Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnLine 54-9 (2021) 297–303

Computing Contraction Metrics for three-dimensional systems

P. Giesl *, **S. Hafstein** **,[1], **I. Mehrabinezhad** ***

* *Department of Mathematics, University of Sussex, Falmer BN1 9QH, United Kingdom, (e-mail: p.a.giesl@sussex.ac.uk).*
** *Faculty of Physical Sciences, University of Iceland, Dunhagi 5, IS-107 Reykjavik, Iceland, (e-mail: shafstein@hi.is)*
*** *Faculty of Physical Sciences, University of Iceland, Dunhagi 5, IS-107 Reykjavik, Iceland, (e-mail: imehrabinzhad@hi.is)*

Abstract: The basin of attraction of an equilibrium can be determined using a contraction metric, which has the advantage of being robust with respect to perturbations of the system. Recently, a novel numerical method to compute and verify a contraction metric was proposed, but so far it has only been applied to two-dimensional systems. In this paper, we apply the method to three-dimensional systems and determine a subsets of their basins of attraction and investigate the sensitivity to perturbations.

Keywords: Contraction Metrics, Lyapunov-like functions, exponential stability, basin of attraction, numerical computation

1. INTRODUCTION

Consider an ordinary differential equation (ODE) of the form

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n \tag{1.1}$$

with a C^s-vector field $f : \mathbb{R}^n \to \mathbb{R}^n$, $s \geq 3$. The solution $x(t)$ with initial value $x(0) = \xi$ is denoted by $S_t\xi := x(t)$ and is assumed to exist for all $t \geq 0$. An equilibrium of the ODE is a point $x_0 \in \mathbb{R}^n$ such that $f(x_0) = 0$, from which $x(t) = S_t x_0 = x_0$ for all $t \in \mathbb{R}$ follows. The equilibrium is said to be *exponentially stable* if there exist $\alpha, \beta, \delta > 0$ such that $\|x(0) - x_0\|_2 < \delta$ implies

$$\|x(t) - x_0\|_2 \leq \alpha \|x(0) - x_0\|_2 e^{-\beta t} \quad \text{for all } t \geq 0,$$

where $\|\cdot\|_2$ denotes the Euclidean norm; note that the definition is independent of the norm. We denote by $\mathcal{A}(x_0) = \{x \in \mathbb{R}^n : \lim_{t\to\infty} S_t x = x_0\}$ its *basin of attraction.* For a given domain in $\mathbb{R}^n$, we are interested in proving the existence, uniqueness and exponential stability of an equilibrium, as well as to determine or estimate its basin of attraction.

One method to achieve this, is to construct a Lyapunov function and determine sub-level sets. However, since the equilibrium must be an isolated local minimum of the Lyapunov function, the same Lyapunov function cannot be used for a perturbed system with a displaced equilibrium. Another method, which is robust with respect to perturbations of the system, including the equilibrium, uses a contraction metric, i.e. a Riemannian metric with respect to which the distance between adjacent solutions decreases as time increases.

Definition 1.1. (Riemannian/contraction metric).
Let K be a compact subset of an open set $G \subset \mathbb{R}^n$ and $M \in C^0(G; \mathbb{S}^{n\times n})$ be a **Riemannian metric**, i.e. a locally Lipschitz continuous matrix-valued function such that $M(x)$ is positive definite for all $x \in G$. Here $\mathbb{S}^{n\times n}$ denotes the space of symmetric $n \times n$ matrices with real entries. For $x \in K, v \in \mathbb{R}^n$ define

$$L_M(x; v) := \frac{1}{2} v^T \big[M(x)Df(x) + Df(x)^T M(x) + M'_+(x)\big] v,$$

where Df denotes the Jacobian matrix of f and the forward orbital derivative $M'_+(x)$ with respect to (1.1) at $x \in G$ is defined by

$$M'_+(x) := \limsup_{h\to 0^+} \frac{M(S_h x) - M(x)}{h}. \tag{1.2}$$

The Riemannian metric is called **contracting** in $K \subset G$ with exponent $-\nu < 0$, or a **contraction metric** on K, if

$$\mathcal{L}_M(x) \leq -\nu \text{ for all } x \in K, \text{ where} \tag{1.3}$$
$$\mathcal{L}_M(x) := \max_{v^T M(x) v = 1} L_M(x; v).$$

Note that the forward orbital derivative (1.2) is formulated using a Dini derivative similar to (Giesl and Hafstein, 2013, Definition 3.1) and always exists for each component M_{ij} in $\mathbb{R} \cup \{\infty\}$. If $M \in C^1(G; \mathbb{S}^{n\times n})$, then we can compute the orbital derivative $M'(x)$ by

$$(M'_+(x))_{ij} = (M'(x))_{ij} = (\nabla M_{ij}(x) \cdot f(x))_{ij}$$

for all $i, j \in \{1, 2, ..., n\}$.

