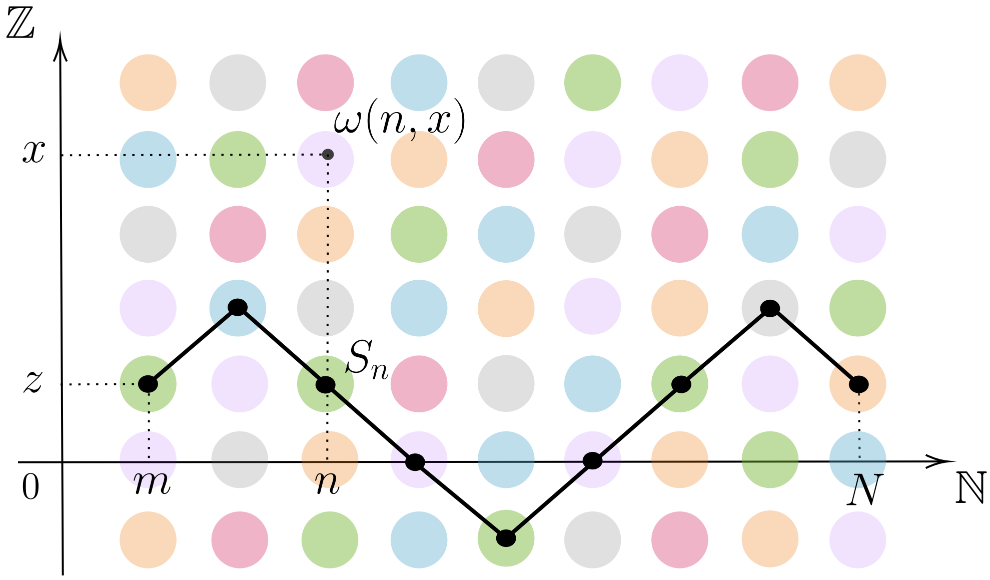

FIGURE 1. Graphic representation for $d = 1$ depicting how the disorder affects one trajectory of the simple random walk with starting point $(m, z) \in \mathbb{N} \times \mathbb{Z}$. Each disorder random variable $\omega(n, x)$ is described by the corresponding dot placed in $(n, x) \in \mathbb{N} \times \mathbb{Z}$ and different colors represent different values attained by the disorder. According to the new law $\mathrm{P}^{\beta}_{N,m,z}$ (see (1.1), the probability of the simple random walk following the trajectory in the picture will depend on those disorder random variables encountered along the path.

between entropy and energy, where the former is maximized when $\beta = 0$ and the latter when $\beta = \infty$.

A key question in the study of disordered systems such as the d-DPRE is to understand whether and how the addition of disorder changes the qualitative behaviour of the pure model. Borrowing terminology from the physics literature, if an arbitrarily small amount of disorder is able to substantially modify the nature of the pure model, such as its large-scale properties, we call the system *disorder relevant*. Otherwise, we talk about *disorder irrelevance* when the disorder has to be strong enough to alter the pure model. In the physiscs literature, a powerful tool to determine the relevance of a system is the *Harris criterion* [**Har74**], whose approach is based on renormalization/coarse graining transformations of models. Denoting by d_{eff} the effective dimension and by ν the correlation length exponent of the pure system, this criterion predicts that the disorder is relevant if $\nu < \frac{2}{d_{\text{eff}}}$ and irrelevant if $\nu > \frac{2}{d_{\text{eff}}}$. When $\nu = \frac{2}{d_{\text{eff}}}$, the disorder is said *marginal*, i.e. its effect depends on the finer details of the model and the Harris criterion is inconclusive.

Regarding the d-DPRE, when the spatial dimension is large, the polymer has much more space to avoid those points where the ω's could be high, hence we suppose that the disorder will not have a significant impact on the pure model when its strength $\beta > 0$ is small enough: we thus expect it to be irrelevant. Otherwise, when the dimension is small, there is not enough space for the polymer to be far away from the disorder, which is therefore expected to be relevant. In this case, for any disorder strength $\beta > 0$, the global configuration of the model drastically changes by favoring those paths which collect a high energy contribution. This heuristic intuition is confirmed by the Harris criterion: indeed, by the diffusivity of the simple random walk on $\mathbb{Z}^d$ it follows that $d_{\text{eff}} = 1 + \frac{d}{2}$ and $\nu = 1$, thus the system is *relevant* when $1 < \frac{2}{1+\frac{d}{2}}$, i.e. $d < 2$, *irrelevant* when $1 > \frac{2}{1+\frac{d}{2}}$, i.e. $d > 2$ and *marginal* (in fact, marginally relevant) when $d = 2$.

Despite its simple formulation, it is not straightforward to apply the Harris criterion in concrete cases, where a deeper and more accurate analysis is required (see [**Gia11**, Chapter 4]). This inspired the study of other strategies to investigate the impact of disorder, such as the approach we will present in Section 1.3, from which our work is strongly inspired.

1.2. A connection to singular stochastic PDEs

Besides the theory of disordered systems, the d-DPRE has attracted more and more attention in recent years because of its connection to some stochastic PDEs, which we briefly recall in this section.

First of all, we define the *space-time white noise* $\dot{W}$ as a distribution-valued centered Gaussian process with covariance

$$\mathbb{E}[\dot{W}(t,x)\,\dot{W}(s,y)] := \delta(t-s)\delta(x-y) \qquad t,s>0 \quad x,y\in\mathbb{R}^d\,,$$

where $\delta(x)$ denotes the Dirac measure centred on some fixed point x. The expression for the covariance is only formal, in fact the white noise cannot be considered as random variable for any fixed coordinates $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$: the correct interpretation is to define it as a *random distribution*, i.e. as a random variable taking values in the space of distributions (generalized functions). Denoting by $\langle \dot{W}, \varphi\rangle$ the duality pairing between $\dot{W}$ and any smooth test function $\varphi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$, then the process $\big(\langle \dot{W}, \varphi\rangle\big)_{\varphi\in C_c^\infty(\mathbb{R}_+\times\mathbb{R}^d)}$ is centered Gaussian with covariance

$$\mathbb{E}\big[\langle \dot{W}, \varphi_1\rangle\,\langle \dot{W}, \varphi_2\rangle\big] := \int_{\mathbb{R}_+\times\mathbb{R}^d} \varphi_1(t,x)\varphi_2(t,x)\,\mathrm{d}t\,\mathrm{d}x \quad \varphi_1,\,\varphi_2 \in C_c^\infty(\mathbb{R}_+\times\mathbb{R}^d)\,.$$

1.2.1. Multiplicative Stochastic Heat Equation.

The first and main equation treated in this thesis is the so-called *Stochastic Heat Equation with multiplicative noise* (mSHE):

$$\begin{cases} \partial_t u(t,x) \;=\; \frac{1}{2}\,\Delta u(t,x) + \beta\, u(t,x)\cdot \dot{W}(t,x) & t\in\mathbb{R}_+\,,\; x\in\mathbb{R}^d\,, \\ u(0,x) \equiv 1\,. \end{cases} \tag{mSHE}$$

As the name suggests, this is the standard heat equation with an additional potential term given by the noise $\dot{W}$ multiplied by u. From a physical point of view, we can interpret $u(t,x)$ as the temperature at time $t>0$ in $x\in\mathbb{R}^d$ as the heat spreads through an environment with random sources and sinks (according to the "sign" of $\dot{W}$), which independently generate or dissipate the heat with rate $\beta|\dot{W}|$.

In general, we stress that the equation written as above is just formal. In fact, the strong irregularity of the white noise (recall that $\dot{W}$ is distribution-valued) could be a priori inherited by the solution itself, therefore we expect u to be a non–smooth function or even a distribution, depending on the space dimension d. For this reason, (mSHE) is said to be *singular*, namely ill-posed, due to the presence of the product $u\cdot\dot{W}$ between a non–smooth function and a distribution, or even between two distributions, which is not canonically defined.

More precisely, the white noise $\dot{W}$ belongs to the (parabolically scaled) Hölder–Besov space of distributions $\mathcal{C}^{\alpha}_{\mathfrak{s}} := \mathcal{B}^{\alpha}_{\infty,\infty}$ with $\alpha=-\frac{d}{2}-1-\kappa$ for all $\kappa>0$, which coincides with the (parabolically scaled) α-Hölder space whenever $\alpha>0$ (see [**CW17**]). Since the smoothing action of the Laplace operator should let the solution of (mSHE) gain two degrees of regularity, we expect that u is an element of $\mathcal{C}^{\alpha'}_{\mathfrak{s}}$ with $\alpha'=-\frac{d}{2}+1-\kappa$ for all $\kappa>0$, ([**CW17**]). Thus, we conclude that u is a continuous (but non–differentiable) random function when $d=1$, while for $d\geq 2$ we expect u to be a random distribution.

We can have an intuitive idea on the difference between the one-dimensional case and the problem in $d\geq 2$ by following a *renormalization procedure*. We look at the noise $\dot{W}$ in (mSHE) as a *perturbation* of the deterministic PDE and we study its qualitative effect both on large and small scale.

In the setting of stochastic PDEs one is usually interested in studying the small scale, thus we first rescale the space-time variables as

$$(t,x)\to(\varepsilon^2 t,\varepsilon x)\,,\qquad \varepsilon>0$$

and we analyze the equation solved by the rescaled solution $u^{\varepsilon}(t,x):=u(\varepsilon^2 t,\varepsilon x)$, namely

$$\partial_t u^{\varepsilon}(t,x) \;=\; \frac{1}{2}\,\Delta u^{\varepsilon}(t,x)+\beta\,\varepsilon^{1-\frac{d}{2}}\,u(t,x)\cdot\dot{W}(t,x)\,, \tag{1.3}$$

where we applied the scaling property of the white noise

$$\dot{W}(t,x)\overset{d}{=}\varepsilon^{1+\frac{d}{2}}\dot{W}\big(\varepsilon^2 t,\varepsilon x\big)\,. \tag{1.4}$$

Notice that (1.3) is still a Stochastic Heat Equation, where the determistic part remains invariant while the noise strength is now rescaled according to $\varepsilon>0$. If we now "zoom in" by sending $\varepsilon\to 0$ we observe different scenarios according to the spatial dimension d. When $d<2$, i.e. $d=1$, the noise strength $\beta\,\varepsilon^{1-\frac{d}{2}}$ vanishes as $\varepsilon\to 0$, thus the noise effect on the small–scale properties of the solution is less and less significant. In the language of stochastic PDEs, this is known as *subcritical regime*. On the other hand, when $d>2$ (also called *supercritical regime*) the noise strongly affects the small–scale properties since its strength $\beta\,\varepsilon^{1-\frac{d}{2}}$ explodes as $\varepsilon \to 0$. When $d = 2$, the noise strength β remains invariant in (1.3), thus this argument is inconclusive because we have no guess on the impact of the noise: this is called *critical regime*.

In order to make a comparison with disordered systems, usually observed on large scale, we now set

$$(t,x) \to \Big(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\Big), \qquad \varepsilon > 0\,.$$

This implies that the rescaled solution $\widetilde{u}^{\varepsilon}(t,x) := e^{\varepsilon^{-1}}(t,x) = u\big(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\big)$ verifies an equation similar to (1.3), where the noise strength is now rescaled as $\beta\,\varepsilon^{\frac{d}{2}-1}$. As a consequence, by sending $\varepsilon \to 0$ we obtain analogous predictions as before, but reversed. The message is that on large scale the noise will be gradually stronger, i.e. *relevant* in $d = 1$, otherwise *irrelevant* in $d > 2$ and *marginal* for $d = 2$. The choice of these adjectives is obviously not a coincidence: one immediately notices that these forecasts are exactly those predicted by the Harris criterion for the d-DPRE. We then guess that there is a parallelism between the notion of *subcriticality* and *criticality* from the theory of stochastic PDEs and the *disorder relevance* and *marginality* from the language of disordered systems, respectively.

Concerning the problem of the rigorous construction of a solution for (mSHE), when $d = 1$ the existence and uniqueness of $u(t,x)$ was already proved (when the space $\mathbb{R}$ is replaced by a bounded interval) by Walsh ([**Wal86**]). Almost ten years later, a Feynman-Kac formula for $u(t,x)$ was given in [**BC95**]. We also stress that the one-dimensional setting is the only case when the solution admits an L^2-convergent Wiener chaos expansion. For higher dimensions $d \geq 2$, the problem becomes more subtle due to the singularity of (mSHE) and a well established solution theory is still missing. Even the breakthrough recent approaches to make sense of singular stochastic PDEs – such as the techniques via regularity structures ([**Hai14**]), via paracontrolled distributions ([**GIP15**]), via energy solutions ([**GJ14**]) or via renormalization ([**Kup16**]) – treat so far examples only in the subcritical regime, thus fail in this critical/supercritical framework.

1.2.2. Mollified multiplicative Stochastic Heat Equation. In order to treat a singular stochastic PDE such as (mSHE) when $d \geq 2$ and try to obtain some notion of solution, the standard approach is to study a similar equation, where the noise term is replaced by a regularized version of it. For instance, one can *mollify* the space-time white noise in space by formally defining

$$\dot{W}^{\varepsilon}(t,x) := (\dot{W}(t,\cdot) * j_{\varepsilon})(x) = \int_{\mathbb{R}^d} j_{\varepsilon}(x-y)\dot{W}(t,y)\mathrm{d}y\,,$$

where $j_{\varepsilon}(x) := \varepsilon^{-d} j\big(x/\varepsilon\big)$ and $j \in C_c^{\infty}(\mathbb{R}^d)$ is a symmetric probability density. Notice that $\dot{W}^{\varepsilon}$ is still a white noise in time, but it is now a smooth function in space: for any fixed x, the process $t \mapsto \int_0^t \dot{W}^{\varepsilon}(s,x)\mathrm{d}s$ is now well–defined and it is a Brownian motion with variance $\|j\|^2_{L^2(\mathbb{R}^d)}$. At this point, we can consider the *regularized equation* obtained by replacing the noise $\dot{W}$ by $\dot{W}^\varepsilon$, namely

$$\begin{cases} \partial_t u^\varepsilon(t,x) \;=\; \frac{1}{2}\,\Delta u^\varepsilon(t,x) + \beta\, u^\varepsilon(t,x)\cdot \dot{W}^\varepsilon(t,x) \qquad t\in\mathbb{R}_+\,,\ x\in\mathbb{R}^d\,, \\ u^\varepsilon(0,x)\equiv 1\,. \end{cases} \tag{1.5}$$

Such regularized equation is well-posed in an integral form, also known as mild formulation, by the Ito-Walsh theory, and by Feynman-Kac we have

$$\begin{aligned} &u^\varepsilon(t,x) \\ &= \mathrm{E}\Bigg[\exp\Bigg\{\int_0^t \beta\,\dot{W}^\varepsilon(t-s,B_s)\,\mathrm{d}s - \frac{\beta^2}{2}\mathbb{E}\Bigg[\Bigg(\int_0^t \dot{W}^\varepsilon(t-s,B_s)\,\mathrm{d}s\Bigg)^2\Bigg]\Bigg\}\,\Bigg|\, B_0=x\Bigg] \\ &= \mathrm{E}\Bigg[\exp\Bigg\{\int_0^t \beta\,\dot{W}^\varepsilon(t-s,B_s)\,\mathrm{d}s - \frac{\beta^2}{2}\|j_\varepsilon\|^2_{L^2(\mathbb{R}^2)}\,t\Bigg\}\,\Bigg|\, B_0=x\Bigg]\,, \end{aligned} \tag{1.6}$$

where E is the expectation with respect to a standard Brownian motion $B=(B_s)_{s\geq 0}$. The goal is then to try to identify a solution for the regularized equation (1.5) once the regularity is removed, i.e. one looks for a scaling limit of u^ε as $\varepsilon\to 0$. If it exists, such a limit will be a natural candidate solution for the singular (mSHE). Actually, as we will discuss in detail for $d=2$ in the next section, it will be necessary to rescale also the noise strength $\beta=\beta_\varepsilon\to 0$ as $\varepsilon\to 0$ to obtain some interesting limit. Despite the fact that the rescaled noise strength vanishes as $\varepsilon\to 0$, the limit will not agree with the solution of the deterministic Heat Equation as one could naively expect: we will see that the mechanism behind this asymptotic behaviour is much more subtle.

We finally stress that the representation of $u^\varepsilon(t,s)$ above shows the link between (mSHE) and the d-DPRE. Indeed, except for a time reversal $s\to t-s$, the expression in (1.6) defines the *partition function of a continuum directed polymer* with length scale ε. The polymer trajectories are here described by the Brownian motion B (instead of the simple random walk S) and are perturbed by the random environment $\dot{W}^\varepsilon$. Therefore, we can look at $Z_N^\beta(m,z)$ *in* (1.2) *as a discrete version of* u^ε; more precisely, for $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$ it is possible to show that the rescaled partition function $Z_N^\beta\big(Nt,\sqrt{N}x\big) := Z_N^\beta\big(\lfloor Nt\rfloor,\lfloor\sqrt{N}x\rfloor\big)$ approximates $u^\varepsilon(t,x)$ (up to a time reversal), where the relation between the two time scales is $N=\varepsilon^{-2}$. This suggests that partition functions of directed polymers provide a natural and alternative regularization of the solution for (mSHE) through *discretization* (rather than mollification) of the noise.

Moreover, this also leads to the key motivation for the problems treated and presented in this work, based on the study of scaling limits of $Z_N^\beta\big(Nt,\sqrt{N}x\big)$ and related random objects as $N\to\infty$, when $d=2$. In the same spirit of the approach via mollification, looking for a suitable large-scale limit for the d-DPRE partition function, which satisfies a discretized version of u in (mSHE), allows to take a step forward in the study of the critical Stochastic Heat Equation with multiplicative noise. Besides, in view of the strict link between discretization and mollification, we will show that all the convergence results obtained for $Z_N^\beta\big(Nt,\sqrt{N}x\big)$ can also be expressed in terms of limit theorems for $u^\varepsilon(t,x)$.

1.2.3. Additive Stochastic Heat Equation. It is worth recalling another example of stochastic PDE, namely the *Stochastic Heat Equation with additive noise*, also called *Edwards–Wilkinson equation* ([**EW**]), with fixed $s,c>0$:

$$\begin{cases} \partial_t v^{(s,c)}(t,x) \;=\; \frac{s}{2}\,\Delta v^{(s,c)}(t,x) + c\,\dot W(t,x) \qquad t\in\mathbb{R}_+\,,\; x\in\mathbb{R}^d\,, \\ v^{(s,c)}(0,x)\equiv 0\,, \end{cases} \tag{EW}$$

which represents one of the simplest and most understoodd stochastic PDEs. Indeed, the presence of an additive rather than multiplicative noise makes the equation no longer singular, and indeed (EW) is well–posed for any spatial dimension $d\geq 1$.

It is possible to show that the solution is given by the *stochastic convolution*

$$v^{(s,c)}(t,x) = c\int_0^t\int_{\mathbb{R}^d} g_{s(t-t')}(x-z)\,\dot W(t',z)\,\mathrm{d}t'\,\mathrm{d}z\,, \tag{1.7}$$

with $g_t(x) := \frac{1}{(2\pi t)^{\frac{d}{2}}}e^{-\frac{|x|^2}{2t}}$. When $d=1$, $v^{(s,c)}(t,x)$ is a random function, α-Hölder continuous for every $\alpha<\frac14$ in time and α-Hölder continuous for every $\alpha<\frac12$ in space (see [**Hai09**]), and the process $(v^{(s,c)}(t,x))_{(t,x)\in\mathbb{R}_+\times\mathbb{R}^d}$ is Gaussian with zero mean and coviariance function

$$\begin{aligned}\mathbb{E}\big[v^{(s,c)}(t,x)v^{(s,c)}(t',y)\big] \;&=\; c^2\int_{|t-t'|}^{(t+t')}\frac{1}{2\sqrt{2\pi s\,u}}e^{-\frac{|x-y|^2}{2su}}\,\mathrm{d}u \\ &=\; \frac{c^2}{2s}\int_{s|t-t'|}^{s(t+t')}\frac{1}{\sqrt{2\pi u}}e^{-\frac{|x-y|^2}{2u}}\,\mathrm{d}u\,.\end{aligned}$$

As in the multiplicative case, when $d\geq 2$ the solution turns out to be a random distribution, thus the expression (1.7) is just formal, since $v^{(s,c)}$ cannot be evaluated pointwise. Nevertheless, up to testing $v^{(s,c)}$ against a smooth function $\varphi\in C_c^\infty(\mathbb{R}_+\times\mathbb{R}^d)$, the random field $(\langle v^{(s,c)},\varphi\rangle)_{\varphi\in C_c^\infty(\mathbb{R}_+\times\mathbb{R}^d)}$ is a centered Gaussian process with covariance

$$\mathbb{E}\big[\langle v^{(s,c)},\varphi_1\rangle\langle v^{(s,c)},\varphi_2\rangle)\big] \;= c^2\int_{\mathbb{R}_+\times\mathbb{R}^d}\varphi_1(t,x)\,K^s_{t,t'}(x,y)\,\varphi_2(t',y)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}t'\,\mathrm{d}y\,,$$

for all $\varphi_1,\varphi_2\in C_c^\infty(\mathbb{R}_+\times\mathbb{R}^d)$. We stress that the kernel

$$K^s_{t,t'}(x,y) := \frac{1}{2s}\int_{s|t-t'|}^{s(t+t')}\frac{1}{(2\pi u)^{\frac{d}{2}}}e^{-\frac{|x-y|^2}{2u}}\,\mathrm{d}u \tag{1.8}$$

diverges on the diagonal for all $d \geq 2$ (logarithmically for $d = 2$), while $K\big((t,x),(t,x)\big)$ is finite for $d = 1$ as expected, indeed in the one-dimensional case

$$v^{(s,c)}(t,x) \sim \mathcal{N}\big(0, c^2 K\big((t,x),(t,x)\big)\big)$$

for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$.

1.2.4. KPZ Equation. The last equation we recall in this overview is the *Kardar-Parisi-Zhang (KPZ) equation*:

$$\begin{cases} \partial_t h(t,x) \;=\; \frac{1}{2}\,\Delta h(t,x) + \frac{1}{2}\,|\nabla h(t,x)|^2 + \beta\,\dot{W}(t,x) & t \in \mathbb{R}_+\,,\ x \in \mathbb{R}^d\,, \\ h(0,x) \equiv 0\,. \end{cases} \tag{KPZ}$$

Originally introduced by the physicists Kardar, Parisi and Zhang ([**KPZ86**]) as a model for the growth of a random d-dimensional interface (embedded in $\mathbb{R}^{1+d}$), the KPZ equation has since become an object of interest and instense research from both a physical and a mathematical point of view.

Unlike (mSHE), the KPZ equation is ill–defined for any space dimension $d \geq 1$ due to its non-linearity. Since h is expected to be rough in space, the non-linear term ∇h is expected to be a distribution, thus the mathematical interpretation of its square is not a priori clear. Even in the simplest one-dimensional case it is difficult to give a meaning of (KPZ): one could try to solve (KPZ) as a perturbation of the corresponding (EW) equation obtained by removing the non-linear term, however the solution of (EW) is continuous, but not differentiable in space, thus it is not obvious how to treat $|\nabla h|^2 = |h'|^2$.

Nevertheless, much work has been achieved in the space dimension $d = 1$. In this case, a direct approach to solve (KPZ) is by considering the Stochastic Heat Equation with multiplicative noise (mSHE), whose solution u is well-defined and strictly positive (see [**Muel91**]) when $d = 1$, and then defining the so–called *Cole–Hopf solution* $h = \log u$, which formally solves the KPZ equation. In other terms, the Cole–Hopf transformation $h \to u = e^h$ formally maps (KPZ) in (mSHE), as already observed in [**KPZ86**]. The first mathematical contribution towards this direction comes from [**BG97**], where the authors studied the solution h of (KPZ) as scaling limit of the fluctuations field of a microscopic interface model, the so-called *weakly asymmetric single step solid on solid process (SOS)*, defined on the one-dimensional lattice. More precisely, through the Cole-Hopf transformation and its discrete analog, the so-called *Gärtner transformation* ([**Gaer88**]), they turned the problem of the convergence of the SOS fluctuations towards (KPZ) into the problem on the convergence of the transformed discrete process to (mSHE). In the same spirit, in the following years several approximations have been provided to give a meaning to the solution h of (KPZ), by showing that the Cole-Hopf solution exhibits the same fluctuations on large space-time scales as several known one-dimensional interface growth models. For a detailed review of such results, we refer to the surveys [**Cor12, QS15**].

In recent years, the aforementioned theories of regularity structures by Hairer, of paracontrolled distributions by Gubinelli-Imkeller-Perkowski, of energy solutions by Goncalves-Jara and the renormalization approach by Kupiainen allowed to develop more robust techniques in order to solve (KPZ), but only in the one-dimensional subcritical case (see [**Hai13, GP17, GJ14, Kup16**]).

In higher dimensions, the situation is much less understood and a robust solution theory has not been achieved yet. A possible approach is to consider the Cole-Hopf solution $h^\varepsilon := \log u^\varepsilon$, where now u^ε is the solution of the mollified Stochastic Heat Equation (1.5), which is strictly positive (see (1.6)). By applying carefully Ito formula to $h^\varepsilon = \log u^\varepsilon$, we derive the following equation solved by h^ε:

$$\begin{cases} \partial_t h^\varepsilon(t,x) \;=\; \frac{1}{2}\,\Delta h^\varepsilon(t,x) + \frac{1}{2}\,|\nabla h^\varepsilon(t,x)|^2 + \beta\,\dot{W}^\varepsilon(t,x) - C_\varepsilon \qquad t \in \mathbb{R}_+\,,\; x \in \mathbb{R}^d\,, \\ h^\varepsilon(0,x) \equiv 0\,, \end{cases} \tag{1.9}$$

which is actually the *mollified KPZ equation* modified by the Ito correction term $C_\varepsilon := \beta^2 \|j_\varepsilon\|^2_{L^2(\mathbb{R}^d)} = \beta^2\varepsilon^{-d}\|j\|^2_{L^2(\mathbb{R}^d)}$, which diverges as $\varepsilon \to 0$. At this point, one tries to characterize the solution of (KPZ) by first replacing $\beta\dot{W}$ by $\beta\dot{W}^\varepsilon - C_\varepsilon$ and then by studying the limit of h^ε in (1.9) as $\varepsilon \to 0$ (up to suitably rescaling $\beta = \beta_\varepsilon$, too).

Recall that the d-DPRE's rescaled partition function provides a discretization for the mollified solution u^ε of (mSHE), up to a time reversal. Therefore, in view of the Cole-Hopf transformation applied in the continuum setting, it is natural to consider the log-partition function $\log Z_N^\beta\big(Nt, \sqrt{N}x\big)$[1] as a discrete approximation of $h^\varepsilon(t,x)$ and then to try to obtain information on (KPZ) by working in the discrete framework and studying the scaling limit of $\log Z_N^\beta\big(Nt, \sqrt{N}x\big)$ as $N \to \infty$.

1.3. Scaling limits for the directed polymer partition function

In this section, we mainly focus on the statistical properties of the d-DPRE partition function $Z_N^\beta(\cdot,\cdot)$ as the polymer length $N \to \infty$.

In the late eighties, Bolthausen ([**Bol89**]) examined the d-DPRE model in the framework of martingales. Since Z_N^β is a positive martingale on $(\widetilde{\Omega}, \mathcal{G}, \mathcal{G}_n, \mathbb{P})$, then it converges $\mathbb{P}$-almost surely

$$\lim_{N\to\infty} Z_N^\beta = Z_\infty^\beta$$

for $\beta \geq 0$ and by Kolmogorov's 0-1 law the following dichotomy holds:

$$\mathbb{P}\big(Z_\infty^\beta = 0\big) = 0 \text{ or } 1\,.$$

The former case is called *weak disorder*, while we refer to the latter as *strong disorder*. This dichotomy was then fully characterized in the early twenty-first

[1]Note that $Z_N^\beta\big(Nt, \sqrt{N}x\big)$ is strictly positive by definition (1.2).

century by Comets, Shiga and Yoshida ([**CSY03, CSY04, CY06**]), who proved the existence of a critical disorder strength $\beta_c = \beta_c(d)$, precisely $\beta_c = 0$ for $d = 1, 2$ ([**CH02**]) and $\beta_c > 0$ for $d \geq 3$ ([**IS88, Bol89**]), such that the weak disorder holds if $\beta \in [0, \beta_c)$, while the strong disorder is verified if $\beta > \beta_c$. In particular, notice that we observe an actual *phase transition* from the weak to the strong disorder regime only in higher dimensions $d \geq 3$.

The existence of a weak disorder was first shown in [**IS88, Bol89**] for sufficiently small β. Under this regime the polymer path, namely the simple random walk under the measure P_N^β is *diffusive*, as it is under the non-perturbed measure P. This was extended to all $\beta < \beta_c$ in [**CY06**]. Heuristically, the disorder tuned by a low value of β does not significantly affect the pure system as $N \to \infty$.

On the other hand, when $d = 1, 2$ for all $\beta > 0$, or for a high disorder strength $\beta > \beta_c$ when $d \geq 3$, the partition function Z_N^β converges almost surely to 0 as $N \to \infty$, despite $\mathbb{E}[Z_N^\beta] = 1$ for all $N \in \mathbb{N}$ and $\beta > 0$. As the chosen terminology suggests, in the strong disorder regime the effect of the disorder considerably alters the pure model, indeed it is expected (but not proved yet) that the polymer path is *superdiffusive* under P_N^β. Moreover, the polymer path localizes in those regions favoured by the random environment (see [**CH02, CSY03, Cha19, BC20, Bat21**]).

The weak/strong disorder dichotomy further confirms the predictions obtained by the Harris criterion: when the disorder is irrelevant ($d \geq 3$) the model preserves its features (weak disorder) until β is large enough to cause a transition to the strong disorder regime. Otherwise, if the disorder is relevant ($d = 1$) or marginally relevant ($d = 2$), the strong disorder regime holds for any value of $\beta > 0$, no matter whether β is low or high.

In the situation of disorder (marginal) relevance, it is natural to zoom around $\beta_c = 0$ in order to try to identify an *intermediate disorder regime* between the weak and strong disorder, where the partition function should admit a non-trivial limit as $N \to \infty$. To be precise, the key observation suggested in [**AKQ14**] is that it should be possible to tune the disorder strength $\beta = \beta_N \to 0$ at a suitable rate as $N \to \infty$ in order to obtain a *non-trivial disordered continuum limit* for $Z_N^{\beta_N}(\cdot, \cdot)$.

In the one-dimensional case, this was achieved by Alberts, Khanin and Quastel ([**AKQ14**]), who proved that by rescaling $\beta = \beta_N := \frac{\hat{\beta}}{N^{1/4}}$ with $\hat{\beta} > 0$ the diffusively rescaled partition function $Z_N^\beta\big(Nt, \sqrt{N}x\big)$ converges in distribution to the solution $u(1-t, x)$ of the Stochastic Heat Equation with multiplicative noise (mSHE) (which is a well-defined random function as long as $d = 1$).

This result was extended by Caravenna, Sun and Zygouras in [**CSZ17a**] to more general disorder relevant systems, including the *pinning model* (see also [**CSZ16**]), the *long-rage directed polymer in* $d = 1$ and the random field *Ising model on* $\mathbb{Z}^2$. The proof techniques are based on *polynomial chaos expansions* for partition functions, which consist of multilinear polynomials of suitable modified versions of the disorder random variables $\omega(n,x)$ (see Section 3.1 of Chapter 3 for more details). The crucial point is to show that such an expansion for $Z_N^{\beta_N}\big(Nt,\sqrt{N}x\big)$ converges to the corresponding Wiener chaos expansion of $u(1-t,x)$, by exploiting as key ingredient an extended version of the *Lindeberg principle* for polynomial chaos (see [**MOO10**]).

Before we focus on the more delicate two-dimensional framework, let us mention additional works towards other directions, where the features of disorder random variables are weakened. Scaling limits of the rescaled 1-DPRE's partition function have been recently investigated by removing the independence hypothesis of the random environment. Depending on whether one considers a family of disorder variables only correlated in space ([**Ran20**]), only in time ([**RSW22+**]) or both in time and space ([**SSSX21**]), it is still possible to obtain a continuum limit towards the solution of a (mSHE) where the noise is white or colored in time/space according to the discrete disorder. We refer to the cited articles for more details on the techniques and on the notions of solution for (mSHE) used.

Alternatively, much attention has been paid to the analysis of intermediate disorder limits beyond the finite second moment assumption of the random environment. By considering a family of i.i.d. centered disorder variables in the domain of attraction of an α-stable law for $\alpha \in (1,2)$, a non-trivial disorder phase exists if and only if the space dimension satisfies $d \leq \frac{2}{\alpha-1}$ (see [**Viv21, Wei16**]). Moreover, as the stength β_N tends to 0 as $N \to \infty$ at some suitable rate depending on α and on d, the scaling limit of the rescaled d-DPRE partition function converges to the solution of a (mSHE) with Lévy noise. See the works of Berger, Chong and Lacoin [**BL21, BL22, BCL21+**] and the references therein for an exhaustive analysis of this problem.

Back to the standard DPRE model, from now on we focus on the marginal case $d = 2$, where the situation is more subtle because we have poor information about the solution of (mSHE) and the Wiener chaos expansion considered in [**AKQ14**] and [**CSZ17a**] diverges for $d \geq 2$. New techniques were developed in [**CSZ17b**], where the authors found that in order to obtain a non-trivial limit for the partition function the correct rescaling for β is logarithmic, namely

$$\beta = \beta_N := \frac{\hat{\beta}}{\sqrt{R_N}} \sim \frac{\hat{\beta}\sqrt{\pi}}{\sqrt{\log N}} \qquad \text{as } N \to \infty\,. \tag{1.10}$$

The parameter $\hat{\beta} > 0$ is fixed, while

$$R_N := \sum_{n=1}^{N} \mathrm{P}(S_n = \widetilde{S}_n) = \sum_{n=1}^{N} \mathrm{P}(S_{2n} = 0)$$

is called *expected replica overlap*, it denotes the expected collision local time between two independent simple sandom walk S and $\widetilde{S}$ and when $d = 2$ it diverges as $R_N = \frac{\log N}{\pi} + O(1)$ as $N \to \infty$ by the local CLT (see also [**ET60**]). Choosing the rescaling (1.10), Caravenna, Sun and Zygouras proved the following pointwise limit in distribution for any fixed $(t,x) \in [0,1] \times \mathbb{R}^2$:

$$Z_N^{\beta_N}\big(Nt, \sqrt{N}x\big) \xrightarrow[N\to\infty]{d} \begin{cases} e^{\sigma(\hat\beta)Y - \frac{1}{2}\sigma(\hat\beta)} & \hat\beta \in (0,1)\,, \\ 0 & \hat\beta \geq 1\,, \end{cases} \tag{1.11}$$

or, equivalently,

$$\log Z_N^{\beta_N}\big(Nt, \sqrt{N}x\big) \xrightarrow[N\to\infty]{d} \begin{cases} \sigma(\hat\beta)Y - \frac{1}{2}\sigma(\hat\beta) & \hat\beta \in (0,1)\,, \\ -\infty & \hat\beta \geq 1\,, \end{cases} \tag{1.12}$$

where $Y \sim \mathcal{N}(0,1)$ and $\sigma(\hat\beta) := \log\big(\frac{1}{1-\hat\beta^2}\big)$. What surprisingly occurs here in contrast to the $d=1$ case is that by zooming around $\beta = 0$ according to (1.10) we observe a new *phase transition* from weak to strong disorder, which reminds the one in $d \geq 3$. The critical point where the transition occurs is here explicit, namely $\hat\beta_c = 1$, and represents the point at which the limit's variance explodes. It can also be guessed by computing the asymptotic variance of the partition function: indeed, one can show that for all $(t,x) \in [0,1] \times \mathbb{R}^2$:

$$\lim_{N\to\infty} \mathbb{V}\mathrm{ar}\Big(Z_N^{\beta_N}\big(Nt, \sqrt{N}x\big)\Big) = \frac{\hat\beta^2}{1-\hat\beta^2}\,,$$

which clearly diverges at $\hat\beta = 1$.

The result (1.11) still relied on polynomial chaos expansion of the partition function, however the continuum limit for $\hat\beta < 1$ is no longer a function of white noise as for $d = 1$, where the result is a Wiener chaos expansion obtained as scaling limit of discrete disorder variables (which approximate the white noise). What was crucially identified in [**CSZ17b**] is a hierarchy of *independent white noises*, which arise as scaling limits of suitable subsets of the polynomial chaos expansion of $Z_N^{\beta_N}\big(Nt, \sqrt{N}x\big)$, determined by the logarithmic scaling (1.10). These independent white noises provide the basic bricks to show the Gaussian result for $\hat\beta < 1$ through the application of the celebrated Fourth Moment Theorem [**NP05, NPR10**], which we will discuss more in detail in the next subsection.

The regime $\hat\beta \in (0,1)$ and $\hat\beta > 1$ are respectively called *subcritical* and *supercritical*, whereas $\hat\beta_c = 1$ characterizes the *critical regime*[2]. While the supecritical regime is poorly understood to the best of our knowledge, the subcritical regime has been throughly investigated in last years and much progress has also been made recently in the analysis of the critical case. We now focus on the latter two regimes for $d = 2$ and we present the related contributions of our work.

[2]There is a conflict of terminology between the subcritical/critical/supercritical regimes just introduced, which refer to β, and the ones related to stochastic PDEs and mentioned in Section 1.2, which instead refer to the class of equations and to the dimension. Despite the common names, there is no analogy between them.

1.4. Main results in this thesis

We now discuss the main original results of our work and how they fit into the existing related literature.

1.4.1. The 2d directed polymer in the subcritical regime. Through the same techniques applied for the pointwise convergence in distribution (1.11) for fixed $(t,x) \in [0,1] \times \mathbb{R}^2$, the authors of [**CSZ17b**] also analized the 2-DPRE partition function as *random field*, by considering the process

$$\left\{ Z_N^{\beta_N}\left(Nt, \sqrt{N}x\right) \,:\, x \in \mathbb{R}^2 \right\},$$

where $t \in [0,1]$ is fixed for simplicity. In particular, they proved that under the subcritical regime (1.10) with $\hat{\beta} \in (0,1)$ the fluctuation of the diffusively rescaled and suitably amplified partition function

$$\left\{ V_N(t,x) := \beta_N^{-1}\left(Z_N^{\beta_N}\left(Nt, \sqrt{N}x\right) - 1 \right) \,:\, x \in \mathbb{R}^2 \right\} \tag{1.13}$$

are Gaussian as $N \to \infty$. To be more precise, this convergence result is meant in the sense of *random distributions on* $\mathbb{R}^2$ (i.e. *generalized functions*), hence for any fixed test function $\varphi \in C_c(\mathbb{R}^2)$ the following convergence in distribution holds:

$$\int_{\mathbb{R}^2} V_N(t,x)\, \varphi(x) \, \mathrm{d}x \xrightarrow[N\to\infty]{d} \langle \widetilde{v}, \varphi \rangle \qquad t \in [0,1]\,, \tag{1.14}$$

where the process $\widetilde{v} = (\widetilde{v}_\varphi := \langle \widetilde{v}, \varphi \rangle)_{\varphi \in C_c(\mathbb{R}^2)}$ is a log-correlated generalized centered Gaussian process which can be characterized as $\widetilde{v}(x) := v^{(s,c)}(1-t,x)$, where $v^{(s,c)}$ is the solution of the Edwards–Wilkinson equation (EW) with $s = \frac{1}{2}$ and $c = c_{\hat{\beta}} = \frac{1}{\sqrt{1-\hat{\beta}^2}}$, which explodes for $\hat{\beta} = 1$.

In [**CSZ20**], the analogous result for the log-partition function was proved: still in the subcritical regime (1.10) with $\hat{\beta} \in (0,1)$, for all $t \in [0,1]$ it holds that

$$\int_{\mathbb{R}^2} \beta_N^{-1}\left(\log Z_N^{\beta_N}\left(Nt, \sqrt{N}x\right) - \mathbb{E}\left[\log Z_N^{\beta_N}\left(Nt, \sqrt{N}x\right)\right]\right) \varphi(x) \, \mathrm{d}x \xrightarrow[N\to\infty]{d} \langle \widetilde{v}, \varphi \rangle\,, \tag{1.15}$$

where $\widetilde{v}$ is the same Gaussian generalized field as above. In order to verify (1.15), the authors developed a non-trivial approach which *linearizes* the log-partition function $\log Z_N^{\beta_N}\left(Nt, \sqrt{N}x\right)$ in terms of $Z_N^{\beta_N}\left(Nt, \sqrt{N}x\right)$ and then applied (1.14) (more details in Subsection 3.2.2 of Chapter 3).

The proofs of the Gaussian limits (1.12), (1.14) and (1.15) under the subcritical regime are all based on the application of the *Fourth Moment Theorem* for polynomial chaos. Formulated in our context in [**NPR10**] and slightly extended in [**CSZ17b**, Theorem 4.2] (see also the previous works [**NP05, dJ90, dJ87, Rot79**]), the Fourth Moment Theorem ensures the convergence of a sequence of centered polynomial chaos of fixed order (see Chapter 2 for more details) towards a Gaussian random variable $\mathcal{N}(0, \sigma^2)$, provided that the second and the fourth moments converge to the corresponding moments of $\mathcal{N}(0, \sigma^2)$. An extension to vectors of polynomial chaos holds if we further assume that each entry of the corresponding covariance matrix tends to some value V_{ij}, so that we obtain the joint convergence towards the multivariate normal random vector $\mathcal{N}\big(0, V = (V_{ij})\big)$.

Although this result is extremely useful and powerful, it is not always immediate to verify the required hypotheses, especially the one concerning the fourth moment. In the framework of polynomial chaos studied in [**CSZ17b**], indeed, the computation of the fourth moment leads to a non-trivial and quite technical combinatorial problem which further complicates the proof structure.

The first main contribution of our thesis in the *subcritical regime* is an alternative proof with refinements of the results (1.12), (1.14) and (1.15), based on a novel and more elementary approach which avoids the application of the Fourth Moment Theorem. In particular, we prove

- a novel Central Limit Theorem (CLT) for polynomial and Wiener chaos, see Theorems 2.2 and 2.9, as a result of independent interest (Chapter 2);

moreover, as an application of our new CLT for polynomial chaos to 2d directed polymers, we present

- a revisited version for the Edwards-Wilkinson fluctuations of the rescaled partition function (1.14) and of the rescaled log-partition function (1.15), see Theorems 3.2 and 3.7 (Chapter 3, Section 3.2);
- a novel Gaussian convergence for a singular product between the rescaled partition function and the disorder, see Theorem 3.13 (Chapter 3, Section 3.3);
- a sharp approximation via polynomial chaos of the log-partition function with fixed starting point, which improves (1.12) (Chapter 4).

Eventually, as an application of our new CLT for Wiener chaos to the 2d (mSHE), we show

- a revisited version for the Edwards-Wilkinson fluctuations of the solution to the mollified multiplicative Stochastic Heat Equation, see Theorem 3.15 (Chapter 3, Section 3.4).

Let us give an overview of these results.

Regarding the CLTs presented in Chapter 2, we identify three sufficient conditions (see (1)–(2)–(3) and Theorem 2.2 in Chapter 2) for the asymptotic Gaussianity of a sequence of polynomial chaos, *only based on second moment computations*. These conditions are indeed much simpler to be verified, both in general and in the 2-DPRE framework, since they do not require estimates for any moments but the second one. In particular, thanks to the nice structure of polynomial chaos written as sums of mutually L^2-orthogonal random variables, working with the second instead of the fourth moment significantly simplifies the computations. As we will show in Chapter 2, the key tool to prove the Gaussianity in Theorem 2.2 is the application of the *Feller-Lindeberg Central Limit Theorem for triangular arrays* (see Theorem 2.14).

We also stress that our criterion applies to superpositions of (even infinite) chaos of different orders, provided that the contribution to the second moment is given by a finite number of chaos of bounded order, up to a negligible error in L^2. In the setting of the 2-DPRE model, this corresponds to work under the subcritical regime (1.10) with $\hat{\beta} \in (0,1)$, since the critical point $\hat{\beta}_c = 1$ is indeed determined by the failure of the latter request. While in the original proofs of (1.12), (1.14) and (1.15) it was necessary to compute the fourth moment of each single chaos of fixed order, our novel criterion allows to deal with the polynomial chaos expansion of $Z_N^{\beta_N}(\cdot,\cdot)$ in its entirety and to estimate (contributions to) the second moment without analysing each fixed chaos separately.

Thanks to this novel general criterion, not only are we able to recover the aforementioned results with alternative and more elementary arguments, but we can also improve them. In particular, we now present how we revisit the results about the Edwards–Wilkinson fluctuations (1.14) and about the asymptotic Gaussianity of the rescaled log-partition function with fixed starting point (1.12), by exploiting our new Theorem 2.2.

In Section 3.2 of Chapter 3, we first show how to recover the Edwards–Wilkinson fluctuations (1.14) through the new CLT for polynomial chaos. However, it is not a priori clear why the fluctuations of the rescaled partition function – which approximates the solution of the Stochastic Heat Equation with *multiplicative* noise (mSHE) – converge (up to a time reversal) to the solution of the Stochastic Heat Equation with *additive* noise (EW). To shed light on this mechanism, in Section 3.3 of Chapter 3 we introduce and analyze the asymptotic behaviour of the following *singular product*

$$\Big\{ \Xi_N(t,x) := \dot{W}_N(t,x)\big(Z_N^{\beta_N}(Nt,\sqrt{N}x) - 1\big) \\ := \beta_N \dot{W}_N(t,x) V_N(t,x) \,:\, (t,x) \in [0,1]\times\mathbb{R}^2 \Big\} \tag{1.16}$$

between the centered and rescaled partition function $V_N(\cdot,\cdot)$ (recall (1.13)) and the rescaled disorder $\dot{W}_N(t,x) := N\eta_N\big(\lfloor Nt\rfloor, \lfloor\sqrt{N}x\rfloor\big)$, where the family $\eta_N = \big(\eta_N(m,z)\big)_{(m,z)\in\mathbb{N}\times\mathbb{Z}^2}$ is a slight modification of ω (see (3.3)). We call Ξ_N *singular* because of the strong irregularity of its factors for large N. In fact, not only is V_N rough for large scales as we discussed earlier, but the same holds for $\dot{W}_N$, which converges in distribution to the white noise as a random distribution, i.e. $\langle \dot{W}_N, \varphi\rangle \xrightarrow{d} \langle \dot{W}, \varphi\rangle \sim \mathcal{N}\big(0, \|\varphi\|_{L^2}^2\big)$ as $N\to\infty$, for all $\varphi \in C_c^\infty([0,1]\times\mathbb{R}^2)$.

From (1.16), we could naively expect Ξ_N to vanish as $N\to\infty$, since $\dot{W}_N$ and V_N converge to the white noise $\dot{W}$ and to $\widetilde{v}$ (see (1.14)) respectively and $\beta_N \to 0$. However, we show that under the subcritical regime (1.10) with $\hat{\beta} \in (0,1)$ the singular product Ξ_N admits a *non-trivial limit*, even jointly with $\dot{W}_N$:

$$(\dot{W}_N, \Xi_N) \xrightarrow[N\to\infty]{d} \left(\dot{W}, \sqrt{c_{\hat{\beta}}^2 - 1}\dot{W}'\right) \tag{1.17}$$

as random distributions on $[0,1]\times\mathbb{R}^2$, where $\dot{W}$ and $\dot{W}'$ are two *independent white noises* on $[0,1]\times\mathbb{R}^2$ and $c_{\hat{\beta}} := \frac{1}{\sqrt{1-\hat{\beta}^2}}$.

At this point, we can figure out why this result (whose proof is based on the application of Theorem 2.2) improves (1.14). Indeed, it is possible to show that V_N formally solves the following *difference equation*

$$-\partial_t^{(N)} V_N = \frac{1}{4}\Delta^{(N)} V_N + \dot{W}_N + \Xi_N\,, \tag{1.18}$$

where $\partial_t^{(N)}$ and $\Delta^{(N)}$ are the discrete time derivative and the discrete Laplace operator[3]. By (formally) sending $N \to \infty$, the result (1.17) then provides an intuitive and heuristic explanation why the random field V_N converges to $\widetilde{v}$, indeed from (1.18) we formally get

$$-\partial_t \widetilde{v} = \frac{1}{4}\Delta\widetilde{v} + \dot{W} + \sqrt{c_{\hat{\beta}}^2 - 1}\,\dot{W}' \overset{d}{=} \frac{1}{4}\Delta\widetilde{v} + c_{\hat{\beta}}\widetilde{\dot{W}}\,,$$

which precisely corresponds to the – up to a time reversal – (EW) equation identified in (1.14), where $\widetilde{\dot{W}}$ is an additional space-time white noise obtained by the sum of the mutually independent noises $\dot{W}$ and $\dot{W}'$.

We now spend a few words about the asymptotic Gaussianity of the log-partition function with fixed starting point. By stationarity, we only deal with $Z_N^{\beta_N} := Z_N^{\beta_N}(0,0)$ where β_N verifies (1.10) and $\hat{\beta} \in (0,1)$. In Chapter 4, we show how we can recover (1.12) by applying our Theorem 2.2 instead of the Fourth Moment Theorem and show that $\log Z_N^{\beta_N}$ converges in law to a normal random variable with mean $-\frac{1}{2}\sigma^2(\hat{\beta})$ and variance $\sigma^2(\hat{\beta}) := \log\frac{1}{1-\hat{\beta}^2}$. The problem in this case is that, unlike $Z_N^{\beta_N}$ and for its linear transformations $V_N(\cdot,\cdot)$ and $\Xi_N(\cdot,\cdot)$, $\log Z_N^{\beta_N}$ *does not admit an explicit polynomial chaos expansion*, which is essential to apply Theorem 2.2. We first solve this issue by presenting a result of independent interest, which provides a *sharp approximation in* L^2 *of* $\log Z_N^{\beta_N}$ *in terms of an explicit polynomial chaos expansion* X_N^{dom} (see (4.5) for a detailed definition):

$$\lim_{N\to\infty}\left\| \log Z_N^{\beta_N} - \left\{X_N^{\mathrm{dom}} - \frac{1}{2}\mathbb{E}\big[\big(X_N^{\mathrm{dom}}\big)^2\big]\right\}\right\|_{L^2} = 0\,. \tag{1.19}$$

We stress that the proof of (1.19) is quite challenging and represents one of the key points in this discussion. Through suitable second moment estimates which underline an exponential time multi-scale, we derive an approximation of the partition function in terms of a *product representation*, which turns out to be extremely convenient when we finally take the logarithm of $Z_N^{\beta_N}$.

[3]A more precise yet still formal version of this difference equation is presented in Section 3.3 of Chapter 3.

Once obtained a polynomial chaos expansion for the log-partition function, by exploiting our CLT for polynomial chaos (Theorem 2.2) we can easily prove that

$$X_N^{\mathrm{dom}} \xrightarrow[N\to\infty]{d} \mathcal{N}\big(0\,,\,\sigma^2(\hat\beta)\big) \quad \text{and} \quad \lim_{N\to\infty} \mathbb{E}\big[\big(X_N^{\mathrm{dom}}\big)^2\big] = \sigma^2(\hat\beta)\,, \tag{1.20}$$

which, together with (1.19), recovers (1.12) for $\hat\beta < 1$.

Before concluding this discussion, we focus more on what happens in the related continuum setting of the singular SPDEs introduced in Section 1.2 of this chapter.

We have already observed that we can interpret the rescaled 2-DPRE partition function $Z_N^{\beta_N}(Nt, \sqrt{N}x)$ as a discretization – up to a time reversal – of the regularized solution $u^\varepsilon(t,x)$ of the Stochastic Heat Equation with multiplicative noise, where $N = \varepsilon^{-2}$. The Cole-Hopf transformation then implies an analog relation between $\log Z_N^{\beta_N}(Nt, \sqrt{N}x)$ and the regularized solution $h^\varepsilon(t,x)$ of the KPZ equation. In particular, in [**CSZ17b**] and [**CSZ20**] the authors also studied the asymptotic behaviour of $u^\varepsilon(t,x)$ and $h^\varepsilon(t,x)$ as $\varepsilon \to 0$ and derived the continuum counterparts of results (1.11)-(1.12)-(1.14)-(1.15), when the noise strength β in the mollified equations (1.5) and (1.9) is rescaled logarithmically as $\beta_\varepsilon := \hat\beta\sqrt{2\pi}/\sqrt{\log \varepsilon^{-1}}$ [4].

In addition to the works by Caravenna, Sun and Zygouras, it is worth mentioning a recent generalization carried out in [**DG22**], which studied the *semilinear regularized (mSHE)*:

$$\begin{cases} \partial_t u_a^\varepsilon(t,x) \;=\; \frac{1}{2}\,\Delta u_a^\varepsilon(t,x) + \frac{1}{\sqrt{\log\varepsilon^{-1}}}\,\sigma(u_a^\varepsilon(t,x)) \cdot \dot W^\varepsilon(t,x) \qquad t \in \mathbb{R}_+\,,\; x \in \mathbb{R}^2\,, \\ u_a^\varepsilon(0,x) \equiv a\,, \end{cases}$$

where $\sigma : [0,\infty) \to [0,\infty)$ is a Lipschitz function with Lipschitz constant $L_\sigma < \sqrt{2\pi}$ and with flat initial condition $a > 0$. In particular, they proved that the limiting distribution of $u_a^\varepsilon(t,x)$ as $\varepsilon \to 0$ is expressed by a forward-backward SDE, which recovers the log-normal distribution (1.11) when σ is the identity function.

Although in this thesis we focus more on the discrete setting, in Chapter 2 we also present the continuum analog of Theorem 2.2, namely a *Central Limit Theorem for Wiener chaos* (see Theorem 2.9). In particular, the sufficient conditions $\widetilde{(1)}$-$\widetilde{(2)}$-$\widetilde{(3)}$ in Chapter 2 for the asymptotic Gaussianity of a given sequence of Wiener chaos (whose definition will be given) are exactly the continuum versions of (1)-(2)-(3). This general criterion provides an alternative approach to recover all convergence results for $u^\varepsilon(t,x)$ and $h^\varepsilon(t,x)$ obtained in [**CSZ17b, CSZ20**], *without the need to return to the discrete setting* as carried out in these works. In Section 3.4 of Chapter 3, by way of example we apply this other strategy to recover

[4]The presence of the factor $\sqrt{2}$ is linked to the periodicity of the simple random walk (see Remark 3.14).

the Edwards–Wilkinson fluctuations for the solution of the regularized (mSHE). The key idea is that the random field $\beta_\varepsilon^{-1}\big(u^\varepsilon(\cdot,\cdot)-1\big)$ is sufficiently regular to admit a Wiener chaos expansion, which easily verifies the assumptions $(\widetilde{1})$-$(\widetilde{2})$-$(\widetilde{3})$ for the asymptotic Gaussianity.

1.4.2. The 2d directed polymer beyond the subcritical regime. We now focus on the more subtle case where we approach the critical point $\hat\beta_c = 1$. We have already observed that by rescaling the disorder strength according to (1.10) with $\hat\beta = 1$ the diffusively rescaled partition function $Z_N^{\beta_N}\big(\sqrt{N}x\big) := Z_N^{\beta_N}\big(\lfloor\sqrt{N}x\rfloor\big)$ converges in law to 0 for all $x \in \mathbb{R}^2$, while $\mathbb{E}\big[Z_N^{\beta_N}\big(\sqrt{N}x\big)\big] \equiv 1$ and the h-th moments $\mathbb{E}\big[Z_N^{\beta_N}\big(\sqrt{N}x\big)^h\big]$ explode as $N\to\infty$ for $h\geq 2$ ([**CSZ19a**]). This singular asymptotic behaviour suggests that the random field

$$\Big\{Z_N^{\beta_N}\big(\sqrt{N}x\big)\ :\ x\in\mathbb{R}^2\Big\} \tag{1.21}$$

becomes rough in space as N is large, thus it is more convenient to study it as a *random distribution on* $\mathbb{R}^2$, i.e. by averaging the partition function in space:

$$Z_N^{\beta_N}(\varphi) := \int_{\mathbb{R}^2} Z_N^{\beta_N}\big(\sqrt{N}x\big)\,\varphi(x)\,\mathrm{d}x\,, \qquad \varphi\in C_c(\mathbb{R}^2)\,, \tag{1.22}$$

where for simplicity we take the initial time $t=0$.

The first contribution in this direction comes from Bertini and Cancrini [**BC98**], where the authors investigated the critical regime for the two-dimensional (mSHE). More precisely, they proved that there exists a critical window around $\hat\beta_c = 1$ where the continuum analog of (1.21) is tight and they explicitely computed the covariance function.

More recently, Caravenna, Sun and Zygouras ([**CSZ19a, CSZ19b**]) recovered results in [**BC98**]. In particular, they identified a critical window (comparable with the aforementioned one) around $\hat\beta_c = 1$ by rescaling the disorder strength $\beta = \beta_N$ such that

$$\beta_N^2 \sim \frac{1}{R_N}\left(1+\frac{\theta+o(1)}{\log N}\right) \quad \text{as } N\to\infty\,, \tag{1.23}$$

where θ is any real fixed parameter, so that $\beta_N \sim \frac{\hat\beta_c\sqrt{\pi}}{\sqrt{\log N}}$ with $\hat\beta_c = 1$. Moreover, they showed that the limiting third moment of (1.22) is bounded as $N\to\infty$, thus all subsequential limits cannot be trivial and admit the same covariance structure identified in [**BC98**].

Inspired by the works of Dell'Antonio-Figari-Teta ([**DFT94**]) and of Dimock-Rajeev ([**DR04**]) on Schrödinger operators with point interactions (also denoted as Delta-Bose gas) in dimension $d=2$, Gu, Quastel and Tsai ([**GQT21**]) computed asymptotically all moments of the averaged solution of the mollified (mSHE), which are bounded as $N\to\infty$. However, these moment estimates grow too fast, thus they are not sufficient to uniquely determine the distribution of the limiting random field.

A decisive step has been made very recently in [**CSZ21+**], where under the critical regime (1.23) the authors finally proved the uniqueness of the limiting random field, which is a *random measure* further studied in [**CSZ22+**] and called the *Critical 2d Stochastic Heat Flow* by the authors for being the good candidate for the solution to the two-dimensional (mSHE). The techniques developed in [**CSZ21+**] are based on polynomial chaos expansions, a coarse graining approach, a time-space renewal theory, a novel Lindeberg principle for multilinear polynomials of dependent variables and functional inequalities for Green's functions of random walks on $\mathbb{Z}^2$.

In the light of what has been said so far, the asymptotic behaviours of the subcritical and of the critical regime are deeply different. In the former, one can show that $Z_N^{\beta_N}(\varphi)$ (without being rescaled and centered) converges in law to a constant, i.e. its mean $\int_{\mathbb{R}^2} \varphi(x)\mathrm{d}x$, and its fluctuations around the mean rescaled by β_N^{-1} converge to a Gaussian limit (recall (1.14)). On the other hand, the situation significantly changes in the critical regime, where $Z_N^{\beta_N}(\varphi)$ converges in law to a random object (thus non constant), *non Gaussian*, withouth the need to rescale and center it.

The analysis carried out in Chapter 5 has the purpose of better investigating the gap between the aforementioned regimes, by studying an *intermediate regime*, namely

$$\beta_N^2 \sim \frac{1}{R_N}\left(1 - \frac{\theta_N}{\log N}\right) \quad \text{as } N \to \infty\,, \tag{1.24}$$

where θ_N is a function which diverges *slower than* $\log N$ as $N \to \infty$. Observe that the above window interpolates between the subcritical (by setting $\theta_N \sim (1-\hat{\beta})\log N$) and the critical regime (where $\theta_N = \theta + o(1)$, $\theta \in \mathbb{R}$). By this choise of β_N, we are approaching the critical point $\hat{\beta}_c = 1$ from below *at an arbitrarily slower rate than in the critical regime*. In fact, for any choice of divergent function $\theta_N = o(\log N)$ we have a family of intermediate regimes which are arbitrarily close to the critical one. For this reason, we choose to call it the *quasi-critical regime* and we investigate the scaling limit of $Z_N^{\beta_N}(\varphi)$ for this choice of β_N.

The second main contribution of our work is then focused on the *quasi-critical regime*, in particular we obtain

- Edwards-Wilkinson fluctuations of the rescaled partition function, see Theorem 5.1 (Chapter 5);
- an exact expression and related upper bounds for the moments with order higher than two of the partition function, with a special focus for the fourth moment, see Proposition 5.4 and Theorems 5.9, 5.11 and 5.13 (Chapter 5, Section 5.3).

Going into more details, in Chapter 5 we show that the rescaled and centered averaged partition function still admits Gaussian fluctuations:

$$\int_{\mathbb{R}^2} \sqrt{\theta_N}\big(Z_N^{\beta_N}(\sqrt{N}x) - 1\big)\,\varphi(x)\,\mathrm{d}x \xrightarrow[N\to\infty]{d} \langle \widetilde{V}, \varphi\rangle \sim \mathcal{N}(0, \sigma_\varphi^2)\,, \tag{1.25}$$

where $\widetilde{V}$ is a generalized Gaussian process with an explicit covariance structure, also characterized as $\widetilde{V}(x) = v^{(s,c)}$, where $v^{(s,c)}(1,x)$ is the solution of (EW) with $s = \frac{1}{2}$ and $c = \pi$. Note that in this case the rescaling $\sqrt{\theta_N}$ is different (it reasonably recovers the one in the subcritical regime when $\theta_N = O(\log N)$) and is arbitrarily slow.

