Journal of Algebraic Combinatorics (2023) 57:329–348
https://doi.org/10.1007/s10801-022-01176-5

On the number of fixed edges of automorphisms of vertex-transitive graphs of small valency

Marco Barbieri[1] · Valentina Grazian[2] · Pablo Spiga[2]

Received: 1 March 2022 / Accepted: 13 September 2022 / Published online: 27 September 2022
© The Author(s) 2022

Abstract
We prove that, if Γ is a finite connected 3-valent vertex-transitive, or 4-valent vertex- and edge-transitive graph, then either Γ is part of a well-understood family of graphs, or every non-identity automorphism of Γ fixes at most $1/3$ of the edges. This answers a question proposed by Primož Potočnik and the third author.

Keywords Valency 3 · Valency 4 · Vertex-transitive · Arc-transitive · fixed-points

Mathematics Subject Classification 05C25 · 20B25

1 Introduction

Potočnik and Spiga have proved in [11] that, if Γ is a finite connected 3-valent vertex-transitive graph, or a 4-valent vertex- and edge-transitive graph then, unless Γ belongs to a well-known family of graphs, every non-identity automorphism of Γ fixes at most $1/3$ of the vertices. In the same work, they have proposed a similar investigation with respect to the edges of the graph, see [11, Problem 1.7]. In this paper we solve this problem.

✉ Marco Barbieri
marco.barbieri07@universitadipavia.it

Valentina Grazian
valentina.grazian@unimib.it

Pablo Spiga
pablo.spiga@unimib.it

1 Dipartimento di Matematica 'Felice Casorati', University of Pavia, Via Ferrata 5, 27100 Pavia, Italy

2 Dipartimento di Matematica e Applicazioni, University of Milano-Bicocca, Via Cozzi 55, 20125 Milan, Italy

Theorem 1 *Let Γ be a finite connected 4-valent vertex- and edge-transitive graph admitting a non-identity automorphism fixing more than $1/3$ of the edges. Then one of the following holds:*

1. *Γ is isomorphic to the complete graph on 5 vertices;*
2. *Γ is isomorphic to a Praeger–Xu graph $C(r, s)$, for some r and s with $3s < 2r - 3$.*

Theorem 2 *Let Γ be a finite connected 3-valent vertex-transitive graph admitting a non-identity automorphism fixing more than $1/3$ of the edges. Then Γ is isomorphic to a split Praeger–Xu graph $SC(r, s)$, for some r and s with $3s < 2r - 2$.*

We refer to Sect. 2.3 for the definition of the ubiquitous Praeger–Xu graphs and for their splitting. The bound in Theorem 2 is sharp. For instance, each 3-valent graph admitting a non-identity automorphism fixing elementwise a complete matching has the aforementioned property. For valency 4, we conjecture that the bound $1/3$ in Theorem 1 can be strengthened to $1/4$, by eventually including some more small exceptional graphs in part (1).

Theorems 1 and 2 rely on the following group-theoretic fact:

Theorem 3 [10, Theorem 1.1] *Let G be a finite transitive permutation group on Ω containing no non-identity normal subgroup of order a power of 2. Suppose there exists $\omega \in \Omega$ such that the stabilizer G_ω of ω in G is a 2-group. Then, every non-identity element of G fixes at most $1/3$ of the points.*

The main results of this paper and the results in [11] show that, besides small exceptions or well-understood families of graphs, non-identity automorphisms of 3-valent or 4-valent vertex-transitive graphs cannot fix many vertices or edges, where “too many” in this context has to be considered as a linear function on the number of vertices (and, even then, with a small caveat for 4-valent graphs, because of the assumption of edge-transitivity). In our opinion, the difficulty in having a unifying theory of vertex-transitive graphs of small valency admitting non-identity automorphisms fixing too many vertices or edges is due to our lack of understanding possible generalizations of Praeger–Xu graphs, that is, a family of vertex-transitive graphs of bounded valency playing the role of Praeger–Xu graphs. It seems to us that this is a recurrent problem in the theory of groups acting on finite graphs of bounded valency. A general investigation in this direction, but with much weaker bounds and only for arc-transitive graphs, is in [7].

Investigations on the number of fixed points of graph automorphisms do have interesting applications. For instance, very recently Potočnik, Toledo and Verret [14] pivoting on the results in [11] have proved remarkable results on the cycle structure of general automorphisms of 3-valent vertex-transitive and 4-valent arc-transitive graphs.