Remark 1.2. Fix $x \in K$. Note that (1.3) is equivalent to

$$M(x)Df(x) + Df(x)^T M(x) + M'_+(x) \preceq -2\nu M(x)$$

where $A \preceq B$ for $A, B \in \mathbb{S}^{n\times n}$ means $A - B$ is negative semi-definite, i.e. $w^T(A - B)w \leq 0$ for all $w \in \mathbb{R}^n$, see (Giesl, 2015, Remark 2.5).

[1] Hafstein's research is partially supported by the Icelandic Research Fund (Rannís), grant number 163074-052, Complete Lyapunov functions: Efficient numerical computation.

10.1016/j.ifacol.2021.06.151

The next theorem from (Giesl et al., 2019, Theorem 2.6) shows how contraction metrics can be used in our study of finding equilibria and their basin of attraction.

Theorem 1.3. Let $\emptyset \neq K \subset \mathbb{R}^n$ be a compact, connected and positively invariant set and M be a Riemannian metric defined on a neighborhood G of K and contracting in K with exponent $-\nu < 0$ as in Definition 1.1. Then there exists one and only one equilibrium x_0 of system (1.1) in K; x_0 is exponentially stable and K is a subset of its basin of attraction $\mathcal{A}(x_0)$.

Hence, in addition to a contraction metric M, we need to determine a positively invariant set K. This can be achieved by a Lyapunov-like function w; note that also this function is robust with respect to perturbations of the system as $w'_+(x) < 0$ is only required in a neighbourhood of the boundary of a compact sub-level set

$$K = \{x \in G \mid w(x) \leq R\}.$$

The following theorem is a slight generalization of a standard result in the Lyapunov stability theory, namely that sub-level sets of Lyapunov functions are positively invariant.

Theorem 1.4. Let $G \subset \mathbb{R}^n$ be a domain, $w : G \to \mathbb{R}$ be locally Lipschitz, and $R \in \mathbb{R}$. Assume that the sub-level set

$$K := \{x \in G \mid w(x) \leq R\}$$

is connected and compact in $\mathbb{R}^n$, and that

$$w'_+(x) < 0$$

holds for all x in a neighbourhood of ∂K. Then K is positively invariant.

Proof: For a Lyapunov function V a corresponding theorem is proved by showing that for an initial value $x_0 \in K$, it leads to a contradiction assuming that $y = S_t x_0 \in G \setminus K$ for some $t > 0$, because $V(y) > R$ but V is strictly decreasing and thus $V(y) = V(S_t x_0) < V(x_0) \leq R$. Now, note that if $w = V$ only violates $V'_+(x) < 0$ outside of a neighbourhood $N \supset \partial K$ of the boundary, then the same proof goes through by considering $x_0^* = S_{t_1} x_0 \in K$ instead of x_0 and $y^* = S_{t_2} x_0 \in G \setminus K$, where t_1, t_2 are chosen such that $S_t y \in N$ for all $t_1 \leq t \leq t_2$. □

1.1 Numerical method

To compute a suitable contraction metric we use the method developed in Giesl et al. (2019). As this paper is focusing on new examples, we only review the essential steps briefly.

For a given pointwise positive definite $C \in C^1(G; \mathbb{S}^{n\times n})$ and using the f from (1.1) we solve numerically the matrix valued linear partial differential equation (PDE)

$$M(x)Df(x) + Df(x)^T M(x) + M'_+(x) = -C(x) \quad (1.4)$$

for M. Our numerical solution method uses collocation with compactly supported radial basis functions (RBF), in particular Wendland functions. For a given set X of collocation points and a fixed Wendland function we obtain a solution denoted by $S(x)$. This method is referred to as the RBF method.

We proceed by triangulating some set D of interest and at the vertices x_k of the triangulation we set $P(x_k) := S(x_k)$. Then we interpolate the values $P(x_k)$ over the set D to obtain a Lipschitz continuous function $P : D \to \mathbb{S}^{n\times n}$. We refer to P as CPA interpolation of the function S, because the function P is continuous and piecewise affine. In the final step, we verify the validity of a finite number of constraints, from which we obtain the information where the function P fulfills the conditions of a contraction metric. In particular we check:

Verification:

(1) whether P is positive definite on the simplex
(2) whether P fulfills the contracting property (1.3), i.e.

$$P(x)Df(x) + Df(x)^T P(x) + P'_+(x) \prec 0$$

This method is referred to as the CPA method.

In Giesl et al. (2019) it is shown that if the system (1.1) possesses a contraction metric in a set $G \subset \mathbb{R}^n$, this method, using dense enough collocation points and a fine enough triangulation, can always deliver a contraction metric $P : D \to \mathbb{S}^{n\times n}$ for any compact $D \subset G$.