Although the result (1.25) is similar to what happens in the subcritical regime (1.14), the proof presented for the quasi-critical case is much more challenging. We still work with the polynomial chaos expansion of the partition function, but contrary to the subcritical regime where the limiting second moment of $Z_N^{\beta_N}(\sqrt{N}x)$ is bounded for any $x \in \mathbb{R}^2$, here it explodes as $N \to \infty$. By suitably rescaling the centered averaged partition function by $\sqrt{\theta_N}$, we are anyway able to obtain a bounded limiting second moment σ_φ^2. However, the key observation is that in this setting *a finite number of fixed chaos no longer gives the main contribution to* σ_φ^2 *as* $N \to \infty$. Therefore, all the standard techniques for proving the Gaussianity, such as the Fourth Moment Theorem and the hypercontractivity property necessary to apply our CLT for polynomial chaos, no longer apply in this context.

The strategy we follow in Sections 5.1, 5.2 and 5.3 is still based on the Feller-Lindeberg CLT for triangular arrays, but we need alternative estimates for the h-th moment with $h > 2$ of the averaged, centered and rescaled partition function. In the subcritical regime we could simply exploit the hypercontractivity for polynomial chaos, which no longer applies here. We present another, more delicate, estimate of the fourth moment, which is a result of independent interest (see Proposition 5.4) and can be potentially generalized for all h-th moments with $h > 2$. The proof is inspired by those for the analog results in [**CSZ21+**, Theorem 6.1] and in [**LZ21+**], based on similar estimates from [**GQT21**] adapted in the discrete setting, where the key ingredient is a functional inequality (precisely, a Hardy-Littlewood-Sobolev type inequality) for the Green's function of multiple random walks on $\mathbb{Z}^2$. These papers work in the critical and subcritical regime, but the same approach can also be applied in the quasi-critical regime that we consider. Also in view of future applications, we also present a refined formulation of this approach in two directions: we first make the strategy explicitly independent of the disorder regime of β and then we separate the exact expression for the h-th moments (see Theorem 5.9) from the upper bounds which can be deduced (see see Theorems 5.11 and 5.13). In particular, in our case we work on a different time scale that requires additional novel ideas to obtain the correct optimal estimate. We prove Proposition 5.4 in Section 5.3 of Chapter 5.

1.5. Conclusions and perspectives

We discussed several novel or revisited convergences to a Gaussian limit for directed polymers, under both subcritical and quasi-critical regimes.

Regarding our general CLT for polynomial chaos, it would be interesting to investigate how far from optimality are our conditions (1)–(2)–(3) in Chapter 2. When the polynomial chaos belongs to a fixed order chaos, the conditions of the Fourth Moment Theorem are known to be *optimal*, i.e. necessary and sufficient for the asymptotic Gaussianity of the chaos sequence. However, a comparison between our conditions and the ones of Fourth Moment Theorem is not in principle straightforward, especially for what concerns condition (3) as we explicitely point out in Remark 2.3.

Another direction of future research is about scaling limits for d-DPRE in higher dimensions $d \geq 3$. The Edwards–Wilkinson fluctuations (1.14) and (1.15) have been proved for $d \geq 3$ in the so-called "L^2 regime" (i.e. where the limiting second moment remains bounded) in [**CN21, LZ22**] and [**CNN22**], sharpening previous work from [**MU18, GRZ18, CCM20, DGRZ20**]; see also [**CCM21+**] for related recent results. Recall that contrary to the two-dimensional setting where the L^2 region agrees with the weak disorder regime $\hat{\beta} \in (0, \hat{\beta}_c)$ with $\hat{\beta}_c = 1$ provided that β_N is rescaled as (1.10), the same behaviour does not hold for $d \geq 3$. Indeed, in higher dimension the L^2 region is just a strict subset of the weak disorder regime, moreover a concrete characterization of the critical value β_c where the weak/strong disorder transition occurs is still missing. In this respect, see [**Jun22, Jun21+, Jun22+**] for recent results in high dimension beyond the L^2 regime. It would be interesting to apply the approach based on our CLT for polynomial chaos (Theorem 2.2) in this higher dimensional context, in order to recover the existing results and to check whether it is possible to go slightly beyond the L^2 regime, in the same spirit as carried out for the quasi-critical regime in $d = 2$.

Moreover, it would be interesting to explore the quasi-critical regime by also extending the results on the singular product (1.17) and on the Edwards-Wilkinson fluctuations for the log-partition function (1.15). Concerning especially the latter result, we expect to recover the analogous Gaussian limit even in the quasi-critical regime. Indeed, the Gaussianity for the subcritical regime eventually follows from the Edwards-Wilkinson fluctuations for the partition function (see (1.14)), which has already been obtained in the quasi-critical regime. This is not sufficient, indeed we will also need to generalize the linearization procedure developed in [**CSZ20**] under the subcritical regime and briefly recalled in Subsection 3.2.2 of Chapter 3. The challenge will be to find a convenient strategy to treat the negative moments estimates required by the approach in [**CSZ20**] under the quasi-critical regime and this is non–trivial a priori.

The situation is more subtle and delicate regarding the approximation of the log-partition function for fixed starting point (1.19)-(1.20) under the quasi-critical regime. Indeed, the proof presented in this thesis to derive the approximation of $\log Z_N^{\beta_N}$ in terms of X_N^{dom} strongly depends on the subcritical structure of the polynomial chaos involved. Therefore, it is not clear how to extend this result in the new framework and novel techniques will be surely nedeed.

CHAPTER 2

Central Limit Theorems for polynomial and Wiener chaos

In this chapter we state our first main results: a general criterion for the convergence in distribution of polynomial chaos or Wiener chaos to a Gaussian limit (see Theorems 2.2 and 2.9).

We start to phrase our converence result in the discrete setting of polynomial chaos, which is more elementary. Then, we will show that the new criterion has a direct translation for the continuum environment of Wiener chaos.

2.1. Polynomial chaos

A polynomial chaos is a *multilinear polynomial* with variables given by *independent random variables.* Also known as *discrete chaos* – due to the direct link with the continuum analogue called Wiener chaos – these multilinear polynomials play a fundamental role in the study of disordered systems and for this reason it is worth treating them separately in a more general context.

In order to define a polynomial chaos, let $\mathbb{T}$ be a countable set. We consider a family $\eta = (\eta_t)_{t\in\mathbb{T}}$ of *independent random variables*, not necessarily identically distributed, defined on the same probability space $(\widetilde{\Omega}, \mathcal{G}, \mathbb{P})$, with zero mean and unit variance:

$$\mathbb{E}[\eta_t] = 0\,, \qquad \mathbb{E}[(\eta_t)^2] = 1\,, \qquad \forall\, t \in \mathbb{T}\,. \tag{2.1}$$

Let $q : \mathcal{P}(\mathbb{T}) \to \mathbb{R}$ be a real-valued *deterministic* function defined on the power set of $\mathbb{T}$ such that $q(A) \neq 0$ only if $0 < |A| < \infty$, where $|A|$ is the cardinality of any arbitrary subset $A \subset \mathbb{T}$.

Definition 2.1. *Given a family of random variables $\eta = (\eta_t)_{t\in\mathbb{T}}$ and a function $q : \mathcal{P}(\mathbb{T}) \to \mathbb{R}$ defined as above, we call* polynomial chaos *a multilinear polynomial in the variables η_t's whose coefficients are given by the function q, namely*

$$X(\eta) := \sum_{A\subset\mathbb{T}} q(A)\,\eta(A)\,, \qquad \textit{with} \qquad \eta(A) := \prod_{t\in A} \eta_t\,, \tag{2.2}$$

where $\eta(A)$ is meant as product of all distinct elements in the subset A.

Notice that, by definition of the coefficient function q, the sum in $X(\eta)$ only ranges over *finite nonempty subsets* $A \subset \mathbb{T}$.

Definition (2.2) is elegant, but quite abstract. We can obtain a more explicit equivalent definition, easier to be managed, by splitting the sum in (2.2) according to the cardinality k of the subset A. If we consider $A = \{t_1, \dots, t_k\}$ with distinct points $t_i \in \mathbb{T}$, we can indeed rewrite (2.2) as

$$X(\eta) = \sum_{k=1}^{\infty} X^k(\eta) = \sum_{k=1}^{\infty} \sum_{\substack{\{t_1,\dots,t_k\}\subset\mathbb{T} \\ t_i\neq t_j\ \forall i\neq j}} q(\{t_1, \dots, t_k\}) \prod_{i=1}^{k} \eta_{t_i}\,, \tag{2.3}$$

where, for each $k \in \mathbb{N}$, the term

$$X^k(\eta) := \sum_{\substack{\{t_1,\dots,t_k\}\subset\mathbb{T} \\ t_i\neq t_j\ \forall i\neq j}} q(\{t_1, \dots, t_k\}) \prod_{i=1}^{k} \eta_{t_i}\,,$$

is referred as *chaos of order k* and simply corresponds to all polynomial terms in the η_t's of degree k.

Assuming that $\sum_{A\subset\mathbb{T}} q(A)^2 < \infty$, the polynomial chaos $X(\eta)$ is a well-defined random variable in $L^2 := L^2(\widetilde{\Omega}, \mathbb{P})$ with

$$\begin{aligned} \mathbb{E}[X(\eta)] &= \sum_{A\subset\mathbb{T}} q(A)\,\mathbb{E}[\eta(A)] = 0\,, \\ \mathbb{E}[X(\eta)^2] &= \sum_{A\subset\mathbb{T}} q(A)^2\,\mathbb{E}[\eta(A)^2] = \sum_{A\subset\mathbb{T}} q(A)^2\,, \end{aligned} \tag{2.4}$$

because by the nice properties of the η_t's we easily see that $(\eta(A))_{A\subset\mathbb{T}}$ are centered and orthogonal random variables in L^2.

To get to the heart of our problem, we allow the η_t's to *depend on* $N \in \mathbb{N}$, i.e. we work with an *array* of random variables $\eta^N = (\eta_t^N)_{t\in\mathbb{T}}$, still independent with zero mean and unit variance. We further require the *uniform integrability of the squares*:

$$\lim_{L\to\infty} \sup_{N\in\mathbb{N},\, t\in\mathbb{T}} \mathbb{E}\Big[|\eta_t^N|^2\,\mathbb{1}_{\{|\eta_t^N|>L\}}\Big] = 0\,, \tag{2.5}$$

which follows from (2.1) if the η_t^N's have the same distribution. In general, a sufficient easy condition for (2.5) is that $\sup_{N,t} \mathbb{E}[|\eta_t^N|^p] < \infty$ for some $p > 2$. We do not necessarily need to impose the latter assumption in order to define a polynomial chaos, however this choice turns out to be convenient when we prove Theorem 2.2. Nevertheless, when we apply our convergence result in the following chapters we will always work with arrays $\eta^N = (\eta_t^N)_{t\in\mathbb{T}}$ of i.i.d. centered random variables with unit variance, which – as already noticed – easily satisfy (2.5).

At this point, consider a *sequence of polynomial chaos* $(X_N)_{N\in\mathbb{N}}$, i.e.

$$X_N = X_N(\eta^N) := \sum_{A\subset\mathbb{T}} q_N(A)\,\eta^N(A)\,, \qquad \text{with} \qquad \eta^N(A) := \prod_{t\in A} \eta_t^N\,, \tag{2.6}$$

or, equivalently,

$$X_N = X_N(\eta^N) := \sum_{k=1}^{\infty} X_N^k = \sum_{k=1}^{\infty} \sum_{\substack{\{t_1,\ldots,t_k\}\subset\mathbb{T} \\ t_i\neq t_j\ \forall i\neq j}} q_N(\{t_1,\ldots,t_k\}) \prod_{i=1}^{k} \eta_{t_i}^N , \tag{2.7}$$

where from now on we drop the dependence of η in X_N to simplify the notation. Moreover, we let the real coefficients $q_N(\cdot)$ depend on $N \in \mathbb{N}$, too.

It is natural to impose also here that $\sum_{A\subset\mathbb{T}} q_N(A)^2 < \infty$, so that $(X_N)_{N\in\mathbb{N}}$ is a sequence of well-defined L^2 random variables with

$$\mathbb{E}[X_N] = 0, \qquad \mathbb{E}[X_N^2] = \sum_{A\subset\mathbb{T}} q_N(A)^2 , \tag{2.8}$$

for all $N \in \mathbb{N}$, in analogy with (2.4).

Our goal is to prove *convergence in distribution as $N \to \infty$ of the sequence X_N toward a Gaussian random variable $X \sim \mathcal{N}(0, \sigma^2)$*, for some finite $\sigma^2 > 0$. In general, since a Gaussian random variable is uniquely characterized by its moments, the *Method of moments* (see, for instance [**NP12**, Theorem A.3.1]) provides a sufficient way to achieve this purpose, by verifying that all moments of X_N converge to the corresponding moments of X. On the other hand, if we deal with a sequence X_N in a *fixed order chaos* (i.e. a single term k in (2.7)), this problem can be significantly simplified by applying the celebrated *Fourth Moment Theorem*. Formulated in our context in [**NPR10**] and slightly extended in [**CSZ17b**, Theorem 4.2] (see also the previous works [**NP05, dJ90, dJ87, Rot79**] and the book [**NP12**]), the Fourth Moment Theorem indeed requires to compute only the *second and fourth moments of X_N* as $N \to \infty$ to show convergence in distribution toward X.

In this connection, our first main result gives sufficient conditions for convergence to a Gaussian limit *purely based on second moment assumptions on X_N*, withouth requiring higher moment bounds. This is indeed convenient, since computing the fourth moment can be a hard and technical combinatorial problem, non-trivial to be solved a priori. Moreover, this novel criterion can be directly applied to a superposition of chaos of different orders.

To see our sufficient conditions in detail, let us introduce the shorthand

$$\sigma_N^2(\mathbb{B}) := \sum_{A\subset\mathbb{B}} q_N(A)^2 \qquad \text{for} \quad \mathbb{B} \subset \mathbb{T} , \tag{2.9}$$

which gives the contribution to the second moment of X_N of the subsets of $\mathbb{B}$ (recall (2.8)). We can formulate our assumptions as follows.

(1) *Limiting second moment*:

$$\lim_{N\to\infty} \sigma_N^2(\mathbb{T}) = \lim_{N\to\infty} \sum_{A\subset\mathbb{T}} q_N(A)^2 \; = \sigma^2 \in (0,\infty) , \tag{2.10}$$

i.e. the second moment of X_N converges to a finite limit.

(2) *Subcriticality*:

$$\lim_{K\to\infty} \limsup_{N\to\infty} \sum_{\substack{A\subset\mathbb{T}\\ |A|>K}} q_N(A)^2 = 0\,, \tag{2.11}$$

i.e. the contribution of high order chaos to the second moment of X_N is negligible.

(3) *Spectral localization*: for any $M, N \in \mathbb{N}$ we can find M *disjoint subsets ("boxes")*:

$$\mathbb{B}_1, \ldots, \mathbb{B}_M \subset \mathbb{T} \qquad \text{with} \qquad \mathbb{B}_i \cap \mathbb{B}_j = \emptyset \quad \text{for } i \neq j\,,$$

(where $\mathbb{B}_i = \mathbb{B}_i^{(N,M)}$ may depend on N, M) such that the following conditions hold (recall (2.9)):

$$\lim_{M\to\infty} \lim_{N\to\infty} \sum_{i=1}^{M} \sigma_N^2(\mathbb{B}_i) \;=\; \sigma^2\,, \tag{2.12}$$

$$\lim_{M\to\infty} \limsup_{N\to\infty} \Big\{ \max_{i=1,\ldots,M} \sigma_N^2(\mathbb{B}_i) \Big\} \;=\; 0\,, \tag{2.13}$$

i.e. the main contribution to the second moment of X_N comes from *subsets contained in one of the boxes* $\mathbb{B}_1, \ldots, \mathbb{B}_M$, whose individual contribution is uniformly small.

Note that conditions (1), (2), (3) are *second moment assumptions.* The name "subcriticality" for condition (2) is inspired by directed polymers, which we discuss in Chapter 3, and more generally by marginally relevant disordered systems (see [**CSZ17b**]), which undergo a phase transition at a critical point determined precisely by the failure of condition (2.11).

We can now state our first main result.

Theorem 2.2 (Gaussian limits for polynomial chaos). *Let X_N be a polynomial chaos as in* (2.6)*, with coefficients $q_N(\cdot)$ satisfying the assumptions* (1)*,* (2)*,* (3) *(see* (2.10)*–*(2.13)*), with respect to independent random variables $\eta^N = (\eta_t^N)_{t\in\mathbb{T}}$ which satisfy* (2.1) *and* (2.5)*. Then as $N \to \infty$ we have the convergence in distribution*

$$X_N \xrightarrow{\;d\;} \mathcal{N}(0, \sigma^2)\,. \tag{2.14}$$

The proof is given in Section 2.3 and comes in two steps:

- first we approximate X_N in L^2 by a sum $\sum_{i=1}^{M} X_{N,i}$ of *independent* random variables, for a suitable $M = M_N \to \infty$;
- then we show that the random variables $(X_{N,i})_{1\le i\le M_N}$ satisfy the assumption of the *Feller-Lindeberg Central Limit Theorem for triangular arrays* (see Theorem 2.14), which eventually yields (2.14).

We will also replace the random variables $(\eta_t^N)_{t\in\mathbb{T}}$ by a family of random variables with bounded moments of some order $p > 2$ (e.g. by Gaussians) to exploit the hypercontractivity of polynomial chaos, see [**MOO10, Jan97**]. The justification of this replacement will be given at the end of the proof exploiting a suitable Lindeberg principle, see [**CSZ17a, MOO10**].

REMARK 2.3. *It would be interesting to investigate how far from optimality are our conditions* (2.10)–(2.13). *When the polynomial chaos X_N belongs to a fixed order chaos, the conditions of the Fourth Moment Theorem are known to be* optimal, *i.e. necessary and sufficient for the asymptotic Gaussianity of X_N. However, in this setting a comparison between our conditions and the Fourth Moment Theorem is not straightforward, due to the freedom in the choice of the boxes $\mathbb{B}_i$ in* (2.12)-(2.13).

2.2. Wiener chaos

We now discuss the continuum setting of Wiener chaos. Let us briefly introduce those notions necessary to state our result. For a complete overview of Wiener Chaos and related topics, see for instance [**Ito51, Jan97, NP12**].

Let $(E, \mathcal{E}, \mu)$ be a Polish (complete separable metric) space, endowed with its Borel σ-field $\mathcal{E}$ and with a non-atomic measure μ. Let $\mathcal{E}^* = \{A \in \mathcal{E} : \ \mu(A) < \infty\}$ be the class of measurable sets with finite measure.

DEFINITION 2.4. *A* Gaussian random measure *on $(E, \mathcal{E}, \mu)$ is a Gaussian process $W = (W(A))_{A\in\mathcal{E}^*}$ defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$, such that*

- $\mathbb{E}[W(A)] = 0$ *for all* $A \in \mathcal{E}^*$*;*
- $\mathbb{C}\mathrm{ov}[W(A), W(B)] = \mu(A \cap B)$ *for all* $A, B \in \mathcal{E}^*$*.*

We often use the informal notation $W(\mathrm{d}x)$. The most important example the reader should keep in mind is given by the *white noise*, which corresponds to $E = \mathbb{R}^d$ with $\mu =$ Lebesgue measure.

We fix a Gaussian random measure $W(\mathrm{d}x)$ on $(E, \mathcal{E}, \mu)$. For every $k \in \mathbb{N}$ and every real function $f \in L^2(E^k, \mu^{\otimes k})$, by [**Ito51, NP12**] it is possible to define the *stochastic multiple Wiener integral*

$$W^{\otimes k}(f) = \int_{E^k} f(x_1, \ldots, x_k)\, W(\mathrm{d}x_1) \cdots W(\mathrm{d}x_k)\,,$$

which is a centered random variable in $L^2(\Omega)$ (non Gaussian as soon as $k > 1$ and $f \not\equiv 0$). For *symmetric* functions $f \in L^2(E^k, \mu^{\otimes k})$ and $g \in L^2(E^{k'}, \mu^{\otimes k'})$ the *Ito isometry* holds:

$$\begin{aligned}\mathbb{E}[W^{\otimes k}(f)\, W^{\otimes k'}(g)] &= \mathbb{1}_{\{k=k'\}}\, k!\, \langle f, g\rangle_{L^2(E^k,\mu^{\otimes k})} \\ &= \mathbb{1}_{\{k=k'\}}\, k! \int_{E^k} f(x_1, \ldots, x_k)\, g(x_1, \ldots, x_k)\, \mu(\mathrm{d}x_1) \cdots \mu(\mathrm{d}x_k)\,.\end{aligned} \tag{2.15}$$

Remark 2.5. *In general, if a function* $f \in L^2(E^k\mu^{\otimes k})$ *is not symmetric, we can define its* symmetrized version *as*

$$\widetilde{f}(x_1,\ldots,x_k) := \frac{1}{k!}\sum_{\pi\in\mathcal{P}(k)} f(x_{\pi(1)},\ldots,x_{\pi(k)})\,, \tag{2.16}$$

where $\mathcal{P}(k)$ *is the set of all permutations of* $\{1,\ldots,k\}$*. Obviously, when* f *is symmetric we have* $\widetilde{f} = f$*.*

Then, it is possible to show that for any $f, g \in L^2(E^k\mu^{\otimes k})$*:*

$$\begin{aligned}\mathbb{E}[W^{\otimes k}(f)\,W^{\otimes k'}(g)] &= \mathbb{1}_{\{k=k'\}}\,k!\,\langle \widetilde{f},\widetilde{g}\rangle_{L^2(E^k,\mu^{\otimes k})}\\ &= \mathbb{1}_{\{k=k'\}}\,k!\int_{E^k}\widetilde{f}(x_1,\ldots,x_k)\,\widetilde{g}(x_1,\ldots,x_k)\,\mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_k)\,,\end{aligned} \tag{2.17}$$

where $\widetilde{f}$ *and* $\widetilde{g}$ *are the symmetrized versions of* f *and* g*, respectively.*

In analogy with the discrete polynomial chaos (2.7), we consider a sequence $(\widetilde{X}_N)_{N\in\mathbb{N}}$ of Wiener chaos with respect to $W(\mathrm{d}x)$. We briefly recall its definition.

Definition 2.6. *Fix* $N \in \mathbb{N}$*. A sequence of* Wiener chaos with respect to $W(\mathrm{d}x)$ *is defined as*

$$\widetilde{X}_N := \sum_{k=1}^{\infty}\widetilde{X}_N^k = \sum_{k=1}^{\infty}\int_{E^k}\widetilde{q}_N(x_1,\ldots,x_k)\,W(\mathrm{d}x_1)\cdots W(\mathrm{d}x_k)\,, \tag{2.18}$$

where $\widetilde{q}_N$ *is a* symmetric L^2 *function defined on* $\bigcup_{k=1}^{\infty}(E^k,\mathcal{E}^{\otimes k},\mu^{\otimes k})$*.*

By applying (2.15), we can easily compute the first and second moment of $\widetilde{X}_N$:

$$\begin{aligned}\mathbb{E}[\widetilde{X}_N] = 0\,,\qquad \mathbb{E}[\widetilde{X}_N^2] &= \sum_{k=1}^{\infty}k!\,\|\widetilde{q}_N\|^2_{L^2(E^k)}\\ &= \sum_{k=1}^{\infty}k!\int_{E^k}\widetilde{q}_N(x_1,\ldots,x_k)^2\,\mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_k)\,.\end{aligned} \tag{2.19}$$

Remark 2.7. *Actually, every centered random variable in* $L^2(\Omega)$*, which is measurable with respect to the* σ*-algebra generated by* W*, admits an expansion like* (2.18)*. See, for instance,* [**Jan97**, Theorem 2.6].

Remark 2.8. *We observe that the factor* $k!$ *in* (2.19) *comes from the fact that* $\widetilde{q}_N$ *in* (2.18) *is a symmetric function of the* ordered *variables* $x_1,\ldots,x_k$*, whereas* q_N *in* (2.7) *is a function of* unordered *variables (i.e. subsets)* $\{t_1,\ldots,t_k\}$*. To formally match* (2.7)-(2.8) *with* (2.18)-(2.19)*, we should identify* q_N *with* $k!\,\widetilde{q}_N$ *and* $\sum_{\{t_1,\ldots,t_k\}\subset\mathbb{T}}\prod_{i=1}^k\eta_{t_i}^N$ *with* $\frac{1}{k!}\int_{E^k}W(\mathrm{d}x_1)\cdots W(\mathrm{d}x_k)$*.*

Mimicking (2.9), we set

$$\widetilde{\sigma}_N^2(\mathbb{B}) := \sum_{k=1}^{\infty}k!\int_{\mathbb{B}^k}\widetilde{q}_N(x_1,\ldots,x_k)^2\,\mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_k)\qquad\text{for measurable } \mathbb{B}\subset E\,, \tag{2.20}$$

which gives the contribution to the second moment of $\widetilde{X}_N$ of subsets in $\mathbb{B}$, see (2.19). We can now formulate our conditions in the continuum setting.

$(\widetilde{1})$ *Limiting second moment*:

$$\lim_{N\to\infty} \widetilde{\sigma}_N^2(E) = \lim_{N\to\infty} \sum_{k=1}^{\infty} k! \, \|\widetilde{q}_N\|^2_{L^2(E^k)} = \sigma^2 \in (0,\infty)\,, \tag{2.21}$$

i.e. the second moment of $\widetilde{X}_N$ converges to a finite limit.

$(\widetilde{2})$ *Subcriticality*:

$$\lim_{K\to\infty} \limsup_{N\to\infty} \sum_{k>K} k! \, \|\widetilde{q}_N\|^2_{L^2(E^k)} = 0\,, \tag{2.22}$$

i.e. the contribution of high order chaos to the second moment of $\widetilde{X}_N$ is negligible.

$(\widetilde{3})$ *Spectral localization*: for any $M, N \in \mathbb{N}$ we can find M *disjoint subsets ("boxes")*:

$$\mathbb{B}_1, \ldots, \mathbb{B}_M \subset E \qquad \text{with} \qquad \mathbb{B}_i \cap \mathbb{B}_j = \emptyset \quad \text{for } i \neq j$$

(where $\mathbb{B}_i = \mathbb{B}_i^{(N,M)}$ may depend on N, M) such that, recalling (2.20),

$$\lim_{M\to\infty} \lim_{N\to\infty} \sum_{i=1}^{M} \widetilde{\sigma}_N^2(\mathbb{B}_i) = \sigma^2\,, \tag{2.23}$$

$$\lim_{M\to\infty} \lim_{N\to\infty} \Big\{ \max_{i=1,\ldots,M} \widetilde{\sigma}_N^2(\mathbb{B}_i) \Big\} = 0\,, \tag{2.24}$$

i.e. the main contribution to the second moment of $\widetilde{X}_N$ comes from *subsets contained in one of the M boxes* $\mathbb{B}_1, \ldots, \mathbb{B}_M$, whose individual contribution is uniformly small.

We can finally state the version of Theorem 2.2 for Wiener chaos.

Theorem 2.9 (Gaussian limits for Wiener chaos). *Let $\widetilde{X}_N$ be a Wiener chaos as in* (2.18)*, with coefficients $\widetilde{q}_N(\cdot)$ satisfying the assumptions* $(\widetilde{1})$, $(\widetilde{2})$, $(\widetilde{3})$ *(see* (2.21)–(2.24)*), with respect to a Gaussian random measure $W(\mathrm{d}x)$ on a Polish measure space $(E, \mathcal{E}, \mu)$. Then as $N \to \infty$ we have the convergence in distribution*

$$\widetilde{X}_N \xrightarrow{d} \mathcal{N}(0, \sigma^2)\,. \tag{2.25}$$

2.3. Central Limit Theorem for polynomial chaos: proof of Theorem 2.2

As a preliminary step to prove Theorem 2.2, *we replace the random variables $(\eta_t^N)_{t\in\mathbb{T}}$ in the definition* (2.6) *of X_N by independent standard Gaussians.* We will show in Subsection 2.3.4 that such a replacement does not affect the asymptotic distribution of X_N as $N \to \infty$.

We therefore assume that $\eta_t^N \sim \mathcal{N}(0,1)$. We then exploit the *hypercontractivity of polynomial chaos*, which allows us to bound moments of order $p > 2$ in terms of second moments, see [**MOO10**, Section 3.2] and [**Jan97**, Theorem 5.1]:

$$\forall p > 2: \qquad \mathbb{E}\Bigg[\bigg|\sum_{A\subset\mathbb{T}} q_N(A)\,\eta^N(A)\bigg|^p\Bigg] \le \bigg(\sum_{A\subset\mathbb{T}} (p-1)^{|A|}\, q_N(A)^2\bigg)^{\frac{p}{2}}. \tag{2.26}$$

Remark 2.10. *Actually, in* [**MOO10**] *and* [**Jan97**], *the hypercontractivity is only shown for chaotic expansions with a finite number of fixed chaos. However, recall that each polynomial chaos is seen as a series* $X_N = \sum_{k=1}^{\infty} X_N^k$ *which converges in* L^2, *thus almost surely up to subsequence. Therefore, the formula in* (2.26) *easily follows by applying Fatou's Lemma.*

Remark 2.11. *The choice of a Gaussian distribution for the* η_t^N*'s is not fundamental here: hypercontractivity of polynomial chaos holds for* arbitrary distributions of the η_t^N's with uniformly bounded moments*: if* $\sup_{N,t}\mathbb{E}[|\eta_t^N|^{\bar{p}}] < \infty$ *for some* $\overline{p} > p$, *then*

$$\mathbb{E}\Bigg[\bigg|\sum_{A\subset\mathbb{T}} q_N(A)\,\eta^N(A)\bigg|^p\Bigg] \le \bigg(\sum_{A\subset\mathbb{T}} C_p^{|A|}\, q_N(A)^2\bigg)^{\frac{p}{2}}, \tag{2.27}$$

for a suitable $C_p < \infty$ *with* $\lim_{p\downarrow 2} C_p = 1$*: see* [**CSZ20**, Theorem B.1].

2.3.1. Preparation. We consider a sequence of polynomial chaos X_N, with coefficients $q_N(\cdot)$ as in (2.6), which satisfy assumptions (1), (2), (3), see the equations (2.10)-(2.13). We now build two suitable diverging sequences of integers $M_N \to \infty$, $K_N \to \infty$.

- We fix $M_N \to \infty$ slowly enough so that assumption (3) still holds with $M = M_N$. More explicitly, for every $N \in \mathbb{N}$ we can find disjoint subsets ("boxes") $\mathbb{B}_i = \mathbb{B}_i^{(N)}$:

$$\mathbb{B}_1, \dots, \mathbb{B}_{M_N} \subset \mathbb{T} \qquad \text{with} \qquad \mathbb{B}_i \cap \mathbb{B}_j = \emptyset \quad \text{for } i \neq j,$$

  such that the following versions of (2.12)-(2.13) hold:

$$\lim_{N\to\infty} \sum_{i=1}^{M_N} \sigma_N^2(\mathbb{B}_i) \,=\, \sigma^2 \qquad \text{and} \qquad \lim_{N\to\infty} \Big\{\max_{i=1,\dots,M_N} \sigma_N^2(\mathbb{B}_i)\Big\} \,=\, 0. \tag{2.28}$$

- By the second relation in (2.28), we can fix $K_N \to \infty$ slowly enough so that

$$\lim_{N\to\infty} 8^{K_N} \max_{i=1,\dots,M_N} \sigma_N^2(\mathbb{B}_i) \,=\, 0. \tag{2.29}$$

  The reason for this specific choice will be clear later, see the discussion after (2.56). Note that by our assumption (2), see (2.11), for any $K_N \to \infty$ we have

$$\lim_{N\to\infty} \sum_{\substack{A\subset\mathbb{T}\\ |A|>K_N}} q_N(A)^2 = 0. \tag{2.30}$$

Remark 2.12. *It is standard to deduce* (2.28) *from* (2.12)-(2.13)*. Indeed, given any real sequence* $a_{N,M}$ *which admits the limits*

$$\lim_{M\to\infty} \limsup_{N\to\infty} a_{N,M} \;=\; \lim_{M\to\infty} \liminf_{N\to\infty} a_{N,M} \;=\; \alpha\,,$$

we can always choose $M = M_N \to \infty$ *slowly enough so that* $\lim_{N\to\infty} a_{N,M_N} = \alpha$*, as one can check directly. Then, to obtain* (2.28) *from* (2.12)-(2.13)*, it suffices to consider*

$$a_{N,M} = \sum_{i=1}^{M} \sigma_N^2\big(\mathbb{B}_i^{(N,M)}\big)\,, \qquad \textit{resp.} \qquad a_{N,M} = \max_{i=1,\ldots,M} \sigma_N^2\big(\mathbb{B}_i^{(N,M)}\big)\,.$$

We next proceed with the actual proof of Theorem 2.2. We follow the steps outlined after the statement of Theorem 2.2:

- first we approximate in L^2 the polynomial chaos X_N in (2.6) by a sum of suitable *independent* random variables

  $$X_{N,1} + \ldots + X_{N,M_N}\,,$$

  for $M_N \to \infty$ as $N \to \infty$ fixed according to (2.28) (see Subsection 2.3.2 and in particular Lemma 2.32);
- then we send $N \to \infty$ and we apply the Feller-Lindeberg CLT to the triangular array $(X_{N,i})_{i=1,\ldots,M_N}$, which ensures the Gaussian limit in distribution of $\sum_{i=1}^{M_N} X_{N,i}$, namely

  $$\sum_{i=1}^{M_N} X_{N,i} \xrightarrow{d} \mathcal{N}(0,\sigma^2)\,,$$

  (see Subsection 2.3.3);
- in conclusion, since X_N and $\sum_{i=1}^{M_N} X_{N,i}$ are close in L^2 as $N \to \infty$, we easily obtain the asymptotic Gaussianity (2.14).

2.3.2. Approximation of X_N. We recall the notation $\eta^N(A) := \prod_{t\in A} \eta_t^N$, see (2.6). We define a triangular array of random variables $(X_{N,i})_{i=1,\ldots,M_N}$ by setting

$$X_{N,i} := \sum_{\substack{A\subset \mathbb{B}_i \\ |A|\le K_N}} q_N(A)\,\eta^N(A) \qquad \text{for } i = 1,\ldots,M_N\,, \tag{2.31}$$

where we recall that $M_N \to \infty$ and $K_N \to \infty$ have been fixed so that (2.28)-(2.30) hold.

We now show that the sum $\sum_{i=1}^{M_N} X_{N,i}$ is a good approximation of X_N.

Lemma 2.13. *The following holds:*

$$\lim_{N\to\infty} \left\| X_N - \sum_{i=1}^{M_N} X_{N,i} \right\|_{L^2} = 0\,. \tag{2.32}$$

Proof. Let us define a modification of the random variables $X_{N,i}$ in (2.31), where we simply remove the constraint $|A| \le K_N$:

$$\widetilde{X}_{N,i} := \sum_{A \subset \mathbb{B}_i} q_N(A)\, \eta^N(A) \qquad \text{for } i = 1, \ldots, M_N\,.$$

We are going to show that

$$\lim_{N\to\infty} \left\| X_N - \sum_{i=1}^{M_N} \widetilde{X}_{N,i} \right\|_{L^2} = 0 \qquad \text{and} \qquad \lim_{N\to\infty} \left\| \sum_{i=1}^{M_N} \widetilde{X}_{N,i} - \sum_{i=1}^{M_N} X_{N,i} \right\|_{L^2} = 0\,. \tag{2.33}$$

The first relation is a direct consequence of our assumptions (1) and (3). Indeed, since the boxes $\mathbb{B}_i$ are disjoint, we have

$$\sum_{i=1}^{M_N} \widetilde{X}_{N,i} = \sum_{i=1}^{M_N} \sum_{A\subset\mathbb{B}_i} q_N(A)\,\eta^N(A) = \sum_{A\subset\bigcup_{i=1}^{M_N}\mathbb{B}_i} q_N(A)\,\eta^N(A)\,,$$

thus the random variable $\sum_{i=1}^{M_N} \widetilde{X}_{N,i}$ is the polynomial chaos where we only sum over subsets $A \subset \bigcup_{i=1}^{M_N} \mathbb{B}_i$. Therefore, the difference $X_N - \sum_{i=1}^{M_N} \widetilde{X}_{N,i}$ is orthogonal in L^2 to $\sum_{i=1}^{M_N} \widetilde{X}_{N,i}$. As a consequence, recalling also (2.9) and the independence of the $\widetilde{X}_{N,i}$'s, we can write

$$\begin{aligned}
\left\| X_N - \sum_{i=1}^{M_N} \widetilde{X}_{N,i} \right\|_{L^2}^2 &= \|X_N\|_{L^2}^2 - \left\| \sum_{i=1}^{M_N} \widetilde{X}_{N,i} \right\|_{L^2}^2 \\
&= \|X_N\|_{L^2}^2 - \sum_{i=1}^{M_N} \left\| \widetilde{X}_{N,i} \right\|_{L^2}^2 \\
&= \sum_{A\subset\mathbb{T}} q_N(A)^2 - \sum_{i=1}^{M_N} \sigma_N^2(\mathbb{B}_i)\,,
\end{aligned}$$

hence by sending $N \to \infty$ the first relation in (2.33) follows by (2.10) and the first relation in (2.28).

The second relation in (2.33) follows by our assumption (2), see (2.30), because

$$\left\| \sum_{i=1}^{M_N} \widetilde{X}_{N,i} - \sum_{i=1}^{M_N} X_{N,i} \right\|_{L^2}^2 = \sum_{i=1}^{M_N} \sum_{\substack{A\subset\mathbb{B}_i \\ |A|>K_N}} q_N(A)^2 \le \sum_{\substack{A\subset\mathbb{T} \\ |A|>K_N}} q_N(A)^2\,.$$

This completes the proof.

□

2.3.3. Asymptotic Gaussianity of X_N. In view of Lemma 2.13, to prove (2.14) it remains to show the convergence in distribution

$$\sum_{i=1}^{M_N} X_{N,i} \xrightarrow[N\to\infty]{d} \mathcal{N}(0,\sigma^2)\,. \tag{2.34}$$

Note that $(X_{N,i})_{i=1,\dots,M_N}$ are *independent* random variables with zero mean and finite variance, see (2.31), because the boxes $\mathbb{B}_i \subset \mathbb{T}$ are disjoint. The key tool to gain the Gaussianity is the *Feller-Lindeberg Central Limit Theorem for triangular arrays* [**Bil95**, Theorem 27.2], which we recall in the following.

THEOREM 2.14. *For $N \in \mathbb{N}$ let $(X_{N,i})_{i=1,\dots,M_N}$ be a triangular array of independent random variables with zero mean and finite variance. Assume*

- *the convergence of the variance:*

$$\lim_{N\to\infty} \mathbb{E}\left[\left(\sum_{i=1}^{M_N} X_{N,i}\right)^2\right] = \sigma^2 , \tag{2.35}$$

- *and the so-called Lindeberg condition:*

$$\forall \epsilon > 0 : \qquad \lim_{N\to\infty} \sum_{i=1}^{M_N} \mathbb{E}\Big[\big(X_{N,i}\big)^2 \mathbb{1}_{\{|X_{N,i}|>\epsilon\}}\Big] \,=\, 0 \,. \tag{2.36}$$

Then,

$$\sum_{i=1}^{M_N} X_{N,i} \xrightarrow[N\to\infty]{d} \mathcal{N}(0,\sigma^2)\,.$$

It is already clear that relation (2.35) directly follows by Lemma 2.13, see (2.32), and our assumption (1), see (2.10). Next we are going to prove the following *Lyapunov condition*:

$$\text{for some } p > 2 : \qquad \lim_{N\to\infty} \sum_{i=1}^{M_N} \mathbb{E}\Big[\big|X_{N,i}\big|^p\Big] \,=\, 0\,, \tag{2.37}$$

which implies Lindeberg's condition (2.36), since

$$\mathbb{E}\big[\big(X_{N,i}\big)^2 \mathbb{1}_{\{|X_{N,i}|>\epsilon\}}\big] \le \mathbb{E}\left[\frac{|X_{N,i}|^p}{|X_{N,i}|^{p-2}} \mathbb{1}_{|X_{N,i}|>\epsilon\}}\right] \le \frac{\mathbb{E}\big[\big|X_{N,i}\big|^p\big]}{\epsilon^{p-2}}\,.$$

To obtain (2.37), we apply the hypercontractivity bound (2.26) to $X_{N,i}$, see (2.31), to get

$$\mathbb{E}\Big[\big|X_{N,i}\big|^p\Big]^{\frac{2}{p}} \,\le\, \sum_{\substack{A\subset\mathbb{B}_i\\|A|\le K_N}} (p-1)^{|A|}\, q_N(A)^2 \,\le\, (p-1)^{K_N}\, \sigma_N^2(\mathbb{B}_i)\,, \tag{2.38}$$

where we recall that $\sigma_N^2(\mathbb{B}_i) = \sum_{A \subset \mathbb{B}_i} q_N(A)^2$. Then by applying the hypercontractivity twice, for any $p > 2$ we can write

$$\begin{aligned}
\sum_{i=1}^{M_N} \mathbb{E}\Big[\big|X_{N,i}\big|^p\Big] &\le \bigg(\max_{i=1,\dots,M_N} \mathbb{E}\Big[\big|X_{N,i}\big|^p\Big]\bigg)^{1-\frac{2}{p}} \sum_{i=1}^{M_N} \mathbb{E}\Big[\big|X_{N,i}\big|^p\Big]^{\frac{2}{p}} \\
&\le \left\{(p-1)^{K_N} \bigg(\max_{i=1,\dots,M_N} \mathbb{E}\Big[\big|X_{N,i}\big|^p\Big]^{\frac{2}{p}}\bigg)^{\frac{p}{2}-1}\right\} \sum_{i=1}^{M_N} \sigma_N^2(\mathbb{B}_i)\,. \\
&\le \left\{(p-1)^{pK_N} \Big(\max_{i=1,\dots,M_N} \sigma_N^2(\mathbb{B}_i)\Big)^{p-2}\right\}^{\frac{1}{2}} \sum_{i=1}^{M_N} \sigma_N^2(\mathbb{B}_i)\,.
\end{aligned} \tag{2.39}$$

If we fix $p = 3$, the term in brackets vanishes as $N \to \infty$ by our choice (2.29) of K_N, while the last sum converges to σ^2 as $N \to \infty$, see (2.28), hence it is uniformly bounded. This completes the proof of (2.37).

2.3.4. Switching to Gaussian random variables. We finally complete the proof of Theorem 2.2 by justifying the preliminary step: we show that replacing the random variables $(\eta_t^N)_{t \in \mathbb{T}}$ in (2.6) by standard Gaussians *does not change the asymptotic distribution of* X_N. More precisely, if $(\hat{\eta}_t)_{t \in \mathbb{T}}$ are independent $\mathcal{N}(0,1)$ and we set

$$\hat{X}_N = \sum_{A \subset \mathbb{T}} q_N(A)\, \hat{\eta}(A)\,, \qquad \text{with} \qquad \hat{\eta}(A) := \prod_{t \in A} \hat{\eta}_t\,, \tag{2.40}$$

it suffices to show that for every bounded and smooth $f : \mathbb{R} \to \mathbb{R}$ we have

$$\lim_{N \to \infty} \big|\, \mathbb{E}[f(X_N)] - \mathbb{E}[f(\hat{X}_N)] \,\big| = 0\,. \tag{2.41}$$

Indeed, since $\hat{X}_N \xrightarrow{d} \mathcal{N}(0,\sigma^2)$ by the first part of the proof, (2.41) implies $X_N \xrightarrow{d} \mathcal{N}(0,\sigma^2)$.

As key ingredient to obtain (2.41), we exploit the Lindeberg principle for polynomial chaos proved in [**CSZ17a**, Theorem 2.6], where the authors generalized [**MOO10**, Theorem 3.18]. In the latter, the random variables need to be centered, with unit variance and finite third moment. On the other hand, the extension in [**CSZ17a**] provides a quantitative estimate (see (2.46)) for the distributional distance in (2.41), which can be taken suitably small for random variables with weaker moments assumptions. In particular, the result requires a condition of uniform integrability of the squares of the random variables, which turns out to be optimal and gives the motivation for assuming (2.5) in Theorem 2.2. We further mention that the authors of [**CSZ17a**] also presented a version of Lindeberg principle covering also the non-zero mean case, but this extension goes beyond our applications.

In order to show that $\mathbb{E}[f(X_N)]$ is close to $\mathbb{E}[f(\hat{X}_N)]$, let us fix $f : \mathbb{R} \to \mathbb{R}$ of class C^3 with

$$C_f := \max\{\|f'\|_\infty, \|f''\|_\infty, \|f'''\|_\infty\} < \infty\,. \tag{2.42}$$

For $L > 0$, denote by $m_2^{>L}$ the second moment tail of the random variables η_t^N and $\hat{\eta}_t$:

$$m_2^{>L} := \sup_{N\in\mathbb{N},\, t\in\mathbb{T}} \max\left\{\mathbb{E}\left[|\eta_t^N|^2 \mathbb{1}_{|\eta_t^N|>L}\right], \mathbb{E}\left[|\hat{\eta}_t|^2 \mathbb{1}_{|\hat{\eta}_t|>L}\right]\right\}. \tag{2.43}$$

Let $\mathsf{C}_{X_N^{\le K}}$, $\mathsf{C}_{X_N^{>K}}$ be the second moments of X_N truncated to chaos of order $\le K$ and $> K$:

$$\mathsf{C}_{X_N^{\le K}} := \sum_{\substack{A\subset\mathbb{T}\\ |A|\le K}} q_N(A)^2, \qquad \mathsf{C}_{X_N^{>K}} := \sum_{\substack{A\subset\mathbb{T}\\ |A|>K}} q_N(A)^2. \tag{2.44}$$

Finally, define the *influence* of the variable $t \in \mathbb{T}$ on X_N by

$$\mathrm{Inf}_t[X_N] := \mathbb{E}\left[\mathbb{V}ar\left[X_N(\eta)|(\eta_s^N)_{s\in\mathbb{T}\setminus t}\right]\right], \tag{2.45}$$

which admits a more explicit and nice expression in this case, provided we write the polynomial chaos X_N as sum of two contributions given by the subsets $A \subset \mathbb{T}$ containing the entry $t \in \mathbb{T}$ or not:

$$X_N = \sum_{\substack{A\subset\mathbb{T}\\ A\ni t}} q_N(A)^2 \prod_{\substack{s\in A\\ s\neq t}} \eta_s^N\, \eta_t^N + \sum_{\substack{A\subset\mathbb{T}\\ A\not\ni t}} q_N(A)^2 \prod_{s\in A} \eta_s^N.$$

Recall that the η_t's are random variables with zero mean and unit variance, thus by definition (2.45) we easily obtain

$$\mathrm{Inf}_t[X_N] = \sum_{\substack{A\subset\mathbb{T}\\ A\ni t}} q_N(A)^2.$$

By applying [**CSZ17a**, Theorem 2.6], for any $L > 0$ such that $m_2^{>L} \le \frac{1}{4}$ and for every $K \in \mathbb{N}$ we have

$$\begin{aligned}\left|\mathbb{E}[f(X_N)] - \mathbb{E}[f(\hat{X}_N)]\right| \le C_f \Bigg\{ & 2\sqrt{\mathsf{C}_{X_N^{>K}}} + 16K^2\, \mathsf{C}_{X_N^{\le K}}\, m_2^{>L} \\ & + 70^{K+1}\, \mathsf{C}_{X_N^{\le K}}\, L^{3K} \max_{t\in\mathbb{T}} \sqrt{\mathrm{Inf}_t[X_N]} \Bigg\}.\end{aligned} \tag{2.46}$$

It only remains to show that the r.h.s. of this expression is small as $N \to \infty$, to prove (2.41). We fix any $\epsilon > 0$ and we argue as follows:

- by assumption (2.11), we can choose $K = K_\epsilon$ such that $\limsup_{N\to\infty} \mathsf{C}_{X_N^{>K}} \le \epsilon$;
- by assumption (2.10), for any $K \in \mathbb{N}$ we can bound $\limsup_{N\to\infty} \mathsf{C}_{X_N^{\le K}} \le \sigma^2$;
- by assumption (2.5), we can choose $L = L_\epsilon$ such that $m_2^{>L_\epsilon} \le \epsilon/(K_\epsilon^2\, \sigma^2)$;
- finally, we show below that

$$\limsup_{N\to\infty} \max_{t\in\mathbb{T}} \sqrt{\mathrm{Inf}_t[X_N]} = 0. \tag{2.47}$$

As a consequence, when we plug $K = K_\epsilon$ and $L = L_\epsilon$ in (2.46) and we let $N \to \infty$, we get

$$\limsup_{N\to\infty} \left|\mathbb{E}[f(X_N)] - \mathbb{E}[f(\hat{X}_N)]\right| \le C_f \left\{ 2\sqrt{\epsilon} + 16\,\epsilon \right\},$$

from which (2.41) follows because $\epsilon > 0$ is arbitrary.

We now prove (2.47). By assumption there are disjoint boxes $\mathbb{B}_1, \ldots, \mathbb{B}_{M_N} \subset \mathbb{T}$, with $M_N \to \infty$, such that relation (2.28) holds. In particular, recalling also (2.9) and (2.10), it follows that *subsets* $A \subset \mathbb{T}$ *not contained in any of the boxes* $\mathbb{B}_i$ *give a negligible contribution*:

$$\Delta_N := \sum_{\substack{A\subset\mathbb{T}:\\ A\not\subset\mathbb{B}_i\,\forall i=1,\ldots,M_N}} q_N(A)^2 = \sigma_N^2(\mathbb{T}) - \sum_{i=1}^{M_N} \sigma_N^2(\mathbb{B}_i) \xrightarrow[N\to\infty]{} 0\,. \tag{2.48}$$

Recall now the definition of influence $\mathrm{Inf}_t[X_N] = \sum_{A\subset\mathbb{T},\, A\ni t} q_N(A)^2$. Fix $t \in \mathbb{T}$ and a subset $A \subset \mathbb{T}$ which contains t, i.e. $A \ni t$. We distinguish two cases:

- if $t \notin \mathbb{B}_i$ for all $i = 1, \ldots, M_N$, then $A \ni t$ implies $A \not\subset \mathbb{B}_i$ for all $i = 1, \ldots, M_N$, hence by (2.48) we can bound $\mathrm{Inf}_t[X_N] \le \Delta_N$;
- if $t \in \mathbb{B}_j$ for some (necessarily unique) $j = 1, \ldots, M_N$, then $A \ni t$ implies that either $A \subset \mathbb{B}_j$, or $A \not\subset \mathbb{B}_i$ for all $i = 1, \ldots, M_N$ (we clearly cannot have $A \subset \mathbb{B}_i$ for some $i \ne j$), hence by (2.9) and (2.48) we can bound $\mathrm{Inf}_t[X_N] \le \sigma_N^2(\mathbb{B}_j) + \Delta_N$.

It follows that

$$\max_{t\in\mathbb{T}} \mathrm{Inf}_t[X_N] \le \max_{j=1,\ldots,M_N} \sigma_N^2(\mathbb{B}_j) + \Delta_N\,,$$

hence (2.47) follows by (2.28) and (2.48). The proof of Theorem 2.2 is complete. □

2.4. Central Limit Theorem for Wiener chaos: proof of Theorem 2.9

Due to the analogy between the discrete and the continuum cases, the proof of Theorem 2.9 follows very closely that of Theorem 2.2 without remarkable differences. Actually, if in the previous section we start by replacing the random variables $(\eta_t)_{t\in\mathbb{T}}$ by standard Gaussians in order to apply the hypercontractivity for polynomial chaos (2.26), here we do not even need this preliminary step. Indeed, the continuum setting is already Gaussian, since Wiener chaos are defined in terms of a Gaussian random measure $W(dx)$ (recall also Remark 2.8), thus the *hypercontractivity for Wiener chaos* holds (see [**Jan97**]):

$$\begin{aligned}\forall p > 2: \quad & \mathbb{E}\Bigg[\Bigg|\sum_{k=1}^{\infty}\int_{E^k}\widetilde{q}_N(x_1,\ldots,x_k)\,W(\mathrm{d}x_1)\cdots W(\mathrm{d}x_k)\Bigg|^p\Bigg] \\ & \le \Bigg(\sum_{k=1}^{\infty}(p-1)^k\,k!\,\|\widetilde{q}_N\|^2_{L^2(E^k)}\Bigg)^{\frac{p}{2}} \qquad (2.49) \\ & = \Bigg(\sum_{k=1}^{\infty}(p-1)^k\,k!\int_{E^k}\widetilde{q}_N(x_1,\ldots,x_k)^2\,\mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_k)\Bigg)^{\frac{p}{2}}.\end{aligned}$$

For completeness, we now briefly illustrate the main steps to prove Theorem 2.9: all the details are just the continuum analog of the arguments followed for polynomial chaos, thus we refer to the previous section for an exhaustive discussion.

We consider a sequence of Wiener chaos $\widetilde{X}_N$, with a symmetric L^2 function $\widetilde{q}_N(\cdot)$ as in (2.18), which satisfy assumptions $(\widetilde{1})$, $(\widetilde{2})$, $(\widetilde{3})$, see the equations (2.21)-(2.24). Following the same argument as in Subsection 2.3.1, we build two suitable diverging sequences of integers $M_N \to \infty$, $K_N \to \infty$.

- We fix $M_N \to \infty$ slowly enough so that assumption $(\widetilde{3})$ still holds with $M = M_N$. More explicitly, for every $N \in \mathbb{N}$ we can find disjoint subsets ("boxes") $\mathbb{B}_i = \mathbb{B}_i^{(N)}$:

  $$\mathbb{B}_1,\ldots,\mathbb{B}_{M_N} \subset \mathbb{T} \qquad \text{with} \qquad \mathbb{B}_i \cap \mathbb{B}_j = \emptyset \quad \text{for } i \neq j\,,$$

  such that the following versions of (2.23)-(2.24) hold:

$$\lim_{N\to\infty}\sum_{i=1}^{M_N}\widetilde{\sigma}_N^2(\mathbb{B}_i) = \sigma^2 \qquad \text{and} \qquad \lim_{N\to\infty}\Big\{\max_{i=1,\ldots,M_N}\widetilde{\sigma}_N^2(\mathbb{B}_i)\Big\} = 0\,. \qquad (2.50)$$

- By the second relation in (2.50), we can fix $K_N \to \infty$ slowly enough so that

  $$\lim_{N\to\infty} 8^{K_N}\max_{i=1,\ldots,M_N}\widetilde{\sigma}_N^2(\mathbb{B}_i) = 0\,. \qquad (2.51)$$

  Note that by our assumption $(\widetilde{2})$, see (2.22), for any $K_N \to \infty$ we have

  $$\lim_{N\to\infty}\sum_{k>K_N}k!\,\|\widetilde{q}_N\|^2_{L^2(E^k)} = 0\,. \qquad (2.52)$$

We next proceed with the actual proof of Theorem 2.9, following the same steps as in the previous section.

First we can approximate in L^2 the Wiener chaos $\widetilde{X}_N$ in (2.6) by a sum of suitable *independent* random variables

$$\widetilde{X}_{N,1} + \ldots + \widetilde{X}_{N,M_N}\,,$$

for $M_N \to \infty$ as $N \to \infty$ fixed according to (2.50), where for $i = 1, \dots, M_N$ we define

$$\widetilde{X}_{N,i} := \sum_{k=1}^{K_N} \int_{\mathbb{B}_i^k} \widetilde{q}_N(x_1, \dots, x_k)\, W(\mathrm{d}x_1) \cdots W(\mathrm{d}x_k)\,, \tag{2.53}$$

where K_N is fixed according to (2.51). Therefore, the following holds.

LEMMA 2.15.

$$\lim_{N\to\infty} \left\| \widetilde{X}_N - \sum_{i=1}^{M_N} \widetilde{X}_{N,i} \right\|_{L^2} = 0\,. \tag{2.54}$$

Proof. See the proof of Lemma 2.13.

□

Then we send $N \to \infty$ and we apply the Feller-Lindeberg CLT (Theorem 2.14) to the triangular array $(\widetilde{X}_{N,i})_{i=1,\dots,M_N}$, provided that (2.35) and (2.36) are verified. Relation (2.35) easily follows from Lemma 2.15, while (2.36) is implied by the Ljapunov condition:

$$\text{for some } p > 2: \qquad \lim_{N\to\infty} \sum_{i=1}^{M_N} \mathbb{E}\Big[\big|\widetilde{X}_{N,i}\big|^p\Big] = 0\,.$$

To obtain the latter, we exploit the hypercontractivity estimate (2.49) to $\widetilde{X}_{N,i}$, see (2.53), and we have To obtain (2.37), we apply the hypercontractivity bound (2.26) to $X_{N,i}$, see (2.31), to get

$$\mathbb{E}\Big[\big|\widetilde{X}_{N,i}\big|^p\Big]^{\frac{2}{p}} \le \sum_{k=1}^{K_N} (p-1)^k\, k!\, \|\widetilde{q}_N\|^2_{L^2(E^k)} \le (p-1)^{K_N}\, \widetilde{\sigma}_N^2(\mathbb{B}_i)\,, \tag{2.55}$$

where we recall that $\widetilde{\sigma}_N^2(\mathbb{B}_i) = \sum_{k=1}^\infty k! \int_{\mathbb{B}_i^k} \widetilde{q}_N(x_1, \dots, x_k)^2\, \mu(\mathrm{d}x_1) \cdots \mu(\mathrm{d}x_k)$. Then by applying the hypercontractivity twice, for any $p > 2$ we can write

$$\begin{aligned} \sum_{i=1}^{M_N} \mathbb{E}\Big[\big|\widetilde{X}_{N,i}\big|^p\Big] &\le \left(\max_{i=1,\dots,M_N} \mathbb{E}\Big[\big|\widetilde{X}_{N,i}\big|^p\Big] \right)^{1-\frac{2}{p}} \sum_{i=1}^{M_N} \mathbb{E}\Big[\big|\widetilde{X}_{N,i}\big|^p\Big]^{\frac{2}{p}} \\ &\le \left\{ (p-1)^{pK_N} \left(\max_{i=1,\dots,M_N} \widetilde{\sigma}_N^2(\mathbb{B}_i) \right)^{p-2} \right\}^{\frac{1}{2}} \sum_{i=1}^{M_N} \widetilde{\sigma}_N^2(\mathbb{B}_i)\,. \end{aligned} \tag{2.56}$$

If we fix $p = 3$, then the term in brackets vanishes as $N \to \infty$ by the choice (2.51) of K_N, on the other hand the last sum is uniformly bounded since it converges to σ^2 as $N \to \infty$. Then, Ljapunov condition holds and we conclude that

$$\sum_{i=1}^{M_N} \widetilde{X}_{N,i} \xrightarrow{d} \mathcal{N}(0, \sigma^2)\,,$$

thus the proof of Theorem 2.9 is complete. □

CHAPTER 3

Gaussian fluctuations in the subcritical regime

3.1. The directed polymer partition function as a polynomial chaos

We now present applications of our convergence results (Theorems 2.2 and 2.9) in Chapter 2 to directed polymers in random environment on $\mathbb{Z}^2$ (2-DPRE). In this section, we briefly recall some essential definition introduced in Chapter 1 for $d = 2$ and we show how the 2-DPRE's partition function can be expressed in terms of a polynomial chaos expansion, which is essential to exploit Theorem 2.2.

Let $S = (S_n)_{n\geq 0}$ be the simple symmetric random walk on $\mathbb{Z}^2$, whose law we denote by P. Let $\omega = (\omega(n,x))_{n\in\mathbb{N},x\in\mathbb{Z}^2}$ be a family of i.i.d. random variables, independent of S with law $\mathbb{P}$ and such that

$$\mathbb{E}[\omega(n,x)] = 0\,, \qquad \mathbb{E}[\omega(n,x)^2] = 1\,, \qquad \lambda(\beta) := \log \mathbb{E}[e^{\beta\omega(n,x)}] < \infty \quad \forall \beta > 0\,. \tag{3.1}$$

Intuitively, trajectories of the random walk S represent polymer configurations, while configurations ω describe the *disorder*, which plays the role of a *random environment*. Given a scale parameter $N \in \mathbb{N}$, a starting time-space point $(m,z) \in \{0,\dots,N\} \times \mathbb{Z}^2$ and an interaction strength $\beta > 0$, the partition function of the directed polymer model is

$$Z_N^{\beta}(m,z) := \mathrm{E}\Big[e^{\sum_{n=m+1}^{N}(\beta\omega(n,S_n)-\lambda(\beta))}\,\Big|\, S_m = z\Big]\,. \tag{3.2}$$

Directed polymers were originally introduced as an effective interface model in the framework of the Ising model with impurities, but over the years they have become an object of independent study and a prototype of a disorder system which is amenable to detailed rigorous investigation. We refer to the monograph by Comets [**Com17**] for a recent account.