1.1 Structure of the paper

In Sect. 2, we introduce some basic terminology and, in particular, we introduce the Praeger–Xu graphs and their splitting. Then, we start in Sect. 3 with some preliminary results. In Sect. 4, we prove Theorem 1 and, in Sect. 5, we prove Theorem 2.

2 The players

2.1 Basic group-theoretic notions

Given a permutation g on a set Ω, we write $\mathrm{Fix}(\Omega, g)$ for the set of **fixed points of** g, i.e.

$$\mathrm{Fix}(\Omega, g) = \left\{\omega \in \Omega \mid \omega^g = \omega\right\},$$

and we write $\mathrm{fpr}(\Omega, g)$ for the **fixed-point ratio of** g, i.e.

$$\mathrm{fpr}(\Omega, g) = \frac{|\mathrm{Fix}(\Omega, g)|}{|\Omega|}.$$

A permutation group G on Ω is said to be **semiregular** if the identity is the only element fixing some point. When G is semiregular and transitive on Ω, the group G is **regular** on Ω.

Given a permutation group G of Ω and a partition Σ of Ω, we say that Σ is G-**invariant** if $\sigma^g \in \pi$, for every $\sigma \in \Sigma$. Given a normal subgroup N of G, the orbits of N on Ω form a G-invariant partition, which we denote by Ω/N.

We present here a useful lemma involving the notion just defined.

Lemma 1 [11, Lemma 1.17] *Let G be a group acting transitively on Ω and let Σ be a G-invariant partition of Ω. For $g \in G$, let g^Σ be the permutation of Σ induced by g. Then* $\mathrm{fpr}(\Omega, g) \leq \mathrm{fpr}(\Sigma, g^\Sigma)$. *In particular, if $N \trianglelefteq G$, then* $\mathrm{fpr}(\Omega, g) \leq \mathrm{fpr}(\Omega/N, Ng)$.

2.2 Basic graph-theoretic notions

In this paper, a **digraph** is a binary relation

$$\Gamma = (V\Gamma, A\Gamma),$$

where $A\Gamma \subseteq V\Gamma \times V\Gamma$. We refer to the elements of $V\Gamma$ as **vertices** and to the elements of $A\Gamma$ as arcs. A **graph** is a finite simple undirected graph, i.e. a pair

$$\Gamma = (V\Gamma, E\Gamma),$$

where $V\Gamma$ is a finite set of **vertices**, and $E\Gamma$ is a set of unordered pairs of $V\Gamma$, called **edges**. In particular, a graph can be thought of as a digraph where the binary relation is symmetric and contains no loops. Given a non-negative integer s, an s-**arc** of Γ is an ordered set of $s+1$ adjacent vertices with any three consecutive elements pairwise distinct. When $s = 0$, an s-arc is simply a vertex of Γ; when $s = 1$, an s-arc is simply an **arc**, that is, an oriented edge.

The **girth** of Γ, denoted by $g(\Gamma)$, is the minimum length of a cycle in Γ.

We denote by $\Gamma(v)$ the **neighbourhood** of the vertex v. The size of $|\Gamma(v)|$ is the **valency** of v. We are mainly dealing with **regular** graphs, that is, with graphs where $|\Gamma(v)|$ is constant as v runs through the elements of $V\Gamma$. In these cases, we refer to the valency of the graph.

Let Γ be a graph, let G be a subgroup of the automorphism group $\mathrm{Aut}(\Gamma)$ of Γ, let $v \in V\Gamma$ and let $w \in \Gamma(v)$. We denote by G_v the **stabilizer** of the vertex v, by $G_{\{v,w\}}$ the setwise stabilizer of the edge $\{v, w\}$, by G_{vw} the pointwise stabilizer of the edge $\{v, w\}$ (that is, the stabilizer of the arc (v, w) underlying the edge $\{v, w\}$). The group G_v acts on $\Gamma(v)$ and we denote by $G_v^{[1]}$ the kernel of the action of G_v on $\Gamma(v)$. Now, the permutation group induced by G_v on $\Gamma(v)$ is denoted by $G_v^{\Gamma(v)}$ and we have

$$G_v^{\Gamma(v)} \cong \frac{G_v}{G_v^{[1]}}.$$

When G acts transitively on the set of s-arcs of Γ, we say that G is s-**arc-transitive**. When $s = 0$, we say that G is **vertex-transitive** and, when $s = 1$, we say that G is **arc-transitive**. Moreover, when G acts regularly on the set of s-arcs of Γ we emphasize this fact by saying that G is s-**arc-regular**.