The procedure to compute a Lyapunov-like function is essentially the same. Indeed, the method in Giesl et al. (2019) to compute contraction metrics was inspired by an analogous method for Lyapunov function in Giesl and Hafstein (2015). For a fixed function $\gamma : G \to \mathbb{R}_+$ we use a set of collocation points and a fixed Wendland function to obtain a numerical solution V_S to the PDE

$$V'(x) = -\gamma(x). \quad (1.5)$$

We then construct a triangulation of the subset of interest and compute the CPA interpolation w of V_S. Again, we can verify the validity of a finite number of constraints to gain information where $w^+(x) < 0$.

2. EXAMPLES

While the method developed in Giesl et al. (2019) has so far only been applied to two-dimensional examples, we will present two three-dimensional examples in this section and study them in some detail. In particular, we will show that the contraction metric is robust with respect to perturbations, which both displace the equilibrium point and alter the dynamics.

In all examples, we choose the identity matrix as the right-hand side of (1.4), i.e. $C(x) = I$ for all x, and we choose

$$\gamma(x) = \sqrt{\delta^2 + \|f(x)\|^2}$$

in (1.5) with $\delta^2 = 10^{-8}$. These choices are rather natural; $C(x) = I$ has the advantage of being very simple and the choice of γ is inspired by considering a Lyapunov function $V'(x) = -1$ for the system

$$\dot{x} = g(x) = \frac{f(x)}{\sqrt{\delta^2 + \|f(x)\|^2}}, \quad (2.1)$$

which has the same trajectories as the system (1.1), cf. e.g. Perko (2001). A Lyapunov function for (2.1) is thus also a Lyapunov function for (1.1), but (2.1) has the advantage that solution trajectories are traversed at (almost) uniform speed because $\|g(x)\| \approx 1$. This has been shown to have certain advantages when computing complete Lyapunov functions, see Argáez et al. (2018), which also apply here.

All examples were computed using a computer with an AMD Ryzen 2700X (8 cores, 3.7 GHz) processor. The method was implemented in C++ and the figures are drawn in MATLAB. Example 2.1 with $N = 1,331$ collocation points and $801^3 \approx 5.1 \cdot 10^8$ vertices was computed in 3 hours and 15 minutes and Example 2.2 with $N = 546$ collocation points and $301^3 \approx 2.7 \cdot 10^6$ vertices in 17 minutes.

Example 2.1. We consider the following system discussed in (Giesl, 2007, Example 6.4), and (Giesl and Wendland, 2019, Example 4.2)

$$\begin{cases} \dot{x} = x(x^2 + y^2 - 1) - y(z^2 + 1) \\ \dot{y} = y(x^2 + y^2 - 1) + x(z^2 + 1) \\ \dot{z} = 10z(z^2 - 1) \end{cases} \tag{2.2}$$

For this example we can analytically determine the basin of attraction of the asymptotically stable equilibrium $(0, 0, 0)$ to be

$$\mathcal{A}(0, 0, 0) = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 < 1, |z| < 1\}$$

and thus can compare the subset obtained by our method to the actual basin of attraction.

We used the Wendland function $\psi_{6,4}(cr)$ as our RBF function, given by

$$\psi_{6,4}(r) = (1 - r)_+^{10} \left(2145(r)^4 + 2250(r)^3 + 1050(r)^2 + 250r + 25 \right)$$

with parameter $c = 0.9$. The corresponding reproducing kernel space (RKHS) is H^σ with $\sigma = 4 + \frac{3+1}{2} = 6$. We used $N = 1,331$ collocation points given by

$$X = 0.13 \cdot \mathbb{Z}^3 \cap [-0.65, 0.65]^3,$$

and a uniform triangulation, cf. Hafstein and Valfells (2019), of the cube $[-0.75, 0.75]^3$ with 801^3 vertices.

The verification shows that P is positive definite in $[-0.65, 0.65]^3$. However, condition (2) is not fulfilled everywhere as shown in Figure 1. In this figure, the black dots are the collocation points, and the green surface is the boundary between the area where the verification condition (2) is satisfied and where it is not satisfied. Hence, P is a contraction metric within the set bounded by the green area.

In Figure 2, we have determined a compact, connected and positively invariant set within the domain of our computed contraction metric. We did this by computing a Lyapunov-like function w using the same set of collocation points and the same triangulation as for the computation of the contraction metric, but using the Wendland function $\psi_{5,3}(r) = (1-r)_+^8 (32(r)^3 + 25(r)^2 + 8r + 1)$ with parameter $c = 0.9$.

In this figure, blue points mark the simplices where the condition $w'_+(x) < 0$ is violated, and the red shape is a level-set of the Lyapunov-like function. The Lyapunov-like function w fulfills $w'_+(x) < 0$ in a neighbourhood of the level-set; hence, it is the boundary of a positively invariant set which is a subset of the basin of attraction by Theorems 1.3 and 1.4.

