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Integrable hierarchies, Frölicher–Nijenhuis bicomplexes and Lauricella bi-flat F-manifolds

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Abstract
Given the Frölicher-Nijenhuis bicomplex (d, d_L) associated with a $(1,1)$-tensor field L with vanishing Nijenhuis torsion, we define a multi-parameter family of bi-flat structures $(\nabla, e, \circ, \nabla^*, *, E)$. This result is obtained by combining the construction of integrable hierarchies of hydrodynamic type starting from Frölicher-Nijenhuis bicomplexes with the construction of flat F-manifold structures from integrable systems of hydrodynamic type. By construction L is the operator of multiplication by the Euler vector field E and the number of parameters coincides with the number of Jordan blocks appearing in its Jordan normal form. We call these structures Lauricella bi-flat structures since in the n-dimensional semisimple case $(n-1)$ flat coordinates of ∇ are Lauricella functions. The $(1,1)$-tensor fields defining the corresponding integrable hierarchies have a similar block diagonal structure.

Keywords: integrable systems, flat F-manifolds, Euler-Poisson-Darboux system, Lauricella functions, Jordan blocks
Mathematics Subject Classification numbers: 53D45, 37K10, 37K25

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Contents

1. Introduction

Given a tensor field L of type $(1,1)$ on a manifold M with vanishing Nijenhuis torsion it is possible to define a bi-differential complex $(d, d_L, \Omega(M))$ called the Frölicher–Nijenhuis bicomplex on the Grasmann algebra $\Omega(M)$ of differential forms on M (see [10]). Such complex plays an important role in the theory of integrable systems, in both finite [25] and infinite dimensional case [19]. More in general, in recent years there has been a growing interest in applications of Nijenhuis geometry to integrable systems of hydrodynamic type (see [7] and references therein).

The present paper is devoted to the study of the relations between the Frölicher–Nijenhuis bicomplex and the theory of bi-flat F-manifolds. In particular, using these relations we construct a class of regular bi-flat F-manifolds for any choice of the Jordan normal form of the operator of multiplication by the Euler vector field.

Flat F-manifolds are Dubrovin–Frobenius manifolds 'without metric': all the axioms involving explicitly the invariant metric (and not just the associated Levi-Civita connection) are dropped. Bi-flat F-manifolds are manifolds equipped with two different flat structures satisfying suitable compatibility conditions. They can be equivalently defined as homogeneous flat F-manifolds, i.e. flat F-manifolds equipped with a linear Euler vector field. Many results in the theory of Dubrovin-Frobenius manifolds, ranging from relations with reflection groups and Painlevé transcendents to relations with enumerative geometry and integrable systems, can be generalized to (bi-)flat F-manifolds.

For the purposes of the present paper it will be crucial the relation between flat F-manifolds and dispersionless integrable systems: given a flat F-manifold, it is possible to define an integrable hierarchy of quasilinear systems of first order evolutionary PDEs called the principal hierarchy [22]. On the other hand it has been proved in [19] that also starting from the Frölicher–Nijenhuis bicomplex it is possible to construct an integrable hierachy of quasilinear systems of PDEs

$$u^i_{t_n} = (V_n(\mathbf{u}))^i_j u^j_x.$$

The tensor fields V_n which define this hierarchy are polynomials in L

$$V_n = L^n + a_0 L^{n-1} + a_1 L^{n-2} + \cdots + a_{n-1} I$$

and the coefficients of these polynomials are obtained recursively from a_0 by means of a generalized Lenard–Magri chain. Moreover it has been proved in [21] that, in the case $L = \mathrm{diag}(u^1, \ldots, u^n)$ and $a_0 = \sum_k \varepsilon_k u^k$, the above hierarchy can be identified with the principal hierarchy of a special kind of bi-flat F-manifolds related to the theory of Lauricella functions [17] and Lauricella connections [18].

Given n real numbers in the interval $(0,1)$, $(\varepsilon_1,\ldots,\varepsilon_n):=\varepsilon$, the *Lauricella function* of weight ε at the point $u:=(u^1,\ldots,u^n)\in\mathbb{C}^n\setminus\mathcal{H}=\cup_{1\leqslant i<j\leqslant n}\{u\in\mathbb{C}^n|u^i=u^j\}$ is defined by

$$\int_{\gamma_u}\eta_u=\int_{\gamma_u}\left(u^1-\zeta\right)^{-\varepsilon_1}\ldots\left(u^n-\zeta\right)^{-\varepsilon_n}\mathrm{d}\zeta.$$

Here γ_u is an oriented piecewise differentiable arc such that the end points of γ_u lie in $\{u^1,\ldots,u^n\}$ (but such that γ_u does not meet this set elsewhere) and a determination of the multivalued differential η_u is fixed. Let δ_k be the oriented piecewise differentiable arc connecting u^{k-1} with u^k and denote by L_u^ε the $(n-1)$-dimensional vector space generated by $\int_{\delta_k}\eta_u$, $k=2,\ldots,n$. Any $f\in L_u^\epsilon$ satisfies the following conditions (see [18] for details):

(1) $e(f)=0$, where $e=\sum_{i=1}^n\frac{\partial}{\partial u^i}$.
(2) f is homogeneous of degree $1-\sum_{i=1}^n\varepsilon_i$.
(3) f satisfies the system of differential equations

$$\left(u^i-u^j\right)\frac{\partial^2 f}{\partial u^i\,\partial u^j}=\varepsilon_j\frac{\partial f}{\partial u^i}-\varepsilon_i\frac{\partial f}{\partial u^j},\quad 1\leqslant i<j\leqslant n. \tag{1.1}$$

The Euler–Poisson–Darboux system (1.1) can be rewritten in the form

$$dd_Lf=da_0\wedge df. \tag{1.2}$$

By definition, the flat coordinates of the connection ∇ of the associated bi-flat F-manifold are the solutions of the Euler-Poisson-Darboux system (1.1) satisfying the condition (1). The homegeneity condition (2) selects $n-1$ flat coordinates that are Lauricella functions, the remaining flat coordinate being the function a_0.

The aim of this paper is to extend the construction of [21] in the non-semisimple regular case that is much less studied in the literature. The extension is based on two main facts:

(1) The construction of integrable hierarchies of hydrodynamic type starting from a $(1,1)$-tensor field L with vanishing Nijenhuis torsion does not require that L is diagonalizable.
(2) In the theory of bi-flat F-manifolds there is a natural tensor field with vanishing Nijenhuis torsion: it is the operator of multiplication by the Euler vector field. Moreover in the regular case there are special coordinates found by David and Hertling in [9] where the unit vector field, the Euler vector field and the product have a canonical form.

The above facts suggest to proceed in the following way:

- Following [19] for each David–Hertling canonical form of L and for a suitable choice of the function a_0 we can construct integrable hierarchies of hydrodynamic type. Assuming that a_0 is a weighted sum of the traces of the blocks appearing in the David–Hertling canonical form of $E\circ$, the freedom reduces to a choice of r parameters (the weights) where r is the number of the blocks. This choice reduces to the usual one in the semisimple case.
- Then, following [23] we can impose that the flows of this hierarchy are symmetries of the principal hierarchy associated with a bi-flat F-manifold.

The main result of the paper is that, for any choice of Jordan block structure of the operator of multiplication by the Euler vector field and for any choice of the weights corresponding to each Jordan block, there is a unique associated bi-flat structure. Following [6] we call bi-flat structures obtained in this way *Lauricella bi-flat structures*.

The paper is organized as follows: in section 2 we recall the main facts about the theory of integrable systems of hydrodynamic type, in section 3 we recall the definition of (bi)-flat F-manifolds and the associated principal hierarchy, in section 4 we recall the construction of integrable systems of hydrodynamic type by means of Frölicher–Nijenhuis bicomplex and in section 5 we recall the definition of Lauricella bi-flat F-manifolds in the semisimple case. Section 6 is devoted to describe non-semisimple regular Lauricella bi-flat F-manifolds in dimensions $2,3,4,5$. This section plays an important role since it suggests a strategy to prove the main result of the paper, first in the case of a single Jordan block of arbitrary size (section 7) and finally in the general regular case (section 8). Even if the results of section 7 follow from the general theorem proved in section 8, we have decided to keep it since it contains the essential ideas of the general proof without the extra non-trivial difficulties due to the high number of subcases that one has to consider in the case of an arbitrary number of Jordan blocks. In section 9 we study the dual structure of a Lauricella bi-flat F-manifold, in section 10 we show that the generalized Lenard-Magri chains associated with the operator of multiplication by the Euler vector field define symmetries of the principal hierarchy and in the last section we draw some conclusions.

2. Integrable systems of hydrodynamic type

Integrable diagonal systems of hydrodynamic type

$$r^i_t = v^i(r)\, r^i_x \qquad i = 1,\dots,n. \tag{2.1}$$

have been studied by Tsarev in [30]. Assuming $v^i \neq v^j$ Tsarev proved that all the information about the integrability of such systems is contained in the $n(n-1)$ functions

$$\Gamma^i_{ij} = \frac{\partial_j v^i}{v^j - v^i}, \qquad i \neq j. \tag{2.2}$$

The system is integrable iff these functions satisfy the conditions

$$\partial_j\left(\frac{\partial_k v^i}{v^k - v^i}\right) = \partial_k\left(\frac{\partial_j v^i}{v^j - v^i}\right), \qquad \forall i \neq j \neq k \neq i. \tag{2.3}$$

Systems satisfying the condition (2.3) are called semi-Hamiltonian systems or rich systems. They possess infinitely many symmetries (depending on n functions of a single variable)

$$r^i_\tau = w^i(r)\, r^i_x \qquad i = 1,\dots,n \tag{2.4}$$

obtained solving the linear system

$$\partial_j w^i = \Gamma^i_{ij}\left(w^j - w^i\right) \tag{2.5}$$

and infinitely many densities of conservation laws obtained solving the linear system

$$\partial_i\partial_j h - \Gamma^i_{ij}\partial_i h - \Gamma^j_{ji}\partial_j h = 0, \qquad i \neq j. \tag{2.6}$$

Tsarev'integrability condition is the compatibility of the systems (2.5) and (2.6).

Let us consider now a general system of hydrodynamic type

$$u^i_t = V^i_j(\mathbf{u})\, u^j_x, \qquad i = 1,\dots,n. \tag{2.7}$$

Assuming that at each point the $(1,1)$-tensor field V has pairwise distict eigenvalues, the diagonalizability of the system is equivalent to the vanishing of the Haantjes tensor of V [11] and the semi-Hamiltonian condition (2.3) is equivalent to the vanishing of a tensor field, called the *semi-Hamiltonian tensor* [28]. The diagonalizing coordinates $(r^1,\dots,r^n)$ are called *Riemann invariants* and the diagonal entries of the $(1,1)$-tensor field V in such coordinates are called *characteristic velocities* of the system. Given a semi-Hamiltonian system and n functional independent solutions $(h^1,\dots,h^n)$ of the system (2.6) we have

$$h^i_t = \partial_x K^i, \qquad i = 1,\dots,n \tag{2.8}$$

for some functions $K^i(r^1,\dots,r^n)$, $i = 1,\dots,n$. In other words the system can be written as a system of conservation laws. It turns out that also the converse statement is true. A system of conservation laws admitting Riemann invariants is semi-Hamiltonian. In this way, following Sevennec, one can equivalently define semi-Hamiltonian systems as systems of hydrodynamic type that can be written both in the diagonal and in the conservative forms (2.1) and (2.8) (we refer for details to [29]).

The main equations of Tsarev's theory can be also formulated in terms of a family of torsionless connections.

Definition 2.1. Given a $(1,1)$-tensor field V, a torsionless connection ∇ satisfying

$$d_\nabla V = 0 \tag{2.9}$$

will be called a *Tsarev's connection* associated with V.

Here $d_\nabla V$ is the exterior covariant derivative of the $(1,1)$-tensor field V:

$$(d_\nabla V)^i_{jk} = \nabla_j V^i_k - \nabla_k V^i_j = \partial_j V^i_k - \partial_k V^i_j + \Gamma^i_{js} V^s_k - \Gamma^i_{ks} V^s_j.$$

Tsarev's connections are not uniquely defined: in the Riemann invariants $(r^1,\dots,r^n)$ where $V = \mathrm{diag}(v^1,\dots,v^n)$ the above condition is equivalent to $\Gamma^i_{jk} = 0$ for pairwise distinct indices and to (2.2). Moreover $\Gamma^i_{ji} = \Gamma^i_{ij}$ due to the vanishing of the torsion. All the remaining Christoffel symbols Γ^i_{jj} and Γ^i_{ii} are free. To prove this fact we have to spell out the condition

$$(d_\nabla V)^i_{jk} = \partial_j V^i_k + \Gamma^i_{mj} V^m_k - \partial_k V^i_j - \Gamma^i_{mk} V^m_j = 0$$

in the Riemann invariants. If all indices are distinct we obtain

$$(d_\nabla V)^i_{jk} = \Gamma^i_{kj}\left(v^k - v^j\right) = 0$$

and as a consequence, taking into account that the characteristic velocities are pairwise distinct, we obtain $\Gamma^i_{kj} = 0$ for $i \neq j \neq k \neq i$. If $i = k$ (or $i = j$) we get

$$(d_\nabla V)^i_{ji} = \partial_j v^i + \Gamma^i_{ij}\left(v^i - v^j\right) = 0.$$

Theorem 2.2. *A diagonalizable system of hydrodynamic type with pairwise distinct characteristic velocities is semi-Hamiltonian iff the Tsarev's connections associated with V satisfy the condition $d^2_\nabla W = 0$ for any $(1,1)$-tensor field W commuting with V.*

Proof. By straightforward computation we get

$$\left[d^2_\nabla W\right]^l_{jik} = R^l_{ijn} W^n_k + R^l_{jkn} W^n_i + R^l_{kin} W^n_j,$$

where R is the Riemann tensor of ∇:

$$R^k_{ijl} = \partial_j \Gamma^k_{il} - \partial_i \Gamma^k_{jl} + \Gamma^k_{js}\Gamma^s_{il} - \Gamma^k_{is}\Gamma^s_{jl}.$$

In the Riemann invariants the set of matrices W are diagonal and the condition $d^2_\nabla W = 0$ reads

$$\left[d^2_\nabla W\right]^l_{jik} = R^l_{ijk}w^k + R^l_{jki}w^i + R^l_{kij}w^j = 0$$

for any $(w^1,\dots,w^n)$. Due to arbitrariness of W this is equivalent to $R^l_{ijk} = 0$ for pairwise distinct indices i,j,k. If at least two of the three indices i,j,k are equal the condition is automatically satisfied since $R^l_{jik} = -R^l_{ijk}$. Thus assuming the indices i,j,k pairwise distinct we need to consider the case $l = i$ (the cases $l = j$ and $l = k$ are equivalent). Due to arbitrariness of W we obtain the conditions

$$R^i_{jki} = 0, \quad R^i_{ijk} = 0.$$

The second condition, also known as *Darboux-Tsarev system*, reads

$$\partial_i\Gamma^k_{kj} + \Gamma^k_{ki}\Gamma^k_{kj} - \Gamma^k_{kj}\Gamma^j_{ji} - \Gamma^k_{ik}\Gamma^i_{ij} = 0, \qquad \forall i \neq j \neq k \neq i \tag{2.10}$$

while the first condition reads

$$\partial_j\Gamma^i_{ik} = \partial_k\Gamma^i_{ij}, \qquad \forall i \neq j \neq k \neq i. \tag{2.11}$$

Clearly (2.11) follows from (2.10). Remarkably, if the functions Γ^i_{ij} are given by (2.2) both conditions (2.3) and (2.10) are equivalent to (2.11) due to the identity [30]

$$\partial_i\Gamma^k_{kj} + \Gamma^k_{ki}\Gamma^k_{kj} - \Gamma^k_{kj}\Gamma^j_{ji} - \Gamma^k_{ki}\Gamma^i_{ij} = \frac{v^i - v^k}{v^j - v^i}\left[\partial_j\left(\frac{\partial_i v^k}{v^i - v^k}\right) - \partial_i\left(\frac{\partial_j v^k}{v^j - v^k}\right)\right]. \tag{2.12}$$

□

The Sevennec's result can be formulated as follows.

Theorem 2.3. *Let V be a $(1,1)$-tensor field with distinct eigenvalues and with vanishing Haantjies tensor, then V defines a semi-Hamiltonian system iff among the associated Tsarev's connections there is a flat connection.*

Proof. Assume that ∇ is a flat Tsarev's connection. In flat coordinates the condition $d_\nabla V = 0$ reads $\partial_k V^i_j = \partial_j V^i_k$ which implies that in flat coordinates (locally) we have $V^i_j = \partial_j X^i$ and thus $V^i_j u^j_x = \partial_x X^i$. Since Riemann invariants exist by hypothesis the Sevennec's result implies that the system defined by V is semi-Hamiltonian.

Assume now that V defines a semihamiltonian system then due to Sevennec's result there exist coordinates $(u^1,\dots,u^n)$ where $V^i_j u^j_x = \partial_x X^i$. This implies $\partial_k V^i_j = \partial_j V^i_k$. Define ∇ in the coordinates $(u^1,\dots,u^n)$ as $\Gamma^i_{jk} = 0$. Then the condition $\partial_k V^i_j = \partial_j V^i_k$ can be written as $d_\nabla V = 0$. In other words ∇ is a flat Tsarev's connection. □

The symmetries

$$u^i_\tau = W^i_j(\mathbf{u})\,u^j_x, \qquad i = 1,\dots,n. \tag{2.13}$$

of the system (2.7) are defined by $(1,1)$-tensor fields $W(\mathbf{u})$ commuting with V and satisfying the condition

$$d_\nabla W = 0. \tag{2.14}$$

The full hierarchy is thus defined by the solutions of the system (2.14).

3. F-manifolds and integrable hierarchies

3.1. Flat and bi-flat F-manifolds

F-manifolds have been introduced by Hertling and Manin in [12].

Definition 3.1. An *F-manifold* is a triple $(M, \circ, e)$, where M is a manifold, $\circ$ is a commutative associative bilinear product on the module of (local) vector fields, satisfying the following identity:

$$\begin{aligned}&[X\circ Y, W\circ Z]-[X\circ Y, Z]\circ W-[X\circ Y, W]\circ Z-X\circ[Y, Z\circ W]+X\circ[Y, Z]\circ W\\&\quad+X\circ[Y, W]\circ Z-Y\circ[X, Z\circ W]+Y\circ[X, Z]\circ W+Y\circ[X, W]\circ Z=0,\end{aligned}\tag{3.1}$$

for all local vector fields X, Y, W, Z, where $[X, Y]$ is the Lie bracket, and e is a distinguished vector field on M such that $e\circ X = X$ for all local vector fields X.

In this paper we will consider F-manifolds equipped with some additional structures.

Definition 3.2 ([26]). A flat F-manifold $(M, \circ, \nabla, e)$ is an F-manifold M equipped with a connection ∇ related to the product $\circ$ and to the unit vector field e by the following axioms:

- **Axiom 1** (the connection and the product). The one parameter family of connections

$$\nabla - \lambda\circ$$

  is flat and torsionless for any λ.
- **Axiom 2** (the connection and the vector field). The vector field e is covariantly constant: $\nabla e = 0$.

Bi-flat F-manifolds are manifolds equipped with two 'compatible' flat structures. They are defined in the following way.

Definition 3.3 ([5]). A *bi-flat* F-manifold $(M, \nabla, \nabla^*, \circ, *, e, E)$ is a manifold M equipped with a pair of connections ∇ and ∇^*, a pair of products $\circ$ and $*$ on the tangent spaces T_uM and a pair of vector fields e and E s.t.:

- $(\nabla, \circ, e)$ defines a flat structure on M.
- $(\nabla^*, *, E)$ defines a flat structure on M.
- The two structures are related by the following conditions

$$X * Y := (E\circ)^{-1} X\circ Y, \qquad [e, E] = e, \qquad \mathrm{Lie}_E\circ = \circ, \qquad (d_\nabla - d_{\nabla^*})(X\circ) = 0,$$

  where X and Y are arbitrary vector fields and at a generic point the operator $E\circ$ is assumed to be invertible.

Notice that not all the axioms are independent. For instance the compatibility between the dual connection and the dual product follows from the other axioms [6]. Notice the dual connection is defined only at the points where the operator $E\circ$ is invertible. At these points the condition

$$(d_\nabla - d_{\nabla^*})(X\circ) = 0 \qquad \forall X \tag{3.2}$$

is equivalent to the condition

$$(d_{\nabla} - d_{\nabla^*})(X*) = 0 \qquad \forall X. \tag{3.3}$$

This implies

$$\Gamma^{*k}_{ij} = \Gamma^k_{ij} - c^{*l}_{ji}\nabla_l E^k. \tag{3.4}$$

Moreover the flatness of the dual connection follows from the linearity of the Euler vector field (see theorem 4.4 in [3] for the semisimple case and lemmas 4.2 and 4.3 in [15] for the general case).

Remark 3.4. The manifolds in the above definitions are real or complex manifolds. In the first case all the geometric data are supposed to be smooth. In the latter case *TM* is intended as the holomorphic tangent bundle and all the geometric data are supposed to be holomorphic.

Dubrovin-Frobenius manifolds are bi-flat F-manifolds equipped with a metric η compatible with the connection ∇ and with the product $\circ$:

$$\nabla_k \eta_{ij} = 0, \qquad \eta_{il} c^l_{kj} = \eta_{kl} c^l_{ij}.$$

Many results in the theory of Dubrovin–Frobenius manifolds admit a natural generalization in this more general setting. We mention: the relation with reflection groups [3, 14, 15], the relation with (generalization of) cohomological field theories and (generalization of) Givental group action [1], the relation with dispersionless and dispersive integrable hierarchies [2, 8, 22] and the relation with Painlevé transcendents [5, 6, 13, 14, 21]. We point out that this relation survives also in the regular three-dimensional non-semisimple cases leading to a correspondence with solutions of the full family of Painlevé IV and V depending on the number of Jordan blocks appearing in the Jordan form of the operator of multiplication by the Euler vector field, while in the case of Dubrovin-Frobenius manifolds the same cases are parametrized by elementary functions [24].

3.2. The principal hierarchy

Given a (bi)-flat F-manifold one can define an integrable hierarchy starting from solutions of the equation (2.14) (in this case ∇ is given). Looking for solutions of the form $W = X\circ$ we obtain the condition

$$(\nabla_Z X) \circ Y = (\nabla_Y X) \circ Z \tag{3.5}$$

for all pairs (Y, Z) of vector fields, that is, in local coordinates,

$$c^i_{jm} \nabla_k X^m = c^i_{km} \nabla_j X^m\ . \tag{3.6}$$

The above condition can be also written

$$d_{\nabla}(X\circ) = 0. \tag{3.7}$$

Let us define now the vector fields $X_{(p,l)}$ as follows: the vector fields $X_{(p,-1)}$ are covariantly constant with respect to ∇, while the others are obtained through the recurrence relation:

$$\nabla X_{(p,l+1)} = X_{(p,l)} \circ\ . \tag{3.8}$$

It is easy to check (see [22] for details) that they are solutions of the system (3.5). As a consequence the corresponding flows

$$u^i_{t_{(p,l)}} = c^i_{jk} X^k_{(p,l)} u^j_x, \qquad i = 1,\dots,n$$

commute and define an integrable hierarchy called *the principal hierarchy*.

4. Frölicher–Nijenhuis bicomplex and integrable systems

Let L be a tensor field of type $(1,1)$ with vanishing Nijenhuis torsion. This means that for any pair of vector fields X and Y we have

$$[LX, LY] - L\,[X, LY] - L\,[LX, Y] + L^2\,[X, Y] = 0. \tag{4.1}$$

Following [19] we recall now a construction of integrable hierarchies starting from the Frölicher–Nijenhuis bicomplex $(d, d_L, \Omega(M))$. The differential d is the usual de Rham differential while the differential d_L is defined as

$$\begin{aligned}(d_L\omega)(X_0,\dots,X_k) = &\sum_{i=0}^{k}(-1)^i\,(LX_i)\left(\omega\left(X_0,\dots,\hat{X}_i,\dots,X_k\right)\right)\\ &+\sum_{0\leqslant i<j\leqslant k}(-1)^{i+j}\omega\left([X_i,X_j]_L, X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots X_k\right),\end{aligned}$$

where $\omega \in \Omega^k(M)$ and

$$[X_i, X_j]_L = [LX_i, X_j] + [X_i, LX_j] - L[X_i, X_j].$$

For $L = I$ the vector field $[X_i, X_j]_L$ reduces to the commutator $[X_i, X_j]$ and the differential d_L to d.

The fact that $d_L^2 = 0$ is equivalent to the vanishing of the Nijenhuis torsion, The differentials d and d_L anticommute and thus the pair (d, d_L) defines a bidifferential complex. In [19] (see also [20]) it has been proved that given any solution of the equation

$$dd_L a_0 = 0 \tag{4.2}$$

and the corresponding sequence of functions $a_1, a_2, a_3, \dots$ defined recursively by

$$da_{k+1} = d_L a_k - a_k da_0$$

the tensor fields of type $(1,1)$ defined by

$$V_{k+1} = V_k L - a_k I,$$

starting from the identity I, that is

$$\begin{aligned}V_0 &= I\\ V_1 &= L - a_0 I\\ V_2 &= L^2 - a_0 L - a_1 I,\\ \cdots &= \cdots\end{aligned}$$

define an integrable hierarchy of hydrodynamic type. Remarkably this construction does not require that L is diagonalizable.

4.1. Examples

4.1.1. Generalized ε-system. The system of hydrodynamic type

$$\begin{bmatrix} u^1_{t_1} \\ u^2_{t_1} \\ \vdots \\ u^n_{t_1} \end{bmatrix} = \begin{bmatrix} u^1 - \sum_{k=1}^n \varepsilon_k u^k & 0 & \dots & 0 \\ 0 & u^2 - \sum_{k=1}^n \varepsilon_k u^k & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & u^n - \sum_{k=1}^n \varepsilon_k u^k \end{bmatrix} \begin{bmatrix} u^1_x \\ u^2_x \\ \vdots \\ u^n_x \end{bmatrix} \tag{4.3}$$

has been obtained in [27] as finite component reduction of an infinite hydrodynamic chain. It can be written as $\mathbf{u}_{t_1} = (L - a_0 I)\mathbf{u}_x$ with $a_0 = \sum_{k=1}^n \varepsilon_k u^k$ and

$$L = \begin{bmatrix} u^1 & 0 & \dots & 0 \\ 0 & u^2 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & u^n \end{bmatrix}. \tag{4.4}$$

For specific values of the constants ε_i it provides well-known examples of integrable systems of hydrodynamic type. The above hierarchy is related to the principal hierarchy associated with Lauricella bi-flat F-manifolds [21, 23].

4.1.2. Kodama-Konopelchenko system. The system of hydrodynamic type [16]

$$\begin{bmatrix} u^1_{t_1} \\ u^2_{t_1} \\ \vdots \\ u^{n-1}_{t_1} \\ u^n_{t_1} \end{bmatrix} = \begin{bmatrix} u^1 & 1 & 0 & \dots & 0 \\ 0 & u^1 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & u^1 & 1 \\ 0 & \dots & 0 & 0 & u^1 \end{bmatrix} \begin{bmatrix} u^1_x \\ u^2_x \\ \vdots \\ u^{n-1}_x \\ u^n_x \end{bmatrix} \tag{4.5}$$

can be written as $\mathbf{u}_{t_1} = (L - a_0 I)\mathbf{u}_x$ with $a_0 = -u^1$ and

$$L = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 1 \\ 0 & \dots & 0 & 0 & 0 \end{bmatrix}.$$

Clearly L has vanishing Nijenhuis torsion and $dd_L a_0 = 0$. Applying the first step of the recursive procedure we have

$$\begin{aligned} \partial_1 a_1 &= -a_0 \partial_1 a_0 = -u^1 \\ \partial_2 a_1 &= \partial_1 a_0 - a_0 \partial_2 a_0 = -1 \\ \partial_3 a_1 &= 0 \\ &\vdots \\ \partial_n a_1 &= 0. \end{aligned}$$

This implies (up to an inessential constant) $a_1 = -u^2 - \frac{(u^1)^2}{2}$. Therefore the first commuting flow $\mathbf{u}_t = (L^2 - a_0 L - a_1 I)\mathbf{u}_x$ is given by

$$\begin{bmatrix} u^1_{t_2} \\ u^2_{t_2} \\ \vdots \\ u^{n-1}_{t_2} \\ u^n_{t_2} \end{bmatrix} = \begin{bmatrix} u^2 + \frac{(u^1)^2}{2} & u^1 & 1 & \dots & 0 \\ 0 & u^2 + \frac{(u^1)^2}{2} & u^1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & u^2 + \frac{(u^1)^2}{2} & u^1 \\ 0 & \dots & 0 & 0 & u^2 + \frac{(u^1)^2}{2} \end{bmatrix} \begin{bmatrix} u^1_x \\ u^2_x \\ \vdots \\ u^{n-1}_x \\ u^n_x \end{bmatrix} \tag{4.6}$$

The higher flows can be obtained in a similar way. The system (4.5) is related to the theory of confluent Lauricella functions (see [16] for details).

5. From integrable hierarchies to Lauricella bi-flat F-manifolds

Let $(\circ, e, E)$ be an F-manifold with Euler vector field. Using Hertling–Manin condition and the properties of the Euler vector field it is easy to check that the operator $E\circ$ has vanishing Nijenhuis torsion (see for instance [4]). Among Tsarev connections of the associated integrable system we consider those satisfying the additional conditions (in [23] they are called *natural connections*):

$$\nabla_j e^i = 0, \qquad \nabla_k c^i_{jl} = \nabla_j c^i_{kl}, \qquad \forall i,j,k,l = 1,\dots,n.$$

In the next two subsections we will show that for special choices of the function a_0 the natural connections defined in this way are flat. Moreover it is possible to define a second compatible flat structure.

5.1. Semisimple Lauricella bi-flat structure

Let us recall the following theorem of [21] (see also [20, 23] for the special case $\varepsilon_1 = \varepsilon_2 = \dots = \varepsilon_n$).

Theorem 5.1. *For any choice of $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ there exists a unique semisimple bi-flat structure $(\nabla, \nabla^*, \circ, *, e, E)$ with canonical coordinates $\{u^1, \dots, u^n\}$ such that $L = E\circ$ and*

$$d_\nabla (L - a_0 I) = 0, \tag{5.1}$$

where $a_0 = \sum_{k=1}^n \varepsilon_k u^k$. Moreover, in canonical coordinates this structure is given by

$$e = \sum_{k=1}^n \partial_k, \quad E = \sum_{k=1}^n u^k \partial_k,$$
$$c^i_{jk} = \delta^i_j \delta^i_k, \qquad c^{*i}_{jk} = \frac{1}{u^i}\delta^i_j \delta^i_k, \qquad \forall i,j,k,$$
$$\Gamma^i_{jk} = 0, \quad \Gamma^{*i}_{jk} = 0, \qquad \forall i \neq j \neq k \neq i$$
$$\Gamma^i_{jj} = -\Gamma^i_{ij}, \quad \Gamma^{*i}_{jj} = -\frac{u^i}{u^j}\Gamma^{*i}_{ij}, \qquad i \neq j$$

$$\Gamma^i_{ij} = \frac{\varepsilon_j}{u^i - u^j}, \quad \Gamma^{*i}_{ij} = \frac{\varepsilon_j}{u^i - u^j}, \qquad i \neq j$$
$$\Gamma^i_{ii} = -\sum_{l\neq i} \Gamma^i_{li}, \quad \Gamma^{*i}_{ii} = -\sum_{l\neq i} \frac{u^l}{u^i}\Gamma^{*i}_{li} - \frac{1}{u^i}.$$

5.2. Non-semisimple Lauricella bi-flat F-manifolds

In this paper we consider a generalization of Lauricella bi-flat structures to the non-semisimple regular case. We will use this important result of [9] about the existence of non-semisimple canonical coordinates.

Definition 5.2 ([9]). An F-manifold with Euler vector field $(M, \circ, e, E)$ is called *regular* if for each $p \in M$ the endomorphism $L_p := E_p\circ : T_pM \to T_pM$ has exactly one Jordan block for each distinct eigenvalue.