When G acts transitively on $E\Gamma$, we say that G is **edge-transitive**. Finally, when G is edge- and vertex-transitive, but not arc-transitive, we say that G is **half-arc-transitive**. This name comes from the fact that G has two orbits on ordered pairs of adjacent vertices of Γ (a.k.a. arcs), each orbit containing precisely one of the two arcs underlying each edge.

We say that Γ is vertex-, edge- or arc-transitive when $\mathrm{Aut}(\Gamma)$ is vertex-, edge- or arc-transitive.

Let G be a finite group and let S be a subset of G. The **Cayley digraph** on G with connection set S is the digraph $\Gamma := \mathrm{Cay}(G, S)$ having vertex set G and where $(g, h) \in A\Gamma$ if and only if $gh^{-1} \in G$. Now, $\mathrm{Cay}(G, S)$ is a symmetric binary relation if and only if S is inverse closed, that is, $S = S^{-1}$ where $S^{-1} := \{s^{-1} \mid s \in S\}$. Observe that the right regular representation of G acts as a group of automorphisms on $\mathrm{Cay}(G, S)$.

2.3 Praeger–Xu graphs

In this and in the next section, we introduce the infinite families of graphs appearing in our main theorems. We introduce the 4-valent **Praeger-Xu graphs** $C(r, s)$ through their directed counterpart defined in [16]. Further details on Praeger–Xu graphs can be found in [2, 4, 17]. We also advertise [5], where the authors have begun a thorough investigation of Praeger–Xu graphs, motivated by the recurrent appearance of these objects in the theory of groups acting on graphs.

Let $r \geq 3$ be an integer. Then $\mathbf{C}(r, 1)$ is the lexicographic product of a directed cycle of length r with the edgeless graph on 2 vertices. In other words, $V\mathbf{C}(r, 1) = \mathbb{Z}_r \times \mathbb{Z}_2$, and the two arcs starting in (x, i) end in $(x + 1, 0)$ and in $(x + 1, 1)$. For any $2 \leq s \leq r - 1$, $V\mathbf{C}(r, s)$ is defined as the set of all $(s - 1)$-arcs of $\mathbf{C}(r, 1)$, and $(v_0, v_1, \ldots, v_{s-1}) \in V\mathbf{C}(r, s)$ is the beginning point of the two arcs ending in

$(v_1, v_2, \ldots, v_{s-1}, u)$ and in $(v_1, v_2, \ldots, v_{s-1}, u')$, where u and u' are the two vertices of $\mathbf{C}(r, 1)$ that prolong the $(s-1)$-arc $(v_1, v_2, \ldots, v_{s-1})$. The Praeger–Xu graph $C(r, s)$ is then defined as the non-oriented underlying graph of $\mathbf{C}(r, s)$. It can be verified that $C(r, s)$ is a connected 4-valent graph with $r2^s$ vertices and $r2^{s+1}$ edges.

We describe the automorphisms of $C(r, s)$. Some automorphism of $C(r, s)$ arises from the action of $\mathrm{Aut}(C(r, 1))$ on the set of s-arcs of $C(r, 1)$. Let $i \in \mathbb{Z}_r$ and let τ_i be the transposition on $V\mathbf{C}(r, 1)$ swapping the vertices $(i, 0)$ and $(i, 1)$ and fixing the remaining vertices. Since τ_i is an automorphism of $\mathbf{C}(r, 1)$, it is immediate to extended the action of τ_i to $C(r, 1)$ and to $C(r, s)$. We define the group

$$K = \langle \tau_i \mid i \in \mathbb{Z}_r \rangle \cong C_2^r,$$

and throughout this paper the symbol K will always refer to this group for some $C(r, s)$. Focusing on the cyclic nature of the Praeger-Xu graphs, it is also natural to define on $V\mathbf{C}(r, 1)$ the permutations ρ and σ as follows

$$(x, i)^\rho = (x+1, i), \qquad \text{and} \qquad (x, i)^\sigma = (-x, i).$$

While ρ is an automorphism of $\mathbf{C}(r, 1)$, σ is an automorphism of $C(r, 1)$ but not of $V\mathbf{C}(r, 1)$. Moreover, observe that the group $\langle \rho, \sigma \rangle$ normalizes K. Define