The advantage of contraction metrics compared to, e.g. Lyapunov functions, is that they, as well as the Lyapunov-like functions, are robust with respect to perturbations. To

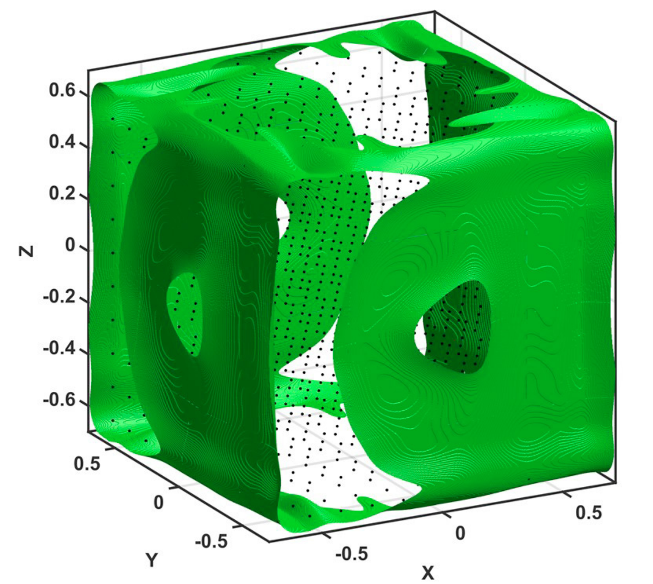

Fig. 1. Example (2.2): the black points are the collocation points and the green surface is the boundary between the area where the verification condition (2) is satisfied and where it is not satisfied. Hence, P is a contraction metric within the area bounded by the green surface.

demonstrate this we consider perturbations of the system (2.2), namely

$$\begin{cases} \dot{x} = (x - \epsilon)(x^2 + y^2 - 1) - (y + \epsilon)(z^2 + 1) \\ \dot{y} = (y + \epsilon)(x^2 + y^2 - 1) + (x - \epsilon)(z^2 + 1) \\ \dot{z} = 10(z - \epsilon)(z^2 - 1). \end{cases} \tag{2.3}$$

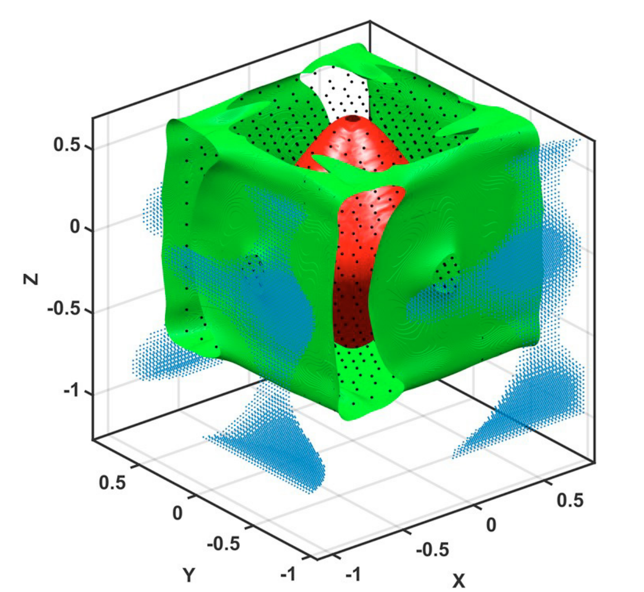

Fig. 2. Example (2.2): the black points are the collocation points and the green surface is the boundary between the area where the verification condition (2) is satisfied and where it is not satisfied. The blue points indicate where the Lyapunov-like function w fails to satisfy $w'_+(x) < 0$, and the red set is a level set of w. The level set is the boundary of the sub-level set which is thus a subset of the basin of attraction.

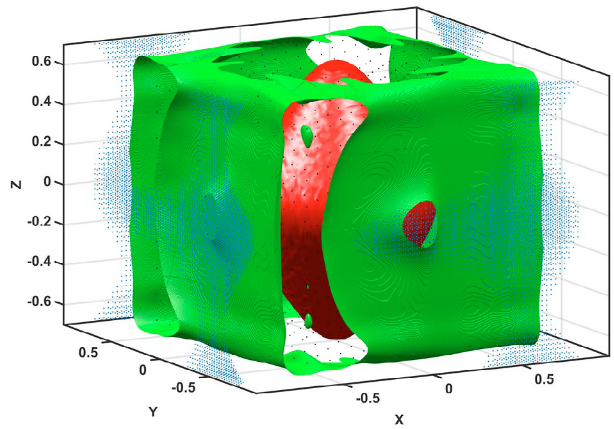

Fig. 3. Example (2.2) with small perturbation $\epsilon = 0.01$, cf. (2.3): the black points are the collocation points, used to compute the metric for the unperturbed system. The green surface is the boundary between the area where the verification condition (2) is satisfied and where it is not satisfied (notice the difference to Figure 1). The same Lyapunov-like function is also valid for the perturbed system. The blue points indicate where the Lyapunov-like function fails to satisfy $w'_+(x) < 0$ and the red set is a level set of w. The level set is the boundary of the sub-level set which is thus a subset of the basin of attraction of the perturbed system's equilibrium.