Theorem 5.3 ([9]). *Let $(M, \circ, e, E)$ be a regular F-manifold of dimension greater or equal to 2 with an Euler vector field E. Furthermore assume that locally around a point $p \in M$, the Jordan canonical form of the operator L has n Jordan blocks $m_1, \dots, m_n$ with distinct eigenvalues. Then there exists locally around p a distinguished system of coordinates $\{u^1, \dots, u^{m_1+\cdots+m_n}\}$ such that*

$$e = \frac{\partial}{\partial u^1} + \frac{\partial}{\partial u^{m_1+1}} + \frac{\partial}{\partial u^{m_1+m_2+1}} + \cdots + \frac{\partial}{\partial u^{m_1+\cdots+m_{n-1}+1}}, \tag{5.2}$$
$$E = \sum_{s=1}^{m_1+\cdots+m_n} u^s \frac{\partial}{\partial u^s}, \tag{5.3}$$
$$c^l_{ij} = \delta^l_{i+j-(m_1+\cdots m_{k-1})-1}, \quad i,j,l = m_1 + \cdots + m_{k-1} + 1, \dots, m_1 + \cdots + m_{k-1} + m_k \tag{5.4}$$
$$c^l_{ij} = 0, \quad \textit{otherwise}. \tag{5.5}$$

We start from the integrable hierarchy associated with the tensor field $L = E\circ$ and with $a_0 = \sum_{i=1}^n \varepsilon_i \mathrm{Tr}(L_i)$. By definition L contains n blocks $L_1, \dots, L_n$ of dimension $m_1, \dots, m_n$ respectively. Each block has the form

$$L_k = \begin{bmatrix} u^{k,1} & 0 & \dots & 0 \\ u^{k,2} & u^{k,1} & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ u^{k,m_k} & \dots & u^{k,2} & u^{k,1} \end{bmatrix} \tag{5.6}$$

where $u^{k,l} = u^{m_1+\cdots+m_{k-1}+l}$. The case $m_1 = \ldots = m_n = 1$ corresponds to the usual generalized ε-system. In the next section we will prove that for any choice of $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ and $m_1, \dots, m_n$ and up to $n = 5$ there exists a unique bi-flat F-structure such that $d_\nabla(L - a_0 I) = 0$. In section 7 we will consider the case $n = 1$ and m_1 arbitrary and in the section 8 we will show that for any choice of $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ there exists a unique regular bi-flat structure such that $L = E\circ$ and

$$d_\nabla (L - a_0 I) = 0. \tag{5.7}$$

Remark 5.4. By straightforward computation one gets

$$(d_\nabla (L - a_0 I))^i_{jk} \partial_i a_0 = L^i_j \nabla_i (da_0)_k - L^i_k \nabla_i (da_0)_j.$$

Therefore the condition (5.7) implies

$$L^i_j \nabla_i (da_0)_k - L^i_k \nabla_i (da_0)_j = 0.$$

6. Bi-flat Lauricella structures in dimension $2, 3, 4, 5$

In this section we provide a complete classification of non-semisimple bi-flat F-manifold structures in 2,3,4 and 5 dimensions.

6.1. 2-dimensional case

6.1.1. 2×2 Jordan block.

$$L=\begin{bmatrix}u^1 & 0\\ u^2 & u^1\end{bmatrix},\quad e=\frac{\partial}{\partial u^1}. \tag{6.1}$$

The non-vanishing Christoffel symbol of ∇ is $\Gamma^2_{22}=-\frac{2\varepsilon_1}{u^2}$.

6.2. 3-dimensional case

6.2.1. 3×3 Jordan block.

$$L=\begin{bmatrix}u^1 & 0 & 0\\ u^2 & u^1 & 0\\ u^3 & u^2 & u^1\end{bmatrix},\quad e=\frac{\partial}{\partial u^1},\quad a_0=3\varepsilon_1 u^1. \tag{6.2}$$

The non-vanishing Christoffel symbols Γ^i_{jk} (up to exchange of j with k) are

$$\Gamma^2_{22}=\Gamma^3_{23}=-\frac{3\varepsilon_1}{u^2},\Gamma^3_{22}=\frac{3\varepsilon_1 u^3}{(u^2)^2}.$$

6.2.2. $2\times 2+1\times 1$ Jordan blocks.

$$L=\begin{bmatrix}u^1 & 0 & 0\\ u^2 & u^1 & 0\\ 0 & 0 & u^3\end{bmatrix},\quad e=\frac{\partial}{\partial u^1}+\frac{\partial}{\partial u^3},\quad a_0=2\varepsilon_1 u^1+\varepsilon_3 u^3. \tag{6.3}$$

The non-vanishing Christoffel symbols Γ^i_{jk} (up to exchange of j with k) are

$$\Gamma^2_{22}=-\frac{2\varepsilon_1}{u^2},\Gamma^1_{13}=\Gamma^2_{23}=-\Gamma^1_{11}=-\Gamma^1_{33}=-\Gamma^2_{12}=\frac{\varepsilon_3}{u^1-u^3},$$
$$\Gamma^3_{11}=\Gamma^3_{33}=-\Gamma^3_{13}=\frac{2\varepsilon_1}{u^1-u^3},\Gamma^2_{11}=\Gamma^2_{33}=-\Gamma^2_{13}=\frac{\varepsilon_3 u^2}{(u^1-u^3)^2}.$$

6.3. Four-dimensional case

6.3.1. 4×4 Jordan block.

$$L=\begin{bmatrix}u^1 & 0 & 0 & 0\\ u^2 & u^1 & 0 & 0\\ u^3 & u^2 & u^1 & 0\\ u^4 & u^3 & u^2 & u^1\end{bmatrix},\quad e=\frac{\partial}{\partial u^1},\quad a_0=4\varepsilon_1 u^1. \tag{6.4}$$

The non-vanishing Christoffel symbols Γ^i_{jk} (up to exchange of j with k) are

$$\Gamma^2_{22}=\Gamma^3_{23}=\Gamma^4_{33}=\Gamma^4_{24}=-\frac{4\varepsilon_1}{u^2},\Gamma^3_{22}=\Gamma^4_{23}=\frac{4\varepsilon_1 u^3}{(u^2)^2},\Gamma^4_{22}=\frac{4\varepsilon_1\left(u^2u^4-(u^3)^2\right)}{(u^2)^3}.$$

6.3.2. $3\times 3+1\times 1$ *Jordan blocks.*

$$L=\begin{bmatrix} u^1 & 0 & 0 & 0\\ u^2 & u^1 & 0 & 0\\ u^3 & u^2 & u^1 & 0\\ 0 & 0 & 0 & u^4\end{bmatrix},\quad e=\frac{\partial}{\partial u^1}+\frac{\partial}{\partial u^4},\quad a_0=3\varepsilon_1 u^1+\varepsilon_4 u^4. \tag{6.5}$$

The non-vanishing Christoffel symbols Γ^i_{jk} (up to exchange of j with k) are

$$\begin{aligned}
&\Gamma^2_{22}=\Gamma^3_{23}=-\frac{3\varepsilon_1}{u^2},\Gamma^3_{11}=\Gamma^3_{44}=-\Gamma^3_{14}=\frac{\varepsilon_4 u^3}{(u^1-u^4)^2}-\frac{\varepsilon_4\left(u^2\right)^2}{(u^1-u^4)^3},\\
&\Gamma^4_{11}=\Gamma^4_{44}=-\Gamma^4_{14}=\frac{3\varepsilon_1}{u^1-u^4},\\
&\Gamma^3_{12}=-\Gamma^3_{24}=-\Gamma^2_{14}=\Gamma^2_{11}=\Gamma^2_{44}=\frac{\varepsilon_4 u^2}{(u^1-u^4)^2},\\
&\Gamma^3_{34}=-\Gamma^3_{13}=-\Gamma^2_{12}=-\Gamma^1_{11}=\Gamma^1_{14}=-\Gamma^1_{44}=\frac{\varepsilon_4}{u^1-u^4},\\
&\Gamma^3_{22}=\frac{3\varepsilon_1 u^3}{\left(u^2\right)^2}-\frac{\varepsilon_4}{u^1-u^4}.
\end{aligned}$$

6.3.3. $2\times 2+2\times 2$ *Jordan blocks.*

$$L=\begin{bmatrix} u^1 & 0 & 0 & 0\\ u^2 & u^1 & 0 & 0\\ 0 & 0 & u^3 & 0\\ 0 & 0 & u^4 & u^3\end{bmatrix},\quad e=\frac{\partial}{\partial u^1}+\frac{\partial}{\partial u^3},\quad a_0=2\varepsilon_1 u^1+2\varepsilon_3 u^3. \tag{6.6}$$

The non-vanishing Christoffel symbols Γ^i_{jk} (up to exchange of j with k) are

$$\begin{aligned}
&\Gamma^2_{22}=-\frac{2\varepsilon_1}{u^2},\Gamma^4_{44}=-\frac{2\varepsilon_3}{u^4},\ \Gamma^1_{13}=-\Gamma^2_{12}=-\Gamma^1_{11}=-\Gamma^1_{33}=\frac{2\varepsilon_3}{u^1-u^3},\\
&\Gamma^3_{11}=\Gamma^4_{34}=\Gamma^3_{33}=-\Gamma^3_{13}=-\Gamma^4_{14}=\frac{2\varepsilon_1}{u^1-u^3},\\
&\Gamma^2_{11}=\Gamma^2_{33}=-\Gamma^2_{13}=\frac{2\varepsilon_3 u^2}{\left(u^1-u^3\right)^2},\Gamma^4_{11}=\Gamma^4_{33}=-\Gamma^4_{13}=\frac{2\varepsilon_1 u^4}{\left(u^1-u^3\right)^2}.
\end{aligned}$$

6.3.4. $2\times 2+1\times 1+1\times 1$ *Jordan blocks.*

$$L=\begin{bmatrix} u^1 & 0 & 0 & 0\\ u^2 & u^1 & 0 & 0\\ 0 & 0 & u^3 & 0\\ 0 & 0 & 0 & u^4\end{bmatrix},\quad e=\frac{\partial}{\partial u^1}+\frac{\partial}{\partial u^3}+\frac{\partial}{\partial u^4},\quad a_0=2\varepsilon_1 u^1+\varepsilon_3 u^3+\varepsilon_4 u^4. \tag{6.7}$$

The non-vanishing Christoffel symbols Γ^i_{jk} (up to exchange of j with k) are

$$\begin{aligned}
&\Gamma^2_{22}=-\frac{2\varepsilon_1}{u^2},\Gamma^1_{13}=\Gamma^2_{23}=-\Gamma^1_{33}=\frac{\varepsilon_3}{u^1-u^3},\Gamma^1_{14}=\Gamma^2_{24}=-\Gamma^1_{44}=\frac{\varepsilon_4}{u^1-u^4},\\
&\Gamma^3_{34}=-\Gamma^3_{44}=\frac{\varepsilon_4}{u^3-u^4},\Gamma^4_{34}=-\frac{\varepsilon_3}{u^3-u^4},\Gamma^3_{13}=-\Gamma^3_{11}=\frac{-2\varepsilon_1}{u^1-u^3},\\
&\Gamma^4_{14}=\frac{-2\varepsilon_1}{u^1-u^4},\Gamma^1_{11}=\Gamma^2_{12}=\Gamma^1_{33}+\Gamma^1_{44},\Gamma^2_{11}=\Gamma^2_{33}+\Gamma^2_{44},
\end{aligned}$$

$$\Gamma^2_{33} = -\Gamma^2_{13} = \frac{\varepsilon_3 u^2}{\left(u^1 - u^3\right)^2}, \Gamma^2_{44} = -\Gamma^2_{14} = \frac{\varepsilon_4 u^2}{\left(u^1 - u^4\right)^2},$$
$$\Gamma^3_{33} = \frac{2\varepsilon_1}{u^1 - u^3} - \frac{\varepsilon_4}{u^3 - u^4}, \Gamma^4_{44} = \frac{2\varepsilon_1}{u^1 - u^4} + \frac{\varepsilon_3}{u^3 - u^4}.$$

6.4. 5-dimensional case

6.4.1. 5×5 *Jordan block.*

$$L = \begin{bmatrix} u^1 & 0 & 0 & 0 & 0 \\ u^2 & u^1 & 0 & 0 & 0 \\ u^3 & u^2 & u^1 & 0 & 0 \\ u^4 & u^3 & u^2 & u^1 & 0 \\ u^5 & u^4 & u^3 & u^2 & u^1 \end{bmatrix}, \quad e = \frac{\partial}{\partial u^1}, \quad a_0 = 5\varepsilon_1 u^1. \tag{6.8}$$

The non-vanishing Christoffel symbols Γ^i_{jk} (up to exchange of j with k) are

$$\Gamma^2_{22} = \Gamma^3_{23} = \Gamma^4_{33} = \Gamma^4_{24} = \Gamma^5_{25} = \Gamma^5_{34} = -\frac{5\varepsilon_1}{u^2},$$
$$\Gamma^3_{22} = \Gamma^4_{23} = \Gamma^5_{24} = \Gamma^5_{33} = \frac{5\varepsilon_1 u^3}{\left(u^2\right)^2}, \Gamma^4_{22} = \Gamma^5_{23} = \frac{5\varepsilon_1\left(u^2u^4 - \left(u^3\right)^2\right)}{\left(u^2\right)^3},$$
$$\Gamma^5_{22} = \frac{5\varepsilon_1\left(\left(u^2\right)^2 u^5 - 2u^2u^3u^4 + \left(u^3\right)^3\right)}{\left(u^2\right)^4}.$$

6.4.2. $4\times 4 + 1\times 1$ *Jordan blocks.*

$$L = \begin{bmatrix} u^1 & 0 & 0 & 0 & 0 \\ u^2 & u^1 & 0 & 0 & 0 \\ u^3 & u^2 & u^1 & 0 & 0 \\ u^4 & u^3 & u^2 & u^1 & 0 \\ 0 & 0 & 0 & 0 & u^5 \end{bmatrix}, \quad e = \frac{\partial}{\partial u^1} + \frac{\partial}{\partial u^5}, \quad a_0 = 4\varepsilon_1 u^1 + \varepsilon_5 u^5. \tag{6.9}$$

The non-vanishing Christoffel symbols Γ^i_{jk} (up to exchange of j with k) are

$$\Gamma^2_{22} = \Gamma^3_{23} = \Gamma^4_{24} = \Gamma^4_{33} = -4\frac{\varepsilon_1}{u^2},$$
$$\Gamma^1_{15} = \Gamma^2_{25} = \Gamma^3_{35} = \Gamma^4_{45} = \frac{\varepsilon_5}{u^1 - u^5}$$
$$\Gamma^1_{55} = \Gamma^2_{12} = \Gamma^1_{11} = \Gamma^3_{13} = \Gamma^4_{14} = \Gamma^4_{41} = -\frac{\varepsilon_5}{u^1 - u^5},$$
$$\Gamma^5_{11} = \Gamma^5_{55} = -\Gamma^5_{15} = \frac{4\varepsilon_1}{u^1 - u^5},$$
$$\Gamma^2_{11} = \Gamma^2_{55} = \Gamma^3_{12} = \Gamma^4_{13} = -\Gamma^2_{15} = -\Gamma^3_{25} = -\Gamma^4_{35} = -\Gamma^4_{53} = \frac{\varepsilon_5 u^2}{\left(u^1 - u^5\right)^2},$$
$$\Gamma^3_{11} = \Gamma^3_{55} = \Gamma^4_{12} = -\Gamma^3_{15} = -\Gamma^4_{25} = \frac{\varepsilon_5 u^3}{\left(u^1 - u^5\right)^2} - \frac{\varepsilon_5\left(u^2\right)^2}{\left(u^1 - u^5\right)^3},$$
$$\Gamma^3_{22} = \Gamma^4_{23} = \frac{4\varepsilon_1 u^3}{\left(u^2\right)^2} - \frac{\varepsilon_5}{u^1 - u^5},$$

$$\Gamma^4_{11} = \Gamma^4_{55} = -\Gamma^4_{15} = \frac{\varepsilon_5\left(u^2\right)^3}{\left(u^1-u^5\right)^4} - \frac{2\varepsilon_5 u^2 u^3}{\left(u^1-u^5\right)^3} + \frac{\varepsilon_5 u^4}{\left(u^1-u^5\right)^2},$$
$$\Gamma^4_{22} = \frac{4\varepsilon_1\left(u^2u^4-\left(u^3\right)^2\right)}{\left(u^2\right)^3} + \frac{\varepsilon_5 u^2}{\left(u^1-u^5\right)^2}.$$

6.4.3. $3\times 3+2\times 2$ *Jordan blocks.*

$$L = \begin{bmatrix} u^1 & 0 & 0 & 0 & 0 \\ u^2 & u^1 & 0 & 0 & 0 \\ u^3 & u^2 & u^1 & 0 & 0 \\ 0 & 0 & 0 & u^4 & 0 \\ 0 & 0 & 0 & u^5 & u^4 \end{bmatrix}, \quad e = \frac{\partial}{\partial u^1} + \frac{\partial}{\partial u^4} \quad a_0 = 3\varepsilon_1 u^1 + 2\varepsilon_4 u^4. \tag{6.10}$$

The non-vanishing Christoffel symbols Γ^i_{jk} (up to exchange of j with k) are

$$\Gamma^1_{14} = \Gamma^2_{24} = \Gamma^3_{34} = -\Gamma^1_{11} = -\Gamma^1_{44} = -\Gamma^2_{12} = -\Gamma^3_{13} = \frac{2\varepsilon_4}{u^1-u^4},$$
$$\Gamma^2_{11} = \Gamma^2_{44} = \Gamma^3_{12} = -\Gamma^2_{14} = -\Gamma^3_{24} = \frac{2u^2\varepsilon_4}{\left(u^1-u^4\right)^2}, \Gamma^2_{22} = \Gamma^3_{23} = -\frac{3\varepsilon_1}{u^2},$$
$$\Gamma^5_{55} = -\frac{2\varepsilon_4}{u^5}, \Gamma^3_{22} = \frac{3\varepsilon_1 u^3}{\left(u^2\right)^2} - \frac{2\varepsilon_4}{u^1-u^4},$$
$$\Gamma^3_{11} = \Gamma^3_{44} = -\Gamma^3_{14} = \frac{2\varepsilon_4\left(u^1u^3-\left(u^2\right)^2-u^3u^4\right)}{\left(u^1-u^4\right)^3},$$
$$\Gamma^4_{11} = \Gamma^4_{44} = -\Gamma^4_{14} = -\Gamma^5_{15} = \Gamma^5_{45} = \frac{3\varepsilon_1}{u^1-u^4},$$
$$\Gamma^5_{11} = \Gamma^5_{44} = -\Gamma^5_{14} = \frac{3u^5\varepsilon_1}{\left(u^1-u^4\right)^2}.$$

6.4.4. $3\times 3+1\times 1+1\times 1$ *Jordan blocks.*

$$L = \begin{bmatrix} u^1 & 0 & 0 & 0 & 0 \\ u^2 & u^1 & 0 & 0 & 0 \\ u^3 & u^2 & u^1 & 0 & 0 \\ 0 & 0 & 0 & u^4 & 0 \\ 0 & 0 & 0 & 0 & u^5 \end{bmatrix}, \quad e = \frac{\partial}{\partial u^1} + \frac{\partial}{\partial u^4} + \frac{\partial}{\partial u^5}, \quad a_0 = 3\varepsilon_1 u^1 + \varepsilon_4 u^4 + \varepsilon_5 u^5. \tag{6.11}$$

The non-vanishing Christoffel symbols Γ^i_{jk} (up to exchange of j with k) are

$$\Gamma^3_{23} = \Gamma^2_{22} = -\frac{3\varepsilon_1}{u^2}, \Gamma^1_{14} = \Gamma^2_{24} = \Gamma^3_{34} = -\Gamma^1_{44} = \frac{\varepsilon_4}{u^1-u^4},$$
$$\Gamma^1_{15} = \Gamma^2_{25} = \Gamma^3_{35} = -\Gamma^1_{55} = \frac{\varepsilon_5}{u^1-u^5}, \Gamma^4_{45} = -\Gamma^4_{55} = \frac{\varepsilon_5}{u^4-u^5},$$
$$\Gamma^5_{44} = -\Gamma^5_{45} = \frac{\varepsilon_4}{u^4-u^5}, \Gamma^4_{11} = -\Gamma^4_{14} = \frac{3\varepsilon_1}{u^1-u^4}, \Gamma^5_{11} = -\Gamma^5_{15} = \frac{3\varepsilon_1}{u^1-u^5},$$
$$\Gamma^2_{11} = \Gamma^3_{12} = \frac{\varepsilon_4 u^2}{\left(u^1-u^4\right)^2} + \frac{\varepsilon_5 u^2}{\left(u^1-u^5\right)^2},$$

$$\Gamma^2_{44} = -\Gamma^2_{14} = -\Gamma^3_{24} = \frac{u^2\varepsilon_4}{\left(u^1-u^4\right)^2}, \Gamma^2_{55} = -\Gamma^2_{15} = -\Gamma^3_{25} = \frac{u^2\varepsilon_5}{\left(u^1-u^5\right)^2},$$
$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^3_{13} = -\frac{\varepsilon_4}{u^1-u^4} - \frac{\varepsilon_5}{u^1-u^5},$$
$$\Gamma^3_{11} = \frac{\varepsilon_4\left(-\left(u^2\right)^2+\left(u^1-u^4\right)u^3\right)}{\left(u^1-u^4\right)^3} + \frac{\varepsilon_5\left(-\left(u^2\right)^2+\left(u^1-u^5\right)u^3\right)}{\left(u^1-u^5\right)^3},$$
$$\Gamma^3_{44} = -\Gamma^3_{14} = \frac{\varepsilon_4\left(u^1u^3-\left(u^2\right)^2-u^3u^4\right)}{\left(u^1-u^4\right)^3}, \Gamma^3_{55} = -\Gamma^3_{15} = \frac{\varepsilon_5\left(u^1u^3-\left(u^2\right)^2-u^3u^5\right)}{\left(u^1-u^5\right)^3},$$
$$\Gamma^4_{44} = \frac{3\varepsilon_1}{u^1-u^4} - \frac{\varepsilon_5}{u^4-u^5}, \Gamma^5_{55} = \frac{3\varepsilon_1}{u^1-u^5} + \frac{\varepsilon_4}{u^4-u^5},$$
$$\Gamma^3_{22} = \frac{3u^3\varepsilon_1}{\left(u^2\right)^2} - \frac{\varepsilon_4}{u^1-u^4} - \frac{\varepsilon_5}{u^1-u^5}.$$

6.4.5. $2\times 2+2\times 2+1\times 1$ *Jordan blocks.*

$$L = \begin{bmatrix} u^1 & 0 & 0 & 0 & 0 \\ u^2 & u^1 & 0 & 0 & 0 \\ 0 & 0 & u^3 & 0 & 0 \\ 0 & 0 & u^4 & u^3 & 0 \\ 0 & 0 & 0 & 0 & u^5 \end{bmatrix}, \quad e = \frac{\partial}{\partial u^1} + \frac{\partial}{\partial u^3} + \frac{\partial}{\partial u^5}, \quad a_0 = 2\varepsilon_1 u^1 + 2\varepsilon_3 u^3 + \varepsilon_5 u^5. \tag{6.12}$$

The non-vanishing Christoffel symbols Γ^i_{jk} (up to exchange of j with k) are

$$\Gamma^1_{11} = \Gamma^2_{12} = -\frac{2\varepsilon_3}{u^1-u^3} - \frac{\varepsilon_5}{u^1-u^5}, \Gamma^3_{33} = \Gamma^4_{34} = \frac{2\varepsilon_1}{u^1-u^3} - \frac{\varepsilon_5}{u^3-u^5},$$
$$-\Gamma^1_{13} = \Gamma^1_{33} = -\frac{2\varepsilon_3}{u^1-u^3}, \Gamma^1_{15} = \Gamma^2_{25} = -\Gamma^1_{55} = \frac{\varepsilon_5}{u^1-u^5},$$
$$\Gamma^2_{11} = \frac{2u^2\varepsilon_3}{\left(u^1-u^3\right)^2} + \frac{u^2\varepsilon_5}{\left(u^1-u^5\right)^2}, \Gamma^2_{33} = -\Gamma^2_{13} = \frac{2u^2\varepsilon_3}{\left(u^1-u^3\right)^2},$$
$$\Gamma^2_{55} = -\Gamma^2_{15} = \frac{u^2\varepsilon_5}{\left(u^1-u^5\right)^2}, \Gamma^2_{22} = -\frac{2\varepsilon_1}{u^2}, \Gamma^4_{44} = -\frac{2\varepsilon_3}{u^4}, \Gamma^2_{23} = \frac{2\varepsilon_3}{u^1-u^3},$$
$$\Gamma^3_{11} = -\Gamma^3_{13} = -\Gamma^4_{14} = \frac{2\varepsilon_1}{u^1-u^3}, \Gamma^3_{35} = \Gamma^4_{45} = -\Gamma^3_{55} = \frac{\varepsilon_5}{u^3-u^5},$$
$$\Gamma^4_{11} = -\Gamma^4_{13} = \frac{2\varepsilon_1 u^4}{\left(u^1-u^3\right)^2}, \Gamma^4_{33} = \frac{2u^4\varepsilon_1}{\left(u^1-u^3\right)^2} + \frac{u^4\varepsilon_5}{\left(u^3-u^5\right)^2},$$
$$\Gamma^4_{55} = -\Gamma^4_{35} = \frac{u^4\varepsilon_5}{\left(u^3-u^5\right)^2}, \Gamma^5_{11} = -\Gamma^5_{15} = \frac{2\varepsilon_1}{u^1-u^5}, \Gamma^5_{33} = -\Gamma^5_{35} = \frac{2\varepsilon_3}{u^3-u^5},$$
$$\Gamma^5_{55} = \frac{2\varepsilon_1}{u^1-u^5} + \frac{\varepsilon_3}{u^3-u^5}.$$

6.4.6. $2\times 2+1\times 1+1\times 1+1\times 1$ *Jordan blocks.*

$$L=\begin{bmatrix} u^1 & 0 & 0 & 0 & 0\\ u^2 & u^1 & 0 & 0 & 0\\ 0 & 0 & u^3 & 0 & 0\\ 0 & 0 & 0 & u^4 & 0\\ 0 & 0 & 0 & 0 & u^5\end{bmatrix},$$
$$e=\frac{\partial}{\partial u^1}+\frac{\partial}{\partial u^3}+\frac{\partial}{\partial u^4}+\frac{\partial}{\partial u^5},\ a_0=2\varepsilon_1 u^1+\varepsilon_3 u^3+\varepsilon_4 u^4+\varepsilon_5 u^5. \tag{6.13}$$

The non-vanishing Christoffel symbols Γ^i_{jk} (up to exchange of j with k) are

$$\Gamma^1_{11}=\Gamma^2_{12}=-\frac{\varepsilon_3}{u^1-u^3}-\frac{\varepsilon_4}{u^1-u^4}-\frac{\varepsilon_5}{u^1-u^5},$$
$$\Gamma^1_{13}=-\Gamma^1_{33}=\frac{\varepsilon_3}{u^1-u^3},\ \Gamma^1_{14}=\frac{\varepsilon_4}{u^1-u^4},\ \Gamma^1_{15}=-\Gamma^1_{55}=\frac{\varepsilon_5}{u^1-u^5},\ \Gamma^1_{44}=-\frac{\varepsilon_4}{u^1-u^4},$$
$$\Gamma^2_{11}=\frac{\varepsilon_3 u^2}{(u^1-u^3)^2}+\frac{\varepsilon_4 u^2}{(u^1-u^4)^2}+\frac{\varepsilon_5 u^2}{(u^1-u^5)^2},$$
$$\Gamma^2_{13}=-\frac{u^2\varepsilon_3}{(u^1-u^3)^2},\ \Gamma^2_{14}=-\frac{u^2\varepsilon_4}{(u^1-u^4)^2},\ \Gamma^2_{15}=-\frac{u^2\varepsilon_5}{(u^1-u^5)^2},$$
$$\Gamma^2_{22}=-\frac{2\varepsilon_1}{u^2},\ \Gamma^2_{23}=\frac{\varepsilon_3}{u^1-u^3},\ \Gamma^2_{24}=\frac{\varepsilon_4}{u^1-u^4},\ \Gamma^2_{25}=\frac{\varepsilon_5}{u^1-u^5},$$
$$\Gamma^2_{33}=\frac{u^2\varepsilon_3}{(u^1-u^3)^2},\ \Gamma^2_{44}=\frac{u^2\varepsilon_4}{(u^1-u^4)^2},\ \Gamma^2_{55}=\frac{u^2\varepsilon_5}{(u^1-u^5)^2},\ \Gamma^3_{11}=-\Gamma^3_{13}=\frac{2\varepsilon_1}{u^1-u^3},$$
$$\Gamma^3_{33}=\frac{2\varepsilon_1}{u^1-u^3}-\frac{\varepsilon_4}{u^3-u^4}-\frac{\varepsilon_5}{u^3-u^5},$$
$$\Gamma^3_{34}=-\Gamma^3_{44}=\frac{\varepsilon_4}{u^3-u^4},\ \Gamma^3_{35}=-\Gamma^3_{55}=\frac{\varepsilon_5}{u^3-u^5},$$
$$\Gamma^4_{11}=-\Gamma^4_{14}=\frac{2\varepsilon_1}{u^1-u^4},\ \Gamma^4_{33}=-\Gamma^4_{34}=\frac{\varepsilon_3}{u^3-u^4},$$
$$\Gamma^4_{44}=\frac{2\varepsilon_1}{u^1-u^4}+\frac{\varepsilon_3}{u^3-u^4}-\frac{\varepsilon_5}{u^4-u^5},$$
$$\Gamma^4_{45}=-\Gamma^4_{55}=\frac{\varepsilon_5}{u^4-u^5},\ \Gamma^5_{11}=-\Gamma^5_{15}=\frac{2\varepsilon_1}{u^1-u^5},$$
$$\Gamma^5_{33}=-\Gamma^5_{35}=\frac{\varepsilon_3}{u^3-u^5},\ \Gamma^5_{44}=-\Gamma^5_{45}=\frac{\varepsilon_4}{u^4-u^5},$$
$$\Gamma^5_{55}=\frac{2\varepsilon_1}{u^1-u^5}+\frac{\varepsilon_3}{u^3-u^5}+\frac{\varepsilon_4}{u^4-u^5}.$$
□

Remark 6.1. Starting from the above formulas and using the definition (5.3) of the Euler vector field, the definition of the dual product

$$X*Y=(E\circ)^{-1}X\circ Y$$

and the formula (3.4) for the dual connection one can reconstruct all the data defining a bi-flat structure.

7. The case of a Jordan block of arbitrary size

This section is devoted to the proof of the following Theorem.

Theorem 7.1. *For any choice of ε there exists a unique non-semisimple regular bi-flat structure $(\nabla,\nabla^*,\circ,*,e,E)$ with canonical coordinates $\{u^1,\dots,u^n\}$ such that $d_\nabla(E\circ -a_0I)=0$, where $E\circ$ has a single Jordan block of size n and $a_0=\varepsilon u^1$.*

Let us start with some preliminary observations. Looking at the formulas for the case of a single Jordan block we observe that in order to pass from the $n\times n$ Jordan block with $n\varepsilon_1=\varepsilon$ to the $(n+1)\times(n+1)$ Jordan block with $(n+1)\varepsilon_1=\varepsilon$ there is a simple rule. The new non-vanishing Christoffel symbols are Γ^{n+1}_{ij} with $i,j\neq 1$ and $n-i-j\geqslant -3$. The Christoffel symbol Γ^{n+1}_{22} is given by the formula

$$\Gamma^{n+1}_{22}=-\frac{1}{u^2}\sum_{s=1}^{n-1}u^{2+s}\Gamma^{n+1-s}_{22} \tag{7.1}$$

while the Christoffel symbols Γ^{n+1}_{ij} with i and j different from 1 and not simultaneously equal to 2 can be written in terms of the Christoffel symbols associated with the $n\times n$ Jordan block:

$$\Gamma^{n+1}_{ij}=\Gamma^{n}_{i-1,j}=\Gamma^{n}_{i,j-1} \tag{7.2}$$

provided that $i-1\neq 1$ or $j-1\neq 1$ (all the Christoffel symbols with i or j equal to 1 vanish). Notice that from the above definition it follows immediately that all the non-vanishing Christoffel symbols can be obtained recursively starting from $\Gamma^2_{22}=-\frac{\varepsilon}{u^2}$. Indeed applying the relation (7.2) $i+j-4$ times we obtain

$$\Gamma^{n+1}_{ij}=\Gamma^{n+5-i-j}_{22} \tag{7.3}$$

and more in general (the above property holds for all n)

$$\Gamma^{k}_{ij}=\Gamma^{k+4-i-j}_{22}\qquad \text{if } k-i-j\geqslant -2,\ i,j\neq 1 \tag{7.4}$$

$$\Gamma^{k}_{ij}=0\qquad \text{if } k-i-j< -2. \tag{7.5}$$

7.1. Technical lemmas

Using the above remarks one can easily prove by induction the following lemmas.