$$H = K\langle \rho, \sigma \rangle, \qquad \text{and} \qquad H^+ = K\langle \rho \rangle,$$

and, as for K, the symbols H and H^+ will always refer to these groups. Clearly $H \cong C_2 \wr D_r$ is a group of automorphisms of $C(r, s)$ and $H^+ \cong C_2 \wr C_r$ is a group of automorphisms of $\mathbf{C}(r, s)$. Moreover, H acts vertex- and edge-transitively on $C(r, s)$ (and so does H^+ on $\mathbf{C}(r, s)$), but not 2-arc-transitively.

Lemma 2 *Using the notation above,* $\mathrm{Aut}(\mathbf{C}(r, s)) = H^+$ *and, if* $r \neq 4$, $\mathrm{Aut}(C(r, s)) = H$. *Moreover,*

$$|\mathrm{Aut}(C(4,1)) : H| = 9, \quad |\mathrm{Aut}(C(4,2)) : H| = 3 \text{ and } |\mathrm{Aut}(C(4,3)) : H| = 2.$$

Proof It follows from [16, Theorem 2.8] and [17, Theorem 2.13] when $p = 2$. □

The Praeger–Xu graphs also admit the following algebraic characterization.

Lemma 3 *Let Γ be a finite connected 4-valent graph and let G be a vertex- and edge-transitive group of automorphisms of Γ. If G has an abelian normal subgroup which is not semiregular on $V\Gamma$, then Γ is isomorphic to a Praeger–Xu graph $C(r, s)$, for some integers r and s.*

Proof It follows by [16, Theorem 2.9] and [17, Theorem 1] upon setting $p = 2$. □

2.4 Split Praeger–Xu graphs

For our purposes, the **split Praeger–Xu graphs** are obtained from the Praeger–Xu graphs via the splitting operation which was introduced in [12, Construction 9], and which we will comment upon in Sect. 5.

Here we give an explicit description of $SC(r, s)$. Split any vertex of $\mathbf{C}(r, s)$ into two copies, say v_+ and v_-. For any arc of $\mathbf{C}(r, s)$ of the form (v, u), let v_+ be adjacent to v_- and u_-. From the complementary perspective, the neighbourhood of v_- is made up of v_+ plus the two vertices w_+ such that (w, v) is an arc of $\mathbf{C}(r, s)$.

3 Preliminary results

3.1 Graph-theoretical considerations

In this section, we develop our tool box that extends outside the scope of proving our main theorems.

Lemma 4 *Let Γ be a connected k-valent graph, with $k \geq 3$, and let G be an s-arc-transitive group of automorphisms of Γ. Then $2s \leq g(\Gamma) + 2$. In particular, the girth of Γ is greater than s.*

Proof The first part of the statement is [1, Proposition 17.2]. The second one is an immediate computation if $s \geq 2$, and it follows from $g(\Gamma) \geq 3$ if $s = 1$. □

Lemma 5 *Let Γ be a finite connected graph and let $v \in V\Gamma$ be a vertex. For each $w \in \Gamma(v)$, suppose there exists t_w automorphism of Γ such that $v^{t_w} = w$. Then $T := \langle t_w \mid w \in \Gamma(v)\rangle$ is vertex-transitive on Γ.*

Proof Let $u \in V\Gamma$. As Γ is connected, we prove the existence of $t_u \in T$ with $v^{t_u} = u$ arguing by induction on the minimal distance $d := d(v, u)$ from v to u in Γ. When $d = 0$, that is, $v = u$, we may take t_u to be the identity of T. Suppose then $d > 0$. Let $v_0, \dots, v_d$ be a path of distance d from $v = v_0$ to $u = v_d$ in Γ. Now, $d(v, v_{d-1}) = d - 1$ and hence, by induction, there exists $t \in T$ with $v^t = v_{d-1}$. Set $u' := u^{t^{-1}}$. As $u = v_d \in \Gamma(v_{d-1})$, we have

$$u' = u^{t^{-1}} \in \Gamma(v_{d-1})^{t^{-1}} = \Gamma(v_{d-1}^{t^{-1}}) = \Gamma(v).$$

By hypothesis, $t_{u'} \in T$ and $v^{t_{u'}} = u'$. Therefore, $v^{t_{u'}t} = u'^t = u$ and we may take $t_u := t_{u'}t$. □

Lemma 6 [3, Lemma 3.3.3] *Let Γ be a finite connected vertex-transitive graph of valency k. Then Γ is k-edge-connected, i.e. Γ remains connected upon eliminating any m edges, with $m \leq k - 1$.*

A general result on the fixed-point ratio of Cayley graphs can be proven regardless of the valency.