Although the definition (3.2) is intuitive from a probabilistic point of view as pointed out in Section 1.1 of Chapter 1, it is not clear to see the dependence on the disorder random variables $\omega(m,z)$. However, we can derive an alternative representation of $Z_N^{\beta_N}(\cdot,\cdot)$, namely its *polynomial chaos expansion*, by means of an explicit function of the disorder. This can be done for a general spatial dimension $d \geq 1$, however we only show the main case we will deal with, namely when $d = 2$. Let us introduce a modified disorder $\eta_N = (\eta_N(m,z))_{m\in\mathbb{N},z\in\mathbb{Z}^2}$, recalling (3.1):

$$\eta_N(m,z) := \frac{e^{\beta_N\omega(m,z)-\lambda(\beta_N)} - 1}{\sigma_N} \quad \text{where} \quad \sigma_N^2 := e^{\lambda(2\beta_N)-2\lambda(\beta_N)} - 1 \underset{N\to\infty}{\sim} \beta_N^2\,. \tag{3.3}$$

For any $N \in \mathbb{N}$, the modified disorder behaves as the original $\omega(m,z)$, indeed $\eta_N = (\eta_N(m,z))_{m\in\mathbb{N},z\in\mathbb{Z}^2}$ is i.i.d. with $\mathbb{E}[\eta_N(m,z)] = 0$ and $\mathbb{E}[\eta_N(m,z)^2] = 1$, see (3.1), and higher moments of η_N are uniformly bounded (see [**CSZ17a**, eq. (6.7)]). Provided that we consider a slighty modification of disorder, we can expand the definition (3.2) as follows:

$$
\begin{aligned}
&Z_N^{\beta_N}(m,z) \\
&= \mathrm{E}\Big[e^{\sum_{n=m+1}^{N}(\beta\omega(n,S_n)-\lambda(\beta))}\,\Big|\, S_m = z\Big] \\
&= \mathrm{E}\Big[\prod_{(n,x)\in\{m+1,\dots,N\}\times\mathbb{Z}^2} e^{(\beta\omega(n,x)-\lambda(\beta))\mathbb{1}_{\{S_n=x\}}}\,\Big|\, S_m = z\Big] \\
&= \mathrm{E}\Big[\prod_{(n,x)\in\{m+1,\dots,N\}\times\mathbb{Z}^2} \big(1+\beta\,\eta_N(n,x)\mathbb{1}_{\{S_n=x\}}\big)\Big|\, S_m = z\Big] \\
&= 1 + \sum_{k=1}^{\infty}(\sigma_N)^k \sum_{\substack{m<n_1<\dots<n_k\le N\\ x_1,\dots,x_k\in\mathbb{Z}^2}} \mathrm{E}\big[\mathbb{1}_{\{S_{n_1}=x_1\}}\cdots\mathbb{1}_{\{S_{n_k}=x_k\}}\big|\, S_m = z\big]\prod_{j=1}^{k}\eta_N(n_j,x_j) \\
&= 1 + \sum_{k=1}^{\infty}(\sigma_N)^k \sum_{\substack{m=:n_0<n_1<\dots<n_k\le N\\ x_0:=z\,,\,x_1,\dots,x_k\in\mathbb{Z}^2}} \prod_{j=1}^{k} q_{n_j-n_{j-1}}(x_j-x_{j-1})\,\eta_N(n_j,x_j)\,,
\end{aligned}
\tag{3.4}
$$

where for $n \in \mathbb{N}$ and $x \in \mathbb{Z}^2$

$$q_n(x) := \mathrm{P}\big(S_n = x \,|\, S_0 = 0\big)$$

is the simple random walk transition kernel. Notice that the last line of (3.4) is an explicit function of disorder η_N and has the structure of a non-centered polynomial chaos with unit mean (cf. the centered version (2.6))

$$Z_N^{\beta_N}(m,z) = 1 + \sum_{A\subset\mathbb{T}} q_N(A)\eta^N(A) = 1 + \sum_{A\subset\mathbb{T}} q_N(A)\prod_{t\in A}\eta_t^N\,,$$

with countable set $\mathbb{T} = \mathbb{N}\times\mathbb{Z}^2$, variables $\eta_t^N = \eta_N(n,x)$ for $t = (n,x) \in \mathbb{N}\times\mathbb{Z}^2$ and coefficient function $q_N(A) = \beta^k \prod_{j=1}^{k} q_{n_j-n_{j-1}}(x_j - x_{j-1})\mathbb{1}_{\{m<n_1<\dots<n_k\le N\}}$ for a subset $A = \{(n_1,x_1),\dots,(n_k,x_k)\} \subset \mathbb{N}\times\mathbb{Z}^2$.

3.2. Edwards–Wilkinson fluctuations revisited

A source of interest for directed polymers is their link with the multiplicative Stochastic Heat Equation (mSHE), which is the stochastic PDE formally written as follows:

$$\partial_t u(t,x) = \frac{1}{2}\Delta_x u(t,x) \;+\; \beta\, u(t,x)\,\dot{W}(t,x)\,, \tag{3.5}$$

where $\beta > 0$ tunes the interaction strength and $\dot{W}(t,x)$ denotes white noise on $\mathbb{R}^+\times\mathbb{R}^2$. In one space dimension $d = 1$, this equation admits a rigorous integral formulation by the classical Ito-Walsh integration. In higher dimensions $d \ge 2$, this approach fails due to strong irregularity of white noise and no obvious meaning can be given to its solution $u(t,x)$.

By the Markov property of simple random walk, the diffusively rescaled partition function

$$U_N(t,x) := Z_N^{\beta}(\lfloor Nt \rfloor, \lfloor \sqrt{N} x \rfloor) \tag{3.6}$$

solves a discretized version of (3.5) (with ∂_t and $\frac{1}{2}\Delta_x$ replaced by $-\partial_t$ and $\frac{1}{4}\Delta_x$, see (3.57) below). This explains the interest for the convergence as $N \to \infty$ of $U_N(t,x)$, possibly for suitable $\beta = \beta_N$, since it provides an approximation of the ill-defined (mSHE) solution $u(t,x)$.

It is also very interesting to look at the *logarithm of the partition function*

$$\log Z_N^{\beta}(\lfloor Nt \rfloor, \lfloor \sqrt{N} x \rfloor)$$

because it provides an approximation for the solution $h(t,x) = \log u(t,x)$ of the Kardar-Parisi-Zhang equation (KPZ), which is the stochastic PDE formally given by

$$\partial_t h(t,x) = \frac{1}{2}\Delta_x h(t,x) \, + \, \frac{1}{2}|\nabla_x h(t,x)|^2 \, + \, \beta \, \dot{W}(t,x) \text{ “} - \infty \text{”}\,, \tag{3.7}$$

where the last term “$-\infty$” indicates a form of renormalization.

Remark 3.1 (Edwards–Wilkinson equation). *The Stochastic Heat Equation* (3.5) *is singular due to the* multiplicative noise term $\dot{W}u$. *The additive version of this equation, known as the* Edwards–Wilkinson equation, *is well-posed and reads as follows:*

$$\partial_t v(t,x) = \frac{s}{2}\Delta_x v(t,x) + c\, \dot{W}(t,x)\,, \tag{3.8}$$

where $s > 0$ and $c \in \mathbb{R}$ are given parameters. Starting from $v(0,\cdot) \equiv 0$, the solution $v = v^{(s,c)}$ is a random distribution *(i.e. generalized function) which is Gaussian with explicit covariance, see* [**CSZ20**, Remark 1.5]. *More precisely, if we denote by $\langle v^{(s,c)}, \varphi \rangle$ the pairing between the distribution $v^{(s,c)}$ and a test function φ, which formally corresponds to*

$$\langle v^{(s,c)}, \varphi \rangle := \int_{[0,\infty)\times\mathbb{R}^2} v^{(s,c)}(t,x)\, \varphi(t,x)\, \mathrm{d}t\, \mathrm{d}x\,, \tag{3.9}$$

then $\langle v^{(s,c)}, \varphi \rangle$ for $\varphi \in C_c^{\infty}([0,\infty)\times\mathbb{R}^2)$ is a centered Gaussian process with

$$\mathbb{C}\mathrm{ov}\left[\langle v^{(s,c)}, \varphi \rangle,\, \langle v^{(s,c)}, \varphi' \rangle\right] = c^2 \int_{([0,\infty)\times\mathbb{R}^2)^2} \varphi(t,x)\, K^s_{t,t'}(x,x')\, \varphi'(t',x')\, \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}t'\, \mathrm{d}x'\,, \tag{3.10}$$

where the covariance kernel is given by

$$K^s_{t,t'}(x,x') := \frac{1}{2s}\int_{s|t-t'|}^{s(t+t')} g_u(x-x')\, \mathrm{d}u\,, \quad \text{where} \quad g_u(y) := \frac{e^{-\frac{|y|^2}{2u}}}{2\pi u}\,. \tag{3.11}$$

Similarly, it is easy to show that for any fixed $t > 0$ and for any $\varphi \in C_c(\mathbb{R}^2)$ the pairing

$$\langle v^{(s,c)}(t,\cdot), \varphi \rangle := \int_{\mathbb{R}^2} v^{(s,c)}(t,x)\, \varphi(x)\, \mathrm{d}x \tag{3.12}$$

is a centered Gaussian process with covariance

$$\mathbb{C}\mathrm{ov}\left[\langle v^{(s,c)}(t,\cdot), \varphi\rangle,\, \langle v^{(s,c)}(t,\cdot), \varphi'\rangle\right] = c^2 \int_{\mathbb{R}^2\times\mathbb{R}^2} \varphi(x)\, K^s_{t,t}(x,x')\, \varphi'(x')\, \mathrm{d}x\, \mathrm{d}x' \,, \tag{3.13}$$

with $K^s_{t,t}(x,x')$ as in (3.11).

At this point, let us define

$$u_n := \sum_{z\in\mathbb{Z}^2} \mathrm{P}(S_n = z)^2 = \mathrm{P}(S_{2n} = 0) \sim \frac{1}{\pi}\frac{1}{n} \,, \tag{3.14}$$

$$R_N := \sum_{n=1}^{N} \sum_{z\in\mathbb{Z}^2} \mathrm{P}(S_n = z)^2 = \sum_{n=1}^{N} u_n \sim \frac{1}{\pi} \log N \,, \tag{3.15}$$

where the asymptotic relations (respectively as $n \to \infty$ and as $N \to \infty$) follow by the *local central limit theorem* ([**LL10**, Theorem 2.1.3]): as $n \to \infty$, uniformly for $y \in \mathbb{Z}^2$,[1]

$$q_n(y) = \frac{1}{n/2}\big(g\big(\tfrac{y}{\sqrt{n/2}}\big) + o(1)\big)\, 2\, \mathbb{1}_{(n,y)\in\mathbb{Z}^3_{\mathrm{even}}} \qquad \text{with} \qquad g(x) := \frac{e^{-|x|^2/2}}{2\pi} \,. \tag{3.16}$$

In particular, for $(n, y) \in \mathbb{Z}^3_{\mathrm{even}}$ in the "diffusive regime" we can write

$$q_n(y) = \frac{4}{n}\, g\big(\tfrac{y}{\sqrt{n/2}}\big)\big(1 + o(1)\big) \qquad \text{for } |y| = O(\sqrt{n}) \,. \tag{3.17}$$

Henceforth we are going to fix $\beta = \beta_N$ given by

$$\beta_N := \frac{\hat\beta}{\sqrt{R_N}} \sim \frac{\hat\beta\,\sqrt{\pi}}{\sqrt{\log N}} \qquad \text{with} \qquad \hat\beta \in (0,1) \,, \tag{3.18}$$

also known as the *sub-critical regime.*

We look at the fluctuations of the diffusively rescaled partition function as random field, encoded by

$$V_N(x) := \frac{1}{\beta_N}\big(Z_N^{\beta_N}(\lfloor\sqrt{N}x\rfloor) - 1\big) \quad \text{for} \quad x \in \mathbb{R}^2 \,, \tag{3.19}$$

(see (1.13), where for simplicity we take the initial time $t = 0$). It was shown in [**CSZ17b**, Theorem 2.13] that the rescaled partition function exhibits *Edwards–Wilkinson fluctuations*, because $V_N(x)$ converges as $N \to \infty$ to a solution of the Edwards–Wilkinson equation (3.8), as we recall in the following.

[1] The scaling factor in (3.16) is $n/2$ because the simple random walk on $\mathbb{Z}^2$ has covariance matrix $\frac{1}{2}I$, while the factor $2\,\mathbb{1}_{(n,y)\in\mathbb{Z}^3_{\mathrm{even}}}$ is due to periodicity.

THEOREM 3.2 (Edwards–Wilkinson fluctuations for $Z_N^{\beta_N}$). *Let $V_N(x)$ be as in* (3.19)*, where $\beta_N = \hat{\beta}/\sqrt{R_N}$ with $\hat{\beta} \in (0,1)$ (cf.* (3.18)*). Then, as $N \to \infty$:*

$$V_N(x) \overset{\mathcal{D}}{\Longrightarrow} \widetilde{v}(x) := v^{(\frac{1}{2}, c_{\hat{\beta}})}(1,x) \qquad \textit{with} \qquad c_{\hat{\beta}} := \sqrt{\frac{1}{1-\hat{\beta}^2}}\,, \tag{3.20}$$

where "$\overset{\mathcal{D}}{\Longrightarrow}$" denotes convergence in law as a random distribution on $\mathbb{R}^2$*: for $\varphi \in C_c(\mathbb{R}^2)$:*

$$\langle V_N, \varphi \rangle := \int_{\mathbb{R}^2} V_N(x)\,\varphi(x)\,\mathrm{d}x \ \xrightarrow[N\to\infty]{d}\ \langle \widetilde{v}, \varphi \rangle \sim \mathcal{N}\big(0, \sigma^2_{\hat{\beta},\varphi}\big)\,, \tag{3.21}$$

with

$$\sigma^2_{\hat{\beta},\varphi} := \frac{1}{1-\hat{\beta}^2} \int_{\mathbb{R}^2\times\mathbb{R}^2} \varphi(x) K(x,x')\varphi(x')\,\mathrm{d}x\,\mathrm{d}x'\,,$$

$$K(x,x') := K^{\frac{1}{2}}_{1,1}(x,x') = \int_0^1 g_u(x-x')\,\mathrm{d}u\,,$$

(cf. (3.13) *and* (3.11)*).*

REMARK 3.3. *The kernel K diverges logarithmically on the diagonal:*

$$K(x,x') \sim C \log \frac{1}{|x-x'|} \qquad \textit{as} \qquad |x-x'| \to 0\,.$$

Observe that such Gaussian fields with logarithmically divergent covariance kernels are of great relevance in the theory of Gaussian Multiplicative Chaos (see, for instance, [**RV14**]*).*

REMARK 3.4. *We stress that relation* (3.77) *implies convergence of all finite-dimensional distributions of the random field $(\langle V_N, \varphi\rangle)_{\varphi \in C_c(\mathbb{R}^2)}$ toward the Gaussian field $(\langle \widetilde{v}, \varphi\rangle)_{\varphi \in C_c(\mathbb{R}^2)}$, namely for any $k \in \mathbb{N}$, for any $\varphi_1, \ldots, \varphi_k$, the the following joint convergence holds:*

$$\big(\langle V_N, \varphi_1\rangle, \ldots, \langle V_N, \varphi_k\rangle\big) \xrightarrow[N\to\infty]{d} \big(\langle \widetilde{v}, \varphi_1\rangle, \ldots, \langle \widetilde{v}, \varphi_k\rangle\big)\,.$$

Indeed, by Cramér-Wold device [**Bil95**, Theorem 29.4]*, the latter expression is equivalent to show that for any $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$:*

$$\sum_{i=1}^k \alpha_i \langle V_N, \varphi_i\rangle \xrightarrow[N\to\infty]{d} \sum_{i=1}^k \alpha_i \langle \widetilde{v}, \varphi_i\rangle\,.$$

The convergence above easily follows from (3.77) *by linearity, observing that*

$$\sum_{i=1}^k \alpha_i \langle V_N, \varphi_i\rangle = \langle V_N, \sum_{i=1}^k \alpha_i\varphi_i\rangle \xrightarrow[N\to\infty]{d} \langle \widetilde{v}, \sum_{i=1}^k \alpha_i\varphi_i\rangle = \sum_{i=1}^k \alpha_i \langle \widetilde{v}, \varphi_i\rangle\,.$$

REMARK 3.5. *Notice that we consider the case $t = 0$ for semplicity. In the original result presented in* [**CSZ17b**]*, the authors consider*

$$V_N(t,x) := \frac{1}{\beta_N}\big(Z_N^{\beta_N}(\lfloor Nt\rfloor, \lfloor\sqrt{N}x\rfloor) - 1\big) \quad \textit{for} \quad (t,x) \in [0,1]\times\mathbb{R}^2 \tag{3.22}$$

and proved the analog of Theorem 3.2, namely

$$V_N(t,x) \overset{\mathcal{D}}{\Longrightarrow} \widetilde{v}(x) := v^{(\frac{1}{2}, c_{\hat{\beta}})}(1-t,x) \tag{3.23}$$

with $c_{\hat{\beta}}$ *as above, equivalently for* $\varphi \in C_c([0,1]\times\mathbb{R}^2)$*:*

$$\langle V_N, \varphi\rangle := \int_{\mathbb{R}^2} V_N(t,x)\,\varphi(t,x)\,\mathrm{d}x \ \xrightarrow[N\to\infty]{d}\ \langle \widetilde{v}, \varphi\rangle \sim \mathcal{N}\big(0, \widetilde{\sigma}^2_{\hat{\beta},\varphi}\big)\,,$$

and

$$\widetilde{\sigma}^2_{\hat{\beta},\varphi} := \frac{1}{1-\hat{\beta}^2}\int_{(\mathbb{R}^2\times[0,1])^2} \varphi(t,x) K^{\frac{1}{2}}_{1-t,1-t'}(x,x')\varphi(t',x')\,\mathrm{d}x\,\mathrm{d}x'\,,$$

where $K^{\frac{1}{2}}_{t,t'}(x,x')$ *is defined in* (3.11)*.*

The convergence (3.20) was proved in [**CSZ17b**] using the Fourth Moment Theorem, based on the polynomial chaos expansion of the partition function, see (3.4). Remarkably, our Theorem 2.2 allows for an *alternative and more elementary proof of* (3.20)*, based on second moments calculations*, which we present in detail below.

Remark 3.6. *The factor* $\frac{1}{2}$ *in the parameters of* $\widetilde{v}(t,x) = v^{(\frac{1}{2}, c_{\hat{\beta}})}(1,x)$*, see* (3.20)*, is due to the fact that* $\mathrm{E}[S_1^{(i)}, S_1^{(j)}] = \frac{1}{2}\mathbb{1}_{\{i=j\}}$ *for* $i,j\in\{1,2\}$*. In view of* (3.8)*, note that* $\widetilde{v}(t,x) := v^{(\frac{1}{2}, c_{\hat{\beta}})}(1-t,x)$ *satisfies*

$$-\partial_t \widetilde{v}(t,x) = \frac{1}{4}\Delta_x \widetilde{v}(t,x) + c_{\hat{\beta}}\,\dot{W}(t,x)\,. \tag{3.24}$$

Edwards–Wilkinson fluctuations also hold for the logarithm of the partition function, suitably centered and rescaled as in (3.19):

$$H_N(x) := \frac{1}{\beta_N}\Big(\log Z_N^{\beta_N}(\lfloor\sqrt{N}x\rfloor) - \mathbb{E}\big[\log Z_N^{\beta_N}(\lfloor\sqrt{N}x\rfloor)\big]\Big)\,. \tag{3.25}$$

Indeed, it was shown in [**CSZ20**, Theorem 1.6] that a precise analogue of (3.20) holds.

Theorem 3.7 (Edwards–Wilkinson fluctuations for $\log Z_N^{\beta_N}$). *Let* $H_N(x)$ *be as in* (3.25)*, where* $\beta_N = \hat{\beta}/\sqrt{R_N}$ *with* $\hat{\beta}\in(0,1)$ *(cf.* (3.18)*). Then, as* $N\to\infty$*:*

$$H_N(x) \overset{\mathcal{D}}{\Longrightarrow} \widetilde{v}(x) = v^{(\frac{1}{2}, c_{\hat{\beta}})}(1,x)\,. \tag{3.26}$$

This convergence was in fact *deduced* in [**CSZ20**] from (3.20) by means of a highly non trivial linearization procedure, which we will briefly recall below for completeness. We cannot avoid to expoilt this linearization, howewer we show below how the alternative proof of (3.20) based on the novel CLT for polynomial chaos can be transferred to yield a proof of (3.26) as well.

Remark 3.8. *A simultaneous and independent proof of* (3.26) *was given in* [**Gu20**] *for small* $\hat{\beta} > 0$ *in a closely related context, namely for the KPZ equation* (3.7) *where the noise* $\dot{W}(t,x)$ *is regularized by mollification (rather than by discretization, as we consider here). Previously, the existence of non-trivial subsequential limits had been shown in* [**CD20**]*. We refer to* [**DG22, NN21+**] *for some recent extensions and generalizations.*

3.2.1. Fluctuations for the partition function: proof of Theorem 3.2. We need to prove that

$$V_N(x) \overset{\mathcal{D}}{\Longrightarrow} \widetilde{v}(x) = v^{(\frac{1}{2}, c_{\hat{\beta}})}(1, x)\,,$$

that is, for any fixed $\varphi \in C_c(\mathbb{R}^2)$ we have

$$\langle V_N, \varphi \rangle \xrightarrow[N\to\infty]{d} \mathcal{N}\big(0, \sigma^2_{\hat{\beta},\varphi}\big)$$

with

$$\sigma^2_{\hat{\beta},\varphi} := \frac{1}{1-\hat{\beta}^2} \int_{\mathbb{R}^2\times\mathbb{R}^2} \varphi(x)\,\varphi(x') \int_0^1 g_u(x-x')\,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}x'\,. \tag{3.27}$$

For convenience, we are going to show the equivalent convergence

$$\widetilde{V}_N(\varphi) := \langle\, \hat{\beta} V_N, \varphi\,\rangle \xrightarrow[N\to\infty]{d} \mathcal{N}\big(0\,,\, \hat{\beta}^2 \sigma^2_{\hat{\beta},\varphi}\big)\,. \tag{3.28}$$

Recall the definition (3.19) of V_N, we can write

$$\begin{aligned}\widetilde{V}_N(\varphi) &= \int_{\mathbb{R}^2} \sqrt{R_N}\big(Z_N^{\beta_N}\big(\lfloor \sqrt{N} x \rfloor\big) - 1\big)\varphi(x)\,\mathrm{d}x \\ &= \frac{1}{N}\int_{\mathbb{R}^2} \sqrt{R_N}\big(Z_N^{\beta_N}\big(\lfloor x \rfloor\big) - 1\big)\varphi\bigg(\frac{x}{\sqrt{N}}\bigg)\,\mathrm{d}x\,.\end{aligned} \tag{3.29}$$

Let us define $\overline{\varphi}_N : \mathbb{Z}^2 \to \mathbb{R}$ as the average of $\varphi\big(\frac{\cdot}{\sqrt{N}}\big)$ over cubes:

$$\overline{\varphi}_N(z) := \int\limits_{(z_1-1,z_1]\times(z_2-1,z_2]} \varphi\big(\tfrac{x}{\sqrt{N}}\big)\,\mathrm{d}x \qquad \text{for} \quad z = (z_1, z_2) \in \mathbb{Z}^2\,. \tag{3.30}$$

Recalling the polynomial chaos expansion (3.4) of $Z_N^{\hat{\beta}}(0,z)$ we can rewrite $\widetilde{V}_N(\varphi)$ as follows:

$$\widetilde{V}_N(\varphi) = \frac{1}{N}\sum_{k=1}^{\infty} \sigma_N^k \sqrt{R_N} \sum_{\substack{0=:n_0<n_1<\cdots<n_k\le N \\ x_1,\ldots,x_k\in\mathbb{Z}^2}} q_{n_1}\big(\overline{\varphi}_N, x_1\big)\,\eta_N(n_1,x_1) \prod_{j=2}^{k} q_{n_j-n_{j-1}}(x_j - x_{j-1})\,\eta_N(n_j,x_j)\,, \tag{3.31}$$

where

$$q_{n_1}\big(\overline{\varphi}_N, x_1\big) := \sum_{x_0\in\mathbb{Z}^2} \overline{\varphi}_N(x_0) q_{n_1}(x_1 - x_0)\,.$$

Note that we can represent $\widetilde{V}_N(\varphi) = \sum_{A\subset\mathbb{T}} q_N(A)\eta^N(A)$ as in (2.6)-(2.7) with the following correspondences:

- the index set is $\mathbb{T} := \mathbb{N}\times\mathbb{Z}^2$;
- the random variables $\eta_t^N = \eta_N(m,z)$, for $t = (m,z)\in\mathbb{T}$, are defined in (3.3): they satisfy (2.1) by construction, while they satisfy (2.5) because $\sup_N \mathbb{E}[|\eta_N(m,z)|^p] < \infty$ for all $p<\infty$ by (3.1) (see [**CSZ17a**, eq. (6.7)]);

- the kernel $q_N(A)$, for $A = \{t_1, \ldots, t_k\} = \{(n_1, x_1), \ldots, (n_k, x_k)\} \subset \mathbb{T}$, is

$$q_N(A) = \frac{1}{N} \sigma_N^k \sqrt{R_N}\, q_{n_1}\big(\overline{\varphi}_N, x_1\big) \prod_{j=2}^{k} q_{n_j - n_{j-1}}(x_j - x_{j-1}) \, \mathbb{1}_{\{0<n_1<\cdots<n_k\le N\}} \,. \tag{3.32}$$

By Theorem 2.2, in order to prove $\widetilde{V}_N(\varphi) \xrightarrow{d} \mathcal{N}(0, \sigma^2)$ with $\sigma^2 := \hat{\beta}^2 \sigma^2_{\hat{\beta},\varphi}$ as in (3.28) we have to check the following conditions.

(1) *Limiting second moment:* we need to prove that $\lim_{N\to\infty} \mathbb{E}[\widetilde{V}_N(\varphi)^2] = \sigma^2$.

(2) *Subcriticality*: we need to show that

$$\lim_{K\to\infty} \limsup_{N\to\infty} \sum_{\substack{A\subset\mathbb{T}\\ |A|>K}} q_N(A)^2 = 0 \,. \tag{3.33}$$

(3) *Spectral localization*: for any $M, N \in \mathbb{N}$ we define the disjoint subsets

$$\mathbb{B}_i := \big(\tfrac{i-1}{M} N, \tfrac{i}{M} N\big] \times \mathbb{Z}^2 \qquad \text{for } i = 1, \ldots, M \,,$$

and, recalling that $\sigma_N^2(\mathbb{B}_i) := \sum_{A\subset\mathbb{B}_i} q_N(A)^2$, we need to show that

$$\lim_{M\to\infty} \sum_{i=1}^{M} \lim_{N\to\infty} \sigma_N^2(\mathbb{B}_i) = \sigma^2 \qquad \text{and} \qquad \lim_{M\to\infty} \Big\{ \max_{i=1,\ldots,M} \limsup_{N\to\infty} \sigma_N^2(\mathbb{B}_i) \Big\} = 0 \,. \tag{3.34}$$

Proof of (2)**.** We start by showing the subcriticality (3.33). For $K \ge 1$ we can write, by (3.31),

$$\sum_{\substack{A\subset\mathbb{T}\\ |A|>K}} q_N(A)^2 = \frac{1}{N^2} \sum_{k>K} (\sigma_N^2)^k R_N \sum_{\substack{0=:n_0<n_1<\cdots<n_k\le N\\ x_1,\ldots,x_k\in\mathbb{Z}^2}} q_{n_1}\big(\overline{\varphi}_N, x_1\big)^2 \prod_{j=2}^{k} q_{n_j-n_{j-1}}(x_j - x_{j-1})^2 \,. \tag{3.35}$$

We can enlarge the sums to $0 < m_j := n_j - n_{j-1} \le N$ and change variables $y_j := x_j - x_{j-1}$, for $j = 2, \ldots, k$, thus we get the following upper bound

$$\begin{aligned}
\sum_{\substack{A\subset\mathbb{T}\\ |A|>K}} q_N(A)^2 &\le \frac{1}{N^2} \sum_{k>K} (\sigma_N^2)^k R_N \sum_{\substack{0<n_1\le N\\ x_1\in\mathbb{Z}^2}} q_{n_1}\big(\overline{\varphi}_N, x_1\big)^2 \prod_{j=2}^{k} \sum_{\substack{0<m_j\le N\\ y_j\in\mathbb{Z}^2}} q_{m_j}(y_j)^2 \\
&= \frac{1}{N^2} \sum_{k>K} (\sigma_N^2)^k R_N \sum_{\substack{0<n_1\le N\\ x_1\in\mathbb{Z}^2}} q_{n_1}\big(\overline{\varphi}_N, x_1\big)^2 \prod_{j=2}^{k} \sum_{0<m_j\le N} u_{m_j} \\
&= \sum_{k>K} (\sigma_N^2 R_N)^k \Big\{ \frac{1}{N^2} \sum_{\substack{0<n_1\le N\\ x_1\in\mathbb{Z}^2}} q_{n_1}\big(\overline{\varphi}_N, x_1\big)^2 \Big\} \\
&\le C \, \frac{(\sigma_N^2 R_N)^K}{1 - \sigma_N^2 R_N} \,,
\end{aligned} \tag{3.36}$$

where we used $\sum_{0<m\le N}\sum_{y\in\mathbb{Z}^2} q_m(y)^2 = \sum_{0<m\le N} u_m = R_N$, see (3.14)-(3.15), and we remark that $\sigma_N^2\, R_N < 1$ for N large enough, because $\sigma_N^2 \sim \hat\beta^2/R_N$, see (3.18), and $\hat\beta < 1$. The term between the curly brackets is bounded by a constant $C = C(\varphi) > 0$, indeed by summing over $x_1 \in \mathbb{Z}^2$ and applying Chapman-Kolmogorov:

$$\begin{aligned}
&\frac{1}{N^2}\sum_{\substack{0<n_1\le N\\ x_1\in\mathbb{Z}^2}} q_{n_1}\big(\overline{\varphi}_N, x_1\big)^2 = \frac{1}{N^2}\sum_{0<n_1\le N}\ \sum_{y,y'\in\mathbb{Z}^2} \overline{\varphi}_N(y)\overline{\varphi}_N(y')q_{2n_1}(y-y') \\
&\le \frac{\|\varphi\|_\infty^2}{N^2}\sum_{0<n_1\le N}\ \sum_{\substack{y\in\mathbb{Z}^2:\\ |y|\le A\sqrt{N}}}\ \sum_{y'\in\mathbb{Z}^2} q_{2n_1}(y-y') \le C(\varphi, A)
\end{aligned} \tag{3.37}$$

for some $A > 0$, since φ is compaclty supported on $\mathbb{R}^2$, thus $\overline{\varphi}_N(y) \neq 0$ only for $|y| \le A\sqrt{N}$. Moreover, recall that $q_{2n_1}(y-y')$ is a probability, then $\sum_{y'\in\mathbb{Z}^2} q_{2n_1}(y-y') = 1$, and, eventually, notice that the number of terms of both sums over $|y| \le A\sqrt{N}$, $y \in \mathbb{Z}^2$ and $1 \le n_1 \le N$ are of order N. Hence, we have

$$\limsup_{N\to\infty} \sum_{\substack{A\subset\mathbb{T}\\ |A|>K}} q_N(A)^2 \le C\,\frac{(\hat\beta^2)^K}{1-\hat\beta^2}\,,$$

from which (3.33) follows, since $\hat\beta^2 < 1$.

Proof of (1) **and** (3)**.** We are going to show that for all $M \in \mathbb{N}$ and $i \in \{1, \dots, M\}$:

$$\lim_{N\to\infty} \sigma_N^2(\mathbb{B}_i) = \sigma_i^2 := \frac{\hat\beta^2}{1-\hat\beta^2}\int_{\mathbb{R}^2\times\mathbb{R}^2} \varphi(x)\,\varphi(x')\int_{\frac{i-1}{M}}^{\frac{i}{M}} g_u(x-x')\,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}x'\,. \tag{3.38}$$

Note that this immediately proves the first relation of (3.34) and also (for $i = M = 1$) $\lim_{N\to\infty}\mathbb{E}[\widetilde{V}_N(\varphi)^2] = \sigma^2$, see also (3.27).

Arguing as in (3.36), we have the following upper bound

$$\begin{aligned}
&\sum_{A\subset\mathbb{B}_i} q_N(A)^2 \\
&= \frac{1}{N^2}\sum_{k=1}^{\infty}(\sigma_N^2)^k R_N \sum_{\substack{\frac{i-1}{M}N<n_1<\dots<n_k\le\frac{i}{M}N\\ x_1,\dots,x_k\in\mathbb{Z}^2}} q_{n_1}\big(\overline{\varphi}_N, x_1\big)^2\prod_{j=2}^{k} q_{n_j-n_{j-1}}(x_j-x_{j-1})^2 \\
&\le \frac{\sigma_N^2\,R_N}{1-\sigma_N^2\,R_N}\left\{\frac{1}{N^2}\sum_{\substack{\frac{i-1}{M}N<n_1\le\frac{i}{M}N\\ x_1\in\mathbb{Z}^2}} q_{n_1}\big(\overline{\varphi}_N, x_1\big)^2\right\}.
\end{aligned} \tag{3.39}$$

We claim that

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{\substack{\frac{i-1}{M}N<n_1\le\frac{i}{M}N \\ x_1\in\mathbb{Z}^2}} q_{n_1}\big(\overline{\varphi}_N, x_1\big)^2 = \int_{\mathbb{R}^2\times\mathbb{R}^2} \varphi(x)\,\varphi(x') \int_{\frac{i-1}{M}}^{\frac{i}{M}} g_u(x-x')\,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}x' \,, \tag{3.40}$$

therefore from (3.39) it follows that

$$\lim_{N\to\infty} \sigma_N^2(\mathbb{B}_i) \le \sigma_i^2 \,.$$

To complete the proof we need to derive a matching lower bound. Let us fix $H \in \mathbb{N}$ large, such that $\frac{1}{H} < \frac{1}{M}$. Starting from the expression in (3.39), we get a lower bound by the following restrictions:

$$1 < k \le H \,, \quad \frac{i-1}{M}N < n_1 \le \Big(\frac{i}{M} - \frac{1}{H}\Big)N \,, \quad 0 < n_j - n_{j-1} \le \frac{N}{H^2} \ \ \forall j = 2,\dots,k \,,$$

which ensures that $n_k \le n_1 + \sum_{j=2}^k (n_j - n_{j-1}) \le \big(\frac{i}{M} - \frac{1}{H}\big)N + H\frac{1}{H^2}N \le \frac{i}{M}N$ as required. Then, similarly to (3.36) and by estimating the first prefactor as $R_N \ge R_{N/H^2}$ (see (3.15)), we obtain the following estimate from below:

$$\begin{aligned} &\sum_{A\subset\mathbb{B}_i} q_N(A)^2 \\ &\ge \frac{\sigma_N^2 R_{N/H^2} - \big(\sigma_N^2 R_{N/H^2}\big)^{H+1}}{1-\sigma_N^2 R_{N/H^2}} \left\{ \frac{1}{N^2} \sum_{\substack{\frac{i-1}{M}N<n_1\le\frac{i}{M}N \\ x_1\in\mathbb{Z}^2}} q_{n_1}\big(\overline{\varphi}_N, x_1\big)^2 \right\}, \end{aligned}$$

where we recall that $\sum_{k=1}^H x^k = \frac{x - x^{H+1}}{1-x}$ for $|x| < 1$. Since $R_{N/H^2} \sim R_N$ for fixed $H \in \mathbb{N}$ and by claim (3.40), we have shown that

$$\liminf_{N\to\infty} \sigma_N^2(\mathbb{B}_i) \ge \frac{\hat\beta^2 - (\hat\beta^2)^{H+1}}{1-\hat\beta^2} \int_{\mathbb{R}^2\times\mathbb{R}^2} \varphi(x)\,\varphi(x') \int_{\frac{i-1}{M}}^{\frac{i}{M}} g_u(x-x')\,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}x' \,.$$

We can finally take the limit $H \to \infty$ to see that (3.38) holds.

We only need to prove the claim (3.40). Since we aim to apply the local central limit theorem in the diffusive regime (recall (5.33)), for any $\delta > 0$ we first consider the contribution of

$$\frac{1}{N^2} \sum_{\substack{\frac{i-1}{M}N<n_1\le\frac{i}{M}N \\ x_1\in\mathbb{Z}^2}} q_{n_1}\big(\overline{\varphi}_N, x_1\big)^2 = \frac{1}{N^2} \sum_{\frac{i-1}{M}N<n_1\le\frac{i}{M}N} \ \sum_{y,y'\in\mathbb{Z}^2} \overline{\varphi}_N(y)\overline{\varphi}_N(y') q_{2n_1}(y-y')$$

where $\frac{i-1}{M}N < n_1 \le \frac{i}{M}N$ and $n_1 > \delta N$ if $i = 1$ (so that $n_1 \to \infty$ as $N \to \infty$) and $|y - y'| \le \frac{1}{\delta}\sqrt{2n_1}$:

$$\frac{1}{N^2} \sum_{\substack{\frac{i-1}{M}N<n_1\le\frac{i}{M}N \\ n_1>\delta N \text{ if } i=1}} \sum_{\substack{y,y'\in\mathbb{Z}^2: \\ |y-y'|\le\frac{1}{\delta}\sqrt{2n_1}}} \overline{\varphi}_N(y)\overline{\varphi}_N(y')q_{2n_1}(y-y')$$

$$= \frac{(1+o(1))}{N^2} \sum_{\substack{\frac{i-1}{M}N<n_1\le\frac{i}{M}N \\ n_1>\delta N \text{ if } i=1}} \sum_{\substack{y,y'\in\mathbb{Z}^2: \\ |y-y'|\le\frac{1}{\delta}\sqrt{2n_1}}} \overline{\varphi}_N(y)\overline{\varphi}_N(y')\, g_{n_1}(y-y')\, 2\,\mathbb{1}_{(2n_1,y-y')\in\mathbb{Z}^3_{\text{even}}}\,. \tag{3.41}$$

Observe that $(2n_1, y - y') \in \mathbb{Z}^3_{\text{even}}$ if and only if $y - y' \in \mathbb{Z}^2_{\text{even}}$, moreover we easily have

$$g_{n_1}(y-y') = \frac{1}{N}\, g_{\frac{n_1}{N}}\left(\frac{y-y'}{\sqrt{N}}\right).$$

By changing variables $u := \frac{n_1}{N}$, $x := \frac{y}{\sqrt{N}}$ and $x' := \frac{y'}{\sqrt{N}}$, a Riemann sum approximation[2] applies (recall also the definition (3.30) of $\overline{\varphi}_N$, which approximates well the continuous function φ for N large enough), thus we obtain

$$\int_{\frac{i-1}{M}\vee\delta}^{\frac{i}{M}} \int_{|x-x'|<\frac{1}{\delta}u} \varphi(x)\varphi(x')\, g_u(x-x')\,\mathrm{d}x\,\mathrm{d}x'\,\mathrm{d}u\,. \tag{3.42}$$

By sending $\delta \to 0$, we conclude the proof of (3.40), provided that the complement of (3.41) gives a negligible contribution as we show below. Indeed, for all $\delta > 0$ and some suitable $A > 0$ we have

$$\begin{aligned}
&\frac{1}{N^2} \sum_{\frac{i-1}{M}N<n_1\le\frac{i}{M}N} \sum_{\substack{y,y'\in\mathbb{Z}^2: \\ |y-y'|>\frac{1}{\delta}\sqrt{2n_1}}} \overline{\varphi}_N(y)\overline{\varphi}_N(y')q_{2n_1}(y-y') \\
&\le \frac{\|\varphi\|_\infty^2}{N^2} \sum_{\frac{i-1}{M}N<n_1\le\frac{i}{M}N} \sum_{|y|\le A\sqrt{N}} \mathrm{P}\left(\left|S_{2n_1}\right| > \frac{1}{\delta}\sqrt{2n_1}\right) \\
&\le C(\varphi, A)\frac{\delta^2}{4n_1^2}\mathrm{E}\left[\left|S_{2n_1}\right|^2\right] \\
&\le C(\varphi, A)\,\delta^2
\end{aligned}$$

where we applied the Markov inequality and the fact that φ is compactly supported. By sending $\delta \to 0$ for arbitrariness, we conclude that the above contribution is negligible. When $i = 1$, arguing as in (3.37) we are able to estimate the additional term

$$\frac{1}{N^2} \sum_{0<n_1\le\delta N} \sum_{y,y'\in\mathbb{Z}^2} \overline{\varphi}_N(y)\overline{\varphi}_N(y')q_{2n_1}(y-y') \le C\delta\,, \tag{3.43}$$

[2]Notice that the factor 2 in the last line of (3.41) is consistent with the Riemann sum approximation since the sum over $x' = \frac{y'}{\sqrt{N}}$ such that $y - y' \in \mathbb{Z}^2_{\text{even}}$ is restricted to a sublattice of $\mathbb{Z}^2$, where each point is the center of a rhombus cell of area $\frac{2}{N}$.

which vanishes as $\delta \to 0$.

To finally complete the proof of Theorem 3.2 we only need to show the second relation of (3.34), namely

$$\lim_{M\to\infty} \Big\{ \max_{i=1,\dots,M} \limsup_{N\to\infty} \sigma_N^2(\mathbb{B}_i) \Big\} = \lim_{M\to\infty} \Big\{ \max_{i=1,\dots,M} \sigma_i^2 \Big\} = 0\,. \tag{3.44}$$

By denoting $f(u) := \frac{\hat\beta^2}{1-\hat\beta^2} \int_{\mathbb{R}^2\times\mathbb{R}^2} \varphi(x)\varphi(x') g_u(x-x')\,\mathrm{d}x\mathrm{d}x'$ which is integrable on $[0,1]$, by uniform integrability for all $\varepsilon > 0$ we can choose $M > 0$ sufficiently large such that

$$\begin{aligned}&\max_{i=1,\dots,M} \int_{\frac{i-1}{M}}^{\frac{i}{M}} f(u)\,\mathrm{d}u \\ &\le \max\Big\{ \int_A |f(u)|\,\mathrm{d}u \,:\, A \subseteq [0,1]\,,\ A \text{ is } \mathcal{L}\text{-measurable, } \mathcal{L}(A) = \frac{1}{M} \Big\} < \varepsilon\,,\end{aligned} \tag{3.45}$$

where $\mathcal{L}$ denotes the Lebesgue measure on $[0,1]$. This shows (3.44) and concludes the proof Theorem 3.2. □

3.2.2. Fluctuations for the log-partition function: proof of Theorem 3.7. Before applying our CLT for polynomial chaos (Theorem 2.2), we need to exploit the linearization procedure developed in [**CSZ20**], which we recall here for sake of completeness. Thanks to this approach, we are able to approximate the centered and rescaled log-partition function $H_N(x)$ (recall (3.25)) in terms of a modification of $V_N(x)$ (recall (3.19)), on which we can then apply our Theorem 2.2 similarly as in the previous subsection.

The key observation made in [**CSZ20**] is that the partition function $Z_N^{\beta_N}(x)$ depends only on the disorder in a neighbourhood of the starting point, negligible on the diffusive scale $(N, \sqrt{N})$. Therefore, in order to obtain a suitable decomposition of the log-partition function, it is convenient to approximate $Z_N^{\beta_N}(x)$ by a modification where the disorder is present only in the window

$$A_N^x := \{(n,z) \in \mathbb{N}\times\mathbb{Z}^2 : n \le N^{1-a_N}\ |z-x| < N^{\frac{1}{2}-\frac{a_N}{4}}\}\,,\ \text{ where } a_N := \frac{1}{(\log N)^{1-\gamma}}\,,$$

with $\gamma \in (0,\gamma^*)$ for a suitable choice of $\gamma^* > 0$ small enough depending on $\hat\beta$. Then, by denoting

$$Z_{N,A}^{\beta_N}(x) := \mathrm{E}\big[e^{H_{A,\beta_N}}\,|S_0 = z\big]\,, \qquad x \in \mathbb{Z}^2 \tag{3.46}$$

where $H_{A,\beta_N} \coloneqq \sum_{(n,x)\in A_N^x} \big(\beta_N\, \omega(n,x) - \lambda(\beta_N)\big) \mathbb{1}_{\{S_n = x\}}$, it is possible to decompose $\log Z_N^{\beta_N}(x)$ as follows:

$$\begin{aligned}\log Z_N^{\beta_N}(x) &= \log \big(Z_{N,A}^{\beta_N}(x) + \hat{Z}_{N,A}^{\beta_N}(x) \big) \\ &= \log Z_{N,A}^{\beta_N}(x) + \log \left(1 + \frac{\hat{Z}_{N,A}^{\beta_N}(x)}{Z_{N,A}^{\beta_N}(x)} \right) \\ &= \log Z_{N,A}^{\beta_N}(x) + \frac{\hat{Z}_{N,A}^{\beta_N}(x)}{Z_{N,A}^{\beta_N}(x)} + O_N(x)\,,\end{aligned}$$

where $\hat{Z}_{N,A}^{\beta_N}(x) \coloneqq Z_N^{\beta_N}(x) - Z_{N,A}^{\beta_N}(x)$ and $O_N(x)$ is the error.

Remarkably, to determine the fluctuations of the rescaled $\log Z_N^{\beta_N}(x)$ averaged against a test function $\varphi \in C_c(\mathbb{R}^2)$, only the remainder $\hat{Z}_{N,A}^{\beta_N}(x)/Z_{N,A}^{\beta_N}(x)$ gives a non-negligible contribution for large N, as pointed out in the following propositions.

Proposition 3.9 ([**CSZ20**, Proposition 2.1]). *For any $\varphi \in C_c(\mathbb{R}^2)$*

$$\int_{\mathbb{R}^2} \sqrt{R_N} \Big(O_N\big(\lfloor Ny \rfloor\big) - \mathbb{E}\big[O_N\big(\lfloor Ny \rfloor\big)\big] \Big)\, \varphi(y)\, \mathrm{d}y \xrightarrow[N\to\infty]{L^2(\mathbb{P})} 0\,. \tag{3.47}$$

Proposition 3.10 ([**CSZ20**, Proposition 2.2]). *For any $\varphi \in C_c(\mathbb{R}^2)$*

$$\int_{\mathbb{R}^2} \sqrt{R_N} \Big(\log Z_{N,A}^{\beta_N}\big(\lfloor Ny \rfloor\big) - \mathbb{E}\big[\log Z_{N,A}^{\beta_N}\big(\lfloor Ny \rfloor\big)\big] \Big)\, \varphi(y)\, \mathrm{d}y \xrightarrow[N\to\infty]{L^2(\mathbb{P})} 0\,. \tag{3.48}$$

Proposition 3.11 ([**CSZ20**, Proposition 2.3]). *For any $\varphi \in C_c(\mathbb{R}^2)$*

$$\int_{\mathbb{R}^2} \sqrt{R_N} \left(\frac{\hat{Z}_{N,A}^{\beta_N}\big(\lfloor Ny \rfloor\big)}{Z_{N,A}^{\beta_N}\big(\lfloor Ny \rfloor\big)} - \Big(Z_{N,B^{\geq}}^{\beta_N}\big(\lfloor Ny \rfloor\big) - 1 \Big) \right) \varphi(y)\, \mathrm{d}y \xrightarrow[N\to\infty]{L^1(\mathbb{P})} 0\,, \tag{3.49}$$

where $Z_{N,B^{\geq}}^{\beta_N}(x)$ is defined similarly to (3.46) *and*

$$B^{\geq} = B_N^{\geq} \coloneqq \Big(\big(N^{1-9a_N/40}, N\big] \cap \mathbb{N} \Big) \times \mathbb{Z}^2\,.$$

Remark 3.12. *Propositions 2.1, 2.2 and 2.3 in* [**CSZ20**] *are actually expressed in the discrete setting, namely*

$$\sqrt{R_N}\, \frac{1}{N} \sum_{z\in\mathbb{Z}^2} \Big(\log Z_{N,A}^{\beta_N}(z) - \mathbb{E}[\log Z_{N,A}^{\beta_N}(z)] \Big)\, \varphi\Big(\frac{z}{\sqrt{N}} \Big) \xrightarrow[N\to\infty]{L^2(\mathbb{P})} 0\,. \tag{3.50}$$

However, we can derive an analogous version of the above expression from (3.48) *by recalling that $\log Z_{N,A}^{\beta_N}\big(\lfloor Ny \rfloor\big)$ is constant on the cubes $C_z \coloneqq (z_1-1, z_1] \times (z_2 - 1, z_2]$ for $z = (z_1, z_2) \in \mathbb{Z}^2$ and arguing as in* (3.29)*. In this way, instead of $\varphi\big(\frac{z}{\sqrt{N}}\big)$ in* (3.50) *we get its average over C_z (check* (3.30)*), but this is not relevant as $N \to \infty$ since φ is smooth enough. Similar observations hold for* (3.47) *and* (3.49)*, too.*

As last step, we need to identify the fluctuations of $Z^{\beta_N}_{N,B\geq}(x)$, namely

$$\widetilde{V}_{N,B\geq}(\varphi) := \int_{\mathbb{R}^2} \sqrt{R_N}\big(Z^{\beta_N}_{N,B\geq}(x) - 1\big)\,\varphi(x)\,\mathrm{d}x \xrightarrow[N\to\infty]{d} \mathcal{N}(0,\sigma^2) \tag{3.51}$$

with

$$\sigma^2 := \frac{\hat{\beta}^2}{1-\hat{\beta}^2} \int_{\mathbb{R}^2\times\mathbb{R}^2} \varphi(x)\,\varphi(x') \int_0^1 g_u(x-x')\,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}x'\,,$$

which finally proves Theorem 3.7.

Repeating the arguments followed in the previous subsection, we can write $\widetilde{V}_{N,B\geq}(\varphi)$ as follows:

$$\widetilde{V}_{N,B\geq}(\varphi) = \frac{1}{N}\sum_{k=1}^{\infty} \sigma_N^k \sqrt{R_N} \sum_{\substack{\varepsilon_N N<n_1<\cdots<n_k\leq N\\ x_1,\dots,x_k\in\mathbb{Z}^2}} q_{n_1}\big(\overline{\varphi}_N, x_1\big)\,\eta_N(n_1,x_1) \prod_{j=2}^{k} q_{n_j-n_{j-1}}(x_j-x_{j-1})\,\eta_N(n_j,x_j)\,, \tag{3.52}$$

where

$$q_{n_1}\big(\overline{\varphi}_N, x_1\big) := \sum_{x_0\in\mathbb{Z}^2} \overline{\varphi}_N(x_0) q_{n_1}(x_1-x_0) \quad \text{and} \quad \varepsilon_N := N^{-9a_N/40} \xrightarrow[N\to\infty]{} 0\,.$$

We have already observed that the main contribution to the limiting variance σ^2 is given, for N large, by those time variables $0 < n_1 < \cdots < n_k \leq N$ such that $n_1 \in (\delta N, N]$, for $\delta > 0$ small and eventually sent to 0 (recall (3.41) and (3.43)). This implies that summing over $\varepsilon_N N < n_1 < \cdots < n_k \leq N$ with $\varepsilon_N := N^{-9a_N/40}$ as in (3.52) will give the main contribution to the limiting variance as $N \to \infty$. Therefore, the convergence in distribution (3.51) of $\widetilde{V}_{N,B\geq}(\varphi)$ follows from the convergence of $\widetilde{V}_N(\varphi)$ (see (3.28)), by adapting the arguments of the previous subsection according to the modified coefficient function (cf (3.32))

$$q_N(A) = \frac{1}{N}\,\sigma_N^k \sqrt{R_N}\, q_{n_1}\big(\overline{\varphi}_N, x_1\big) \prod_{j=2}^{k} q_{n_j-n_{j-1}}(x_j-x_{j-1})\, \mathbb{1}_{\{\varepsilon_N N<n_1<\cdots<n_k\leq N\}}\,.$$

□

3.3. Gaussian limit for a singular product

In this section, we exploit Theorem 2.2 to prove a new Gaussian convergence result related to the partition function, which we now describe. We have already mentioned that the diffusively rescaled partition function $U_N(t,x)$ in (3.6) approximates the solution of the Stochastic Heat Equation (3.5) with *multiplicative* noise. It is not clear a priori why the fluctuations of $U_N(t,x)$, encoded by $V_N(t,x)$ in (3.22), converge to $\widetilde{v}(t,x)$ which solves the Stochastic Heat Equation with *additive* noise, see (3.24), with an intensity $c_{\hat{\beta}}$ which *explodes* as $\hat{\beta} \uparrow 1$. We now present a result which sheds light on the mechanism which leads to (3.24).

We denote by $\dot{W}_N(t,x)$, for $t > 0$, $x \in \mathbb{R}^2$, the diffusively rescaled version of η_N:

$$\dot{W}_N(t,x) := N\, \eta_N(\lfloor Nt \rfloor, \lfloor \sqrt{N} x \rfloor)\,. \tag{3.53}$$

Recall that for all $N \in \mathbb{N}$, the modified disorder $\eta_N = (\eta_N(m,z))_{m\in\mathbb{N}, z\in\mathbb{Z}^2}$ is i.i.d. with $\mathbb{E}[\eta_N(m,z)] = 0$ and $\mathbb{E}[\eta_N(m,z)^2] = 1$, see (3.1) and (3.3), and higher moments of η_N are uniformly bounded (see [**CSZ17a**, eq. (6.7)]). It follows that $\dot{W}_N$ converges in law to the white noise:

$$\dot{W}_N(t,x) \overset{\mathcal{D}}{\Longrightarrow} \dot{W}(t,x)\,, \tag{3.54}$$

that is $\langle \dot{W}_N, \psi\rangle \xrightarrow{d} \langle \dot{W}, \psi \rangle \sim \mathcal{N}(0, \|\psi\|_{L^2}^2)$ as $N \to \infty$, for $\psi \in C_c^\infty([0,1]\times\mathbb{R}^2)$.

For $(t,x) \in [0,1]\times\mathbb{Z}^2$ we now consider the product between $\dot{W}_N$ and $U_N(t,x)-1$, i.e. the centered and diffusively rescaled partition function $Z_N^{\beta_N}(\lfloor Nt\rfloor, \lfloor \sqrt{N}x\rfloor) - 1$, see (3.6):

$$\begin{aligned} \Xi_N(t,x) &:= \dot{W}_N(t,x)\,\big(U_N(t,x)-1\big) \\ &= \beta_N\, \dot{W}_N(t,x)\, V_N(t,x)\,, \end{aligned} \tag{3.55}$$

where we recall that $V_N(t,x) = \beta_N^{-1}(U_N(t,x)-1)$, (cf. (3.22)). We know that $V_N \overset{\mathcal{D}}{\Longrightarrow} \widetilde{v}$ and $\dot{W}_N \overset{\mathcal{D}}{\Longrightarrow} W$ as $N\to\infty$, see (3.23) and (3.54). Since $\beta_N \to 0$, one could expect that $\Xi_N \overset{\mathcal{D}}{\Longrightarrow} 0$, but *this turns out to be false.* The point is that V_N and $\dot{W}_N$ only converge as random distributions, and the product of distributions is not a continuous operation (it is generally not even defined). The following result shows that Ξ_N has in fact a non-trivial limit as $N\to\infty$.

THEOREM 3.13 (White noise from singular product). *Let $\beta = \beta_N$ be fixed as in (3.18), and set $c_{\hat\beta} := (1-\hat\beta^2)^{-1/2}$. As $N\to\infty$, we have the joint convergence in law:*

$$(\dot{W}_N, \Xi_N) \overset{\mathcal{D}}{\Longrightarrow} \Big(\dot{W},\ \sqrt{c_{\hat\beta}^2 - 1}\,\dot{W}'\Big)\,,$$

where $\dot{W}$ and $\dot{W}'$ denote two independent white noises on $[0,1]\times\mathbb{R}^2$. More precisely, for any $\psi \in C_c^\infty([0,1]\times\mathbb{R}^2)$, the following joint convergence in distribution holds:

$$\big(\langle \dot{W}_N, \psi\rangle,\, \langle \Xi_N, \psi\rangle\big) \xrightarrow{d} \mathcal{N}\big(0, \|\psi\|_{L^2}^2\,\Sigma_{\hat\beta}\big) \qquad \textit{where} \qquad \Sigma_{\hat\beta} = \begin{pmatrix} 1 & 0 \\ 0 & c_{\hat\beta}^2 - 1\end{pmatrix}.$$

We prove Theorem 3.13 below as an application of our Theorem 2.2.

At this point, we can finally give a heuristic explanation for equation (3.24). One can check that $Z_N^{\beta_N}(m,z)$ in (3.2) solves the following *difference equation*, for $m \le N$ and $z \in \mathbb{Z}^2$:

$$Z_N^{\beta_N}(m-1,z) - Z_N^{\beta_N}(m,z) = \frac{1}{4}\Delta_{\mathbb{Z}^2} Z_N^{\beta_N}(m,z) + \sigma_N \frac{1}{4}\sum_{z'\sim z} \eta_N(m,z')\, Z_N^{\beta_N}(m,z')\,, \tag{3.56}$$

where $z' \sim z$ means $z' \in \{z\pm(1,0), z\pm(0,1)\}$ and $\Delta_{\mathbb{Z}^2} f(z) := \sum_{z'\sim z}\{f(z') - f(z)\}$ denotes the lattice Laplacian (we recall that σ_N and $\eta_N(m,z)$ are defined in (3.3)).

By (3.22) and (3.53), we can rewrite (3.56) as follows, for $(t,x) \in ((0,1] \cap \frac{\mathbb{Z}}{N}) \times (\mathbb{R}^2 \cap \frac{\mathbb{Z}^2}{\sqrt{N}})$:

$$- \partial_t^{(N)} U_N(t,x) = \frac{1}{4}\Delta_x^{(N)} U_N(t,x) \;+\; \sigma_N \, \frac{1}{4} \sum_{x' \overset{N}{\sim} x} \dot{W}_N(t,x')\, U_N(t,x') \,, \tag{3.57}$$

where $x' \overset{N}{\sim} x$ means $x' \in \{x \pm (\frac{1}{\sqrt{N}}, 0), x \pm (0, \frac{1}{\sqrt{N}})\}$ and we define the rescaled operators

$$\begin{aligned} \partial_t^{(N)} f(t,x) &:= N\big\{ f(t,x) - f(t - \tfrac{1}{N}, x)\big\} \,, \\ \Delta_x^{(N)} f(t,x) &:= N \sum_{x' \overset{N}{\sim} x} \big\{ f(t,x') - f(t,x)\big\} \,. \end{aligned}$$

Note that (3.57) is a discretization of the (time reversed) Stochastic Heat Equation (3.5), with the factor $\frac{1}{4}$ instead of $\frac{1}{2}$ (see Remark 3.6) and with $\sigma_N \sim \beta_N$ in place of β.

We now consider $V_N(t,x) = \beta_N^{-1}(U_N(t,x) - 1)$, see (3.20). By (3.57) we obtain

$$-\partial_t^{(N)} V_N(t,x) = \frac{1}{4}\Delta_x^{(N)} V_N(t,x) + \frac{\sigma_N}{\beta_N} \, \frac{1}{4} \sum_{x' \overset{N}{\sim} x} \Big\{ \dot{W}_N(t,x') + \beta_N \, \dot{W}_N(t,x')\, V_N(t,x') \Big\} \,. \tag{3.58}$$

The last term $\beta_N \, \dot{W}_N(t,x')\, V_N(t,x')$ is nothing but $\Xi_N(t,x')$ in (3.55), which formally vanishes as $N \to \infty$ but actually *converges to an independent white noise* $\sqrt{c_{\hat\beta}^2 - 1}\, \dot{W}'(t,x)$, by Theorem 3.13 (note that $x' \overset{N}{\sim} x$ implies $|x' - x| = 1/\sqrt{N} \to 0$). If we assume that $V_N(t,x)$ converges to a limit $\widetilde{v}(t,x)$, by taking the formal limit of (3.58) we finally obtain

$$- \partial_t \widetilde{v}(t,x) = \frac{1}{4}\Delta_x \widetilde{v}(t,x) \;+\; \dot{W}(t,x) \;+\; \sqrt{c_{\hat\beta}^2 - 1}\, \dot{W}'(t,x) \,. \tag{3.59}$$

Note that *this is equivalent to* (3.24), because $\dot{W}(t,x) + \sqrt{c_{\hat\beta}^2 - 1}\, \dot{W}'(t,x) \overset{d}{=} c_{\hat\beta}\, \dot{W}(t,x)$.

In conclusion, Theorem 3.13 provides an intuitive explanation why the random field $\widetilde{v}(t,x)$ to which $V_N(t,x)$ converges should satisfy the equation (3.24), or more precisely (3.59). The factor $c_{\hat\beta}$ in (3.24) arises from the *singular product* $\Xi_N(t,x) = \beta_N \, \dot{W}_N(t,x)\, V_N(t,x)$ which gives rise to an *independent white noise*, by Theorem 3.13.

This result is the first step toward a *"robust analysis"* of the two-dimensional SHE (3.5), which would allow for a rigorous derivation of (3.59) from (3.58).