Lemma 7.2. *The Christoffel symbols Γ^k_{ij} associated with the $n\times n$ Jordan block, defined recursively as explained above starting from the 2×2 Jordan block satisfy the following identity:*

$$\frac{\partial\Gamma^k_{ij}}{\partial u^l}=\frac{\partial\Gamma^{k-1}_{ij}}{\partial u^{l-1}},\qquad l>2. \tag{7.6}$$

Proof. It is sufficient to prove the lemma in the case $i=j=2$. Indeed, if Γ^k_{ij} vanishes it is easy to check that also Γ^{k-1}_{ij} vanishes. If Γ^k_{ij} does not vanish then there exists $2\leqslant h\leqslant n+1$ satisfying $i+j-k=4-h$. Moreover $\Gamma^k_{ij}=\Gamma^h_{22}$ and $\Gamma^{k-1}_{ij}=\Gamma^{h-1}_{22}$. For $h=2$ we have to prove that

$$\frac{\partial\Gamma^2_{22}}{\partial u^l}=\frac{\partial\Gamma^1_{22}}{\partial u^{l-1}},\qquad l>2. \tag{7.7}$$

The l.h.s. vanishes since Γ^2_{22} depends only on u^2 while the r.h.s vanishes since $\Gamma^1_{22}=0$. For $h>2$ we have to prove that

$$\frac{\partial\Gamma^h_{22}}{\partial u^l}=\frac{\partial\Gamma^{h-1}_{22}}{\partial u^{l-1}},\qquad l>2. \tag{7.8}$$

For $h=3$ it is true. Assume that it is true for a given $h\geqslant 3$. Then using (7.1) and the inductive hypothesis we get

$$\frac{\partial\Gamma^{h+1}_{22}}{\partial u^l}=-\frac{1}{u^2}\sum_{s=1}^{h-1}\left(\frac{\partial\Gamma^{h-s+1}_{22}}{\partial u^l}\right)u^{s+2}-\frac{1}{u^2}\Gamma^{h-l+3}_{22}=-\frac{1}{u^2}\sum_{s=1}^{h-1}\left(\frac{\partial\Gamma^{h-s}_{22}}{\partial u^{l-1}}\right)u^{s+2}-\frac{1}{u^2}\Gamma^{h-l+3}_{22},$$

$$\frac{\partial\Gamma^{h}_{22}}{\partial u^{l-1}}=\frac{\delta^3_l}{(u^2)^2}\sum_{s=1}^{h-2}\Gamma^{h-s}_{22}u^{s+2}-\frac{1}{u^2}\sum_{s=1}^{h-2}\left(\frac{\partial\Gamma^{h-s}_{22}}{\partial u^{l-1}}\right)u^{s+2}-\frac{1-\delta^l_3}{u^2}\Gamma^{h-l+3}_{22}$$

thus

$$\begin{aligned}\frac{\partial\Gamma^{h+1}_{22}}{\partial u^l}-\frac{\partial\Gamma^{h}_{22}}{\partial u^{l-1}}&=-\frac{\delta^3_l}{(u^2)^2}\sum_{s=1}^{h-2}\Gamma^{h-s}_{22}u^{s+2}-\frac{1}{u^2}\left(\frac{\partial\Gamma^1_{22}}{\partial u^{l-1}}\right)u^{h+1}-\frac{\delta^3_l}{u^2}\Gamma^{h-l+3}_{22}\\&=-\frac{\delta^3_l}{(u^2)^2}\sum_{s=1}^{h-2}\Gamma^{h-s}_{22}u^{s+2}+\frac{\delta^3_l}{(u^2)^2}\sum_{s=1}^{h-2}\Gamma^{h-s}_{22}u^{s+2}=0.\end{aligned}$$

□

Lemma 7.3. *The Christoffel symbols Γ^k_{ij} associated with the $n\times n$ Jordan block, defined recursively as explained above starting from the 2×2 Jordan block satisfy the following identities:*

$$\sum_{k=1}^{n}\Gamma^i_{jk}u^k=\sum_{k=2}^{n}\Gamma^i_{jk}u^k=0,\qquad i\neq j\quad or\quad i=1\quad or\quad j=1 \tag{7.9}$$

$$\sum_{k=1}^{n}\Gamma^i_{jk}u^k=\sum_{k=2}^{n}\Gamma^i_{jk}u^k=-\varepsilon,\qquad i=j\neq 1 \tag{7.10}$$

Proof. For $i=1$ or $j=1$ the first identity is trivially satisfied since all the summands vanish. Thus we can assume both indices different from 1. For $i=j=2$ the only term surviving in the sum is $\Gamma^2_{22}u^2$ and the result follows from the definition of Γ^2_{22}. For $j=2$ and $i\neq 2$ the result follows from the formula (7.1). Indeed, we can rewrite the identity

$$\Gamma^i_{22}=-\frac{1}{u^2}\sum_{s=1}^{i-2}u^{2+s}\Gamma^{i-s}_{22}$$

as

$$\sum_{s=2}^{i}u^s\Gamma^i_{2s}=0.$$

Taking into account that for $s>i$ $\Gamma^i_{2s}=0$ and $\Gamma^i_{21}=0$ we have

$$\sum_{s=1}^{n}u^s\Gamma^i_{2s}=0.$$

The cases $i>2$ and $j>2$ can be reduced to the above cases using the identity $\Gamma^i_{jk}=\Gamma^{i-1}_{j-1,k}$. For instance, if $i=j$ applying $(i-2)$-times this identity we reduce to the case $i=j=2$. □

Lemma 7.4. *The connection* ∇ *associated with the* $n\times n$ *Jordan block satisfies the condition*

$$\nabla_j E^i=(1-\varepsilon)\,\delta^i_j+\varepsilon e^i e^j. \tag{7.11}$$

Proof. Since $\nabla_j E^i=\delta^i_j+\Gamma^i_{jk}u^k$ the result follows from the previous lemma. □

Lemma 7.5. *The component of* E^{-1} *are defined recursively by*

$$\left(E^{-1}\right)^1=\frac{1}{u^1},\qquad \left(E^{-1}\right)^{m+1}=-\frac{1}{u^1}\sum_{s=1}^{m}u^{1+s}\left(E^{-1}\right)^{m+1-s} \tag{7.12}$$

Proof. By definition $E^{-1}\circ E=e$. In canonical coordinates we obtain

$$\left(E^{-1}\right)^{i+1-k}u^k=\delta^i_1.$$

In the one component case we obtain $(E^{-1})^1=\frac{1}{u^1}$. In the $(n+1)$-component case we obtain

$$\begin{aligned}
&\left(E^{-1}\right)^1u^1=1\\
&\left(E^{-1}\right)^2u^1+\left(E^{-1}\right)^1u^2=0\\
&\qquad\vdots\\
&\left(E^{-1}\right)^{n+1}u^1+\left(E^{-1}\right)^n u^2+\cdots+\left(E^{-1}\right)^1u^{n+1}=0.
\end{aligned}$$

The first n equations coincide with the equations for the n-component case. □

Lemma 7.6. *The dual connection* ∇^* *is defined by*

$$\Gamma^{*k}_{ij}=\Gamma^k_{ij}+(\varepsilon-1)\left(E^{-1}\right)^{k-i-j+2}-\varepsilon\left(E^{-1}\right)^1\delta^k_1\delta^1_i\delta^1_j \tag{7.13}$$

where it is understood that $(E^{-1})^{k-i-j+2}=0$ *if* $k-i-j<-1$.

Proof. From (3.4) and (7.11) it follows that

$$\Gamma^{*k}_{ij}=\Gamma^k_{ij}-c^{*l}_{ji}\left((1-\varepsilon)\,\delta^k_l+\varepsilon e^k e^l\right)=\Gamma^k_{ij}-(1-\varepsilon)\,c^{*k}_{ji}-\varepsilon c^{*1}_{ji}\delta^k_1.$$

By definition

$$c^{*i}_{jk}=c^i_{jl}c^l_{km}\left(E^{-1}\right)^m=\delta^i_{j+l-1}\delta^l_{k+m-1}\left(E^{-1}\right)^m=\delta^i_{j+l-1}\left(E^{-1}\right)^{l+1-k}=\left(E^{-1}\right)^{2+i-j-k}.$$

Substituting in Γ^{*k}_{ij} we get the result. □

Using the above lemmas we can prove theorem 7.1. The proof can be divided in the following steps

(1) Flatness of ∇ and $\nabla e=0$.
(2) Compatibility of ∇ and $\circ$.
(3) Linearity of the Euler vector field.
(4) Uniqueness.

7.2. Flatness of ∇

We already know that the connection ∇ is flat for $n = 2,3,4,5$. We need to prove that if the connection ∇ associated with the $n\times n$ Jordan block is flat, that is

$$\partial_k\Gamma^i_{hj} - \partial_h\Gamma^i_{kj} - \sum_{l=1}^{n}\left(\Gamma^i_{hl}\Gamma^l_{kj} - \Gamma^i_{kl}\Gamma^l_{hj}\right) = 0, \qquad i,j,h,k = 1,\dots,n \tag{7.14}$$

then also the connection associated with the $(n+1)\times(n+1)$ Jordan block is flat:

$$R^i_{hkj} = \partial_k\Gamma^i_{hj} - \partial_h\Gamma^i_{kj} - \sum_{l=1}^{n+1}\left(\Gamma^i_{hl}\Gamma^l_{kj} - \Gamma^i_{kl}\Gamma^l_{hj}\right) = 0, \qquad i,j,h,k = 1,\dots,n+1. \tag{7.15}$$

Since $\Gamma^i_{j,n+1} = 0$ if $i \neq n+1$ we need only to check the case $i = n+1$. We have three interesting subcases: $h > 2, k > 2, j > 2$, $h = 2, k > 2, j > 2$ and $h = j = 2$, $k > 2$. In the first case using the lemma and the definition of the Christoffel symbols we have (notice that $\Gamma^2_{ij} = 0$ if i or j are greater than 2)

$$\begin{aligned}
R^{n+1}_{hkj} &= \partial_{k-1}\Gamma^n_{hj} - \partial_{h-1}\Gamma^n_{kj} - \sum_{l=3}^{n}\left(\Gamma^{n+1}_{hl}\Gamma^l_{kj} - \Gamma^{n+1}_{kl}\Gamma^l_{hj}\right)\\
&= \partial_{k-1}\Gamma^{n-1}_{h-1,j} - \partial_{h-1}\Gamma^{n-1}_{k-1,j} - \sum_{l=3}^{n}\left(\Gamma^{n-1}_{h-1,l-1}\Gamma^{l-1}_{k-1,j} - \Gamma^{n-1}_{k-1,l-1}\Gamma^{l-1}_{h-1,j}\right)\\
&= \partial_{k-1}\Gamma^{n-1}_{h-1,j} - \partial_{h-1}\Gamma^{n-1}_{k-1,j} - \sum_{l=1}^{n}\left(\Gamma^{n-1}_{h-1,l}\Gamma^{l}_{k-1,j} - \Gamma^{n-1}_{k-1,l}\Gamma^{l}_{h-1,j}\right)\\
&= R^{n-1}_{h-1,k-1,j}.
\end{aligned}$$

The quantity $R^{n-1}_{h-1,k-1,j}$ vanishes by hypothesis if $j = 1,\dots,n$. For $j = n+1$ it vanishes since each term in the sum contains a Christoffel symbol of the form $\Gamma^r_{s,n+1}$ with $r < n+1$. In the second case we have

$$\begin{aligned}
R^{n+1}_{2kj} &= \partial_k\Gamma^{n+1}_{2j} - \partial_2\Gamma^{n+1}_{kj} - \sum_{l=3}^{n+1}\left(\Gamma^{n+1}_{2l}\Gamma^l_{kj} - \Gamma^{n+1}_{kl}\Gamma^l_{2j}\right)\\
&= \partial_{k-1}\Gamma^n_{2j} - \partial_2\Gamma^n_{k-1,j} - \sum_{l=3}^{n+1}\left(\Gamma^n_{2,l-1}\Gamma^{l-1}_{k-1,j} - \Gamma^n_{k-1,l}\Gamma^l_{2j}\right)\\
&= \partial_{k-1}\Gamma^n_{2j} - \partial_2\Gamma^n_{k-1,j} - \sum_{l=1}^{n}\left(\Gamma^n_{2l}\Gamma^l_{k-1,j} - \Gamma^n_{k-1,l}\Gamma^l_{2j}\right) = R^n_{2,k-1,j} = 0.
\end{aligned}$$

Finally in the last case we have

$$\begin{aligned}
R^{n+1}_{2k2} &= \partial_k\Gamma^{n+1}_{22} - \partial_2\Gamma^{n+1}_{k2} - \sum_{l=3}^{n+1}\Gamma^{n+1}_{2l}\Gamma^l_{k2} + \sum_{l=2}^{n+1}\Gamma^{n+1}_{kl}\Gamma^l_{22}\\
&= \partial_{k-1}\Gamma^n_{22} - \partial_2\Gamma^n_{k-1,2} - \sum_{l=3}^{n+1}\Gamma^n_{2,l-1}\Gamma^{l-1}_{k-1,2} + \sum_{l=2}^{n}\Gamma^n_{k-1,l}\Gamma^l_{22}\\
&= \partial_{k-1}\Gamma^n_{22} - \partial_2\Gamma^n_{k-1,2} - \sum_{l=1}^{n}\Gamma^n_{2l}\Gamma^l_{k-1,2} + \sum_{l=1}^{n}\Gamma^n_{k-1,l}\Gamma^l_{22} = R^n_{2,k-1,2} = 0.
\end{aligned}$$

The condition $\nabla e = 0$ is equivalent to $\Gamma^i_{j1} = 0$.

7.3. *Compatibility of ∇ and $\circ$*

In canonical coordinates this means that

$$\Gamma^k_{i+j-1,l} - \Gamma^{k-i+1}_{lj} = \Gamma^k_{l+j-1,i} - \Gamma^{k-l+1}_{ij}.$$

Let us prove this condition by induction. We already know that it is satisfied up to $n = 5$. Let us suppose that it is satisfied for a given n. We have to prove that

$$\Gamma^{n+1}_{i+j-1,l} - \Gamma^{n+2-i}_{lj} = \Gamma^{n+1}_{l+j-1,i} - \Gamma^{n+2-l}_{ij},$$

where it is understood that $\Gamma^k_{ij} = 0$ if i,j or k are greater than $n+1$. If $i = 1$ or $l = 1$ the above condition is trivially satisfied. If $i,l \neq 1$ we have $\Gamma^{n+1}_{i+j-1,l} = \Gamma^{n+1}_{l+j-1,i}$ (if $i + j \geqslant n+1$ both sides vanish). If $j = 1$ the remaining condition is trivially satisfied. If also $j \neq 1$ the remaining condiiton is satisfied since $\Gamma^{n+2-i}_{lj} = \Gamma^{n+1}_{l+i-1,j}$ and $\Gamma^{n+2-l}_{ij} = \Gamma^{n+1}_{i+l-1,j}$ (if $i + l \geqslant n+2$ both Christoffel symbols vanish).

7.4. *Linearity of the Euler vector field*

We need to prove $\nabla\nabla E = 0$. In local coordinates we have:

$$\begin{aligned}\nabla_i\nabla_j E^k &= \Gamma^k_{il}\nabla_j E^l - \Gamma^l_{ij}\nabla_l E^k\\ &= \Gamma^k_{il}\left((1-\varepsilon)\,\delta^l_j + \varepsilon e^l e^j\right) - \Gamma^l_{ij}\left((1-\varepsilon)\,\delta^k_l + \varepsilon e^k e^l\right) = 0.\end{aligned}$$

The flatness of ∇^* and the additional properties follows from the linearity of E (here we are using the already mentioned result of [15] which holds true also in the non-semisimple setting) and from the definition of ∇^* and $*$.

7.5. $d_\nabla(E\circ - a_0 I) = 0$

It is easy to prove by induction that the Christoffel symbols obtained using conditions (7.1) and (7.2) satisfy the condition $d_\nabla(L - a_0 I) = 0$ with $L = E\circ$. Indeed, let us denote by $V_{(n)}$ the tensor field V in the n-dimensional case. For $n = 2$ the condition $(d_{\nabla_{(2)}} V_{(2)})^k_{ij} = 0$ is satisfied. Let us assume that the Christoffel symbols Γ^i_{jk}, $1 \leqslant i,j,k \leqslant n$ of the connection $\nabla_{(n)}$ associated with the $n \times n$ block obtained applying the formulas (7.1) and (7.2) satisfy the condition $(d_{\nabla_{(n)}} V_{(n)})^k_{ij} = 0$. We have to show the Christoffel symbols associated with the $(n+1)\times(n+1)$ Jordan block obtained applying the formulas (7.1) and (7.2) satisfy the condition $(d_{\nabla_{(n+1)}} V_{(n+1)})^k_{ij} = 0$. Let us consider the case where $1 \leqslant i,j \leqslant n$ and $k = n+1$:

$$\left(d_{\nabla_{(n+1)}} V_{(n+1)}\right)^{n+1}_{ij} = \frac{\partial V^{n+1}_j}{\partial u^i} + \Gamma^{n+1}_{il} V^l_j - \frac{\partial V^{n+1}_i}{\partial u^j} - \Gamma^{n+1}_{jl} V^l_i = \sum_{l=j}^{n+1} \Gamma^{n+1}_{il} V^l_j - \sum_{l=i}^{n+1} \Gamma^{n+1}_{jl} V^l_i.$$

For i or j equal to 1 it is immediate to check that the above expression vanishes. For i and j different from 1 we have (replacing l with $k = l - j + 1$ in the first sum and l with $k = l - i + 1$ in the second sum)

$$\sum_{l=j}^{n+1} \Gamma^{n+1}_{il} V^l_j - \sum_{l=i}^{n+1} \Gamma^{n+1}_{jl} V^l_i = \sum_{l=j+1}^{3+n-i} \Gamma^{n+5-i-l}_{22} u^{l-j+1} - \sum_{l=i+1}^{3+n-j} \Gamma^{n+5-j-l}_{22} u^{l-i+1} = 0.$$

The proof in the remaining cases $1 \leqslant i,j,k \leqslant n$, $1 \leqslant i,k \leqslant n, j = n+1$ and $1 \leqslant i \leqslant n, k = j = n+1$ is straightforward.

7.6. Uniqueness

The connection ∇ is uniquely determined by the conditions

$$\nabla_j e^i = \partial_j e^i + \Gamma^i_{jl} e^l = 0$$
$$(d_\nabla V)^k_{ij} = \frac{\partial V^k_j}{\partial u^i} + \Gamma^k_{il} V^l_j - \frac{\partial V^k_i}{\partial u^j} - \Gamma^k_{jl} V^l_i = 0.$$

where $V = L - a_0 I$. Indeed, in the case of a single Jordan block in David–Hertling coordinates the $(1,1)$-tensor field V has the form

$$V = \begin{bmatrix} du_1 & 0 & 0 & \dots & 0 \\ u^2 & du_1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ u^{n-1} & \dots & u^2 & du_1 & 0 \\ u^n & \dots & u^3 & u^2 & du_1 \end{bmatrix}, \tag{7.16}$$

where $d = 1 - \varepsilon$. We have that :

- The vanishing of $(d_\nabla V)^i_{n,n-1}$ defines uniquely Γ^i_{nn}.
- Using the previous condition the vanishing of $(d_\nabla V)^i_{n-2,n}$ defines uniquely $\Gamma^i_{n-1,n}$.
- More in general using the previous conditions the vanishing of $(d_\nabla V)^i_{k,n}$ defines uniquely $\Gamma^i_{k+1,n}$.
- Similarly the vanishing of $(d_\nabla V)^i_{n-2,n-1}$ defines uniquely $\Gamma^i_{n-1,n-1}$ and the vanishing of $(d_\nabla V)^i_{k,n-1}$ defines uniquely $\Gamma^i_{k+1,n-1}$.

In general, the vanishing of $(d_\nabla V)^i_{n-j-1,n-j}$ defines uniquely $\Gamma^i_{n-j,n-j}$ and the vanishing of $(d_\nabla V)^i_{k,n-j}$ defines uniquely $\Gamma^i_{k+1,n-j}$ taking into account all the previous conditions starting from $j = n-1, k = n$. In this way get all the Christoffel symbols apart from $\Gamma^i_{1j} = \Gamma^i_{j1}$ which vanish due to the condition $\nabla e = 0$. This means that the connection constructed above is unique and thus coincide with the connection obtained using conditions (7.1) and (7.2).

Remark 7.7. Alternatively one could also prove uniqueness observing that using the properties of ∇ and the condition $d_\nabla(E\circ -a_0 I) = 0$ one obtains conditions (7.1) and (7.2).

8. The case of an arbitrary number of Jordan blocks

Theorem 7.1 can be extended to the general case where the operator $L = E\circ$ has a block diagonal form.

Theorem 8.1. *For any choice of $\varepsilon_1, \dots, \varepsilon_r$ there exists a unique regular bi-flat structure $(\nabla, \nabla^*, \circ, *, e, E)$ with canonical coordinates $\{u^1, \dots, u^n\}$ such that $d_\nabla(E\circ -a_0 I) = 0$, where r is the number of the Jordan blocks (of sizes $m_1, \dots, m_r$) of $E\circ$ and, set $m_0 = 0$,*

$$a_0 = \sum_{\alpha=1}^r m_\alpha \varepsilon_\alpha u^{1(\alpha)} = \sum_{\alpha=1}^r m_\alpha \varepsilon_\alpha u^{m_0+m_1+\dots+m_{\alpha-1}+1}.$$

In order to prove this theorem, the crucial Lemmas 6.2–6.6 must be suitably extended too and some new preliminary results must be taken into account.

Let $(M,\circ,e,E)$ be a regular F-manifold of dimension $n\geqslant 2$ with an Euler vector field E. Around a point $p\in M$, let the canonical form of the operator $L=E\circ$ have r Jordan blocks $L_1,\dots,L_r$ of sizes $m_1,\dots,m_r$ with distinct eigenvalues. Let us define $a_0=\sum_{\alpha=1}^r\varepsilon_\alpha\,\mathrm{Tr}(L_\alpha)$. Our final goal is to prove that for any choice of $\varepsilon_1,\dots,\varepsilon_r$ there exists a unique regular bi-flat F-structure with non-semisimple canonical coordinates such that $L=E\circ$ and $d_\nabla(L-a_0I)=0$.

Any set of coordinates $u^1,\dots,u^n$ for M can be re-labelled by means of the following notation: for each $\alpha\in\{2,\dots,r\}$ and for each $j\in\{1,\dots,m_\alpha\}$ we write

$$j(\alpha)=m_1+\cdots+m_{\alpha-1}+j \tag{8.1}$$

(for $\alpha=1$ we set $j(\alpha)=j$) so that $u^{j(\alpha)}$ denotes the j-th coordinate associated to the α-th Jordan block. From now on, we will write u^i when seeing the coordinate as running from 1 to the dimension of the manifold and we will write $u^{i(\alpha)}$ when in need to highlight the Jordan block to which the coordinate refers. According to this notation, ∂_i and $\partial_{i(\alpha)}$ will denote the partial derivative with respect to u^i and $u^{i(\alpha)}$ respectively.

Let us recall that in these coordinates we have

- $e=\sum\limits_{\alpha=1}^r\partial_{1(\alpha)}$;
- $E=\sum\limits_{s=1}^n u^s\,\partial_s=\sum\limits_{\sigma=1}^r\sum\limits_{s=1}^{m_\sigma}u^{s(\sigma)}\,\partial_{s(\sigma)}$;
- $c^{k(\gamma)}_{i(\alpha)j(\beta)}=\delta^\gamma_\alpha\delta^\gamma_\beta\delta^k_{i+j-1}$ for any $\alpha,\beta,\gamma\in\{1,\dots,r\}$, $i\in\{1,\dots,m_\alpha\}$, $j\in\{1,\dots,m_\beta\}$, $k\in\{1,\dots,m_\gamma\}$.

Remark 8.2. Due to regularity condition we are implicitly assuming that $u^{2(\alpha)}\neq 0$ and $u^{1(\alpha)}\neq u^{1(\beta)}$ if $\alpha\neq\beta$.

8.1. The Christoffel symbols

The following proposition plays the role of conditions (7.1) and (7.2) in the case of a single block of arbitrary size.

Proposition 8.3. *Let α, β, γ be pairwise distinct. Then there exists a unique torsionless connection ∇ satisfying the following conditions:*

1. *For each value of i,j,k*
$$\Gamma^{k(\gamma)}_{i(\alpha)j(\beta)}=0. \tag{8.2}$$
2. *For every j,k when $i\geqslant 2$*
$$\Gamma^{k(\alpha)}_{i(\beta)j(\alpha)}=0, \tag{8.3}$$
and when $i=1$

$$\Gamma^{k(\alpha)}_{1(\beta)j(\alpha)}=\Gamma^{(k-j+1)(\alpha)}_{1(\beta)1(\alpha)}=\begin{cases}0 & \text{if } k<j,\\ \frac{m_\beta\varepsilon_\beta}{u^{1(\alpha)}-u^{1(\beta)}} & \text{if } k=j,\\ -\frac{1}{u^{1(\alpha)}-u^{1(\beta)}}\sum\limits_{s=2}^{k-j+1}\Gamma^{(k-j-s+2)(\alpha)}_{1(\beta)1(\alpha)}u^{s(\alpha)} & \text{if } k>j.\end{cases} \tag{8.4}$$

3. *For each k when* $i+j\geqslant 3$

$$\Gamma^{k(\alpha)}_{i(\beta)j(\beta)}=0, \tag{8.5}$$

and when $i=j=1$

$$\Gamma^{k(\alpha)}_{1(\beta)1(\beta)}=-\Gamma^{k(\alpha)}_{1(\beta)1(\alpha)}. \tag{8.6}$$

4. *The Christoffel symbols* $\Gamma^{k(\alpha)}_{i(\alpha)j(\alpha)}$ *are defined by the following formulas*

$$\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}=-\frac{m_\alpha\varepsilon_\alpha}{u^{2(\alpha)}}, \tag{8.7}$$

and

$$\Gamma^{k(\alpha)}_{1(\alpha)j(\alpha)}=-\sum_{\sigma\neq\alpha}\Gamma^{k(\alpha)}_{1(\sigma)j(\alpha)}=\begin{cases}0 & \text{if } k<j,\\ -\sum_{\sigma\neq\alpha}\frac{m_\sigma\varepsilon_\sigma}{u^{1(\alpha)}-u^{1(\sigma)}} & \text{if } k=j,\\ \sum_{\sigma\neq\alpha}\frac{1}{u^{1(\alpha)}-u^{1(\sigma)}}\sum_{s=2}^{k-j+1}\Gamma^{(k-j-s+2)(\alpha)}_{1(\sigma)1(\alpha)}u^{s(\alpha)} & \text{if } k>j,\end{cases} \tag{8.8}$$

and (for $k\geqslant 3$*)*

$$\Gamma^{k(\alpha)}_{2(\alpha)2(\alpha)}=\Gamma^{(k-2)(\alpha)}_{1(\alpha)1(\alpha)}-\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\frac{u^{k(\alpha)}}{u^{2(\alpha)}}-\frac{1}{u^{2(\alpha)}}\sum_{l=1}^{k-3}\left(\Gamma^{(l+2)(\alpha)}_{2(\alpha)2(\alpha)}-\Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)}\right)u^{(k-l)(\alpha)}, \tag{8.9}$$

and (for $i,j\geqslant 2$*)*

$$\Gamma^{k(\alpha)}_{i(\alpha)j(\alpha)}=\begin{cases}0 & \text{if } k-i-j\leqslant -3,\\ \Gamma^{(k-i-j+4)(\alpha)}_{2(\alpha)2(\alpha)} & \text{if } k-i-j\geqslant -2.\end{cases} \tag{8.10}$$

Proof. The above formulas uniquely determine the expressions for all of the Christoffel symbols. By (8.2) all of those Christoffel symbols whose indices correspond to pairwise distinct Jordan blocks vanish. One therefore only needs expressions for the ones whose indices correspond to at most two different Jordan blocks. Let us first explain how to construct Christoffel symbols whose indices correspond to two distinct Jordan blocks, which we label by α and β. By (8.4) we determine $\Gamma^{k(\alpha)}_{1(\beta)j(\alpha)}$ for $k\leqslant j$ and starting from these functions we determine the Christoffel symbols

$$\left\{\Gamma^{k(\alpha)}_{1(\beta)j(\alpha)}\,|\,j=1,\dots,m_\alpha,\,k=1,\dots,m_\alpha\right\}.$$

By (8.3) we determine

$$\Gamma^{k(\alpha)}_{i(\beta)j(\alpha)}=0$$

for $i\geqslant 2$ for each $j,k=1,\dots,m_\alpha$. By (8.5) we determine

$$\Gamma^{k(\alpha)}_{i(\beta)j(\beta)}=0$$

when $i+j \geqslant 3$ for each $k=1,\dots,m_\alpha$. By (8.6) we determine

$$\Gamma^{k(\alpha)}_{1(\beta)1(\beta)} = -\Gamma^{k(\alpha)}_{1(\beta)1(\alpha)}$$

for each $k=1,\dots,m_\alpha$. Let us now explain how to construct Christoffel symbols whose indices correspond to a single Jordan block, which we label by α. By (8.7) we determine $\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}$. By (8.8), for each $j,k=1,\dots,m_\alpha$ we determine $\Gamma^{k(\alpha)}_{1(\alpha)j(\alpha)}$. By (8.9), for each $k \geq 3$ we can determine recursively $\Gamma^{k(\alpha)}_{2(\alpha)2(\alpha)}$ (since the formula for $\Gamma^{k(\alpha)}_{2(\alpha)2(\alpha)}$ involves the Christoffel symbols $\{\Gamma^{t(\alpha)}_{1(\alpha)1(\alpha)} \,|\, t=1,\dots,k-2\}$ that we already know from above and $\{\Gamma^{t(\alpha)}_{2(\alpha)2(\alpha)} \,|\, t=2,\dots,k-1\}$). By (8.10), for each $i,j \geqslant 2$ and for each $k=1,\dots,m_\alpha$ we determine $\Gamma^{k(\alpha)}_{i(\alpha)j(\alpha)}$ in terms of the Christoffel symbols $\{\Gamma^{t(\alpha)}_{2(\alpha)2(\alpha)} \,|\, t=2,\dots,m_\alpha\}$ that we know from above. □

Example 8.4. The above proposition allows to recover all the formulas in section 6. For instance, let us reconstruct the Christoffel symbols in the five-dimensional case of $3\times 3+2\times 2$ Jordan blocks by means of the above formulas. In this case we have

$$u^1 = u^{1(1)},\ u^2 = u^{2(1)},\ u^3 = u^{3(1)},\ u^4 = u^{1(2)},\ u^5 = u^{2(2)}.$$

By (8.4) we get

$$\begin{aligned}
\Gamma^1_{14} &= \frac{2\varepsilon_4}{u^1-u^4} = \Gamma^2_{24} = \Gamma^3_{34}\\
\Gamma^4_{14} &= -\frac{3\varepsilon_1}{u^1-u^4} = \Gamma^5_{15}\\
\Gamma^2_{14} &= -\frac{1}{u^1-u^4}\Gamma^1_{14}u^2 = -\frac{2\varepsilon_4}{(u^1-u^4)^2}u^2 = \Gamma^3_{24}\\
\Gamma^3_{14} &= -\frac{1}{u^1-u^4}\left(\Gamma^2_{14}u^2+\Gamma^1_{14}u^3\right) = \frac{2\varepsilon_4}{(u^1-u^4)^3}\left(\left(u^2\right)^2 - u^1u^3 + u^3u^4\right)\\
\Gamma^5_{14} &= \frac{1}{u^1-u^4}\Gamma^4_{14}u^5 = -\frac{3\varepsilon_1}{(u^1-u^4)^2}u^5\\
\Gamma^4_{15} &= 0\\
\Gamma^1_{24} &= 0 = \Gamma^1_{34} = \Gamma^2_{34}\\
\Gamma^1_{25} &= 0 = \Gamma^1_{35} = \Gamma^2_{35}.
\end{aligned}$$

By (8.3) we get

$$\begin{aligned}
&\Gamma^k_{i5} = 0 \qquad \text{for } i,k=1,2,3\\
&\Gamma^k_{2j} = \Gamma^k_{3j} = 0 \qquad \text{for } j,k=4,5.
\end{aligned}$$

By (8.5) we get

$$\begin{aligned}
\Gamma^4_{22} &= 0 = \Gamma^4_{23} = \Gamma^4_{33} = \Gamma^5_{22} = \Gamma^5_{23} = \Gamma^5_{33}\\
\Gamma^1_{55} &= 0 = \Gamma^2_{55} = \Gamma^3_{55}.
\end{aligned}$$

By (8.6) and (8.8) we get

$$\Gamma^1_{12} = -\Gamma^1_{24} = 0$$
$$\Gamma^2_{12} = -\Gamma^2_{24} = -\frac{2\varepsilon_4}{u^1 - u^4}$$
$$\Gamma^3_{12} = -\Gamma^3_{24} = \frac{2\varepsilon_4}{(u^1 - u^4)^2} u^2$$
$$\Gamma^4_{12} = -\Gamma^4_{24} = 0$$
$$\Gamma^5_{12} = -\Gamma^5_{24} = 0$$
$$\Gamma^1_{13} = -\Gamma^1_{34} = 0$$
$$\Gamma^2_{13} = -\Gamma^2_{34} = 0$$
$$\Gamma^3_{13} = -\Gamma^3_{34} = -\frac{2\varepsilon_4}{u^1 - u^4}$$
$$\Gamma^4_{13} = -\Gamma^4_{34} = 0$$
$$\Gamma^5_{13} = -\Gamma^5_{34} = 0$$
$$\Gamma^1_{45} = -\Gamma^1_{15} = 0$$
$$\Gamma^2_{45} = -\Gamma^2_{15} = 0$$
$$\Gamma^3_{45} = -\Gamma^3_{15} = 0$$
$$\Gamma^4_{45} = -\Gamma^4_{15} = 0$$
$$\Gamma^5_{45} = -\Gamma^5_{15} = \frac{3\varepsilon_1}{u^1 - u^4}$$
$$\Gamma^1_{11} = -\Gamma^1_{14} = -\frac{2\varepsilon_4}{u^1 - u^4}$$
$$\Gamma^2_{11} = -\Gamma^2_{14} = \frac{2\varepsilon_4}{(u^1 - u^4)^2} u^2$$
$$\Gamma^3_{11} = -\Gamma^3_{14} = -\frac{2\varepsilon_4}{(u^1 - u^4)^3} \left((u^2)^2 - u^1 u^3 + u^3 u^4 \right)$$
$$\Gamma^4_{11} = -\Gamma^4_{14} = \frac{3\varepsilon_1}{u^1 - u^4}$$
$$\Gamma^5_{11} = -\Gamma^5_{14} = \frac{3\varepsilon_1}{(u^1 - u^4)^2} u^5$$
$$\Gamma^1_{44} = -\Gamma^1_{14} = -\frac{2\varepsilon_4}{u^1 - u^4}$$
$$\Gamma^2_{44} = -\Gamma^2_{14} = \frac{2\varepsilon_4}{(u^1 - u^4)^2} u^2$$
$$\Gamma^3_{44} = -\Gamma^3_{14} = -\frac{2\varepsilon_4}{(u^1 - u^4)^3} \left((u^2)^2 - u^1 u^3 + u^3 u^4 \right)$$
$$\Gamma^4_{44} = -\Gamma^4_{14} = \frac{3\varepsilon_1}{u^1 - u^4}$$
$$\Gamma^5_{44} = -\Gamma^5_{14} = \frac{3\varepsilon_1}{(u^1 - u^4)^2} u^5.$$

By (8.7) we get

$$\Gamma^2_{22} = -\frac{3\varepsilon_1}{u^2}$$
$$\Gamma^5_{55} = -\frac{2\varepsilon_4}{u^5}.$$

By (8.9) we get

$$\Gamma^3_{22} = \Gamma^1_{11} - \Gamma^2_{22}\frac{u^3}{u^2} = -\frac{2\varepsilon_4}{u^1 - u^4} + \frac{3\varepsilon_1}{(u^2)^2}u^3.$$

By (8.10) we get

$$\Gamma^1_{22} = 0 = \Gamma^1_{23} = \Gamma^1_{33} = \Gamma^2_{23} = \Gamma^2_{33} = \Gamma^3_{33}$$
$$\Gamma^3_{23} = \Gamma^2_{22} = -\frac{3\varepsilon_1}{u^2}$$
$$\Gamma^4_{55} = 0.$$

We obtained the same expressions as the ones exemplified is section 6.