In particular, we consider a small perturbation with $\epsilon = 0.01$, and a large one with $\epsilon = 0.2$; note that in both cases the position of the equilibrium changes. We use the contraction metric and Lyapunov-like function that were computed for the unperturbed system (2.2) and check, whether and where they are still valid for the perturbed system. In particular, we check the verification condition (2), where f is replaced by the right-hand side of the perturbed system (2.3) (and so on for its derivatives); note that condition (1), the positive definiteness of P, trivially holds as it is the same metric for the unperturbed system.

For the small perturbation $\epsilon = 0.01$, both the contraction metric and the Lyapunov-like function with the same level set satisfy the conditions, and thus the same sub-level set is also a subset of the basin of attraction for the perturbed system (see Figure 3). For the larger perturbation $\epsilon = 0.2$, the contraction metric still remains valid in a large area, while the Lyapunov-like function fails to satisfy $w'_+(x) < 0$ in many more points and we are not able to find a sub-level set, to which Theorem 1.4 is applicable. Hence, we have kept the unperturbed contraction metric, but calculated a new Lyapunov-like function for the perturbed system; the results are shown in Figure 4 and 5.

Example 2.2. (balsam fir tree, moose, and wolf). In this section we consider the following system discussed in (Agarwal et al., 2019, Example 7.10)

$$\begin{cases} \dot{x} = x(1-x) - xy \\ \dot{y} = y(1-y) + xy - yz \\ \dot{z} = z(1-z) + yz \end{cases} \tag{2.4}$$

in which we denote by $x(t), y(t), z(t)$, respectively, the populations of trees, moose, and wolves at time t. The fir trees are eaten by the moose, moose are eaten by wolves, and the change in the wolf population affects the trees. The model assumes that these populations in

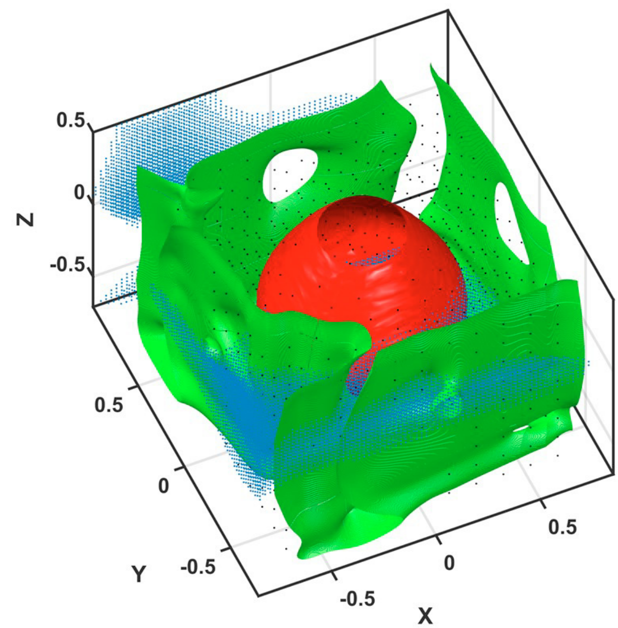

Fig. 4. Example (2.2) with large perturbation $\epsilon = 0.2$, cf. (2.3): the black points are the collocation points, used to compute the metric for the unperturbed system. The green surface is the boundary between the area where the verification condition (2) is satisfied and where it is not satisfied (notice the difference to Figure 1 and 3). The blue points are indicate where the Lyapunov-like function fails to satisfy $w'_+(x) < 0$, and the red set is a level set of w, failing to deliver a subset of the basin of attraction of the perturbed system's equilibrium because Theorem 1.4 is not applicable.

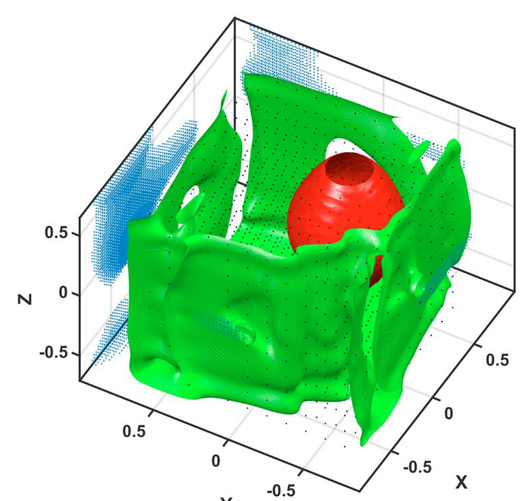

Fig. 5. Example (2.2) with large perturbation $\epsilon = 0.2$: the black points are the collocation points, used to compute the metric for the unperturbed system. The green surface is the boundary between the area where the verification condition (2) is satisfied and where it is not satisfied (notice the difference to Figure 1 and 3). A new Lyapunov-like function w_ϵ was calculated for the perturbed system. The blue points are indicate where the new Lyapunov-like function fails to satisfy $(w_\epsilon)'_+(x) < 0$, and the red set is a level set of w_ϵ. The level set is the boundary of the sub-level set which is thus a subset of the basin of attraction of the perturbed system's equilibrium.

isolation are subject to logistic growth and the effect of interaction between species is proportional to the product of the populations. Further, for simplicity, we assume that all the parameters, namely, the growth rate, the carrying capacity, and the effect of iteration, on which the behaviour of solutions depends, are 1.