3.3.1. Proof of Theorem 3.13. Similarly as in the proof of Theorem 3.2, we have to show that

$$(\dot{W}_N, \Xi_N) \overset{\mathcal{D}}{\Longrightarrow} \Big(\dot{W}, \, \sqrt{c_{\hat\beta}^2 - 1}\, \dot{W}'\Big) \,,$$

that is, for any fixed $\varphi \in C_c^\infty([0,1] \times \mathbb{R}^2)$ we have

$$\big(\langle \dot{W}_N, \varphi\rangle,\, \langle \Xi_N, \varphi\rangle\big) \xrightarrow{d} \mathcal{N}\big(0, \|\varphi\|_{L^2}^2\, \Sigma_{\hat\beta}\big) \qquad \text{where} \qquad \Sigma_{\hat\beta} = \begin{pmatrix} 1 & 0 \\ 0 & c_{\hat\beta}^2 - 1 \end{pmatrix}. \tag{3.60}$$

By the Cramér-Wold device [**Bil95**, Theorem 29.4], it suffices to show that for all $\lambda, \mu \in \mathbb{R}$

$$X_N := \mu\, \langle \dot{W}_N, \varphi\rangle + \lambda\, \langle \Xi_N, \varphi\rangle \xrightarrow{d} \mathcal{N}\Big(0, \sigma^2 :=\ \|\varphi\|_{L^2}^2 \big(\mu^2 + \lambda^2\,(c_{\hat\beta}^2 - 1)\big)\Big). \tag{3.61}$$

To this purpose we are going to apply Theorem 2.2.

Recall the definitions (3.53) and (3.55) of $\dot{W}_N$ and Ξ_N (see also (3.19)), we can write

$$\begin{aligned} X_N &= N \int\limits_{(0,1]\times\mathbb{R}^2} \varphi(t,x)\, \eta_N\big(\lfloor Nt\rfloor, \lfloor\sqrt{N}x\rfloor\big) \Big\{\mu + \lambda\big(Z_N^{\beta_N}(\lfloor Nt\rfloor, \lfloor \sqrt{N}x\rfloor) - 1\big)\Big\}\, \mathrm{d}t\, \mathrm{d}x \\ &= \frac{1}{N} \int\limits_{(0,N]\times\mathbb{R}^2} \varphi\big(\tfrac{t}{N}, \tfrac{x}{\sqrt{N}}\big)\, \eta_N\big(\lfloor t\rfloor, \lfloor x\rfloor\big) \Big\{\mu + \lambda\big(Z_N^{\beta_N}(\lfloor t\rfloor, \lfloor x\rfloor) - 1\big)\Big\}\, \mathrm{d}t\, \mathrm{d}x\,. \end{aligned} \tag{3.62}$$

We recall the definition of $\overline{\varphi}_N : \mathbb{N}\times\mathbb{Z}^2 \to \mathbb{R}$:

$$\overline{\varphi}_N(n,z) := \int\limits_{(n-1,n]\times\{(z_1-1,z_1]\times(z_2-1,z_2]\}} \varphi\big(\tfrac{t}{N}, \tfrac{x}{\sqrt{N}}\big)\, \mathrm{d}t\, \mathrm{d}x \qquad \text{for} \quad (n,z) \in \mathbb{N}\times\mathbb{Z}^2\,.$$

Plugging the polynomial chaos expansion (3.4) of $Z_N^{\beta_N}(m,z)$ in (3.62), we can rewrite X_N as follows:

$$\begin{aligned} X_N = \frac{1}{N} \sum_{n_0=1}^{N} \sum_{x_0\in\mathbb{Z}^2} \overline{\varphi}_N(n_0,x_0)\, \eta_N(n_0,x_0) \\ \Bigg\{\mu\, +\, \lambda \sum_{k=1}^{\infty} (\sigma_N)^k \sum_{\substack{n_0<n_1<\ldots<n_k\leq N \\ x_0,x_1,\ldots,x_k\in\mathbb{Z}^2}} \prod_{j=1}^{k} q_{n_j-n_{j-1}}(x_j - x_{j-1})\, \eta_N(n_j,x_j)\Bigg\}. \end{aligned}$$

Renaming $(n_0,\ldots,n_k)$ as $(n_1,\ldots,n_{k+1})$ and similarly $(x_0,\ldots,x_k)$ as $(x_1,\ldots,x_{k+1})$, and subsequently renaming $k+1$ as k, we obtain the compact expression

$$X_N = \frac{1}{N} \sum_{k=1}^{\infty} (\sigma_N)^{k-1} \sum_{\substack{0<n_1<\ldots<n_k\leq N \\ x_1,\ldots,x_k\in\mathbb{Z}^2}} f_N(n_1,x_1,\ldots,n_k,x_k) \prod_{j=1}^{k} \eta_N(n_j,x_j)\,, \tag{3.63}$$

where we set

$$f_N(n_1,x_1,\ldots,n_k,x_k) \,:=\, \big\{\mu\, \mathbb{1}_{\{k=1\}} + \lambda\, \mathbb{1}_{\{k\geq 2\}}\big\}\, \overline{\varphi}_N(n_1,x_1) \prod_{j=2}^{k} q_{n_j-n_{j-1}}(x_j - x_{j-1})\,. \tag{3.64}$$

In conclusion, we can write $X_N = \sum_{A\subset\mathbb{T}} q_N(A)\, \eta^N(A)$ as in (2.6)-(2.7), with the following correspondences:

- the index set is $\mathbb{T} := \mathbb{N} \times \mathbb{Z}^2$;
- the random variables $\eta_t^N = \eta_N(m,z)$, for $t = (m,z) \in \mathbb{T}$, are defined in (3.3): they satisfy (2.1) by construction, while they satisfy (2.5) because $\sup_N \mathbb{E}[|\eta_N(m,z)|^p] < \infty$ for all $p < \infty$ by (3.1) (see [**CSZ17a**, eq. (6.7)]);
- the kernel $q_N(A)$, for $A := \{t_1, \ldots, t_k\} = \{(n_1, x_1), \ldots, (n_k, x_k)\} \subseteq \mathbb{T}$, is

$$q_N(A) = \frac{1}{N} (\sigma_N)^{k-1} f_N(n_1, x_1, \ldots, n_k, x_k) \mathbb{1}_{\{0<n_1<\ldots<n_k \le N\}} .$$

By Theorem 2.2, to prove $X_N \xrightarrow{d} \mathcal{N}(0, \sigma^2)$ as in (3.61), we check the following conditions.

(1) *Limiting second moment:* we need to prove that $\lim_{N\to\infty} \mathbb{E}[X_N^2] = \sigma^2$.

(2) *Subcriticality*: we need to show that

$$\lim_{K\to\infty} \limsup_{N\to\infty} \sum_{\substack{A\subset\mathbb{T} \\ |A|>K}} q_N(A)^2 = 0 . \tag{3.65}$$

(3) *Spectral localization*: for any $M, N \in \mathbb{N}$ we define the disjoint subsets

$$\mathbb{B}_i := \left(\tfrac{i-1}{M} N, \tfrac{i}{M} N\right] \times \mathbb{Z}^2 \qquad \text{for } i = 1, \ldots, M ,$$

and, recalling that $\sigma_N^2(\mathbb{B}_i) := \sum_{A\subset\mathbb{B}_i} q_N(A)^2$, we need to show that

$$\lim_{M\to\infty} \sum_{i=1}^{M} \lim_{N\to\infty} \sigma_N^2(\mathbb{B}_i) = \sigma^2 \qquad \text{and} \qquad \lim_{M\to\infty} \left\{ \max_{i=1,\ldots,M} \limsup_{N\to\infty} \sigma_N^2(\mathbb{B}_i) \right\} = 0 . \tag{3.66}$$

Proof of (2)**.** We need to prove (3.65). For $K \ge 1$ we can write, by (3.63)-(3.64),

$$\sum_{\substack{A\subset\mathbb{T} \\ |A|>K}} q_N(A)^2 = \frac{\lambda^2}{N^2} \sum_{k>K} (\sigma_N^2)^{k-1} \sum_{\substack{0<n_1<\ldots<n_k\le N \\ x_1,\ldots,x_k\in\mathbb{Z}^2}} \overline{\varphi}_N(n_1, x_1)^2 \prod_{j=2}^{k} q_{n_j - n_{j-1}}(x_j - x_{j-1})^2. \tag{3.67}$$

We can enlarge the sums to $0 < m_j := n_j - n_{j-1} \le N$ and change variables $y_j := x_j - x_{j-1}$, for $j = 2, \ldots, k$, to get the upper bound

$$\begin{aligned} \sum_{\substack{A\subset\mathbb{T} \\ |A|>K}} q_N(A)^2 &\le \frac{\lambda^2}{N^2} \sum_{k>K} (\sigma_N^2)^{k-1} \sum_{\substack{0<n_1\le N \\ x_1\in\mathbb{Z}^2}} \overline{\varphi}_N(n_1, x_1)^2 \prod_{j=2}^{k} \left\{ \sum_{\substack{0<m_j\le N \\ y_j\in\mathbb{Z}^2}} q_{m_j}(y_j)^2 \right\} \\ &= \lambda^2 \left\{ \frac{1}{N^2} \sum_{\substack{0<n_1\le N \\ x_1\in\mathbb{Z}^2}} \overline{\varphi}_N(n_1, x_1)^2 \right\} \frac{(\sigma_N^2 R_N)^K}{1 - \sigma_N^2 R_N} , \end{aligned} \tag{3.68}$$

where we used $\sum_{0<m\le N} \sum_{y\in\mathbb{Z}^2} q_m(y)^2 = \sum_{0<m\le N} u_m = R_N$, see (3.14)-(3.15), and we remark that $\sigma_N^2 R_N < 1$ for N large enough, because $\sigma_N^2 \sim \hat{\beta}^2 / R_N$, see (3.18), and $\hat\beta < 1$. Then, by Riemann sum approximation, from (3.30) we get

$$\limsup_{N\to\infty} \sum_{\substack{A\subset\mathbb{T}\\ |A|>K}} q_N(A)^2 \le \lambda^2 \left\{ \int_{[0,1]\times\mathbb{R}^2} \varphi(t,x)^2 \,\mathrm{d}t\,\mathrm{d}x \right\} \frac{(\hat\beta^2)^K}{1-\hat\beta^2} = \lambda^2 \|\varphi\|_{L^2}^2 \frac{(\hat\beta^2)^K}{1-\hat\beta^2}, \tag{3.69}$$

from which (3.65) follows.

Proof of (1) **and** (3)**.** We are going to show that for all $M \in \mathbb{N}$ and $i \in \{1,\ldots,M\}$

$$\lim_{N\to\infty} \sigma_N^2(\mathbb{B}_i) = \big(\mu^2 + \lambda^2(c_{\hat\beta}^2 - 1)\big) \int_{(\frac{i-1}{M},\frac{i}{M}]\times\mathbb{R}^2} \varphi(t,x)^2 \,\mathrm{d}t\,\mathrm{d}x\,. \tag{3.70}$$

Note that this proves (3.66) (the second expression can be verified similarly as in (3.45)) and also (for $i = M = 1$) $\lim_{N\to\infty} \mathbb{E}[X_N^2] = \sigma^2$, see (3.61).

To compute $\sigma_N^2(\mathbb{B}_i) := \sum_{A\subset\mathbb{B}_i} q_N(A)^2$ we first consider the contribution of sets $A \subset \mathbb{B}_i$ with $|A| = 1$, that is $A = \{(n_1,x_1)\}$. Since $f_N(n_1,x_1) = \mu\,\overline{\varphi}_N(n_1,x_1)$, see (3.64), we get

$$\sum_{\substack{A\subset\mathbb{B}_i\\ |A|=1}} q_N(A)^2 = \frac{\mu^2}{N^2} \sum_{\substack{\frac{i-1}{M}N<n_1\le\frac{i}{M}N\\ x_1\in\mathbb{Z}^2}} \overline{\varphi}_N(n_1,x_1)^2 \xrightarrow{N\to\infty} \mu^2 \int_{(\frac{i-1}{M},\frac{i}{M}]\times\mathbb{R}^2} \varphi(t,x)^2 \,\mathrm{d}t\,\mathrm{d}x\,,$$

by Riemann sum approximation. Note that this matches with the first term in (3.70).

We next focus on sets $A \subset \mathbb{B}_i$ with $|A| > 1$. Note that $\sum_{A\subset\mathbb{B}_i,\,|A|>1} q_N(A)^2$ is given by (3.67) with $K = 1$ and with the sum restricted to $\frac{i-1}{M}N < n_1 < \ldots < n_k \le \frac{i}{M}N$. Then, arguing as in (3.68), we obtain an analogue of (3.69):

$$\limsup_{N\to\infty} \sum_{A\subset\mathbb{B}_i,\,|A|>1} q_N(A)^2 \le \lambda^2 \left\{ \int_{(\frac{i-1}{M},\frac{i}{M}]\times\mathbb{R}^2} \varphi(t,x)^2 \,\mathrm{d}t\,\mathrm{d}x \right\} \frac{\hat\beta^2}{1-\hat\beta^2},$$

which agrees with the second term in (3.70) because $\frac{\hat\beta^2}{1-\hat\beta^2} = c_{\hat\beta}^2 - 1$, see (3.20). To complete the proof, it suffices to prove a matching lower bound, that is

$$\liminf_{N\to\infty} \sum_{A\subset\mathbb{B}_i,\,|A|>1} q_N(A)^2 \ge \lambda^2 \left\{ \int_{(\frac{i-1}{M},\frac{i}{M}]\times\mathbb{R}^2} \varphi(t,x)^2 \,\mathrm{d}t\,\mathrm{d}x \right\} \frac{\hat\beta^2}{1-\hat\beta^2}. \tag{3.71}$$

Let us fix $H \in \mathbb{N}$ large, such that $\frac{1}{H} < \frac{1}{M}$. Starting from the expression (3.67) for $K = 1$ and with $\frac{i-1}{M}N < n_1 < \ldots < n_k \le \frac{i}{M}N$, we get a lower bound by the following restrictions:

$$1 < k \le H\,, \qquad \tfrac{i-1}{M}N < n_1 \le \big(\tfrac{i}{M} - \tfrac{1}{H}\big)N\,, \qquad 0 < n_j - n_{j-1} \le \tfrac{1}{H^2}N \quad \forall j = 2,\ldots,k\,,$$

which ensure that $n_k \le n_1 + \sum_{j=2}^k (n_j - n_{j-1}) \le (\frac{j}{M} - \frac{1}{H})N + H\frac{1}{H^2}N \le \frac{i}{M}N$ as required. Then, similarly to (3.68), we get the following lower bound on

$\sum_{A\subset \mathbb{B}_i,\, |A|>1} q_N(A)^2$:

$$\begin{aligned} &\frac{\lambda^2}{N^2}\sum_{k=2}^{H}(\sigma_N^2)^{k-1}\sum_{\substack{\frac{i-1}{M}<n_1\le(\frac{i}{M}-\frac{1}{H})N\\ x_1\in\mathbb{Z}^2}}\overline{\varphi}_N(n_1,x_1)^2\prod_{j=2}^{k}\Bigg\{\sum_{\substack{0<m_j\le\frac{1}{H^2}N\\ y_j\in\mathbb{Z}^2}} q_{m_j}(y_j)^2\Bigg\}\\ &\qquad=\Bigg\{\frac{\lambda^2}{N^2}\sum_{\substack{\frac{i-1}{M}<n_1\le(\frac{i}{M}-\frac{1}{H})N\\ x_1\in\mathbb{Z}^2}}\overline{\varphi}_N(n_1,x_1)^2\Bigg\}\frac{\sigma_N^2\,R_{N/H^2}-(\sigma_N^2\,R_{N/H^2})^H}{1-\sigma_N^2\,R_{N/H^2}}\,, \end{aligned} \tag{3.72}$$

where we recall that $\sum_{k=2}^{H} x^{k-1}=\frac{x-x^H}{1-x}$ for $|x|<1$. Since $R_{N/H^2}\sim R_N$ for fixed $H\in\mathbb{N}$, we have shown that

$$\liminf_{N\to\infty}\sum_{A\subset\mathbb{B}_i,\,|A|>1} q_N(A)^2\ge\lambda^2\Bigg\{\int_{(\frac{i-1}{M},\frac{i}{M}-\frac{1}{H}]\times\mathbb{R}^2}\varphi(t,x)^2\,\mathrm{d}t\,\mathrm{d}x\Bigg\}\frac{\hat\beta^2-(\hat\beta^2)^H}{1-\hat\beta^2}\,.$$

We can finally take the limit $H\to\infty$ to see that (3.71) holds. □

3.4. Fluctuations for the mollified Stochastic Heat Equation

As already stressed in Chapter 1, all Gaussian limits presented in previous sections can be transferred in the continuum framework and applied to the mollified 2d Stochastic Heat Equation with multiplicative white noise (mSHE):

$$\partial_t u^\varepsilon(t,x)=\frac{1}{2}\Delta_x u^\varepsilon(t,x)\,+\,\beta_\varepsilon\,u^\varepsilon(t,x)\,\dot W^\varepsilon(t,x)\qquad u(0,x)\equiv 1\,, \tag{3.73}$$

thanks to the connection between the solution $u^\varepsilon(t,x)$ and the diffusively rescaled partition function $Z_N^\beta(\lfloor Nt\rfloor,\lfloor\sqrt{N}x\rfloor)$ of the 2d directed polymer. In [**CSZ17b**] this was achieved by discretizing $u^\varepsilon(t,x)$ in terms of the partition function and consequently by exploiting the corresponding convergence results already proved in the discrete setting. In this section, we illustrate how our novel CLT for Wiener chaos (Theorem 2.9) provides an alternative and straightforward approach to recover these limits in the continuum setting, *without the need to discretize and involve the directed polymer model.* As a way of example, we apply our strategy to recover the Edwards–Wilkinson fluctuations for $u^\varepsilon(t,x)$ [**CSZ17b**, Theorem 2.17].

For $\varepsilon>0$, in analogy with the subcritical regime (3.18) for directed polymers, we rescale the noise strength in (3.73) as

$$\beta_\varepsilon:=\frac{\hat\beta\sqrt{2\pi}}{\sqrt{\log\varepsilon^{-1}}}\,,\qquad\text{with}\qquad\hat\beta\in(0,1). \tag{3.74}$$

Remark 3.14. *By comparing the continuum and discrete rescalings of β*

$$\beta_\varepsilon:=\frac{\hat\beta\sqrt{2\pi}}{\sqrt{\log\varepsilon^{-1}}}\qquad\textit{and}\qquad\beta_N:=\frac{\hat\beta}{\sqrt{R_N}}\sim\frac{\hat\beta\sqrt{\pi}}{\sqrt{\log N}}\,,$$

where $N = \varepsilon^{-2}$, *we realize that they are equivalent* up to a factor 2 *arising in the continuum setting, indeed*

$$\beta_\varepsilon := \frac{\hat\beta\sqrt{2\pi}}{\sqrt{\log \varepsilon^{-1}}} = \frac{\hat\beta\sqrt{2\pi}}{\sqrt{\log N^{\frac{1}{2}}}} = \frac{\hat\beta\, 2\sqrt{\pi}}{\sqrt{\log N}} .$$

This comes from the fact that $u^\varepsilon(t,x)$ *is close in distribution to the diffusively rescaled and time reversed partition function of a directed polymer associated with an* aperiodic random walk $\widetilde{S}$ with covariance I instead of the standard simple random walk with period 2 and covariance $\frac{1}{2}I$. *With this identification, the local central limit theorem (cf. with* (3.16)-(5.33)*) becomes*

$$\widetilde{q}_n(y) := \mathrm{P}(\widetilde{S}_n = y) = \frac{1}{n}\big(g\big(\tfrac{y}{\sqrt{n}}\big) + o(1)\big) \qquad \text{with} \qquad g(x) := \frac{e^{-|x|^2/2}}{2\pi}$$

as $n \to \infty$ *uniformly in* $y \in \mathbb{Z}^2$ *and, as a consequence, the corresponding overlap behaves as*

$$\widetilde{R}_N := \sum_{n=1}^{N}\sum_{y\in\mathbb{Z}^2} \widetilde{q}_n(y)^2 = \sum_{n=1}^{N} \widetilde{q}_{2n}(0) \sim \frac{1}{4\pi}\log N \qquad \text{as } N \to \infty .$$

Therefore, to gain a perfect match with the discrete framework we need to compare β_ε *in* (3.74) *with the rescaled disorder strength of a directed polymer with the aforementioned random walk* $\widetilde{S}$, *namely*

$$\beta_N := \frac{\hat\beta}{\sqrt{\widetilde{R}_N}} \sim \frac{\hat\beta\, 2\sqrt{\pi}}{\sqrt{\log N}} .$$

We now recall the analogue of Theorem 3.2 on the limiting fluctuations of the rescaled $u^\varepsilon(t,x)$, see [**CSZ17b**, Theorem 2.17]. To be precise, the original result involves a space–time average, but the analogous theorem for the space average presented below follows by the same arguments.

THEOREM 3.15 (Edwards–Wilkinson fluctuations for u^ε). *For* $(t,x) \in [0,1]\times\mathbb{R}^2$ *let*

$$\nu_\varepsilon(t,x) := \frac{1}{\beta_\varepsilon}\big(u^\varepsilon(t,x) - 1\big) = \frac{\sqrt{\log \varepsilon^{-1}}}{\hat\beta\, 2\sqrt{\pi}}\big(u^\varepsilon(t,x) - 1\big) \tag{3.75}$$

be the centered and rescaled solution of the mollified (mSHE) (3.73) *and* β_ε *is defined as* (3.74) *with* $\hat\beta \in (0,1)$. *Then, for any* $t \in [0,1]$ *as* $\varepsilon \to 0$*:*

$$\nu_\varepsilon(t,x) \overset{\mathcal{D}}{\Longrightarrow} \widetilde{\widetilde{v}}(t,x) := v^{(1,c_{\hat\beta})}(t,x) \qquad \text{with} \qquad c_{\hat\beta} := \sqrt{\frac{1}{1-\hat\beta^2}}\,, \tag{3.76}$$

where $v^{(1,c_{\hat\beta})}(t,x)$ *is the solution of the Edwards-Wilkinson equation* (3.8) *with* $s = 1$ *and* $c = c_{\hat\beta}$ *and "*$\overset{\mathcal{D}}{\Longrightarrow}$*" denotes convergence in law* as a random distribution on $\mathbb{R}^2$, *namely for* $t \in [0,1]$ *and for* $\varphi \in C_c(\mathbb{R}^2)$

$$\langle \nu_\varepsilon(t,\cdot)\,,\varphi\rangle := \int_{\mathbb{R}^2} \nu_\varepsilon(t,x)\,\varphi(x)\,\mathrm{d}x \;\xrightarrow[\varepsilon\to 0]{d}\; \langle\, \widetilde{\widetilde{v}}, \varphi\rangle \;\sim\; \mathcal{N}\big(0, \sigma^2_{\hat\beta,\varphi}\big)\,, \tag{3.77}$$

with

$$\sigma^2_{\hat\beta,\varphi} := \frac{1}{1-\hat\beta^2}\int_{\mathbb{R}^2\times\mathbb{R}^2} \varphi(x)K_t(x,x')\varphi(x')\,\mathrm{d}x\,\mathrm{d}x'\,,$$

$$K_t(x,x') := K^1_{t,t}(x,x') = \int_0^t g_{2u}(x-x')\,\mathrm{d}u\,.$$

REMARK 3.16. *Notice that for a matter of periodicity (see Remark 3.14) the limiting random field* $\widetilde{\widetilde{v}}$ *is the solution of the Edwards-Wilkinson equation with* $s=1$*, contrary to the discrete case where* $s=\frac{1}{2}$*.*

3.4.1. The solution as a Wiener chaos. Before illustrating the alternative proof based on Theorem 2.9, we need to derive a Wiener chaos expansion for $u^\varepsilon(t,x)$, which is essential in order to apply our strategy. Given a symmetric probability density $j\in C_c^\infty(\mathbb{R}^d)$ and $\varepsilon>0$, we recall that the mollified white noise is formally defined as

$$\dot W^\varepsilon(t,x) := (\dot W(t,\cdot)*j_\varepsilon)(x) = \int_{\mathbb{R}^d} j_\varepsilon(x-y)\dot W(t,y)\mathrm{d}y\,,$$

where $j_\varepsilon(x) := \varepsilon^{-d}j\big(x/\varepsilon\big)$, so that for any function $f\in L^2(\mathbb{R}\times\mathbb{R}^2)$:

$$\int_{\mathbb{R}\times\mathbb{R}^2} f(t,x)\,\dot W^\varepsilon(t,x)\,\mathrm{d}t\,\mathrm{d}x := \int_{\mathbb{R}\times\mathbb{R}^2}\bigg(\int_{\mathbb{R}^2} f(t,x)j_\varepsilon(x-y)\mathrm{d}x\bigg)\dot W(t,y)\,\mathrm{d}t\,\mathrm{d}y\,.$$

By [**BC95**] the solution of (3.73) admits the following Feynman-Kac representation:

$$u^\varepsilon(t,x) = \mathrm{E}_x\bigg[\exp\bigg\{\beta_\varepsilon\int_0^t \dot W^\varepsilon(t-s,B_s)\,\mathrm{d}s - \frac{1}{2}\beta_\varepsilon^2\,\mathbb{E}\bigg[\bigg(\int_0^t \dot W^\varepsilon(t-s,B_s)\,\mathrm{d}s\bigg)^2\bigg]\bigg\}\bigg]\,,$$

where E_x and $\mathbb{E}$ are respectively the expectations with respect to a standard Brownian motion $(B_s)_{s\geq 0}$ in $\mathbb{R}^2$ starting from $B_0=x$ and with respect to the white noise $\dot W$. Since $\dot W^\varepsilon$ is invariant under time–reversal, the solution $u^\varepsilon(t,x)$ is equal in law for fixed $(t,x)\in[0,1]\times\mathbb{R}^2$ to

$$\begin{aligned} u(t,x) &\stackrel{d}{=} \mathrm{E}_x\bigg[\exp\bigg\{\beta_\varepsilon\int_0^t \dot W^\varepsilon(s,B_s)\,\mathrm{d}s - \frac{1}{2}\beta_\varepsilon^2\,\mathbb{E}\bigg[\bigg(\int_0^t \dot W^\varepsilon(s,B_s)\,\mathrm{d}s\bigg)^2\bigg]\bigg\}\bigg] \\ &= \mathrm{E}_x\bigg[\exp\bigg\{\beta_\varepsilon\int_0^t\int_{\mathbb{R}^2}\varepsilon^{-2}j\bigg(\frac{B_s-y}{\varepsilon}\bigg)\dot W(s,y)\,\mathrm{d}y\,\mathrm{d}s - \frac{1}{2}\beta_\varepsilon^2 t\varepsilon^{-2}\|j\|^2_{L^2(\mathbb{R}^2)}\bigg]\bigg\}\bigg] \\ &= \mathrm{E}_{\varepsilon^{-1}x}\bigg[\exp\bigg\{\beta_\varepsilon\int_0^{\varepsilon^{-2}t}\int_{\mathbb{R}^2} j\big(B_{\widetilde s}-\widetilde y\big)\dot{\widetilde W}(\widetilde s,\widetilde y)\,\mathrm{d}\widetilde y\,\mathrm{d}\widetilde s - \frac{1}{2}\beta_\varepsilon^2 t\varepsilon^{-2}\|j\|^2_{L^2(\mathbb{R}^2)}\bigg]\bigg\}\bigg]\,, \end{aligned} \tag{3.78}$$

where we changed variables $\widetilde y := \frac{y}{\varepsilon}$ and $\widetilde s := \frac{s}{\varepsilon^{-2}}$,

$$\dot{\widetilde W}(\widetilde s,\widetilde y)\,\mathrm{d}\widetilde y\,\mathrm{d}\widetilde s := \varepsilon^{-2}\dot W(\varepsilon^2\widetilde s,\varepsilon\widetilde y)\mathrm{d}(\varepsilon^2\widetilde s)\,\mathrm{d}(\varepsilon\widetilde y)$$

is a new space–time white noise and $\varepsilon^{-1}B_s = \varepsilon^{-1}B_{s\varepsilon^2\varepsilon^{-2}} \overset{d}{=} B_{\widetilde{s}}$. Moreover, we have applied Ito isometry to obtain

$$\mathbb{E}\bigg[\bigg(\int_0^t \dot{W}^\varepsilon(s,B_s)\,\mathrm{d}s\bigg)^2\bigg] = \int_0^t\int_{\mathbb{R}^2} j_\varepsilon(B_s-y)^2\,\mathrm{d}y\,\mathrm{d}s = t\varepsilon^{-2}\|j\|^2_{L^2(\mathbb{R}^2)}\,.$$

We can write the expression (3.78) as

$$u(t,x) \overset{d}{=} \mathrm{E}_{\varepsilon^{-1}x}\bigg[\,:\exp\bigg\{\beta_\varepsilon\int_0^{\varepsilon^{-2}t}\int_{\mathbb{R}^2} j\big(B_s-y\big)\dot{W}(s,y)\,\mathrm{d}y\,\mathrm{d}s\bigg\}:\,\bigg]\,, \tag{3.79}$$

where : exp : is the Wick exponential (see [**Jan97**, Chapter 3, Section 2]). Let us denote

$$\xi := \beta_\varepsilon \int_0^{\varepsilon^{-2}t}\int_{\mathbb{R}^2} j\big(B_s-y\big)\dot{W}(s,y)\,\mathrm{d}y\,\mathrm{d}s\,,$$

then by definition of the Wick exponential (see also [**Jan97**, Theorem 7.26]), we can expand (3.79) as follows:

$$\begin{aligned}
&u(t,x) \overset{d}{=} \mathrm{E}_{\varepsilon^{-1}x}\big[\,:\exp\xi:\,\big]\\
&= \mathrm{E}_{\varepsilon^{-1}x}\Big[1+\sum_{k=1}^\infty \frac{1}{k!}:\xi^k:\Big]\\
&= 1+\sum_{k=1}^\infty \beta_\varepsilon^k\frac{1}{k!}\int_{[0,\varepsilon^{-2}t]^k}\int_{(\mathbb{R}^2)^k}\mathrm{E}_{\varepsilon^{-1}x}\Big[\prod_{i=1}^k j(B_{t_i}-x_i)\Big]\prod_{i=1}^k \dot{W}(t_i,x_i)\,\mathrm{d}t_i\,\mathrm{d}x_i\\
&= 1+\sum_{k=1}^\infty \beta_\varepsilon^k\int_{0<t_1<\cdots<t_k<\varepsilon^{-2}t}\int_{(\mathbb{R}^2)^k}\mathrm{E}_{\varepsilon^{-1}x}\Big[\prod_{i=1}^k j(B_{t_i}-x_i)\Big]\prod_{i=1}^k \dot{W}(t_i,x_i)\,\mathrm{d}t_i\,\mathrm{d}x_i\\
&= 1+\sum_{k=1}^\infty \beta_\varepsilon^k\int_{0<t_1<\cdots<t_k<\varepsilon^{-2}t}\int_{(\mathbb{R}^2\times\mathbb{R}^2)^k}\prod_{i=1}^k g_{t_i-t_{i-1}}(y_i-y_{i-1})j(y_i-x_i)\prod_{i=1}^k\mathrm{d}y_i\times\\
&\qquad\qquad\times\prod_{i=1}^k \dot{W}(t_i,x_i)\,\mathrm{d}t_i\,\mathrm{d}x_i\,,
\end{aligned} \tag{3.80}$$

where $t_0 := 0$, $y_0 := \varepsilon^{-1}x$ and : ξ^k : is the Wick product (see [**Jan97**, Chapter 3, Section 1]). The final line in (3.80) represents the *Wiener chaos expansion of the solution* $u^\varepsilon(t,x)$ and it is indeed a continuum version of the polynomial chaos expansion (3.4) for the 2d directed polymer.

We are now ready to present the alternative approach to prove Theorem 3.15.

3.4.2. Proof of Theorem 3.15. We need to show that for fixed $t\in[0,1]$ and $\varphi\in C_c(\mathbb{R}^2)$:

$$\nu_\varepsilon(t,\varphi) := \langle\,\nu_\varepsilon(t,\cdot)\,,\varphi\rangle \xrightarrow[\varepsilon\to 0]{d} \mathcal{N}(0,\sigma^2) \tag{3.81}$$

with

$$\sigma^2 := \sigma^2_{\hat\beta,\varphi} = \frac{1}{1-\hat\beta^2}\int_{\mathbb{R}^2\times\mathbb{R}^2}\varphi(x)\,\varphi(x')\int_0^t g_{2u}(x-x')\,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}x'\,. \tag{3.82}$$

In order to apply Theorem 2.9, we first express $\nu_\varepsilon(\varphi)$ in terms of a Wiener chaos expansion. Recalling (3.75) and (3.80), we can write

$$\nu_\varepsilon(t,\varphi) = \sum_{k=1}^{\infty} \beta_\varepsilon^{k-1} \int_{0<t_1<\dots<t_k<\varepsilon^{-2}t} \int_{(\mathbb{R}^2)^k} \prod_{i=1}^{k} \dot{W}(t_i,x_i)\,\mathrm{d}t_i\,\mathrm{d}x_i \times \\ \times \left\{ \int_{\mathbb{R}^2} \varphi(x)\,\mathrm{d}x \int_{(\mathbb{R}^2)^k} \prod_{i=1}^{k} g_{t_i-t_{i-1}}(y_i-y_{i-1}) j(y_i-x_i) \prod_{i=1}^{k} \mathrm{d}y_i \right\},$$

where $t_0 \coloneqq 0$ and $y_0 \coloneqq \varepsilon^{-1}x$.

We are finally able to represent

$$\nu_\varepsilon(t,\varphi) = \sum_{k=1}^{\infty} \int_{E^k} \widetilde{q}_\varepsilon(z_1,\dots,z_k)\, W(\mathrm{d}z_1)\cdots W(\mathrm{d}z_k)$$

as in (2.18) (here $\varepsilon = N^{-\frac{1}{2}}$) with the following correspondences:

- $E \coloneqq \mathbb{R}_+ \times \mathbb{R}^2$;
- as Gaussian random measure $W(\mathrm{d}x)$ on the Polish space $(E,\mathcal{E},\mu)$ we consider the time–space white noise $\dot{W}$ on $\big(\mathbb{R}_+ \times \mathbb{R}^2, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^2), \mathcal{L}\big)$ where $\mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^2)$ is the Borel σ-algebra on $\mathbb{R}_+ \times \mathbb{R}^2$ and $\mathcal{L}$ is the Lebesgue measure on $\mathbb{R}_+ \times \mathbb{R}^2$;
- the kernel $\widetilde{q}_\varepsilon(z_1,\dots,z_k)$ for $\{z_1,\dots,z_k\} \coloneqq \{(t_1,x_k),\dots,(t_k,x_k)\}$ is

$$\widetilde{q}_\varepsilon(t_1,x_k,\dots,t_k,x_k) \coloneqq q_\varepsilon(t_1,x_k,\dots,t_k,x_k)\mathbb{1}_{\{0=:t_0<t_1<\dots<t_k<\varepsilon^{-2}t\}} \\ \coloneqq \beta_\varepsilon^{k-1}\left\{ \int_{\mathbb{R}^2} \varphi(x)\,\mathrm{d}x \int_{(\mathbb{R}^2)^k} \prod_{i=1}^{k} g_{t_i-t_{i-1}}(y_i-y_{i-1}) j(y_i-x_i) \prod_{i=1}^{k} \mathrm{d}y_i \right\} \times \\ \times \mathbb{1}_{\{0=:t_0<t_1<\dots<t_k<\varepsilon^{-2}t\}}\,.$$

We easily have

$$\mathbb{E}\big[\nu_\varepsilon(t,\varphi)\big] = 0\,.$$

Regarding the second moment, we need to be careful because $\widetilde{q}_\varepsilon$ is not symmetric in this case. However, by Remark 2.5 we can show that the Ito isometry holds without the prefactor $k!$, since $\widetilde{q}_\varepsilon$ vanishes for unordered times by definition. Then, recalling (2.16) and (2.17), we have

$$
\begin{aligned}
\widetilde{\sigma}^2_\varepsilon(E) &:= \mathbb{E}\big[\nu_\varepsilon(t,\varphi)^2\big] \\
&= \sum_{k=1}^{\infty} \frac{1}{k!} \int_{E^k} \Bigg(\sum_{\pi\in\mathcal{P}(k)} q_\varepsilon(t_{\pi(1)}, x_{\pi(1)}, \ldots, t_{\pi(k)}, x_{\pi(k)}) \mathbb{1}_{\{0=:t_0<t_{\pi(1)}<\ldots<t_{\pi(k)}<\varepsilon^{-2}t\}} \Bigg)^2 \times \\
&\qquad\qquad\qquad\qquad \times \mathrm{d}t_{\pi(1)}\,\mathrm{d}x_{\pi(1)} \ldots \mathrm{d}t_{\pi(k)}\,\mathrm{d}x_{\pi(k)} \\
&= \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{\pi\in\mathcal{P}(k)} \int_{E^k} \Big(q_\varepsilon(t_{\pi(1)}, x_{\pi(1)}, \ldots, t_{\pi(k)}, x_{\pi(k)}) \mathbb{1}_{\{0=:t_0<t_{\pi(1)}<\ldots<t_{\pi(k)}<\varepsilon^{-2}t\}} \Big)^2 \times \\
&\qquad\qquad\qquad\qquad \times \mathrm{d}t_{\pi(1)}\,\mathrm{d}x_{\pi(1)} \ldots \mathrm{d}t_{\pi(k)}\,\mathrm{d}x_{\pi(k)} \\
&= \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{\pi\in\mathcal{P}(k)} \|\widetilde{q}_\varepsilon\|^2_{L^2(E^k)} \\
&= \sum_{k=1}^{\infty} \|\widetilde{q}_\varepsilon\|^2_{L^2(E^k)} \,.
\end{aligned}
$$

Hence, by Theorem 2.9, in order to show that $\nu_\varepsilon(t,\varphi) \xrightarrow{d} \mathcal{N}(0,\sigma^2)$ we need to verify the following assumptions.

$(\widetilde{1})$ *Limiting second moment:* $\lim_{\varepsilon\to 0} \mathbb{E}[\nu_\varepsilon(t,\varphi)^2] = \sigma^2$.

$(\widetilde{2})$ *Subcriticality*: we need to show that

$$
\lim_{K\to\infty} \limsup_{\varepsilon\to 0} \sum_{k>K} \|\widetilde{q}_\varepsilon\|^2_{L^2(E^k)} = 0\,. \tag{3.83}
$$

$(\widetilde{3})$ *Spectral localization*: for any $M \in \mathbb{N}$ and $\varepsilon > 0$ we define the disjoint subsets

$$
\mathbb{B}_j := \big(\tfrac{j-1}{M}\varepsilon^{-2}t,\ \tfrac{j}{M}\varepsilon^{-2}t \big] \times \mathbb{R}^2 \qquad \text{for } j = 1, \ldots, M\,,
$$

and, denoting

$$
\widetilde{\sigma}^2_\varepsilon(\mathbb{B}_j) := \sum_{k=1}^{\infty} \int_{(\mathbb{B}_j)^k} \widetilde{q}_\varepsilon(t_1, x_1, \ldots, t_k, x_k)^2 \,\mathrm{d}t_1\,\mathrm{d}x_1, \ldots, \mathrm{d}t_k\,\mathrm{d}x_k\,,
$$

we need to show that

$$
\lim_{M\to\infty} \sum_{i=1}^{M} \lim_{\varepsilon\to 0} \sigma^2_N(\mathbb{B}_i) = \sigma^2 \qquad \text{and} \qquad \lim_{M\to\infty} \Big\{ \max_{i=1,\ldots,M} \limsup_{\varepsilon\to 0} \sigma^2_N(\mathbb{B}_i) \Big\} = 0\,. \tag{3.84}
$$

Due to the connection to the setting of directed polymer, the proofs of $(\widetilde{1})$–$(\widetilde{2})$–$(\widetilde{3})$ are reasonably similar to those of (1)–(2)–(3) in Subsection 3.2.1. Nevertheless, we show them below for completeness. In particular, in the definition of $\widetilde{q}_\varepsilon(\cdot)$ we can notice that the product of the heat kernels $\prod_{i=1}^k g_{t_i-t_{i-1}}(y_i-y_{i-1})$ (which in the discrete framework corresponds to the product of the simple random walk transition kernels $\prod_{i=1}^k q_{n_i-n_{i-1}}(x_i - x_{i-1})$) is further integrated against the functions j, arising from the mollification procedure applied here. This will require some additional step, not necessary for the discrete estimates, to obtain the desired bounds.

Proof of $\widetilde{(1)}$ **and** $\widetilde{(3)}$**.** We are going to show that for all $M \in \mathbb{N}$ and $i \in \{1, \dots, M\}$:

$$\lim_{\varepsilon\to 0} \sigma_\varepsilon^2(\mathbb{B}_j) = \sigma_j^2 := \frac{1}{1-\hat{\beta}^2} \int_{\mathbb{R}^2\times\mathbb{R}^2} \varphi(x)\,\varphi(x') \int_{\frac{j-1}{M}t}^{\frac{j}{M}t} g_{2u}(x-x')\,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}x' . \tag{3.85}$$

Notice that this immediately implies the first relation of (3.84) and also (for $j = M = 1$) $\lim_{N\to\infty} \mathbb{E}[\nu_\varepsilon(t,\varphi)^2] = \sigma^2$, see also (3.82). The second limit in (3.84) easily follows by slightly modifying the bound (3.45).

By Ito isometry, we have

$$\sigma_\varepsilon^2(\mathbb{B}_j) = \mathbb{E}\big[\nu_\varepsilon(t,\varphi)^2\big] = \int_{\mathbb{R}^2\times\mathbb{R}^2} \varphi(x)\,\mathcal{K}_t^\varepsilon(x,x')\,\varphi(x')\,\mathrm{d}x\,\mathrm{d}x' , \tag{3.86}$$

where

$$\mathcal{K}_t^\varepsilon(x,x') := \sum_{k=1}^\infty (\beta_\varepsilon^2)^{k-1} \int_{\frac{j-1}{M}\varepsilon^{-2}t<t_1<\dots<t_k<\frac{j}{M}\varepsilon^{-2}t} \prod_{i=1}^k \mathrm{d}t_i \int_{(\mathbb{R}^2)^k} \prod_{i=1}^k \mathrm{d}x_i \times$$
$$\times \int_{(\mathbb{R}^2)^k\times(\mathbb{R}^2)^k} \prod_{i=1}^k g_{t_i-t_{i-1}}(y_i-y_{i-1}) g_{t_i-t_{i-1}}(\widetilde{y}_i-\widetilde{y}_{i-1})\, j(y_i-x_i) j(\widetilde{y}_i-x_i) \prod_{i=1}^k \mathrm{d}y_i \mathrm{d}\widetilde{y}_i , \tag{3.87}$$

with $y_0 := \varepsilon^{-1}x$ and $\widetilde{y}_0 := \varepsilon^{-1}x'$.

In order to verify (3.85) we are going to show that

$$\limsup_{\varepsilon\to 0} \sigma_\varepsilon^2(\mathbb{B}_j) \le \frac{1}{1-\hat{\beta}^2} \int_{\mathbb{R}^2\times\mathbb{R}^2} \varphi(x)\,\varphi(x') \int_{\frac{j-1}{M}t}^{\frac{j}{M}t} g_{2u}(x-x')\,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}x' \tag{3.88}$$

and

$$\liminf_{\varepsilon\to 0} \sigma_\varepsilon^2(\mathbb{B}_j) \ge \frac{1}{1-\hat{\beta}^2} \int_{\mathbb{R}^2\times\mathbb{R}^2} \varphi(x)\,\varphi(x') \int_{\frac{j-1}{M}t}^{\frac{j}{M}t} g_{2u}(x-x')\,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}x' . \tag{3.89}$$

We denote $J := j * j$. Since

$$\int_{(\mathbb{R}^2)} \prod_{i=1}^k \mathrm{d}x_i\, j(y_i-x_i) j(\widetilde{y}_i-x_i) = \prod_{i=1}^k \int_{\mathbb{R}^2} j(y_i-\widetilde{y}_i-z)\, j(z)\,\mathrm{d}z = \prod_{i=1}^k J(y_i-\widetilde{y}_i) ,$$

then from (3.87) we get

$$\mathcal{K}_t^\varepsilon(x,x') = \sum_{k=1}^\infty (\beta_\varepsilon^2)^{k-1} \int_{\frac{j-1}{M}\varepsilon^{-2}t<t_1<\dots<t_k<\frac{j}{M}\varepsilon^{-2}t} \prod_{i=1}^k \mathrm{d}t_i \times$$
$$\times \int_{(\mathbb{R}^2)^k\times(\mathbb{R}^2)^k} \prod_{i=1}^k g_{t_i-t_{i-1}}(y_i-y_{i-1}) g_{t_i-t_{i-1}}(\widetilde{y}_i-\widetilde{y}_{i-1})\, J(y_i-\widetilde{y}_i) \prod_{i=1}^k \mathrm{d}y_i \mathrm{d}\widetilde{y}_i .$$

We now change variables $z_i := y_i - \widetilde{y}_i$ and $w_i := y_i + \widetilde{y}_i$ and we denote $z_0 := \varepsilon^{-1}(x-x')$, thus since $\mathrm{d}y_i\,\mathrm{d}\widetilde{y}_i = \frac{1}{4}\mathrm{d}z_i\,\mathrm{d}w_i$ and

$$g_{t_i-t_{i-1}}(y_i-y_{i-1}) g_{t_i-t_{i-1}}(\widetilde{y}_i-\widetilde{y}_{i-1}) = 4\, g_{2(t_i-t_{i-1})}(z_i-z_{i-1}) g_{2(t_i-t_{i-1})}(w_i-w_{i-1}) ,$$

we obtain

$$
\begin{aligned}
\mathcal{K}_t^\varepsilon(x,x') &= \sum_{k=1}^{\infty}(\beta_\varepsilon^2)^{k-1}\int_{\frac{j-1}{M}\varepsilon^{-2}t<t_1<\dots<t_k<\frac{j}{M}\varepsilon^{-2}t}\prod_{i=1}^{k}\mathrm{d}t_i\times \\
&\times\int_{(\mathbb{R}^2)^k\times(\mathbb{R}^2)^k}\prod_{i=1}^{k} g_{2(t_i-t_{i-1})}(z_i-z_{i-1})g_{2(t_i-t_{i-1})}(w_i-w_{i-1})\,J(z_i)\prod_{i=1}^{k}\mathrm{d}z_i\mathrm{d}w_i \\
&= \sum_{k=1}^{\infty}(\beta_\varepsilon^2)^{k-1}\int_{\frac{j-1}{M}\varepsilon^{-2}t<t_1<\dots<t_k<\frac{j}{M}\varepsilon^{-2}t}\prod_{i=1}^{k}\mathrm{d}t_i\times \\
&\qquad\times\int_{(\mathbb{R}^2)^k}\prod_{i=1}^{k} g_{2(t_i-t_{i-1})}(z_i-z_{i-1})\,J(z_i)\prod_{i=1}^{k}\mathrm{d}z_i\,.
\end{aligned}
\tag{3.90}
$$

To get an upper bound, we change variables $m_i := t_i - t_{i-1}$ for $i = 2,\dots,k$ and we enlarge the integrals over $\frac{j-1}{M}\varepsilon^{-2}t < t_1 < \frac{j}{M}\varepsilon^{-2}t$ and $0 < m_i < \varepsilon^{-2}t$ for $i = 2,\dots,k$, therefore

$$
\begin{aligned}
\mathcal{K}_t^\varepsilon(x,x') &\le \sum_{k=1}^{\infty}(\beta_\varepsilon^2)^{k-1}\int_{\frac{j-1}{M}\varepsilon^{-2}t}^{\frac{j}{M}\varepsilon^{-2}t}\int_{\mathbb{R}^2} g_{2t_1}\big(z_1-\varepsilon^{-1}(x-x')\big)J(z_1)\mathrm{d}t_1\,\mathrm{d}z_1\times \\
&\quad\times\int_{[0,\,\varepsilon^{-2}t]^{k-1}}\int_{(\mathbb{R}^2)^{k-1}}\prod_{i=2}^{k}\mathrm{d}t_i\mathrm{d}z_i\prod_{i=2}^{k}g_{2(t_i-t_{i-1})}(z_i-z_{i-1})\,J(z_i) \\
&= \sum_{k=1}^{\infty}(\beta_\varepsilon^2)^{k-1}\int_{\frac{j-1}{M}t}^{\frac{j}{M}t}\int_{\mathbb{R}^2} g_{2s}\big(\varepsilon z_1-(x-x')\big)J(z_1)\mathrm{d}s\,\mathrm{d}z_1\times \\
&\quad\times\int_{[0,\,\varepsilon^{-2}t]^{k-1}}\int_{(\mathbb{R}^2)^{k-1}}\prod_{i=2}^{k}\mathrm{d}t_i\mathrm{d}z_i\prod_{i=2}^{k}g_{2(t_i-t_{i-1})}(z_i-z_{i-1})\,J(z_i)\,,
\end{aligned}
$$

where the last equality holds since we changed variable $u := \varepsilon^2 t_1$ and

$$
g_{2t_1}\big(z_1-\varepsilon^{-1}(x-x')\big)\,\mathrm{d}t_1 = g_{2u}\big(\varepsilon z_1-(x-x')\big)\mathrm{d}u\,.
$$

Notice that

$$
g_{2m}(y-x) = \frac{1}{4\pi m} + O\bigg(\frac{1}{m^2}\bigg) \quad \text{as } m\to\infty\,,\ \text{uniformly in } x,y\in\mathrm{supp}J\,. \tag{3.91}
$$

Therefore, by denoting

$$
u(m) := \sup_{z'\in\mathrm{supp}J}\int_{\mathbb{R}^2} g_{2m}(z-z')\,J(z)\,\mathrm{d}z\,, \tag{3.92}
$$

we have that for some suitable constants $m_0 > 0$, $C > 0$ and for ε small enough such that $m_0 < \varepsilon^{-2}t$ it holds

$$\begin{aligned}
&\int_0^{\varepsilon^{-2}t} u(m)\,\mathrm{d}m \\
&= \int_0^{m_0} \sup_{z'\in\mathrm{supp}J} \int_{\mathbb{R}^2} g_{2m}(z-z')J(z)\mathrm{d}z\mathrm{d}m + \int_{m_0}^{\varepsilon^{-2}t} \sup_{z'\in\mathrm{supp}J} \int_{\mathbb{R}^2} g_{2m}(z-z')J(z)\mathrm{d}z\mathrm{d}m \\
&\le \|J\|_\infty m_0 + \int_{m_0}^{\varepsilon^{-2}t} \Big(\frac{1}{4\pi m} + \frac{C}{m^2}\Big)\mathrm{d}m \\
&= \frac{\log\varepsilon^{-1}}{2\pi} + O(1) \qquad \text{as } \varepsilon \to 0\,,
\end{aligned} \tag{3.93}$$

where we used that $\int_{\mathbb{R}^2} J(z)\mathrm{d}z = 1$, since $J = j * j$ and j is a probability density. By applying (3.93), we obtain that

$$\begin{aligned}
&\mathcal{K}_t^\varepsilon(x,x') \\
&\le \sum_{k=1}^\infty \Big(\beta_\varepsilon^2\Big(\frac{\log\varepsilon^{-1}}{2\pi} + O(1)\Big)\Big)^{k-1} \int_{\frac{j-1}{M}t}^{\frac{j}{M}t} \int_{\mathbb{R}^2} g_{2u}\big(\varepsilon z_1 - (x-x')\big)J(z_1)\mathrm{d}u\mathrm{d}z_1 \\
&= \sum_{k=1}^\infty \Big(\hat\beta^2\big(1+o(1)\big)\Big)^{k-1} \int_{\frac{j-1}{M}t}^{\frac{j}{M}t} \int_{\mathbb{R}^2} g_{2u}\big(\varepsilon z_1 - (x-x')\big)J(z_1)\mathrm{d}u\mathrm{d}z_1 \\
&= \frac{1}{1-\hat\beta^2\big(1+o(1)\big)} \int_{\frac{j-1}{M}t}^{\frac{j}{M}t} \int_{\mathbb{R}^2} g_{2u}\big(\varepsilon z_1 - (x-x')\big)J(z_1)\mathrm{d}u\mathrm{d}z_1\,,
\end{aligned} \tag{3.94}$$

since for ε small enough $\beta_\varepsilon^2\Big(\frac{\log\varepsilon^{-1}}{2\pi} + O(1)\Big) = \hat\beta^2\big(1+o(1)\big) < 1$ (recall (3.74)).

Hence, from (3.86) and (3.94) we have the following upper bound

$$\begin{aligned}
\sigma_\varepsilon^2(\mathbb{B}_j) \le \frac{1}{1-\hat\beta^2 + O\big(\frac{1}{\log\varepsilon^{-1}}\big)} \times \\
\times \int_{\mathbb{R}^2\times\mathbb{R}^2} \varphi(x) \int_{\frac{j-1}{M}t}^{\frac{j}{M}t} \int_{\mathbb{R}^2} g_{2u}\big(\varepsilon z_1 - (x-x')\big)J(z_1)\mathrm{d}u\mathrm{d}z_1\, \varphi(x')\,\mathrm{d}x\,\mathrm{d}x'\,,
\end{aligned}$$

which implies (3.88) by sending $\varepsilon \to 0$.

On the other hand, let us fix $L \in \mathbb{N}$ large enough such that $\frac{1}{L} < \frac{1}{M}$. We obtain a lower bound for (3.90) with the following restrictions: we sum over $1 \le k \le H$ and we integrate over $\frac{j-1}{M}\varepsilon^{-2}t < t_1 < \big(\frac{j}{M} - \frac{1}{L}\big)\varepsilon^{-2}t$ and $0 < m_i := t_i - t_{i-1} < \frac{1}{L^2}\varepsilon^{-2}t$ for $i = 2,\dots,k$. This ensures that $\frac{j-1}{M}\varepsilon^{-2}t < t_1 < \dots < t_k$ and

$$t_k \le t_1 + \sum_{i=2}^k m_i \le \Big(\frac{j}{M} - \frac{1}{L}\Big)\varepsilon^{-2}t + L\frac{1}{L^2}\varepsilon^{-2}t \le \frac{j}{M}\varepsilon^{-2}t\,,$$

as required. Therefore, since it still holds that

$$\int_0^{\frac{1}{L^2}\varepsilon^{-2}t} u(m)\,\mathrm{d}m = \frac{\log\varepsilon^{-1}}{2\pi} + O(1) \qquad \text{as } \varepsilon \to 0\,,$$

(recall (3.91), (3.92) and (3.93)), we can repeat the same arguments we followed to get the upper bound and prove that

$$\begin{aligned}
&\mathcal{K}_t^\varepsilon(x,x') \\
&\geq \sum_{k=1}^{L} \Big(\hat\beta^2\big(1+o(1)\big)\Big)^{k-1} \int_{\frac{j-1}{M}t}^{\left(\frac{j}{M}-\frac{1}{L}\right)t} \int_{\mathbb{R}^2} g_{2u}\big(\varepsilon z_1 - (x-x')\big) J(z_1) \mathrm{d}u \mathrm{d}z_1 \\
&= \frac{\Big(\hat\beta^2\big(1+o(1)\big)\Big)^{L}}{1-\hat\beta^2\big(1+o(1)\big)} \int_{\frac{j-1}{M}t}^{\left(\frac{j}{M}-\frac{1}{L}\right)t} \int_{\mathbb{R}^2} g_{2u}\big(\varepsilon z_1 - (x-x')\big) J(z_1) \mathrm{d}u \mathrm{d}z_1 \,.
\end{aligned}$$

We take the limit $\varepsilon \to 0$ to obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \sigma_\varepsilon^2(\mathbb{B}_j) \geq \frac{(\hat\beta^2)^L}{1-\hat\beta^2} \times \\
&\qquad \times \int_{\mathbb{R}^2\times\mathbb{R}^2} \varphi(x) \int_{\frac{j-1}{M}t}^{\left(\frac{j}{M}-\frac{1}{L}\right)t} \int_{\mathbb{R}^2} g_{2u}\big(\varepsilon z_1 - (x-x')\big) J(z_1) \mathrm{d}u \mathrm{d}z_1 \, \varphi(x') \mathrm{d}x \mathrm{d}x'
\end{aligned}$$

and by eventually sending $L \to \infty$ we get (3.89).

Proof of $(\widetilde{2})$**.** We can write

$$\begin{aligned}
&\sum_{k>K} \|\widetilde{q}_\varepsilon\|^2_{L^2(E^k)} \\
&= \int_{\mathbb{R}^2\times\mathbb{R}^2} \varphi(x)\, \mathcal{K}^\varepsilon_{t,K}(x,x')\, \varphi(x') \, \mathrm{d}x \, \mathrm{d}x' \,,
\end{aligned} \tag{3.95}$$

where

$$\begin{aligned}
\mathcal{K}^\varepsilon_{t,K}(x,x') &:= \sum_{k=1}^{K} (\beta_\varepsilon^2)^{k-1} \int_{\frac{j-1}{M}\varepsilon^{-2}t < t_1 < \cdots < t_k < \frac{j}{M}\varepsilon^{-2}t} \prod_{i=1}^{k} \mathrm{d}t_i \int_{(\mathbb{R}^2)^k} \prod_{i=1}^{k} \mathrm{d}x_i \times \\
\times \int_{(\mathbb{R}^2)^k\times(\mathbb{R}^2)^k} &\prod_{i=1}^{k} g_{t_i-t_{i-1}}(y_i-y_{i-1}) g_{t_i-t_{i-1}}(\widetilde{y}_i-\widetilde{y}_{i-1})\, j(y_i-x_i) j(\widetilde{y}_i-x_i) \prod_{i=1}^{k} \mathrm{d}y_i \mathrm{d}\widetilde{y}_i \,,
\end{aligned}$$

with $y_0 := \varepsilon^{-1}x$ and $\widetilde{y}_0 := \varepsilon^{-1}x'$. At this point, the limit (3.83) easily follows by repeating the same arguments used in the previous proof. Indeed, for some finite constant $C>0$ we get

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \sum_{k>K} \|\widetilde{q}_\varepsilon\|^2_{L^2(E^k)} \\
&\leq \frac{\big(\hat\beta^2\big)^K}{1-\hat\beta^2} \int_{\mathbb{R}^2\times\mathbb{R}^2} \varphi(x)\, \varphi(x') \int_{\frac{j-1}{M}t}^{\frac{j}{M}t} g_{2u}(x-x') \, \mathrm{d}u \, \mathrm{d}x \, \mathrm{d}x' \\
&\leq C \, \frac{\big(\hat\beta^2\big)^K}{1-\hat\beta^2} \,,
\end{aligned}$$

which implies (3.83) by sending $K \to \infty$.

CHAPTER 4

Approximation of the directed polymer log–partition function in the subcritical regime

So far we have discussed the distribution of the partition function $Z_N^{\beta_N}(m,z)$, suitably rescaled, as a *random field*, i.e. averaging over the starting point (m,z) against a continuous and compactly supported test function. In this chapter, we look at the distribution of $Z_N^{\beta_N}(m,z)$ for a *fixed* starting point: we fix $(m,z) = (0,0)$ by stationarity and we set

$$Z_N^{\beta_N} := Z_N^{\beta_N}(0,0)\,. \tag{4.1}$$

It was shown in [**CSZ17b**, Theorem 2.8] that $Z_N^{\beta_N}$ is *asymptotically log-normal*:

$$\log Z_N^{\beta_N} \xrightarrow{d} \mathcal{N}\big(-\tfrac{1}{2}\sigma_{\hat\beta}^2,\, \sigma_{\hat\beta}^2\big) \qquad \text{where} \qquad \sigma_{\hat\beta}^2 = \sigma^2(\hat\beta) = \log c_{\hat\beta}^2 = \log \tfrac{1}{1-\hat\beta^2}\,. \tag{4.2}$$

The original proof of this result, based on the Fourth Moment Theorem, is long and technical. Our goal is to provide a less technical and more insightful proof, based on second moment computation, exploiting our Theorem 2.2. The problem is that, unlike for $Z_N^{\beta_N}$, *we do not have a polynomial chaos expansion for* $\log Z_N^{\beta_N}$, which is essential for Theorem 2.2. We solve this problem by first proving a result of independent interest, which shows that $\log Z_N^{\beta_N}$ is sharply approximated in L^2 by an explicit polynomial chaos expansion X_N^{dom}.