Remark 8.5. All the Christoffel symbols can be obatained from the functions $\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} = -\frac{m_\alpha \varepsilon_\alpha}{u^{2(\alpha)}}$ and $\Gamma^{1(\alpha)}_{1(\beta)1(\alpha)} = \frac{m_\beta \varepsilon_\beta}{u^{1(\alpha)} - u^{1(\beta)}}$ (with $\beta \neq \alpha$). The last functions appear only in the case of multiple Jordan blocks and are at the origin of the additional difficulties one meets in the proof of the general case. However, exactly as in the case of a single block, increasing the size of a block $m_\alpha = N \to m_\alpha = N+1$ and rescaling the corresponding weight $\varepsilon_\alpha \to \frac{N+1}{N}\varepsilon_\alpha$ does not affect the definition of the Christoffels symbols $\Gamma^{k(\beta)}_{i(\sigma)j(\tau)}$ for the original range of the indices.

Remark 8.6. It is easy to observe that a_0 is a flat coordinate for ∇, namely $\nabla(da_0) = 0$.

Remark 8.7. It is likewise easy to check that $dd_L a_0 = 0$.

8.2. *Technical lemmas*

The results of this subsection follow from the above expressions for the Christoffel symbols and play a crucial role in the proof of the main theorem. We omit the details.

Lemma 8.8. *For every choice of α, β, γ, $\delta \in \{1,\ldots,r\}$ we have*

$$\frac{\partial \Gamma^{k(\gamma)}_{i(\alpha)j(\beta)}}{\partial u^{l(\delta)}} = \frac{\partial \Gamma^{(k-1)(\gamma)}_{i(\alpha)j(\beta)}}{\partial u^{(l-1)(\delta)}} \tag{8.11}$$

for all $k \in \{2,\ldots,m_\gamma\}$ and $l \in \{3,\ldots,m_\delta\}$. Moreover, if $\beta \neq \alpha = \gamma = \delta$ then (8.11) holds for $l = 2$ as well.

Lemma 8.9. *For each α, $\beta \in \{1,\ldots,r\}$, $i \in \{1,\ldots,m_\alpha\}$, $j \in \{1,\ldots,m_\beta\}$ we have*

$$\sum_{k=1}^{n} \Gamma^{i(\alpha)}_{j(\beta)k} u^k = \begin{cases} 0 & \text{if } i \neq j \\ -\delta^\alpha_\beta \sum\limits_{\sigma\neq\alpha} m_\sigma \varepsilon_\sigma + \left(1 - \delta^\alpha_\beta\right) m_\beta \varepsilon_\beta & \text{if } i = j = 1 \\ -\delta^\alpha_\beta \sum\limits_{\tau=1}^{r} m_\tau \varepsilon_\tau & \text{if } i = j \neq 1. \end{cases} \tag{8.12}$$

Lemma 8.10. *For each* α, $\beta \in \{1,\ldots,r\}$, $i \in \{1,\ldots,m_\alpha\}$, $j \in \{1,\ldots,m_\beta\}$ *we have*

$$\nabla_{j(\beta)} E^{i(\alpha)} = \begin{cases} 0 & \text{if } i \neq j \\ \delta^\alpha_\beta \left(1 - \sum_{\sigma\neq\alpha} m_\sigma \varepsilon_\sigma\right) + \left(1 - \delta^\alpha_\beta\right) m_\beta \varepsilon_\beta & \text{if } i = j = 1 \\ \delta^\alpha_\beta \left(1 - \sum_{\tau=1}^{r} m_\tau \varepsilon_\tau\right) & \text{if } i = j \neq 1. \end{cases} \tag{8.13}$$

Lemma 8.11. *For each* $\alpha \in \{1,\ldots,r\}$ *and* $l \in \{3,\ldots,m_\alpha - 1\}$, *we have*

$$A^{l(\alpha)} := \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} \left(\frac{u^{3(\alpha)}}{u^{2(\alpha)}} u^{l(\alpha)} - u^{(l+1)(\alpha)}\right) - \sum_{s=2}^{l-1} \left(\Gamma^{(s+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{s(\alpha)}_{1(\alpha)1(\alpha)}\right) u^{(l-s+1)(\alpha)} = 0. \tag{8.14}$$

Lemma 8.12. *For each* $\alpha, \sigma \in \{1,\ldots,r\}$ *with* $\alpha \neq \sigma$ *and* $l \in \{1,\ldots,m_\alpha - 2\}$, *we have*

$$\partial_{1(\sigma)} \left(\Gamma^{(l+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)}\right) = 0. \tag{8.15}$$

Lemma 8.13. *Given* $\alpha, \beta, \epsilon \in \{1,\ldots,r\}$ *with* $\alpha \neq \beta \neq \epsilon \neq \alpha$ *and* $s \in \{1,\ldots,m_\alpha - 1\}$,

$$B^{s(\alpha)}_{\beta\epsilon} := -\sum_{t=1}^{s+1} \Gamma^{(s-t+2)(\alpha)}_{1(\epsilon)1(\alpha)} \Gamma^{t(\alpha)}_{1(\beta)1(\alpha)} + \Gamma^{(s+1)(\alpha)}_{1(\beta)1(\alpha)} \Gamma^{1(\beta)}_{1(\epsilon)1(\beta)} + \Gamma^{(s+1)(\alpha)}_{1(\epsilon)1(\alpha)} \Gamma^{1(\epsilon)}_{1(\beta)1(\epsilon)} = 0. \tag{8.16}$$

Lemma 8.14. *Given* $\alpha, \beta \in \{1,\ldots,r\}$ *with* $\alpha \neq \beta$ *and* $s \in \{0,\ldots,m_\alpha - 2\}$[3], *we have*

$$\begin{aligned} C^{s(\alpha)}_{\beta} := \sum_{l=2}^{s+1} \left(\Gamma^{(s-l+4)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(s-l+2)(\alpha)}_{1(\alpha)1(\alpha)}\right) \Gamma^{l(\alpha)}_{1(\beta)1(\alpha)} + \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} \Gamma^{(s+2)(\alpha)}_{1(\beta)1(\alpha)} \\ + \Gamma^{(s+1)(\alpha)}_{1(\beta)1(\alpha)} \Gamma^{1(\beta)}_{1(\alpha)1(\beta)} = 0. \end{aligned} \tag{8.17}$$

We have all the ingredients to prove theorem 8.1. The proof is divided in the following steps:

(1) $\nabla e = 0$.
(2) $d_\nabla (E \circ - a_0 I) = 0$.
(3) Compatibility between ∇ and $\circ$.
(4) Linearity of the Euler vector field.
(5) Flatness of ∇.
(6) Uniqueness.

8.3. $\nabla e = 0$

The condition $\nabla e = 0$ is equivalent to the request that for every $\alpha, \beta \in \{1,\ldots,r\}$ and $i \in \{1,\ldots,m_\alpha\}$, $j \in \{1,\ldots,m_\beta\}$

[3] The summation is intended to be non-zero when $s \geqslant 1$.

$$\sum_{\sigma=1}^{r}\Gamma^{i(\alpha)}_{1(\sigma)j(\beta)}=0. \tag{8.18}$$

This condition is verified by the Christoffel symbols defined above.

8.4. $d_\nabla(E\circ -a_0 I)=0$

Let us consider now the condition

$$d_\nabla\left(L-a_0 I\right)=0 \tag{8.19}$$

for $L=E\circ$. For each $\alpha,\beta,\gamma=1,\dots,r$ and $i=1,\dots,m_\alpha, j=1,\dots,m_\beta, k=1,\dots,m_\gamma$ we have

$$\begin{aligned}
\left(d_\nabla\left(L-a_0 I\right)\right)^{i(\alpha)}_{j(\beta)k(\gamma)} &= \partial_{j(\beta)}\left(L-a_0 I\right)^{i(\alpha)}_{k(\gamma)}-\partial_{k(\gamma)}\left(L-a_0 I\right)^{i(\alpha)}_{j(\beta)}\\
&\quad+\Gamma^{i(\alpha)}_{j(\beta)l(\delta)}\left(L-a_0 I\right)^{l(\delta)}_{k(\gamma)}-\Gamma^{i(\alpha)}_{k(\gamma)l(\delta)}\left(L-a_0 I\right)^{l(\delta)}_{j(\beta)}\\
&=\delta^\alpha_\gamma\delta^\alpha_\beta\delta^{i-k+1}_j-\delta^\alpha_\gamma\delta^i_k m_\beta\varepsilon_\beta\delta^1_j-\delta^\alpha_\beta\delta^\alpha_\gamma\delta^{i-j+1}_k+\delta^\alpha_\beta\delta^i_j m_\gamma\varepsilon_\gamma\delta^1_k\\
&\quad+\sum_{\delta=1}^{r}\sum_{l=1}^{m_\delta}\Gamma^{i(\alpha)}_{j(\beta)l(\delta)}\delta_{\delta\gamma}u^{(l-k+1)(\gamma)}\mathbb{1}_{\{l\geqslant k\}}\\
&\quad-\sum_{\delta=1}^{r}\sum_{l=1}^{m_\delta}\Gamma^{i(\alpha)}_{k(\gamma)l(\delta)}\delta_{\delta\beta}u^{(l-j+1)(\beta)}\mathbb{1}_{\{l\geqslant j\}}\\
&=\delta^\alpha_\beta\delta^i_j m_\gamma\varepsilon_\gamma\delta^1_k-\delta^\alpha_\gamma\delta^i_k m_\beta\varepsilon_\beta\delta^1_j+\sum_{l=k}^{m_\gamma}\Gamma^{i(\alpha)}_{j(\beta)l(\gamma)}u^{(l-k+1)(\gamma)}\\
&\quad-\sum_{l=j}^{m_\beta}\Gamma^{i(\alpha)}_{k(\gamma)l(\beta)}u^{(l-j+1)(\beta)}
\end{aligned}$$

as

$$\left(L-a_0 I\right)^{a(\eta)}_{b(\mu)}=L^{a(\eta)}_{b(\mu)}-a_0\,\delta^{a(\eta)}_{b(\mu)}=\delta^\eta_\mu\sum_{s=1}^{m_\eta}u^{s(\eta)}\,\delta^{a+b-1}_s-\delta^\eta_\mu\,\delta^a_b\sum_{\alpha=1}^{r}m_\alpha\varepsilon_\alpha u^{1(\alpha)}$$

for each $\eta,\mu=1,\dots,r$ and $a=1,\dots,m_\eta$, $b=1,\dots,m_\beta$. Therefore (8.19) amounts to

$$\delta^\alpha_\beta\delta^i_j m_\gamma\varepsilon_\gamma\delta^1_k-\delta^\alpha_\gamma\delta^i_k m_\beta\varepsilon_\beta\delta^1_j+\sum_{l=k}^{m_\gamma}\Gamma^{i(\alpha)}_{j(\beta)l(\gamma)}u^{(l-k+1)(\gamma)}-\sum_{l=j}^{m_\beta}\Gamma^{i(\alpha)}_{k(\gamma)l(\beta)}u^{(l-j+1)(\beta)}=0 \tag{8.20}$$

for each $\alpha,\beta,\gamma=1,\dots,r$ and $i=1,\dots,m_\alpha$, $j=1,\dots,m_\beta$, $k=1,\dots,m_\gamma$. We split the proof in the following cases:

1. $\alpha=\beta=\gamma$
2. $\alpha=\beta\neq\gamma$ (this also covers $\alpha=\gamma\neq\beta$)
3. $\alpha\neq\beta=\gamma$
4. α,β,γ are pairwise distinct.

Case 1: $\alpha=\beta=\gamma$. Condition (8.20) becomes

$$m_\alpha\varepsilon_\alpha\left(\delta^i_j\delta^1_k-\delta^i_k\delta^1_j\right)+\sum_{l=k}^{m_\alpha}\Gamma^{i(\alpha)}_{j(\alpha)l(\alpha)}u^{(l-k+1)(\alpha)}-\sum_{l=j}^{m_\alpha}\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}u^{(l-j+1)(\alpha)}=0 \tag{8.21}$$

which is trivially satisfied if $j=k=1$ due to the symmetry between the indices j, k. If both j and k are greater or equal than 2, the left hand side term of (8.21) reads

$$\sum_{l=k}^{m_\alpha}\Gamma^{i(\alpha)}_{j(\alpha)l(\alpha)}u^{(l-k+1)(\alpha)}-\sum_{l=j}^{m_\alpha}\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}u^{(l-j+1)(\alpha)}$$
$$\overset{(8.10)}{=}\sum_{l=k}^{i-j+2}\Gamma^{(i-j-l+4)(\alpha)}_{2(\alpha)2(\alpha)}u^{(l-k+1)(\alpha)}-\sum_{l=j}^{i-k+2}\Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)}u^{(l-j+1)(\alpha)}$$

which vanishes by changing the variables in the two summations. Let us then consider the case where $j=1$ and $k\geqslant 2$ (this covers the case where $j\geqslant 2$ and $k=1$ as well). The left hand side term of (8.21) reads

$$-\delta^i_k m_\alpha\varepsilon_\alpha+\sum_{l=k}^{m_\alpha}\Gamma^{i(\alpha)}_{1(\alpha)l(\alpha)}u^{(l-k+1)(\alpha)}-\sum_{l=1}^{m_\alpha}\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}u^{l(\alpha)}$$
$$\underset{(8.10)}{\overset{(8.8)}{=}}-\delta^i_k m_\alpha\varepsilon_\alpha+\sum_{l=k}^{i}\Gamma^{i(\alpha)}_{1(\alpha)l(\alpha)}u^{(l-k+1)(\alpha)}-\Gamma^{i(\alpha)}_{1(\alpha)k(\alpha)}u^{1(\alpha)}-\sum_{l=2}^{i-k+2}\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}u^{l(\alpha)}$$
$$\underset{(8.10)}{\overset{(8.8)}{=}}-\delta^i_k m_\alpha\varepsilon_\alpha+\sum_{l=k+1}^{i}\Gamma^{(i-l+1)(\alpha)}_{1(\alpha)1(\alpha)}u^{(l-k+1)(\alpha)}-\sum_{l=2}^{i-k+2}\Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)}u^{l(\alpha)} \tag{8.22}$$

which trivially vanishes if $i<k$ and which vanishes by means of (8.7) if $i=k$. Let us then fix $i>k$. (8.22) becomes

$$\sum_{l=k+1}^{i}\Gamma^{(i-l+1)(\alpha)}_{1(\alpha)1(\alpha)}u^{(l-k+1)(\alpha)}-\sum_{l=2}^{i-k+2}\Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)}u^{l(\alpha)}$$
$$=\sum_{s=2}^{i-k+1}\Gamma^{(i-k-s+2)(\alpha)}_{1(\alpha)1(\alpha)}u^{s(\alpha)}-\sum_{l=2}^{i-k+2}\Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)}u^{l(\alpha)}$$
$$=-\sum_{l=2}^{i-k+1}\left(\Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)}-\Gamma^{(i-k-l+2)(\alpha)}_{1(\alpha)1(\alpha)}\right)u^{l(\alpha)}-\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}u^{(i-k+2)(\alpha)}$$
$$=-\sum_{s=1}^{i-k}\left(\Gamma^{(s+2)(\alpha)}_{2(\alpha)2(\alpha)}-\Gamma^{s(\alpha)}_{1(\alpha)1(\alpha)}\right)u^{(i-k-s+2)(\alpha)}-\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}u^{(i-k+2)(\alpha)}$$
$$\overset{(8.14)}{=}-\left(\Gamma^{3(\alpha)}_{2(\alpha)2(\alpha)}-\Gamma^{1(\alpha)}_{1(\alpha)1(\alpha)}\right)u^{(i-k+1)(\alpha)}-\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\frac{u^{3(\alpha)}}{u^{2(\alpha)}}u^{(i-k+1)(\alpha)}\overset{(8.9)}{=}0.$$

Case 2: $\alpha=\beta\neq\gamma$. Condition (8.20) becomes

$$\delta^i_j m_\gamma\varepsilon_\gamma\delta^1_k+\sum_{l=k}^{m_\gamma}\Gamma^{i(\alpha)}_{j(\alpha)l(\gamma)}u^{(l-k+1)(\gamma)}-\sum_{l=j}^{m_\alpha}\Gamma^{i(\alpha)}_{k(\gamma)l(\alpha)}u^{(l-j+1)(\alpha)}=0$$

that, by means of (8.3) and (8.4), is

$$\delta^1_k\left(\delta^i_j m_\gamma \varepsilon_\gamma + \Gamma^{(i-j+1)(\alpha)}_{1(\alpha)1(\gamma)}u^{1(\gamma)} - \sum_{l=j}^{i}\Gamma^{(i-l+1)(\alpha)}_{1(\gamma)1(\alpha)}u^{(l-j+1)(\alpha)}\right) = 0 \tag{8.23}$$

which is trivially satisfied for $k \geqslant 2$. Let us then fix $k = 1$. If $i < j$ then (8.23) is satisfied by means of (8.4). If $i = j$ then the left hand side of (8.23) reads

$$m_\gamma \varepsilon_\gamma + \Gamma^{1(\alpha)}_{1(\alpha)1(\gamma)}u^{1(\gamma)} - \Gamma^{1(\alpha)}_{1(\gamma)1(\alpha)}u^{1(\alpha)} \overset{(8.4)}{=} 0.$$

If $i > j$ then the left hand side of (8.23) reads

$$\begin{aligned}
&\Gamma^{(i-j+1)(\alpha)}_{1(\alpha)1(\gamma)}u^{1(\gamma)} - \sum_{l=j}^{i}\Gamma^{(i-l+1)(\alpha)}_{1(\gamma)1(\alpha)}u^{(l-j+1)(\alpha)}\\
&\quad = -\Gamma^{(i-j+1)(\alpha)}_{1(\gamma)1(\alpha)}\left(u^{1(\alpha)} - u^{1(\gamma)}\right) - \sum_{l=j+1}^{i}\Gamma^{(i-l+1)(\alpha)}_{1(\gamma)1(\alpha)}u^{(l-j+1)(\alpha)}\\
&\quad = -\Gamma^{(i-j+1)(\alpha)}_{1(\gamma)1(\alpha)}\left(u^{1(\alpha)} - u^{1(\gamma)}\right) - \sum_{s=2}^{i-j+1}\Gamma^{(i-j-s+2)(\alpha)}_{1(\gamma)1(\alpha)}u^{s(\alpha)} \overset{(8.4)}{=} 0.
\end{aligned}$$

Case 3: $\alpha \neq \beta = \gamma$. Condition (8.20) becomes

$$\sum_{l=k}^{m_\beta}\Gamma^{i(\alpha)}_{j(\beta)l(\beta)}u^{(l-k+1)(\beta)} - \sum_{l=j}^{m_\beta}\Gamma^{i(\alpha)}_{k(\beta)l(\beta)}u^{(l-j+1)(\beta)} = 0 \tag{8.24}$$

where, by means of (8.5), the two sums survive only if $j = k = 1$ (and with the only $l = 1$ term), in which case they mutually cancel out.
Case 4: $\alpha \neq \beta \neq \gamma \neq \alpha$. Condition (8.20) is trivially satisfied by means of (8.2).

8.5. Compatibility between ∇ and $\circ$

We are now going to prove that

$$\nabla_{i(\alpha)}c^{l(\epsilon)}_{j(\beta)k(\gamma)} = \nabla_{j(\beta)}c^{l(\epsilon)}_{i(\alpha)k(\gamma)}$$

which is equivalent to

$$\Gamma^{l(\epsilon)}_{i(\alpha)s(\sigma)}c^{s(\sigma)}_{j(\beta)k(\gamma)} - \Gamma^{s(\sigma)}_{i(\alpha)k(\gamma)}c^{l(\epsilon)}_{j(\beta)s(\sigma)} = \Gamma^{l(\epsilon)}_{j(\beta)s(\sigma)}c^{s(\sigma)}_{i(\alpha)k(\gamma)} - \Gamma^{s(\sigma)}_{j(\beta)k(\gamma)}c^{l(\epsilon)}_{i(\alpha)s(\sigma)}$$

and

$$\Gamma^{l(\epsilon)}_{i(\alpha)(j+k-1)(\beta)}\delta_{\beta\gamma} - \Gamma^{(l-j+1)(\beta)}_{i(\alpha)k(\gamma)}\delta^{\epsilon}_{\beta} = \Gamma^{l(\epsilon)}_{j(\beta)(i+k-1)(\alpha)}\delta_{\alpha\gamma} - \Gamma^{(l-i+1)(\alpha)}_{j(\beta)k(\gamma)}\delta^{\varepsilon}_{\alpha} \tag{8.25}$$

for all $\alpha, \beta, \gamma, \epsilon \in \{1, \ldots, r\}$ and any suitable choice of the indices i, j, k, l. The possible cases are the following ones:

1. $\alpha = \beta = \gamma = \epsilon$	4. $\alpha = \gamma = \epsilon \neq \beta$	7. $\alpha = \gamma \neq \beta = \epsilon$
2. $\alpha = \beta = \gamma \neq \epsilon$	5. $\beta = \gamma = \epsilon \neq \alpha$	8. $\alpha = \epsilon \neq \beta = \gamma$
3. $\alpha = \beta = \epsilon \neq \gamma$	6. $\alpha = \beta \neq \gamma = \epsilon$	9. otherwise.

Case 1: $\alpha = \beta = \gamma = \epsilon$. (8.25) becomes

$$\Gamma^{l(\alpha)}_{i(\alpha)(j+k-1)(\alpha)} - \Gamma^{(l-j+1)(\alpha)}_{i(\alpha)k(\alpha)} = \Gamma^{l(\alpha)}_{j(\alpha)(i+k-1)(\alpha)} - \Gamma^{(l-i+1)(\alpha)}_{j(\alpha)k(\alpha)} \tag{8.26}$$

If $i = 1$ (or equivalently $j = 1$) then this is

$$\Gamma^{l(\alpha)}_{1(\alpha)(j+k-1)(\alpha)} - \Gamma^{(l-j+1)(\alpha)}_{1(\alpha)k(\alpha)} = \Gamma^{l(\alpha)}_{j(\alpha)k(\alpha)} - \Gamma^{l(\alpha)}_{j(\alpha)k(\alpha)}$$

where both the left and the right-hand sides vanish, as

$$\begin{aligned}\Gamma^{l(\alpha)}_{1(\alpha)(j+k-1)(\alpha)} - \Gamma^{(l-j+1)(\alpha)}_{1(\alpha)k(\alpha)} &\overset{(8.18)}{=} -\sum_{\sigma\neq\alpha}\left(\Gamma^{l(\alpha)}_{1(\sigma)(j+k-1)(\alpha)} - \Gamma^{(l-j+1)(\alpha)}_{1(\sigma)k(\alpha)}\right)\\ &\overset{(8.4)}{=} -\sum_{\sigma\neq\alpha}\left(\Gamma^{(l-j+1)(\alpha)}_{1(\sigma)k(\alpha)} - \Gamma^{(l-j+1)(\alpha)}_{1(\sigma)k(\alpha)}\right) = 0.\end{aligned}$$

If $k = 1$ then (8.26) reads

$$\Gamma^{l(\alpha)}_{i(\alpha)j(\alpha)} - \Gamma^{(l-j+1)(\alpha)}_{i(\alpha)1(\alpha)} = \Gamma^{l(\alpha)}_{j(\alpha)i(\alpha)} - \Gamma^{(l-i+1)(\alpha)}_{j(\alpha)1(\alpha)}$$

that holds true, as

$$\begin{aligned}\Gamma^{(l-j+1)(\alpha)}_{i(\alpha)1(\alpha)} &\overset{(8.18)}{=} -\sum_{\sigma\neq\alpha}\Gamma^{(l-j+1)(\alpha)}_{i(\alpha)1(\sigma)} \overset{(8.4)}{=} -\sum_{\sigma\neq\alpha}\Gamma^{(l-i-j+2)(\alpha)}_{1(\alpha)1(\sigma)}\\ &\overset{(8.4)}{=} -\sum_{\sigma\neq\alpha}\Gamma^{(l-i+1)(\alpha)}_{j(\alpha)1(\sigma)} \overset{(8.18)}{=} \Gamma^{(l-i+1)(\alpha)}_{j(\alpha)1(\alpha)}.\end{aligned}$$

If all of i, j and k are greater or equal then 2 then, by (8.10) and (8.26) reads

$$\Gamma^{(l-i-j-k+5)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(l-i-j-k+5)(\alpha)}_{2(\alpha)2(\alpha)} = \Gamma^{(l-i-j-k+5)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(l-i-j-k+5)(\alpha)}_{2(\alpha)2(\alpha)}$$

which is trivially verified.

Case 2: $\alpha = \beta = \gamma \neq \epsilon$. (8.25) becomes

$$\Gamma^{l(\epsilon)}_{i(\alpha)(j+k-1)(\alpha)} = \Gamma^{l(\epsilon)}_{j(\alpha)(i+k-1)(\alpha)}$$

which is true by means of (8.5) and (8.6).

Case 3: $\alpha = \beta = \epsilon \neq \gamma$. (8.25) becomes

$$-\Gamma^{(l-j+1)(\alpha)}_{i(\alpha)k(\gamma)} = -\Gamma^{(l-i+1)(\alpha)}_{j(\alpha)k(\gamma)}$$

which is true by means of (8.3) and (8.4).

Case 4: $\alpha = \gamma = \epsilon \neq \beta$. (8.25) becomes

$$0 = \Gamma^{l(\alpha)}_{j(\beta)(i+k-1)(\alpha)} - \Gamma^{(l-i+1)(\alpha)}_{j(\beta)k(\alpha)}$$

which is true by means of (8.3) and (8.4).

Case 5: $\beta=\gamma=\epsilon\neq\alpha$. (8.25) becomes

$$\Gamma^{l(\beta)}_{i(\alpha)(j+k-1)(\beta)}-\Gamma^{(l-j+1)(\beta)}_{i(\alpha)k(\beta)}=0$$

which is true by means of (8.3) and (8.4).

Case 6: $\alpha=\beta\neq\gamma=\epsilon$. (8.25) becomes $0=0$.

Case 7: $\alpha=\gamma\neq\beta=\epsilon$. (8.25) becomes

$$-\Gamma^{(l-j+1)(\beta)}_{i(\alpha)k(\alpha)}=\Gamma^{l(\beta)}_{j(\beta)(i+k-1)(\alpha)}$$

which is true by means of (8.3)–(8.6).

Case 8: $\alpha=\epsilon\neq\beta=\gamma$. (8.25) becomes

$$\Gamma^{l(\alpha)}_{i(\alpha)(j+k-1)(\beta)}=-\Gamma^{(l-i+1)(\alpha)}_{j(\beta)k(\beta)}$$

which is true by means of (8.3)–(8.6).

Case 9: at least three among α, β, γ, ε are pairwise distinct. (8.25) becomes trivially $0=0$.

This proves compatibility between ∇ and $\circ$.

8.6. Linearity of the Euler vector field

We are now going to show that ∇, ∇^* are flat connections. We know that if we take the flatness of ∇ as already verified and assume $\nabla\nabla E=0$, then we deduce that $(\nabla,\nabla^*,\circ,*,e,E)$ define a bi-flat structure on M. It is then enough for us to only prove the flatness of ∇ and to verify the condition $\nabla\nabla E=0$.

Let us start by proving $\nabla\nabla E=0$. We have

$$\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)}=\partial_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)}+\Gamma^{k(\gamma)}_{i(\alpha)l(\sigma)}\nabla_{j(\beta)}E^{l(\sigma)}-\Gamma^{l(\sigma)}_{i(\alpha)j(\beta)}\nabla_{l(\sigma)}E^{k(\gamma)}$$

where, by means of (8.13), $\nabla_{j(\beta)}E^{k(\gamma)}$ is constant and $\nabla_{j(\beta)}E^{l(\sigma)}$, $\nabla_{l(\sigma)}E^{k(\gamma)}$ vanish respectively whenever $l\neq j$, $l\neq k$. Thus

$$\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)}=\Gamma^{k(\gamma)}_{i(\alpha)j(\sigma)}\nabla_{j(\beta)}E^{j(\sigma)}-\Gamma^{k(\sigma)}_{i(\alpha)j(\beta)}\nabla_{k(\sigma)}E^{k(\gamma)}. \tag{8.27}$$

The possible cases are:

1. $\alpha=\beta=\gamma$ 3. $\alpha=\gamma\neq\beta$ 5. $\alpha\neq\beta\neq\gamma\neq\alpha$.
2. $\alpha=\beta\neq\gamma$ 4. $\beta=\gamma\neq\alpha$

Case 1: $\alpha=\beta=\gamma$.

$$\begin{aligned}\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)}&=\Gamma^{k(\alpha)}_{i(\alpha)j(\sigma)}\nabla_{j(\alpha)}E^{j(\sigma)}-\Gamma^{k(\sigma)}_{i(\alpha)j(\alpha)}\nabla_{k(\sigma)}E^{k(\alpha)}\\&=\Gamma^{k(\alpha)}_{i(\alpha)j(\alpha)}\left(\nabla_{j(\alpha)}E^{j(\alpha)}-\nabla_{k(\alpha)}E^{k(\alpha)}\right)\\&\quad+\sum_{\sigma\neq\alpha}\left(\Gamma^{k(\alpha)}_{i(\alpha)j(\sigma)}\nabla_{j(\alpha)}E^{j(\sigma)}-\Gamma^{k(\sigma)}_{i(\alpha)j(\alpha)}\nabla_{k(\sigma)}E^{k(\alpha)}\right).\end{aligned} \tag{8.28}$$

If $j = k = 1$ then

$$\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} = \Gamma^{1(\alpha)}_{i(\alpha)1(\alpha)}\left(\nabla_{1(\alpha)}E^{1(\alpha)} - \nabla_{1(\alpha)}E^{1(\alpha)}\right) + \sum_{\sigma\neq\alpha}\left(\Gamma^{1(\alpha)}_{i(\alpha)1(\sigma)}\nabla_{1(\alpha)}E^{1(\sigma)} - \Gamma^{1(\sigma)}_{i(\alpha)1(\alpha)}\nabla_{1(\sigma)}E^{1(\alpha)}\right)$$

which vanishes by (8.3) and (8.5) if $i \geqslant 2$ and by (8.2), (8.4), (8.13) and (8.18) if $i = 1$. If both j and k are greater or equal then 2 then (8.28) vanishes by (8.13). If $j = 1$ and $k \geqslant 2$ then (8.28) vanishes by (8.13) and (8.18). If $j \geqslant 2$ and $k = 1$ then (8.28) becomes

$$\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} \overset{(8.13)}{=} -\Gamma^{1(\alpha)}_{i(\alpha)j(\alpha)}m_\alpha\varepsilon_\alpha - \sum_{\sigma\neq\alpha}\Gamma^{1(\sigma)}_{i(\alpha)j(\alpha)}m_\sigma\varepsilon_\sigma$$

which is

$$\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} = -\Gamma^{1(\alpha)}_{1(\alpha)j(\alpha)}m_\alpha\varepsilon_\alpha - \sum_{\sigma\neq\alpha}\Gamma^{1(\sigma)}_{1(\alpha)j(\alpha)}m_\sigma\varepsilon_\sigma \underset{(8.2)}{\overset{(8.18)}{=}} 0$$

if $i = 1$ and

$$\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} = -\Gamma^{1(\alpha)}_{i(\alpha)j(\alpha)}m_\alpha\varepsilon_\alpha - \sum_{\sigma\neq\alpha}\Gamma^{1(\sigma)}_{i(\alpha)j(\alpha)}m_\sigma\varepsilon_\sigma \underset{(8.10)}{\overset{(8.5)}{=}} 0$$

(as $1 - i - j \leqslant 1 - 2 - 2 = -3$) if $i \geqslant 2$.