Lemma 7 *Let G be a finite group, let S be an inverse closed non-empty subset of G, let $\Gamma := \mathrm{Cay}(G, S)$ and let $g \in G \setminus \{1\}$. If $\mathrm{fpr}(E\Gamma, g) \neq 0$, then $g^2 = 1$ and*

$$\mathrm{fpr}(E\Gamma, g) = \frac{|g^G \cap S|}{|S||g^G|},$$

where $g^G := \{hgh^{-1} \mid h \in G\}$ is the conjugacy class of g in G. In particular, $\mathrm{fpr}(E\Gamma, g) \leq 1/|S|$ and the equality is attained if and only if $g^G \subseteq S$.

Proof Suppose $\mathrm{fpr}(E\Gamma, g) \neq 0$. We let $\mathbf{C}_G(g)$ denote the centralizer of g in G.

For each $s \in S$, let $E_s := \{\{x, sx\} \mid x \in G\}$. Observe that E_s is a complete matching of Γ and that $\{E_s \mid s \in S\}$ is a partition of the edge set $E\Gamma$.

Let $s \in S$. Suppose $E_s \cap \mathrm{Fix}(E\Gamma, g) \neq \emptyset$ and fix $\{\bar{x}, s\bar{x}\} \in E_s \cap \mathrm{Fix}(E\Gamma, g)$. As g fixes the edge $\{\bar{x}, s\bar{x}\}$, we have $\bar{x}g = s\bar{x}$ and $s\bar{x}g = \bar{x}$. We deduce $g^2 = 1$ and $s = \bar{x}g\bar{x}^{-1}$. In other words, g has order 2 and g has a conjugate in S. Now, for every $\{x, sx\} \in E_s$, with a similar computation, we obtain that $\{x, sx\} \in \mathrm{Fix}(E\Gamma, g)$ if and only if $s = xgx^{-1}$. Thus $\bar{x}g\bar{x}^{-1} = xgx^{-1}$ and $x \in \bar{x}\mathbf{C}_G(g)$. In particular, $E_s \cap \mathrm{Fix}(E\Gamma, g) = \{\{\bar{x}h, s\bar{x}h\} \mid h \in \mathbf{C}_G(x)\}$ and hence

$$|E_s \cap \mathrm{Fix}(E\Gamma, g)| = \frac{|\mathbf{C}_G(g)|}{2}.$$

The previous paragraph has established that g has order 2. Moreover, for each $s \in S$, $E_s \cap \mathrm{Fix}(E\Gamma, g) \neq \emptyset$ if and only if $s \in g^G$. Furthermore, in the case that $s \in g^G$, the cardinality of $E_s \cap \mathrm{Fix}(E\Gamma, g)$ does not depend on s and equals $|\mathbf{C}_G(g)|/2$. Therefore,

$$\begin{aligned}\mathrm{fpr}(E\Gamma, g) &= \frac{|g^G \cap S||\mathbf{C}_G(g)|/2}{|E\Gamma|}\\ &= \frac{|g^G \cap S||\mathbf{C}_G(g)|/2}{|S||G|/2}\\ &= \frac{|g^G \cap S|}{|S||G : \mathbf{C}_G(g)|}\\ &= \frac{|g^G \cap S|}{|S||g^G|}.\end{aligned}$$

Since $|g^G \cap S| \leq |g^G|$, we have $\mathrm{fpr}(E\Gamma, S) \leq 1/|S|$. Moreover, the equality is attained if and only if $g^G \cap S = g^G$, that is, $g^G \subseteq S$. □

The next lemma studies the nature of fixed edges in a Praeger–Xu graph.