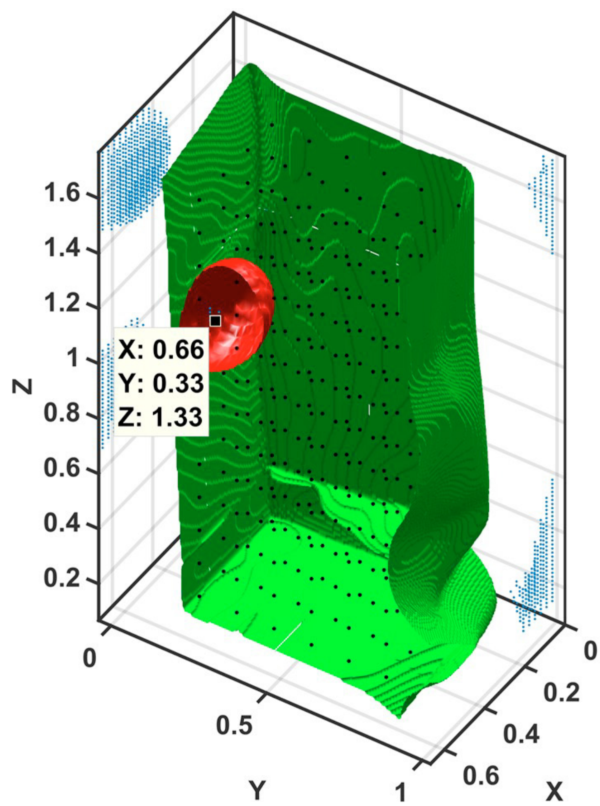

Fig. 6. Example (2.4): the black points are the collocation points and the green surface is the boundary between the area where the verification condition (2) is satisfied and where it is not satisfied. Hence, P is a contraction metric within the area bounded by the green set. Additionally, the blue points indicate where the Lyapunov-like function fails to satisfy $w'_+(x) < 0$, and the red set is a level set of w. The level set is the boundary of the sub-level set which is thus a subset of the basin of attraction of the equilibrium at $(2/3, 1/3, 4/3)$. The figure displays only a part of the phase space to show more details.

The equilibria of the system are $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 0, 1)$, and $(2/3, 1/3, 4/3)$. The last one has all populations present, and turns out to be stable. We are interested in its basin of attraction.

For the calculation of the contraction metric we have used the Wendland function $\psi_{6,4}(cr)$ with $c = 0.9$, as in the previous example, and $N = 576$ collocation points,

$$X = 0.12 \cdot \mathbb{Z}^3 \cap \left\{(0.15, 0.85)^2 \times (0.15, 2.15)\right\},$$

and for the verification we have used a uniform triangulation of the area

$$[-0.05, 1.1] \times [-0.05, 1.1] \times [0.15, 2.5]$$

with 301^3 vertices. The Lyapunov-like function was computed as in the last example and with the same parameters.

Figure 6 shows the collocation points in black and the boundary of the area where the verification condition (2) is satisfied in green; condition (1) holds in the whole area displayed. The blue points indicate where the Lyapunov-like function w fails to satisfy the condition $w'_+(x) < 0$ and the red set is a level-set of w. The red set is thus the boundary of a sub-level set which is a subset of the basin of attraction of the equilibrium, cf. Theorem 1.4