Case 2: $\alpha = \beta \neq \gamma$.

$$\begin{aligned}\nabla_{i(\alpha)}\nabla_{j(\alpha)}E^{k(\gamma)} &= \Gamma^{k(\gamma)}_{i(\alpha)j(\sigma)}\nabla_{j(\alpha)}E^{j(\sigma)} - \Gamma^{k(\sigma)}_{i(\alpha)j(\alpha)}\nabla_{k(\sigma)}E^{k(\gamma)}\\ &\overset{(8.2)}{=} \Gamma^{k(\gamma)}_{i(\alpha)j(\alpha)}\nabla_{j(\alpha)}E^{j(\alpha)} - \Gamma^{k(\alpha)}_{i(\alpha)j(\alpha)}\nabla_{k(\alpha)}E^{k(\gamma)} + \Gamma^{k(\gamma)}_{i(\alpha)j(\gamma)}\nabla_{j(\alpha)}E^{j(\gamma)}\\ &\quad - \Gamma^{k(\gamma)}_{i(\alpha)j(\alpha)}\nabla_{k(\gamma)}E^{k(\gamma)} - \sum_{\sigma\neq\alpha,\gamma}\Gamma^{k(\sigma)}_{i(\alpha)j(\alpha)}\nabla_{k(\sigma)}E^{k(\gamma)}.\end{aligned} \tag{8.29}$$

If $j \geqslant 2$ and $k \geqslant 2$ then it vanishes by (8.13). If $j = k = 1$ then (8.29) becomes

$$\begin{aligned}\nabla_{i(\alpha)}\nabla_{j(\alpha)}E^{k(\gamma)} &\underset{(8.2)}{\overset{(8.18)}{=}} -\Gamma^{1(\gamma)}_{i(\alpha)1(\gamma)}\nabla_{1(\alpha)}E^{1(\alpha)} + \Gamma^{1(\alpha)}_{i(\alpha)1(\gamma)}\nabla_{1(\alpha)}E^{1(\gamma)}\\ &\quad + \sum_{\sigma\neq\alpha,\gamma}\Gamma^{1(\alpha)}_{i(\alpha)1(\sigma)}\nabla_{1(\alpha)}E^{1(\gamma)}\\ &\quad + \Gamma^{1(\gamma)}_{i(\alpha)1(\gamma)}\nabla_{1(\alpha)}E^{1(\gamma)} + \Gamma^{1(\gamma)}_{i(\alpha)1(\gamma)}\nabla_{1(\gamma)}E^{1(\gamma)} + \sum_{\sigma\neq\alpha,\gamma}\Gamma^{1(\sigma)}_{i(\alpha)1(\sigma)}\nabla_{1(\sigma)}E^{1(\gamma)}\\ &\overset{(8.13)}{=} \Gamma^{1(\gamma)}_{i(\alpha)1(\gamma)}m_\gamma\varepsilon_\gamma + \Gamma^{1(\alpha)}_{i(\alpha)1(\gamma)}m_\alpha\varepsilon_\alpha + \sum_{\sigma\neq\alpha,\gamma}\Gamma^{1(\alpha)}_{i(\alpha)1(\sigma)}m_\alpha\varepsilon_\alpha\\ &\quad + \sum_{\sigma\neq\alpha,\gamma}\Gamma^{1(\sigma)}_{i(\alpha)1(\sigma)}m_\sigma\varepsilon_\sigma\end{aligned}$$

which vanishes by (8.3) and (8.4) both if $i \geqslant 2$ and if $i=1$. If $j=1$ and $k \geqslant 2$ then (8.29) vanishes by (8.2), (8.13) and (8.18). If $j \geqslant 2$ and $k=1$ then (8.29) vanishes by (8.4), (8.5), (8.8) and (8.10).

Case 3: $\alpha=\gamma\neq\beta$.

$$\begin{aligned}\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} &= \Gamma^{k(\alpha)}_{i(\alpha)j(\sigma)}\nabla_{j(\beta)}E^{j(\sigma)} - \Gamma^{k(\sigma)}_{i(\alpha)j(\beta)}\nabla_{k(\sigma)}E^{k(\alpha)} \\ &\overset{(8.2)}{=} \Gamma^{k(\alpha)}_{i(\alpha)j(\alpha)}\nabla_{j(\beta)}E^{j(\alpha)} - \Gamma^{k(\alpha)}_{i(\alpha)j(\beta)}\nabla_{k(\alpha)}E^{k(\alpha)} \\ &\quad + \Gamma^{k(\alpha)}_{i(\alpha)j(\beta)}\nabla_{j(\beta)}E^{j(\beta)} - \Gamma^{k(\beta)}_{i(\alpha)j(\beta)}\nabla_{k(\beta)}E^{k(\alpha)} \\ &\quad + \sum_{\sigma\neq\alpha,\beta}\Gamma^{k(\alpha)}_{i(\alpha)j(\sigma)}\nabla_{j(\beta)}E^{j(\sigma)}. \end{aligned} \tag{8.30}$$

If both $j \geqslant 2$ and $k \geqslant 2$ then it vanishes by (8.13). If $j=k=1$ then (8.30) becomes

$$\begin{aligned}\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} = {}& \Gamma^{1(\alpha)}_{i(\alpha)1(\alpha)}\nabla_{1(\beta)}E^{1(\alpha)} - \Gamma^{1(\alpha)}_{i(\alpha)1(\beta)}\nabla_{1(\alpha)}E^{1(\alpha)} \\ & + \Gamma^{1(\alpha)}_{i(\alpha)1(\beta)}\nabla_{1(\beta)}E^{1(\beta)} - \Gamma^{1(\beta)}_{i(\alpha)1(\beta)}\nabla_{1(\beta)}E^{1(\alpha)} \\ & + \sum_{\sigma\neq\alpha,\beta}\Gamma^{1(\alpha)}_{i(\alpha)1(\sigma)}\nabla_{1(\beta)}E^{1(\sigma)}\end{aligned}$$

which trivially vanishes if $i \geqslant 2$ (by (8.3), (8.4) and (8.8)) and is

$$\begin{aligned}\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} &\underset{(8.18)}{\overset{(8.13)}{=}} -\Gamma^{1(\alpha)}_{1(\beta)1(\alpha)}m_\beta\varepsilon_\beta - \sum_{\sigma\neq\alpha,\beta}\Gamma^{1(\alpha)}_{1(\sigma)1(\alpha)}m_\beta\varepsilon_\beta - \Gamma^{1(\alpha)}_{1(\alpha)1(\beta)}\left(1-\sum_{\sigma\neq\alpha}m_\sigma\varepsilon_\sigma\right) \\ &\quad + \Gamma^{1(\alpha)}_{1(\alpha)1(\beta)}\left(1-\sum_{\sigma\neq\beta}m_\sigma\varepsilon_\sigma\right) - \Gamma^{1(\beta)}_{1(\alpha)1(\beta)}m_\beta\varepsilon_\beta + \sum_{\sigma\neq\alpha,\beta}\Gamma^{1(\alpha)}_{1(\alpha)1(\sigma)}m_\beta\varepsilon_\beta \\ &= -\Gamma^{1(\alpha)}_{1(\beta)1(\alpha)}m_\alpha\varepsilon_\alpha - \Gamma^{1(\beta)}_{1(\alpha)1(\beta)}m_\beta\varepsilon_\beta \overset{(8.4)}{=} 0\end{aligned}$$

if $i=1$. If $j=1$ and $k \geqslant 2$ then (8.30) becomes

$$\begin{aligned}\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} &\overset{(8.13)}{=} -\Gamma^{k(\alpha)}_{i(\alpha)1(\beta)}m_\beta\varepsilon_\beta - \sum_{\sigma\neq\alpha,\beta}\Gamma^{k(\alpha)}_{i(\alpha)1(\sigma)}m_\beta\varepsilon_\beta - \Gamma^{k(\alpha)}_{i(\alpha)1(\beta)}\left(1-\sum_{\tau=1}^{r}m_\tau\varepsilon_\tau\right) \\ &\quad + \Gamma^{k(\alpha)}_{i(\alpha)1(\beta)}\left(1-\sum_{\sigma\neq\beta}m_\sigma\varepsilon_\sigma\right) + \sum_{\sigma\neq\alpha,\beta}\Gamma^{k(\alpha)}_{i(\alpha)1(\sigma)}m_\beta\varepsilon_\beta \\ &= \Gamma^{k(\alpha)}_{i(\alpha)1(\beta)}\left(-m_\beta\varepsilon_\beta + \sum_{\tau=1}^{r}m_\tau\varepsilon_\tau - \sum_{\sigma\neq\beta}m_\sigma\varepsilon_\sigma\right) = 0.\end{aligned}$$

If $j \geqslant 2$ and $k=1$ then (8.30) vanishes by (8.3), (8.4) and (8.13).

Case 4: $\beta = \gamma \neq \alpha$.

$$\begin{aligned}\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} &= \Gamma^{k(\beta)}_{i(\alpha)j(\sigma)}\nabla_{j(\beta)}E^{j(\sigma)} - \Gamma^{k(\sigma)}_{i(\alpha)j(\beta)}\nabla_{k(\sigma)}E^{k(\beta)} \\ &\overset{(8.2)}{=} \Gamma^{k(\beta)}_{i(\alpha)j(\alpha)}\nabla_{j(\beta)}E^{j(\alpha)} - \Gamma^{k(\alpha)}_{i(\alpha)j(\beta)}\nabla_{k(\alpha)}E^{k(\beta)} \\ &\quad + \Gamma^{k(\beta)}_{i(\alpha)j(\beta)}\nabla_{j(\beta)}E^{j(\beta)} - \Gamma^{k(\beta)}_{i(\alpha)j(\beta)}\nabla_{k(\beta)}E^{k(\beta)}. \end{aligned} \tag{8.31}$$

If both $j \geqslant 2$ and $k \geqslant 2$ then it vanishes by (8.13). If $j = k = 1$ then (8.31) becomes

$$\begin{aligned}\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} &\underset{(8.18)\,(8.2)}{\overset{(8.13)}{=}} -\Gamma^{1(\beta)}_{i(\alpha)1(\beta)}m_\beta\varepsilon_\beta - \Gamma^{1(\alpha)}_{i(\alpha)1(\beta)}m_\alpha\varepsilon_\alpha \\ &\quad + \Gamma^{1(\beta)}_{i(\alpha)1(\beta)}\left(1 - \sum_{\sigma\neq\beta} m_\sigma\varepsilon_\sigma\right) - \Gamma^{1(\beta)}_{i(\alpha)1(\beta)}\left(1 - \sum_{\sigma\neq\beta} m_\sigma\varepsilon_\sigma\right) \overset{(8.4)}{=} 0.\end{aligned}$$

If $j = 1$ and $k \geqslant 2$ then (8.31) becomes

$$\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} \overset{(8.13)}{=} \Gamma^{k(\beta)}_{i(\alpha)1(\beta)}\left(-m_\beta\varepsilon_\beta + 1 - \sum_{\sigma\neq\beta} m_\sigma\varepsilon_\sigma - 1 + \sum_{\tau=1}^{r} m_\tau\varepsilon_\tau\right) = 0.$$

If $j \geqslant 2$ and $k = 1$ then (8.31) vanishes by (8.3), (8.4) and (8.13).

Case 5: $\alpha \neq \beta \neq \gamma \neq \alpha$.

$$\begin{aligned}\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} &= \Gamma^{k(\gamma)}_{i(\alpha)j(\sigma)}\nabla_{j(\beta)}E^{j(\sigma)} - \Gamma^{k(\sigma)}_{i(\alpha)j(\beta)}\nabla_{k(\sigma)}E^{k(\gamma)} \overset{(8.2)}{=} \Gamma^{k(\gamma)}_{i(\alpha)j(\alpha)}\nabla_{j(\beta)}E^{j(\alpha)} \\ &\quad - \Gamma^{k(\alpha)}_{i(\alpha)j(\beta)}\nabla_{k(\alpha)}E^{k(\gamma)} - \Gamma^{k(\beta)}_{i(\alpha)j(\beta)}\nabla_{k(\beta)}E^{k(\gamma)} + \Gamma^{k(\gamma)}_{i(\alpha)j(\gamma)}\nabla_{j(\beta)}E^{j(\gamma)}.\end{aligned} \tag{8.32}$$

If both $j \geqslant 2$ and $k \geqslant 2$ then it vanishes by (8.13). If $j = k = 1$ then (8.32) becomes

$$\begin{aligned}\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} &\underset{(8.18)\,(8.2)}{\overset{(8.13)}{=}} -\Gamma^{1(\gamma)}_{i(\alpha)1(\gamma)}m_\beta\varepsilon_\beta - \Gamma^{1(\alpha)}_{i(\alpha)1(\beta)}m_\alpha\varepsilon_\alpha \\ &\quad - \Gamma^{1(\beta)}_{i(\alpha)1(\beta)}m_\beta\varepsilon_\beta + \Gamma^{1(\gamma)}_{i(\alpha)1(\gamma)}m_\beta\varepsilon_\beta \overset{(8.4)}{=} 0.\end{aligned}$$

If $j = 1$ and $k \geqslant 2$ then (8.32) vanishes by (8.13). If $j \geqslant 2$ and $k = 1$ then (8.32) vanishes by (8.3) and (8.13). This proves that $\nabla\nabla E = 0$.

8.7. Flatness of ∇

We are now left with proving the flatness of ∇, i.e. $R = 0$. By the symmetries of

$$\begin{aligned}R^{i(\alpha)}_{h(\epsilon)k(\gamma)j(\beta)} &= \partial_{k(\gamma)}\Gamma^{i(\alpha)}_{h(\epsilon)j(\beta)} - \partial_{h(\epsilon)}\Gamma^{i(\alpha)}_{k(\gamma)j(\beta)} \\ &\quad + \sum_{\sigma=1}^{r}\sum_{l=1}^{m_\sigma}\left(\Gamma^{i(\alpha)}_{k(\gamma)l(\sigma)}\Gamma^{l(\sigma)}_{h(\epsilon)j(\beta)} - \Gamma^{i(\alpha)}_{h(\epsilon)l(\sigma)}\Gamma^{l(\sigma)}_{k(\gamma)j(\beta)}\right)\end{aligned} \tag{8.33}$$

the cases to be considered are the following ones:

1. $\alpha=\beta=\gamma=\epsilon$
2. $\alpha=\beta=\gamma\neq\epsilon$
3. $\alpha=\gamma=\epsilon\neq\beta$
4. $\beta=\gamma=\epsilon\neq\alpha$
5. $\alpha=\beta\neq\gamma=\epsilon$
6. $\alpha=\gamma\neq\beta=\epsilon$
7. $\alpha=\beta\notin\{\gamma,\epsilon\},\gamma\neq\epsilon$
8. $\alpha=\gamma\notin\{\beta,\epsilon\},\beta\neq\epsilon$
9. $\beta=\gamma\notin\{\alpha,\epsilon\},\alpha\neq\epsilon$
10. $\gamma=\epsilon\notin\{\alpha,\beta\},\alpha\neq\beta$
11. α,β,γ and ϵ are pairwise distinct.

Case 1: $\alpha=\beta=\gamma=\epsilon$. Our goal is to prove that

$$\begin{aligned}R^{i(\alpha)}_{h(\alpha)k(\alpha)j(\alpha)} &= \partial_{k(\alpha)}\Gamma^{i(\alpha)}_{h(\alpha)j(\alpha)}-\partial_{h(\alpha)}\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)}\\ &\quad+\sum_{l=1}^{m_\alpha}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\alpha)j(\alpha)}-\Gamma^{i(\alpha)}_{h(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)j(\alpha)}\right)\\ &\quad+\sum_{\sigma\neq\alpha}\sum_{l=1}^{m_\sigma}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{h(\alpha)j(\alpha)}-\Gamma^{i(\alpha)}_{h(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{k(\alpha)j(\alpha)}\right)\end{aligned} \tag{8.34}$$

vanishes. Let us first note that for each integer $N\geqslant 2$ it is possible to recover part of the Christoffel symbols for the case where $m_\alpha=N+1$ starting from the ones for the case where $m_\alpha=N$. More precisely, let us denote by $(\Gamma^{N+1})^k_{ij}$ the Christoffel symbols in the case where $m_\alpha=N+1$ and by $(\Gamma^N)^k_{ij}$ the Christoffel symbols in the case where $m_\alpha=N$, where the sizes m_σ of the remaining blocks $\sigma\neq\alpha$ are the same and where the constant ε_α has been replaced by $\frac{N+1}{N}\varepsilon_\alpha$.

$$\left(\Gamma^{N+1}\right)^{k(\beta)}_{i(\sigma)j(\tau)}=\left(\Gamma^{N}\right)^{k(\beta)}_{i(\sigma)j(\tau)} \tag{8.35}$$

for any possible choice of the indices in the right hand side (see remark 8.5). In the wake of this property, we will proceed by induction over m_α. Let us first consider the case[4] where $m_\alpha=2$, so that the indices i, j, k and h run from 1 to 2. In particular, since $R^{i(\alpha)}_{h(\alpha)k(\alpha)j(\alpha)}$ automatically vanishes when $k=h$, the only relevant cases are the one where $k=1$, $h=2$ and the one where $k=2$, $h=1$. By using the symmetries of R, we only consider the case where $k=1$ and $h=2$, hence obtaining

$$\begin{aligned}R^{i(\alpha)}_{2(\alpha)1(\alpha)j(\alpha)} &= \partial_{1(\alpha)}\Gamma^{i(\alpha)}_{2(\alpha)j(\alpha)}-\partial_{2(\alpha)}\Gamma^{i(\alpha)}_{1(\alpha)j(\alpha)}+\sum_{l=1}^{2}\left(\Gamma^{i(\alpha)}_{1(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{2(\alpha)j(\alpha)}\right.\\ &\quad\left.-\Gamma^{i(\alpha)}_{2(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)j(\alpha)}\right)+\sum_{\sigma\neq\alpha}\sum_{l=1}^{m_\sigma}\left(\Gamma^{i(\alpha)}_{1(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{2(\alpha)j(\alpha)}-\Gamma^{i(\alpha)}_{2(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{1(\alpha)j(\alpha)}\right)\end{aligned} \tag{8.36}$$

where both $\Gamma^{i(\alpha)}_{1(\alpha)l(\sigma)}$ and $\Gamma^{i(\alpha)}_{2(\alpha)l(\sigma)}$ survive only for $l=1$ by (8.3). This yields

$$\begin{aligned}R^{i(\alpha)}_{2(\alpha)1(\alpha)j(\alpha)} &= \partial_{1(\alpha)}\Gamma^{i(\alpha)}_{2(\alpha)j(\alpha)}-\partial_{2(\alpha)}\Gamma^{i(\alpha)}_{1(\alpha)j(\alpha)}+\Gamma^{i(\alpha)}_{1(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{2(\alpha)j(\alpha)}-\Gamma^{i(\alpha)}_{2(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{1(\alpha)j(\alpha)}\\ &\quad+\Gamma^{i(\alpha)}_{1(\alpha)2(\alpha)}\Gamma^{2(\alpha)}_{2(\alpha)j(\alpha)}-\Gamma^{i(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{2(\alpha)}_{1(\alpha)j(\alpha)}\\ &\quad+\sum_{\sigma\neq\alpha}\left(\Gamma^{i(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)j(\alpha)}-\Gamma^{i(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)j(\alpha)}\right).\end{aligned} \tag{8.37}$$

[4] The case where $m_\alpha=1$ is trivial, as $k=h=1$ directly implies $R^{i(\alpha)}_{h(\alpha)k(\alpha)j(\alpha)}=0$.

If $i=1$ we get

$$R^{1(\alpha)}_{2(\alpha)1(\alpha)j(\alpha)} \overset{(8.4)}{=} \partial_{1(\alpha)}\Gamma^{1(\alpha)}_{2(\alpha)j(\alpha)} - \partial_{2(\alpha)}\Gamma^{1(\alpha)}_{1(\alpha)j(\alpha)} + \Gamma^{1(\alpha)}_{1(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{2(\alpha)j(\alpha)} - \Gamma^{1(\alpha)}_{2(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{1(\alpha)j(\alpha)} + \Gamma^{1(\alpha)}_{1(\alpha)2(\alpha)}\Gamma^{2(\alpha)}_{2(\alpha)j(\alpha)} - \Gamma^{1(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{2(\alpha)}_{1(\alpha)j(\alpha)} + \sum_{\sigma\neq\alpha}\Gamma^{1(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)j(\alpha)}$$

that becomes

$$\begin{aligned} R^{1(\alpha)}_{2(\alpha)1(\alpha)1(\alpha)} &= \partial_{1(\alpha)}\Gamma^{1(\alpha)}_{2(\alpha)1(\alpha)} - \partial_{2(\alpha)}\Gamma^{1(\alpha)}_{1(\alpha)1(\alpha)} + \Gamma^{1(\alpha)}_{1(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{2(\alpha)1(\alpha)} - \Gamma^{1(\alpha)}_{2(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{1(\alpha)1(\alpha)} \\ &\quad + \Gamma^{1(\alpha)}_{1(\alpha)2(\alpha)}\Gamma^{2(\alpha)}_{2(\alpha)1(\alpha)} - \Gamma^{1(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{2(\alpha)}_{1(\alpha)1(\alpha)} + \sum_{\sigma\neq\alpha}\Gamma^{1(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)1(\alpha)} \\ &\underset{(8.18)}{\overset{(8.4),(8.10)}{=}} \sum_{\sigma\neq\alpha}\partial_{2(\alpha)}\Gamma^{1(\alpha)}_{1(\sigma)1(\alpha)} \overset{(8.4)}{=} \sum_{\sigma\neq\alpha}\partial_{2(\alpha)}\left(\frac{m_\sigma\varepsilon_\sigma}{u^{1(\alpha)}-u^{1(\sigma)}}\right) = 0 \end{aligned}$$

when $j=1$ and $R^{1(\alpha)}_{2(\alpha)1(\alpha)2(\alpha)} = 0$ by (8.5), (8.8) and (8.10) when $j=2$. If $i=2$ then (8.37) reads

$$\begin{aligned} R^{2(\alpha)}_{2(\alpha)1(\alpha)j(\alpha)} &= \partial_{1(\alpha)}\Gamma^{2(\alpha)}_{2(\alpha)j(\alpha)} - \partial_{2(\alpha)}\Gamma^{2(\alpha)}_{1(\alpha)j(\alpha)} + \Gamma^{2(\alpha)}_{1(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{2(\alpha)j(\alpha)} \\ &\quad - \Gamma^{2(\alpha)}_{2(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{1(\alpha)j(\alpha)} + \Gamma^{2(\alpha)}_{1(\alpha)2(\alpha)}\Gamma^{2(\alpha)}_{2(\alpha)j(\alpha)} \\ &\quad - \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{2(\alpha)}_{1(\alpha)j(\alpha)} + \sum_{\sigma\neq\alpha}\left(\Gamma^{2(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)j(\alpha)} - \Gamma^{2(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)j(\alpha)}\right) \end{aligned}$$

that becomes

$$\begin{aligned} R^{2(\alpha)}_{2(\alpha)1(\alpha)1(\alpha)} &\underset{(8.18)}{\overset{(8.5),(8.8)}{=}} \partial_{1(\alpha)}\Gamma^{2(\alpha)}_{2(\alpha)1(\alpha)} - \partial_{2(\alpha)}\Gamma^{2(\alpha)}_{1(\alpha)1(\alpha)} \\ &\quad + \sum_{\sigma\neq\alpha}\left(\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{2(\alpha)}_{1(\sigma)1(\alpha)} + \Gamma^{2(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)1(\sigma)}\right) \\ &\underset{(8.7)}{\overset{(8.4)}{=}} \partial_{1(\alpha)}\Gamma^{2(\alpha)}_{2(\alpha)1(\alpha)} - \partial_{2(\alpha)}\Gamma^{2(\alpha)}_{1(\alpha)1(\alpha)} \\ &\quad + \sum_{\sigma\neq\alpha}\left(\frac{m_\alpha\varepsilon_\alpha}{u^{2(\alpha)}}\,\frac{1}{u^{1(\alpha)}-u^{1(\sigma)}}\,\Gamma^{1(\alpha)}_{1(\sigma)1(\alpha)}\,u^{2(\alpha)} - \Gamma^{1(\alpha)}_{1(\alpha)1(\sigma)}\,\frac{m_\alpha\varepsilon_\alpha}{u^{1(\alpha)}-u^{1(\sigma)}}\right) \\ &\overset{(8.18)}{=} \sum_{\sigma\neq\alpha}\left(-\partial_{1(\alpha)}\Gamma^{2(\alpha)}_{2(\alpha)1(\sigma)} + \partial_{2(\alpha)}\Gamma^{2(\alpha)}_{1(\sigma)1(\alpha)}\right) \\ &\overset{(8.4)}{=} \sum_{\sigma\neq\alpha}\left[\frac{m_\sigma\varepsilon_\sigma}{(u^{1(\alpha)}-u^{1(\sigma)})^2} - \frac{1}{u^{1(\alpha)}-u^{1(\sigma)}}\,\frac{m_\sigma\varepsilon_\sigma}{u^{1(\alpha)}-u^{1(\sigma)}}\right] = 0 \end{aligned}$$

when $j=1$ and

$$R^{2(\alpha)}_{2(\alpha)1(\alpha)2(\alpha)} \underset{(8.8)\,(8.18)}{\overset{\substack{(8.4)\\(8.5),(8.10)}}{=}} \partial_{1(\alpha)}\left(-\frac{m_\alpha\varepsilon_\alpha}{u^{2(\alpha)}}\right) + \sum_{\sigma\neq\alpha}\partial_{2(\alpha)}\left(\frac{m_\sigma\varepsilon_\sigma}{u^{1(\alpha)}-u^{1(\sigma)}}\right) = 0$$

when $j=2$. Therefore we proved that (8.34) vanishes when $m_\alpha=2$. Given an integer $N\geqslant 2$, let us now suppose that (8.34) vanishes for $m_\alpha=N$ and show it vanishes for $m_\alpha=N+1$ as well. In other words, we are supposing that

$$\begin{aligned}\left(R^N\right)^{i(\alpha)}_{h(\alpha)k(\alpha)j(\alpha)} :=& \partial_{k(\alpha)}\Gamma^{i(\alpha)}_{h(\alpha)j(\alpha)}-\partial_{h(\alpha)}\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)}\\ &+\sum_{l=1}^{N}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\alpha)j(\alpha)}-\Gamma^{i(\alpha)}_{h(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)j(\alpha)}\right)\\ &+\sum_{\sigma\neq\alpha}\sum_{l=1}^{m_\sigma}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{h(\alpha)j(\alpha)}-\Gamma^{i(\alpha)}_{h(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{k(\alpha)j(\alpha)}\right)=0 \end{aligned} \tag{8.38}$$

for every $i,j,k,h\in\{1,\dots,m_\alpha\}$ for each $m_\alpha\leqslant N$ and we want to prove that

$$\begin{aligned}\left(R^{N+1}\right)^{i(\alpha)}_{h(\alpha)k(\alpha)j(\alpha)} :=& \partial_{k(\alpha)}\Gamma^{i(\alpha)}_{h(\alpha)j(\alpha)}-\partial_{h(\alpha)}\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)}\\ &+\sum_{l=1}^{N+1}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\alpha)j(\alpha)}-\Gamma^{i(\alpha)}_{h(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)j(\alpha)}\right)\\ &+\sum_{\sigma\neq\alpha}\sum_{l=1}^{m_\sigma}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{h(\alpha)j(\alpha)}-\Gamma^{i(\alpha)}_{h(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{k(\alpha)j(\alpha)}\right)=0 \end{aligned} \tag{8.39}$$

for every $i,j,k,h\in\{1,\dots,N+1\}$ for $m_\alpha=N+1$. Notice that due to the property (8.35) and replacing ε_α with $\frac{N+1}{N}\varepsilon_\alpha$ we can use in both cases the same notation for the Christoffel symbols. Let us start by considering the case where $i\leqslant N$ and observe that

$$\Gamma^{i(\alpha)}_{j(\alpha)(N+1)(\alpha)}=0\qquad \forall j\in\{1,\dots,N+1\} \tag{8.40}$$

as $i-j-(N+1)\leqslant N-2-N-1=-3$ for $j\geqslant 2$ (we recall (8.10)) and

$$\Gamma^{i(\alpha)}_{1(\alpha)(N+1)(\alpha)}\overset{(8.18)}{=}-\sum_{\sigma\neq\alpha}\Gamma^{i(\alpha)}_{1(\sigma)(N+1)(\alpha)}\overset{(8.4)}{=}0$$

$(i\leqslant N<N+1)$ for $j=1$. If all of j,k,h are less or equal than N then

$$\begin{aligned}\left(R^{N+1}\right)^{i(\alpha)}_{h(\alpha)k(\alpha)j(\alpha)} =& \partial_{k(\alpha)}\Gamma^{i(\alpha)}_{h(\alpha)j(\alpha)}-\partial_{h(\alpha)}\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)}+\sum_{l=1}^{N+1}\Big(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\alpha)j(\alpha)}\\ &-\Gamma^{i(\alpha)}_{h(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)j(\alpha)}\Big)+\sum_{\sigma\neq\alpha}\sum_{l=1}^{m_\sigma}\Big(\Gamma^{i(\alpha)}_{k(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{h(\alpha)j(\alpha)}\\ &-\Gamma^{i(\alpha)}_{h(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{k(\alpha)j(\alpha)}\Big)\end{aligned}$$

where in the first summation only the terms for $l\leqslant N$ survive, as both $\Gamma^{i(\alpha)}_{k(\alpha)(N+1)(\alpha)}$ and $\Gamma^{i(\alpha)}_{h(\alpha)(N+1)(\alpha)}$ vanish due to (8.40). This yields

$$\left(R^{N+1}\right)^{i(\alpha)}_{h(\alpha)k(\alpha)j(\alpha)} = \partial_{k(\alpha)}\Gamma^{i(\alpha)}_{h(\alpha)j(\alpha)} - \partial_{h(\alpha)}\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)} + \sum_{l=1}^{N}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\alpha)j(\alpha)} - \Gamma^{i(\alpha)}_{h(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)j(\alpha)}\right) + \sum_{\sigma\neq\alpha}\sum_{l=1}^{m_\sigma}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{h(\alpha)j(\alpha)} - \Gamma^{i(\alpha)}_{h(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{k(\alpha)j(\alpha)}\right) = (R^N)^{i(\alpha)}_{h(\alpha)k(\alpha)j(\alpha)} \overset{(8.38)}{=} 0.$$

If $k, h \leqslant N$ and $j = N+1$ then

$$\left(R^{N+1}\right)^{i(\alpha)}_{h(\alpha)k(\alpha)(N+1)(\alpha)} \overset{(8.40)}{=} \sum_{\sigma\neq\alpha}\sum_{l=1}^{m_\sigma}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{h(\alpha)(N+1)(\alpha)} - \Gamma^{i(\alpha)}_{h(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{k(\alpha)(N+1)(\alpha)}\right).$$

This vanishes by (8.5). If $k, j \leqslant N$ and $h = N+1$ (due to the symmetries of R, this covers the case where $h, j \leqslant N$ and $k = N+1$ as well) then

$$\left(R^{N+1}\right)^{i(\alpha)}_{(N+1)(\alpha)k(\alpha)j(\alpha)} \overset{(8.40)}{=} -\partial_{(N+1)(\alpha)}\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)} + \sum_{\sigma\neq\alpha}\sum_{l=1}^{m_\sigma}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{(N+1)(\alpha)j(\alpha)} - \Gamma^{i(\alpha)}_{(N+1)(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{k(\alpha)j(\alpha)}\right).$$

This yields

$$\left(R^{N+1}\right)^{i(\alpha)}_{(N+1)(\alpha)k(\alpha)j(\alpha)} \underset{(8.5)}{\overset{(8.4)}{=}} -\partial_{(N+1)(\alpha)}\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)}$$

that becomes

$$\left(R^{N+1}\right)^{i(\alpha)}_{(N+1)(\alpha)k(\alpha)1(\alpha)} = -\partial_{(N+1)(\alpha)}\Gamma^{i(\alpha)}_{k(\alpha)1(\alpha)} = 0$$

($\Gamma^{i(\alpha)}_{k(\alpha)1(\alpha)} \overset{(8.18)}{=} -\sum_{\sigma\neq\alpha}\Gamma^{i(\alpha)}_{k(\alpha)1(\sigma)}$ only depends on $\{u^{1(\alpha)} - u^{1(\sigma)}\,|\,\sigma\neq\alpha\}$ and $\{u^{s(\alpha)}\,|\,2\leqslant s\leqslant i-k+1\}$ by (8.4), thus it does not depend on $u^{(N+1)(\alpha)}$ as $i-k+1\leqslant N-1+1 = N < N+1$) when $j=1$,

$$\left(R^{N+1}\right)^{i(\alpha)}_{(N+1)(\alpha)1(\alpha)j(\alpha)} = -\partial_{(N+1)(\alpha)}\Gamma^{i(\alpha)}_{1(\alpha)j(\alpha)} = 0$$

(analogously) when $k=1$ and

$$\left(R^{N+1}\right)^{i(\alpha)}_{(N+1)(\alpha)k(\alpha)j(\alpha)} = -\partial_{(N+1)(\alpha)}\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)} = 0$$

when both j and k are greater or equal than 2, as[5]

[5] Without loss of generality we assume $i-j-k\geqslant -2$, as $\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)} = 0$ automatically when $i-j-k\leqslant -3$.