Before stating our result, we need some setup. We recall that the modified disorder $(\eta_N(n,x))_{n\in\mathbb{N},x\in\mathbb{Z}^2}$ was defined in (3.3). We also recall the transition kernel of the simple random walk:

$$q_n(x) := \mathrm{P}(S_n = x \,|\, S_0 = 0) \tag{4.3}$$

and the polynomial chaos expansion of the partition function [**CSZ17a**]:

$$Z_N^{\beta_N}(m,z) := 1 + \sum_{k=1}^{\infty} (\sigma_N)^k \sum_{\substack{m=n_0<n_1<\ldots<n_k\le N \\ x_0:=z,\, x_1,\ldots,x_k\in\mathbb{Z}^2}} \prod_{i=1}^{k} q_{n_i-n_{i-1}}(x_i - x_{i-1})\, \eta_N(n_i,x_i)\,. \tag{4.4}$$

We define a new polynomial chaos expansion X_N^{dom}, obtained from the centered partition function $Z_N^{\beta_N} - 1 = Z_N^{\beta_N}(0,0) - 1$ imposing the constraint that *all increments $n_i - n_{i-1}$ for $i \geq 2$ are dominated by the first time n_1*:

$$X_N^{\mathrm{dom}} := \sum_{k=1}^{\infty} (\sigma_N)^k \sum_{\substack{0=n_0<n_1<\ldots<n_k\leq N: \\ \max\{n_2-n_1, n_3-n_2, \ldots, n_k-n_{k-1}\}\leq n_1 \\ x_0:=0,\, x_1,\ldots,x_k\in\mathbb{Z}^2}} \prod_{i=1}^{k} q_{n_i-n_{i-1}}(x_i - x_{i-1})\, \eta_N(n_i, x_i)\,. \tag{4.5}$$

Our key approximation result shows that X_N^{dom} *is a sharp approximation of* $\log Z_N^{\beta_N}$. The reason why this approximation is possible will be clear in the proof, but one can already give a look at equation (4.10), which shows that a natural approximation of $Z_N^{\beta_N}$ has a *product structure*, where (a restricted version of) X_N^{dom} appears.

THEOREM 4.1 (Polynomial chaos for $\log Z$). *Set $\beta = \beta_N$ as in* (3.18). *Then*

$$\lim_{N\to\infty} \left\| \log Z_N^{\beta_N} - \left\{ X_N^{\mathrm{dom}} - \tfrac{1}{2}\,\mathbb{E}[(X_N^{\mathrm{dom}})^2] \right\} \right\|_{L^2} = 0\,. \tag{4.6}$$

We then show, by our general Theorem 2.2, that X_N^{dom} is asymptotically Gaussian.

THEOREM 4.2 (Asymptotic Gaussianity of X_N^{dom}). *Set $\beta = \beta_N$ as in* (3.18). *Then*

$$\lim_{N\to\infty} \mathbb{E}\big[(X_N^{\mathrm{dom}})^2\big] = \sigma_{\hat\beta}^2 = \log \tfrac{1}{1-\hat\beta^2} \qquad \textit{and} \qquad X_N^{\mathrm{dom}} \xrightarrow{d} \mathcal{N}\big(0, \sigma_{\hat\beta}^2\big)\,. \tag{4.7}$$

Note that relations (4.6) and (4.7) together provide a strengthening of the asymptotic log-normality of $Z_N^{\beta_N}$, see (4.2).

4.1. Polynomial chaos for the log–partition function: proof of Theorem 4.1

The proof is self-contained but long, therefore it is organized in four parts: we give different approximations of the partition function $Z_N^{\beta_N}$ and of its logarithm, which will lead us to the proof of our goal (4.6). Let us first present a general overview of the strategy and then show the technical proof below.

Part 1 (record times). Let us define a "constrained version" $X_{N,[a,b;b']}^{\mathrm{dom}}(x,z;z')$ of X_N^{dom} from (4.5), where we fix $(n_0, n_1; n_k) = (a, b; b')$ and $(x_0, x_1; x_k) = (x, z; z')$:

$$\begin{aligned} X_{N,[a,b;b']}^{\mathrm{dom}}(x,z;z') &:= \sum_{k=1}^{\infty} (\sigma_N)^k\, q_{b-a}(z-x)\, \eta_N(b,z) \times \\ &\times \sum_{\substack{b=:n_1<n_2<\ldots<n_{k-1}<n_k=:b' \\ \max\{n_2-n_1,\ldots,n_k-n_{k-1}\}\leq b}} \; \sum_{\substack{x_1=z,\, x_k=z', \\ x_2,\ldots,x_{k-1}\in\mathbb{Z}^2}} \prod_{i=2}^{k} q_{n_i-n_{i-1}}(x_i - x_{i-1})\, \eta_N(n_i, x_i)\,. \end{aligned} \tag{4.8}$$

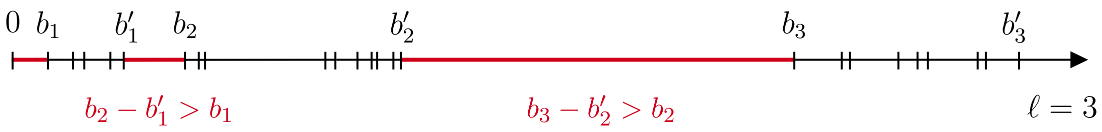

FIGURE 1. An example of the variables b_i, b_i' in (4.9). These correspond to *record times* which satisfy $b_i - b_{i-1}' > b_{i-1}$, see subsection 4.1.1.

(Note that if $b = b'$ only the terms $k = 1$ contributes to the sum — and we must have $z = z'$, otherwise the sum vanishes — while if $b < b'$ only the terms $k \geq 2$ give a contribution.)

We first show that the partition function $Z_N^{\beta_N}$ in (4.4) can be written as a concatenation of products of $X^{\mathrm{dom}}_{N,[a,b;b']}(x, z; z')$'s corresponding to suitable *record times*, see Figure 1. The next result is proved in subsection 4.1.1.

LEMMA 4.3 (Record times). *The following equality holds, with* $(b_0', z_0') := (0, 0)$*:*

$$Z_N^{\beta_N} = 1 + \sum_{\ell=1}^{\infty} \sum_{\substack{0<b_1\leq b_1'<\ldots<b_\ell\leq b_\ell'\leq N: \\ b_i-b_{i-1}'>b_{i-1}\ \forall i=2,\ldots,\ell}} \ \sum_{\underline{z},\underline{z}'\in(\mathbb{Z}^2)^\ell} \prod_{i=1}^{\ell} X^{\mathrm{dom}}_{N,[b_{i-1}',b_i;b_i']}(z_{i-1}', z_i; z_i'), \tag{4.9}$$

where we use the shortcuts $\underline{z} = (z_1, \ldots, z_\ell)$ *and* $\underline{z}' = (z_1', \ldots, z_\ell')$*.*

Part 2 (coarse-graining and diffusive approximation). We fix a large parameter $M \in \mathbb{N}$ and we define an approximation $Z_{N,M}^{(\mathrm{diff})}$ of the partition function $Z_N^{\beta_N}$ from (4.9), as follows:[1]

1. we set $b_{i-1}' = 0$, $z_{i-1}' = 0$ in each $X^{\mathrm{dom}}_{N,[b_{i-1}',b_i;b_i']}(z_{i-1}', z_i; z_i')$;
2. we impose that each pair $b_i \leq b_i'$ belongs to the same interval $(N^{\frac{j-1}{M}}, N^{\frac{j}{M}}]$, for some $j = 1, \ldots, M$, and we ignore the constraint $b_i - b_{i-1}' > b_{i-1}$.

This yields the following definition of $Z_{N,M}^{(\mathrm{diff})}$:

$$Z_{N,M}^{(\mathrm{diff})} := 1 + \sum_{\ell=1}^{\infty} \sum_{1\leq j_1<\ldots<j_\ell\leq M} \prod_{i=1}^{\ell} X^{\mathrm{dom}}_{N,M}(j_i) = \prod_{j=1}^{M} \left(1 + X^{\mathrm{dom}}_{N,M}(j)\right), \tag{4.10}$$

where we set

$$X^{\mathrm{dom}}_{N,M}(j) := \sum_{b\leq b'\in(N^{\frac{j-1}{M}},N^{\frac{j}{M}}]} \ \sum_{z,z'\in\mathbb{Z}^2} X^{\mathrm{dom}}_{N,[0,b;b']}(0, z; z') \qquad \text{for } j = 1, \ldots, M. \tag{4.11}$$

We prove that $Z_{N,M}^{(\mathrm{diff})}$ is close to $Z_N^{\beta_N}$ in L^2 for $N \gg M \gg 1$, in the following sense.

[1]Heuristically, these are good approximations because the main contribution to (4.9) will be shown to come from $b_{i-1}' \approx N^{\alpha_{i-1}'}$ and $b_i \approx N^{\alpha_i}$ with $\alpha_{i-1}' < \alpha_i$, hence $b_{i-1}' \ll b_i$.

Lemma 4.4 (Coarse-graining and diffusive approximation). *The following holds:*

$$\limsup_{M\to\infty} \limsup_{N\to\infty} \left\| Z_N^{\beta_N} - Z_{N,M}^{(\mathrm{diff})} \right\|_{L^2} = 0 \,. \tag{4.12}$$

The proof of this result is given in subsection 4.1.2 below.

Part 3 (log approximation). The product form of $Z_{N,M}^{(\mathrm{diff})}$ in (4.10) is especially suitable to take the logarithm. We thus prove a preliminary version of our goal (4.6), where we replace $\log Z_N^{\beta_N}$ by $\log Z_{N,M}^{(\mathrm{diff})}$ (and convergence in L^2 by convergence in probability). To this purpose, we define the event

$$A_{N,M} := \bigcap_{j=1}^{M} \left\{ |X_{N,M}^{\mathrm{dom}}(j)| \le \tfrac{1}{2} \right\}, \tag{4.13}$$

which ensures that $Z_{N,M}^{(\mathrm{diff})} > 0$, see (4.10).

Lemma 4.5 (log approximation). *Recall X_N^{dom} from* (4.5)*. For any $\epsilon > 0$ we have*

$$\lim_{M\to\infty} \limsup_{N\to\infty} \mathbb{P}\Big(\big| \log Z_{N,M}^{(\mathrm{diff})} - \big\{ X_N^{\mathrm{dom}} - \tfrac{1}{2}\mathbb{E}[(X_N^{\mathrm{dom}})^2] \big\} \big| > \epsilon, \, A_{N,M} \Big) = 0 \,, \tag{4.14}$$

for $A_{N,M} \subseteq \{Z_{N,M}^{(\mathrm{diff})} > 0\}$ defined in (4.13) *(so that $\log Z_{N,M}^{(\mathrm{diff})}$ is well-defined) which satisfies*

$$\lim_{M\to\infty} \limsup_{N\to\infty} \mathbb{P}\big((A_{N,M})^c \big) \;=\; 0 \,. \tag{4.15}$$

The proof of this result is given in subsection 4.1.3 below.

Part 4 (final approximation). At last, we complete the proof of Theorem 4.1. Our final goal (4.6) is a consequence of the next lemma, where we prove convergence in probability and boundedness in L^p for some $p > 2$.

Lemma 4.6 (Final approximation). *Recall X_N^{dom} from* (4.5)*. For any $\epsilon > 0$ we have*

$$\lim_{N\to\infty} \mathbb{P}\big(\big| \log Z_N^{\beta_N} - \big\{ X_N^{\mathrm{dom}} - \tfrac{1}{2}\mathbb{E}[(X_N^{\mathrm{dom}})^2] \big\} \big| > \epsilon \big) = 0 \,. \tag{4.16}$$

Moreover, for some $p > 2$ we have

$$\sup_{N\in\mathbb{N}} \mathbb{E}\big[\big| \log Z_N^{\beta_N} \big|^p \big] < \infty \,, \qquad \sup_{N\in\mathbb{N}} \mathbb{E}\big[\big| X_N^{\mathrm{dom}} \big|^p \big] < \infty \,. \tag{4.17}$$

Notice that, once we have convergence in probability (4.16), to obtain convergence in L^2 it suffices to show *uniform integrability of the squares of* $\log Z_N^{\beta_N}$ *and* X_N^{dom}, which is in turn implied by boundedness in L^p for some $p > 2$, as in (4.17).

Intuitively, we can deduce (4.16) from (4.14) by exploiting the approximation (4.12), but some care is needed to handle the logarithm.

The proof of Lemma 4.6, given in subsection 4.1.4, concludes the proof of Theorem 4.1. □

4.1.1. Proof of Lemma 4.3. We rewrite the sum over $n_1, \ldots, n_k$ in (4.4) according to suitable *record times*. The first record time is n_1; the second record time is the smallest n_i for which the previous jump $n_i - n_{i-1}$ exceeds n_1; and so on. More precisely, the record times are $n_{j_1}, n_{j_2}, \ldots, n_{j_\ell}$ where we define $j_1 := 1$ and, assuming that $j_r < \infty$, we set $j_{r+1} := \min\{i \in \{j_r + 1, \ldots, k\} : n_i - n_{i-1} > n_{j_r}\}$, where we agree that $\min \emptyset := \infty$. The number of record times is therefore $\ell := \min\{r \geq 1 : \; j_{r+1} = \infty\}$.

If we rename the record times as $b_r := n_{j_r}$, and we also set $b'_{r-1} := n_{j_r - 1}$, we have by construction $b_2 - b'_1 > b_1$ and, more generally, $b_i - b'_{i-1} > b_{i-1}$ for $i = 2, \ldots, \ell$ (see Figure 1). If we name the corresponding space variables $z_r := x_{b_r}$ and $z'_{r-1} := x_{b'_{r-1}}$, then we can rewrite (4.4) equivalently as (4.9), with $X^{\mathrm{dom}}_{N,[a,b;b']}(x, z; z')$ defined in (4.8). □

4.1.2. Proof of Lemma 4.4. The proof, which is long and structured, is based on explicit L^2 computations. A key observation is that, by the expression (4.9) for $Z_N^{\beta_N}$, we can write

$$\mathbb{E}\Big[\big(Z_N^{\beta_N}\big)^2\Big] = 1 + \sum_{\ell=1}^{\infty} \sum_{\substack{0<b_1 \leq b'_1 < \ldots < b_\ell \leq b'_\ell \leq N: \\ b_i - b'_{i-1} > b_{i-1} \, \forall i=2,\ldots,\ell}} \sum_{\underline{z}, \underline{z}' \in (\mathbb{Z}^2)^\ell} \prod_{i=1}^{\ell} \mathbb{E}\Big[\big(X^{\mathrm{dom}}_{N,[b'_{i-1}, b_i; b'_i]}(z'_{i-1}, z_i; z'_i)\big)^2\Big] . \tag{4.18}$$

To see why this holds, note that by (4.4) we can write

$$\mathbb{E}\big[\big(Z_N^{\beta_N}\big)^2\big] = 1 + \sum_{k=1}^{\infty} (\sigma_N^2)^k \sum_{\substack{0=:n_0<n_1<\ldots<n_k \leq N \\ x_0:=0, \; x_1,\ldots,x_k \in \mathbb{Z}^2}} \prod_{j=1}^{k} q_{n_j - n_{j-1}}(x_j - x_{j-1})^2 , \tag{4.19}$$

with $q_n(x) = \mathrm{P}(S_n = x \,|\, S_0 = 0)$, see (4.3), and σ_N as in (3.3). Similarly, by (4.8),

$$\mathbb{E}\Big[\big(X^{\mathrm{dom}}_{N,[a,b,b']}(x, z; z')\big)^2\Big] = \sum_{k=1}^{\infty} (\sigma_N^2)^k \, q_{b-a}(z - x)^2 \times \\ \times \sum_{\substack{b=:n_1<n_2<\ldots<n_{k-1}<n_k=b' \\ \max\{n_2-n_1,\ldots,n_k-n_{k-1}\} \leq b}} \sum_{\substack{x_1=z; \, x_k=z' \\ x_2,\ldots,x_{k-1} \in \mathbb{Z}^2}} \prod_{i=2}^{k} q_{n_i - n_{i-1}}(x_i - x_{i-1})^2. \tag{4.20}$$

When we plug (4.20) into (4.18) we obtain (4.19) by the same argument in the proof of Lemma 4.3, see subsection 4.1.1, because the sum over n_j, x_j in (4.19) can be rewritten in terms of record times, which lead to the variables b_r, b'_r and z_r, z'_r in (4.18).

We now turn to the proof of (4.12). We will define two "coarse-grained approximations" $Z^{(\mathrm{cg})}_{N,K,M}$ and $Z^{(\mathrm{cg}')}_{N,K,M}$, which depend on a further parameter $K \in \mathbb{N}$, and we will show that

$$Z_N^{\beta_N} \approx Z^{(\mathrm{cg})}_{N,K,M}\,, \qquad Z^{(\mathrm{cg})}_{N,K,M} \approx Z^{(\mathrm{cg}')}_{N,K,M}\,, \qquad Z^{(\mathrm{cg}')}_{N,K,M} \approx Z^{(\mathrm{diff})}_{N,M}\,,$$

where $\approx$ denotes closeness in L^2 when we let $N \to \infty$, then $K \to \infty$ and finally $M \to \infty$. More precisely, we are going to prove the following relations:

$$\limsup_{M\to\infty}\ \limsup_{K\to\infty}\ \limsup_{N\to\infty}\ \big\| Z_N^{\beta_N} - Z^{(\mathrm{cg})}_{N,K,M} \big\|_{L^2} = 0\,, \tag{4.21}$$

$$\limsup_{M\to\infty}\ \limsup_{K\to\infty}\ \limsup_{N\to\infty}\ \big\| Z^{(\mathrm{cg})}_{N,K,M} - Z^{(\mathrm{cg}')}_{N,K,M} \big\|_{L^2} = 0\,, \tag{4.22}$$

$$\limsup_{M\to\infty}\ \limsup_{K\to\infty}\ \limsup_{N\to\infty}\ \big\| Z^{(\mathrm{cg}')}_{N,K,M} - Z^{(\mathrm{diff})}_{N,M} \big\|_{L^2} = 0\,, \tag{4.23}$$

which together yield (4.12). We accordingly split the proof in three steps.

Step 1: definition of $Z^{(\mathrm{cg})}_{N,K,M}$ and proof of (4.21)**.** Let us fix $M, K, N \in \mathbb{N}$ with $1 \ll M \ll K \ll N$. Our first coarse-graining approximation $Z^{(\mathrm{cg})}_{N,K,M}$ of the partition function $Z_N^{\beta_N}$ in (4.9) is obtained by *suitably restricting the sums over* $\underline{b}, \underline{b}'$ *and* $\underline{z}, \underline{z}'$:

$$Z^{(\mathrm{cg})}_{N,K,M} := 1 + \sum_{\ell=1}^{\infty} \ \sum_{\underline{j} \in \{1,\ldots,M\}^\ell_{\ll}} \ \sum_{(\underline{b},\underline{b}') \in \mathcal{B}^\ell(\underline{j})} \ \sum_{(\underline{z},\underline{z}') \in \mathcal{S}^\ell(\underline{b},\underline{b}')} \ \prod_{i=1}^{\ell} X^{\mathrm{dom}}_{N,[b'_{i-1},b_i;b'_i]}(z'_{i-1}, z_i; z'_i)\,, \tag{4.24}$$

where we sum over $\underline{j} = (j_1, \ldots, j_\ell)$ in the following set:

$$\{1,\ldots,M\}^\ell_{\ll} := \Big\{ 1 \le j_1 < \ldots < j_\ell \le M : \quad j_i - j_{i-1} \ge 2 \ \ \forall i = 2, \ldots, \ell \Big\}\,, \tag{4.25}$$

then, given $\underline{j} = (j_1, \ldots, j_\ell)$, we sum over $(\underline{b}, \underline{b}')$ in the set

$$\mathcal{B}^\ell(\underline{j}) := \Big\{ (\underline{b}, \underline{b}') \in \mathbb{N}^\ell \times \mathbb{N}^\ell : b_i \in (N^{\frac{j_i - 1}{M}}, \tfrac{1}{K} N^{\frac{j_i}{M}}]\,,\ b'_i \in [b_i, K b_i] \quad \forall i = 1, \ldots, \ell \ \Big\}, \tag{4.26}$$

and finally, given $(\underline{b}, \underline{b}')$, we sum over $\underline{z}, \underline{z}'$ in the "diffusive set"

$$\mathcal{S}^\ell(\underline{b}, \underline{b}') := \Big\{ (\underline{z}, \underline{z}') \in (\mathbb{Z}^2)^\ell \times (\mathbb{Z}^2)^\ell : |z_i| \le K\sqrt{b_i}\,,\ |z'_i| \le K^2 \sqrt{b_i} \quad \forall i = 1, \ldots, \ell \Big\}.$$

To see that $Z^{(\mathrm{cg})}_{N,K,M}$ *in* (4.24) *is a restriction of* $Z_N^{\beta_N}$ *in* (4.9), note that for $(\underline{b}, \underline{b}') \in \mathcal{B}^\ell(\underline{j})$ we have $0 < b_1 \le b'_1 < \ldots < b_\ell \le b'_\ell \le N$, and for large N we also have $b_i - b'_{i-1} > b_{i-1}$ for $i \ge 2$, because $b_i > N^{\frac{j_i - 1}{M}} \ge N^{\frac{j_{i-1}+1}{M}} \ge K N^{\frac{1}{M}} b_{i-1}$ (recall that $j_i - j_{i-1} \ge 2$) hence

$$b_i - b'_{i-1} > K N^{\frac{1}{M}} b_{i-1} - K b_{i-1} = (N^{\frac{1}{M}} - 1) K\, b_{i-1} > b_{i-1} \qquad \text{for } N > 2^M\,.$$

Thus the range of the sums in (4.24) is included in the range of the sums in (4.9). Since the terms in the polynomial chaos (4.4) are orthogonal in L^2, it follows that

$$\big\|Z_N^{\beta_N} - Z_{N,K,M}^{(\mathrm{cg})}\big\|_{L^2}^2 = \big\|Z_N^{\beta_N}\big\|_{L^2}^2 - \big\|Z_{N,K,M}^{(\mathrm{cg})}\big\|_{L^2}^2 \,, \tag{4.27}$$

hence to prove (4.21) it suffices to show that

$$\limsup_{N\to\infty}\ \mathbb{E}\big[\big(Z_N^{\beta_N}\big)^2\big] \le \frac{1}{1-\hat\beta^2}\,, \tag{4.28}$$

$$\liminf_{M\to\infty}\ \liminf_{K\to\infty}\ \liminf_{N\to\infty}\ \mathbb{E}\big[\big(Z_{N,K,M}^{(\mathrm{cg})}\big)^2\big] \ge \frac{1}{1-\hat\beta^2}\,. \tag{4.29}$$

Relation (4.28) can be easily deduced from the expression (4.19). Indeed, enlarging the sums to $1 \le n_j - n_{j-1} \le N$ and recalling the definition (3.15) of R_N, we get

$$\begin{aligned}\mathbb{E}\big[\big(Z_N^{\beta_N}\big)^2\big] &\le 1 + \sum_{k=1}^{\infty}(\sigma_N^2)^k \sum_{\substack{1\le n_j-n_{j-1}\le N\\ j=1,\dots,k}} \ \sum_{x_0:=0,\ x_1,\dots,x_k\in\mathbb{Z}^2}\ \prod_{j=1}^{k} q_{n_j-n_{j-1}}(x_j-x_{j-1})^2\\ &= 1+\sum_{k=1}^{\infty}(\sigma_N^2)^k\left(\sum_{n=1}^{N}\sum_{x\in\mathbb{Z}^2} q_n(x)^2\right)^k = 1+\sum_{k=1}^{\infty}\big(\sigma_N^2 R_N\big)^k = \frac{1}{1-\sigma_N^2 R_N}.\end{aligned} \tag{4.30}$$

Since $\sigma_N \sim \beta_N \sim \hat\beta\sqrt{\pi}/\sqrt{\log N}$, see (3.3) and (3.18), and since $R_N \sim \frac{1}{\pi}\log N$, see (3.15), we see that (4.28) is proved.

We next prove (4.29). By definition (4.24) of $Z_{N,K,M}^{(\mathrm{cg})}$, in analogy with (4.18), we have

$$\mathbb{E}\Big[\big(Z_{N,K,M}^{(\mathrm{cg})}\big)^2\Big] = 1+\sum_{\ell=1}^{\infty}\ \sum_{\underline{j}\in\{1,\dots,M\}^\ell_{\ll}}\ \sum_{\substack{(\underline{b},\underline{b}')\in\mathcal{B}^\ell(\underline{j})\\ (\underline{z},\underline{z}')\in\mathcal{S}^\ell(\underline{b},\underline{b}')}}\ \prod_{i=1}^{\ell}\mathbb{E}\Big[\big(X^{\mathrm{dom}}_{N,[b'_{i-1},b_i;b'_i]}(z'_{i-1},z_i;z'_i)\big)^2\Big]. \tag{4.31}$$

We now give a lower bound on $\mathbb{E}\big[\big(X^{\mathrm{dom}}_{N,[b'_{i-1},b_i;b'_i]}(z'_{i-1},z_i;z'_i)\big)^2\big]$ when we sum over b_i, b'_i and z_i, z'_i in the sets $\mathcal{B}^\ell(\underline{j})$ and $\mathcal{S}^\ell(\underline{b},\underline{b}')$. The next result is proved below in subsubsection 4.1.5.1.

Lemma 4.7. *For $N, M, K \in \mathbb{N}$ and $j \in \{1,\dots,M\}$, define*

$$\Xi_{N,M,K}(j) := \inf_{\substack{0\le a\le N^{\frac{(j-2)^+}{M}}\\ |x|\le K^2\sqrt{a}}}\ \sum_{\substack{b\in(N^{\frac{j-1}{M}},\frac{1}{K}N^{\frac{j}{M}}]\\ b'\in[b,Kb]}}\ \sum_{\substack{|z|\le K\sqrt{b}\\ |z'|\le K^2\sqrt{b}}}\ \mathbb{E}\Big[\big(X^{\mathrm{dom}}_{N,[a,b,b']}(x,z;z')\big)^2\Big]. \tag{4.32}$$

Then, for any $M \in \mathbb{N}$ and $j \in \{1,\dots,M\}$, we have

$$\liminf_{K\to\infty}\ \liminf_{N\to\infty}\ \Xi_{N,M,K}(j)\ =\ I_M(j) := \int_{\frac{j-1}{M}}^{\frac{j}{M}} \frac{\hat\beta^2}{1-\hat\beta^2 s}\mathrm{d}s\,. \tag{4.33}$$

Coming back to (4.31), by definition (4.32) of $\Xi_{N,M,K}(j)$, we have the lower bound

$$\mathbb{E}\Big[\big(Z^{(\mathrm{cg})}_{N,K,M}\big)^2\Big] \;\ge\; 1 + \sum_{\ell=1}^{\infty} \sum_{\underline{j}\in\{1,\dots,M\}^\ell_{\ll}} \prod_{i=1}^{\ell} \Xi_{N,M,K}(j_i)\,, \tag{4.34}$$

which yields, by (4.33),

$$\liminf_{K\to\infty}\, \liminf_{N\to\infty}\, \mathbb{E}\Big[\big(Z^{(\mathrm{cg})}_{N,K,M}\big)^2\Big] \;\ge\; 1 + \sum_{\ell=1}^{\infty} \sum_{\underline{j}\in\{1,\dots,M\}^\ell_{\ll}} \prod_{i=1}^{\ell} I_M(j_i)\,. \tag{4.35}$$

Recalling the definition (4.25) of $\{1,\dots,M\}^\ell_{\ll}$, we can rewrite the r.h.s. of (4.35) as

$$1 + \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \Bigg\{ \bigg(\sum_{j=1}^{M} I_M(j) \bigg)^{\ell} - \sum_{\substack{j_1,\dots,j_\ell\in\{1,\dots,M\} \\ \exists h\neq k:\ |j_h-j_k|\le 1}} I_M(j_1)\,\cdots\, I_M(j_\ell) \Bigg\}.$$

The second term gives a vanishing contribution as $M\to\infty$, because

$$\max_{1\le j\le M} I_M(j) \le \frac{C}{M}\,,$$

with $C := \frac{\hat\beta^2}{1-\hat\beta^2} < \infty$, hence

$$\sum_{\ell=1}^{\infty} \frac{1}{\ell!} \sum_{\substack{j_1,\dots,j_\ell\in\{1,\dots,M\} \\ \exists h\neq k:\ |j_h-j_k|\le 1}} I_M(j_1)\,\cdots\, I_M(j_\ell) \le \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \frac{C^\ell}{M^\ell} \binom{\ell}{2} 3M^{\ell-1} \;=\; \frac{C'}{M} \xrightarrow{M\to\infty} 0\,,$$

where $\binom{\ell}{2}$ is the number of pairs $\{h,k\}$ with $h\neq k$ and $3M^{\ell-1}$ bounds the number of choices of $j_1,\dots,j_\ell$ with $j_h\in\{j_k-1,\, j_k,\, j_k+1\}$. Since $\sum_{j=1}^{M} I_M(j) = \int_0^1 \frac{\hat\beta^2}{1-\hat\beta^2 s}\mathrm{d}s = \log\frac{1}{1-\hat\beta^2}$, we have finally shown that

$$\liminf_{M\to\infty}\, \liminf_{K\to\infty}\, \liminf_{N\to\infty}\, \mathbb{E}\Big[\big(Z^{(\mathrm{cg})}_{N,K,M}\big)^2\Big] \;\ge\; 1 + \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \Big(\log\tfrac{1}{1-\hat\beta^2}\Big)^{\ell} \;=\; \frac{1}{1-\hat\beta^2}\,, \tag{4.36}$$

which is (4.29). This completes the proof of (4.21). □

Step 2: definition of $Z^{(\mathrm{cg}')}_{N,K,M}$ and proof of (4.22)**.** Starting from $Z^{(\mathrm{cg})}_{N,K,M}$ in (4.24), we set $b'_{i-1}=0$ and $z'_{i-1}=0$ inside each X^{dom}_N to obtain our second approximation:

$$Z^{(\mathrm{cg}')}_{N,K,M} := 1 + \sum_{\ell=1}^{\infty} \sum_{\underline{j}\in\{1,\dots,M\}^\ell_{\ll}} \sum_{(\underline{b},\underline{b}')\in\mathcal{B}^\ell(\underline{j})} \sum_{(\underline{z},\underline{z}')\in\mathcal{S}^\ell(\underline{b},\underline{b}')} \prod_{i=1}^{\ell} X^{\mathrm{dom}}_{N,[0,b_i;b'_i]}(0,z_i;z'_i)\,. \tag{4.37}$$

Heuristically, the reason why we set $b'_{i-1}=0$ is that $b_i \gg b'_{i-1}$, hence $b_i - b'_{i-1}\approx b_i$ (indeed, note that $b_i \ge N^{\frac{j_i-1}{M}} \gg N^{\frac{j_{i-1}}{M}} \ge b'_{i-1}$ since $j_i-1 > j_{i-1}$, see (4.26) and (4.25)).

We need to prove (4.22). Given $\underline{b}, \underline{b}'$ and $\underline{z}, \underline{z}'$, let us introduce the shortcuts

$$X_i := X^{\mathrm{dom}}_{N,[b'_{i-1},b_i;b'_i]}(z'_{i-1}, z_i; z'_i)\,, \qquad Y_i := X^{\mathrm{dom}}_{N,[0,b_i;b'_i]}(0, z_i; z'_i)\,, \tag{4.38}$$

so that, comparing (4.24) and (4.37), we can write

$$\begin{aligned} & Z^{(\mathrm{cg}')}_{N,K,M} - Z^{(\mathrm{cg})}_{N,K,M} \\ &= \sum_{\ell=1}^{\infty} \sum_{\underline{j}\in\{1,\ldots,M\}^{\ell}_{\ll}} \sum_{(\underline{b},\underline{b}')\in\mathcal{B}^{\ell}(\underline{j})} \sum_{(\underline{z},\underline{z}')\in\mathcal{S}^{\ell}(\underline{b},\underline{b}')} \left(\prod_{i=1}^{\ell} Y_i - \prod_{i=1}^{\ell} X_i \right) \\ &= \sum_{\ell=1}^{\infty} \sum_{\underline{j}\in\{1,\ldots,M\}^{\ell}_{\ll}} \sum_{(\underline{b},\underline{b}')\in\mathcal{B}^{\ell}(\underline{j})} \sum_{(\underline{z},\underline{z}')\in\mathcal{S}^{\ell}(\underline{b},\underline{b}')} \sum_{h=1}^{\ell} \left\{ \prod_{i=1}^{h-1} Y_i \right\} (Y_h - X_h) \left\{ \prod_{i=h+1}^{\ell} X_i \right\}, \end{aligned}$$

and note that different terms in the sums are orthogonal in L^2. We justify below the following key estimate, see Lemma 4.9: for any $\varepsilon > 0$, for N large enough, we can bound for all $i = 1, \ldots, \ell$

$$\mathbb{E}\big[(Y_i - X_i)^2\big] \le \epsilon^2\, \mathbb{E}[Y_i^2]\,. \tag{4.39}$$

By the triangle inequality, this implies $\mathbb{E}[X_i^2]^{1/2} \le (1+\epsilon)\mathbb{E}[Y_i^2]^{1/2} \le 2\,\mathbb{E}[Y_i^2]^{1/2}$, hence

$$\begin{aligned} & \mathbb{E}\big[\big(Z^{(\mathrm{cg}')}_{N,K,M} - Z^{(\mathrm{cg})}_{N,K,M}\big)^2\big] \\ &\le \sum_{\ell=1}^{\infty} \sum_{\underline{j}\in\{1,\ldots,M\}^{\ell}_{\ll}} \sum_{(\underline{b},\underline{b}')\in\mathcal{B}^{\ell}(\underline{j})} \sum_{(\underline{z},\underline{z}')\in\mathcal{S}^{\ell}(\underline{b},\underline{b}')} \left(\epsilon^2 \sum_{h=1}^{\ell} 2^{2(\ell-h)} \right) \prod_{i=1}^{\ell} \mathbb{E}[Y_i^2] \\ &\le \epsilon^2 \sum_{\ell=1}^{\infty} 4^{\ell} \sum_{\underline{j}\in\{1,\ldots,M\}^{\ell}_{\ll}} \sum_{(\underline{b},\underline{b}')\in\mathcal{B}^{\ell}(\underline{j})} \sum_{(\underline{z},\underline{z}')\in\mathcal{S}^{\ell}(\underline{b},\underline{b}')} \prod_{i=1}^{\ell} \mathbb{E}[Y_i^2]\,, \end{aligned}$$

because $\sum_{h=1}^{\ell} 2^{2(\ell-h)} = \frac{4^{\ell}-1}{4-1} \le 4^{\ell}$. We now enlarge the sum ranges to obtain the factorization

$$\begin{aligned} & \mathbb{E}\big[\big(Z^{(\mathrm{cg}')}_{N,K,M} - Z^{(\mathrm{cg})}_{N,K,M}\big)^2\big] \\ &\quad \le \epsilon^2 \sum_{\ell=1}^{\infty} 4^{\ell} \sum_{1\le j_1<j_2<\ldots<j_{\ell}\le M} \prod_{i=1}^{\ell} \left\{ \sum_{b_i\le b'_i\in(N^{\frac{j_i-1}{M}}, N^{\frac{j_i}{M}}]} \sum_{z_i,z'_i\in\mathbb{Z}^2} \mathbb{E}[Y_i^2] \right\}. \end{aligned} \tag{4.40}$$

The following asymptotics on the term in brackets is proved in subsubsection 4.1.5.2.

Lemma 4.8. *For any $M \in \mathbb{N}$ and $j \in \{1, \ldots, M\}$ we have*

$$\lim_{N\to\infty} \left\{ \sum_{\substack{b\le b'\in(N^{\frac{j-1}{M}}, N^{\frac{j}{M}}] \\ z,z'\in\mathbb{Z}^2}} \mathbb{E}\big[X^{\mathrm{dom}}_{N,[0,b;b']}(0, z; z')^2\big] \right\} = I_M(j) = \int_{\frac{j-1}{M}}^{\frac{j}{M}} \frac{\hat{\beta}^2}{1-\hat{\beta}^2 s}\, \mathrm{d}s\,. \tag{4.41}$$

We can plug (4.41) into (4.40) (where the sum is finite: it can be stopped at $\ell = M$, since for $\ell > M$ there is no choice of $1 \le j_1 < j_2 < \ldots < j_\ell \le M$), which yields

$$\limsup_{N\to\infty} \mathbb{E}\big[\big(Z^{(\mathrm{cg}')}_{N,K,M} - Z^{(\mathrm{cg})}_{N,K,M}\big)^2\big] \le \epsilon^2 \sum_{\ell=1}^{\infty} 4^\ell \sum_{1\le j_1<j_2<\ldots<j_\ell\le M} \prod_{i=1}^{\ell} I_M(j_i)$$
$$\le \epsilon^2 \sum_{\ell=1}^{\infty} \frac{4^\ell}{\ell!}\bigg(\sum_{j=1}^{M} I_M(j)\bigg)^\ell \le \epsilon^2 \exp\bigg(4\sum_{j=1}^{M} I_M(j)\bigg) = \frac{\epsilon^2}{(1-\hat\beta^2)^4}\,. \tag{4.42}$$

This completes the proof of (4.22), since we can take $\epsilon > 0$ as small as we wish.

It only remains to justify (4.39). The following result is proved in subsubsection 4.1.5.3.

Lemma 4.9. *Given $K, M \in \mathbb{N}$ and $\epsilon > 0$, there exists $N_0 = N_0(\epsilon, M, K) < \infty$ such that for all $N > N_0$ the following bound holds:*

$$\mathbb{E}\big[\big(X^{\mathrm{dom}}_{N,[a,b;b']}(x,z;z') - X^{\mathrm{dom}}_{N,[0,b;b']}(0,z;z')\big)^2\big] \le \varepsilon^2\, \mathbb{E}\big[X^{\mathrm{dom}}_{N,[0,b;b']}(0,z;z')^2\big]\,, \tag{4.43}$$

uniformly for $(a,x),(b,z),(b',z') \in \mathbb{Z}^3_{\mathrm{even}} = \{y \in \mathbb{Z}^3 :\ y_1+y_2+y_3 \text{ is even}\}$ such that, for some $j \in \{1,\ldots,M\}$,

$$a \in [0, N^{\frac{(j-2)^+}{M}}]\,, \quad b \in (N^{\frac{j-1}{M}}, N^{\frac{j}{M}}]\,, \quad |x| \le K^2\sqrt{a}\,, \quad |z| \le K\sqrt{b}\,. \tag{4.44}$$

Step 3: proof of (4.23)**.** Recalling (4.11), we can rewrite $Z^{(\mathrm{diff})}_{N,M}$ in (4.10) as follows:

$$Z^{(\mathrm{diff})}_{N,M} = 1 + \sum_{\ell=1}^{\infty} \sum_{1\le j_1<j_2<\ldots<j_\ell\le M} \sum_{\substack{\underline{b},\underline{b}'\in\mathbb{N}^\ell:\\ b_i\le b_i'\in(N^{\frac{j_i-1}{M}},N^{\frac{j_i}{M}}]}} \sum_{\underline{z},\underline{z}'(\mathbb{Z}^2)^\ell} \prod_{i=1}^{\ell} X^{\mathrm{dom}}_{N,[0,b_i;b_i']}(0,z_i;z_i')\,. \tag{4.45}$$

By (4.37), we see that $Z^{(\mathrm{cg}')}_{N,K,M}$ is a *restriction* of the sum which defines $Z^{(\mathrm{diff})}_{N,M}$, therefore

$$\big\|Z^{(\mathrm{cg}')}_{N,K,M} - Z^{(\mathrm{diff})}_{N,M}\big\|^2_{L^2} = \big\|Z^{(\mathrm{diff})}_{N,M}\big\|^2_{L^2} - \big\|Z^{(\mathrm{cg}')}_{N,K,M}\big\|^2_{L^2}\,.$$

Then, to prove (4.23), it is enough to show that

$$\liminf_{M\to\infty}\ \liminf_{K\to\infty}\ \liminf_{N\to\infty}\ \mathbb{E}\big[\big(Z^{(\mathrm{cg}')}_{N,K,M}\big)^2\big] \ge \frac{1}{1-\hat\beta^2}\,, \tag{4.46}$$

$$\forall M \in \mathbb{N}: \qquad \limsup_{N\to\infty}\ \mathbb{E}\big[\big(Z^{(\mathrm{diff})}_{N,M}\big)^2\big] \le \frac{1}{1-\hat\beta^2}\,. \tag{4.47}$$

We first consider (4.46). Recalling (4.37), in analogy with (4.18), we can write

$$\begin{aligned}
&\mathbb{E}\big[\big(Z^{(\mathrm{cg}')}_{N,K,M}\big)^2\big] \\
&= 1 + \sum_{\ell=1}^{\infty} \sum_{\underline{j}\in\{1,\ldots,M\}^{\ell}_{\ll}} \sum_{(\underline{b},\underline{b}')\in\mathcal{B}^{\ell}(\underline{j})} \sum_{(\underline{z},\underline{z}')\in\mathcal{S}^{\ell}(\underline{b},\underline{b}')} \prod_{i=1}^{\ell} \mathbb{E}\big[X^{\mathrm{dom}}_{N,[0,b_i;b_i']}(0,z_i;z_i')^2\big] .
\end{aligned}$$

We can now use the quantity $\Xi_{N,M,K}(j_i)$ defined in (4.32) to bound

$$\mathbb{E}\big[\big(Z^{(\mathrm{cg}')}_{N,K,M}\big)^2\big] \;\ge\; 1 + \sum_{\ell=1}^{\infty} \sum_{\underline{j}\in\{1,\ldots,M\}^{\ell}_{\ll}} \prod_{i=1}^{\ell} \Xi_{N,M,K}(j_i) \,,$$

which coincides with the r.h.s. of (4.34). As a consequence, the bounds from (4.35) to (4.36) apply verbatim to $\mathbb{E}\big[\big(Z^{(\mathrm{cg}')}_{N,K,M}\big)^2\big]$ and show that (4.46) holds.

We finally consider (4.47), which we have essentially already proved. Indeed, note that $\mathbb{E}\big[\big(Z^{(\mathrm{diff})}_{N,K,M}\big)^2\big]$ is given by the second line of (4.40) where we replace ϵ^2 and 4^{ℓ} by 1. When we apply the limit (4.41), we obtain an analogue of (4.42), again with ϵ^2 and 4^{ℓ} replaced by 1, which yields precisely (4.47). This completes the proof of Lemma 4.8. □

4.1.3. Proof of Lemma 4.5. We recall that the event $A_{N,M}$ was defined in (4.13). In order to prove (4.14), it is enough to show that the following three relations hold:

$$\lim_{M\to\infty} \limsup_{N\to\infty} \mathbb{P}\Bigg(\bigg|\log Z^{(\mathrm{diff})}_{N,M} - \sum_{j=1}^{M} \big\{X^{\mathrm{dom}}_{N,M}(j) - \tfrac{1}{2} X^{\mathrm{dom}}_{N,M}(j)^2\big\}\bigg| > \varepsilon\,,\ A_{N,M}\Bigg) = 0\,, \tag{4.48}$$

$$\lim_{M\to\infty} \limsup_{N\to\infty} \bigg\|\sum_{j=1}^{M} X^{\mathrm{dom}}_{N,M}(j) - X^{\mathrm{dom}}_{N}\bigg\|_{L^2} = 0\,, \tag{4.49}$$

$$\lim_{M\to\infty} \limsup_{N\to\infty} \bigg\|\sum_{j=1}^{M} X^{\mathrm{dom}}_{N,M}(j)^2 - \mathbb{E}[(X^{\mathrm{dom}}_{N})^2]\bigg\|_{L^1} = 0\,. \tag{4.50}$$

We are going to exploit the following result.

Lemma 4.10. *Fix $\hat\beta < 1$. For every $M \in \mathbb{N}$ and $j \in \{1,\ldots,M\}$ we have*

$$\lim_{N\to\infty} \mathbb{E}\big[X^{\mathrm{dom}}_{N,M}(j)^2\big] \;=\; \int_{\frac{j-1}{M}}^{\frac{j}{M}} \frac{\hat\beta^2}{1-\hat\beta^2 s}\,\mathrm{d}s \;\le\; \frac{c}{M}\,, \qquad \text{with} \quad c = c_{\hat\beta} := \tfrac{\hat\beta^2}{1-\hat\beta^2}\,. \tag{4.51}$$

Moreover, there exist $p_{\hat\beta} > 2$ and $C = C_{\hat\beta} < \infty$ such that for all $2 < p \le p_{\hat\beta}$

$$\forall M \in \mathbb{N}\,,\ \forall j \in \{1,\ldots,M\}: \qquad \limsup_{N\to\infty} \mathbb{E}\,\big[|X^{\mathrm{dom}}_{N,M}(j)|^p\big] \le \frac{C}{M^{\frac{p}{2}}}\,. \tag{4.52}$$

Proof. Relation (4.51) is already proved in (4.41), by the definition (4.11) of $X_{N,M}^{\mathrm{dom}}(j)$.

Intuitively, the bound (4.52) holds because $\mathbb{E}\big[|X_{N,M}^{\mathrm{dom}}(j)|^p\big] \le C\,\mathbb{E}\big[X_{N,M}^{\mathrm{dom}}(j)^2\big]^{\frac{p}{2}}$ by the *hypercontractivity of polynomial chaos*. The details are presented in subsubsection 4.1.5.4. □

It only remains to prove (4.15) and the three relations (4.48)-(4.50).

Proof of (4.15). For any $p > 2$ we can bound, by Markov's inequality,

$$\mathbb{P}\big((A_{N,M})^c\big) \le \sum_{j=1}^{M} \mathbb{P}\big(|X_{N,M}^{\mathrm{dom}}(j)| > \tfrac{1}{2}\big) \le M\,2^p \max_{j\in\{1,\dots,M\}} \mathbb{E}\big[|X_{N,M}^{\mathrm{dom}}(j)|^p\big]\,,$$

and relation (4.15) follows directly by (4.52). □

Proof of (4.48). By (4.10) we can write $\log Z_{N,M}^{(\mathrm{diff})} = \sum_{j=1}^{M} \log(1 + X_{N,M}^{\mathrm{dom}}(j))$. If we fix $2 < p < \min\{3, p_{\hat\beta}\}$, with $p_{\hat\beta}$ as in Lemma 4.10, we can bound $|\log(1+x) - \{x - \frac{1}{2}x^2\}| \le c|x|^p$ for $|x| \le \frac{1}{2}$, hence

$$\begin{aligned}\mathbb{E}\Bigg[\bigg|\log Z_{N,M}^{(\mathrm{diff})} - \sum_{j=1}^{M}\big\{X_{N,M}^{\mathrm{dom}}(j) - \tfrac{1}{2}X_{N,M}^{\mathrm{dom}}(j)^2\big\}\bigg|\,\mathbb{1}_{A_{N,M}}\Bigg] &\le c\sum_{j=1}^{M}\mathbb{E}\big[|X_{N,M}^{\mathrm{dom}}(j)|^p\big] \\ &\le c\,\frac{C}{M^{\frac{p}{2}-1}}\,,\end{aligned}$$

which proves (4.48), by Markov's inequality. □

Proof of (4.49). The polynomial chaos $\sum_{j=1}^{M} X_{N,M}^{\mathrm{dom}}(j)$ contains less terms than X_N^{dom}, therefore to prove (4.49) it is enough to show that for any fixed $M \in \mathbb{N}$

$$\lim_{N\to\infty} \mathbb{E}\Bigg[\bigg(\sum_{j=1}^{M} X_{N,M}^{\mathrm{dom}}(j)\bigg)^2\Bigg] = \lim_{N\to\infty} \mathbb{E}\big[\big(X_N^{\mathrm{dom}}\big)^2\big] = \int_0^1 \frac{\hat\beta^2}{1-\hat\beta^2 s}\,\mathrm{d}s \tag{4.53}$$

where the last equality follows by (4.51), because X_N^{dom} equals $X_{N,M}^{\mathrm{dom}}(j)$ for $M = j = 1$ (cf. (4.5) with (4.11) and (4.8)). Since the variables $X_{N,M}^{\mathrm{dom}}(j)$'s are centered and independent, a further application of (4.51) yields

$$\mathbb{E}\Bigg[\bigg(\sum_{j=1}^{M} X_{N,M}^{\mathrm{dom}}(j)\bigg)^2\Bigg] = \sum_{j=1}^{M}\mathbb{E}\big[X_{N,M}^{\mathrm{dom}}(j)^2\big] \xrightarrow{N\to\infty} \sum_{j=1}^{M} I_M(j) = \int_0^1 \frac{\hat\beta^2}{1-\hat\beta^2 s}\,\mathrm{d}s\,, \tag{4.54}$$

as desired. This completes the proof. □

Proof of (4.50). In view of the first equalities in (4.53) and (4.54), it suffices to show that

$$\lim_{M\to\infty}\ \limsup_{N\to\infty}\ \bigg\|\sum_{j=1}^{M}\big\{X_{N,M}^{\mathrm{dom}}(j)^2 - \mathbb{E}\big[X_{N,M}^{\mathrm{dom}}(j)^2\big]\big\}\bigg\|_{L^1} = 0\,. \tag{4.55}$$

This is a weak law of large numbers for the independent random variables $W_j := X^{\mathrm{dom}}_{N,M}(j)^2$, which satisfy the following Lyapunov condition (by (4.52) with $q := p/2$):

$$\exists q = q_{\hat\beta} > 1,\ C = C_{\hat\beta} < \infty : \qquad \forall M \in \mathbb{N} \qquad \limsup_{N\to\infty} \max_{j\in\{1,\ldots,M\}} \mathbb{E}[W_j^q] \le \frac{C}{M^q}\,. \tag{4.56}$$

We prove (4.55) by truncation at level $T_M := M^{-\alpha}$, for an arbitrary $\alpha \in (\frac{1}{2}, 1)$. Note that

$$\begin{aligned}\Bigg\| \sum_{j=1}^{M} W_j \, \mathbb{1}_{\{W_j > T_M\}} \Bigg\|_{L^1} &= \sum_{j=1}^{M} \mathbb{E}\big[W_j \, \mathbb{1}_{\{W_j > T_M\}}\big] \\ &\le \sum_{j=1}^{M} \frac{\mathbb{E}[W_j^q]}{T_M^{q-1}} \le M^{1+\alpha(q-1)} \max_{j\in\{1,\ldots,M\}} \mathbb{E}[W_j^q]\,,\end{aligned}$$

which, by (4.56), vanishes as $N \to \infty$ followed by $M \to \infty$ provided $1 + \alpha(q - 1) - q < 0$, that is $\alpha < 1$. To prove (4.55) it only remains to show that

$$\lim_{M\to\infty} \limsup_{N\to\infty} \Bigg\| \sum_{j=1}^{M} \Big\{ W_j \, \mathbb{1}_{\{W_j \le T_M\}} - \mathbb{E}\big[W_j \, \mathbb{1}_{\{W_j \le T_M\}}\big] \Big\} \Bigg\|_{L^1} = 0\,.$$

It is simpler to prove convergence in L^2, because this follows by a variance computation:

$$\mathbb{V}\mathrm{ar}\Bigg(\sum_{j=1}^{M} W_j \, \mathbb{1}_{\{W_j \le T_M\}} \Bigg) = \sum_{j=1}^{M} \mathbb{V}\mathrm{ar}\big(W_j \, \mathbb{1}_{\{W_j \le T_M\}}\big) \le M\, T_M^2 = M^{1-2\alpha}\,,$$

which vanishes as $M \to \infty$ provided $1 - 2\alpha < 0$, that is $\alpha > \frac{1}{2}$. □

4.1.4. Proof of Lemma 4.6. We first prove (4.16). In view of (4.14) and (4.15), it suffices to show that

$$\forall \epsilon > 0 : \qquad \lim_{N\to\infty} \mathbb{P}\big(\big| \log Z_N^{\beta_N} - \log Z_{N,M}^{(\mathrm{diff})} \big| > \epsilon\,,\ A_{N,M} \big) = 0\,, \tag{4.57}$$

where we recall that the event $A_{N,M} \subseteq \{Z_{N,M}^{(\mathrm{diff})} > 0\}$ was defined in (4.13).

For any $a, b \in \mathbb{R}$ and $\epsilon, \eta \in (0, 1)$ we have the following inclusion:

$$\{|\log a - \log b| > \varepsilon\} \subseteq \{b < 2\eta\varepsilon\} \cup \{|a - b| > \eta\epsilon^2\}\,.$$

Indeed, if both $b \ge 2\eta\epsilon$ and $|a - b| \le \eta\epsilon^2$, then $a \ge b - \eta\epsilon^2 \ge 2\eta\epsilon - \eta\epsilon^2 \ge \eta\epsilon$, so that both $a, b \in [\eta\epsilon, \infty)$, hence $|\log a - \log b| = |\int_a^b \frac{1}{x} \mathrm{d}x| \le \frac{1}{\eta\epsilon}|b - a| \le \frac{1}{\eta\epsilon}\eta\epsilon^2 = \epsilon$. It follows that

$$\begin{aligned}\mathbb{P}\big(\big| \log Z_N^{\beta_N} - \log Z_{N,M}^{(\mathrm{diff})} \big| > \epsilon,\ A_{N,M} \big) \le{}& \mathbb{P}\big(Z_{N,M}^{(\mathrm{diff})} < 2\eta\epsilon,\ A_{N,M} \big) \\ &+ \mathbb{P}\big(\big| Z_N^{\beta_N} - Z_{N,M}^{(\mathrm{diff})} \big| > \eta\epsilon^2 \big)\end{aligned}$$

and note that the second term in the r.h.s. vanishes as $N \to \infty$ followed by $M \to \infty$, for any fixed $\epsilon, \eta \in (0,1)$, thanks to (4.12). It remains to show that

$$\forall \epsilon > 0 : \qquad \lim_{\eta \downarrow 0} \limsup_{M\to\infty} \limsup_{N\to\infty} \mathbb{P}\big(Z^{(\mathrm{diff})}_{N,M} < 2\eta\epsilon,\, A_{N,M}\big) \,=\, 0\,.$$

To this purpose, we can bound

$$\begin{aligned}\mathbb{P}\big(Z^{(\mathrm{diff})}_{N,M} < 2\eta\epsilon,\, A_{N,M}\big) \le \mathbb{P}\Big(\big|\log Z^{(\mathrm{diff})}_{N,M} - \big\{X^{\mathrm{dom}}_N - \tfrac{1}{2}\mathbb{E}[(X^{\mathrm{dom}}_N)^2]\big\}\big| > 1,\, A_{N,M}\Big) \\ + \mathbb{P}\Big(X^{\mathrm{dom}}_N - \tfrac{1}{2}\mathbb{E}[(X^{\mathrm{dom}}_N)^2] < \log(2\eta\epsilon) + 1\Big)\end{aligned}$$

and note that the first term in the r.h.s. vanishes as $N \to \infty$ followed by $M \to \infty$, by (4.14). To show that the second term vanishes as $N \to \infty$ followed by $\eta \downarrow 0$, we fix $\eta > 0$ small, so that $\log(2\eta\epsilon) + 1 < 0$, and we apply Markov's inequality to bound, for some $C < \infty$,

$$\begin{aligned}\mathbb{P}\Big(X^{\mathrm{dom}}_N - \tfrac{1}{2}\mathbb{E}[(X^{\mathrm{dom}}_N)^2] < \log(2\eta\epsilon) + 1\Big) &\le \frac{\mathbb{E}\big[\big(X^{\mathrm{dom}}_N - \tfrac{1}{2}\mathbb{E}[(X^{\mathrm{dom}}_N)^2]\big)^2\big]}{|\log(2\eta\epsilon) + 1|^2} \\ &\le \frac{C}{|\log(2\eta\epsilon) + 1|^2}\,,\end{aligned}$$

because $\mathbb{E}\big[\big(X^{\mathrm{dom}}_N - \tfrac{1}{2}\mathbb{E}[(X^{\mathrm{dom}}_N)^2]\big)^2\big]$ converges to a finite limit as $N \to \infty$, see (4.53).

It only remains to prove (4.17). The second bound in (4.17) follows by (4.52), because we already remarked that $X^{\mathrm{dom}}_N = X^{\mathrm{dom}}_{N,M}(j)$ with $j = M = 1$, see (4.5) and (4.11), (4.8). The first bound in (4.17) was proved in [**CSZ20**] (see equations (3.12), (3.14) and the lines following (3.16)) exploiting *concentration of measure for the left tail of* $\log Z_N$. ☐

4.1.5. Technical results. We collect below the proofs of some technical results we applied in the previous subsections.

4.1.5.1. *Proof of Lemma 4.7.* We are going to prove that there is a constant $C < \infty$ such that, for any given $M, K \in \mathbb{N}$ and $j \in \{1, \ldots, M\}$, we have

$$\liminf_{N\to\infty} \,\Xi_{N,M,K}(j) \,\ge\, \big(1 - (\hat\beta^2)^K\big) \int_{\frac{j-1}{M}}^{\frac{j}{M}} \frac{\hat\beta^2(1 - \frac{C}{K^2})}{1 - \hat\beta^2(1 - \frac{C}{K^2})\, s}\, \mathrm{d}s\,, \tag{4.58}$$

which clearly implies (4.33).

Given $a, b \in \mathbb{N}_0$ as in the range of the sums (4.32), we note that for large N:

$$a \le \tfrac{1}{4}K^{-2}b\,. \tag{4.59}$$

This clearly holds if $a = 0$, hence for $j = 1$, because $a \le N^{\frac{(j-2)^+}{M}} = 0$, while for $j \ge 2$ from $a \le N^{\frac{j-2}{M}}$ and $b > N^{\frac{j-1}{M}}$ we get $a \le N^{-\frac{1}{M}}b \le \frac{1}{4}K^{-2}b$ for large N, say $N \geq (2K)^{2M}$. By (4.20), for fixed a, b and x, the sums over $b' \in [b, Kb]$ and $z, z' \in \mathbb{Z}^2$ in (4.32) equal

$$\sum_{b' \in [b,Kb]} \sum_{\substack{|z| \leq K\sqrt{b} \\ |z'| \leq K^2\sqrt{b}}} \mathbb{E}\Big[\big(X^{\mathrm{dom}}_{N,[a,b,b']}(x,z;z') \big)^2 \Big]$$

$$= \sum_{k=1}^{\infty} (\sigma_N^2)^k \sum_{|x_1| \leq K\sqrt{b}} q_{b-a}(x_1 - x)^2 \sum_{\substack{b < n_2 < \ldots < n_k \leq Kb: \\ \max\{n_2 - b, \ldots, n_k - n_{k-1}\} \leq b \\ x_2, \ldots, x_k \in \mathbb{Z}^2 :\ |x_k| \leq K^2\sqrt{b}}} \prod_{i=2}^{k} q_{n_i - n_{i-1}}(x_i - x_{i-1})^2. \tag{4.60}$$

We get a lower bound by keeping just the first K terms in the sum over $k \in \mathbb{N}$. Moreover:

- we remove the constraint $n_k \leq Kb$ (because $\max\{n_2 - b, \ldots, n_k - n_{k-1}\} \leq b$ already yields $n_k = b + \sum_{i=2}^{k}(n_i - n_{i-1}) \leq Kb$) and sum freely over the increments

$$m_i := n_i - n_{i-1} \in \{1, \ldots, b\} \qquad \text{for } i = 2, \ldots, k\,; \tag{4.61}$$

- we change variables to $y_1 := x_1 - x$ and $y_i := x_i - x_{i-1}$ for $i \geq 2$, that we restrict to

$$|y_1| \leq \tfrac{1}{2}K\sqrt{b-a} \qquad \text{and} \qquad |y_i| \leq \tfrac{1}{2}K\sqrt{m_i} \qquad \text{for } i \geq 2\,,$$

which imply both $|x_1| \leq K\sqrt{b}$ and $|x_k| \leq K^2\sqrt{b}$ as required by (4.60). Indeed, recalling that $|x| \leq K^2\sqrt{a} \leq \frac{1}{2}K\sqrt{b}$ by (4.32) and (4.59), we obtain

$$|x_1| \leq |y_1| + |x| \leq \frac{1}{2}K\sqrt{b-a} + \frac{1}{2}K\sqrt{b} \leq K\sqrt{b}\,,$$

$$|x_k| \leq |x_1| + \sum_{i=2}^{k} |y_i| \leq K\sqrt{b} + (K-1)\frac{1}{2}K\sqrt{b} \leq K^2\sqrt{b}\,.$$

These restrictions yield the following lower bound on (4.60):

$$\sum_{k=1}^{K} (\sigma_N^2)^k \Bigg(\sum_{|y_1| \leq \frac{1}{2}K\sqrt{b-a}} q_{b-a}(y_1)^2 \Bigg) \prod_{i=2}^{k} \Bigg(\sum_{m_i=1}^{b} \sum_{|y_i| \leq \frac{1}{2}K\sqrt{m_i}} q_{m_i}(y_i)^2 \Bigg). \tag{4.62}$$

Recalling that u_n and R_N are defined in (3.14) and (3.15), we define restricted versions

$$u_n^{(K)} := \sum_{|y| \leq \frac{1}{2}K\sqrt{n}} q_n(y)^2\,, \qquad R_N^{(K)} := \sum_{m=1}^{N} u_m^{(K)} = \sum_{m=1}^{N} \sum_{|y| \leq \frac{1}{2}K\sqrt{m}} q_m(y)^2\,, \tag{4.63}$$

so that we can rewrite (4.62) more compactly as follows:

$$\sum_{k=1}^{K} (\sigma_N^2)^k\, u_{b-a}^{(K)} \big(R_b^{(K)}\big)^{k-1} \;=\; \sigma_N^2\, u_{b-a}^{(K)} \frac{1 - \big(\sigma_N^2 R_b^{(K)}\big)^K}{1 - \sigma_N^2 R_b^{(K)}}\,.$$

Bounding $(\sigma_N^2 R_b^{(K)})^K \le (\sigma_N^2 R_N)^K$ in the numerator and recalling (4.32), we obtain

$$\Xi_{N,M,K}(j) \;\ge\; \big(1-\big(\sigma_N^2 R_N\big)^K\big) \inf_{0\le a\le N^{\frac{(j-2)^+}{M}}} \sum_{b\in(N^{\frac{j-1}{M}}+\log N,\frac{1}{K}N^{\frac{j}{M}}]} \frac{\sigma_N^2\, u_{b-a}^{(K)}}{1-\sigma_N^2\, R_b^{(K)}}\,, \tag{4.64}$$

where we restricted the sum range to $b \in (N^{\frac{j-1}{M}}+\log N, \frac{1}{K}N^{\frac{j}{M}}]$ for later convenience.