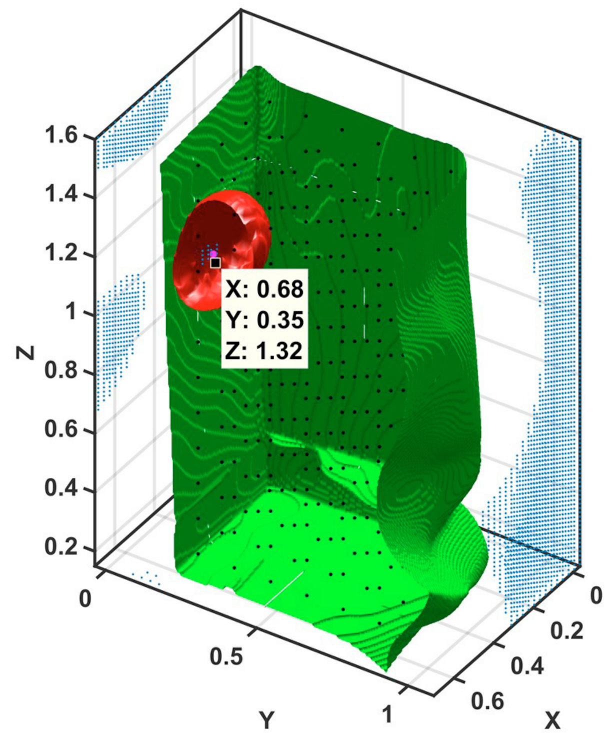

Fig. 7. Example (2.5) with small perturbation $\epsilon = -0.03$, cf. (2.5): the black points are the collocation points and the green surface is the boundary between the area where the verification condition (2) is satisfied and where it is not satisfied. The magenta point is the equilibrium point of the unperturbed system. The same Lyapunov-like function is also valid for the perturbed system. The blue points indicate where the Lyapunov-like function fails to satisfy $w'_+(x) < 0$, and the red set is a level set of w. The level set is the boundary of the sub-level set which is thus a subset of the basin of attraction of the perturbed system's equilibrium. The figure displays only a part of the phase space to show more details.

Then, we consider the perturbed system

$$\begin{cases} \dot{x} = x(1-x) - x(y+\epsilon) \\ \dot{y} = y(1-y) + (x+\epsilon)y - y(z+\epsilon) \\ \dot{z} = z(1-z) + (y+\epsilon)z \end{cases} \qquad (2.5)$$

with $\epsilon = -0.03$, and -0.1. The equilibrium point moves to $(0.68, 0.35, 1.32)$ and $(0.7, 0.4, 1.3)$, respectively.

Similar to the previous example, we keep the computed contraction metric and Lyapunov-like function of the unperturbed system and verify where they are valid for the perturbed system.

In the case of $\epsilon = -0.03$, both methods show robustness with respect to the perturbation (see Figure 7), however, when $\epsilon = -0.1$, the failing points of the Lyapunov-like

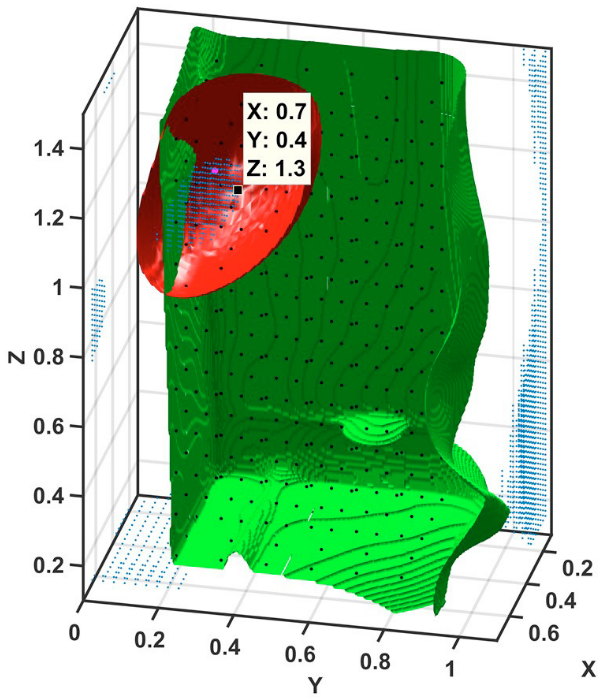

Fig. 8. Example (2.5) with large perturbation $\epsilon = -0.1$, cf. (2.5): the black points are the collocation points and the green surface is the boundary between the area where the verification condition (2) is satisfied and where it is not satisfied. Hence, P is a contraction metric within the area bounded by the green set. The magenta point is the equilibrium point of the unperturbed system. The blue points are indicate where the Lyapunov-like function fails to satisfy $w'_+(x) < 0$, and the red set is a level set of w, failing to deliver a subset of the basin of attraction of the perturbed system's equilibrium. The figure displays only a part of the phase space to show more details.

function near the equilibrium point continue toward the green boundary of the contraction metric. Therefore, it is impossible to find a level set containing the equilibrium point and the failing points around it inside the suitable area suggested by the contraction metric. As a result, we calculated a new Lyapunov-like function (see Figures 8 and 9).