$$\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)} \overset{(8.10)}{=} \Gamma^{(i-j-k+4)(\alpha)}_{2(\alpha)2(\alpha)} \overset{(8.9)}{=} \Gamma^{(i-j-k+2)(\alpha)}_{1(\alpha)1(\alpha)} - \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} \frac{u^{(i-j-k+4)(\alpha)}}{u^{2(\alpha)}} - \frac{1}{u^{2(\alpha)}} \sum_{l=1}^{i-j-k+1} \left(\Gamma^{(l+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)}\right) u^{(i-j-k+4-l)(\alpha)}$$

does not depend on $u^{(N+1)(\alpha)}$ ($\Gamma^{(i-j-k+2)(\alpha)}_{1(\alpha)1(\alpha)}$ only depends on $\{u^{1(\alpha)} - u^{1(\sigma)} \,|\, \sigma \neq \alpha\}$ and $\{u^{s(\alpha)} \,|\, 2 \leqslant s \leqslant i-j-k+2\}$ by (8.8) where $i-j-k+2 \leqslant N-2-2+2 = N-2 < N+1$, $\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}$ only depends on $u^{2(\alpha)}$, $i-j-k+4 \leqslant N-2-2+4 = N < N+1$ and for every $1 \leqslant l \leqslant i-j-k+1$ we have $i-j-k+4-l < i-j-k+4 < N+1$ and $\Gamma^{(l+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)}$ only depends on quantities that correspond to lower indices so it does not depend on $u^{(N+1)(\alpha)}$ *a fortiori*). If $k = h = N+1$ then $(R^{N+1})^{i(\alpha)}_{h(\alpha)k(\alpha)j(\alpha)} = 0$ for every value of j due to the symmetries of R. If $k \leqslant n$ and $j = h = N+1$ (due to the symmetries of R, this covers the case where $h \leqslant N$ and $j = k = N+1$ as well) then

$$\left(R^{N+1}\right)^{i(\alpha)}_{(N+1)(\alpha)k(\alpha)(N+1)(\alpha)} \overset{(8.40)}{=} \sum_{\sigma\neq\alpha} \sum_{l=1}^{m_\sigma} \left(\Gamma^{i(\alpha)}_{k(\alpha)l(\sigma)} \Gamma^{l(\sigma)}_{(N+1)(\alpha)(N+1)(\alpha)} - \Gamma^{i(\alpha)}_{(N+1)(\alpha)l(\sigma)} \Gamma^{l(\sigma)}_{k(\alpha)(N+1)(\alpha)}\right).$$

This vanishes by (8.5). We have therefore proved (8.39) under the assumption that $i \leqslant N$. Let us now fix $i = N+1$. We have

$$\left(R^{N+1}\right)^{(N+1)(\alpha)}_{h(\alpha)k(\alpha)j(\alpha)} = \partial_{k(\alpha)} \Gamma^{(N+1)(\alpha)}_{h(\alpha)j(\alpha)} - \partial_{h(\alpha)} \Gamma^{(N+1)(\alpha)}_{k(\alpha)j(\alpha)} + \sum_{l=1}^{N+1} \left(\Gamma^{(N+1)(\alpha)}_{k(\alpha)l(\alpha)} \Gamma^{l(\alpha)}_{h(\alpha)j(\alpha)} - \Gamma^{(N+1)(\alpha)}_{h(\alpha)l(\alpha)} \Gamma^{l(\alpha)}_{k(\alpha)j(\alpha)}\right) + \sum_{\sigma\neq\alpha} \sum_{l=1}^{m_\sigma} \left(\Gamma^{(N+1)(\alpha)}_{k(\alpha)l(\sigma)} \Gamma^{l(\sigma)}_{h(\alpha)j(\alpha)} - \Gamma^{(N+1)(\alpha)}_{h(\alpha)l(\sigma)} \Gamma^{l(\sigma)}_{k(\alpha)j(\alpha)}\right)$$

where in the last summation only survive the terms for $l = 1$ by (8.3), yielding

$$\left(R^{N+1}\right)^{(N+1)(\alpha)}_{h(\alpha)k(\alpha)j(\alpha)} = \partial_{k(\alpha)} \Gamma^{(N+1)(\alpha)}_{h(\alpha)j(\alpha)} - \partial_{h(\alpha)} \Gamma^{(N+1)(\alpha)}_{k(\alpha)j(\alpha)} + \sum_{l=1}^{N+1} \left(\Gamma^{(N+1)(\alpha)}_{k(\alpha)l(\alpha)} \Gamma^{l(\alpha)}_{h(\alpha)j(\alpha)} - \Gamma^{(N+1)(\alpha)}_{h(\alpha)l(\alpha)} \Gamma^{l(\alpha)}_{k(\alpha)j(\alpha)}\right) + \sum_{\sigma\neq\alpha} \left(\Gamma^{(N+1)(\alpha)}_{k(\alpha)1(\sigma)} \Gamma^{1(\sigma)}_{h(\alpha)j(\alpha)} - \Gamma^{(N+1)(\alpha)}_{h(\alpha)1(\sigma)} \Gamma^{1(\sigma)}_{k(\alpha)j(\alpha)}\right). \tag{8.41}$$

We distinguish between the following subcases:

a. both k and h are greater or equal than 3
b. $k = 1$, $h \geqslant 3$ (this covers $h = 1$, $k \geqslant 3$ as well)

c. $k=2$, $h\geqslant 3$ (this covers $h=2$, $k\geqslant 3$ as well)
d. $k=1$, $h=2$ (this covers $h=1$, $k=2$ as well)

observing that $(R^{N+1})^{(N+1)(\alpha)}_{h(\alpha)k(\alpha)j(\alpha)}=0$ automatically whenever $k=h$.

Subcase a: both k and h are greater or equal than 3. We have

$$\left(R^{N+1}\right)^{(N+1)(\alpha)}_{h(\alpha)k(\alpha)j(\alpha)} \underset{(8.4),(8.10)}{\overset{(8.11)}{=}} \partial_{(k-1)(\alpha)}\Gamma^{N(\alpha)}_{h(\alpha)j(\alpha)}-\partial_{(h-1)(\alpha)}\Gamma^{N(\alpha)}_{k(\alpha)j(\alpha)} + \sum_{l=2}^{N+1}\left(\Gamma^{N(\alpha)}_{(k-1)(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\alpha)j(\alpha)}-\Gamma^{N(\alpha)}_{(h-1)(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)j(\alpha)}\right) + \sum_{\sigma\neq\alpha}\left(\Gamma^{N(\alpha)}_{(k-1)(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{h(\alpha)j(\alpha)}-\Gamma^{N(\alpha)}_{(h-1)(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{k(\alpha)j(\alpha)}\right). \tag{8.42}$$

If $j\leqslant N-1$ we get

$$\left(R^{N+1}\right)^{(N+1)(\alpha)}_{h(\alpha)k(\alpha)j(\alpha)} \overset{(8.10)}{=} \partial_{(k-1)(\alpha)}\Gamma^{N(\alpha)}_{(h-1)(\alpha)(j+1)(\alpha)}-\partial_{(h-1)(\alpha)}\Gamma^{N(\alpha)}_{(k-1)(\alpha)(j+1)(\alpha)} + \sum_{l=1}^{N+1}\left(\Gamma^{N(\alpha)}_{(k-1)(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{(h-1)(\alpha)(j+1)(\alpha)}-\Gamma^{N(\alpha)}_{(h-1)(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{(k-1)(\alpha)(j+1)(\alpha)}\right) + \sum_{\sigma\neq\alpha}\left(\Gamma^{N(\alpha)}_{(k-1)(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{(h-1)(\alpha)(j+1)(\alpha)}-\Gamma^{N(\alpha)}_{(h-1)(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{(k-1)(\alpha)(j+1)(\alpha)}\right)$$
$$\overset{(8.10)}{=} (R^N)^{N(\alpha)}_{(h-1)(\alpha)(k-1)(\alpha)(j+1)(\alpha)}+\Gamma^{N(\alpha)}_{(k-1)(\alpha)(N+1)(\alpha)}\Gamma^{(N+1)(\alpha)}_{(h-1)(\alpha)(j+1)(\alpha)} -\Gamma^{N(\alpha)}_{(h-1)(\alpha)(N+1)(\alpha)}\Gamma^{(N+1)(\alpha)}_{(k-1)(\alpha)(j+1)(\alpha)} \underset{(8.40)}{\overset{(8.38)}{=}} 0.$$

If $j=N$ then (8.42) vanishes by (8.5), (8.10) and (8.40). If $j=N+1$ then (8.42) vanishes by (8.5), (8.40).

Subcase b: $k=1$, $h\geqslant 3$. We have

$$\left(R^{N+1}\right)^{(N+1)(\alpha)}_{h(\alpha)1(\alpha)j(\alpha)} \overset{(8.11)}{=} \partial_{1(\alpha)}\Gamma^{N(\alpha)}_{(h-1)(\alpha)j(\alpha)}-\partial_{(h-1)(\alpha)}\Gamma^{N(\alpha)}_{1(\alpha)j(\alpha)}+\sum_{l=1}^{N+1}\Gamma^{(N+1)(\alpha)}_{1(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\alpha)j(\alpha)} -\sum_{l=1}^{N+1}\Gamma^{(N+1)(\alpha)}_{h(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)j(\alpha)}+\sum_{\sigma\neq\alpha}\left(\Gamma^{(N+1)(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{h(\alpha)j(\alpha)}-\Gamma^{(N+1)(\alpha)}_{h(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)j(\alpha)}\right) +\sum_{\sigma\neq\alpha}\left(\Gamma^{N(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{(h-1)(\alpha)j(\alpha)}-\Gamma^{N(\alpha)}_{(h-1)(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)j(\alpha)}\right) -\sum_{\sigma\neq\alpha}\left(\Gamma^{N(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{(h-1)(\alpha)j(\alpha)}-\Gamma^{N(\alpha)}_{(h-1)(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)j(\alpha)}\right)$$

where added and subtracted the quantity $\sum_{\sigma\neq\alpha}\Big(\Gamma^{N(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{(h-1)(\alpha)j(\alpha)}$ $-\Gamma^{N(\alpha)}_{(h-1)(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)j(\alpha)}\Big)$ and where the terms

$$\partial_{1(\alpha)}\Gamma^{N(\alpha)}_{(h-1)(\alpha)j(\alpha)},\ -\partial_{(h-1)(\alpha)}\Gamma^{N(\alpha)}_{1(\alpha)j(\alpha)},$$

$$\sum_{l=2}^{N+1}\Gamma^{(N+1)(\alpha)}_{1(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\alpha)j(\alpha)}\overset{(8.10)}{\underset{(8.8)}{=}}\sum_{l=2}^{N+1}\Gamma^{N(\alpha)}_{1(\alpha)(l-1)(\alpha)}\Gamma^{(l-1)(\alpha)}_{(h-1)(\alpha)j(\alpha)}=\sum_{l=1}^{N}\Gamma^{N(\alpha)}_{1(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{(h-1)(\alpha)j(\alpha)},$$

$$-\sum_{l=1}^{N}\Gamma^{(N+1)(\alpha)}_{h(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)j(\alpha)}\overset{(8.10)}{\underset{(8.8)}{=}}-\sum_{l=1}^{N}\Gamma^{N(\alpha)}_{(h-1)(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)j(\alpha)}$$

and

$$\sum_{\sigma\neq\alpha}\left(\Gamma^{N(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{(h-1)(\alpha)j(\alpha)}-\Gamma^{N(\alpha)}_{(h-1)(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)j(\alpha)}\right)$$

combine to form $(R^N)^{N(\alpha)}_{(h-1)(\alpha)1(\alpha)j(\alpha)}$, which[6] vanishes by (8.38). Thus

$$\left(R^{N+1}\right)^{(N+1)(\alpha)}_{h(\alpha)1(\alpha)j(\alpha)}\overset{(8.4)\,(8.5)}{\underset{(8.10)\,(8.8)}{=}}\sum_{\sigma\neq\alpha}\left(-\Gamma^{N(\alpha)}_{(h-1)(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)j(\alpha)}+\Gamma^{N(\alpha)}_{(h-1)(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)j(\alpha)}\right)=0.$$

Subcase c: $k=2$, $h\geqslant 3$. We have

$$\begin{aligned}\left(R^{N+1}\right)^{(N+1)(\alpha)}_{h(\alpha)2(\alpha)j(\alpha)}\overset{(8.10)\,(8.8)}{\underset{(8.11)}{=}}&\ \partial_{2(\alpha)}\Gamma^{N(\alpha)}_{(h-1)(\alpha)j(\alpha)}-\partial_{(h-1)(\alpha)}\Gamma^{N(\alpha)}_{2(\alpha)j(\alpha)}+\sum_{l=1}^{N+1}\Gamma^{(N+1)(\alpha)}_{2(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\alpha)j(\alpha)}\\&-\sum_{l=1}^{N+1}\Gamma^{(N+1)(\alpha)}_{h(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{2(\alpha)j(\alpha)}+\sum_{\sigma\neq\alpha}\left(\Gamma^{(N+1)(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{h(\alpha)j(\alpha)}-\Gamma^{(N+1)(\alpha)}_{h(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)j(\alpha)}\right)\\&+\sum_{\sigma\neq\alpha}\left(\Gamma^{N(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{(h-1)(\alpha)j(\alpha)}-\Gamma^{N(\alpha)}_{(h-1)(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)j(\alpha)}\right)\\&-\sum_{\sigma\neq\alpha}\left(\Gamma^{N(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{(h-1)(\alpha)j(\alpha)}-\Gamma^{N(\alpha)}_{(h-1)(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)j(\alpha)}\right)\end{aligned}$$

where we added and subtracted the quantity $\sum_{\sigma\neq\alpha}\Big(\Gamma^{N(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{(h-1)(\alpha)j(\alpha)}-$ $\Gamma^{N(\alpha)}_{(h-1)(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)j(\alpha)}\Big)$ and where the terms

[6] This only holds for $j\leqslant N$. Still, for $j=N+1$ each of these addends vanishes by itself, by means of (8.5), (8.10) and (8.40).

$$\partial_{2(\alpha)}\Gamma^{N(\alpha)}_{(h-1)(\alpha)j(\alpha)},\ -\partial_{(h-1)(\alpha)}\Gamma^{N(\alpha)}_{2(\alpha)j(\alpha)},$$

$$\sum_{l=2}^{N+1}\Gamma^{(N+1)(\alpha)}_{2(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\alpha)j(\alpha)} \overset{(8.10)\,(8.8)}{=} \sum_{l=2}^{N+1}\Gamma^{N(\alpha)}_{2(\alpha)(l-1)(\alpha)}\Gamma^{(l-1)(\alpha)}_{(h-1)(\alpha)j(\alpha)} = \sum_{l=1}^{N}\Gamma^{N(\alpha)}_{2(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{(h-1)(\alpha)j(\alpha)},$$

$$-\sum_{l=1}^{N}\Gamma^{(N+1)(\alpha)}_{h(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{2(\alpha)j(\alpha)} \overset{(8.10)\,(8.8)}{=} -\sum_{l=1}^{N}\Gamma^{N(\alpha)}_{(h-1)(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{2(\alpha)j(\alpha)}$$

and

$$\sum_{\sigma\neq\alpha}\left(\Gamma^{N(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{(h-1)(\alpha)j(\alpha)} - \Gamma^{N(\alpha)}_{(h-1)(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)j(\alpha)}\right)$$

combine to form $(R^N)^{N(\alpha)}_{(h-1)(\alpha)2(\alpha)j(\alpha)}$, which[7] vanishes by (8.38). Thus $(R^{N+1})^{(N+1)(\alpha)}_{h(\alpha)2(\alpha)j(\alpha)}$ vanishes by (8.5), (8.8) and (8.10).

Subcase d: $k=1$, $h=2$. We have

$$\begin{aligned}\left(R^{N+1}\right)^{(N+1)(\alpha)}_{2(\alpha)1(\alpha)j(\alpha)} &= \partial_{1(\alpha)}\Gamma^{(N+1)(\alpha)}_{2(\alpha)j(\alpha)} - \partial_{2(\alpha)}\Gamma^{(N+1)(\alpha)}_{1(\alpha)j(\alpha)} + \sum_{l=1}^{N+1}\left(\Gamma^{(N+1)(\alpha)}_{1(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{2(\alpha)j(\alpha)}\right.\\ &\left. - \Gamma^{(N+1)(\alpha)}_{2(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)j(\alpha)}\right) + \sum_{\sigma\neq\alpha}\left(\Gamma^{(N+1)(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)j(\alpha)} - \Gamma^{(N+1)(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)j(\alpha)}\right).\end{aligned} \tag{8.43}$$

If $j\geqslant 3$ we get

$$\begin{aligned}\left(R^{N+1}\right)^{(N+1)(\alpha)}_{2(\alpha)1(\alpha)j(\alpha)} &\overset{(8.10)}{\underset{(8.8)}{=}} \partial_{1(\alpha)}\Gamma^{N(\alpha)}_{2(\alpha)(j-1)(\alpha)} - \partial_{2(\alpha)}\Gamma^{N(\alpha)}_{1(\alpha)(j-1)(\alpha)} + \sum_{l=1}^{N+1}\Gamma^{(N+1)(\alpha)}_{1(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{2(\alpha)j(\alpha)}\\ &-\sum_{l=1}^{N+1}\Gamma^{(N+1)(\alpha)}_{2(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)j(\alpha)} + \sum_{\sigma\neq\alpha}\left(\Gamma^{(N+1)(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)j(\alpha)} - \Gamma^{(N+1)(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)j(\alpha)}\right)\\ &+\sum_{\sigma\neq\alpha}\left(\Gamma^{N(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)(j-1)(\alpha)} - \Gamma^{N(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)(j-1)(\alpha)}\right)\\ &-\sum_{\sigma\neq\alpha}\left(\Gamma^{N(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)(j-1)(\alpha)} - \Gamma^{N(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)(j-1)(\alpha)}\right)\end{aligned}$$

where we added and subtracted the quantity $\sum_{\sigma\neq\alpha}\left(\Gamma^{N(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)(j-1)(\alpha)} - \Gamma^{N(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)(j-1)(\alpha)}\right)$ and where the terms

[7] This only holds for $j\leqslant N$. Still, for $j=N+1$ each of these addends vanishes by itself, by means of (8.5), (8.10) and (8.40).

$$\partial_{1(\alpha)}\Gamma^{N(\alpha)}_{2(\alpha)(j-1)(\alpha)},\ -\partial_{2(\alpha)}\Gamma^{N(\alpha)}_{1(\alpha)(j-1)(\alpha)},$$

$$\sum_{l=2}^{N+1}\Gamma^{(N+1)(\alpha)}_{1(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{2(\alpha)j(\alpha)} \underset{(8.8)}{\overset{(8.10)}{=}} \sum_{l=2}^{N+1}\Gamma^{N(\alpha)}_{1(\alpha)(l-1)(\alpha)}\Gamma^{(l-1)(\alpha)}_{2(\alpha)(j-1)(\alpha)} = \sum_{l=1}^{N}\Gamma^{N(\alpha)}_{1(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{2(\alpha)(j-1)(\alpha)},$$

$$-\sum_{l=2}^{N+1}\Gamma^{(N+1)(\alpha)}_{2(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)j(\alpha)} \underset{(8.8)}{\overset{(8.10)}{=}} -\sum_{l=2}^{N+1}\Gamma^{N(\alpha)}_{2(\alpha)(l-1)(\alpha)}\Gamma^{(l-1)(\alpha)}_{1(\alpha)(j-1)(\alpha)} = -\sum_{l=1}^{N}\Gamma^{N(\alpha)}_{2(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)(j-1)(\alpha)}$$

and

$$\sum_{\sigma\neq\alpha}\left(\Gamma^{N(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)(j-1)(\alpha)} - \Gamma^{N(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)(j-1)(\alpha)}\right)$$

combine to form $(R^N)^{N(\alpha)}_{2(\alpha)1(\alpha)(j-1)(\alpha)}$, which vanishes by (8.38). Thus $(R^{N+1})^{(N+1)(\alpha)}_{2(\alpha)1(\alpha)j(\alpha)}$ vanishes by (8.5), (8.8) and (8.10). If $j=2$ then (8.43) becomes

$$\begin{aligned}\left(R^{N+1}\right)^{(N+1)(\alpha)}_{2(\alpha)1(\alpha)2(\alpha)} &= \partial_{1(\alpha)}\Gamma^{(N+1)(\alpha)}_{2(\alpha)2(\alpha)} - \partial_{2(\alpha)}\Gamma^{(N+1)(\alpha)}_{1(\alpha)2(\alpha)}\\ &\quad + \sum_{l=1}^{N+1}\left(\Gamma^{(N+1)(\alpha)}_{1(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(N+1)(\alpha)}_{2(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)2(\alpha)}\right)\\ &\quad + \sum_{\sigma\neq\alpha}\left(\Gamma^{(N+1)(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)2(\alpha)} - \Gamma^{(N+1)(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)2(\alpha)}\right)\end{aligned}$$

where in the first summation only the terms for $l\geqslant 2$ survive, as $\Gamma^{1(\alpha)}_{2(\alpha)2(\alpha)}=0$ by (8.10) and $\Gamma^{1(\alpha)}_{1(\alpha)2(\alpha)} \overset{(8.18)}{=} -\sum_{\sigma\neq\alpha}\Gamma^{1(\alpha)}_{1(\sigma)2(\alpha)} \overset{(8.4)}{=} 0$. This yields

$$\begin{aligned}\left(R^{N+1}\right)^{(N+1)(\alpha)}_{2(\alpha)1(\alpha)2(\alpha)} &\underset{(8.18)}{\overset{(8.5)}{=}} \partial_{1(\alpha)}\Gamma^{(N+1)(\alpha)}_{2(\alpha)2(\alpha)} - \partial_{2(\alpha)}\Gamma^{N(\alpha)}_{1(\alpha)1(\alpha)}\\ &\quad -\sum_{\sigma\neq\alpha}\sum_{l=2}^{N+1}\Gamma^{(N-l+2)(\alpha)}_{1(\sigma)1(\alpha)}\Gamma^{l(\alpha)}_{2(\alpha)2(\alpha)} - \sum_{l=2}^{N+1}\Gamma^{(N-l+3)(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)2(\alpha)}\end{aligned}$$

where the last two sums cancel out by (8.18). Thus

$$\left(R^{N+1}\right)^{(N+1)(\alpha)}_{2(\alpha)1(\alpha)2(\alpha)} = \partial_{1(\alpha)}\left[\Gamma^{(N-1)(\alpha)}_{1(\alpha)1(\alpha)} + \left(\Gamma^{(N+1)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(N-1)(\alpha)}_{1(\alpha)1(\alpha)}\right)\right] - \partial_{2(\alpha)}\Gamma^{N(\alpha)}_{1(\alpha)1(\alpha)}$$

where we added and subtract the quantity $\Gamma^{(N-1)(\alpha)}_{1(\alpha)1(\alpha)}$ and where

$$\partial_{2(\alpha)}\Gamma^{N(\alpha)}_{1(\alpha)1(\alpha)} \overset{(8.18)}{=} -\sum_{\sigma\neq\alpha}\partial_{2(\alpha)}\Gamma^{N(\alpha)}_{1(\sigma)1(\alpha)} \overset{(8.11)}{=} -\sum_{\sigma\neq\alpha}\partial_{1(\alpha)}\Gamma^{(N-1)(\alpha)}_{1(\sigma)1(\alpha)} \overset{(8.18)}{=} \partial_{1(\alpha)}\Gamma^{(N-1)(\alpha)}_{1(\alpha)1(\alpha)}.$$

This implies

$$\left(R^{N+1}\right)^{(N+1)(\alpha)}_{2(\alpha)1(\alpha)2(\alpha)} = \partial_{1(\alpha)}\left(\Gamma^{(N+1)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(N-1)(\alpha)}_{1(\alpha)1(\alpha)}\right) = 0$$

because

$$\Gamma^{(N+1)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(N-1)(\alpha)}_{1(\alpha)1(\alpha)} \overset{(8.9)}{=} -\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} \frac{u^{(N+1)(\alpha)}}{u^{2(\alpha)}} - \frac{1}{u^{2(\alpha)}} \sum_{l=1}^{N-2} \left(\Gamma^{(l+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)} \right) u^{(N-l+1)(\alpha)}$$

does not depend on $u^{1(\alpha)}$ (it only depends on $\{u^{1(\sigma)} \,|\, \sigma \neq \alpha\}$ and $\{u^{2(\alpha)} \,|\, 2 \leqslant s \leqslant N\}$). If $j = 1$ then (8.43) becomes

$$\left(R^{N+1}\right)^{(N+1)(\alpha)}_{2(\alpha)1(\alpha)1(\alpha)} \overset{(8.5)}{=} \partial_{1(\alpha)} \Gamma^{(N+1)(\alpha)}_{2(\alpha)1(\alpha)} - \partial_{2(\alpha)} \Gamma^{(N+1)(\alpha)}_{1(\alpha)1(\alpha)} + \sum_{l=1}^{N+1} \left(\Gamma^{(N+1)(\alpha)}_{1(\alpha)l(\alpha)} \Gamma^{l(\alpha)}_{2(\alpha)1(\alpha)} - \Gamma^{(N+1)(\alpha)}_{2(\alpha)l(\alpha)} \Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)} \right) - \sum_{\sigma \neq \alpha} \Gamma^{(N+1)(\alpha)}_{2(\alpha)1(\sigma)} \Gamma^{1(\sigma)}_{1(\alpha)1(\alpha)}$$

where

$$\partial_{1(\alpha)} \Gamma^{(N+1)(\alpha)}_{2(\alpha)1(\alpha)} \overset{(8.8)}{=} \partial_{1(\alpha)} \Gamma^{N(\alpha)}_{1(\alpha)1(\alpha)}$$

and

$$\partial_{2(\alpha)} \Gamma^{(N+1)(\alpha)}_{1(\alpha)1(\alpha)} \overset{(8.18)}{=} -\sum_{\sigma \neq \alpha} \partial_{2(\alpha)} \Gamma^{(N+1)(\alpha)}_{1(\sigma)1(\alpha)} \overset{(8.11)}{=} -\sum_{\sigma \neq \alpha} \partial_{1(\alpha)} \Gamma^{N(\alpha)}_{1(\sigma)1(\alpha)} \overset{(8.18)}{=} \partial_{1(\alpha)} \Gamma^{N(\alpha)}_{1(\alpha)1(\alpha)}$$

mutually cancel out. This yields

$$\begin{aligned}
\left(R^{N+1}\right)^{(N+1)(\alpha)}_{2(\alpha)1(\alpha)1(\alpha)} &\underset{(8.8)}{\overset{(8.10)}{=}} -\Gamma^{(N+1)(\alpha)}_{2(\alpha)1(\alpha)} \Gamma^{1(\alpha)}_{1(\alpha)1(\alpha)} + \sum_{l=1}^{N} \Gamma^{N(\alpha)}_{1(\alpha)l(\alpha)} \Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)} \\
&\quad - \sum_{l=2}^{N+1} \Gamma^{(N-l+3)(\alpha)}_{2(\alpha)2(\alpha)} \Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)} - \sum_{\sigma \neq \alpha} \Gamma^{N(\alpha)}_{1(\alpha)1(\sigma)} \Gamma^{1(\sigma)}_{1(\alpha)1(\alpha)} \\
&\overset{(8.8)}{=} -\Gamma^{N(\alpha)}_{1(\alpha)1(\alpha)} \Gamma^{1(\alpha)}_{1(\alpha)1(\alpha)} + \Gamma^{N(\alpha)}_{1(\alpha)1(\alpha)} \Gamma^{1(\alpha)}_{1(\alpha)1(\alpha)} + \sum_{l=2}^{N} \Gamma^{N(\alpha)}_{1(\alpha)l(\alpha)} \Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)} \\
&\quad - \sum_{l=2}^{N} \Gamma^{(N-l+3)(\alpha)}_{2(\alpha)2(\alpha)} \Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)} - \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} \Gamma^{(N+1)(\alpha)}_{1(\alpha)1(\alpha)} - \sum_{\sigma \neq \alpha} \Gamma^{N(\alpha)}_{1(\alpha)1(\sigma)} \Gamma^{1(\sigma)}_{1(\alpha)1(\alpha)} \\
&\underset{(8.2)}{\overset{(8.18)}{=}} \sum_{l=2}^{N} \left(\Gamma^{N(\alpha)}_{1(\alpha)l(\alpha)} - \Gamma^{(N-l+3)(\alpha)}_{2(\alpha)2(\alpha)} \right) \Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)} - \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} \Gamma^{(N+1)(\alpha)}_{1(\alpha)1(\alpha)} \\
&\quad + \sum_{\sigma \neq \alpha} \Gamma^{N(\alpha)}_{1(\alpha)1(\sigma)} \Gamma^{1(\sigma)}_{1(\alpha)1(\sigma)} \overset{(8.18)}{=} \sum_{\sigma \neq \alpha} \Big[\Gamma^{N(\alpha)}_{1(\alpha)1(\sigma)} \Gamma^{1(\sigma)}_{1(\alpha)1(\sigma)} + \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} \Gamma^{(N+1)(\alpha)}_{1(\sigma)1(\alpha)} \\
&\quad + \sum_{l=2}^{N} \left(\Gamma^{(N-l+3)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(N-l+1)(\alpha)}_{1(\alpha)1(\alpha)} \right) \Gamma^{l(\alpha)}_{1(\sigma)1(\alpha)} \Big] = 0
\end{aligned}$$

because for each $\sigma \neq \alpha$

$$\Gamma^{N(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)1(\sigma)} + \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{(N+1)(\alpha)}_{1(\sigma)1(\alpha)} + \sum_{l=2}^{N}\left(\Gamma^{(N-l+3)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(N-l+1)(\alpha)}_{1(\alpha)1(\alpha)}\right)\Gamma^{l(\alpha)}_{1(\sigma)1(\alpha)}$$

can be rewritten, due to (8.4), (8.7) and (8.9) as

$$\frac{1}{u^{1(\alpha)} - u^{1(\sigma)}}\sum_{l=3}^{N-2}\Gamma^{(N-l+1)(\alpha)}_{1(\sigma)1(\alpha)}\left[\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\left(\frac{u^{3(\alpha)}}{u^{2(\alpha)}}u^{l(\alpha)} - u^{(l+1)(\alpha)}\right)\right.$$
$$\left. -\sum_{s=2}^{l-1}\left(\Gamma^{(s+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{s(\alpha)}_{1(\alpha)1(\alpha)}\right)u^{(l-s+1)(\alpha)}\right] \overset{(8.14)}{=} 0.$$

Case 2: $\alpha = \beta = \gamma \neq \epsilon$. Our goal is to prove that

$$R^{i(\alpha)}_{h(\epsilon)k(\alpha)j(\alpha)} = \partial_{k(\alpha)}\Gamma^{i(\alpha)}_{h(\epsilon)j(\alpha)} - \partial_{h(\epsilon)}\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)} + \sum_{\sigma=1}^{r}\sum_{l=1}^{m_\sigma}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{h(\epsilon)j(\alpha)} - \Gamma^{i(\alpha)}_{h(\epsilon)l(\sigma)}\Gamma^{l(\sigma)}_{k(\alpha)j(\alpha)}\right) \tag{8.44}$$

vanishes. Let us first consider the case where $h \geqslant 2$. We have

$$R^{i(\alpha)}_{h(\epsilon)k(\alpha)j(\alpha)} \overset{(8.3)}{=} \partial_{k(\alpha)} - \partial_{h(\epsilon)}\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)} + \sum_{\sigma=1}^{r}\sum_{l=1}^{m_\sigma}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{h(\epsilon)j(\alpha)} - \Gamma^{i(\alpha)}_{h(\epsilon)l(\sigma)}\Gamma^{l(\sigma)}_{k(\alpha)j(\alpha)}\right)$$

where $\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)}$ only depends on $\{u^{s(\alpha)}\,|\,1 \leqslant s \leqslant m_\alpha\}$ and $\{u^{1(\sigma)}\,|\,\sigma \neq \alpha\}$ (thus it does not depend on $u^{h(\epsilon)}$ as $h \geqslant 2$, that is $\partial_{h(\epsilon)}\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)} = 0$). This yields

$$R^{i(\alpha)}_{h(\epsilon)k(\alpha)j(\alpha)} = \sum_{\sigma=1}^{r}\sum_{l=1}^{m_\sigma}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{h(\epsilon)j(\alpha)} - \Gamma^{i(\alpha)}_{h(\epsilon)l(\sigma)}\Gamma^{l(\sigma)}_{k(\alpha)j(\alpha)}\right)$$

where the terms $\Gamma^{l(\sigma)}_{h(\epsilon)j(\alpha)}$ and $\Gamma^{i(\alpha)}_{h(\epsilon)l(\sigma)}$ trivially vanish for $\sigma \notin \{\alpha, \epsilon\}$ by (8.2). Thus

$$R^{i(\alpha)}_{h(\epsilon)k(\alpha)j(\alpha)} \underset{(8.5)}{\overset{(8.3)}{=}} \sum_{l=h}^{m_\epsilon}\Gamma^{i(\alpha)}_{k(\alpha)l(\epsilon)}\Gamma^{l(\epsilon)}_{h(\epsilon)j(\alpha)} = 0$$

as $\Gamma^{i(\alpha)}_{k(\alpha)l(\epsilon)} = 0$ by (8.4) for every $l \geqslant h$ ($h \geqslant 2$ implies $l \geqslant 2$). Let us now fix $h = 1$. We have (the terms $\Gamma^{l(\sigma)}_{1(\epsilon)j(\alpha)}$ and $\Gamma^{i(\alpha)}_{1(\epsilon)l(\sigma)}$ trivially vanish for $\sigma \notin \{\alpha, \epsilon\}$ by (8.2))

$$R^{i(\alpha)}_{1(\epsilon)k(\alpha)j(\alpha)} = \partial_{k(\alpha)}\Gamma^{i(\alpha)}_{1(\epsilon)j(\alpha)} - \partial_{1(\epsilon)}\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)} + \sum_{l=1}^{m_\alpha}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\epsilon)j(\alpha)}\right.$$
$$\left. - \Gamma^{i(\alpha)}_{1(\epsilon)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)j(\alpha)}\right) + \sum_{l=1}^{m_\epsilon}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\epsilon)}\Gamma^{l(\epsilon)}_{1(\epsilon)j(\alpha)} - \Gamma^{i(\alpha)}_{1(\epsilon)l(\epsilon)}\Gamma^{l(\epsilon)}_{k(\alpha)j(\alpha)}\right)$$

$$\stackrel{(8.4)}{\underset{(8.5)}{=}} \partial_{k(\alpha)}\Gamma^{i(\alpha)}_{1(\epsilon)j(\alpha)} - \partial_{1(\epsilon)}\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)} + \sum_{l=1}^{m_\alpha}\Big(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\epsilon)j(\alpha)} - \Gamma^{i(\alpha)}_{1(\epsilon)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)j(\alpha)}\Big) + \Gamma^{i(\alpha)}_{k(\alpha)1(\epsilon)}\Gamma^{1(\epsilon)}_{1(\epsilon)j(\alpha)} - \Gamma^{i(\alpha)}_{1(\epsilon)1(\epsilon)}\Gamma^{1(\epsilon)}_{k(\alpha)j(\alpha)}. \tag{8.45}$$