We now claim that for some $C<\infty$ we have, for n, N large enough,

$$u_n^{(K)} \ge (1-\tfrac{C}{K^2})\,\frac{1}{\pi}\,\frac{1}{n} \qquad \Longrightarrow \qquad R_N^{(K)} \ge (1-\tfrac{C}{K^2})\,\frac{1}{\pi}\,\log N\,. \tag{4.65}$$

This follows by (4.63) writing $u_n^{(K)} = u_n - \sum_{|y|>\frac{1}{2}K\sqrt{n}} q_n(y)^2$, recalling that $u_n \sim \frac{1}{\pi}\frac{1}{n}$ by (3.14), bounding $\sup_{y\in\mathbb{Z}^2} q_n(y) \le \frac{c_1}{n}$ by the local limit theorem (see (3.16) below) and then estimating

$$\sum_{|y|>\frac{1}{2}K\sqrt{n}} q_n(y) = \mathrm{P}(|S_n| > \tfrac{1}{2}K\sqrt{n}) \le 4\,\frac{\mathrm{E}[|S_n|^2]}{K^2\,n} = \frac{4}{K^2}\,.$$

We can plug the bounds (4.65) into (4.64) because, uniformly for a, b in the sum range, we have $b \ge b-a \ge \log N \to \infty$ as $N\to\infty$. Since $\sigma_N^2 \sim \beta_N^2 \sim \pi\hat\beta^2/\log N$, see (3.18) and (3.3), for large N we have (possibly enlarging C)

$$\frac{\sigma_N^2\, u_{b-a}^{(K)}}{1-\sigma_N^2 R_b^{(K)}} \;\ge\; (1-\tfrac{C}{K^2})\,\frac{1}{b-a}\,\frac{\frac{\hat\beta^2}{\log N}}{1-\frac{\hat\beta^2}{\log N}\,(1-\frac{C}{K^2})\,\log b}\,. \tag{4.66}$$

The r.h.s. is a decreasing function of $b-a$, hence we get a lower bound setting $a=0$. By monotonicity in b, we can then bound the sum in (4.64) by an integral:

$$\Xi_{N,M,K}(j) \;\ge\; (1-\tfrac{C}{K^2})\,\big(1-(\hat\beta^2)^K\big) \int_{\lceil N^{\frac{j-1}{M}}+\log N\rceil}^{\frac{1}{K}N^{\frac{j}{M}}} \frac{1}{x}\,\frac{\frac{\hat\beta^2}{\log N}}{1-\frac{\hat\beta^2}{\log N}\,(\log x)\,(1-\frac{C}{K^2})}\,\mathrm{d}x\,.$$

With the change of variable $x=N^s$, the integral equals

$$\int_{a_N}^{b_N} \frac{\hat\beta^2}{1-\hat\beta^2 s\,(1-\frac{C}{K^2})}\,\mathrm{d}s \quad \text{with} \quad a_N := \frac{\log\lceil N^{\frac{j-1}{M}}+\log N\rceil}{\log N}\,,\; b_N := \frac{\log(\frac{1}{K}N^{\frac{j}{M}})}{\log N}\,.$$

Since $\lim_{N\to\infty} a_N = \frac{j-1}{M}$ and $\lim_{N\to\infty} b_N = \frac{j}{M}$, we have proved (4.58). □

4.1.5.2. *Proof of Lemma 4.8.* A lower bound for (4.41) is already provided by (4.33), hence it suffices to prove a matching upper bound. By (4.20) with

$(a,x) = (0,0)$, we can write

$$\sum_{b \le b' \in (N^{\frac{j-1}{M}}, N^{\frac{j}{M}}]} \sum_{z,z' \in \mathbb{Z}^2} \mathbb{E}\big[X^{\mathrm{dom}}_{N,[0,b;b']}(0,z;z')^2\big] \le \sum_{k=1}^{\infty} (\sigma_N^2)^k \sum_{b \in (N^{\frac{j-1}{M}}, N^{\frac{j}{M}}]} \sum_{z \in \mathbb{Z}^2} q_b(z)^2$$
$$\times \sum_{\substack{b=:n_1<n_2<\ldots<n_k<\infty \\ \max\{n_2-n_1,\ldots,n_k-n_{k-1}\} \le b}} \sum_{\substack{x_1:=z \\ x_2,\ldots,x_k \in \mathbb{Z}^2}} \prod_{i=2}^{k} q_{n_i-n_{i-1}}(x_i - x_{i-1})^2 . \tag{4.67}$$

We can sum over the space variables: by (3.14) and (3.15), the r.h.s. equals

$$\sum_{k=1}^{\infty} (\sigma_N^2)^k \sum_{b \in (N^{\frac{j-1}{M}}, N^{\frac{j}{M}}]} u_b \, (R_b)^{k-1} = \sum_{b \in (N^{\frac{j-1}{M}}, N^{\frac{j}{M}}]} \frac{\sigma_N^2 \, u_b}{1 - \sigma_N^2 \, R_b} . \tag{4.68}$$

Since $\sigma_N^2 \, u_b \sim \frac{\hat\beta^2}{\log N} \frac{1}{b}$ and $\sigma_N^2 \, R_b \sim \frac{\hat\beta^2}{\log N} \log b$, as $N \to \infty$ the r.h.s. of (4.68) is asymptotic to

$$\sum_{b \in (N^{\frac{j-1}{M}}, N^{\frac{j}{M}}]} \frac{\frac{\hat\beta^2}{\log N} \, \frac{1}{b}}{1 - \frac{\hat\beta^2}{\log N} \log b} \sim \int_{N^{\frac{j-1}{M}}}^{N^{\frac{j}{M}}} \frac{\frac{\hat\beta^2}{\log N} \, \frac{1}{x}}{1 - \frac{\hat\beta^2}{\log N} \log x} \, \mathrm{d}x = \int_{\frac{j-1}{M}}^{\frac{j}{M}} \frac{\hat\beta^2}{1 - \hat\beta^2 \, s} \, \mathrm{d}s , \tag{4.69}$$

by the change of variable $x = N^s$. This completes the proof of (4.41). □

4.1.5.3. *Proof of Lemma 4.9.* We can assume that $j \ge 2$, because if $j = 1$ we have $a = 0$ and $x = 0$, see (4.44), hence (4.43) trivially holds.

Note that by (4.8) we can write

$$\mathbb{E}\big[X^{\mathrm{dom}}_{N,[a,b;b']}(x,z;z')^2\big] = q_{b-a}(z-x)^2 \, F_{N,[b;b']}(z;z') ,$$

where we set

$$F_{N,[b;b']}(z;z') := \sum_{k=1}^{\infty} (\sigma_N^2)^k \sum_{\substack{b=:n_1<n_2<\ldots<n_{k-1}<n_k=b' \\ 1 \le n_2-n_1,\ldots,n_k-n_{k-1} \le b}} \sum_{\substack{x_1:=z, \, x_k:=z' \\ x_2,\ldots,x_{k-1} \in \mathbb{Z}^2}} \prod_{i=2}^{k} q_{n_i-n_{i-1}}(x_i - x_{i-1})^2 .$$

The key point is that $F_{N,[b;b']}(z;z')$ does not depend on (a,x). It follows that

$$\mathbb{E}\Big[\big(X^{\mathrm{dom}}_{N,[a,b;b']}(x,z;z') - X^{\mathrm{dom}}_{N,[0,b;b']}(0,z;z')\big)^2\Big] = \big(q_{b-a}(z-x) - q_b(z)\big)^2 \, F_{N,[b;b']}(z;z') ,$$

therefore, to prove (4.43), it is enough to show that for $K, M \in \mathbb{N}$ and $\epsilon > 0$ there is $N_0 = N_0(\epsilon, M, K) < \infty$ such that, for $N > N_0$ and for a, b, x, z as in (4.44), we have

$$\left| 1 - \frac{q_b(z)}{q_{b-a}(z-x)} \right| \le \epsilon . \tag{4.70}$$

We recall the local limit theorem [**LL10**, Theorem 2.1.3]: as $n \to \infty$, uniformly for $y \in \mathbb{Z}^2$,

$$q_n(y) = \frac{1}{n/2}\big(g\big(\tfrac{y}{\sqrt{n/2}}\big) + o(1)\big)\, 2\,\mathbb{1}_{(n,y)\in\mathbb{Z}^3_{\rm even}} \qquad \text{with} \qquad g(x) := \frac{e^{-|x|^2/2}}{2\pi}\,.$$

In particular, for $(n, y) \in \mathbb{Z}^3_{\rm even}$ in the "diffusive regime" we can write

$$q_n(y) = \frac{4}{n}\, g\big(\tfrac{y}{\sqrt{n/2}}\big)\big(1 + o(1)\big) \qquad \text{for } |y| = O(\sqrt{n})\,.$$

Note that a, b, x, z as in (4.44) satisfy (recall that $j \geq 2$)

$$0 \leq a \leq N^{\frac{j-2}{M}} \leq N^{-\frac{1}{M}} b\,, \qquad |z| \leq K\sqrt{b}\,, \qquad |x| \leq K^2\sqrt{a} \leq K^2\sqrt{N^{-\frac{1}{M}}}\,\sqrt{b}\,. \tag{4.71}$$

It follows that for any $K, M \in \mathbb{N}$, uniformly for a, b, x, z as in (4.44), we have as $N \to \infty$

$$a = o(b)\,, \qquad |z| = O(\sqrt{b})\,, \qquad |x| = o(\sqrt{b})\,,$$

which in turn imply that $|z - x| \leq |z| + |x| = O(\sqrt{b}) = O(\sqrt{b-a})$ and hence, by (5.33),

$$\frac{q_b(z)}{q_{b-a}(z-x)} = \frac{b-a}{b}\exp\left(\frac{|z-x|^2}{b-a} - \frac{|z|^2}{b}\right)(1 + o(1)) \xrightarrow[N\to\infty]{} 1\,.$$

This completes the proof of (4.70), hence of (4.43). □

4.1.5.4. *Proof of* (4.52). The random variables η_N in (3.3) satisfy $\sup_N \mathbb{E}[|\eta_N|^{\overline{p}}] < \infty$ for all $\overline{p} < \infty$, by the assumption (3.1) (see [**CSZ17a**, eq. (6.7)]). We can then estimate $\mathbb{E}\big[|X^{\rm dom}_{N,M}(j)|^p\big]^{\frac{2}{p}}$ by the hypercontractive bound (2.27), which gives rise to the r.h.s. of (4.67) with σ_N^2 replaced by $C_p\,\sigma_N^2$. We can then follow the proof of Lemma 4.8 in Appendix 4.1.5.2 verbatim though (4.68) and (4.69), where we note that the replacement of σ_N^2 by $C_p\,\sigma_N^2$ amounts to replace $\hat\beta^2$ by $C_p\,\hat\beta^2$, by (3.3) and (3.18). Since $\hat\beta < 1$ and $\lim_{p\downarrow 2} C_p = 1$, see [**CSZ20**, Theorem B.1], we can fix $p_{\hat\beta} > 2$ and $\widetilde{c} = \widetilde{c}_{\hat\beta} < 1$ such that for all $2 < p \leq p_{\hat\beta}$ we can bound $C_p\hat\beta^2 \leq \widetilde{c} < 1$, hence

$$\limsup_{N\to\infty}\ \mathbb{E}\big[|X^{\rm dom}_{N,M}(j)|^p\big]^{\frac{2}{p}} \leq \int_{\frac{j-1}{M}}^{\frac{j}{M}} \frac{C_p\hat\beta^2}{1 - C_p\hat\beta^2 s}\,\mathrm{d}s \leq \frac{\widetilde{c}/(1-\widetilde{c})}{M}\,, \tag{4.72}$$

which completes the proof. □

4.2. Asymptotic Gaussianity: proof of Theorem 4.2

We have already noticed in (4.53) that

$$\lim_{N\to\infty} \mathbb{E}\big[(X_N^{\rm dom})^2\big] = \sigma^2 := \log\tfrac{1}{1-\hat\beta^2}\,, \tag{4.73}$$

which follows by (4.51), because $X_N^{\mathrm{dom}} = X_{N,1}^{\mathrm{dom}}(1)$ (see (4.5) and (4.11), (4.8)). Therefore we only need to prove that

$$X_N^{\mathrm{dom}} \xrightarrow{d} \mathcal{N}\big(0, \sigma^2\big)\,. \tag{4.74}$$

We can apply Theorem 2.2 to the polynomial chaos X_N^{dom} defined in (4.5). As in the proof of Theorem 3.13, we can cast X_N^{dom} in the form (2.7) with $\mathbb{T} := \mathbb{N} \times \mathbb{Z}^2$ and $\eta_t^N = \eta_N(m,z)$ defined in (3.3), while for $A := \{t_1, \ldots, t_k\} = \{(n_1, x_1), \ldots, (n_k, x_k)\} \subseteq \mathbb{T}$ we set

$$q_N(A) = (\sigma_N)^k \, \mathbb{1}_{\left\{\substack{0 =: n_0 < n_1 < \ldots < n_k \le N \\ \max\{n_2 - n_1, \ldots, n_k - n_{k-1}\} \le n_1 - n_0}\right\}} \prod_{j=1}^k q_{n_j - n_{j-1}}(x_j - x_{j-1})\,.$$

By Theorem 2.2, to prove (4.74) we need to verify the following conditions:

(1) *Limiting second moment:* we already showed that $\lim_{N\to\infty} \mathbb{E}[(X_N^{\mathrm{dom}})^2] = \sigma^2$, see (4.73).

(2) *Subcriticality*: we need to show that

$$\lim_{K\to\infty} \limsup_{N\to\infty} \sum_{\substack{A \subset \mathbb{T} \\ |A| \ge K}} q_N(A)^2 = 0\,. \tag{4.75}$$

Arguing as in (4.30), we can enlarge the sums to $1 \le n_j - n_{j-1} \le N$ and remove the constraint $\max\{n_2 - n_1, \ldots, n_k - n_{k-1}\} \le n_1 - n_0$, to get the bound

$$\begin{aligned} \sum_{\substack{A \subset \mathbb{T} \\ |A| \ge K}} q_N(A)^2 &\le \sum_{k=K}^{\infty} (\sigma_N^2)^k \sum_{\substack{1 \le n_j - n_{j-1} \le N \\ j=1,\ldots,k}} \sum_{\substack{x_1, \ldots, x_k \in \mathbb{Z}^2 \\ x_0 := 0}} \prod_{j=1}^k q_{n_j - n_{j-1}}(x_j - x_{j-1})^2 \\ &= \sum_{k=K}^{\infty} (\sigma_N^2)^k \Big(\sum_{n=1}^N \sum_{x \in \mathbb{Z}^2} q_n(x)^2 \Big)^k = \sum_{k=K}^{\infty} (\sigma_N^2\, R_N)^k \xrightarrow{N\to\infty} \frac{(\hat\beta^2)^K}{1 - \hat\beta^2}\,, \end{aligned}$$

from which (4.75) follows.

(3) *Spectral localization*: given $M, N \in \mathbb{N}$, we define disjoint subsets $\mathbb{B}_j \subseteq \mathbb{T}$ by

$$\mathbb{B}_j := \big((N^{\frac{j-1}{M}}, N^{\frac{j}{M}}] \cap \mathbb{N} \big) \times \mathbb{Z}^2 \qquad \text{for } j = 1, \ldots, M\,,$$

and, recalling that $\sigma_N^2(\mathbb{B}_j) := \sum_{A \subset \mathbb{B}_j} q_N(A)^2$, see (2.9), we need to show that

$$\lim_{M\to\infty} \sum_{j=1}^M \lim_{N\to\infty} \sigma_N^2(\mathbb{B}_j) = \sigma^2 \quad \text{and} \quad \lim_{M\to\infty} \Big\{ \max_{j=1,\ldots,M} \limsup_{N\to\infty} \sigma_N^2(\mathbb{B}_j) \Big\} = 0\,.$$

For this it suffices to note that $\sigma_N^2(\mathbb{B}_j) = \mathbb{E}[X_{N,M}^{\mathrm{dom}}(j)^2]$ and then to apply (4.51).

The proof of Theorem 4.2 is completed. □

CHAPTER 5

Gaussian fluctuations in the quasi–critical regime

So far we have presented and discussed results for polynomial and Wiener chaos expansions under the assumption of *subcriticality*. In a general context, we showed that this is equivalent to require that only a finite number of fixed chaos contibutes to the whole limiting second moment of the expansion. For 2d directed polymers, this terminology has a precise meaning, closely linked to the choice of how we rescale the disorder strength $\beta = \beta_N \to 0$ as $N \to \infty$. By tuning the interaction strength as

$$\beta_N \sim \frac{\pi\,\hat{\beta}}{\sqrt{\log N}} \qquad \text{as } N \to \infty\,,$$

the *subcritical regime* corresponds to take the disorder parameter $\hat{\beta} \in (0,1)$.

The goal of this chapter is to explore and study the 2d directed polymer and its partition function *beyond the subcritical regime*, where the setting is more subtle and many tools exploited in the previous chapter no longer apply.

For convenience and completeness, let us recall the main notations. We consider the partition function of the $2d$ directed polymer in random environment:

$$Z^{\omega}_{N,\beta}(z) := \mathrm{E}\big[e^{\sum_{n=1}^{N}\{\beta\omega(n,S_n)-\lambda(\beta)\}}\,\big|\, S_0 = z\big]\,, \tag{5.1}$$

where $N \in \mathbb{N}$ is the system size, $\beta > 0$ is the disorder strength, and:

- $S = (S_n)_{n\geq 0}$ is the simple random walk on $\mathbb{Z}^2$ with law P;
- $\omega = (\omega(n,z))_{n\in\mathbb{N},\, z\in\mathbb{Z}^2}$ are i.i.d. random variables with law $\mathbb{P}$, independent of S, with

$$\mathbb{E}[\omega] = 0\,, \qquad \mathbb{E}[\omega^2] = 1\,, \qquad \lambda(\beta) := \log \mathbb{E}[e^{\beta\omega}] < \infty \quad \text{for } \beta > 0\,, \tag{5.2}$$

  which play the role of *disorder* (or *random environment*).

Note that we subtract $\lambda(\beta)$ in (5.1) to fix the expectation of $Z^{\omega}_{N,\beta}(z)$, namely

$$\mathbb{E}\big[Z^{\omega}_{N,\beta}(z)\big] = 1\,. \tag{5.3}$$

It is known since [**CSZ17b**] that a phase transition is observed when the disorder strength $\beta = \beta_N$ is suitably rescaled as $N \to \infty$. Using the more convenient parameter

$$\sigma_\beta := \sqrt{\mathbb{V}\mathrm{ar}[e^{\beta\omega-\lambda(\beta)}]} = \sqrt{e^{\lambda(2\beta)-2\lambda(\beta)} - 1}\,, \tag{5.4}$$

with the same leading behavior $\sigma_\beta \sim \beta$ as $\beta \downarrow 0$ (since $\lambda(\beta) \sim \frac{1}{2}\beta^2$), we consider $\beta = \beta_N$ such that

$$\sigma_{\beta_N}^2 = \frac{\hat\beta^2}{R_N} \sim \frac{\hat\beta^2\,\pi}{\log N}\,, \qquad \text{with} \quad \hat\beta \in (0,\infty)\,, \tag{5.5}$$

where R_N denotes the expected replica overlap of two independent random walks S, S':

$$R_N := \mathrm{E}^{\otimes 2}\bigg[\sum_{n=1}^N \mathbb{1}_{\{S_n = S'_n\}}\bigg] = \sum_{n=1}^N \mathrm{P}(S_{2n}=0) = \frac{\log N}{\pi} + O(1) \tag{5.6}$$

(the last equality follows by the local limit theorem (5.32)). As mentioned above and in Chapter 3, it was shown in [**CSZ17b**] that in the so-called *subcritical regime* $\hat\beta < 1$ the partition function has *Edwards-Wilkinson fluctuations*: for any $\varphi \in C_c(\mathbb{R}^2)$, the diffusively rescaled and averaged partition function

$$Z_{N,\beta}^\omega(\varphi) := \int_{\mathbb{R}^2} Z_{N,\beta}^\omega\big(\lfloor Nx \rfloor\big)\,\varphi(x)\,\mathrm{d}x \tag{5.7}$$

has Gaussian fluctuations as $N \to \infty$:

$$\forall \hat\beta \in (0,1): \qquad \sqrt{R_N}\,\big\{Z_{N,\beta_N}^\omega(\varphi) - \mathbb{E}[Z_{N,\beta_N}^\omega(\varphi)]\big\} \xrightarrow[N\to\infty]{d} \mathcal{N}\big(0\,,\sigma_{\varphi,\hat\beta}^2\big)\,, \tag{5.8}$$

for an explicit limiting variance $\sigma_{\varphi,\hat\beta}^2 > 0$. To be precise, the result proved in [**CSZ17b**, Theorem 2.13] involves a space-time average, but the analogous result for the space average as in (5.7) follows by the same arguments (see [**CSZ20**]).

We point out that the restriction in (5.8) to the *subcritcal regime* $\hat\beta < 1$ is necessary, because the limiting variance $\sigma_{\varphi,\hat\beta}^2$ diverges as $\hat\beta \uparrow 1$. Indeed, in the *critical regime* $\hat\beta = 1$ a very different picture emerges, as recently shown in [**CSZ21+**]: the averaged partition function $Z_{N,\beta_N}^\omega(\varphi)$ in (5.7), *with no need of rescaling and centering*, converges in distribution as $N \to \infty$ to a *non-Gaussian* limit $\mathscr{Z}(\varphi)$ (indeed, we have $\mathscr{Z}(\varphi) \geq 0$ for $\varphi \geq 0$). The same result was proven, more generally, in the *critical window* $\hat\beta^2 = 1 + O(\frac{1}{\log N})$ around $\hat\beta = 1$.

In view of this discrepancy, it is natural to wonder what happens between the *subcritcal regime* $\hat\beta < 1$ and the *critcal regime* $\hat\beta = 1$ (before the critical window). To explore this gap, we should let $\hat\beta \uparrow 1$ *but slower than* $\hat\beta^2 = 1 + O(\frac{1}{\log N})$: recalling (5.5), we then consider

$$\sigma_{\beta_N}^2 = \frac{1}{R_N}\bigg(1 - \frac{\theta_N}{\log N}\bigg) \qquad \text{for some} \quad 1 \ll \theta_N \ll \log N\,, \tag{5.9}$$

where we recall that R_N is defined in (5.6). We call this regime of $\beta = \beta_N$ *quasi-critical*, as it interpolates between the subcritcal and the critical regimes. Indeed, by choosing $\theta_N = (1-\hat\beta^2)\log N$ we come back to the subcritical regime (recall (5.5)); on the other hand if we set $\theta_N = \theta + o(1)$ with $\theta \in \mathbb{R}$ we reach the critical regime explored in [**CSZ21+**]. In the middle, the quasi-critical regime allows us to approach the critical value $\hat\beta = 1$ *arbitrarily slowly* according to the (almost) free choice of θ_N, which has to diverge as $N \to \infty$ slower than $\log N$. In this way, we are able to fully study the asymptotic behaviour closer and closer the critical regime, without ever reaching it.

In particular, we are going to prove that *in the whole quasi-critical regime* (5.9) *of* $\beta = \beta_N$, the averaged partition function $Z^{\omega}_{N,\beta_N}(\varphi)$ in (5.7) has *Gaussian fluctuations* after centering and suitable rescaling, i.e. replacing $\sqrt{R_N}$ in (5.8) by the smaller factor $\sqrt{\theta_N} \ll \sqrt{R_N}$. This is our main result.

THEOREM 5.1 (Quasi-critical Edwards-Wilkinson fluctuations). *Let* $Z^{\omega}_{N,\beta}(\varphi)$ *denote the diffusively rescaled and averaged partition function of the* 2d *directed polymer model, see* (5.1) *and* (5.7), *for disorder variables* ω *which satisfy* (5.2). *Then, for* $(\beta_N)_{N\in\mathbb{N}}$ *in the quasi-critical regime, see* (5.4) *and* (5.9), *we have the convergence in distribution*

$$\sqrt{\theta_N}\,\{Z^{\omega}_{N,\beta_N}(\varphi) - \mathbb{E}[Z^{\omega}_{N,\beta_N}(\varphi)]\} \xrightarrow[N\to\infty]{d} \mathcal{N}\big(0\,,\sigma^2_{\varphi}\big)\,, \tag{5.10}$$

for any $\varphi \in C_c(\mathbb{R}^2)$, *where the limiting variance is given by*

$$\sigma^2_{\varphi} := \int\limits_{\mathbb{R}^2\times\mathbb{R}^2} \varphi(x)\,K(x,x')\,\varphi(x')\,\mathrm{d}x\,\mathrm{d}x'\,, \qquad \textit{with} \quad K(x,x') := \int_0^1 \frac{1}{2u}\,e^{-\frac{|x-x'|^2}{2u}}\,\mathrm{d}u\,. \tag{5.11}$$

REMARK 5.2 (Comparison with the subcritical regime and the Edwards-Wilkinson equation). *The convergence stated in Theorem 5.1 is clearly comparable and similar to the Gaussian fluctuations proved in Theorem 3.2 of Chapter 3 under the subcritical regime. In that case, we recall that*

$$\sqrt{R_N}\,\{Z^{\omega}_{N,\beta_N}(\varphi) - \mathbb{E}[Z^{\omega}_{N,\beta_N}(\varphi)]\} \xrightarrow[N\to\infty]{d} \mathcal{N}\bigg(0\,,\,\frac{\hat\beta^2}{1-\hat\beta^2}\,\frac{\sigma^2_{\varphi}}{\pi}\bigg)\,,$$

where $\beta_N = \hat\beta/\sqrt{R_N}$, $\hat\beta \in (0,1)$ *and* σ^2_{φ} *is the same as in* (5.11) *(since* $\frac{1}{2u}e^{-\frac{|x|^2}{2u}} = \pi g_u(x)$*) or, equivalently,*

$$\frac{\sqrt{\pi\,(1-\hat\beta^2)}}{\hat\beta}\,\sqrt{R_N}\,\{Z^{\omega}_{N,\beta_N}(\varphi) - \mathbb{E}[Z^{\omega}_{N,\beta_N}(\varphi)]\} \xrightarrow[N\to\infty]{d} \mathcal{N}\big(0\,,\sigma^2_{\varphi}\big)\,. \tag{5.12}$$

Then, the limit (5.12) *explains the reason why we now choose to rescale the averaged and centered partition function by the prefactor* θ_N *in* (5.10). *In the quasi-critical regime, the disorder parameter* $\hat\beta$ *is set equal to*

$$\hat\beta = \hat\beta_N = \sqrt{1 - \frac{\theta_N}{\log N}}\,,$$

which is still strictly smaller than 1 *for fixed* $N \in \mathbb{N}$, *but then converges to* 1 *as* $N \to \infty$. *Heuristically, by plugging the new definition of* $\hat\beta_N$ *into the prefactor of*

(5.12), *we obtain*

$$\frac{\sqrt{\pi\,(1-\hat{\beta}^2)}}{\hat{\beta}}\,\sqrt{R_N} = \frac{\sqrt{\pi}\,\sqrt{R_N}\,\sqrt{\frac{\theta_N}{\log N}}}{\sqrt{1-\frac{\theta_N}{\log N}}} \sim \frac{\sqrt{\theta_N}}{\sqrt{1-\frac{\theta_N}{\log N}}} \sim \sqrt{\theta_N} \qquad \text{as } N\to\infty\,,$$

(recall (5.6)*).*

Moreover, according to the notation used in Theorem 3.2, we can rephrase Theorem 5.1 in terms of the solution $v^{(s,c)}$ of the 2d Stochastic Heat Equation with additive noise

$$\partial_t v(t,x) = \frac{s}{2}\,\Delta v(t,x) + c\,\dot{W}(t,x)$$

and flat initial condition $v(0,\cdot)\equiv 0$. By recalling (3.12)*,* (3.13) *and* (3.11) *and by denoting*

$$V_N^{Q\text{-}C}(x) := \sqrt{\theta_N}\left(Z^{\omega}_{N,\beta_N}\big(\lfloor\sqrt{N}x\rfloor\big)-1\right), \quad x\in\mathbb{R}^2$$

with β_N according to (5.4) *and* (5.9) *the convergence* (5.10) *can be rewritten as*

$$V_N^{Q\text{-}C}(x) \overset{\mathcal{D}}{\Longrightarrow} \widetilde{v}(x) := v^{(\frac{1}{2},\pi)}(1,x) \qquad \text{as } N\to\infty\,. \tag{5.13}$$

Our strategy to prove Theorem 5.1 is inspired by the results presented in Chapter 2: we obtain (5.10) exploiting a Central Limit Theorem under a Lyapunov condition (see Theorem 2.14 and (2.37) and more details below), which requires to estimate *moments of the partition function of order higher than two.* To fulfil this condition, a key point in the subcritical regime is the application of the *hypercontractivity for polynomial chaos* (recall (2.26) and (2.27)). The latter allows to control the high moments of the partition function in terms of its second moment, up to a constant which depends on the order of the chaos involved. In the subcritical regime the main contribution to the limiting (convergent) second moment is given by a *finite number* of fixed chaos up to a negligible L^2 error, then we were able to properly bound the aforementioned constant and this was sufficient to verify the Ljapunov condition.

As explained below with more details, such a property *fails in the quasi-critical regime*, where all chaos components contribute to the limiting second moment (see Subsection 5.2). This is the key technical difficulty that we face in this chapter, for which model-specific arguments are required to estimate high moments. To this purpose, we exploit the general strategy developed in [**CSZ21+, LZ21+**], which extends the approach in [**GQT21**], but novel quantitative estimates are required in our setting (see Subsection 5.3).

5.1. Fluctuations for the partition function: proof of Theorem 5.1

We show in this section that Theorem 5.1 follows by two key steps, see Propositions 5.3 and 5.4, which will be proved in the next sections.

Let us call X_N the LHS of (2.14): recalling (5.7) and (5.3), we can write

$$\begin{aligned} X_N &:= \sqrt{\theta_N}\,\{Z^\omega_{N,\beta_N}(\varphi) - \mathbb{E}[Z^\omega_{N,\beta_N}(\varphi)]\} \\ &= \sqrt{\theta_N}\int_{\mathbb{R}^2}\Big\{Z^\omega_{N,\beta_N}\big(\lfloor\sqrt{N}x\rfloor\big) - 1\Big\}\,\varphi(x)\,\mathrm{d}x \\ &= \frac{\sqrt{\theta_N}}{N}\int_{\mathbb{R}^2}\Big\{Z^\omega_{N,\beta_N}\big(\lfloor x\rfloor\big) - 1\Big\}\,\varphi\Big(\frac{x}{\sqrt{N}}\Big)\,\mathrm{d}x\,. \end{aligned} \tag{5.14}$$

We denote $\varphi_N : \mathbb{Z}^2 \to \mathbb{R}$ the average of $\varphi\big(\frac{\cdot}{\sqrt{N}}\big)$ over cubes:

$$\varphi_N(z) := \int_{(z^1-1,z^1]\times(z^2-1,z^2]} \varphi\Big(\frac{x}{\sqrt{N}}\Big)\,\mathrm{d}x \qquad \text{for } z = (z^1,z^2)\in\mathbb{Z}^2\,, \tag{5.15}$$

thus since $Z^\omega_{N,\beta_N}\big(\lfloor x\rfloor\big)$ is constant over all cubes $(z^1-1,z^1]\times(z^2-1,z^2]$ we can express X_N in (5.14) as

$$X_N = \frac{\sqrt{\theta_N}}{N}\sum_{z\in\mathbb{Z}^2}\big\{Z^\omega_{N,\beta_N}(z) - 1\big\}\,\overline{\varphi}_N(z)\,. \tag{5.16}$$

We prove Theorem 5.1 via the following two main steps:

(1) we first approximate X_N in L^2 by a sum $\sum_{i=1}^M X^{(i)}_{N,M}$ of *independent* random variables, for $M = M_N \to \infty$ slowly enough;
(2) we then show that the random variables $(X^{(i)}_{N,M})_{1\le i\le M}$ for $M = M_N$ satisfy the assumptions of the classical *Central Limit Theorem* for triangular arrays (see Theorem 2.14 in Chapter 2).

Let us describe more precisely these steps and how they yield the proof of Theorem 5.1.

First step. In order to define the random variables $X^{(i)}_{N,M}$, for $M\in\mathbb{N}$ and $1\le i\le M$, we introduce a variation of (5.1), for $-\infty < A < B < \infty$:

$$Z^\omega_{(A,B],\beta}(z) := \mathrm{E}\big[e^{\sum_{n\in(A,B]\cap\mathbb{N}}\{\beta\omega(n,S_n)-\lambda(\beta)\}}\,\big|\,S_0 = z\big]\,. \tag{5.17}$$

We then define $X^{(i)}_{N,M}$ replacing $Z^\omega_{N,\beta}$ by $Z^\omega_{(\frac{i-1}{M}N,\frac{i}{M}N],\beta}$ in the definition (5.16) of X_N:

$$X^{(i)}_{N,M} = \frac{\sqrt{\theta_N}}{N}\sum_{z\in\mathbb{Z}^2}\big\{Z^\omega_{(\frac{i-1}{M}N,\frac{i}{M}N],\beta_N}(z) - 1\big\}\,\varphi_N(z)\,. \tag{5.18}$$

Note that $X^{(i)}_{N,M}$ for $1\le i\le M$ are *independent* and *centered* random variables (because $Z^\omega_{(A,B],\beta}(z)$ only depends on $\omega(n,x)$ for $A < n\le B$, and $\mathbb{E}[Z^\omega_{(A,B],\beta}(z)] = 1$ as in (5.3)).

The core of this first step is the following approximation result, proved in Subsection 5.2.

PROPOSITION 5.3 (L^2 approximation). *For $(\beta_N)_{N\in\mathbb{N}}$ in the quasi-critical regime, see* (5.4) *and* (5.9)*, the following relations hold for any $\varphi \in C_c(\mathbb{R}^2)$, with σ_φ^2 as in* (5.11)*:*

$$\lim_{N\to\infty} \mathbb{E}\big[X_N^2\big] = \sigma_\varphi^2 \,, \qquad \forall M \in \mathbb{N}: \quad \lim_{N\to\infty} \Bigg\| X_N - \sum_{i=1}^{M} X_{N,M}^{(i)} \Bigg\|_{L^2} = 0 \,. \tag{5.19}$$

It follows from the second relation in (5.19) that, for any $(M_N)_{N\in\mathbb{N}}$ with $M_N \to \infty$ slowly enough as $N \to \infty$, we have (see Remark 2.12):

$$\lim_{N\to\infty} \Bigg\| X_N - \sum_{i=1}^{M_N} X_{N,M_N}^{(i)} \Bigg\|_{L^2} = 0 \,, \tag{5.20}$$

that is we approximate X_N in L^2 by a sum of independent and centered random variables. Then, by the first relation in (5.19), it follows that

$$\lim_{N\to\infty} \mathbb{E}\Bigg[\Bigg(\sum_{i=1}^{M_N} X_{N,M_N}^{(i)}\Bigg)^2\Bigg] = \lim_{N\to\infty} \sum_{i=1}^{M_N} \mathbb{E}\Big[\big(X_{N,M_N}^{(i)}\big)^2\Big] = \sigma_\varphi^2 \,. \tag{5.21}$$

Second step. Recalling (5.16), we can rephrase our goal (5.10) as

$$X_N \xrightarrow{d} \mathcal{N}(0, \sigma_\varphi^2) \,.$$

In view of (5.20), this follows if we prove the convergence in distribution

$$\sum_{i=1}^{M_N} X_{N,M_N}^{(i)} \xrightarrow[N\to\infty]{d} \mathcal{N}\big(0 \,, \sigma_\varphi^2\big) \,. \tag{5.22}$$

Since $(X_{N,M_N}^{(i)})_{1\le i\le M_N}$ are independent and centered, we prove (5.22) by the classical Central Limit Theorem for triangular arrays, see e.g. [**Bil95**, Theorem 27.3] and Theorem 2.14: since we have convergence of the variance by (5.21), it is enough to check the Lyapunov condition

$$\text{for some } p > 2 : \qquad \lim_{N\to\infty} \sum_{i=1}^{M_N} \mathbb{E}\Big[\big|X_{N,M_N}^{(i)}\big|^p\Big] = 0 \,. \tag{5.23}$$

This follows by the next result, proved in Subsection 5.3, where we focus on the case $p = 4$.

PROPOSITION 5.4 (Fourth moment bound). *For $(\beta_N)_{N\in\mathbb{N}}$ in the quasi-critical regime, see* (5.4) *and* (5.9)*, and for any $\varphi \in C_c(\mathbb{R}^2)$, there is a constant $C < \infty$ and, for every $M \in \mathbb{N}$, a constant $\overline{N} = \overline{N}(M) < \infty$ such that*

$$\mathbb{E}\Big[\big(X_{N,M}^{(i)}\big)^4\Big] \le \frac{C}{M^2} \qquad \textit{for all } \; M \in \mathbb{N}, \;\; 1 \le i \le M \;\; \textit{and} \;\; N \ge \overline{N} \;. \tag{5.24}$$

As in the first step, we can take $M_N \to \infty$ as slowly as we wish, so that (5.24) applies with $M = M_N$ for all $i = 1, \ldots, M_N$. This shows that (5.23) holds with $p = 4$, since the sum therein vanishes as $N \to \infty$:

$$\sum_{i=1}^{M_N} \mathbb{E}\Big[\big|X^{(i)}_{N,M_N}\big|^p\Big] \le \frac{C}{M_N} \xrightarrow[N\to\infty]{} 0\,.$$

The proof of Theorem 5.1 is then completed, once we prove Propositions 5.3 and 5.4. The next sections are devoted to this task. □

5.2. Second moment bounds

We are going to prove Proposition 5.3 exploiting a polynomial chaos expansion of the partition function, that we first describe.

We fix $(\beta_N)_{N\in\mathbb{N}}$ in the quasi-critical regime, see (5.4) and (5.9), and $\varphi \in C_b(\mathbb{R}^2)$. We will denote by $C, C', \ldots$ generic finite constants (that may vary from place to place).

5.2.1. Polynomial chaos expansion. The partition function admits a key polynomial chaos expansion (see [**CSZ17a**] and Section 3.1 of Chapter 3). Let us define, for $\beta > 0$,

$$\xi_\beta(n,x) := e^{\beta\omega(n,x)-\lambda(\beta)} - 1\,, \qquad \text{for } n \in \mathbb{N}\,,\ x \in \mathbb{Z}^2\,. \tag{5.25}$$

Recalling (5.4), we note that $(\xi_\beta(n,x))_{n\in\mathbb{N},x\in\mathbb{Z}^2}$ are independent random variables with

$$\mathbb{E}[\xi_\beta] = 0\,, \qquad \mathbb{E}[\xi_\beta^2] = \sigma_\beta^2\,, \qquad \mathbb{E}[|\xi_\beta|^k] \le C_k\,\sigma_\beta^k \quad \forall k \ge 3\,, \tag{5.26}$$

for some $C_k < \infty$ (for the bound on $\mathbb{E}[|\xi_\beta|^k]$ see, e.g., [**CSZ17a**, eq. (6.7)]).

We denote by $q_n(x)$ the random walk transition kernel:

$$q_n(x) := \mathrm{P}(S_n = x \,|\, S_0 = 0)\,. \tag{5.27}$$

Then, writing $e^{\sum_n \{\beta\omega(n,x)-\lambda(\beta)\}} = \prod_n (1+\xi_\beta(n,x))$ and expanding the product, we can write $Z^\omega_{(A,B],\beta}(z)$ in (5.17) as the following polynomial chaos expansion:

$$\begin{aligned} Z^\omega_{(A,B],\beta}(z) = 1 + \sum_{k=1}^{\infty} \sum_{\substack{A<n_1<\ldots<n_k\le B \\ x_1,\ldots,x_k\in\mathbb{Z}^2}} & q_{n_1}(x_1 - z)\,\xi_\beta(n_1,x_1)\times \\ & \times \prod_{j=2}^{k} q_{n_j-n_{j-1}}(x_j - x_{j-1})\,\xi_\beta(n_j,x_j)\,, \end{aligned} \tag{5.28}$$

where we agree that the time variables $n_1 < \ldots < n_k$ are summed in the set $(A, B] \cap \mathbb{Z}$ (in particular, the seemingly infinite sum over k can be stopped at $B - A$).

Plugging (5.28) into (5.16), we obtain a corresponding polynomial chaos expansion for X_N: defining the averaged random walk transition kernel

$$q_n^f(x) := \sum_{z\in\mathbb{Z}^2} q_n(x-z)\, f(z)\,, \qquad \text{for } \; f:\mathbb{Z}^2\to\mathbb{R}\,, \tag{5.29}$$

we obtain

$$X_N = \frac{\sqrt{\theta_N}}{N} \sum_{k=1}^{\infty} \sum_{\substack{0<n_1<\ldots<n_k\le N \\ x_1,\ldots,x_k\in\mathbb{Z}^2}} q_{n_1}^{\varphi_N}(x_1)\, \xi_{\beta_N}(n_1,x_1) \times \\ \times \prod_{j=2}^{k} q_{n_j-n_{j-1}}(x_j-x_{j-1})\, \xi_{\beta_N}(n_j,x_j)\,. \tag{5.30}$$

The analogous polynomial chaos expansion for the random variables $X_{N,M}^{(i)}$, see (5.18), is obtained from (5.30) restricting the sum to $\frac{i-1}{M}N < n_1 < \ldots < n_k \le \frac{i}{M}N$:

$$X_{N,M}^{(i)} = \frac{\sqrt{\theta_N}}{N} \sum_{k=1}^{\infty} \sum_{\substack{\frac{i-1}{M}N<n_1<\ldots<n_k\le\frac{i}{M}N \\ x_1,\ldots,x_k\in\mathbb{Z}^2}} q_{n_1}^{\varphi_N}(x_1)\, \xi_{\beta_N}(n_1,x_1) \times \\ \times \prod_{j=2}^{k} q_{n_j-n_{j-1}}(x_j-x_{j-1})\, \xi_{\beta_N}(n_j,x_j)\,. \tag{5.31}$$

Since the random variables $(\xi_\beta(n,x))_{n\in\mathbb{N},x\in\mathbb{Z}^2}$ are independent and centered, see (5.25), *the terms in the sums in* (5.28)*,* (5.30)*,* (5.31) *are orthogonal in* L^2. Also note that, for any $M\in\mathbb{N}$, the random variables $X_{N,M}^{(i)}$ for $1\le i\le M$ are independent.

We finally recall the local limit theorem for the simple random walk on $\mathbb{Z}^2$, see [**LL10**, Theorem 2.1.3]: as $n\to\infty$, uniformly for $x\in\mathbb{Z}^2$ we have[1]

$$q_n(x) = \frac{1}{n/2}\Big(g\Big(\tfrac{x}{\sqrt{n/2}}\Big) + o(1)\Big)\, 2\,\mathbb{1}_{(n,x)\in\mathbb{Z}^3_{\text{even}}}\,, \qquad \text{where} \qquad g(y) := \frac{e^{-\frac{1}{2}|y|^2}}{2\pi}\,, \tag{5.32}$$

and we set $\mathbb{Z}^3_{\text{even}} := \big\{\, y = \big(y_1,\, y_2,\, y_3\big) \in \mathbb{Z}^3 \colon y_1+y_2+y_3\in 2\mathbb{Z}\,\big\}$. In particular, in the "diffusive regime" we can turn the additive error term $o(1)$ in a multiplicative one:

$$q_n(x) = \frac{4}{n}\, g\Big(\tfrac{x}{\sqrt{n/2}}\Big)\, \mathbb{1}_{(n,x)\in\mathbb{Z}^3_{\text{even}}}\big(1+o(1)\big) \qquad \text{for } |x| = O(\sqrt{n})\,. \tag{5.33}$$

[1]The scaling factor in (3.16) is $n/2$ because the covariance matrix of the simple random walk on $\mathbb{Z}^2$ is $\frac{1}{2}I$, while the factor $2\mathbb{1}_{(m,z)\in\mathbb{Z}^3_{\text{even}}}$ is due to periodicity.

5.2.2. Proof of Proposition 5.3. Looking at (5.31), we see that $\sum_{i=1}^M X_{N,M}^{(i)}$ is a polynomial chaos where the time variables $n_1 < \ldots < n_k$ must all belong to *one of the intervals* $(\frac{i-1}{M}N\,,\,\frac{i}{M}N]$, for $i = 1, \ldots, M$, while X_N from (5.30) is the analogous polynomial chaos where we sum over time variables $n_1 < \ldots < n_k$ in *the whole interval* $(0, N] = \bigcup_{i=1}^M (\frac{i-1}{M}N\,,\,\frac{i}{M}N]$. It follows that X_N is a *larger polynomial chaos*, i.e. it contains all the terms from $\sum_{i=1}^M X_{N,M}^{(i)}$, plus additional terms. Since all terms in the polynomial chaos are orthogonal in L^2, because the $\xi_\beta(n,x)$'s are independent and centered, it follows that *the difference* $X_N - \sum_{i=1}^M X_{N,M}^{(i)}$ *is orthogonal in* L^2 *to* $\sum_{i=1}^M X_{N,M}^{(i)}$, therefore

$$\Big\| X_N - \sum_{i=1}^M X_{N,M}^{(i)} \Big\|_{L^2}^2 = \|X_N\|_{L^2}^2 - \Big\| \sum_{i=1}^M X_{N,M}^{(i)} \Big\|_{L^2}^2 = \|X_N\|_{L^2}^2 - \sum_{i=1}^M \|X_{N,M}^{(i)}\|_{L^2}^2 .$$

As a consequence, to prove our goal (5.19) it is enough to show that

$$\lim_{N\to\infty} \mathbb{E}\big[X_N^2\big] = \sigma_\varphi^2\,, \qquad \forall M \in \mathbb{N}: \quad \lim_{N\to\infty} \sum_{i=1}^M \mathbb{E}\Big[\big(X_{N,M}^{(i)}\big)^2\Big] = \sigma_\varphi^2\,, \tag{5.34}$$

where we recall that σ_φ^2 is defined in (5.11). The first relation in (5.34) follows from the second one, because $X_N = X_{N,1}^{(1)}$. Then the proof is completed by the next result. ☐

LEMMA 5.5 (Quasi-critical variance). *Fix* $(\beta_N)_{N\in\mathbb{N}}$ *in the quasi-critical regime, see* (5.4) *and* (5.9), *and* $\varphi \in C_c(\mathbb{R}^2)$. *For any* $M \in \mathbb{N}$, *the following holds for all* $i = 1, \ldots, M$:

$$\lim_{N\to\infty} \mathbb{E}\big[\big(X_{N,M}^{(i)}\big)^2\big] \;=\; \sigma^2_{\varphi,(\frac{i-1}{M},\frac{i}{M}]} := \int_{\mathbb{R}^2\times\mathbb{R}^2} \varphi(x)\,\varphi(x') \left(\int_{\frac{i-1}{M}}^{\frac{i}{M}} \frac{1}{2u}\, e^{-\frac{|x-x'|^2}{2u}}\,\mathrm{d}u \right) \mathrm{d}x\,\mathrm{d}x' . \tag{5.35}$$

Proof. Let us fix $M \in \mathbb{N}$ and $1 \le i \le M$. We split the proof of (5.35) in the two bounds

$$\limsup_{N\to\infty} \mathbb{E}\Big[\big(X_{N,M}^{(i)}\big)^2\Big] \le \sigma^2_{\varphi,(\frac{i-1}{M},\frac{i}{M}]} \tag{5.36}$$

and

$$\liminf_{N\to\infty} \mathbb{E}\Big[\big(X_{N,M}^{(i)}\big)^2\Big] \ge \sigma^2_{\varphi,(\frac{i-1}{M},\frac{i}{M}]}\,. \tag{5.37}$$

We first obtain an exact expression for the second moment of $X_{N,M}^{(i)}$ by (5.31): since the random variables $\xi_\beta(n,x)$ are independent with zero mean and variance σ_β^2, we have

$$\mathbb{E}\Big[\big(X_{N,M}^{(i)}\big)^2\Big] = \frac{\theta_N}{N^2} \sum_{k=1}^\infty (\sigma_{\beta_N}^2)^k \sum_{\substack{\frac{i-1}{M}N < n_1 < \ldots < n_k \le \frac{i}{M}N \\ x_1,\ldots,x_k\in\mathbb{Z}^2}} q_{n_1}^{\varphi_N}(x_1)^2 \prod_{j=2}^k q_{n_j-n_{j-1}}(x_j - x_{j-1})^2 .$$

We can sum the space variables $x_k, x_{k-1}, \ldots, x_2$ because $\sum_{x\in\mathbb{Z}^2} q_n(x)^2 = q_{2n}(0)$, see (5.27), while to handle the sum over x_1 we note that, recalling (5.29),

$$\sum_{x\in\mathbb{Z}^2} q_n^f(x)^2 = q_{2n}^{f,f} \qquad \text{where we set} \quad q_m^{f,f} := \sum_{z,z'\in\mathbb{Z}^2} q_m(z-z')\, f(z)\, f(z') \,. \quad (5.38)$$

We then obtain

$$\mathbb{E}\Big[\big(X_{N,M}^{(i)}\big)^2\Big] = \frac{\theta_N}{N^2} \sum_{k=1}^{\infty} (\sigma_{\beta_N}^2)^k \sum_{\frac{i-1}{M}N < n_1 < \ldots < n_k \le \frac{i}{M}N} q_{2n_1}^{\varphi_N,\varphi_N} \prod_{j=2}^{k} q_{2(n_j-n_{j-1})}(0) \,. \quad (5.39)$$

We then prove the upper bound (5.36). We rename $n_1 = n$ and enlarge the sum over the other time variables $n_2, \ldots, n_k$, by letting each increment $m_j := n_j - n_{j-1}$ for $j = 2, \ldots, k$ vary in the whole interval $(0, N]$: since $\sum_{m=1}^{N} q_{2m}(0) = R_N$, see (5.6), we obtain

$$\begin{aligned} \mathbb{E}\Big[\big(X_{N,M}^{(i)}\big)^2\Big] &\le \frac{\theta_N}{N^2} \sum_{\frac{i-1}{M}N < n \le \frac{i}{M}N} q_{2n}^{\varphi_N,\varphi_N} \sum_{k=1}^{\infty} (\sigma_{\beta_N}^2)^k (R_N)^{k-1} \\ &= \theta_N \left\{ \frac{1}{N^2} \sum_{\frac{i-1}{M}N < n \le \frac{i}{M}N} q_{2n}^{\varphi_N,\varphi_N} \right\} \frac{\sigma_{\beta_N}^2}{1 - \sigma_{\beta_N}^2 R_N} \,, \end{aligned} \quad (5.40)$$

where we summed the geometric series since $\sigma_{\beta_N}^2 R_N = 1 - \frac{\theta_N}{\log N} < 1$ for large N, by (5.9). We will prove the following Riemann sum approximation, for any given $0 \le a < b \le 1$:

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{aN < n \le bN} q_{2n}^{\varphi_N,\varphi_N} = \int_{\mathbb{R}^2\times\mathbb{R}^2} \varphi(x)\, \varphi(x') \left(\int_a^b \frac{1}{u}\, g\Big(\frac{x-x'}{\sqrt{u}}\Big)\, \mathrm{d}u \right) \mathrm{d}x\, \mathrm{d}x' \,, \quad (5.41)$$

where $g(y) = \frac{1}{2\pi} e^{-\frac{1}{2}|y|^2}$ is the standard Gaussian density on $\mathbb{R}^2$, see (5.32). Plugging this into (5.40), since $1 - \sigma_{\beta_N}^2 R_N = \frac{\theta_N}{\log N}$ and $\sigma_{\beta_N}^2 \sim \frac{1}{R_N} \sim \frac{\pi}{\log N}$ as $N \to \infty$ by (5.9) and (5.6), we obtain precisely the upper bound (5.36) (note that $\pi \frac{1}{u} g(\frac{x-x'}{\sqrt{u}}) = \frac{1}{2u} \exp(-\frac{|x-x'|^2}{2u})$).

Let us now prove (5.41). This is based on the local limit theorem (5.32) as $n \to \infty$, hence the case $a = 0$ could be delicate, as the sum in (5.41) starts from $n = 1$ and, therefore, n needs not be large in this case. For this reason, we first show that small values of n are negligible for (5.41). Since φ is compactly supported, when we plug $f = \varphi_N$ into $q_{2n}^{f,f}$, see (5.38), we can restrict the sums to $|z'| \le C\sqrt{N}$, which yields the following *uniform bound*:

$$\forall m \in \mathbb{N}: \qquad |q_m^{\varphi_N,\varphi_N}| \le \|\varphi\|_\infty^2 \sum_{|z'|\le C\sqrt{N}} \sum_{z\in\mathbb{Z}^2} q_m(z-z') \le C' \, \|\varphi\|_\infty^2 \, N \,. \quad (5.42)$$

In particular, the contribution of $n \le \epsilon N$ to the LHS of (5.41) is $O(\epsilon)$. As a consequence, it is enough to prove (5.41) when $a > 0$, which we assume henceforth.

Recalling (5.38) and applying (5.32), we can write the LHS of (5.41) as follows:

$$\frac{1}{N^2}\sum_{aN<n\le bN} q_{2n}^{\varphi_N,\varphi_N} = \frac{1}{N^2}\sum_{aN<n\le bN}\ \sum_{\substack{z,z'\in\mathbb{Z}^2:\\ (n,z-z')\in\mathbb{Z}^3_{\rm even}}} \frac{2}{n}\Big(g\big(\tfrac{z-z'}{\sqrt{n}}\big)+o(1)\Big)\,\varphi_N(z)\varphi_N(z')\,,$$

where $o(1)\to 0$ as $N\to\infty$ (because $n>aN\to\infty$ and we assume $a>0$). The additive term $o(1)$ gives a vanishing contribution as $N\to\infty$, because we can bound $\frac{2}{n}\le\frac{2}{aN}$ and $|\varphi_N(\cdot)|\le\|\varphi\|_\infty$, and the sums contain $O(N^3)$ terms (since $|z|,|z'|\le C\sqrt{N}$). Introducing the rescaled variables $u:=\frac{n}{N}$ and $x:=\frac{z}{\sqrt{N}}$, $x':=\frac{z'}{\sqrt{N}}$, we can then rewrite the RHS as

$$\begin{aligned}
&\frac{1}{N^3}\sum_{u\in(a,b]\cap\frac{\mathbb{N}}{N}}\ \sum_{\substack{x,x'\in\frac{\mathbb{Z}^2}{\sqrt{N}}:\\ (Nu,\sqrt{N}(x-x'))\in\mathbb{Z}^3_{\rm even}}} \frac{2}{u}\,g\big(\tfrac{x-x'}{\sqrt{u}}\big)\,\varphi_N\big(\sqrt{N}x\big)\,\varphi_N\big(\sqrt{N}x'\big)+o(1)\\
&=\frac{1}{N}\sum_{u\in(a,b]\cap\frac{\mathbb{N}}{N}}\ \sum_{\substack{x,x'\in\frac{\mathbb{Z}^2}{\sqrt{N}}:\\ (Nu,\sqrt{N}(x-x'))\in\mathbb{Z}^3_{\rm even}}} \frac{2}{u}\,g\big(\tfrac{x-x'}{\sqrt{u}}\big)\int_{\left(x-\frac{1}{\sqrt{N}},x\right]\times\left(x'-\frac{1}{\sqrt{N}},x'\right]}\varphi(y)\varphi(y')\,\mathrm{d}y\,\mathrm{d}y'\\
&\qquad\qquad+o(1)\,.
\end{aligned}\tag{5.43}$$

Since $(y,y')\in\big(x-\frac{1}{\sqrt{N}},x\big]\times\big(x'-\frac{1}{\sqrt{N}},x'\big]$, we can write

$$g\big(\tfrac{x-x'}{\sqrt{u}}\big)=g\Big(\tfrac{y-y'}{\sqrt{u}}+O\Big(\frac{1}{\sqrt{N}}\Big)\Big)=g\big(\tfrac{y-y'}{\sqrt{u}}\big)+o(1)\qquad\text{as } N\to\infty\,,$$

then the expression (5.43) becomes

$$\frac{1}{N}\sum_{u\in(a,b]\cap\frac{\mathbb{N}}{N}}\ \sum_{\substack{x,x'\in\frac{\mathbb{Z}^2}{\sqrt{N}}:\\ (Nu,\sqrt{N}(x-x'))\in\mathbb{Z}^3_{\rm even}}} \frac{2}{u}\int_{\left(x-\frac{1}{\sqrt{N}},x\right]\times\left(x'-\frac{1}{\sqrt{N}},x'\right]} g\big(\tfrac{y-y'}{\sqrt{u}}\big)\varphi(y)\varphi(y')\,\mathrm{d}y\,\mathrm{d}y' + o(1)\,,$$

which is a Riemann sum for the integral in the RHS of (5.41). Note that the restriction $(Nu,\sqrt{N}(x-x'))\in\mathbb{Z}^3_{\rm even}$ effectively *halves* the range of the sum: indeed, for any given u and x, the sum over $x'=\frac{z'}{\sqrt{N}}\in\frac{\mathbb{Z}^2}{\sqrt{N}}$ is restricted to points $z'\in\mathbb{Z}^2$ with a fixed parity (even or odd, depending on u,x). This restriction is compensated by the multiplicative factor 2, which disappears as we let $N\to\infty$. This completes the proof of (5.41).

We finally prove the lower bound (5.37). We fix $\epsilon>0$ small enough and we bound the RHS of (5.39) from below as follows:

- we rename $n=n_1$ and we restrict its sum to the interval

$$\Big(\frac{i-1}{M}N\,,\,(1-\epsilon)\frac{i}{M}N\Big];$$

- for $k \geq 2$, we introduce the "displacements" $m_j := n_j - n_1$ from n_1, for $j = 2, \ldots, k$, and we restrict the sum over $n_2, \ldots, n_k$ to the set $0 < m_2 < \ldots < m_k \leq \epsilon \frac{i}{M} N$.