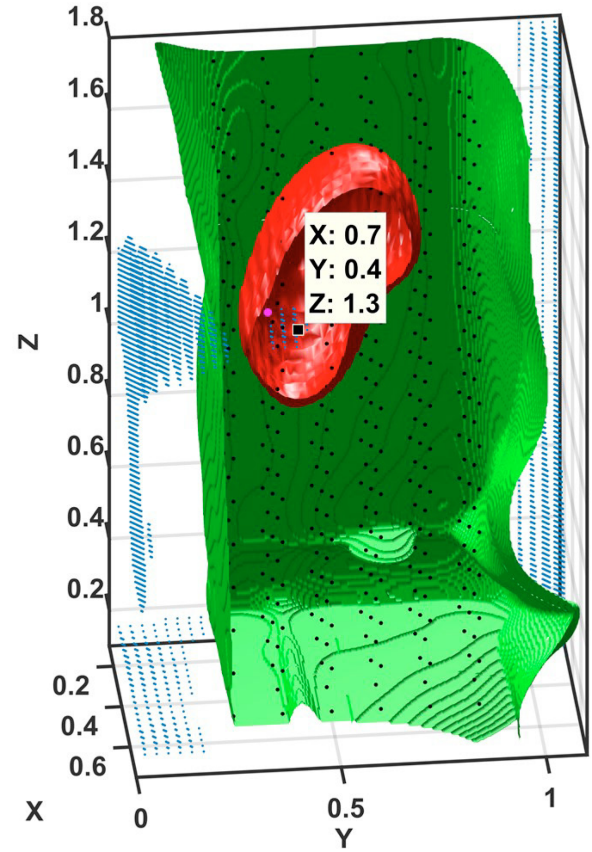

Fig. 9. Example (2.2) with large perturbation $\epsilon = -0.1$, cf. (2.5), and new Lyapunov-like function: the black points are the collocation points and the green surface is the boundary between the area where the verification condition (2) is satisfied and where it is not satisfied. Hence, P is a contraction metric within the area bounded by the green set. The magenta point is the equilibrium point of the unperturbed system. A new Lyapunov-like function w_ϵ was calculated for the perturbed system. The blue points indicate where the new Lyapunov-like function fails to satisfy $(w_\epsilon)'_+(x) < 0$, the red set is a level set of w_ϵ and the corresponding sub-level set is a subset of the basin of attraction of the perturbed system's equilibrium.

3. CONCLUSION

In this paper we have used a method to construct and verify a contraction metric and a positively invariant set, and thus were able to determine an area containing exactly one exponentially stable equilibrium, which is a subset of its basin of attraction. The advantage of this method is that it is robust with respect to perturbations of the dynamical system, including perturbing the position of the equilibrium.

We have demonstrated the ability of this method to determine a large subset of the basin of attraction for the first time for three-dimensional examples. We have also shown that the contraction metric and the Lyapunov-like function can both be used for systems with a small perturbation, and the contraction metric is even robust to larger perturbations, for which the original Lyapunov-like function fails.

The method combines the RBF method, which is fast and constructs a contraction metric by approximately solving a matrix-valued PDE with mesh-free collocation, with the CPA method, which interpolates the RBF metric by a continuous function, which is affine on each simplex of a fixed triangulation. The CPA method enables a rigorous verification that the computed metric is in fact a contraction metric.

When compared to other methods to determine the basin of attraction of an equilibrium, e.g. Lyapunov functions, the computation of a contraction metric is computationally more demanding as we construct a matrix-valued function, but it is robust with respect to perturbations of the system. Even for three-dimensional examples, however, the computations are not too demanding. Further work will include optimization of the numerical code to tackle higher-dimensional examples.

REFERENCES

Agarwal, R.P., Hodis, S., and O'Regan, D. (2019). *500 Examples and Problems of Applied Differential Equations*. Springer.

Argáez, C., Giesl, P., and Hafstein, S. (2018). Computational approach for complete Lyapunov functions. In *Dynamical Systems in Theoretical Perspective. Springer Proceedings in Mathematics & Statistics. ed. Awrejcewicz, J.*, volume 248, 1–11.

Giesl, P. (2007). *Construction of Global Lyapunov Functions Using Radial Basis Functions*, volume 1904 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin.

Giesl, P. (2015). Converse theorems on contraction metrics for an equilibrium. *J. Math. Anal. Appl.*, (424), 1380–1403.

Giesl, P. and Hafstein, S. (2013). Construction of a CPA contraction metric for periodic orbits using semidefinite optimization. *Nonlinear Anal.*, 86, 114–134.

Giesl, P. and Hafstein, S. (2015). Computation and verification of Lyapunov functions. *SIAM Journal on Applied Dynamical Systems*, 14(4), 1663–1698.

Giesl, P. and Wendland, H. (2019). Construction of a contraction metric by meshless collocation. *Discrete Contin. Dyn. Syst. Ser. B*, 24(8), 3843–3863.

Giesl, P., Hafstein, S., and Mehrabinezhad, I. (2019). Computation and verification of contraction metrics for exponentially stable equilibria. *arXiv preprint arXiv:1909.10334*.

Hafstein, S. and Valfells, A. (2019). Efficient computation of Lyapunov functions for nonlinear systems by integrating numerical solutions. *Nonlinear Dynamics*, 97(3), 1895–1910.

Perko, L. (2001). *Differential Equations and Dynamical Systems*, volume 7 of *Texts in Applied Mathematics*. Springer, 3rd edition.