We distinguish between the following subcases:

$a.\, j \geqslant 2, k \geqslant 2 \qquad c.\, j = 1, k \geqslant 2$
$b.\, j = k = 1 \qquad d.\, j \geqslant 2, k = 1.$

Subcase a: both j and k are greater or equal than 2. Let us first claim that in this case

$$\partial_{k(\alpha)}\Gamma^{i(\alpha)}_{1(\epsilon)j(\alpha)} - \partial_{1(\epsilon)}\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)} = 0. \tag{8.46}$$

Indeed, if $i<j$ then both $\Gamma^{i(\alpha)}_{1(\epsilon)j(\alpha)}$ and $\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)}$ trivially vanish by (8.3) and (8.10) respectively. If $i=j$ then

$$\partial_{k(\alpha)}\Gamma^{j(\alpha)}_{1(\epsilon)j(\alpha)} - \partial_{1(\epsilon)}\Gamma^{j(\alpha)}_{k(\alpha)j(\alpha)} \stackrel{(8.4)}{\underset{(8.10)}{=}} \partial_{k(\alpha)}\left[\frac{m_\epsilon \varepsilon_\epsilon}{u^{1(\alpha)}-u^{1(\epsilon)}}\right] - \partial_{1(\epsilon)}\Gamma^{(4-k)(\alpha)}_{2(\alpha)2(\alpha)} \stackrel{(8.10)}{=} -\partial_{1(\epsilon)}\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\,\delta^2_k \stackrel{(8.7)}{=} 0.$$

For $i>j$ (8.46) can be proved by induction over i (starting from the case $i=j$ that we just handled), using (8.4), (8.8), (8.10), (8.15) and (8.18). Thus (8.45) becomes

$$R^{i(\alpha)}_{1(\epsilon)k(\alpha)j(\alpha)} \stackrel{(8.3)}{\underset{(8.5)}{=}} \sum_{l=1}^{m_\alpha}\Big(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\epsilon)j(\alpha)} - \Gamma^{i(\alpha)}_{1(\epsilon)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)j(\alpha)}\Big) + \Gamma^{i(\alpha)}_{k(\alpha)1(\epsilon)}\Gamma^{1(\epsilon)}_{1(\epsilon)j(\alpha)} - \Gamma^{i(\alpha)}_{1(\epsilon)1(\epsilon)}\Gamma^{1(\epsilon)}_{k(\alpha)j(\alpha)} \stackrel{(8.4)}{=} \sum_{l=j}^{i}\Big(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\epsilon)j(\alpha)} - \Gamma^{i(\alpha)}_{1(\epsilon)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)j(\alpha)}\Big)$$

which trivially vanishes for $i<j$. For $i \geqslant j$ we have

$$R^{i(\alpha)}_{1(\epsilon)k(\alpha)j(\alpha)} \stackrel{(8.4)}{\underset{(8.10)}{=}} \sum_{l=j}^{i}\Big(\Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{(l-j+1)(\alpha)}_{1(\epsilon)1(\alpha)} - \Gamma^{(i-l+1)(\alpha)}_{1(\epsilon)1(\alpha)}\Gamma^{(l-k-j+4)(\alpha)}_{2(\alpha)2(\alpha)}\Big)$$
$$= \sum_{l=j}^{i}\Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{(l-j+1)(\alpha)}_{1(\epsilon)1(\alpha)} - \sum_{l=j}^{i}\Gamma^{(i-l+1)(\alpha)}_{1(\epsilon)1(\alpha)}\Gamma^{(l-k-j+4)(\alpha)}_{2(\alpha)2(\alpha)} = 0.$$

Subcase b: $j=k=1$. We have

$$R^{i(\alpha)}_{1(\epsilon)1(\alpha)1(\alpha)} = \partial_{1(\alpha)}\Gamma^{i(\alpha)}_{1(\epsilon)1(\alpha)} - \partial_{1(\epsilon)}\Gamma^{i(\alpha)}_{1(\alpha)1(\alpha)} + \sum_{l=1}^{m_\alpha}\Big(\Gamma^{i(\alpha)}_{1(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)} - \Gamma^{i(\alpha)}_{1(\epsilon)l(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)}\Big) + \Gamma^{i(\alpha)}_{1(\alpha)1(\epsilon)}\Gamma^{1(\epsilon)}_{1(\epsilon)1(\alpha)} - \Gamma^{i(\alpha)}_{1(\epsilon)1(\epsilon)}\Gamma^{1(\epsilon)}_{1(\alpha)1(\alpha)}$$

where in the summation only the terms for $l \leqslant i$ survive (by (8.4) and (8.8)) and

$$\partial_{1(\epsilon)}\Gamma^{i(\alpha)}_{1(\alpha)1(\alpha)} \stackrel{(8.18)}{=} -\partial_{1(\epsilon)}\Gamma^{i(\alpha)}_{1(\epsilon)1(\alpha)} = \partial_{1(\alpha)}\Gamma^{i(\alpha)}_{1(\epsilon)1(\alpha)}$$

as $\Gamma^{i(\alpha)}_{1(\epsilon)1(\alpha)}$ only depends on $u^{1(\epsilon)}$ and $u^{1(\alpha)}$ by means of the term $u^{1(\alpha)}-u^{1(\epsilon)}$. Thus

$$R^{i(\alpha)}_{1(\epsilon)1(\alpha)1(\alpha)} \overset{(8.18)}{=} -\sum_{l=1}^{i}\left(\Gamma^{i(\alpha)}_{1(\epsilon)l(\alpha)}\Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)}-\Gamma^{i(\alpha)}_{1(\epsilon)l(\alpha)}\Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)}\right)$$
$$-\sum_{\sigma\notin\{\alpha,\epsilon\}}\sum_{l=1}^{i}\left(\Gamma^{i(\alpha)}_{1(\sigma)l(\alpha)}\Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)}-\Gamma^{i(\alpha)}_{1(\epsilon)l(\alpha)}\Gamma^{l(\alpha)}_{1(\sigma)1(\alpha)}\right)\overset{(8.4)}{=}0.$$

Subcase c: $j=1$, $k\geqslant 2$. We have

$$R^{i(\alpha)}_{1(\epsilon)k(\alpha)1(\alpha)} \underset{(8.4)\,(8.10)}{\overset{(8.18),(8.3)}{=}} \partial_{k(\alpha)}\Gamma^{i(\alpha)}_{1(\epsilon)1(\alpha)}+\partial_{1(\epsilon)}\Gamma^{(i-k+1)(\alpha)}_{1(\epsilon)1(\alpha)}+\sum_{l=1}^{i}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)}\right.$$
$$\left.-\Gamma^{i(\alpha)}_{1(\epsilon)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)1(\alpha)}\right)+\Gamma^{i(\alpha)}_{k(\alpha)1(\epsilon)}\Gamma^{1(\epsilon)}_{1(\epsilon)1(\alpha)}.$$

Let us first claim that in this case

$$\partial_{k(\alpha)}\Gamma^{i(\alpha)}_{1(\epsilon)1(\alpha)}+\partial_{1(\epsilon)}\Gamma^{(i-k+1)(\alpha)}_{1(\epsilon)1(\alpha)}=0. \tag{8.47}$$

This condition can be proved by induction over i (starting from the case $i=1$ where (8.47) holds by (8.4)), using (8.11). Thus

$$R^{i(\alpha)}_{1(\epsilon)k(\alpha)1(\alpha)} = \sum_{l=1}^{i}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)}-\Gamma^{i(\alpha)}_{1(\epsilon)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)1(\alpha)}\right)+\Gamma^{i(\alpha)}_{k(\alpha)1(\epsilon)}\Gamma^{1(\epsilon)}_{1(\epsilon)1(\alpha)}$$
$$\underset{(8.8)}{\overset{(8.10)}{=}}\sum_{l=1}^{i-k+2}\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)}-\sum_{l=k}^{i}\Gamma^{i(\alpha)}_{1(\epsilon)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)1(\alpha)}+\Gamma^{i(\alpha)}_{k(\alpha)1(\epsilon)}\Gamma^{1(\epsilon)}_{1(\epsilon)1(\alpha)}$$

which trivially vanishes for $i<k$ by (8.4) and (8.8). For $i\geqslant k$ we get

$$R^{i(\alpha)}_{1(\epsilon)k(\alpha)1(\alpha)} \underset{(8.10)}{\overset{(8.4)}{=}} \Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{1(\epsilon)1(\alpha)}+\sum_{l=2}^{i-k+2}\Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)}$$
$$-\sum_{l=k}^{i}\Gamma^{(i-l+1)(\alpha)}_{1(\epsilon)1(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)1(\alpha)}+\Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\epsilon)}\Gamma^{1(\epsilon)}_{1(\epsilon)1(\alpha)}$$
$$=\Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{1(\epsilon)1(\alpha)}+\sum_{l=2}^{i-k+2}\Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)}$$
$$-\sum_{t=2}^{i-k+1}\Gamma^{t(\alpha)}_{1(\epsilon)1(\alpha)}\Gamma^{(i-t+1)(\alpha)}_{k(\alpha)1(\alpha)}+\Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\epsilon)}\Gamma^{1(\epsilon)}_{1(\epsilon)1(\alpha)}$$
$$=\sum_{l=2}^{i-k+1}\Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)}\left(\Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)}-\Gamma^{(i-l+1)(\alpha)}_{k(\alpha)1(\alpha)}\right)$$
$$+\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{(i-k+2)(\alpha)}_{1(\epsilon)1(\alpha)}+\Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\epsilon)}\Gamma^{1(\epsilon)}_{1(\epsilon)1(\alpha)}$$

$$\overset{(8.8)}{=}\sum_{l=2}^{i-k+1}\Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)}\left(\Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)}-\Gamma^{(i-k-l+2)(\alpha)}_{1(\alpha)1(\alpha)}\right)$$
$$+\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{(i-k+2)(\alpha)}_{1(\epsilon)1(\alpha)}+\Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\epsilon)}\Gamma^{1(\epsilon)}_{1(\epsilon)1(\alpha)}$$
$$\underset{(8.7)}{\overset{(8.4)}{=}}\sum_{l=2}^{i-k+1}\Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)}\left(\Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)}-\Gamma^{(i-k-l+2)(\alpha)}_{1(\alpha)1(\alpha)}\right)$$
$$-\frac{\Gamma^{1(\epsilon)}_{1(\alpha)1(\epsilon)}}{u^{2(\alpha)}}\sum_{s=3}^{i-k+2}\Gamma^{(i-k-s+3)(\alpha)}_{1(\epsilon)1(\alpha)}u^{s(\alpha)}$$
$$=\sum_{l=2}^{i-k+1}\Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)}\left(\Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)}-\Gamma^{(i-k-l+2)(\alpha)}_{1(\alpha)1(\alpha)}\right)$$
$$-\frac{\Gamma^{1(\epsilon)}_{1(\alpha)1(\epsilon)}}{u^{2(\alpha)}}\sum_{s=3}^{i-k+2}\Gamma^{(i-k-s+3)(\alpha)}_{1(\epsilon)1(\alpha)}u^{s(\alpha)}.$$

The vanishing of this quantity can be proved by induction over i (starting from the trivial case where $i=k$), using (8.4), (8.7), (8.9) and (8.14).

Subcase d: $j\geqslant 2$, $k=1$. We have

$$R^{i(\alpha)}_{1(\epsilon)1(\alpha)j(\alpha)}\underset{(8.5)}{\overset{(8.3)}{=}}\partial_{1(\alpha)}\Gamma^{i(\alpha)}_{1(\epsilon)j(\alpha)}-\partial_{1(\epsilon)}\Gamma^{i(\alpha)}_{1(\alpha)j(\alpha)}$$
$$+\sum_{l=1}^{m_\alpha}\left(\Gamma^{i(\alpha)}_{1(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\epsilon)j(\alpha)}-\Gamma^{i(\alpha)}_{1(\epsilon)l(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)j(\alpha)}\right)$$

where

$$\partial_{1(\epsilon)}\Gamma^{i(\alpha)}_{1(\alpha)j(\alpha)}\overset{(8.18)}{=}-\partial_{1(\epsilon)}\Gamma^{i(\alpha)}_{1(\epsilon)j(\alpha)}-\sum_{\sigma\notin\{\alpha,\epsilon\}}\partial_{1(\epsilon)}\Gamma^{i(\alpha)}_{1(\sigma)j(\alpha)}=\partial_{1(\alpha)}\Gamma^{i(\alpha)}_{1(\epsilon)j(\alpha)}$$

as $\Gamma^{i(\alpha)}_{1(\epsilon)j(\alpha)}$ only depends on $u^{1(\epsilon)}$ and $u^{1(\alpha)}$ by means of the term $u^{1(\alpha)}-u^{1(\epsilon)}$. Thus

$$R^{i(\alpha)}_{1(\epsilon)1(\alpha)j(\alpha)}\underset{(8.8)}{\overset{(8.4)}{=}}\sum_{l=j}^{i}\Gamma^{(i-l+1)(\alpha)}_{1(\alpha)1(\alpha)}\Gamma^{(l-j+1)(\alpha)}_{1(\epsilon)1(\alpha)}-\sum_{l=j}^{i}\Gamma^{(i-l+1)(\alpha)}_{1(\epsilon)1(\alpha)}\Gamma^{(l-j+1)(\alpha)}_{1(\alpha)1(\alpha)}=0.$$

Case 3: $\alpha=\gamma=\epsilon\neq\beta$. Our goal is to prove that

$$R^{i(\alpha)}_{h(\alpha)k(\alpha)j(\beta)}\overset{(8.2)}{=}\partial_{k(\alpha)}\Gamma^{i(\alpha)}_{h(\alpha)j(\beta)}-\partial_{h(\alpha)}\Gamma^{i(\alpha)}_{k(\alpha)j(\beta)}+\sum_{l=1}^{m_\alpha}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\alpha)j(\beta)}\right.$$
$$\left.-\Gamma^{i(\alpha)}_{h(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)j(\beta)}\right)+\sum_{l=1}^{m_\beta}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\beta)}\Gamma^{l(\beta)}_{h(\alpha)j(\beta)}-\Gamma^{i(\alpha)}_{h(\alpha)l(\beta)}\Gamma^{l(\beta)}_{k(\alpha)j(\beta)}\right)\tag{8.48}$$

vanishes. If $j\geqslant 2$ we get

$$R^{i(\alpha)}_{h(\alpha)k(\alpha)j(\beta)}\overset{(8.3)}{=}\Gamma^{i(\alpha)}_{k(\alpha)1(\beta)}\Gamma^{1(\beta)}_{h(\alpha)j(\beta)}-\Gamma^{i(\alpha)}_{h(\alpha)1(\beta)}\Gamma^{1(\beta)}_{k(\alpha)j(\beta)}\overset{(8.4)}{=}0.$$

Let us then fix $j = 1$. We have

$$R^{i(\alpha)}_{h(\alpha)k(\alpha)1(\beta)} = \partial_{k(\alpha)}\Gamma^{i(\alpha)}_{h(\alpha)1(\beta)} - \partial_{h(\alpha)}\Gamma^{i(\alpha)}_{k(\alpha)1(\beta)} + \sum_{l=1}^{m_\alpha}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\alpha)1(\beta)} - \Gamma^{i(\alpha)}_{h(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)1(\beta)}\right) + \sum_{l=1}^{m_\beta}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\beta)}\Gamma^{l(\beta)}_{h(\alpha)1(\beta)} - \Gamma^{i(\alpha)}_{h(\alpha)l(\beta)}\Gamma^{l(\beta)}_{k(\alpha)1(\beta)}\right) \tag{8.49}$$

where, by (8.47),

$$\partial_{t(\alpha)}\Gamma^{i(\alpha)}_{1(\epsilon)1(\alpha)} = -\partial_{1(\epsilon)}\Gamma^{(i-t+1)(\alpha)}_{1(\epsilon)1(\alpha)} \tag{8.50}$$

for every $\epsilon \neq \alpha$ and $t \geqslant 2$. We distinguish between the following subcases:

a. both h and k are greater or equal than 2
b. $h = 1$, $k \geqslant 2$ (this covers $h \geqslant 2$, $k = 1$ as well)

observing that $R^{i(\alpha)}_{h(\epsilon)k(\gamma)1(\beta)} = 0$ automatically whenever $k = h$.
Subcase a: both k and h are greater or equal than 2. We have

$$R^{i(\alpha)}_{h(\alpha)k(\alpha)1(\beta)} \overset{(8.3),(8.4)}{\underset{(8.10)}{=}} \partial_{k(\alpha)}\Gamma^{i(\alpha)}_{h(\alpha)1(\beta)} - \partial_{h(\alpha)}\Gamma^{i(\alpha)}_{k(\alpha)1(\beta)} + \sum_{l=h}^{i-k+2}\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\alpha)1(\beta)} - \sum_{l=k}^{i-h+2}\Gamma^{i(\alpha)}_{h(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)1(\beta)}$$

where

$$\partial_{k(\alpha)}\Gamma^{i(\alpha)}_{h(\alpha)1(\beta)} - \partial_{h(\alpha)}\Gamma^{i(\alpha)}_{k(\alpha)1(\beta)} \overset{(8.4)}{=} \partial_{k(\alpha)}\Gamma^{(i-h+1)(\alpha)}_{1(\alpha)1(\beta)} - \partial_{h(\alpha)}\Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\beta)} \overset{(8.50)}{=} 0.$$

This yields

$$R^{i(\alpha)}_{h(\alpha)k(\alpha)1(\beta)} = \sum_{l=h}^{i-k+2}\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\alpha)1(\beta)} - \sum_{l=k}^{i-h+2}\Gamma^{i(\alpha)}_{h(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)1(\beta)} \tag{8.51}$$

which vanishes automatically for $i < k$ and by (8.4) and (8.10) for $i \geqslant k$.
Subcase b: $h = 1$, $k \geqslant 2$. We have

$$R^{i(\alpha)}_{1(\alpha)k(\alpha)1(\beta)} \overset{(8.3),(8.10)}{\underset{(8.8)}{=}} \partial_{k(\alpha)}\Gamma^{i(\alpha)}_{1(\alpha)1(\beta)} - \partial_{1(\alpha)}\Gamma^{i(\alpha)}_{k(\alpha)1(\beta)} + \sum_{l=1}^{i-k+2}\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)1(\beta)} - \sum_{l=k}^{i}\Gamma^{i(\alpha)}_{1(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)1(\beta)} + \Gamma^{i(\alpha)}_{k(\alpha)1(\beta)}\Gamma^{1(\beta)}_{1(\alpha)1(\beta)}$$

$$\overset{(8.4)\,(8.10)}{\underset{(8.8)\,(8.50)}{=}} \Gamma^{i(\alpha)}_{k(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{1(\alpha)1(\beta)} + \sum_{l=2}^{i-k+2}\Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)1(\beta)}$$
$$-\sum_{l=k}^{i}\Gamma^{(i-l+1)(\alpha)}_{1(\alpha)1(\alpha)}\Gamma^{(l-k+1)(\alpha)}_{1(\alpha)1(\beta)} + \Gamma^{i(\alpha)}_{k(\alpha)1(\beta)}\Gamma^{1(\beta)}_{1(\alpha)1(\beta)}$$
$$\overset{(8.8)}{=} \Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{1(\alpha)1(\beta)} + \sum_{l=2}^{i-k+2}\Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)1(\beta)}$$
$$-\Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{1(\alpha)1(\beta)} - \sum_{l=2}^{i-k+1}\Gamma^{(i-k-l+2)(\alpha)}_{1(\alpha)1(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)1(\beta)} + \Gamma^{i(\alpha)}_{k(\alpha)1(\beta)}\Gamma^{1(\beta)}_{1(\alpha)1(\beta)}$$
$$\overset{(8.4)}{=} \sum_{l=2}^{i-k+1}\left(\Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(i-k-l+2)(\alpha)}_{1(\alpha)1(\alpha)}\right)\Gamma^{l(\alpha)}_{1(\alpha)1(\beta)} + \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{(i-k+2)(\alpha)}_{1(\alpha)1(\beta)}$$
$$+\Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\beta)}\Gamma^{1(\beta)}_{1(\alpha)1(\beta)} \overset{(8.17)}{=} 0. \tag{8.52}$$

Case 4: $\beta = \gamma = \epsilon \neq \alpha$. Our goal is to prove that

$$R^{i(\alpha)}_{h(\beta)k(\beta)j(\beta)} \overset{(8.2)}{=} \partial_{k(\beta)}\Gamma^{i(\alpha)}_{h(\beta)j(\beta)} - \partial_{h(\beta)}\Gamma^{i(\alpha)}_{k(\beta)j(\beta)} + \sum_{l=1}^{m_\alpha}\left(\Gamma^{i(\alpha)}_{k(\beta)l(\alpha)}\Gamma^{l(\alpha)}_{h(\beta)j(\beta)} - \Gamma^{i(\alpha)}_{h(\beta)l(\alpha)}\Gamma^{l(\alpha)}_{k(\beta)j(\beta)}\right) + \sum_{l=1}^{m_\beta}\left(\Gamma^{i(\alpha)}_{k(\beta)l(\beta)}\Gamma^{l(\beta)}_{h(\beta)j(\beta)} - \Gamma^{i(\alpha)}_{h(\beta)l(\beta)}\Gamma^{l(\beta)}_{k(\beta)j(\beta)}\right)$$

vanishes. Without loss of generality, by the symmetries of R, we can set $h > k$. In particular, $h \geqslant 2$. We get

$$R^{i(\alpha)}_{h(\beta)k(\beta)j(\beta)} \overset{(8.5)}{=} -\partial_{h(\beta)}\Gamma^{i(\alpha)}_{k(\beta)j(\beta)} + \sum_{l=1}^{m_\beta}\Gamma^{i(\alpha)}_{k(\beta)l(\beta)}\Gamma^{l(\beta)}_{h(\beta)j(\beta)} \tag{8.53}$$

which vanishes by (8.5) for $k \geqslant 2$. For $k = 1$ (8.53) becomes

$$R^{i(\alpha)}_{h(\beta)1(\beta)j(\beta)} \overset{(8.18)\,(8.2)}{\underset{(8.3)}{=}} \partial_{h(\beta)}\Gamma^{i(\alpha)}_{1(\alpha)j(\beta)} - \Gamma^{i(\alpha)}_{1(\alpha)1(\beta)}\Gamma^{1(\beta)}_{h(\beta)j(\beta)}$$

which vanishes by (8.3) and (8.10) for $j \geqslant 2$. For $j = 1$ it reads

$$R^{i(\alpha)}_{h(\beta)1(\beta)1(\beta)} = \partial_{h(\beta)}\Gamma^{i(\alpha)}_{1(\alpha)1(\beta)} - \Gamma^{i(\alpha)}_{1(\alpha)1(\beta)}\Gamma^{1(\beta)}_{h(\beta)1(\beta)} = 0$$

as $\Gamma^{i(\alpha)}_{1(\alpha)1(\beta)}$ does not depend on any of the $u^{h(\beta)}$s for $h \geqslant 2$ and $(1 < 2 \leqslant h)$

$$\Gamma^{1(\beta)}_{h(\beta)1(\beta)} \overset{(8.18)}{=} -\sum_{\sigma\neq\beta}\Gamma^{1(\beta)}_{h(\beta)1(\sigma)} \overset{(8.4)}{=} 0.$$

Case 5: $\alpha=\beta\neq\gamma=\epsilon$. Our goal is to prove that

$$R^{i(\alpha)}_{h(\gamma)k(\gamma)j(\alpha)} \overset{(8.2)}{=} \partial_{k(\gamma)}\Gamma^{i(\alpha)}_{h(\gamma)j(\alpha)} - \partial_{h(\gamma)}\Gamma^{i(\alpha)}_{k(\gamma)j(\alpha)} + \sum_{l=1}^{m_\alpha}\left(\Gamma^{i(\alpha)}_{k(\gamma)l(\alpha)}\Gamma^{l(\alpha)}_{h(\gamma)j(\alpha)} - \Gamma^{i(\alpha)}_{h(\gamma)l(\alpha)}\Gamma^{l(\alpha)}_{k(\gamma)j(\alpha)}\right) + \sum_{l=1}^{m_\gamma}\left(\Gamma^{i(\alpha)}_{k(\gamma)l(\gamma)}\Gamma^{l(\gamma)}_{h(\gamma)j(\alpha)} - \Gamma^{i(\alpha)}_{h(\gamma)l(\gamma)}\Gamma^{l(\gamma)}_{k(\gamma)j(\alpha)}\right) \tag{8.54}$$

vanishes. Without loss of generality, by the symmetries of R, we can set $h>k$. In particular, $h\geqslant 2$. We get

$$R^{i(\alpha)}_{h(\gamma)k(\gamma)j(\alpha)} \underset{(8.5)}{\overset{(8.3)}{=}} -\partial_{h(\gamma)}\Gamma^{i(\alpha)}_{k(\gamma)j(\alpha)} + \sum_{l=1}^{m_\gamma}\Gamma^{i(\alpha)}_{k(\gamma)l(\gamma)}\Gamma^{l(\gamma)}_{h(\gamma)j(\alpha)}$$

where

$$\sum_{l=1}^{m_\gamma}\Gamma^{i(\alpha)}_{k(\gamma)l(\gamma)}\Gamma^{l(\gamma)}_{h(\gamma)j(\alpha)} \underset{(8.4)}{\overset{(8.3)}{=}} \sum_{l=h}^{m_\gamma}\Gamma^{i(\alpha)}_{k(\gamma)l(\gamma)}\Gamma^{l(\gamma)}_{h(\gamma)j(\alpha)} \overset{(8.5)}{=} 0.$$

As $\Gamma^{i(\alpha)}_{k(\gamma)j(\alpha)}$ does not depend on any of the $u^{h(\gamma)}$s for $h\geqslant 2$, $R^{i(\alpha)}_{h(\gamma)k(\gamma)j(\alpha)}=0$.

Case 6: $\alpha=\gamma\neq\beta=\epsilon$. Our goal is to prove that

$$R^{i(\alpha)}_{h(\beta)k(\alpha)j(\beta)} \overset{(8.2)}{=} \partial_{k(\alpha)}\Gamma^{i(\alpha)}_{h(\beta)j(\beta)} - \partial_{h(\beta)}\Gamma^{i(\alpha)}_{k(\alpha)j(\beta)} + \sum_{l=1}^{m_\alpha}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\beta)j(\beta)} - \Gamma^{i(\alpha)}_{h(\beta)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)j(\beta)}\right) + \sum_{l=1}^{m_\beta}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\beta)}\Gamma^{l(\beta)}_{h(\beta)j(\beta)} - \Gamma^{i(\alpha)}_{h(\beta)l(\beta)}\Gamma^{l(\beta)}_{k(\alpha)j(\beta)}\right) + \sum_{\sigma\notin\{\alpha,\beta\}}\sum_{l=1}^{m_\sigma}\Gamma^{i(\alpha)}_{k(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{h(\beta)j(\beta)} \tag{8.55}$$

vanishes. For $j\geqslant 2$ (8.55) vanishes by (8.3)–(8.5), (8.10) and (8.18). Let us then fix $j=1$. (8.55) becomes

$$\begin{aligned} R^{i(\alpha)}_{h(\beta)k(\alpha)1(\beta)} = {} & \partial_{k(\alpha)}\Gamma^{i(\alpha)}_{h(\beta)1(\beta)} - \partial_{h(\beta)}\Gamma^{i(\alpha)}_{k(\alpha)1(\beta)} \\ & + \sum_{l=1}^{m_\alpha}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\beta)1(\beta)} - \Gamma^{i(\alpha)}_{h(\beta)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)1(\beta)}\right) \\ & + \sum_{l=1}^{m_\beta}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\beta)}\Gamma^{l(\beta)}_{h(\beta)1(\beta)} - \Gamma^{i(\alpha)}_{h(\beta)l(\beta)}\Gamma^{l(\beta)}_{k(\alpha)1(\beta)}\right) \\ & + \sum_{\sigma\notin\{\alpha,\beta\}}\sum_{l=1}^{m_\sigma}\Gamma^{i(\alpha)}_{k(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{h(\beta)1(\beta)}. \end{aligned} \tag{8.56}$$

We distinguish between the following subcases:

a. $h\geqslant 2, k\geqslant 2$ *c*. $h\geqslant 2, k=1$
b. $h=k=1$ *d*. $h=1, k\geqslant 2$.

Subcase a: both k and h are greater or equal than 2. The vanishing of (8.56) follows from (8.3)–(8.5) and (8.18).

Subcase b: $h = k = 1$. The vanishing of (8.56) follows from (8.3)–(8.6), (8.16) and (8.18).

Subcase c: $h \geqslant 2$, $k = 1$. The argument of *subcase a* applies here as well.

Subcase d: $h = 1$, $k \geqslant 2$. By (8.3)–(8.5), (8.10), (8.18) and (8.47), we get

$$R^{i(\alpha)}_{1(\beta)k(\alpha)1(\beta)} = \sum_{l=1}^{i-k+2} \Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\beta)1(\beta)} - \sum_{l=k}^{i} \Gamma^{i(\alpha)}_{1(\beta)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)1(\beta)} + \Gamma^{i(\alpha)}_{k(\alpha)1(\beta)}\Gamma^{1(\beta)}_{1(\beta)1(\beta)} + \sum_{\sigma\notin\{\alpha,\beta\}} \Gamma^{i(\alpha)}_{k(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\beta)1(\beta)}.$$

This quantity can be proved to vanish by (8.16)–(8.18) and other conditions from Proposition 2.2.

Case 7: $\alpha = \beta \notin \{\gamma, \epsilon\}$, $\gamma \neq \epsilon$. By (8.18) and (8.2)–(8.4), $R^{i(\alpha)}_{h(\epsilon)k(\gamma)j(\alpha)} = 0$ trivially.

Case 8: $\alpha = \gamma \notin \{\beta, \epsilon\}$, $\beta \neq \epsilon$. $R^{i(\alpha)}_{h(\epsilon)k(\alpha)j(\beta)}$ vanishes by (8.2)–(8.4) and (8.16).

Case 9: $\beta = \gamma \notin \{\alpha, \epsilon\}$, $\alpha \neq \epsilon$. $R^{i(\alpha)}_{h(\epsilon)k(\beta)j(\beta)}$ vanishes by (8.2), (8.3), (8.5), (8.6) and (8.16).

Case 10: $\gamma = \epsilon \notin \{\alpha, \beta\}$, $\alpha \neq \beta$. $R^{i(\alpha)}_{h(\gamma)k(\gamma)j(\beta)}$ trivially vanishes by (8.2)–(8.5).

Case 11: α, β, γ and ε are pairwise distinct. $R^{i(\alpha)}_{h(\epsilon)k(\gamma)j(\beta)}$ trivially vanishes by (8.2).

This concludes the proof of the flatness of ∇.

8.8. Uniqueness

In order to prove uniqueness we have to prove that (part of the) conditions (1)–(5) force ∇ to be of the form given in Proposition 2.2. We have seen that the condition $\nabla e = 0$ is equivalent to (8.18). Let α, β, γ be pairwise distinct. Using the condition $\nabla_i c^l_{jk} = \nabla_j c^l_{ik}$ $\forall i, j, k, l \in \{1, \ldots, n\}$ after a long but straightforward computation one obtains:

(i) $\Gamma^{l(\beta)}_{i(\alpha)(j+k)(\alpha)} = \Gamma^{l(\beta)}_{(i+k)(\alpha)j(\alpha)}$;
(ii) $\Gamma^{k(\beta)}_{i(\alpha)j(\beta)} = \Gamma^{l(\beta)}_{i(\alpha)m(\beta)}$ when $k - j = l - m$;
(iii) $\Gamma^{k(\alpha)}_{(i-1)(\alpha)j(\alpha)} = \Gamma^{k(\alpha)}_{i(\alpha)(j-1)(\alpha)}$ when the lower indices are both different from 1 and not simultaneously equal to 2;
(iv) $\Gamma^{k(\alpha)}_{i(\alpha)j(\alpha)} = \Gamma^{(k+1)(\alpha)}_{i(\alpha)(j+1)(\alpha)}$ when $j \neq 1$;
(v) $\Gamma^{k(\alpha)}_{i(\alpha)j(\beta)} = -\Gamma^{(k-i+1)(\alpha)}_{1(\beta)j(\beta)}$;
(vi) $\Gamma^{k(\gamma)}_{i(\alpha)j(\beta)} = 0$.