We thus obtain by (5.39)

$$\mathbb{E}\Big[\big(X_{N,M}^{(i)}\big)^2\Big] \geq \frac{\theta_N}{N^2} \sum_{\frac{i-1}{M} N < n \leq (1-\epsilon)\frac{i}{M} N} q_{2n}^{\varphi_N, \varphi_N} \times \\ \times \left(\sigma_{\beta_N}^2 + \sum_{k=2}^{\infty} (\sigma_{\beta_N}^2)^k \sum_{0 < m_2 < \ldots < m_k \leq \epsilon \frac{i}{M} N} q_{2m_2}(0) \prod_{j=3}^{k} q_{2(m_j - m_{j-1})}(0) \right). \tag{5.44}$$

We now give a probabilistic interpretation to the sum over $m_2, \ldots, m_k$: following [**CSZ19a**] and recalling (3.15), given $N \in \mathbb{N}$ we define i.i.d. random variables $(T_i^{(N)})_{i \in \mathbb{N}}$ with distribution

$$\mathrm{P}\big(T_i^{(N)} = n\big) = \frac{q_{2n}(0)}{R_N} \mathbb{1}_{\{1, \ldots, N\}}(n), \tag{5.45}$$

so that the second line of (5.44) can be written, renaming $\ell = k - 1$, as

$$\sigma_{\beta_N}^2 \left(1 + \sum_{\ell=1}^{\infty} (\sigma_{\beta_N}^2 R_N)^\ell \, \mathrm{P}\Big(T_1^{(N)} + \ldots + T_\ell^{(N)} \leq \varepsilon \tfrac{i}{M} N \Big) \right) \\ = \sigma_{\beta_N}^2 \left(\frac{1}{1 - \sigma_{\beta_N}^2 R_N} - \sum_{\ell=1}^{\infty} (\sigma_{\beta_N}^2 R_N)^\ell \, \mathrm{P}\Big(T_1^{(N)} + \ldots + T_\ell^{(N)} > \varepsilon \tfrac{i}{M} N \Big) \right).$$

Plugging this into (5.44) and recalling (5.42), we obtain

$$\mathbb{E}\Big[\big(X_{N,M}^{(i)}\big)^2\Big] \geq \theta_N \left\{ \frac{1}{N^2} \sum_{\frac{i-1}{M} N < n \leq (1-\epsilon)\frac{i}{M} N} q_{2n}^{\varphi_N, \varphi_N} \right\} \frac{\sigma_{\beta_N}^2}{1 - \sigma_{\beta_N}^2 R_N} \\ - \big(C' \, \|\varphi\|_\infty^2\big) \, \theta_N \, \sigma_{\beta_N}^2 \sum_{\ell=1}^{\infty} (\sigma_{\beta_N}^2 R_N)^\ell \, \mathrm{P}\Big(T_1^{(N)} + \ldots + T_\ell^{(N)} > \tfrac{\varepsilon}{M} N \Big). \tag{5.46}$$

The first term in the RHS is similar to (5.40), just with $(1-\epsilon)\frac{i}{M}$ instead of $\frac{i}{M}$, therefore *we already proved that it converges to* $\sigma^2_{\varphi, (\frac{i-1}{M}, (1-\epsilon)\frac{i}{M}]}$ *as* $N \to \infty$, see (5.41) and the following lines (recall also (5.35)). Letting $\epsilon \downarrow 0$ after $N \to \infty$ we recover $\sigma^2_{\varphi, (\frac{i-1}{M}, \frac{i}{M}]}$, hence to prove (5.37) we just need to show that the second term in the RHS of (5.46) is negligible:

$$\lim_{N \to \infty} \theta_N \, \sigma_{\beta_N}^2 \sum_{\ell=1}^{\infty} (\sigma_{\beta_N}^2 R_N)^\ell \, \mathrm{P}\Big(T_1^{(N)} + \ldots + T_\ell^{(N)} > \tfrac{\varepsilon}{M} N \Big) = 0. \tag{5.47}$$

Recall that the random variables $(T_i^{(N)})_{i\in\mathbb{N}}$ are i.i.d. with distribution (5.45). Since $q_{2n}(0) \le \frac{C}{n}$ by the local limit theorem (5.32), we have

$$\mathrm{E}[T_i^{(N)}] = \frac{1}{R_N} \sum_{n=1}^{N} n\, q_{2n}(0) \le C \frac{N}{R_N}$$

and, by Markov's inequality, we can bound

$$\mathrm{P}\Big(T_1^{(N)} + \ldots + T_\ell^{(N)} > \tfrac{\varepsilon}{M} N\Big) \le \frac{\mathrm{E}\big[T_1^{(N)} + \ldots + T_\ell^{(N)}\big]}{\frac{\varepsilon}{M} N} \le \frac{C\,\ell}{\frac{\epsilon}{M}\, R_N} .$$

Since $\sum_{\ell=1}^\infty \ell\, x^\ell = \frac{x}{(1-x)^2}$, we obtain

$$\begin{aligned} &\theta_N\, \sigma_{\beta_N}^2 \sum_{\ell=1}^{\infty} (\sigma_{\beta_N}^2 R_N)^\ell\, \mathrm{P}\Big(T_1^{(N)} + \ldots + T_\ell^{(N)} > \tfrac{\varepsilon}{M} N\Big) \\ &\le \theta_N\, \sigma_{\beta_N}^2 \, \frac{C}{\frac{\epsilon}{M}\, R_N} \, \frac{\sigma_{\beta_N}^2 R_N}{(1-\sigma_{\beta_N}^2 R_N)^2} \\ &= \frac{C\, M}{\epsilon} \, \frac{\theta_N\, (\sigma_{\beta_N}^2)^2}{(1-\sigma_{\beta_N}^2 R_N)^2} . \end{aligned}$$

Note that $1 - \sigma_{\beta_N}^2 R_N = \frac{\theta_N}{\log N}$ and $\sigma_{\beta_N}^2 \sim \frac{1}{R_N} \sim \frac{\pi}{\log N}$ by (5.9) and (5.6), hence the last term is asymptotically equivalent to

$$\frac{CM}{\epsilon} \, \frac{\pi^2}{\theta_N} \to 0 \qquad \text{as } N \to \infty ,$$

since $\theta_N \to \infty$, see (5.9). This shows that (5.47) holds and completes the proof of Proposition 5.3. ☐

5.3. Fourth moment bounds

In this section we prove Proposition 5.4, refining the approach in [**CSZ21+**, Theorem 6.1] and [**LZ21+**, Theorem 1.3] to bound high moments of the partition function. These papers deal with the critical and subcritical regime, but the same approach can also be applied in the quasi-critical regime that we consider. In fact, also in view of future applications, we present below a refined formulation of this approach:

- we make the approach explicitly *independent of the regime of* β;
- we give an *exact expansion* for the moments, see Theorem 5.9, from which we deduce *upper bounds*, see Theorems 5.11 and 5.13, which depend on explicit quantities, that we call *boundary terms*, *Green's function terms* and *bulk terms*;
- we obtain explicit estimates on the boundary, Green's function and bulk terms, which plugged in Theorem 5.13 yield explicit estimates on the moments: these will be applied to obtain the proof of Proposition 5.4.

Subsection 5.3.3 is devoted to the proof of Proposition 5.4. The key difficulty is that our goal (5.24) involves not only the boundedness of the fourth moment, but also *the optimal* $1/M^2$ *dependence* on the width of the time interval $(\frac{i-1}{M}N, \frac{i}{M}N]$ (recall the definition (5.31) of the random variable $X^{(i)}_{N,M}$). This requires sharp ad hoc estimates that are specific to the quasi-critical regime.

5.3.1. Moment expansion and upper bounds. The partition function $Z^{\omega}_{(A,B],\beta}(z)$ in (5.17) is called "point-to-plane", since random walk paths start at $S_0 = z$ but have no constrained endpoint. We introduce a "point-to-point" version, for simplicity when $(A,B] = (0,L]$ for $L \in \mathbb{N}$, restricting to random walk paths with a fixed endpoint $S_L = w$:

$$Z^{\omega}_{L,\beta}(z,w) := \mathrm{E}\Big[e^{\sum_{n=1}^{L-1}\{\beta\omega(n,S_n)-\lambda(\beta)\}}\,\mathbb{1}_{\{S_L=w\}}\,\Big|\,S_0 = z\Big] \tag{5.48}$$

(we stop the sum at $n = L-1$ for later convenience).

Given two "boundary conditions" $f, g : \mathbb{Z}^2 \to \mathbb{R}$, we define the averaged version

$$Z^{\omega}_{L,\beta}(f,g) \coloneqq \sum_{z,w\in\mathbb{Z}^2} f(z)\, Z^{\omega}_{L,\beta}(z,w)\, g(w)\,. \tag{5.49}$$

We focus on the *centered moments* of $Z^{\omega}_{L,\beta}(f,g)$, that we denote by

$$\mathcal{M}^h_{L,\beta}(f,g) \coloneqq \mathbb{E}\Big[\big(Z^{\omega}_{L,\beta}(f,g) - \mathbb{E}[Z^{\omega}_{L,\beta}(f,g)]\big)^h\Big] \qquad \text{for } h \in \mathbb{N}\,. \tag{5.50}$$

Remark 5.6. *Recalling* (5.17), (5.18) *and* (5.15), (5.29), *by translation invariance we have*

$$\mathbb{E}\left[(X^{(i)}_{N,M})^4\right] = \frac{\theta_N^2}{N^4}\,\mathcal{M}^4_{\frac{N}{M},\beta_N}(f_N,\mathbb{1})\,, \qquad \textit{where} \quad \begin{cases} f_N(z) := q^{\varphi_N}_{\frac{i-1}{M}N}(z)\,, \\ \mathbb{1}(w) \equiv 1\,, \end{cases} \tag{5.51}$$

hence to prove Proposition 5.4 we can focus on $\mathcal{M}^4_{L,\beta}(f,g)$.

Henceforth we fix $h \in \mathbb{N}$ with $h \geq 2$ (the non-trivial case is $h \geq 3$, and we are interested in $h = 4$). We give an *exact expression* for $\mathcal{M}^h_{L,\beta}(f,g)$, see Theorem 5.9, from which we derive *sharp upper bounds, see Theorems 5.11 and 5.13.* We first need some notation.

We denote by $I \vdash \{1,\ldots,h\}$ a *partition of* $\{1,\ldots,h\}$, i.e. a family $I = \{I^1,\ldots,I^m\}$ of non–empty disjoint subsets $I^j \subseteq \{1,\ldots,h\}$ with $I^1 \cup \ldots \cup I^m = \{1,\ldots,h\}$. We single out:

- the unique partition $I = * := \{\{1\},\{2\},\ldots,\{h\}\}$ composed by all singletons;
- the $\binom{h}{2}$ partitions with one single pair and then all singletons, namely of the form $I = \{\{a,b\},\{c\}\colon c \neq a, c \neq b\}$, that we call *pairs*.

Example 5.7 (Cases $h = 2, 3, 4$). *All partitions* $I \vdash \{1, 2\}$ *are* $I = *$ *and* $I = \{\{1, 2\}\}$.

All partitions $I \vdash \{1, 2, 3\}$ *are* $I = *$*, three* pairs $I = \{\{a, b\}, \{c\}\}$ *and* $I = \{\{1, 2, 3\}\}$.

All partitions $I \vdash \{1, 2, 3, 4\}$ *are* $I = *$*, six* pairs $I = \{\{a, b\}, \{c\}, \{d\}\}$*, six* double pairs $I = \{\{a, b\}, \{c, d\}\}$*, four* triples $I = \{\{a, b, c\}, \{d\}\}$ *and the* quadruple $I = \{\{1, 2, 3, 4\}\}$.

Given a partition $I = \{I^1, \ldots, I^m\} \vdash \{1, \ldots, h\}$, we define for $\mathbf{x} = (x^1, \ldots, x^h) \in (\mathbb{Z}^2)^h$

$$\mathbf{x} \sim I \text{ if and only if } \begin{cases} x^a = x^b & \text{if } a, b \in I^i \text{ for some } i\,, \\ x^a \neq x^b & \text{if } a \in I^i, b \in I^j \text{ for some } i \neq j \text{ with } |I^i|, |I^j| \geq 2\,. \end{cases} \tag{5.52}$$

For instance, $\mathbf{x} \sim \{\{1, 2\}, \{3\}, \{4\}\}$ means $x^1 = x^2$, $\mathbf{x} \sim \{\{1, 2\}, \{3, 4\}\}$ means $x^1 = x^2$ and $x^3 = x^4$ with $x^1 \neq x^3$, while $\mathbf{x} \sim *$ imposes no constraint. We correspondingly define

$$(\mathbb{Z}^2)^h_I := \big\{\mathbf{x} \in (\mathbb{Z}^2)^h \colon \mathbf{x} = (x^1, \ldots, x^h) \sim I\big\}\,, \tag{5.53}$$

which is essentially a copy of $(\mathbb{Z}^2)^m$ embedded in $(\mathbb{Z}^2)^h$.

A family $I_1, \ldots, I_r$ of partitions $I_i = \{I_i^1, \ldots, I_i^{m_i}\} \vdash \{1, \ldots, h\}$ is said to have *full support* if any $a \in \{1, \ldots, h\}$ belongs to some partition I_i not as a singleton, i.e. $a \in I_i^j$ with $|I_i^j| \geq 2$.

Example 5.8 (Full support for $h = 4$). *A single partition* $I_1 \vdash \{1, 2, 3, 4\}$ *with full support is either the quadruple* $I_1 = \{\{1, 2, 3, 4\}\}$ *or a double pair* $I_1 = \{\{a, b\}, \{c, d\}\}$*. There are many families of two partitions* $I_1, I_2 \vdash \{1, 2, 3, 4\}$ *with full support, for instance two non overlapping pairs such as* $I_1 = \{\{1, 3\}, \{2\}, \{4\}\}$, $I_2 = \{\{2, 4\}, \{1\}, \{3\}\}$.

We now introduce h-fold analogues of the random walk transition kernel (5.27) and of its averaged version (5.29): given partitions $I, J \vdash \{1, \ldots, h\}$, we define for $\mathbf{x}, \mathbf{z} \in (\mathbb{Z}^2)^h$

$$\mathsf{Q}_n^{I,J}(\mathbf{z}, \mathbf{x}) := \mathbb{1}_{\{\mathbf{z} \sim I, \mathbf{x} \sim J\}} \prod_{i=1}^h q_n(x^i - z^i)\,, \qquad \mathsf{q}_n^{f,J}(\mathbf{x}) := \mathbb{1}_{\{\mathbf{x} \sim J\}} \prod_{i=1}^h q_n^f(x^i)\,. \tag{5.54}$$

For $n < m \in \mathbb{Z}$ and $J \vdash \{1, \ldots, h\} \neq *$, we define for $\mathbf{x}, \mathbf{z} \in (\mathbb{Z}^2)^h$ the operator

$$\mathsf{U}^J_{m-n,\beta}(\mathbf{z}, \mathbf{x}) := \begin{cases} \displaystyle\sum_{k=1}^\infty \mathbb{E}[\xi_\beta^J]^k \sum_{\substack{n =: n_0 < n_1 < \cdots < n_k := m \\ \mathbf{y}_1, \ldots, \mathbf{y}_{k-1} \in (\mathbb{Z}^2)^h \\ \mathbf{y}_0 := \mathbf{z},\ \mathbf{y}_k := \mathbf{x}}} \prod_{i=1}^k \mathsf{Q}^{J,J}_{n_i - n_{i-1}}(\mathbf{y}_{i-1}, \mathbf{y}_i) & \text{if } n < m\,, \\ \mathbb{1}_{\{\mathbf{z} = \mathbf{x} \sim J\}} & \text{if } n = m\,, \end{cases} \tag{5.55}$$

where for $J = \{J^1, \dots, J^m\} \vdash \{1, \dots, h\}$ with $J \neq *$ we define

$$\mathbb{E}[\xi_\beta^J] := \prod_{i:\, |J^i| \geq 2} \mathbb{E}[\xi_\beta^{|J^i|}] \,. \tag{5.56}$$

For instance, if J is a pair, then $\mathbb{E}[\xi_\beta^J] = \sigma_\beta^2$, see (5.26).

Given two functions $\mathsf{q}^f(\mathbf{x})$, $\mathsf{q}^g(\mathbf{x})$ and a family of matrices $\mathsf{U}_i(\mathbf{z}, \mathbf{x})$, $\mathsf{Q}_i(\mathbf{z}, \mathbf{x})$ for $\mathbf{x}, \mathbf{z} \in \mathbb{T}$, where $\mathbb{T}$ is a countable set, we use the standard notation

$$\begin{aligned} &\left\langle \mathsf{q}^f,\, \mathsf{U}_1 \left\{ \prod_{i=2}^r \mathsf{Q}_i \, \mathsf{U}_i \right\} \mathsf{q}^g \right\rangle \\ &:= \sum_{\substack{\mathbf{z}_1, \dots, \mathbf{z}_r \in \mathbb{T} \\ \mathbf{z}'_1, \dots, \mathbf{z}'_r \in \mathbb{T}}} \mathsf{q}^f(\mathbf{z}_1) \, \mathsf{U}_1(\mathbf{z}_1, \mathbf{z}'_1) \left\{ \prod_{i=2}^r \mathsf{Q}_i(\mathbf{z}'_{i-1}, \mathbf{z}_i) \, \mathsf{U}_i(\mathbf{z}_i, \mathbf{z}'_i) \right\} \mathsf{q}^g(\mathbf{z}'_r) \,. \end{aligned}$$

We can now give the announced expansion for $\mathcal{M}^h_{L,\beta}(f, g)$, that we prove in Subsection 5.3.4.

Theorem 5.9 (Moment expansion). *Let $Z^\omega_{L,\beta}(f, g)$ be the averaged partition function in* (5.49) *with centered moments $\mathcal{M}^h_{L,\beta}(f, g)$, see* (5.50). *For any $h \in \mathbb{N}$ with $h \geq 2$ we have*

$$\begin{aligned} \mathcal{M}^h_{L,\beta}(f, g) = \sum_{r=1}^\infty \; \sum_{0 < n_1 \leq m_1 < \dots < n_r \leq m_r < L} \; \sum_{\substack{I_1, \dots, I_r \vdash \{1, \dots, h\} \\ \text{with full support} \\ \text{and } I_i \neq I_{i-1},\ I_i \neq * \ \forall i}} \left\{ \prod_{i=1}^r \mathbb{E}[\xi_\beta^{I_i}] \right\} \times \\ \times \left\langle \mathsf{q}^{f, I_1}_{n_1} \,,\; \mathsf{U}^{I_1}_{m_1 - n_1, \beta} \left\{ \prod_{i=2}^r \mathsf{Q}^{I_{i-1}, I_i}_{n_i - m_{i-1}} \; \mathsf{U}^{I_i}_{m_i - n_i, \beta} \right\} \mathsf{q}^{g, I_r}_{L - m_r} \right\rangle . \end{aligned} \tag{5.57}$$

Remark 5.10 (Sanity check). *In case $h = 2$, the conditions $I_i \neq I_{i-1}$ and $I_i \neq *$ in* (5.57) *force $r = 1$ and $I_1 = \{\{1, 2\}\}$, hence formula* (5.57) *reduces to*

$$\mathcal{M}^2_{L,\beta}(f, g) = \mathbb{V}\mathrm{ar}[Z^\omega_{L,\beta}(f, g)] = \sigma_\beta^2 \sum_{\substack{0 < n \leq m < L \\ z, x \in \mathbb{Z}^2}} q^f_n(z)^2 \, \mathsf{U}_{m-n,\beta}(z, x) \, q^g_{L-m}(x)^2 \,,$$

which is a classical expansion for the variance of the partition function, see [**CSZ21+**, Equation 3.51].

We next obtain an upper bound from (5.57). For $L \in \mathbb{N}$ we define the summed kernels

$$\widehat{\mathsf{Q}}^{I,J}_L(\mathbf{z}, \mathbf{x}) := \sum_{n=1}^L \mathsf{Q}^{I,J}_n(\mathbf{z}, \mathbf{x}) \,, \qquad \widehat{\mathsf{q}}^{f,J}_L(\mathbf{x}) := \sum_{n=1}^L \mathsf{q}^{f,J}_n(\mathbf{x}) \,. \tag{5.58}$$

Recalling (5.55) and (5.56) we set, with some abuse of notation,

$$|\mathsf{U}|^J_{m-n,\beta}(\mathbf{z}, \mathbf{x}) := \mathsf{U}^J_{m-n,\beta}(\mathbf{z}, \mathbf{x}) \text{ from (5.55) with } \mathbb{E}[\xi_\beta^J] \text{ replaced by } |\mathbb{E}[\xi_\beta^J]| \,, \tag{5.59}$$

and then for $L \in \mathbb{N}$ and $\lambda \geq 0$ we define the Laplace sum

$$|\widehat{\mathsf{U}}|^{J}_{L,\lambda,\beta}(\mathbf{z},\mathbf{x}) := \mathbb{1}_{\{\mathbf{z}=\mathbf{x}\sim J\}} + \sum_{m=1}^{L} e^{-\lambda m}\, |\mathsf{U}|^{J}_{m,\beta}(\mathbf{z},\mathbf{x})\,. \tag{5.60}$$

Finally, we introduce a *uniform bound* on the right boundary function $\mathsf{q}^{g,I_r}_{L-m_r}$ in (5.57):

$$\overline{\mathsf{q}}^{g,I}_{L}(\mathbf{z}) := \max_{1\leq n\leq L} \mathsf{q}^{g,I}_{n}(\mathbf{z})\,. \tag{5.61}$$

We can now state our first moment upper bound.

THEOREM 5.11 (Moment upper bound, I). *Let $Z^{\omega}_{L,\beta}(f,g)$ denote the averaged partition function in* (5.49) *with centred moments $\mathcal{M}^{h}_{L,\beta}(f,g)$, see* (5.50)*. Given $\lambda \geq 0$, for any $h \in \mathbb{N}$ with $h \geq 2$ we have*

$$\begin{aligned}\big|\mathcal{M}^{h}_{L,\beta}(f,g)\big| \leq e^{\lambda L} \sum_{r=1}^{\infty} \sum_{\substack{I_1,\ldots,I_r \vdash \{1,\ldots,h\} \\ \text{with full support} \\ \text{and } I_i \neq I_{i-1},\ I_i \neq *\ \forall i}} \Big\{ \prod_{i=1}^{r} \big|\mathbb{E}[\xi^{I_i}_{\beta}]\big| \Big\} \times \\ \times \Big\langle\, \widehat{\mathsf{q}}^{|f|,I_1}_{L}\ ,\ |\widehat{\mathsf{U}}|^{I_1}_{L,\lambda,\beta} \Big\{ \prod_{i=2}^{r} \widehat{\mathsf{Q}}^{I_{i-1},I_i}_{L}\ |\widehat{\mathsf{U}}|^{I_i}_{L,\lambda,\beta} \Big\}\, \overline{\mathsf{q}}^{|g|,I_r}_{L} \,\Big\rangle\,.\end{aligned} \tag{5.62}$$

Proof. Replacing $\mathbb{E}[\xi^{I_i}_{\beta}]$, f, g, U in (5.57) respectively by $|\mathbb{E}[\xi^{I_i}_{\beta}]|$, $|f|$, $|g|$, $|\mathsf{U}|$, every term becomes non-negative. We next replace $\mathsf{q}^{|g|,I_r}_{L-m_r}$ by the uniform bound $\overline{\mathsf{q}}^{|g|,I^r}_{L}$ and then enlarge the sum in (5.57), allowing all increments $n_i - m_{i-1}$ and $m_i - n_i$ to vary freely in $\{1,\ldots,L\}$. Plugging $1 \leq e^{\lambda L}\, e^{-\lambda m_r} \leq e^{\lambda L}\, e^{-\lambda \sum_{i=1}^{r}(m_i-n_i)}$, we obtain (5.62). □

In order to bound the scalar product in (5.62), let us recall some basic functional analysis. Given a countable set $\mathbb{T}$ and a function $f : \mathbb{T} \to \mathbb{R}$, we define for $p \in [1,\infty)$

$$\|f\|_{\ell^p(\mathbb{T})} = \|f\|_{\ell^p} := \Big(\sum_{z\in\mathbb{T}} |f(z)|^p \Big)^{\frac{1}{p}} \quad \text{for } p \in [1,\infty)\,. \tag{5.63}$$

Given any $p,q \in (1,\infty)$ with $\frac{1}{p}+\frac{1}{q}=1$ and a linear operator $A : \ell^q(\mathbb{T}) \to \ell^q(\mathbb{T}')$, we set

$$\|A\|_{\ell^q\to\ell^q} := \sup_{g\not\equiv 0} \frac{\|A\,g\|_{\ell^q(\mathbb{T}')}}{\|g\|_{\ell^q(\mathbb{T})}} = \sup_{\|f\|_{\ell^p(\mathbb{T}')}\leq 1,\, \|g\|_{\ell^q(\mathbb{T})}\leq 1} \langle f, A\,g\rangle\,. \tag{5.64}$$

Since $|\langle g,h\rangle| \leq \|g\|_{\ell^p}\, \|h\|_{\ell^q}$ by Hölder's inequality, we can bound (5.62) by

$$\begin{aligned}\big|\mathcal{M}^{h}_{L,\beta}(f,g)\big| \leq e^{\lambda L} \sum_{r=1}^{\infty} \sum_{\substack{I_1,\ldots,I_r \vdash \{1,\ldots,h\} \\ \text{with full support} \\ \text{and } I_i \neq I_{i-1},\ I_i \neq *\ \forall i}} \Big\{ \prod_{i=1}^{r} \big|\mathbb{E}[\xi^{I_i}_{\beta}]\big| \Big\} \times \\ \times \big\|\widehat{\mathsf{q}}^{|f|,I_1}_{L}\big\|_{\ell^p}\, \big\||\widehat{\mathsf{U}}|^{I_1}_{L,\lambda,\beta}\big\|_{\ell^q\to\ell^q} \Big\{ \prod_{i=2}^{r} \big\|\widehat{\mathsf{Q}}^{I_{i-1},I_i}_{L}\big\|_{\ell^q\to\ell^q}\, \big\||\widehat{\mathsf{U}}|^{I_i}_{L,\lambda,\beta}\big\|_{\ell^q\to\ell^q} \Big\}\, \big\|\overline{\mathsf{q}}^{|g|,I_r}_{L}\big\|_{\ell^q}\,.\end{aligned} \tag{5.65}$$

REMARK 5.12 (Restricted ℓ^q spaces). *Due to the constraint* $\mathbb{1}_{\{\mathbf{z}\sim I,\mathbf{x}\sim J\}}$ *in* (5.54), *we may regard* $\widehat{\mathsf{Q}}_L^{I,J}$ *as a linear operator from* $\ell^q((\mathbb{Z}^2)^h_J)$ *to* $\ell^q((\mathbb{Z}^2)^h_I)$, *see* (5.53). *Similarly, we may view* $|\widehat{\mathsf{U}}|^J_{L,\lambda,\beta}$ *as a linear operator from* $\ell^q((\mathbb{Z}^2)^h_J)$ *to itself.*

To make the bound (5.65) more useful, we fix a *weight function* $\mathcal{W} : (\mathbb{Z}^2)^h \to (0,\infty)$, that we identify with the diagonal operator $\mathcal{W}(\mathbf{x})\,\mathbb{1}_{\{\mathbf{x}=\mathbf{y}\}}$. In particular, we have

$$\big(\mathcal{W}\,A\,\tfrac{1}{\mathcal{W}}\big)(\mathbf{x},\mathbf{y}) := \mathcal{W}(\mathbf{x})\,A(\mathbf{x},\mathbf{y})\,\frac{1}{\mathcal{W}(\mathbf{y})}$$

for any linear operator $A = A(\mathbf{x},\mathbf{y})$. Inserting $(\mathcal{W}\,\frac{1}{\mathcal{W}})$ between each pair of adjacent operators in (5.62), we obtain an improved version of the bound (5.65):

$$\begin{aligned}
&\big|\mathcal{M}^h_{L,\beta}(f,g)\big| \\
&\le e^{\lambda L}\sum_{r=1}^{\infty}\ \sum_{\substack{I_1,\ldots,I_r\vdash\{1,\ldots,h\}\\ \text{with full support}\\ \text{and } I_i\neq I_{i-1},\ I_i\neq *\ \forall i}} \bigg\{\prod_{i=1}^r \big|\mathbb{E}[\xi_\beta^{I_i}]\big|\bigg\}\,\big\|\widehat{\mathsf{q}}_L^{|f|,I_1}\,\tfrac{1}{\mathcal{W}}\big\|_{\ell^p}\big\|\mathcal{W}\,|\widehat{\mathsf{U}}|^{I_1}_{L,\lambda,\beta}\,\tfrac{1}{\mathcal{W}}\big\|_{\ell^q\to\ell^q}\times \\
&\times\bigg\{\prod_{i=2}^r \big\|\mathcal{W}\,\widehat{\mathsf{Q}}_L^{I_{i-1},I_i}\,\tfrac{1}{\mathcal{W}}\big\|_{\ell^q\to\ell^q}\,\big\|\mathcal{W}\,|\widehat{\mathsf{U}}|^{I_i}_{L,\lambda,\beta}\,\tfrac{1}{\mathcal{W}}\big\|_{\ell^q\to\ell^q}\bigg\}\,\big\|\mathcal{W}\,\overline{\mathsf{q}}_L^{|g|,I_r}\big\|_{\ell^q}\,.
\end{aligned} \tag{5.66}$$

This leads to our second moment upper bound.

THEOREM 5.13 (Moment upper bound, II). *Let* $Z^\omega_{L,\beta}(f,g)$ *be the averaged partition function in* (5.49) *with centred moments* $\mathcal{M}^h_{L,\beta}(f,g)$, *see* (5.50). *Given* $\lambda\ge 0$, $q\in(1,\infty)$ *and a weight function* $\mathcal{W} : (\mathbb{Z}^2)^h \to (0,\infty)$, *we define*

$$\mathsf{C}^q_{\mathcal{W},L} := \max_{\substack{I,J\neq *\\ I\neq J}} \big\|\mathcal{W}\,\widehat{\mathsf{Q}}_L^{I,J}\,\tfrac{1}{\mathcal{W}}\big\|_{\ell^q\to\ell^q}\,,\qquad \rho^q_{\mathcal{W},L,\lambda,\beta} := \sum_{I\neq *} \big|\mathbb{E}[\xi^I_\beta]\big|\,\big\|\mathcal{W}\,|\widehat{\mathsf{U}}|^I_{L,\lambda,\beta}\,\tfrac{1}{\mathcal{W}}\big\|_{\ell^q\to\ell^q}\,, \tag{5.67}$$

and we assume that

$$\mathsf{C}^q_{\mathcal{W},L}\,\rho^q_{\mathcal{W},L,\lambda,\beta} < 1\,.$$

Then for any $h\in\mathbb{N}$ *with* $h\ge 2$ *we have, with* $p = (1-\frac{1}{q})^{-1}$,

$$\big|\mathcal{M}^h_{L,\beta}(f,g)\big| \le e^{\lambda L}\Big(\max_{I\neq *}\big\|\widehat{\mathsf{q}}_L^{|f|,I}\,\tfrac{1}{\mathcal{W}}\big\|_{\ell^p}\Big)\,\frac{\rho^q_{\mathcal{W},L,\lambda,\beta}}{1-\mathsf{C}^q_{\mathcal{W},L}\,\rho^q_{\mathcal{W},L,\lambda,\beta}}\,\Big(\max_{J\neq *}\big\|\mathcal{W}\,\overline{\mathsf{q}}_L^{|g|,J}\big\|_{\ell^q}\Big)\,. \tag{5.68}$$

Proof. From (5.66) we can bound

$$\begin{aligned}
\big|\mathcal{M}^h_{L,\beta}(f,g)\big| &\le e^{\lambda L}\Big(\max_{I\neq *}\big\|\widehat{\mathsf{q}}_L^{|f|,I}\,\tfrac{1}{\mathcal{W}}\big\|_{\ell^p}\Big)\bigg\{\sum_{r=1}^\infty \big(\mathsf{C}^q_{\mathcal{W},L}\big)^{r-1}\big(\rho^q_{\mathcal{W},L,\lambda,\beta}\big)^r\bigg\}\times \\
&\qquad\times\Big(\max_{J\neq *}\big\|\mathcal{W}\,\overline{\mathsf{q}}_L^{|g|,J}\big\|_{\ell^q}\Big)\,,
\end{aligned} \tag{5.69}$$

hence (5.68) follows by summing the geometric series. □

REMARK 5.14. *The bound obtained in* (5.62) *holds in general* for any disorder regime of $\beta > 0$. *However, in order to achieve the optimal* $\frac{1}{M^2}$ *dependence on the width of the time interval* $\left(\frac{i-1}{M}N\,,\,\frac{i}{M}N\right]$ *required by Proposition 5.4, it will be necessary to derive estimates which are tailored to the quasi-critical regime. In particular, the terms* $r = 1$ *and* $r \geq 3$ *of the geometric series arisen in* (5.69) *will be shown to be negligible as* $N \to \infty$.

5.3.2. General estimates. In this subsection we obtain universal estimates on the quantities in (5.68). These will be later specified in our context, in order to prove Proposition 5.4.

For $t \in \mathbb{R}$ we define the weights $w_t : \mathbb{Z}^2 \to (0, \infty)$ and $\mathcal{W}_t : (\mathbb{Z}^2)^h \to (0, \infty)$ by

$$w_t(x) := e^{-t|x|}\,, \qquad \mathcal{W}_t(\mathbf{x}) := \prod_{i=1}^{h} w_t(x^i) = \prod_{i=1}^{h} e^{-t|x^i|}\,, \tag{5.70}$$

and note that by the triangle inequality we can bound

$$\frac{\mathcal{W}_t(\mathbf{z})}{\mathcal{W}_t(\mathbf{x})} \leq \prod_{i=1}^{h} e^{|t||z^i - x^i|}\,. \tag{5.71}$$

Given a *pair partition*, or simply *pair*, $I = \{\{a,b\},\{c\}\colon c \neq a,b\}$, we also define $\mathcal{V}_s : (\mathbb{Z}^2)^h \to (0,\infty)$ by

$$\mathcal{V}_s^I(\mathbf{x}) := w_s(x^a - x^b) = e^{-s|x^a - x^b|}\,, \tag{5.72}$$

and by $|z^a - z^b| \leq |z^a - x^a| + |x^a - x^b| + |x^b - z^b|$ we bound

$$\frac{\mathcal{V}_s^I(\mathbf{z})}{\mathcal{V}_s^I(\mathbf{x})} \leq e^{|s||z^a - x^a| + |s||z^b - x^b|}\,. \tag{5.73}$$

We start with some basic random walk estimates.

LEMMA 5.15 (Weighted random walk bounds). *There is* $c \in [1, \infty)$ *such that for all* $t \in \mathbb{R}$ *and* $n \in \mathbb{N}$, *writing* $x = (x^1, x^2)$, *we have*

$$\forall a = 1,2: \qquad \sum_{x \in \mathbb{Z}^2} e^{t x^a} q_n(x) \leq e^{\frac{t^2}{2} n}\,, \qquad \sum_{x \in \mathbb{Z}^2} e^{t x^a} \frac{q_n(x)^2}{q_{2n}(0)} \leq e^{c\,\frac{t^2}{2} n}\,, \tag{5.74}$$

therefore

$$\sum_{x \in \mathbb{Z}^2} e^{t|x|}\, q_n(x) \leq 2\, e^{2\,t^2 n}\,. \tag{5.75}$$

Proof. We bound $\sum_{x \in \mathbb{Z}^2} e^{t|x|}\, q_n(x) \leq \sum_{x \in \mathbb{Z}^2} e^{2t|x^1|}\, q_n(x)$ applying $|x| \leq |x^1| + |x^2|$, Cauchy-Scwharz and symmetry. Since $e^{|z|} \leq e^z + e^{-z}$, the first bound in (5.74) yields (5.75).

To prove the first bound in (5.74), we first compute

$$\sum_{x \in \mathbb{Z}^2} e^{t x^a}\, q_1(x) = \frac{1}{2}(1 + \cosh(t)).$$

Since $\cosh(t) \le \exp(t^2/2)$, the first bound in (5.74) holds for $n = 1$, hence it holds for any $n \in \mathbb{N}$ by independence of increments of the random walk.

To prove the second bound in (5.74), we first note that $q_n(x)^2/q_{2n}(0) \le c\, q_n(x)$ for some $c \in [1, \infty)$, because $q_n(x)^2 \le \|q_n\|_\infty\, q_n(x)$ and $\|q_n\|_\infty \le c\, q_{2n}(0)$ by the local limit theorem (5.32) (for the simple random walk we have $\|q_n\|_\infty = q_n(0)$). Since $q_n(x) = q_n(-x)$, we get

$$\sum_{x\in\mathbb{Z}^2} e^{tx^a}\, \frac{q_n(x)^2}{q_{2n}(0)} - 1 = \sum_{x\in\mathbb{Z}^2} \left(\tfrac{e^{tx^a}+e^{-tx^a}}{2} - 1\right) \frac{q_n(x)^2}{q_{2n}(0)} \le c \sum_{x\in\mathbb{Z}^2} \left(\tfrac{e^{tx^a}+e^{-tx^a}}{2} - 1\right) q_n(x)$$
$$= c\left(e^{\frac{t^2}{2}n} - 1\right) = c \sum_{k=1}^{\infty} \tfrac{1}{k!}\left(\tfrac{t^2}{2}n\right)^k \le \sum_{k=1}^{\infty} \tfrac{1}{k!}\left(c\tfrac{t^2}{2}n\right)^k = e^{c\,\frac{t^2}{2}n} - 1\,,$$

which proves the second bound in (5.74). □

Proposition 5.16 (Left boundary estimate). *For any $f \in \ell^1$, $t \in \mathbb{R}$, $L \in \mathbb{N}$ we have*

$$\max_{I\neq *} \left\| \frac{\widehat{\mathsf{q}}_L^{|f|,I}}{\mathcal{W}_t} \right\|_{\ell^2} \le L\, 2^{\frac{h}{2}}\, e^{(4h^3)t^2L}\, \|f\|_\infty^{\frac{h+1}{2}} \left\| \frac{f}{w_{2|t|h}} \right\|_{\ell^1}^{\frac{h-1}{2}} . \tag{5.76}$$

Moreover, for any pair partition $J = \{\{a,b\},\{c\} : c \neq a, c\neq b\}$ and for any $s > 0$ we have

$$\max_{\substack{I\neq *\\ I\neq J\\ I \text{ is a pair}}} \left\| \frac{\widehat{\mathsf{q}}_L^{|f|,I}}{\mathcal{W}_t} \mathcal{V}_s^J \right\|_{\ell^2} \le L\, 2^{\frac{h-1}{2}}\, e^{80h(t^2+s^2)L}\, \|f\|_\infty^{\frac{h+1}{2}} \left\| \frac{f}{w_{4|t|}} \right\|_{\ell^1}^{\frac{h-3}{2}} \left\| \frac{f}{w_{4|t|}} \right\|_{\ell^2} \frac{C}{s}, \tag{5.77}$$

for some finite constant $C > 0$.

Proof. Recalling (5.58) and (5.53), (5.54), we may write

$$\left\| \frac{\widehat{\mathsf{q}}_L^{|f|,I}}{\mathcal{W}_t} \right\|_{\ell^2}^2 = \sum_{\mathbf{x}=(x^1,\dots,x^h)\in(\mathbb{Z}^2)^h} \frac{\widehat{\mathsf{q}}_L^{f,I}(\mathbf{x})^2}{\mathcal{W}_t(\mathbf{x})^2} = \sum_{\mathbf{x}\in(\mathbb{Z}^2)^h_I} \sum_{n=1}^{L} \sum_{\ell=1}^{L} \prod_{i=1}^{h} q_n^f(x^i)\, q_\ell^f(x^i)\, e^{2t|x^i|} .$$

We can write $I = \{I^1, \dots, I^m\}$ with $I^j \subseteq \{1,\dots,h\}$ for some $m \le h-1$ (the case $m = h$ is excluded because $I \neq *$); notice that the I^j's are mutually disjoint by construction. By definition of $(\mathbb{Z}^2)^h_I$, see (5.52)-(5.53), the coordinates of $\mathbf{x} \in (\mathbb{Z}^2)^h_I$ must agree to each other according to the partition I, namely $x^a = x^b =: y^i$ for all $a, b \in I^i$ and $i = 1, \dots, m$. Therefore, the sum over $\mathbf{x} = (x^1, \dots, x^h) \in (\mathbb{Z}^2)^h_I$ has actually $m < h$ degrees of freedom, thus it can be rearranged by summing over m free variables $y^1, \dots, y^m \in \mathbb{Z}^2$ to get

$$\left\| \frac{\widehat{\mathsf{q}}_L^{|f|,I}}{\mathcal{W}_t} \right\|_{\ell^2}^2 \le \sum_{\mathbf{y}=(y^1,\dots,y^m)\in(\mathbb{Z}^2)^m} \sum_{n=1}^{L} \sum_{\ell=1}^{L} \prod_{j=1}^{m} \left\{ q_n^f(y^j)^{|I^j|}\, q_\ell^f(y^j)^{|I^j|}\, e^{2|t||I^j|\,|y^j|} \right\} .$$

Note that for any $k \in \{1, \dots, h\}$ we can bound

$$\begin{aligned} \sum_{y \in \mathbb{Z}^2} q_n^{|f|}(y)^k \, q_\ell^{|f|}(y)^k \, e^{2|t|k\,|y|} &\le \|f\|_\infty^{2k-1} \sum_{y \in \mathbb{Z}^2} q_n^{|f|}(y) \, e^{2|t|k\,|y|} \\ &\le 2\, e^{8h^2t^2n} \, \|f\|_\infty^{2k-1} \left\| \frac{f}{w_{2|t|h}} \right\|_{\ell^1}, \end{aligned} \tag{5.78}$$

because $q_n^{|f|}(y) \le \|q_n^f\|_\infty \le \|f\|_\infty$ by (5.29) and, by (5.75), for $u = 2|t|\,k \le 2|t|\,h$

$$\sum_{y \in \mathbb{Z}^2} q_n^{|f|}(y) \, e^{u|y|} \le \sum_{z \in \mathbb{Z}^2} e^{u|z|} \, |f(z)| \left(\sum_{y \in \mathbb{Z}^2} e^{u|y-z|} \, q_n(y-z) \right) = \left\| \frac{f}{w_u} \right\|_{\ell^1} 2\, e^{2\,u^2 n}.$$

Since $\sum_{j=1}^m |I^j| = h$, we can bound

$$\left\| \frac{\widehat{\mathsf{q}}_L^{|f|,I}}{\mathcal{W}_t} \right\|_{\ell^2}^2 \le L^2 \, 2^m \, e^{(8h^2m)t^2L} \, \|f\|_\infty^{2h} \left(\frac{1}{\|f\|_\infty} \left\| \frac{f}{w_{2|t|h}} \right\|_{\ell^1} \right)^m,$$

and since $\|\frac{f}{w_s}\|_{\ell^1} \ge \|f\|_{\ell^1} \ge \|f\|_\infty$, for $m \le h-1$ we obtain (5.76).

In order to verify (5.77), we assume without loss of generality that

$$J = \{\{1\}, \{2\}, \dots, \{h-2\}, \{h-1, h\}\}.$$

Then, for any pair I such that $I \ne J$ we write

$$\begin{aligned} \left\| \frac{\widehat{\mathsf{q}}_L^{|f|,I}}{\mathcal{W}_t} \mathcal{V}_s^J \right\|_{\ell^2}^2 &= \sum_{\mathbf{x}=(x^1,\dots,x^h) \in (\mathbb{Z}^2)^h} \frac{\widehat{\mathsf{q}}_L^{f,I}(\mathbf{x})^2}{\mathcal{W}_t(\mathbf{x})^2} \mathcal{V}_s^J(\mathbf{x})^2 \\ &= \sum_{\mathbf{x}=(x^1,\dots,x^h) \in (\mathbb{Z}^2)_I^h} \sum_{n=1}^L \sum_{\ell=1}^L \prod_{i=1}^h q_n^f(x^i) \, q_\ell^f(x^i) \, e^{2t|x^i|} \, e^{-2s|x^h - x^{h-1}|}. \end{aligned}$$

We now write $I = \{I^1, \dots, I^{h-1}\}$ where $I^j \subseteq \{1, \dots, h\}$ is either a singleton or the unique pair in I, which cannot be equal to $\{h-1, h\}$ since $I \ne J$. This implies that the weight $\mathcal{V}_s^J$ cannot be identically equal to 1 (recall 5.72). Without loss of generality, we assume that $h-1 \in I^{h-2}$ and $h \in I^{h-1}$ with $|I^{h-2}|$, $|I^{h-1}| \le 2$. Therefore, we can sum over $h-1$ free variables $y^1, \dots, y^{h-1} \in \mathbb{Z}^2$ and obtain

$$\begin{aligned} \left\| \frac{\widehat{\mathsf{q}}_L^{|f|,I}}{\mathcal{W}_t} \mathcal{V}_s^J \right\|_{\ell^2}^2 \le \sum_{\mathbf{y}=(y^1,\dots,y^{h-1}) \in (\mathbb{Z}^2)^{h-1}} \sum_{n=1}^L \sum_{\ell=1}^L \prod_{j=1}^{h-1} \Big\{ q_n^f(y^j)^{|I^j|} \, q_\ell^f(y^j)^{|I^j|} \, e^{2|t||I^j|\,|y^j|} \Big\} \times \\ \times \, e^{-2s|y^{h-1} - y^{h-2}|}. \end{aligned} \tag{5.79}$$

We first sum over $y^{h-2}, y^{h-1} \in \mathbb{Z}^2$ and we bound

$$\sum_{y^{h-2},y^{h-1}\in\mathbb{Z}^2} q_n^f(y^{h-2})^{|I^{h-2}|}\, q_\ell^f(y^{h-2})^{|I^{h-2}|}\, q_n^f(y^{h-1})^{|I^{h-1}|}\, q_\ell^f(y^{h-1})^{|I^{h-1}|}\, e^{2|t|\,|I^{h-2}|\,|y^{h-2}|}\times$$
$$\times\, e^{2|t|\,|I^{h-1}|\,|y^{h-1}|}\, e^{-2s|y^{h-1}-y^{h-2}|}$$
$$\le \|f\|_\infty^{2(|I^{h-2}|+|I^{h-1}|-1)} \sum_{y^{h-2},y^{h-1}\in\mathbb{Z}^2} q_n^f(y^{h-2})\, q_n^f(y^{h-1})\, e^{2|t|\,|I^{h-2}|\,|y^{h-2}|}\, e^{2|t|\,|I^{h-1}|\,|y^{h-1}|}\times$$
$$\times\, e^{-2s|y^{h-1}-y^{h-2}|}\,,$$

because $q_n^f(y) \le \|q_n^f\|_\infty \le \|f\|_\infty$. By (5.29), the triangular inequality similar to (5.73), Cauchy-Schwarz and (5.75) and by recalling that $|I^{h-2}|$, $|I^{h-1}| \le 2$, we have

$$\sum_{y^{h-2},y^{h-1}\in\mathbb{Z}^2} q_n^f(y^{h-2})\, q_n^f(y^{h-1})\, e^{2|t|\,|I^{h-2}|\,|y^{h-2}|}\, e^{2|t|\,|I^{h-1}|\,|y^{h-1}|}\, e^{-2s|y^{h-1}-y^{h-2}|}$$
$$\le \sum_{z,z'\in\mathbb{Z}^2} |f(z)|\,|f(z')|\, e^{4|t||z|}\, e^{4|t||z'|}\, e^{-2s|z-z'|}\times$$
$$\times \Big\{ \sum_{y^{h-2}\in\mathbb{Z}^2} e^{(4|t|+2s)|y^{h-2}-z'|}\, q_n(y^{h-2}-z') \Big\} \Big\{ \sum_{y^{h-1}\in\mathbb{Z}^2} e^{(4|t|+2s)|y^{h-1}-z|}\, q_n(y^{h-1}-z) \Big\}$$
$$\le 4e^{4(4|t|+2s)^2 n} \Big(\sum_{z,z'\in\mathbb{Z}^2} \big(|f(z)|e^{4|t||z|}\big)^2 e^{-2s|z-z'|} \Big)^{\frac12} \Big(\sum_{z,z'\in\mathbb{Z}^2} \big(|f(z')|e^{4|t||z'|}\big)^2 e^{-2s|z-z'|} \Big)^{\frac12}$$
$$\le 4e^{128(t^2+s^2)n} \frac{C}{s^2} \Big\| \frac{f}{w_{4|t|}} \Big\|_{\ell^2}^2 ,$$

where the last inequality holds because $\sum_{z\in\mathbb{Z}^2} e^{-s|z|} \le \frac{C}{s^2}$ for some $C < \infty$. The remaining sum over $y^{h-3}, \dots, y^1 \in \mathbb{Z}^2$ can be treated as in (5.78) with $k = |I^j| \le 2$, then we obtain the bound

$$2e^{32t^2 n}\, \|f\|_\infty^{2|I^j|-1} \Big\| \frac{f}{w_{4|t|}} \Big\|_{\ell^1} \qquad j = 1, \dots, h-3\,.$$

Since $\sum_{j=1}^{h-1} |I^j| = h$, from (5.79) we get

$$\Big\| \frac{\widehat{\mathsf{q}}_L^{|f|,I}}{\mathcal{W}_t} \Big\|_{\ell^2}^2 \le L^2\, 2^{h-1}\, e^{32(h-3)t^2 L + 128(t^2+s^2)L}\, \|f\|_\infty^{h+1} \Big\| \frac{f}{w_{4|t|}} \Big\|_{\ell^1}^{h-3} \Big\| \frac{f}{w_{4|t|}} \Big\|_{\ell^2}^2 \frac{C}{s^2}$$
$$\le L^2\, 2^{h-1}\, e^{160h(t^2+s^2)L}\, \|f\|_\infty^{h+1} \Big\| \frac{f}{w_{4|t|}} \Big\|_{\ell^1}^{h-3} \Big\| \frac{f}{w_{4|t|}} \Big\|_{\ell^2}^2 \frac{C}{s^2}\,,$$

which yields (5.77).

□

In the next bound we can place the weight $\mathcal{V}_s$ in the denominator on both sides.

Proposition 5.17 (Green's function estimate). *There are constants $C, c < \infty$ such that, for all $t \in \mathbb{R}$ and $L \in \mathbb{N}$,*

$$\max_{I,J\neq *,\, I\neq J} \big\| \mathcal{W}_t\, \widehat{\mathsf{Q}}_L^{I,J} \tfrac{1}{\mathcal{W}_t} \big\|_{\ell^2\to\ell^2} \le C\, e^{\frac{h}{2}\, c t^2 L} \tag{5.80}$$

Moreover, for partitions that are pairs, for all $s, t \in \mathbb{R}$ and $L \in \mathbb{N}$ we can bound

$$\max_{I,J \text{ pairs}, I \neq J} \left\| \tfrac{\mathcal{W}_t}{\mathcal{V}_s^J} \, \widehat{\mathsf{Q}}_L^{I,J} \, \tfrac{1}{\mathcal{V}_s^I \mathcal{W}_t} \right\|_{\ell^2 \to \ell^2} \leq C \, e^{4h \, c \, (t^2+s^2) L} \tag{5.81}$$

Proof. We are going to use a crucial functional inequality proved in [**CSZ21+**, Lemma 6.8]: for some constant $C = C_h < \infty$ we have

$$\sum_{\mathbf{z} \in (\mathbb{Z}^2)^h_I, \, \mathbf{x} \in (\mathbb{Z}^2)^h_J} \frac{f(\mathbf{z}) \, g(\mathbf{x})}{(1 + |\mathbf{x} - \mathbf{z}|^2)^{h-1}} \leq C \, \|f\|_{\ell^2} \, \|g\|_{\ell^2} \, . \tag{5.82}$$

We first state a basic random walk bound: since $\widehat{\mathsf{Q}}_L^{*,*}(\mathbf{z}, \mathbf{x}) = \sum_{n=1}^L \prod_{i=1}^h q_n(x^i - z^i)$ (the partition $*$ imposes no constraints on $\mathbf{z}, \mathbf{x}$), it follows by [**CSZ21+**, Lemma 6.7] that

$$\widehat{\mathsf{Q}}_L^{*,*}(\mathbf{z}, \mathbf{x}) \leq \begin{cases} \dfrac{C}{(1 + |\mathbf{x} - \mathbf{z}|^2)^{h-1}} & \text{for all } \mathbf{x}, \mathbf{z} \in (\mathbb{Z}^2)^h \, , \\ \dfrac{C}{L^{h-1}} \, e^{-\frac{|\mathbf{x}-\mathbf{z}|^2}{CL}} & \text{for } |\mathbf{x} - \mathbf{z}| > \sqrt{L} \, . \end{cases}$$

(Note that the first line is the behaviour of the Green's function of a random walk of dimension $2h$.) We can combine the estimates in the two lines as follows, for a suitable $c < \infty$:

$$\widehat{\mathsf{Q}}_L^{*,*}(\mathbf{z}, \mathbf{x}) \leq \frac{c}{(1 + |\mathbf{x} - \mathbf{z}|^2)^{h-1}} \; e^{-\frac{|\mathbf{x}-\mathbf{z}|^2}{2CL}} \, . \tag{5.83}$$

Indeed, for $|\mathbf{x} - \mathbf{z}| \leq \sqrt{L}$ we have $1 \leq e^{\frac{1}{2C}} \, e^{-\frac{|\mathbf{x}-\mathbf{z}|^2}{2CL}}$, while for $|\mathbf{x} - \mathbf{z}| > \sqrt{L}$ we bound

$$\frac{1}{L^{h-1}} = \frac{(2C)^{h-1}}{(1 + |\mathbf{x} - \mathbf{z}|^2)^{h-1}} \left(\tfrac{1+|\mathbf{x}-\mathbf{z}|^2}{2CL} \right)^{h-1} \leq \frac{C'}{(1 + |\mathbf{x} - \mathbf{z}|^2)^{h-1}} \, e^{\frac{|\mathbf{x}-\mathbf{z}|^2}{2CL}} \, ,$$

where we used $a^{h-1} \leq (h-1)! \, e^a$ for $a \geq 0$ and we set $C' = (2C)^{h-1} (h-1)! \, e^{\frac{1}{2C}}$.

Thanks to (5.83) and (5.71), since $\widehat{\mathsf{Q}}_L^{I,J}(\mathbf{z}, \mathbf{x}) = \widehat{\mathsf{Q}}_L^{*,*}(\mathbf{z}, \mathbf{x}) \, \mathbb{1}_{\{\mathbf{z} \sim I, \mathbf{x} \sim J\}}$ we can estimate

$$\big(\mathcal{W}_t \, \widehat{\mathsf{Q}}_L^{I,J} \tfrac{1}{\mathcal{W}_t} \big)(\mathbf{z}, \mathbf{x}) \leq \frac{c \, \mathbb{1}_{\{\mathbf{z} \sim I, \mathbf{x} \sim J\}}}{(1 + |\mathbf{x} - \mathbf{z}|^2)^{h-1}} \prod_{i=1}^h e^{|t| |z^i - x^i| - \frac{1}{2CL} |z^i - x^i|^2} \leq \frac{c \, e^{\frac{h}{2} C t^2 L} \, \mathbb{1}_{\{\mathbf{z} \sim I, \mathbf{x} \sim J\}}}{(1 + |\mathbf{x} - \mathbf{z}|^2)^{h-1}} \, ,$$

because $\max_{a \in \mathbb{R}} \{ |t| a - \frac{1}{2CL} a^2 \} = \frac{1}{2} C L t^2$. Applying (5.82), we have proved (5.80).

Finally, let I, J be pairs, say $I = \{\{a, b\}, \{c\} \colon c \neq a, c \neq b\}$ and $J = \{\{\widetilde{a}, \widetilde{b}\}, \{c\} \colon c \neq \widetilde{a}, c \neq \widetilde{b}\}$. For $\mathbf{z} \sim I$ and $\mathbf{x} \sim J$ we have $z^a = z^b$, hence

$$\frac{1}{\mathcal{V}_s^I(\mathbf{x})} \leq e^{|s| |x^a - x^b|} \leq e^{|s| \{|x^a - z^a| + |z^a - z^b| + |z^b - x^b|\}} = e^{|s| |x^a - z^a|} \, e^{|s| |z^b - x^b|} \, ,$$

and similarly $\frac{1}{\mathcal{V}_s^J(\mathbf{z})} \leq e^{|s| |x^{\widetilde{a}} - z^{\widetilde{a}}|} \, e^{|s| |z^{\widetilde{b}} - x^{\widetilde{b}}|}$. Arguing as above, we obtain

$$\begin{aligned} \big(\tfrac{\mathcal{W}_t}{\mathcal{V}_s^J} \, \widehat{\mathsf{Q}}_L^{I,J} \tfrac{1}{\mathcal{V}_s^I \mathcal{W}_t} \big)(\mathbf{z}, \mathbf{x}) &\leq \frac{c \, \mathbb{1}_{\{\mathbf{z} \sim I, \mathbf{x} \sim J\}}}{(1 + |\mathbf{x} - \mathbf{z}|^2)^{h-1}} \prod_{i=1}^h e^{(|t| + 2|s|) |z^i - x^i| - \frac{1}{2CL} |z^i - x^i|^2} \\ &\leq \frac{c \, e^{\frac{h}{2} C (|t| + 2|s|)^2 L} \, \mathbb{1}_{\{\mathbf{z} \sim I, \mathbf{x} \sim J\}}}{(1 + |\mathbf{x} - \mathbf{z}|^2)^{h-1}} \, , \end{aligned}$$

and (5.81) follows because $(|t|+2|s|)^2 \le 2(t^2+4s^2)$. □

In the next result we assume that $|\mathbb{E}[\xi_\beta^I]| \le \sigma_\beta^2$. This always holds if I is a pair, while if $I \ne *$ is not a pair *it holds for* $\beta > 0$ *small enough*, see (5.56), since $|\mathbb{E}[\xi_\beta^k]| \le C_k\, \sigma_\beta^k$ by (5.26).

PROPOSITION 5.18 (Bulk estimate). *Fix any partition* $I \ne *$ *and assume that* $|\mathbb{E}[\xi_\beta^I]| \le \sigma_\beta^2$. *Then for any* $h \in \mathbb{N}$, $s,t \in \mathbb{R}$, $L \in \mathbb{N}$, $\beta > 0$ *and for any* pair J *we have*

$$\big\| \, \mathcal{V}_s^J \mathcal{W}_t \, |\widehat{\mathsf{U}}|_{L,0,\beta}^I \, \tfrac{1}{\mathcal{V}_s^J \mathcal{W}_t} \, \big\|_{\ell^2 \to \ell^2} \le 1 + \Big\{ 2^h \, e^{8c\, h\, (t^2+s^2) L} \Big\} \, \frac{\sigma_\beta^2 \, R_L}{1-\sigma_\beta^2 \, R_L} \,. \tag{5.84}$$

REMARK 5.19. *Notice that by taking* $-s$ *instead of* $s \in \mathbb{R}$ *in* (5.84) *(recall* (5.72)*), for any* $h \in \mathbb{N}$, $s,t \in \mathbb{R}$, $L \in \mathbb{N}$, $\beta > 0$ *and for any pair* J *we also obtain*

$$\big\| \, \tfrac{1}{\mathcal{V}_s^J} \mathcal{W}_t \, |\widehat{\mathsf{U}}|_{L,0,\beta}^I \, \tfrac{\mathcal{V}_s^J}{\mathcal{W}_t} \, \big\|_{\ell^2 \to \ell^2} \le 1 + \Big\{ 2^h \, e^{8c\, h\, (t^2+s^2) L} \Big\} \, \frac{\sigma_\beta^2 \, R_L}{1-\sigma_\beta^2 \, R_L} \,. \tag{5.85}$$

Proof. Let us omit the subscript from $|\widehat{\mathsf{U}}|^I$ for a moment: starting from the formula

$$\big\| |\widehat{\mathsf{U}}|^I \big\|_{\ell^2 \to \ell^2} := \sup_{\mathsf{f},\mathsf{g}:\ \|\mathsf{f}\|_{\ell^2} \le 1,\ \|\mathsf{g}\|_{\ell^2} \le 1} \ \sum_{\mathbf{z},\mathbf{x} \in (\mathbb{Z}^2)_I^h} \mathsf{f}(\mathbf{z}) \, |\widehat{\mathsf{U}}|^I(\mathbf{z},\mathbf{x}) \, \mathsf{g}(\mathbf{x}) \,,$$

we can bound

$$\sum_{\mathbf{z},\mathbf{x}} \mathsf{f}(\mathbf{z}) \, |\widehat{\mathsf{U}}|^I(\mathbf{z},\mathbf{x}) \, \mathsf{g}(\mathbf{x}) \le \Big(\sum_{\mathbf{z},\mathbf{x}} \mathsf{f}(\mathbf{z})^2 \, |\widehat{\mathsf{U}}|^I(\mathbf{z},\mathbf{x}) \Big)^{1/2} \Big(\sum_{\mathbf{z},\mathbf{x}} |\widehat{\mathsf{U}}|^I(\mathbf{z},\mathbf{x}) \, \mathsf{g}(\mathbf{x})^2 \Big)^{1/2}$$

by Cauchy-Schwarz, hence

$$\big\| |\widehat{\mathsf{U}}|^I \big\|_{\ell^2 \to \ell^2} \le \max \Bigg\{ \sup_{\mathbf{z} \in (\mathbb{Z}^2)_I^h} \sum_{\mathbf{x} \in (\mathbb{Z}^2)_I^h} |\widehat{\mathsf{U}}|^I(\mathbf{z},\mathbf{x}) \,, \ \sup_{\mathbf{x} \in (\mathbb{Z}^2)_I^h} \sum_{\mathbf{z} \in (\mathbb{Z}^2)_I^h} |\widehat{\mathsf{U}}|^I(\mathbf{z},\mathbf{x}) \Bigg\} \,. \tag{5.86}$$

We will prove (5.84) exploiting this bound.

We need some preliminary definitions. Let us set for $n \in \mathbb{N}$, $\beta > 0$ and $x \in \mathbb{Z}^2$

$$U_{n,\beta}(x) := \sum_{k=1}^{\infty} (\sigma_\beta^2)^k \sum_{\substack{0 =: n_0 < n_1 < \cdots < n_k := n \\ x_0 := 0,\ x_1, \ldots, x_{k-1} \in \mathbb{Z}^2,\ x_k := x}} \ \prod_{i=1}^{k} q_{n_i - n_{i-1}}(x_i - x_{i-1})^2 \,, \tag{5.87}$$

and also

$$U_{n,\beta} := \sum_{x \in \mathbb{Z}^2} U_{n,\beta}(x) = \sum_{k=1}^{\infty} (\sigma_\beta^2)^k \sum_{0 =: n_0 < n_1 < \cdots < n_k := n} \ \prod_{i=1}^{k} q_{2(n_i - n_{i-1})}(0) \,. \tag{5.88}$$

When we sum $U_{n,\beta}$ for $n = 1, \ldots, L$, if we enlarge the sum range in (5.88) by letting each increment $m_i := n_i - n_{i-1}$ vary freely in $\{1, \ldots, M\}$, we obtain

$$\sum_{n=1}^{L} U_{n,\beta} \le \sum_{k=1}^{\infty} (\sigma_\beta^2)^k \Bigg(\sum_{m=1}^{L} q_{2m}(0) \Bigg)^k = \sum_{k=1}^{\infty} (\sigma_\beta^2 \, R_L)^k = \frac{\sigma_\beta^2 \, R_L}{1 - \sigma_\beta^2 \, R_L} \,, \tag{5.89}$$

where we recall that $R_L = \sum_{n=1}^{L} q_{2n}(0)$, see (5.6).

We next estimate the exponential spatial moments of $U_{n,\beta}(x)$. Plugging the second bound from (5.74) into (5.87) yields, writing $x = (x^1, x^2)$ and $x^a = \sum_{i=1}^k (x_i^a - x_{i-1}^a)$,

$$\forall a = 1,2 : \qquad \sum_{x\in\mathbb{Z}^2} e^{tx^a}\, U_{n,\beta}(x) \le e^{c\,\frac{t^2}{2}\,n}\, U_{n,\beta}\,.$$

From this we deduce that

$$\sum_{x\in\mathbb{Z}^2} e^{t|x|}\, U_{n,\beta}(x) \le 2\, e^{2c\,t^2\,n}\, U_{n,\beta} \tag{5.90}$$

by $|x| \le |x^1| + |x^2|$, applying Cauchy-Schwarz and then $e^{t|x^a|} \le e^{tx^a} + e^{-tx^a}$.