The above quantities must be considered non-null when the indices do not exceed the size of the corresponding block. Using condition (8.20) by straightforward computation one obtains:

1. $\Gamma^{k(\alpha)}_{i(\alpha)j(\alpha)} = \begin{cases} \Gamma^{(k-i-j+4)(\alpha)}_{2(\alpha)2(\alpha)} & \text{if } k-i-j \geqslant -2 \\ 0 & \text{if } k-i-j \leqslant -3 \end{cases}$ when $i, j > 1$
2. $\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} = -\frac{m_\alpha \varepsilon_\alpha}{u^{2(\alpha)}}$
3. $\Gamma^{1(\alpha)}_{1(\alpha)j(\alpha)} = 0$ when or $j \geqslant 2$

4. $\Gamma^{k(\alpha)}_{i(\alpha)j(\beta)} = 0$ when $k < i$ or $j \geqslant 2$
5. $\Gamma^{i(\alpha)}_{i(\alpha)1(\beta)} = \frac{m_\beta \varepsilon_\beta}{u^{1(\alpha)} - u^{1(\beta)}}$
6. $\Gamma^{(i+h)(\alpha)}_{i(\alpha)1(\beta)} = -\frac{1}{u^{1(\alpha)} - u^{1(\beta)}} \sum_{s=2}^{h+1} \Gamma^{(i+h-s+1)(\alpha)}_{i(\alpha)1(\beta)} u^{s(\alpha)}$ for $h \geqslant 1$
7. $\Gamma^{n(\alpha)}_{2(\alpha)2(\alpha)} = \Gamma^{(n-2)(\alpha)}_{1(\alpha)1(\alpha)} - \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} \frac{u^{n(\alpha)}}{u^{2(\alpha)}} - \frac{1}{u^{2(\alpha)}} \sum_{l=1}^{n-3} \left(\Gamma^{(l+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)}\right) u^{(n-l)(\alpha)}$ for $n \geqslant 3$[8].

Collecting all the conditions above one obtains Christoffel symbols of the form given in proposition 2.2.

9. The dual structure

In this section we study in more detail the dual structure $(\nabla^*, *, E)$ where:

- E is the Euler vector field,
- $*$ is the dual product defined by the formula

$$X * Y = E^{-1} \circ X \circ Y$$

  for arbitrary vector fields X and Y,
- ∇^* is the connection defined by the Christoffel symbols

$$\Gamma^{*k}_{ij} = \Gamma^k_{ij} - c^{*l}_{ji} \nabla_l E^k.$$

Proposition 9.1. *For each $\alpha \in \{1, \ldots, r\}$ the components of the inverse of the Euler vector field E are given by*

$$\left(E^{-1}\right)^{1(\alpha)} = \frac{1}{u^{1(\alpha)}} \tag{9.1}$$

$$\left(E^{-1}\right)^{(k+1)(\alpha)} = -\frac{1}{u^{1(\alpha)}} \sum_{s=1}^{k} \left(E^{-1}\right)^{(k-s+1)(\alpha)} u^{(s+1)(\alpha)} \qquad \textit{for } 1 \leqslant k \leqslant m_\alpha - 1. \tag{9.2}$$

Proof. This can be proved by straightforward computation. □

Remark 9.2. By definition, the dual product $*$ must verify the following relation:

$$X * Y = E^{-1} \circ X \circ Y \tag{9.3}$$

for X, Y arbitrary vector fields. This means that

$$X^j c^{*i}_{jk} Y^k = \left(E^{-1}\right)^a c^i_{ab} X^j c^b_{jk} Y^k \qquad \forall X, Y$$

i.e.

$$c^{*i}_{jk} = \left(E^{-1}\right)^a c^i_{ab} c^b_{jk}.$$

[8] The last summation is not to be considered for $n = 2, 3$.

Therefore

$$
\begin{aligned}
c^{*i(\alpha)}_{\ \ j(\beta)k(\gamma)} &= \sum_{\sigma,\tau=1}^{r}\sum_{a=1}^{m_\sigma}\sum_{b=1}^{m_\tau}\left(E^{-1}\right)^{a(\sigma)} c^{i(\alpha)}_{a(\sigma)b(\tau)}c^{b(\tau)}_{j(\beta)k(\gamma)} = \sum_{a,b=1}^{m_\alpha}\left(E^{-1}\right)^{a(\alpha)} c^{i(\alpha)}_{a(\alpha)b(\alpha)}c^{b(\alpha)}_{j(\beta)k(\gamma)}\\
&= \sum_{a,b=1}^{m_\alpha}\left(E^{-1}\right)^{a(\alpha)}\delta^{i}_{a+b-1}\,\delta^\alpha_\beta\delta^\alpha_\gamma\delta^b_{j+k-1} = \delta^\alpha_\beta\delta^\alpha_\gamma\sum_{b=1}^{m_\alpha}\left(E^{-1}\right)^{(i-b+1)(\alpha)}\delta^b_{j+k-1}\\
&= \delta^\alpha_\beta\delta^\alpha_\gamma\left(E^{-1}\right)^{(i-j-k+2)(\alpha)}. \qquad (9.4)
\end{aligned}
$$

Proposition 9.3. *The Christoffel symbols of the dual connection ∇^* are given by*

$$
\begin{aligned}
\Gamma^{*k(\alpha)}_{\ \ i(\beta)j(\gamma)} &= \Gamma^{k(\alpha)}_{i(\beta)j(\gamma)} - \delta^\alpha_\beta\delta^\alpha_\gamma\left(E^{-1}\right)^{(k-i-j+2)(\alpha)}\left[\delta^k_1\left(1-\sum_{\sigma\neq\alpha}m_\sigma\varepsilon_\sigma\right)\right.\\
&\quad\left.+\left(1-\delta^k_1\right)\left(1-\sum_{\tau=1}^{r}m_\tau\varepsilon_\tau\right)\right] - \left(1-\delta^\alpha_\beta\right)\delta_{\beta\gamma}\,\delta^1_i\delta^1_j\delta^k_1\,\frac{m_\beta\varepsilon_\beta}{u^{1(\beta)}} \qquad (9.5)
\end{aligned}
$$

for every choice of α, β, $\gamma\in\{1,\ldots,r\}$ and every $k\in\{1,\ldots,m_\alpha\}$, $i\in\{1,\ldots,m_\beta\}$, $j\in\{1,\ldots,m_\gamma\}$.

Proof. The proof follows from (3.4) and lemma 8.10. □

10. Generalized Lenard–Magri chains and the principal hierarchy

The main idea at the origin of the present paper was to identify the integrable hierarchy obtained applying the construction of [19] starting from a $(1,1)$-tensor field L with vanishing Nijenhuis torsion with a set of symmetries of the principal hierarchy associated with a bi-flat F-manifold, for each of the canonical forms found by David and Hertling in [9] for $L=E\circ$ in the case of regular F-manifolds with Euler vector field. This amounts to require that all the tensor fields

$$
\begin{aligned}
V_0 &= X_{(0)}\circ = e\circ = I\\
V_1 &= X_{(1)}\circ = (E-a_0\,e)\circ = L-a_0I\\
V_2 &= X_{(2)}\circ = (E\circ E-a_0\,E-a_1\,e)\circ = L^2-a_0L-a_1I\\
&\ \ \vdots\\
V_{k+1} &= X_{(k+1)}\circ = LV_k-a_kI = \left(E^{k+1}-a_0E^k-a_1E^{k-1}+\cdots-a_ke\right)\circ\\
&= L^{k+1}-a_0L^k-a_1L^{k-1}+\cdots-a_kI
\end{aligned}
$$

defined recursively by

$$
da_{k+1} = d_La_k - a_kda_0, \qquad a_0 = \sum_{\alpha=1}^{r} m_\alpha\varepsilon_\alpha u^{1(\alpha)},
$$

satisfy the condition $d_\nabla V_k = 0$ (see (3.7)). Actually, in order to get the connection ∇, we needed to impose only the first non-trivial condition

$$
d_\nabla\left(L-a_0I\right) = 0.
$$

In this section we will prove that the same condition is satisfied by all tensor fields V_k. In other words, as it is natural to expect, the connection ∇ is associated to the full hierarchy and not to a single special flow. To prove this fact we will use the commutativity of the associated flows [19]. According to the results of [22] the commutativity of the flows associated with V_α and V_β can be written as

$$c^r_{is}\left[\left(\mathrm{Lie}_{X_{(\alpha)}}c\right)^i_{jk}X^k_{(\beta)}-\left(\mathrm{Lie}_{X_{(\beta)}}c\right)^i_{jk}X^k_{(\alpha)}+c^i_{jk}\left[X_{(\alpha)},X_{(\beta)}\right]^k\right]$$
$$+c^r_{ij}\left[\left(\mathrm{Lie}_{X_{(\alpha)}}c\right)^i_{sk}X^k_{(\beta)}-\left(\mathrm{Lie}_{X_{(\beta)}}c\right)^i_{sk}X^k_{(\alpha)}+c^i_{sk}\left[X_{(\alpha)},X_{(\beta)}\right]^k\right]=0\,.$$

We have the following lemma.

Lemma 10.1. *The commutativity condition can be written as*

$$V^s_i\left(d_\nabla W\right)^l_{js}+V^s_j\left(d_\nabla W\right)^l_{is}+W^s_i\left(d_\nabla V\right)^l_{js}+W^s_j\left(d_\nabla V\right)^l_{is}=0. \tag{10.1}$$

where $V=X_{(\alpha)}\circ$ *and* $W=X_{(\beta)}\circ$.

The proof is a straightforward computation.

Using this lemma we can prove the following proposition.

Proposition 10.2. *The tensor fields* V_β *satisfy the condition*

$$d_\nabla V_\beta=0,\qquad \beta=2,3,4\ldots.$$

Proof. Due to the previous lemma and taking into account that $d_\nabla V_1=0$ we can assume the validity of the equation

$$V^s_i\left(d_\nabla W\right)^l_{js}+V^s_j\left(d_\nabla W\right)^l_{is}=0, \tag{10.2}$$

with $V=L-a_0I$ and $W=V_\beta$ for some fixed $\beta\geqslant 2$. We recall that

$$V^{a(\alpha)}_{b(\beta)}=\delta^\alpha_\beta\left(u^{(a-b+1)(\alpha)}\mathbb{1}_{\{a\geqslant b\}}-\delta^a_b a_0\right).$$

In particular, $V^{a(\alpha)}_{b(\beta)}=0$ whenever $\alpha\neq\beta$. Using these facts it is immediate to check that the condition (10.2) in David–Hertling canonical coordinates reads:

$$V^{s(\sigma)}_{j(\alpha)}\left(d_\nabla W\right)^{i(\gamma)}_{l(\beta)s(\sigma)}+V^{s(\sigma)}_{l(\beta)}\left(d_\nabla W\right)^{i(\gamma)}_{j(\alpha)s(\sigma)}=0.$$

Let us study its consequences. We consider the following cases:

1. $\boldsymbol{\alpha=\beta}$
2. $\boldsymbol{\alpha\neq\beta}$.

Case 1: $\alpha=\beta$. We fix the index i. In this case we have

$$0=V^{s(\sigma)}_{j(\alpha)}\left(d_\nabla W\right)^{i(\gamma)}_{l(\alpha)s(\sigma)}+V^{s(\sigma)}_{l(\alpha)}\left(d_\nabla W\right)^{i(\gamma)}_{j(\alpha)s(\sigma)}$$
$$=V^{s(\alpha)}_{j(\alpha)}\left(d_\nabla W\right)^{i(\gamma)}_{l(\alpha)s(\alpha)}+V^{s(\alpha)}_{l(\alpha)}\left(d_\nabla W\right)^{i(\gamma)}_{j(\alpha)s(\alpha)}\,.$$

We show that $\left(d_\nabla W\right)^{i(\gamma)}_{(m_\alpha-q)(\alpha)(m_\alpha-h)(\alpha)}=0$ by a double procedure of induction, over q and h. By taking $j=m_\alpha$ we get

$$0=V^{s(\alpha)}_{m_\alpha(\alpha)}\left(d_\nabla W\right)^{i(\gamma)}_{l(\alpha)s(\alpha)}+V^{s(\alpha)}_{l(\alpha)}\left(d_\nabla W\right)^{i(\gamma)}_{m_\alpha(\alpha)s(\alpha)}$$

which gives

$$\begin{aligned}0 &= V^{s(\alpha)}_{m_\alpha(\alpha)}\,(d_\nabla W)^{i(\gamma)}_{(m_\alpha-1)(\alpha)s(\alpha)} + V^{s(\alpha)}_{(m_\alpha-1)(\alpha)}\,(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)s(\alpha)}\\ &= V^{m_\alpha(\alpha)}_{m_\alpha(\alpha)}\,(d_\nabla W)^{i(\gamma)}_{(m_\alpha-1)(\alpha)m_\alpha(\alpha)} + V^{(m_\alpha-1)(\alpha)}_{(m_\alpha-1)(\alpha)}(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)(m_\alpha-1)(\alpha)}\\ &\quad + V^{m_\alpha(\alpha)}_{(m_\alpha-1)(\alpha)}(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)m_\alpha(\alpha)} = u^{2(\alpha)}(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)m_\alpha(\alpha)}\end{aligned}$$

thus $(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)m_\alpha(\alpha)} = 0$ for $l = m_\alpha - 1$ (we already knew it, due to the antisymmetry of $d_\nabla W$ in the lower indices),

$$\begin{aligned}0 &= V^{s(\alpha)}_{m_\alpha(\alpha)}\,(d_\nabla W)^{i(\gamma)}_{(m_\alpha-2)(\alpha)s(\alpha)} + V^{s(\alpha)}_{(m_\alpha-2)(\alpha)}\,(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)s(\alpha)}\\ &= V^{m_\alpha(\alpha)}_{m_\alpha(\alpha)}\,(d_\nabla W)^{i(\gamma)}_{(m_\alpha-2)(\alpha)m_\alpha(\alpha)} + V^{(m_\alpha-2)(\alpha)}_{(m_\alpha-2)(\alpha)}(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)(m_\alpha-2)(\alpha)}\\ &\quad + V^{(m_\alpha-1)(\alpha)}_{(m_\alpha-2)(\alpha)}(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)(m_\alpha-1)(\alpha)} + V^{m_\alpha(\alpha)}_{(m_\alpha-2)(\alpha)}(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)m_\alpha(\alpha)}\\ &= u^{2(\alpha)}(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)(m_\alpha-1)(\alpha)}\end{aligned}$$

thus $(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)(m_\alpha-1)(\alpha)} = 0$ for $l = m_\alpha - 2$ and

$$\begin{aligned}0 &= V^{s(\alpha)}_{m_\alpha(\alpha)}\,(d_\nabla W)^{i(\gamma)}_{(m_\alpha-h-1)(\alpha)s(\alpha)} + V^{s(\alpha)}_{(m_\alpha-h-1)(\alpha)}\,(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)s(\alpha)}\\ &= V^{m_\alpha(\alpha)(\alpha)}_{m_\alpha(\alpha)}\,(d_\nabla W)^{i(\gamma)}_{(m_\alpha-h-1)(\alpha)m_\alpha(\alpha)(\alpha)} + V^{(m_\alpha-h-1)(\alpha)}_{(m_\alpha-h-1)(\alpha)}(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)(m_\alpha-h-1)(\alpha)}\\ &\quad + \sum_{s=m_\alpha-h}^{m_\alpha} V^{s(\alpha)}_{(m_\alpha-h-1)(\alpha)}(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)s(\alpha)}\\ &= V^{(m_\alpha-h)(\alpha)}_{(m_\alpha-h-1)(\alpha)}(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)(m_\alpha-h)(\alpha)} + \sum_{s=m_\alpha-h+1}^{m_\alpha} V^{s(\alpha)}_{(m_\alpha-h-1)(\alpha)}(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)s(\alpha)}\end{aligned}$$

for $l = m_\alpha - h - 1$ (for a given $h \geqslant 1$). This last condition, if we inductively assume that $(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)(m_\alpha-r)(\alpha)} = 0$ for each $r \leqslant h-1$, yields $(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)(m_\alpha-h)(\alpha)} = 0$. We have just proved that $(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)j(\alpha)} = 0$ for every choice of j. We want to prove now that $(d_\nabla W)^{i(\gamma)}_{(m_\alpha-q)(\alpha)j(\alpha)} = 0$ for each choice of j for a given $q \geqslant 1$. We inductively assume that $(d_\nabla W)^{i(\gamma)}_{(m_\alpha-r)(\alpha)j(\alpha)} = 0$ for each choice of j and $r \leqslant q-1$. By taking $l = m_\alpha - q$ we get

$$0 = V^{s(\alpha)}_{j(\alpha)}\,(d_\nabla W)^{i(\gamma)}_{(m_\alpha-q)(\alpha)s(\alpha)} + V^{s(\alpha)}_{(m_\alpha-q)(\alpha)}\,(d_\nabla W)^{i(\gamma)}_{j(\alpha)s(\alpha)}$$

where $(d_\nabla W)^{i(\gamma)}_{(m_\alpha-q)(\alpha)s(\alpha)} = 0$ and $(d_\nabla W)^{i(\gamma)}_{j(\alpha)s(\alpha)} = 0$ for each $s \geqslant m_\alpha - q + 1$ thus

$$\begin{aligned}0 &= \sum_{s=j}^{m_\alpha-q} V^{s(\alpha)}_{j(\alpha)}\,(d_\nabla W)^{i(\gamma)}_{(m_\alpha-q)(\alpha)s(\alpha)} + \sum_{s=m_\alpha-q}^{m_\alpha-q} V^{s(\alpha)}_{(m_\alpha-q)(\alpha)}\,(d_\nabla W)^{i(\gamma)}_{j(\alpha)s(\alpha)}\\ &= \sum_{s=j}^{m_\alpha-q} V^{s(\alpha)}_{j(\alpha)}\,(d_\nabla W)^{i(\gamma)}_{(m_\alpha-q)(\alpha)s(\alpha)} + V^{(m_\alpha-q)(\alpha)}_{(m_\alpha-q)(\alpha)}(d_\nabla W)^{i(\gamma)}_{j(\alpha)(m_\alpha-q)(\alpha)}\end{aligned}$$

$$= V^{j(\alpha)}_{j(\alpha)}(d_\nabla W)^{i(\gamma)}_{(m_\alpha-q)(\alpha)j(\alpha)} + \sum_{s=j+1}^{m_\alpha-q} V^{s(\alpha)}_{j(\alpha)}(d_\nabla W)^{i(\gamma)}_{(m_\alpha-q)(\alpha)s(\alpha)}$$
$$+ V^{(m_\alpha-q)(\alpha)}_{(m_\alpha-q)(\alpha)}(d_\nabla W)^{i(\gamma)}_{j(\alpha)(m_\alpha-q)(\alpha)} = \sum_{s=j+1}^{m_\alpha-q} V^{s(\alpha)}_{j(\alpha)}(d_\nabla W)^{i(\gamma)}_{(m_\alpha-q)(\alpha)s(\alpha)}$$

which is trivially verified whenever $j \geqslant m_\alpha - q$ and gives

$$0 = V^{(m_\alpha-q)(\alpha)}_{(m_\alpha-q-1)(\alpha)}(d_\nabla W)^{i(\gamma)}_{(m_\alpha-q)(\alpha)(m_\alpha-q)(\alpha)}$$

thus $(d_\nabla W)^{i(\gamma)}_{(m_\alpha-q)(\alpha)(m_\alpha-q)(\alpha)} = 0$ for $j = m_\alpha - q - 1$ and

$$0 = \sum_{s=m_\alpha-t}^{m_\alpha-q} V^{s(\alpha)}_{(m_\alpha-t-1)(\alpha)}(d_\nabla W)^{i(\gamma)}_{(m_\alpha-q)(\alpha)s(\alpha)}$$
$$= V^{(m_\alpha-t)(\alpha)}_{(m_\alpha-t-1)(\alpha)}(d_\nabla W)^{i(\gamma)}_{(m_\alpha-q)(\alpha)(m_\alpha-t)(\alpha)} + \sum_{s=m_\alpha-t+1}^{m_\alpha-q} V^{s(\alpha)}_{(m_\alpha-t-1)(\alpha)}(d_\nabla W)^{i(\gamma)}_{(m_\alpha-q)(\alpha)s(\alpha)}$$

for $j = m_\alpha - t - 1$ (given some $t \geqslant 1$). This last condition, together with the inductive assumption of $(d_\nabla W)^{i(\gamma)}_{(m_\alpha-q)(\alpha)(m_\alpha-r)(\alpha)} = 0$ for each choice of $r \leqslant t-1$, yields $(d_\nabla W)^{i(\gamma)}_{(m_\alpha-q)(\alpha)(m_\alpha-t)(\alpha)} = 0$. This proves that $(d_\nabla W)^{i(\gamma)}_{(m_\alpha-q)(\alpha)j(\alpha)} = 0$ for each choice of j and in turn that $(d_\nabla W)^{i(\gamma)}_{j(\alpha)l(\alpha)} = 0$ for each choice of j and l.

Case 2: $\alpha \neq \beta$. We fix the index i. In this case we have

$$0 = V^{s(\sigma)}_{j(\alpha)}(d_\nabla W)^{i(\gamma)}_{l(\beta)s(\sigma)} + V^{s(\sigma)}_{l(\beta)}(d_\nabla W)^{i(\gamma)}_{j(\alpha)s(\sigma)} = \sum_{s=j}^{m_\alpha} V^{s(\alpha)}_{j(\alpha)}(d_\nabla W)^{i(\gamma)}_{l(\beta)s(\alpha)}$$
$$+ \sum_{s=l}^{m_\beta} V^{s(\beta)}_{l(\beta)}(d_\nabla W)^{i(\gamma)}_{j(\alpha)s(\beta)} = \left(u^{1(\alpha)} - a_0\right)(d_\nabla W)^{i(\gamma)}_{l(\beta)j(\alpha)} + \sum_{s=j+1}^{m_\alpha} V^{s(\alpha)}_{j(\alpha)}(d_\nabla W)^{i(\gamma)}_{l(\beta)s(\alpha)}$$
$$+ \left(u^{1(\beta)} - a_0\right)(d_\nabla W)^{i(\gamma)}_{j(\alpha)l(\beta)} + \sum_{s=l+1}^{m_\beta} V^{s(\beta)}_{l(\beta)}(d_\nabla W)^{i(\gamma)}_{j(\alpha)s(\beta)}$$
$$= \left(u^{1(\alpha)} - u^{1(\beta)}\right)(d_\nabla W)^{i(\gamma)}_{l(\beta)j(\alpha)} + \sum_{s=j+1}^{m_\alpha} V^{s(\alpha)}_{j(\alpha)}(d_\nabla W)^{i(\gamma)}_{l(\beta)s(\alpha)} + \sum_{s=l+1}^{m_\beta} V^{s(\beta)}_{l(\beta)}(d_\nabla W)^{i(\gamma)}_{j(\alpha)s(\beta)}$$

which is trivially verified when $\gamma \neq \alpha$, $\gamma \neq \beta$ since

$$(d_\nabla W)^{i(\gamma)}_{j(\alpha)l(\beta)} = \partial_{j(\alpha)} W^{i(\gamma)}_{l(\beta)} + \Gamma^{i(\gamma)}_{j(\alpha)s(\sigma)} W^{s(\sigma)}_{l(\beta)} - \partial_{l(\beta)} W^{i(\gamma)}_{j(\alpha)} - \Gamma^{i(\gamma)}_{l(\beta)s(\sigma)} W^{s(\sigma)}_{j(\alpha)}$$

where $W^{a(\mu)}_{b(\nu)} = 0$ whenever $\mu \neq \nu$ (because $W = V_k$ is a polynomial in L and $L^{a(\mu)}_{b(\nu)} = 0$ whenever $\mu \neq \nu$) and $\Gamma^{a(\mu)}_{b(\nu)c(\tau)} = 0$ whenever μ, ν and τ are pairwise distinct. We are then left to consider the case where $\gamma = \alpha \neq \beta$ (due to the antisymmetry of $d_\nabla W$ in the lower indices, this covers the case $\gamma = \beta \neq \alpha$ as well). We have

$$0=\left(u^{1(\alpha)}-u^{1(\beta)}\right)(d_\nabla W)^{i(\alpha)}_{l(\beta)j(\alpha)}$$
$$+\sum_{s=j+1}^{m_\alpha}V^{s(\alpha)}_{j(\alpha)}(d_\nabla W)^{i(\alpha)}_{l(\beta)s(\alpha)}+\sum_{s=l+1}^{m_\beta}V^{s(\beta)}_{l(\beta)}(d_\nabla W)^{i(\alpha)}_{j(\alpha)s(\beta)}$$

where

$$\begin{aligned}(d_\nabla W)^{i(\alpha)}_{j(\alpha)l(\beta)}&=\partial_{j(\alpha)}W^{i(\alpha)}_{l(\beta)}+\Gamma^{i(\alpha)}_{j(\alpha)s(\beta)}W^{s(\beta)}_{l(\beta)}-\partial_{l(\beta)}W^{i(\alpha)}_{j(\alpha)}-\Gamma^{i(\alpha)}_{l(\beta)s(\alpha)}W^{s(\alpha)}_{j(\alpha)}\\&=\Gamma^{i(\alpha)}_{j(\alpha)1(\beta)}W^{1(\beta)}_{l(\beta)}-\partial_{l(\beta)}W^{i(\alpha)}_{j(\alpha)}-\Gamma^{i(\alpha)}_{l(\beta)s(\alpha)}W^{s(\alpha)}_{j(\alpha)}\end{aligned}$$

trivially vanishes whenever $i<j$ ($W^{i(\alpha)}_{j(\alpha)}=0$ for $i<j$ because $W=V_k$ is a polynomial in L and $L^{i(\alpha)}_{j(\alpha)}=0$ for $i<j$). We are then left to consider $i\geqslant j$. For $i=j$ we get

$$\begin{aligned}0&=\left(u^{1(\alpha)}-u^{1(\beta)}\right)(d_\nabla W)^{j(\alpha)}_{l(\beta)j(\alpha)}\\&\quad+\sum_{s=j+1}^{m_\alpha}V^{s(\alpha)}_{j(\alpha)}(d_\nabla W)^{j(\alpha)}_{l(\beta)s(\alpha)}+\sum_{s=l+1}^{m_\beta}V^{s(\beta)}_{l(\beta)}(d_\nabla W)^{j(\alpha)}_{j(\alpha)s(\beta)}\\&=\left(u^{1(\alpha)}-u^{1(\beta)}\right)(d_\nabla W)^{j(\alpha)}_{l(\beta)j(\alpha)}+\sum_{s=l+1}^{m_\beta}V^{s(\beta)}_{l(\beta)}(d_\nabla W)^{j(\alpha)}_{j(\alpha)s(\beta)}\end{aligned}$$

which gives

$$0=\left(u^{1(\alpha)}-u^{1(\beta)}\right)(d_\nabla W)^{j(\alpha)}_{m_\beta(\beta)j(\alpha)}$$

thus $(d_\nabla W)^{j(\alpha)}_{m_\beta(\beta)j(\alpha)}=0$ for $l=m_\beta$ and

$$0=\left(u^{1(\alpha)}-u^{1(\beta)}\right)(d_\nabla W)^{j(\alpha)}_{(m_\beta-h)(\beta)j(\alpha)}+\sum_{s=m_\beta-h+1}^{m_\beta}V^{s(\beta)}_{(m_\beta-h)(\beta)}(d_\nabla W)^{j(\alpha)}_{j(\alpha)s(\beta)}$$

for $l=m_\beta-h$ (for a given $h\geqslant 1$). This last condition, together with the inductive assumption of $(d_\nabla W)^{j(\alpha)}_{j(\alpha)(m_\beta-r)(\beta)}=0$ for each choice of $r\leqslant h-1$, yields $(d_\nabla W)^{j(\alpha)}_{j(\alpha)(m_\beta-h)(\beta)}=0$. This proves $(d_\nabla W)^{j(\alpha)}_{j(\alpha)l(\beta)}=0$ for every choice of l. We inductively assume that $(d_\nabla W)^{(j+t)(\alpha)}_{j(\alpha)l(\beta)}=0$ for every l and for every $t\leqslant p-1$ (for a fixed $p\geqslant 1$). We want to show that $(d_\nabla W)^{(j+p)(\alpha)}_{j(\alpha)l(\beta)}=0$ for every l. For $i=j+p$ we get

$$0=\left(u^{1(\alpha)}-u^{1(\beta)}\right)(d_\nabla W)^{(j+p)(\alpha)}_{l(\beta)j(\alpha)}$$
$$+\sum_{s=j+1}^{m_\alpha}V^{s(\alpha)}_{j(\alpha)}(d_\nabla W)^{(j+p)(\alpha)}_{l(\beta)s(\alpha)}+\sum_{s=l+1}^{m_\beta}V^{s(\beta)}_{l(\beta)}(d_\nabla W)^{(j+p)(\alpha)}_{j(\alpha)s(\beta)}$$

where $(d_\nabla W)^{(j+p)(\alpha)}_{l(\beta)s(\alpha)}=0$ for every $s\geqslant j+1$ by the inductive hypothesis, so

$$0=\left(u^{1(\alpha)}-u^{1(\beta)}\right)(d_\nabla W)^{(j+p)(\alpha)}_{l(\beta)j(\alpha)}+\sum_{s=l+1}^{m_\beta}V^{s(\beta)}_{l(\beta)}(d_\nabla W)^{(j+p)(\alpha)}_{j(\alpha)s(\beta)}$$

which gives

$$0=\left(u^{1(\alpha)}-u^{1(\beta)}\right)(d_{\nabla}W)^{(j+p)(\alpha)}_{m_\beta(\beta)j(\alpha)}$$

thus $(d_{\nabla}W)^{(j+p)(\alpha)}_{m_\beta(\beta)j(\alpha)}=0$ for $l=m_\beta$ and

$$0=\left(u^{1(\alpha)}-u^{1(\beta)}\right)(d_{\nabla}W)^{(j+p)(\alpha)}_{(m_\beta-h)(\beta)j(\alpha)}+\sum_{s=m_\beta-h+1}^{m_\beta}V^{s(\beta)}_{(m_\beta-h)(\beta)}(d_{\nabla}W)^{(j+p)(\alpha)}_{j(\alpha)s(\beta)}$$

for $l=m_\beta-h$ (for a fixed $h\geqslant 1$). This last condition, together with the inductive assumption of $(d_{\nabla}W)^{(j+p)(\alpha)}_{(m_\beta-r)(\beta)j(\alpha)}=0$ for each choice of $r\leqslant h-1$, yields $(d_{\nabla}W)^{(j+p)(\alpha)}_{(m_\beta-h)(\beta)j(\alpha)}=0$. This proves $(d_{\nabla}W)^{(j+p)(\alpha)}_{j(\alpha)l(\beta)}=0$ for every choice of l and in turn $(d_{\nabla}W)^{i(\alpha)}_{j(\alpha)l(\beta)}=0$ for every choice of i,j and l.

This concludes the proof of the fact that (10.2) implies $d_{\nabla}W=0$ for the choice of $V=L-a_0I$ and $W=V_k$. □

11. Conclusions

In this paper we have presented a general construction of multiparameter families of regular bi-flat F-structures $(\nabla,\nabla^*,\circ,*,e,E)$ starting from a $(1,1)$-tensor field L with vanishing Nijenhuis torsion and a special solution of the equation $dd_La_0=0$.

By construction:

- the tensor field L coincides with the operator of multiplication by the Euler vector field;
- the flows defined by the tensor fields

$$\begin{aligned}V_0&=e\circ=I\\ V_1&=(E-a_0e)\circ=L-a_0I\\ V_2&=(E\circ E-a_0E-a_1e)\circ=L^2-a_0L-a_1I,\\ \cdots&=\cdots\end{aligned}$$

  with $a_1,a_2,\ldots$ defined recursively by

$$da_{k+1}=d_La_k-a_kda_0,$$

  define symmetries of the associated principal hierarchy.

Due to David-Hertling's result we have assumed L to have block-diagonal form with each block of the form

$$L_\alpha=\begin{bmatrix}u^{1(\alpha)}&0&\ldots&0\\ u^{2(\alpha)}&u^{1(\alpha)}&\ldots&0\\ \vdots&\ddots&\ddots&\vdots\\ u^{m_\alpha(\alpha)}&\ldots&u^{2(\alpha)}&u^{1(\alpha)}\end{bmatrix}. \qquad (11.1)$$

Moreover motivated by the example of semisimple Lauricella structures we have chosen a_0 as a weighted sum of the traces of the blocks defining L:

$$a_0 = \sum_{\alpha=1}^{r} \varepsilon_\alpha \mathrm{Tr}\left(L_\alpha\right).$$

Our construction works for any number of Jordan blocks and for any choice of the weights ε_α.

Some preliminary computations suggest that this construction could be further generalized considering an arbitrary linear function a_0 as starting point:

$$a_0 = \sum_{\alpha=1}^{r} \sum_{i=1}^{m_r} \varepsilon_{i(\alpha)} u^{i(\alpha)}.$$

Unfortunately in this case the computations become much more involved due to more complicated 'interaction' between different blocks and the strategy used in the present paper does not seem to work anymore.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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