We are now ready to prove (5.84). *Let I be a pair*, say $I = \{\{a,b\},\{c\}\colon c \ne a,b\}$. For $\mathbf{z},\mathbf{x} \in (\mathbb{Z}^2)^h_I$ we have $z^a = z^b$ and $x^a = x^b$, see (5.52), hence

$$\mathsf{Q}_n^{I,I}(\mathbf{z},\mathbf{x}) = q_n(x^a - z^a)^2 \prod_{c\ne a,b} q_n(x^c - z^c)\,,$$

and since $\mathbb{E}[\xi_\beta^I] = \sigma_\beta^2$ we obtain from (5.55) and (5.59), recalling (5.87) and using Chapman-Kolmogorov,

$$|\mathsf{U}|^I_{n,\beta}(\mathbf{z},\mathbf{x}) = \mathsf{U}^I_{n,\beta}(\mathbf{z},\mathbf{x}) = U_{n,\beta}(x^a - z^a) \prod_{c\ne a,b} q_n(x^c - z^c)\,. \tag{5.91}$$

For any pair J, by (5.71) and (5.73) we have the rough bound

$$\frac{\mathcal{W}_t(\mathbf{z})\,\mathcal{V}_s(\mathbf{z})}{\mathcal{W}_t(\mathbf{x})\,\mathcal{V}_s(\mathbf{x})} \le e^{2(|t|+|s|)|x^a - z^a|} \prod_{c\ne a,b} e^{(|t|+|s|)|x^c - z^c|}\,. \tag{5.92}$$

We next multiply (5.91) and (5.92) ans sum over $\mathbf{x}$: by (5.90) and the bound in (5.75), since $8 + 2(h-2) = 4 + 2h \le 4h$ for $h \ge 2$ and $(|t|+|s|)^2 \le 2(t^2+s^2)$, we obtain

$$\text{if } I \text{ is a pair:} \qquad \sum_{\mathbf{x}\in(\mathbb{Z}^2)^h_I} \left(|\mathsf{U}|^I_{n,\beta}(\mathbf{z},\mathbf{x})\, \frac{\mathcal{W}_t(\mathbf{x})\,\mathcal{V}_s(\mathbf{x})}{\mathcal{W}_t(\mathbf{z})\,\mathcal{V}_s(\mathbf{z})} \right) \le 2^h\, e^{8hc\,(t^2+s^2)\,n}\, U_{n,\beta}\,. \tag{5.93}$$

Let now $I = \{I^1,\ldots,I^m\} \ne *$ not be a pair. We may order $|I^1| \ge |I^2| \ge \ldots \ge |I^m|$, therefore $|I^1| \ge 2$, and for $\mathbf{z},\mathbf{x} \in (\mathbb{Z}^2)^h_I$ we can write, with self-explaining notation,

$$\mathsf{Q}_n^{I,I}(\mathbf{z},\mathbf{x}) = q_n(x^{I_1} - z^{I_1})^{|I_1|} \prod_{j=2}^m q_n(x^{I_j} - z^{I_j})^{|I_j|}\,.$$

Bounding $q_n(\cdot)^{|I_1|} \le \|q_n\|_\infty^{|I_1|-2}\, q_n(\cdot)^2$ and, for $j \ge 2$, $q_n(\cdot)^{|I_j|} \le \|q_n\|_\infty^{|I_j|-1}\, q_n(\cdot)$, we obtain

$$\mathsf{Q}_n^{I,I}(\mathbf{z},\mathbf{x}) \le \|q_n\|_\infty^{h-m-1}\, q_n(x^{I_1} - z^{I_1})^2 \prod_{j=2}^m q_n(x^{I_j} - z^{I_j})\,.$$

Replacing $|\mathbb{E}[\xi_\beta^I]|$ in (5.55) by σ_β^2 (since we assume that $|\mathbb{E}[\xi_\beta^I]| \le \sigma_\beta^2$), we can bound

$$|\mathsf{U}|^I_{n,\beta}(\mathbf{z},\mathbf{x}) \le \|q_n\|_\infty^{h-m-1}\, U_{n,\beta}(x^{I_1} - z^{I_1}) \prod_{j=2}^m q_n(x^{I_j} - z^{I_j})\,.$$

Note that $m \le h-2$ when I is not a pair, hence $\|q_n\|_\infty^{h-m-1} \le \|q_n\|_\infty$. We then obtain a modification of (5.93) (note that $\|q_n\|_\infty \le 1$):

$$\text{if } I \text{ is not a pair:} \quad \sum_{\mathbf{x}\in(\mathbb{Z}^2)^h_I} \left(|\mathsf{U}|^I_{n,\beta}(\mathbf{z},\mathbf{x}) \, \frac{\mathcal{W}_t(\mathbf{x})\,\mathcal{V}_s(\mathbf{x})}{\mathcal{W}_t(\mathbf{z})\,\mathcal{V}_s(\mathbf{z})} \right) \le \|q_n\|_\infty \, 2^h \, e^{8hc\,(t^2+s^2)\,n} \, U_{n,\beta} \,. \tag{5.94}$$

Overall, recalling (5.60) and (5.70), for any partition $I \neq *$ we have

$$\sup_{\mathbf{z}\in(\mathbb{Z}^2)^h_I} \sum_{\mathbf{x}\in(\mathbb{Z}^2)^h_I} |\widehat{\mathsf{U}}|^J_{\mathcal{W}_t\mathcal{V}_s^J,L,0,\beta}(\mathbf{z},\mathbf{x}) \le 1 + 2^h \, e^{8hc\,(t^2+s^2)L} \sum_{n=1}^{L} U_{n,\beta} \,, \tag{5.95}$$

and the same holds exchanging $\mathbf{x}$ and $\mathbf{z}$ by symmetry (note that the bound (5.92) is symmetric in $\mathbf{x} \leftrightarrow \mathbf{z}$). Recalling (5.89) and (5.86), we obtain (5.84). $\square$

Proposition 5.20 (Right boundary estimate). *There is $C<\infty$ such that, for any $t>0$ and $L\in\mathbb{N}$, we have*

$$\max_{J\neq *} \big\| \overline{\mathsf{q}}_L^{|g|,J} \, \mathcal{W}_t \big\|_{\ell^2} \le C \, \frac{\|g\|_\infty^h}{t^{h-1}} \,. \tag{5.96}$$

Moreover, for any pair partition I and for $s>0$, we have

$$\max_{\substack{J\neq * \\ J\neq I \\ J \text{ is a pair}}} \big\| \overline{\mathsf{q}}_L^{|g|,J} \, \mathcal{W}_t \, \mathcal{V}_s^I \big\|_{\ell^2} \le C \frac{\|g\|_\infty^h}{t^{h-2} s} \,. \tag{5.97}$$

Proof. By (5.54) we can bound $\mathsf{q}_n^{|g|,J}(\mathbf{x}) \le \|g\|_\infty^h \, \mathbb{1}_{\{\mathbf{x}\sim J\}}$, hence also $\overline{\mathsf{q}}_L^{|g|,J}(\mathbf{x}) \le \|g\|_\infty^h \, \mathbb{1}_{\{\mathbf{x}\sim J\}}$, see (5.61). It follows that

$$\big\| \overline{\mathsf{q}}_L^{|g|,J} \, \mathcal{W}_t \big\|_{\ell^2}^2 \le \|g\|_\infty^{2h} \sum_{\mathbf{x}\in(\mathbb{Z}^2)^h_J} \mathcal{W}_t(\mathbf{x})^2 \,.$$

Writing $J = \{J^1, \ldots, J^m\}$ we get

$$\big\| \overline{\mathsf{q}}_L^{|g|,J} \, \mathcal{W}_t \big\|_{\ell^2}^2 \le \|g\|_\infty^{2h} \sum_{\mathbf{y}\in(\mathbb{Z}^2)^m} \prod_{j=1}^m \mathcal{W}_t(y^j)^{2|J^j|} \le \|g\|_\infty^{2h} \prod_{j=1}^m \sum_{y^j\in\mathbb{Z}^2} e^{-2t|y^j|} \,, \tag{5.98}$$

where we bounded $|J^j| \ge 1$ in the last inequality. Since $\sum_{y\in\mathbb{Z}^2} e^{-s|y|} \le \frac{C}{s^2}$ for some $C<\infty$, and since $m \le h-1$ for $J \neq *$, we obtain (5.96).

In order verify (5.97) we assume withouth loss of generality that

$$I = \{1\} \cup \cdots \cup \{h-2\} \cup \{h-1,\, h\}.$$

This implies that $h-1$ and h cannot form the unique pair in J, since $I \neq J$, thus the weight $\mathcal{V}_s^I$ is not identically equal to 1 (recall (5.72)). Then, by writing

$J = \{J^1, \dots, J^{h-1}\}$ and by assuming without loss of generality that $h-1 \in J^{h-2}$ and $h \in J^{h-1}$ with $|I^{h-2}|$, $|I^{h-1}| \geq 1$, we get

$$\big\| \overline{\mathsf{q}}_L^{|g|,J} \, \mathcal{W}_t \, \mathcal{V}_s^I \big\|_{\ell^2}^2 \leq \|g\|_\infty^{2h} \sum_{\boldsymbol{y}=(y^1,\dots,y^{h-1}) \in (\mathbb{Z}^2)^{h-1}} \prod_{j=1}^{h-1} \mathcal{W}_t(y^j)^2 \mathcal{V}_s^I(\boldsymbol{y})$$

$$\leq \|g\|_\infty^{2h} \sum_{\boldsymbol{y}=(y^1,\dots,y^{h-1}) \in \mathbb{Z}^2} \prod_{j=1}^{h-1} e^{-2t|y^j|} \, e^{-2s|y^{h-1}-y^{h-2}|} \, .$$

For fixed $y^{h-2} \in \mathbb{Z}^2$ we can bound

$$\sum_{y^{h-1} \in \mathbb{Z}^2} e^{-2t|y^{h-1}|} \, e^{-2s|y^{h-1}-y^{h-2}|} \leq \sum_{y^{h-1} \in \mathbb{Z}^2} e^{-2s|y^{h-1}-y^{h-2}|} \leq \frac{C}{s^2}$$

for some constant $C < \infty$, while the remaining sum over $y^{h-2}, \dots, y^1$ can be treated as in (5.98). This completes the proof. $\square$

5.3.3. Proof of Proposition 5.4. By formula (5.51) from Remark 5.6, we can write

$$\mathbb{E}\left[(X_{N,M}^{(i)})^4 \right] = \frac{\theta_N^2}{N^4} \, \mathcal{M}_{L,\beta}^4(f,g) \tag{5.99}$$

where L, β, f, g are given as follows (for $i = 1, \dots, M$):

$$L = \frac{N}{M} \, , \qquad \beta = \beta_N \ \text{ in (5.9)} \, , \qquad f(\cdot) = q^{\varphi_N}_{\frac{i-1}{M}N}(\cdot) \ \text{ in (5.15)-(5.29)} \, , \qquad g(\cdot) \equiv 1 \, . \tag{5.100}$$

From (5.99) we can bound $\mathcal{M}_{\frac{N}{M},\beta_N}(f,g)$ exploiting (5.62) with

$$h = 4 \, , \qquad p = q = 2 \, , \qquad \lambda = 0 \, .$$

and obtain

$$\begin{aligned} \mathbb{E}\left[(X_{N,M}^{(i)})^4 \right] &\leq \frac{\theta_N^2}{N^4} \sum_{r=1}^{\infty} \sum_{\substack{I_1,\dots,I_r \vdash \{1,2,3,4\} \\ \text{with full support} \\ \text{and } I_i \neq I_{i-1}, \, I_i \neq * \ \forall i}} \left\{ \prod_{i=1}^{r} \left| \mathbb{E}[\xi_{\beta_N}^{I_i}] \right| \right\} \times \\ &\qquad \times \left\langle \widehat{\mathsf{q}}_{\frac{N}{M}}^{|f|,I_1} \, , \, |\widehat{\mathsf{U}}|^{I_1}_{\frac{N}{M},0,\beta_N} \left\{ \prod_{i=2}^{r} \widehat{\mathsf{Q}}_{\frac{N}{M}}^{I_{i-1},I_i} \, |\widehat{\mathsf{U}}|^{I_i}_{\frac{N}{M},0,\beta_N} \right\} \overline{\mathsf{q}}_{\frac{N}{M}}^{1,I_r} \right\rangle \\ &= \Xi_{N,M}^{r=1} + \Xi_{N,M}^{r=2} + \Xi_{N,M}^{r\geq 3} \, , \end{aligned} \tag{5.101}$$

where $\Xi^{r=1}$, $\Xi^{r=2}$ and $\Xi^{r\geq 3}$ are respectively the terms $r=1$, $r=2$ and $r \geq 3$ of the series above, precisely:

$$\Xi_{N,M}^{r=1} := \frac{\theta_N^2}{N^4} \sum_{\substack{I \vdash \{1,2,3,4\} \\ \text{with full support} \\ I \neq *}} \left| \mathbb{E}[\xi_{\beta_N}^I] \right| \left\langle \widehat{\mathsf{q}}_{\frac{N}{M}}^{|f|,I} \, , \, |\widehat{\mathsf{U}}|^I_{\frac{N}{M},0,\beta_N} \, \overline{\mathsf{q}}_{\frac{N}{M}}^{1,I} \right\rangle ,$$

$$\Xi^{r=2}_{N,M} := \frac{\theta_N^2}{N^4} \sum_{\substack{I,J \vdash \{1,2,3,4\} \\ \text{with full support} \\ I\neq J,\ I,J\neq *}} \big|\mathbb{E}[\xi^I_{\beta_N}]\big|\,\big|\mathbb{E}[\xi^J_{\beta_N}]\big| \Big\langle \widehat{\mathsf{q}}^{|f|,I}_{\frac{N}{M}}\ ,\ |\widehat{\mathsf{U}}|^I_{\frac{N}{M},0,\beta_N}\, \widehat{\mathsf{Q}}^{I,J}_{\frac{N}{M}}\, |\widehat{\mathsf{U}}|^J_{\frac{N}{M},0,\beta_N}\, \overline{\mathsf{q}}^{1,J}_{\frac{N}{M}} \Big\rangle ,$$

and

$$\Xi^{r\geq 3}_{N,M} := \frac{\theta_N^2}{N^4} \sum_{r=3}^{\infty} \sum_{\substack{I_1,\dots,I_r \vdash \{1,2,3,4\} \\ \text{with full support} \\ \text{and } I_i\neq I_{i-1},\ I_i\neq *\ \forall i}} \Big\{ \prod_{i=1}^r \big|\mathbb{E}[\xi^{I_i}_{\beta_N}]\big| \Big\} \times$$

$$\times \Big\langle \widehat{\mathsf{q}}^{|f|,I_1}_{\frac{N}{M}}\ ,\ |\widehat{\mathsf{U}}|^{I_1}_{\frac{N}{M},0,\beta_N} \Big\{ \prod_{i=2}^r \widehat{\mathsf{Q}}^{I_{i-1},I_i}_{\frac{N}{M}}\ |\widehat{\mathsf{U}}|^{I_i}_{\frac{N}{M},0,\beta_N} \Big\} \overline{\mathsf{q}}^{1,I_r}_{\frac{N}{M}} \Big\rangle .$$

We now show that, when $N\to\infty$, the non-negligible terms in (5.101) are only those in $\Xi^{r=2}$ (with *both partition* I_1 *and* I_2 *pairs*), namely for any $M\in\mathbb{N}$ and N large enough:

$$\Xi^{r=2}_{N,M} \leq \frac{C}{M^2}\,, \tag{5.102}$$

while

$$\lim_{N\to\infty} \Xi^{r=1} = 0 \qquad \text{and} \qquad \lim_{N\to\infty} \Xi^{r\geq 3} = 0\,. \tag{5.103}$$

From the expressions in (5.102) and (5.103), we finally prove Proposition 5.4.

□

5.3.3.1. ***Terms*** $r\geq 3$. We bound $\Xi^{r\geq3}$ by (5.66), where we fix the weight $\mathcal{W}_t$ as in (5.70). Then, by arguing as in the proof of Theorem 5.13, we have

$$\Xi^{r\geq 3}_{N,M} \leq \frac{\theta_N^2}{N^4} \Big(\max_{I\neq *} \big\| \widehat{\mathsf{q}}^{|f|,I}_{\frac{N}{M}} \tfrac{1}{\mathcal{W}_t} \big\|_{\ell^2} \Big) \frac{(\mathsf{C}^2_{\mathcal{W}_t,\frac{N}{M}})^2 (\rho^2_{\mathcal{W}_t,\frac{N}{M},0,\beta_N})^3}{1-\mathsf{C}^2_{\mathcal{W}_t,\frac{N}{M}}\,\rho^2_{\mathcal{W}_t,\frac{N}{M},0,\beta_N}} \Big(\max_{J\neq *} \big\| \mathcal{W}_t\, \overline{\mathsf{q}}^{|1|,J}_{\frac{N}{M}} \big\|_{\ell^2} \Big)\,, \tag{5.104}$$

where we recall that

$$\mathsf{C}^2_{\mathcal{W}_t,\frac{N}{M}} := \max_{\substack{I,J\neq * \\ I\neq J}} \big\| \mathcal{W}_t\, \widehat{\mathsf{Q}}^{I,J}_{\frac{N}{M}}\, \tfrac{1}{\mathcal{W}_t} \big\|_{\ell^2\to\ell^2}\,,$$

and

$$\rho^2_{\mathcal{W}_t,\frac{N}{M},0,\beta_N} := \sum_{I\neq *} |\mathbb{E}[\xi^I_{\beta_N}]|\, \big\| \mathcal{W}_t\, |\widehat{\mathsf{U}}|^I_{\frac{N}{M},0,\beta_N} \tfrac{1}{\mathcal{W}_t} \big\|_{\ell^2\to\ell^2}\,.$$

Therefore, we need to estimate the four quantities

$$\max_{I\neq *} \big\| \widehat{\mathsf{q}}^{|f|,I}_{\frac{N}{M}} \tfrac{1}{\mathcal{W}_t} \big\|_{\ell^2} \qquad \mathsf{C}^2_{\mathcal{W}_t,\frac{N}{M}} \qquad \rho^2_{\mathcal{W}_t,\frac{N}{M},0,\beta_N} \qquad \max_{J\neq *} \big\| \mathcal{W}_t\, \overline{\mathsf{q}}^{|1|,J}_{\frac{N}{M}} \big\|_{\ell^2}\,.$$

We are going to exploit (5.76), (5.80), (5.84) and (5.96) with

$$t = \frac{1}{\sqrt{N}}\,, \qquad L = \frac{N}{M}\,, \qquad \beta = \beta_N \text{ in (5.9)}\,.$$

For convenience, we write $a \lesssim b$ whenever $a \leq C\,b$ for some constant $0<C<\infty$.

Estimate of $\max_{I\neq *} \big\| \widehat{\mathsf{q}}^{|f|,I}_{\frac{N}{M}} \frac{1}{\mathcal{W}_t} \big\|_{\ell^2}$. For $f(\cdot) = q^{\varphi_N}_{\frac{i-1}{M}N}(\cdot)$ as in (5.100) we bound $\|f\|_\infty \leq \|\varphi_N\|_\infty \leq \|\varphi\|_\infty$ and we assume that φ is supported in the ball $B(0,R)$

for $R > 0$ (then φ_N is supported in $B(0, R\sqrt{N})$, see (5.15)) and w_t is defined as in (5.70). For $t = \frac{1}{\sqrt{N}}$ and $h = 4$, we have

$$\left\| \frac{f}{w_{2|t|h}} \right\|_{\ell^1} = \left\| \frac{q^{\varphi_N}_{\frac{i-1}{M}N}}{w_{\frac{8}{\sqrt{N}}}} \right\|_{\ell^1} \le 2\, e^{32} \left\| \frac{\varphi_N}{w_{\frac{8}{\sqrt{N}}}} \right\|_{\ell^1} \le 2\, e^{32}\, e^{8R} \|\varphi_N\|_{\ell^1} \le 2\, e^{32}\, e^{8R}\, N\, \|\varphi\|_{L^1}\,, \tag{5.105}$$

where the last inequality above follows from the definition (5.15) of φ_N, while the first one holds by (5.75):

$$\begin{aligned} \left\| \frac{q^{\varphi_N}_{\frac{i-1}{M}N}}{w_{\frac{8}{\sqrt{N}}}} \right\|_{\ell^1} &= \sum_{z,y\in\mathbb{Z}^2} e^{\frac{8}{\sqrt{N}}|z|}\, q_{\frac{i-1}{M}N}(y-z)\, |\varphi_N(y)| \\ &\le \sum_{y\in\mathbb{Z}^2} e^{\frac{8}{\sqrt{N}}|y|}\, |\varphi_N(y)| \sum_{z\in\mathbb{Z}^2} e^{\frac{8}{\sqrt{N}}|y-z|}\, q_{\frac{i-1}{M}N}(y-z) \\ &\le 2\, e^{32\frac{i-1}{M}} \left\| \frac{\varphi_N}{w_{\frac{8}{\sqrt{N}}}} \right\|_{\ell^1} \\ &\le 2\, e^{32} \left\| \frac{\varphi_N}{w_{\frac{8}{\sqrt{N}}}} \right\|_{\ell^1} . \end{aligned}$$

Then, by applying (5.76) we have:

$$\max_{I\neq *} \left\| \frac{\widehat{\mathsf{q}}^{|f|,I}_{\frac{N}{M}}}{\mathcal{W}_t} \right\|_{\ell^2} \le \frac{N}{M}\, 4\, e^{256\frac{1}{M}}\, \|f\|_\infty^{\frac{5}{2}} \left\| \frac{f}{w_{\frac{8}{\sqrt{N}}}} \right\|_{\ell^1}^{\frac{3}{2}} \lesssim \frac{N^{\frac{5}{2}}}{M}\,. \tag{5.106}$$

Estimate of $\mathsf{C}^2_{\mathcal{W}_t, \frac{N}{M}}$. From the bound (5.80) we obtain, for some finite constant $\widetilde{C} < \infty$,

$$\mathsf{C}^2_{\mathcal{W}_t, \frac{N}{M}} = \max_{I,J\neq *,\, I\neq J} \big\| \mathcal{W}_t\, \widehat{\mathsf{Q}}^{I,J}_L \tfrac{1}{\mathcal{W}_t} \big\|_{\ell^2\to\ell^2} \le C\, e^{2c\frac{1}{M}} \le \widetilde{C}\,. \tag{5.107}$$

Estimate of $\rho^2_{\mathcal{W}_t, \frac{N}{M}, 0, \beta_N}$. For $L \le N$ as in (5.100) and $\beta = \beta_N$ as in (5.9), see (5.6), we have

$$\mathbb{E}\big[\xi^I_\beta\big] \le \sigma^2_\beta = \sigma^2_{\beta_N} \le \frac{1}{R_N}\,, \qquad \sigma^2_\beta\, R_L \le 1\,, \qquad 1 - \sigma^2_\beta\, R_L \ge 1 - \sigma^2_{\beta_N}\, R_N \ge \frac{\theta_N}{\log N}\,, \tag{5.108}$$

therefore from (5.84) for $s = 0$ we obtain,

$$\rho^2_{\mathcal{W}_t, \frac{N}{M}, 0, \beta_N} \le \sum_{I\neq *} \sigma^2_\beta \left(1 + 16\, e^{32\frac{c}{M}}\, \frac{\sigma^2_\beta\, R_L}{1 - \sigma^2_\beta\, R_L} \right) \le \frac{1}{R_N} \left(1 + 16\, e^{32c} \frac{\log N}{\theta_N} \right), \tag{5.109}$$

and since $1 \ll \theta_N \ll \log N$ and $R_N \sim \frac{\log N}{\pi}$, see (5.9) and (5.6), we finally get

$$\rho^2_{\mathcal{W}_t, \frac{N}{M}, 0, \beta_N} \lesssim \frac{1}{\theta_N}\,. \tag{5.110}$$

Estimate of $\max_{J\neq *} \left\| \mathcal{W}_t \, \overline{\mathsf{q}}^{|1|,J}_{\frac{N}{M}} \right\|_{\ell^2}$. By applying (5.96) we obtain

$$\max_{J\neq *} \left\| \overline{\mathsf{q}}^{|g|,J}_{\frac{N}{M}} \right\|_{\ell^2} \leq C\, N^{\frac{3}{2}} \,. \tag{5.111}$$

Conclusion for $r \geq 3$. From (5.104), we now apply (5.106), (5.107), (5.110) and (5.111), then (up to some finite constant) we obtain

$$\Xi^{r\geq 3}_{N,M} \leq \frac{\theta_N^2}{N^4} \frac{N^{\frac{5}{2}}}{M} \frac{\frac{1}{\theta_N^3}}{1 - \frac{\widetilde{C}}{\theta_N}} N^{\frac{3}{2}} = \frac{1}{\theta_N\, M} \frac{1}{1 - \frac{\widetilde{C}}{\theta_N}} \xrightarrow[N\to\infty]{} 0 \,,$$

where we used that θ_N diverges as $N \to \infty$ and thus $\mathsf{C}^2_{\mathcal{W}_t, \frac{N}{M}} \, \rho^2_{\mathcal{W}_t, \frac{N}{M}, 0, \beta_N} \leq \frac{\widetilde{C}}{\theta_N} < 1$ for N large enough.

Notice that the same arguments can be applied to show that only the first $\left\lfloor \frac{h}{2} \right\rfloor$ terms in the h-th moment of $X^{(i)}_{N,M}$ are non–negligible as $N \to \infty$ under the quasi-critical regime.

5.3.3.2. ***Terms*** $r = 1$**.** We consider the terms with $r = 1$:

$$\begin{aligned}
\Xi^{r=1}_{N,M} &= \frac{\theta_N^2}{N^4} \sum_{\substack{I \vdash \{1,2,3,4\} \\ \text{with full support} \\ I \neq *}} \left| \mathbb{E}[\xi^I_{\beta_N}] \right| \left\langle \widehat{\mathsf{q}}^{|f|,I}_{\frac{N}{M}} \;,\; |\widehat{\mathsf{U}}|^I_{\frac{N}{M},0,\beta_N} \, \overline{\mathsf{q}}^{1,I}_{\frac{N}{M}} \right\rangle \\
&= \frac{\theta_N^2}{N^4} \sum_{\substack{I \vdash \{1,2,3,4\} \\ \text{with full support} \\ I \neq *}} \left| \mathbb{E}[\xi^I_{\beta}] \right| \sum_{\mathbf{x},\mathbf{y} \in (\mathbb{Z}^2)^4} \widehat{\mathsf{q}}^{|f|,I}_{\frac{N}{M}}(\mathbf{x}) \, |\widehat{\mathsf{U}}|^I_{\frac{N}{M},0,\beta_N}(\mathbf{x},\mathbf{y}) \, \overline{\mathsf{q}}^{1,I}_{\frac{N}{M}}(\mathbf{y}) \\
&\lesssim \frac{\theta_N^2}{N^4} \sigma^2_{\beta_N} \sum_{\substack{I \vdash \{1,2,3,4\} \\ \text{with full support} \\ I \neq *}} \sum_{\mathbf{x},\mathbf{y} \in (\mathbb{Z}^2)^4} \widehat{\mathsf{q}}^{|f|,I}_{\frac{N}{M}}(\mathbf{x}) \, |\widehat{\mathsf{U}}|^I_{\frac{N}{M},0,\beta_N}(\mathbf{x},\mathbf{y}) \, \overline{\mathsf{q}}^{1,I}_{\frac{N}{M}}(\mathbf{y}) \,,
\end{aligned} \tag{5.112}$$

where we used that $\left| \mathbb{E}[\xi^I_{\beta_N}] \right| \lesssim \sigma^2_{\beta_N}$ for N large. By definition (recall (5.61) and (5.54)), we have

$$\overline{\mathsf{q}}^{1,I}_{\frac{N}{M}}(\mathbf{y}) \leq 1 \,, \qquad \mathbf{y} \in (\mathbb{Z}^2)^4$$

and by arguing as in the proof of Proposition 5.18 with $t = 0$ and $s = 0$ we get

$$\sum_{\mathbf{y} \in (\mathbb{Z}^2)^4} |\widehat{\mathsf{U}}|^I_{\frac{N}{M},0,\beta_N}(\mathbf{x},\mathbf{y}) \leq 1 + 16 \, \frac{\sigma^2_{\beta_N} R_{\frac{N}{M}}}{1 - \sigma^2_{\beta_N} R_{\frac{N}{M}}} \lesssim \frac{\log N}{\theta_N}$$

for N large enough (see (5.108)). Moreover, setting $f(\cdot) = q^{\varphi_N}_{\frac{i-1}{M}N}(\cdot)$ and assuming that φ_N is supported in $B(0, R\sqrt{N})$, we can bound

$$\begin{aligned}\sum_{x\in\mathbb{Z}^2} q_n^{|f|}(x) &= \sum_{x,y,z\in\mathbb{Z}^2} |\varphi_N(z)|\, q_n(y-x) q_{\frac{i-1}{M}N}(z-y)\\ &= \sum_{x,z\in\mathbb{Z}^2} |\varphi_N(z)| q_{n+\frac{i-1}{M}N}(z-x)\\ &\le \|\varphi\|_\infty \sum_{|z|\le R\sqrt{N}} \sum_{x\in\mathbb{Z}^2} q_{n+\frac{i-1}{M}N}(z-x)\\ &\lesssim \|\varphi\|_\infty\, N\,.\end{aligned}$$

Since the partitions I in (5.112) have full support, they can only be either the quadruple $\{1,2,3,4\}$ or one of the possible six double pair $\{\{a,b\},\{c,d\}\}$. This implies that summing over $\mathbf{x} = (x^1, x^2, x^3, x^4) \in (\mathbb{Z}^2)^4_I$ is equivalent to summing over *at most two* free space variables $x, x' \in \mathbb{Z}^2$. Therefore, from (5.112) we bound $\Xi^{r=1}_{N,M}$ by

$$\begin{aligned}\Xi^{r=1}_{N,M} &\le \frac{\theta_N}{N^4}\sigma^2_{\beta_N}\, \log N \sum_{\substack{I\vdash\{1,2,3,4\}\\ \text{with full support}}} \sum_{\mathbf{x}\in(\mathbb{Z}^2)^4} \widehat{\mathsf{q}}^{|f|,I}_{\frac{N}{M}}(\mathbf{x})\\ &= \frac{\theta_N}{N^4}\sigma^2_{\beta_N}\, \log N \sum_{\substack{I\vdash\{1,2,3,4\}\\ \text{with full support}}} \sum_{\mathbf{x}=(x^1,x^2,x^3,x^4)\in(\mathbb{Z}^2)^4_I} \sum_{n=1}^{\frac{N}{M}} \prod_{j=1}^4 q_n^{|f|}(x^i)\\ &\le 7\,\frac{\theta_N}{N^4}\sigma^2_{\beta_N}\, \log N\, \|f\|^2_\infty \sum_{n=1}^{\frac{N}{M}} \Big(\sum_{x\in\mathbb{Z}^2} q_n^{|f|}(x)\Big)\Big(\sum_{x'\in\mathbb{Z}^2} q_n^{|f|}(x')\Big)\\ &\lesssim \frac{\theta_N}{N^4}\,\frac{N}{M}\,N^2\\ &= \frac{\theta_N}{M\,N} \xrightarrow[N\to\infty]{} 0\,,\end{aligned}$$

since θ_N diverges more slowly than $\log N$ (see also definition (5.54) and (5.4)-(5.6)-(5.9)).

5.3.3.3. ***Terms*** $r = 2$. We consider the terms for $r = 2$ in (5.101) and we further insert $\big(\frac{1}{\mathcal{W}_t}\mathcal{W}_t\big)$ (recall (5.70)) between each pair of adjacent operators. Therefore, we have

$$\begin{aligned}\Xi^{r=2}_{N,M} = \frac{\theta_N^2}{N^4} &\sum_{\substack{I,J\vdash\{1,2,3,4\}\\ \text{with full support}\\ I\neq J,\ I,J\neq *}} \big|\mathbb{E}[\xi^I_{\beta_N}]\big|\,\big|\mathbb{E}[\xi^J_{\beta_N}]\big|\\ &\times \Big\langle \widehat{\mathsf{q}}^{|f|,I}_{\frac{N}{M}}\tfrac{1}{\mathcal{W}_t}\,,\ \mathcal{W}_t\,|\widehat{\mathsf{U}}|^I_{\frac{N}{M},0,\beta_N}\tfrac{1}{\mathcal{W}_t}\,\mathcal{W}_t\widehat{\mathsf{Q}}^{I,J}_{\frac{N}{M}}\tfrac{1}{\mathcal{W}_t}\,\mathcal{W}_t|\widehat{\mathsf{U}}|^J_{\frac{N}{M},0,\beta_N}\tfrac{1}{\mathcal{W}_t}\,\mathcal{W}_t\overline{\mathsf{q}}^{1,J}_{\frac{N}{M}}\Big\rangle.\end{aligned} \tag{5.113}$$

For any fixed partitions I, J in the sum above, we distinguish two cases: either both I and J are pairs $\{\{a,b\},\{c,d\}\}$ or at least one, say J, is not a pair.

We start assuming that *at least* J *is not a pair*. Then, by Hölder's inequality we bound this contribution to (5.113) by

$$\begin{aligned}\frac{\theta_N^2}{N^4}\max_{I\neq *}\Big\|\widehat{\mathsf{q}}^{|f|,I}_{\frac{N}{M}}\tfrac{1}{\mathcal{W}_t}\Big\|_{\ell^2}\max_{I,J\,\text{pairs},\,I\neq J}\big\|\mathcal{W}_t\widehat{Q}_{\frac{N}{M}}{}^{I,J}\tfrac{1}{\mathcal{W}_t}\big\|_{\ell^2\to\ell^2}\max_{J\neq *}\Big\|\mathcal{W}_t\overline{\mathsf{q}}^{1,J}_{\frac{N}{M}}\Big\|_{\ell^2}\times\\ \times\,\rho^2_{\mathcal{W}_t,\frac{N}{M},0,\beta_N}\,\overline{\rho}^2_{\mathcal{W}_t,\frac{N}{M},0,\beta_N}\,,\end{aligned}\tag{5.114}$$

where we recall

$$\rho^2_{\mathcal{W}_t,\frac{N}{M},0,\beta_N}:=\sum_{I\neq *}\big|\mathbb{E}[\xi^I_{\beta_N}]\big|\,\big\|\mathcal{W}_t\,|\widehat{\mathsf{U}}|^I_{\frac{N}{M},0,\beta_N}\tfrac{1}{\mathcal{W}_t}\big\|_{\ell^2\to\ell^2}\,,$$

while

$$\overline{\rho}^2_{\mathcal{W}_t,\frac{N}{M},0,\beta_N}:=\sum_{\substack{J\neq *\\ J\text{ is not a pair}}}\big|\mathbb{E}[\xi^J_{\beta_N}]\big|\big\|\mathcal{W}_t\,|\widehat{\mathsf{U}}|^J_{\frac{N}{M},0,\beta_N}\tfrac{1}{\mathcal{W}_t}\big\|_{\ell^2\to\ell^2}\,.$$

Recalling the definition (5.56) and since $\mathbb{E}\big[|\xi^k_{\beta_N}\big]\le C_k\sigma^k_{\beta_N}$ for all $k\ge 3$ (see (5.26)) and $\sigma_\beta<1$ for N large, we bound the disorder moments associated with J when J is not a pair as follows

$$\big|\mathbb{E}[\xi^J_{\beta_N}]\big|=\begin{cases}\mathbb{E}[\xi^4_{\beta_N}] & \le\sigma^4_{\beta_N}\quad\text{if } J \text{ is a quadruple}\\ \mathbb{E}[\xi^2_{\beta_N}]\,\mathbb{E}[\xi^2_{\beta_N}] & \le\sigma^4_{\beta_N}\quad\text{if } J\text{ is a double pair}\\ \big|\mathbb{E}[\xi^3_{\beta_N}]\big| & \le\sigma^3_{\beta_N}\quad\text{if } J \text{ is a triple}\end{cases}\le\sigma^3_{\beta_N}\lesssim\frac{1}{(R_N)^{3/2}}\,,$$

(recall (5.9)). Thus, applying (5.84) with $s=0$ and arguing as in (5.109), we have

$$\begin{aligned}\overline{\rho}^2_{\mathcal{W}_t,\frac{N}{M},0,\beta_N}&\le\sum_{\substack{J\neq *\\ J\text{is not a pair}}}\sigma^3_{\beta_N}\left(1+16e^{32c}\frac{\log N}{\theta_N}\right)\\ &\lesssim\frac{1}{\sqrt{R_N}}\frac{1}{R_N}\left(1+16e^{32c}\frac{\log N}{\theta_N}\right)\\ &\lesssim\frac{1}{\sqrt{\log N}\,\theta_N}\,,\end{aligned}\tag{5.115}$$

since $1\ll\theta_N\ll\log N$ and $R_N\sim\frac{\log N}{\pi}$, see (5.9) and (5.6). At this point, we easily show that (5.114) vanishes as $N\to\infty$ by applying respectively (5.106), (5.107), (5.110), (5.115) and (5.111) with $t=\frac{1}{\sqrt{N}}$, $f(\cdot)\equiv q^{\varphi_N}_{\frac{i-1}{M}N}(\cdot)$ and $g\equiv 1$ to the quantities

$$\max_{I\neq *}\Big\|\widehat{\mathsf{q}}^{|f|,I}_{\frac{N}{M}}\tfrac{1}{\mathcal{W}_t}\Big\|_{\ell^2}\,,\qquad\Big\|\mathcal{W}_t\widehat{\mathsf{Q}}^{I,J}_{\frac{N}{M}}\tfrac{1}{\mathcal{W}_t}\Big\|_{\ell^2\to\ell^2}\,,$$

$$\rho^2_{\mathcal{W}_t,\frac{N}{M},0,\beta_N}\,,\qquad\overline{\rho}^2_{\mathcal{W}_t,\frac{N}{M},0,\beta_N}\,,\qquad\max_{J\neq *}\Big\|\mathcal{W}_t\overline{\mathsf{q}}^{1,J}_{\frac{N}{M}}\Big\|_{\ell^2}\,,$$

indeed we obtain

$$\frac{\theta_N^2}{N^4}\,\frac{N^{\frac{5}{2}}}{M}\,\frac{1}{\theta_N}\,\frac{1}{\sqrt{\log N}\,\theta_N}\,N^{\frac{3}{2}}=\frac{1}{M\sqrt{\log N}}\xrightarrow[N\to\infty]{}0\,.\tag{5.116}$$

We only need to deal with the case when *both I and J are two pairs.* For these terms in (5.113), after inserting the weights $(\frac{1}{\mathcal{W}_t}\mathcal{W}_t)$ as done above we further *insert the weights $\mathcal{V}_s^J \frac{1}{\mathcal{V}_s^J}$ and $\frac{1}{\mathcal{V}_s^I}\mathcal{V}_s^I$* as in (5.72):

$$\begin{aligned}
&\frac{\theta_N^2}{N^4} \sum_{\substack{I,J\vdash\{1,2,3,4\} \\ \text{with full support} \\ I\neq J,\, I,J\neq * \\ I,J \text{ pairs}}} \left|\mathbb{E}[\xi_{\beta_N}^I]\right| \left|\mathbb{E}[\xi_{\beta_N}^J]\right| \\
&\times \left\langle \widehat{\mathsf{q}}^{|f|,I}_{\frac{N}{M}} \frac{1}{\mathcal{W}_t}\mathcal{V}_s^J \,,\; \frac{\mathcal{W}_t}{\mathcal{V}_s^J}\, |\widehat{\mathsf{U}}|^I_{\frac{N}{M},0,\beta_N} \frac{\mathcal{V}_s^J}{\mathcal{W}_t}\; \frac{\mathcal{W}_t}{\mathcal{V}_s^J} \widehat{\mathsf{Q}}^{I,J}_{\frac{N}{M}} \frac{1}{\mathcal{V}_s^I\mathcal{W}_t}\; \mathcal{V}_s^I\mathcal{W}_t |\widehat{\mathsf{U}}|^J_{\frac{N}{M},0,\beta_N} \frac{1}{\mathcal{V}_s^I\mathcal{W}_t}\; \mathcal{W}_t\mathcal{V}_s^I\, \overline{\mathsf{q}}^{1,J}_{\frac{N}{M}} \right\rangle \\
&\le \frac{\theta_N^2}{N^4} \max_{\substack{I,J\neq * \\ I\neq J \\ I,J \text{ pairs}}} \left\| \widehat{\mathsf{q}}^{|f|,I}_{\frac{N}{M}} \frac{1}{\mathcal{W}_t}\mathcal{V}_s^J \right\|_{\ell^2} \max_{\substack{I,J\neq * \\ I\neq J \\ I,J \text{ pairs}}} \left\| \frac{\mathcal{W}_t}{\mathcal{V}_s^J} \widehat{\mathsf{Q}}^{I,J}_{\frac{N}{M}} \frac{1}{\mathcal{V}_s^I\mathcal{W}_t} \right\|_{\ell^2\to\ell^2} \max_{\substack{I,J\neq * \\ I\neq J \\ I,J \text{ pairs}}} \left\| \mathcal{W}_t\mathcal{V}_s^I\, \overline{\mathsf{q}}^{1,J}_{\frac{N}{M}} \right\|_{\ell^2} \times \\
&\qquad \times \rho^2_{\mathcal{W}_t,\mathcal{V}_s,\frac{N}{M},0,\beta_N}\; \widetilde{\rho}^2_{\mathcal{W}_t,\mathcal{V}_s,\frac{N}{M},0,\beta_N} \,,
\end{aligned}$$

where

$$\rho^2_{\mathcal{W}_t,\mathcal{V}_s,\frac{N}{M},0,\beta_N} := \sum_{\substack{J\neq * \\ J \text{ pair}}} \left|\mathbb{E}[\xi_{\beta_N}^J]\right| \max_{\substack{I\neq * \\ I \text{ pair}}} \left\| \mathcal{V}_s^I\mathcal{W}_t |\widehat{\mathsf{U}}|^J_{\frac{N}{M},0,\beta_N} \frac{1}{\mathcal{V}_s^I\mathcal{W}_t} \right\|_{\ell^2\to\ell^2}$$

and

$$\widetilde{\rho}^2_{\mathcal{W}_t,\mathcal{V}_s,\frac{N}{M},0,\beta_N} := \sum_{\substack{I\neq * \\ I \text{ pair}}} \left|\mathbb{E}[\xi_{\beta_N}^I]\right| \max_{\substack{J\neq * \\ J \text{ pair}}} \left\| \frac{\mathcal{W}_t}{\mathcal{V}_s^J}\, |\widehat{\mathsf{U}}|^I_{\frac{N}{M},0,\beta_N} \frac{\mathcal{V}_s^J}{\mathcal{W}_t} \right\|_{\ell^2\to\ell^2} .$$

To get the desired bound, we only need to estimate the five quantities

$$\max_{\substack{I,J\neq * \\ I\neq J \\ I,J \text{ pairs}}} \left\| \widehat{\mathsf{q}}^{|f|,I}_{\frac{N}{M}} \frac{1}{\mathcal{W}_t}\mathcal{V}_s^J \right\|_{\ell^2} , \qquad \max_{\substack{I,J\neq * \\ I\neq J \\ I,J \text{ pairs}}} \left\| \frac{\mathcal{W}_t}{\mathcal{V}_s^J} \widehat{\mathsf{Q}}^{I,J}_{\frac{N}{M}} \frac{1}{\mathcal{V}_s^I\mathcal{W}_t} \right\|_{\ell^2\to\ell^2} ,$$

$$\rho^2_{\mathcal{W}_t,\mathcal{V}_s,\frac{N}{M},0,\beta_N} , \qquad \widetilde{\rho}^2_{\mathcal{W}_t,\mathcal{V}_s,\frac{N}{M},0,\beta_N} , \qquad \max_{\substack{I,J\neq * \\ I\neq J \\ I,J \text{ pairs}}} \left\| \mathcal{W}_t\mathcal{V}_s^I\, \overline{\mathsf{q}}^{1,J}_{\frac{N}{M}} \right\|_{\ell^2} .$$

We are going to apply (5.77), (5.81), (5.84), (5.85) and (5.97) with

$$h = 4 , \qquad t = \frac{1}{\sqrt{N}} , \qquad s = \frac{1}{\sqrt{L}} = \frac{\sqrt{M}}{\sqrt{N}} .$$

Estimate of $\max_{I,J} \left\| \widehat{\mathsf{q}}^{|f|,I}_{\frac{N}{M}} \frac{1}{\mathcal{W}_t}\mathcal{V}_s^J \right\|_{\ell^2}$. For $f(\cdot) = q^{\varphi_N}_{\frac{i-1}{M}N}(\cdot)$, we bound $\|f\|_\infty \le \|\varphi_N\|_\infty \le \|\varphi\|_\infty$. Moreover let us fix the weight functions $\mathcal{W}_t$, w_t and $\mathcal{V}_s$ as in

(5.70) and (5.72). Then, by applying (5.77) we get

$$\begin{aligned}
\max_{\substack{I,J\neq * \\ I\neq J \\ I,J \text{ pairs}}} \big\|\widehat{\mathsf{q}}^{|f|,I}_{\frac{N}{M}} \tfrac{1}{\mathcal{W}_t}\mathcal{V}_s^J\big\|_{\ell^2} &\lesssim \sqrt{8}\,\frac{N}{M}\, e^{320(1+\frac{1}{M})}\,\|\varphi\|_\infty^{\frac{5}{2}} \left\|\frac{f}{w_{\frac{4}{\sqrt{N}}}}\right\|_{\ell^1}^{\frac{1}{2}} \left\|\frac{f}{w_{\frac{4}{\sqrt{N}}}}\right\|_{\ell^2} \frac{N^{\frac{1}{2}}}{M^{\frac{1}{2}}} \\
&\lesssim \frac{N^{\frac{3}{2}}}{M^{\frac{3}{2}}} \left\|\frac{f}{w_{\frac{4}{\sqrt{N}}}}\right\|_{\ell^1}^{\frac{1}{2}} \left\|\frac{f}{w_{\frac{4}{\sqrt{N}}}}\right\|_{\ell^2} \\
&\lesssim \frac{N^2}{M^{\frac{3}{2}}} \left\|\frac{f}{w_{\frac{4}{\sqrt{N}}}}\right\|_{\ell^2},
\end{aligned}$$

where the last inequality for $\left\|\frac{f}{w_{\frac{4}{\sqrt{N}}}}\right\|_{\ell^1}$ holds similarly as in (5.105). Eventually, we assume again that φ is supported in the ball $B(0,R)$, then φ_N is supported in $B(0,R\sqrt{N})$ by definition (see (5.15)), we set $n := \frac{i-1}{M}N \le N$ and recall the definition (5.29), from the bound (5.75) we have

$$\begin{aligned}
\left\|\frac{f}{w_{\frac{4}{\sqrt{N}}}}\right\|_{\ell^2}^2 &= \sum_{z\in\mathbb{Z}^2}\sum_{y,y'\in\mathbb{Z}^2} \varphi_N(y)\,\varphi_N(y')\,q_n(y-z)\,q_n(y'-z)\,e^{\frac{8}{\sqrt{N}}|z|} \\
&\le \|\varphi\|_\infty \sum_{y\in\mathbb{Z}^2}\varphi_N(y)\,e^{\frac{8}{\sqrt{N}}|y|}\sum_{z\in\mathbb{Z}^2} q_n(y-z)\,e^{\frac{8}{\sqrt{N}}|y-z|}\sum_{y'\in\mathbb{Z}^2} q_n(y'-z) \\
&\le \|\varphi\|_\infty\, 2\, e^{\frac{32}{N}n}\, e^{8R}\|\varphi_N\|_{\ell^1} \\
&\le \|\varphi\|_\infty\, 2\, e^{32+8R}\, N\,\|\varphi\|_{L^1}\,.
\end{aligned}$$

Therefore, we finally obtain

$$\max_{\substack{I,J\neq * \\ I\neq J \\ I,J \text{ pairs}}} \big\|\widehat{\mathsf{q}}^{|f|,I}_{\frac{N}{M}} \tfrac{1}{\mathcal{W}_t}\mathcal{V}_s^J\big\|_{\ell^2} \lesssim \frac{N^{\frac{5}{2}}}{M^{\frac{3}{2}}}\,. \tag{5.117}$$

Estimate of $\max_{I,J}\big\|\frac{\mathcal{W}_t}{\mathcal{V}_s^J}\widehat{\mathsf{Q}}^{I,J}_{\frac{N}{M}}\frac{1}{\mathcal{V}_s^I\mathcal{W}_t}\big\|_{\ell^2\to\ell^2}$. From the bound (5.81) we obtain

$$\max_{\substack{I,J\neq * \\ I\neq J \\ I,J \text{ pairs}}} \big\|\tfrac{\mathcal{W}_t}{\mathcal{V}_s^J}\widehat{\mathsf{Q}}^{I,J}_{\frac{N}{M}}\tfrac{1}{\mathcal{V}_s^I\mathcal{W}_t}\big\|_{\ell^2\to\ell^2} \le C\, e^{16c\left(\frac{1}{N}+\frac{M}{N}\right)\frac{N}{M}} \le C\,, \tag{5.118}$$

for some constant $0 < C < \infty$.

Estimate of $\rho^2_{\mathcal{W}_t,\mathcal{V}_s,\frac{N}{M},0,\beta_N}$. By exploiting (5.84) (recalling also (5.108)) we get

$$\rho^2_{\mathcal{W}_t,\mathcal{V}_s,\frac{N}{M},0,\beta_N} \le \sum_{\substack{J\neq * \\ J \text{ pair}}} \sigma^2_{\beta_N}\left(1+16\,e^{32c\left(1+\frac{1}{M}\right)}\frac{\sigma^2_{\beta_N}R_{\frac{N}{M}}}{1-\sigma^2_{\beta_N}R_{\frac{N}{M}}}\right) \le \frac{1}{R_N}\left(1+16e^{64c}\frac{\log N}{\theta_N}\right),$$

and since $1 \ll \theta_N \ll \log N$ and $R_N \sim \frac{\log N}{\pi}$, see (5.9) and (5.6), we finally get

$$\rho^2_{\mathcal{W}_t,\mathcal{V}_s,\frac{N}{M},0,\beta_N} \lesssim \frac{1}{\theta_N}\,. \tag{5.119}$$

Estimate of $\widetilde{\rho}^2_{\mathcal{W}_t,\mathcal{V}_s,\frac{N}{M},0,\beta_N}$. Similarly as done above for the bound of $\rho^2_{\mathcal{W}_t,\mathcal{V}_s,\frac{N}{M},0,\beta_N}$, by applying (5.85) we get

$$\widetilde{\rho}^2_{\mathcal{W}_t,\mathcal{V}_s,\frac{N}{M},0,\beta_N} \lesssim \frac{1}{\theta_N}\,. \tag{5.120}$$

Estimate of $\max_{I,J} \big\| \mathcal{W}_t \mathcal{V}_s^I \, \overline{\mathsf{q}}^{1,J}_{\frac{N}{M}} \big\|_{\ell^2}$. By the bound (5.97), we obtain

$$\max_{\substack{I,J\neq * \\ I\neq J \\ I,J \text{ pairs}}} \big\| \mathcal{W}_t \mathcal{V}_s^I \, \overline{\mathsf{q}}^{1,J}_{\frac{N}{M}} \big\|_{\ell^2} \lesssim \frac{N^{\frac{3}{2}}}{M^{\frac{1}{2}}}\,. \tag{5.121}$$

Conclusion for $r=2$. We are finally able to show that only $\Xi^{r=2}_{N,M}$ with both I and J pairs give a non–negligible contribution to $\mathbb{E}\big[(X^{(i)}_{N,M})^4\big]$ as $N\to\infty$. Indeed, applying (5.117), (5.118), (5.119), (5.120), (5.121) and (5.116), we can finally show that for any $M\in\mathbb{N}$ the term $\Xi^{r=2}_{N,M}$ is controlled (up to some finite constant) by

$$\Xi^{r=2}_{N,M} \lesssim \frac{\theta_N^2}{N^4}\frac{N^{\frac{5}{2}}}{M^{\frac{3}{2}}}\frac{1}{\theta_N^2}\frac{N^{\frac{3}{2}}}{M^{\frac{1}{2}}} + \frac{1}{M\sqrt{\log N}} \leq \frac{1}{M^2}\,,$$

for N large enough. This completes the proof of Proposition 5.4.

5.3.4. Proof of Theorem 5.9. We recall that the averaged partition function $Z^{\omega}_{L,\beta}(f,g)$ is defined in (5.48)-(5.49). In analogy with (5.28) and (5.30), by (5.48)-(5.49) we can write

$$Z^{\omega}_{L,\beta}(f,g) - \mathbb{E}[Z^{\omega}_{L,\beta}(f,g)] = \sum_{k=1}^{\infty} \sum_{\substack{0<n_1<\ldots<n_k<L \\ x_1,\ldots,x_k\in\mathbb{Z}^2}} q^f_{n_1}(x_1)\,\xi_\beta(n_1,x_1)\times \\ \times \left\{ \prod_{j=2}^{k} q_{n_j-n_{j-1}}(x_j-x_{j-1})\,\xi_\beta(n_j,x_j) \right\} q^g_{L-n_k}(x_k)\,, \tag{5.122}$$

where we recall the random walk kernels (5.27) and (5.29). Recalling (5.50), we obtain

$$\mathcal{M}^h_{L,\beta}(f,g) = \mathbb{E}\Bigg[\Bigg(\sum_{k=1}^{\infty} \sum_{\substack{0<n_1<\ldots<n_k<L \\ x_1,\ldots,x_k\in\mathbb{Z}^2}} q^f_{n_1}(x_1)\,\xi_\beta(n_1,x_1)\times \\ \times \left\{ \prod_{j=2}^{k} q_{n_j-n_{j-1}}(x_j-x_{j-1})\,\xi_\beta(n_j,x_j) \right\} q^g_{L-n_k}(x_k) \Bigg)^h\Bigg]\,. \tag{5.123}$$

When we expand the h-th power, we obtain a sum over h families of space-time points $A_i := \{(n^i_1,x^i_1),\ldots,(n^i_{k_i},x^i_{k_i})\}$ for $i=1,\ldots,h$. These points must *match at least in pairs*, i.e. any point (n^i_ℓ,x^i_ℓ) in any family A_i must coincide with at least another point (n^j_m,x^j_m) in a different family A_j for $j\neq i$, otherwise the expectation vanishes (since $\xi_\beta(n,x)$ are independent and centered). In order to handle this constraint, following [**CSZ21+**, Theorem 6.1], we rewrite (5.123) by first *summing over the set of all space-time points*

$$A := \bigcup_{i=1}^{h} A_i = \bigcup_{i=1}^{h} \{(n_1^i, x_1^i), \ldots, (n_{k_i}^i, x_{k_i}^i)\} \subseteq \mathbb{N} \times \mathbb{Z}^2$$

and then specifying *which families* each point $(n, x) \in A$ belongs to.

Let us fix the *time coordinates* $n_1 < \ldots < n_r$ of the points in A. For each such time $n \in \{n_1, \ldots, n_r\}$, we have $(n, x) \in A$ for one or more $x \in \mathbb{Z}^2$ (there are at most $h/2$ such x, by the matching constraint described above). We then make the following observations:

- if $(n, x) = (n_j^i, x_j^i)$ belongs to the family A_i, then we have in (5.123) the product of a random walk kernel "entering" (n, x) and another one "exiting" (n, x):

$$q_{n-n_{j-1}^i}(x - x_{j-1}^i) \cdot q_{n_{j+1}^i - n}(x_{j+1}^i - x)\,;$$

- if (n, x) does *not* belong to the family A_i, then we have in (5.123) a random walk kernel "jumping over time n", say $q_{n_j^i - n_{j-1}^i}(x_j - x_{j-1})$ with $n_{j-1}^i < n < n_j^i$: we can split this kernel at time n by Chapman-Kolmogorov, writing

$$q_{n_j^i - n_{j-1}^i}(x_j^i - x_{j-1}^i) = \sum_{z \in \mathbb{Z}^2} q_{n-n_{j-1}^i}(z - x_{j-1}^i) \cdot q_{n_j^i - n}(x_j^i - z)\,. \tag{5.124}$$

Then, to each time $n \in \{n_1, \ldots, n_r\}$, we can associate a vector $\mathbf{y} = (y^1, \ldots, y^h) \in (\mathbb{Z}^2)^h$ with h space coordinates, where $y^i = x$ if the family A^i contains (n, x) and $y^i = z$ from (5.124) otherwise. The constraint that a point $(n, x) \in A$ belongs to two families A^i and $A^{i'}$ means that the corresponding coordinates of the vector $\mathbf{y}$ must coincide: $y^i = y^{i'}$. In order to specify which families A^i share the same points, we assign a *partition* $I \vdash \{1, \ldots, h\}$ to each time $n \in \{n_1, \ldots, n_r\}$ and we require that $\mathbf{y} \sim I$, see (5.52).

We are now ready to provide a convenient rewriting of (5.123) by first summing over the number $r \geq 1$ and the time coordinates $n_1 < \ldots < n_r$, then on the corresponding space coordinates $\mathbf{y}_1, \ldots, \mathbf{y}_r$ and partitions $I_1, \ldots, I_r \vdash \{1, \ldots, h\}$ with $\mathbf{y}_i \sim I_i$. Defining for $\mathbf{x}, \mathbf{z} \in (\mathbb{Z}^2)^h$ the h-component random walk kernels, see (5.27) and (5.29), by

$$\mathsf{Q}_n(\mathbf{z}, \mathbf{x}) := \prod_{i=1}^{h} q_n(x^i - z^i)\,, \qquad \mathsf{q}_n^f(\mathbf{x}) := \prod_{i=1}^{h} q_n^f(x^i)\,, \tag{5.125}$$

we can finally rewrite (5.123) as follows:

$$\mathcal{M}^h_{L,\beta}(f,g) = \sum_{r=1}^{\infty} \sum_{\substack{0<n_1<\cdots<n_r<L \\ \mathbf{y}_1,\ldots,\mathbf{y}_r\in(\mathbb{Z}^2)^h}} \sum_{\substack{I_1,\ldots,I_r\vdash\{1,\ldots,h\} \\ \text{with full support} \\ \text{and } I_i\neq * \ \forall i}} \mathsf{q}^f_{n_1}(\mathbf{y}_1)\, \mathbb{1}_{\{\mathbf{y}_1\sim I_1\}}\, \mathbb{E}[\xi^{I_1}_\beta] \times \\ \times \left\{ \prod_{i=2}^{r} \mathsf{Q}_{n_i-n_{i-1}}(\mathbf{y}_{i-1},\mathbf{y}_i)\, \mathbb{1}_{\{\mathbf{y}_i\sim I_i\}}\, \mathbb{E}[\xi^{I_i}_\beta] \right\} \mathsf{q}^g_{L-n_k}(\mathbf{y}_r)\,. \tag{5.126}$$

We can obtain a more compact expression absorbing the constraints $\mathbf{y}_i \sim I_i$ in the random walk kernels: recalling the definitions of $\mathsf{Q}^{I,J}_n$ and $\mathsf{q}^{f,J}_n$ from (5.54), we have

$$\mathcal{M}^h_{L,\beta}(f,g) = \sum_{r=1}^{\infty} \sum_{\substack{0<n_1<\cdots<n_r<L \\ \mathbf{y}_1,\ldots,\mathbf{y}_r\in(\mathbb{Z}^2)^h}} \sum_{\substack{I_1,\ldots,I_r\vdash\{1,\ldots,h\} \\ \text{with full support} \\ \text{and } I_i\neq * \ \forall i}} \mathsf{q}^{f,I_1}_{n_1}(\mathbf{y}_1)\, \mathbb{E}[\xi^{I_1}_\beta] \times \\ \times \left\{ \prod_{i=2}^{r} \mathsf{Q}^{I_{i-1},I_i}_{n_i-n_{i-1}}(\mathbf{y}_{i-1},\mathbf{y}_i)\, \mathbb{E}[\xi^{I_i}_\beta] \right\} \mathsf{q}^{g,I_r}_{L-n_r}(\mathbf{y}_r)\,. \tag{5.127}$$

Finally, formula (5.57) follows from (5.127) after we group together stretches of *consecutive repeated partitions*, i.e. when $I_i = J$ for consecutive indexes i. The kernel $\mathsf{U}^J_{m-n,\beta}(\mathbf{z},\mathbf{x})$ from (5.55) does exactly this job, which leads precisely to (5.57). □

REMARK 5.21. *Formula* (5.57) *still contains the product of* $\mathbb{E}[\xi^{I_i}_\beta]$ *because these factors from* (5.127) *are only partially absorbed in* $\mathsf{U}^J_{m-n,\beta}(\mathbf{z},\mathbf{x})$*: indeed, in* (5.55) *we have* $k+1$ *points* $n_0 < n_1 < \ldots < n_k$*, but the factor* $\mathbb{E}[\xi^J_\beta]$ *therein is only raised to the power* k*.*

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