Lemma 8 *Let $\Gamma = C(r, s)$ be a Praeger–Xu graph and let $g \in \mathrm{Aut}(\Gamma)$ with $g \neq 1$ and with $\mathrm{fpr}(E\Gamma, g) > 1/3$. Then $3s < 2r - 3$ and, either $g \in K$ or $(r, s) = (4, 1)$. In particular, g fixes an edge if and only if g fixes both of its ends. (The group K is defined in Sect. 2.3.)*

Proof The lexicographic product $C(4, 1) \cong K_{4,4}$ admits automorphisms h fixing 8 edges and hence $\mathrm{fpr}(E\Gamma, h) = 8/16 = 1/2 > 1/3$. (The non-identity elements h in $\mathrm{Aut}(C(4, 1))$ with $\mathrm{fpr}(E\Gamma, h) > 1/3$ are not necessarily in K, but they fix an edge if and only if they fix both of its ends.) Similarly, it can be verified that, for every $h \in \mathrm{Aut}(C(4, 2))$ with $h \neq 1$, we have $\mathrm{fpr}(E\Gamma, h) \leq 8/32 = 1/4$. Furthermore, for every $h \in \mathrm{Aut}(C(4, 3))$ with $h \neq 1$, we have $\mathrm{fpr}(E\Gamma, h) = 8/64 = 1/8$. In particular, when $r = 4$, the result follows from these computations.

Suppose $r \neq 4$. By Lemma 2, $\mathrm{Aut}(\Gamma) = H = K\langle \rho, \sigma\rangle$. In particular,

$$g = \tau\rho^i\sigma^\varepsilon, \qquad \text{for some } \tau \in K, i \in \mathbb{Z}_r, \varepsilon \in \mathbb{Z}_2.$$

Denote by Δ_x the set of $(s-1)$-arcs in $\mathbf{C}(r, 1)$ starting at $(x, 0)$ or at $(x, 1)$. From the definition of the vertex set of $C(r, s)$, we have $\Delta_x \subseteq VC(r, s)$, $|\Delta_x| = 2^s$ and

$$VC(r, s) = \bigcup_{x\in\mathbb{Z}_r} \Delta_x.$$

We claim that the subgraph induced by Γ on $\Delta_x \cup \Delta_{x+1}$ is the disjoint union of cycles of length 4. In fact, consider the $(s-1)$-arcs in Δ_x parameterized as

$$\begin{aligned}&((x, 0), (x + 1, y_1), (x + 2, y_2), \ldots, (x + s - 1, y_{s-1})) \quad \text{and}\\ &((x, 1), (x + 1, y_1), (x + 2, y_2), \ldots, (x + s - 1, y_{s-1}))\,,\end{aligned}$$

for some $y_i \in \mathbb{Z}_2$. In Γ, they are both adjacent to

$$\begin{aligned}&((x + 1, y_1), (x + 2, y_2), \ldots, (x + s - 1, y_{s-1}), (x + s, 0)) \quad \text{and}\\ &((x + 1, y_1), (x + 2, y_2), \ldots, (x + s - 1, y_{s-1}), (x + s, 1))\,.\end{aligned}$$

Since the induced subgraph is 2-valent, these elements form a cycle of length 4, which is a connected component of the induced graph. Moreover, Δ_x is a K-orbit, and, for any $x \in \mathbb{Z}_r$,

$$\Delta_x^\rho = \Delta_{x+1}, \qquad\qquad \Delta_x^\sigma = \Delta_{-x-s+1}. \tag{3.1}$$

We start by proving that $g \in K$.

Suppose $\varepsilon = 0$. Let $\{a, b\} \in \mathrm{Fix}(E\Gamma, g)$. Replacing a with b if necessary, we may suppose that $a \in \Delta_x$ and $b \in \Delta_{x+1}$, for some $x \in \mathbb{Z}_r$. If $a^g = a$ and $b^g = b$, we have $\Delta_x^g = \Delta_x$ and $\Delta_{x+1}^g = \Delta_{x+1}$. Now, (3.1) yields $x + i = x$ and $(x + 1) + i = x + 1$, that is, $i = 0$. Therefore $g \in K$. Similarly, if $a^g = b$ and $b^g = a$, we have $\Delta_x^g = \Delta_{x+1}$ and $\Delta_{x+1}^g = \Delta_x$. Now, (3.1) yields $x + i = x + 1$ and $(x + 1) + i = x$, that is, $2 = 0$. However, this implies $r = 2$, which is a contradiction because $r \geq 3$.

Suppose $\varepsilon = 1$. Since $\langle \rho, \sigma\rangle$ is a dihedral group of order $2r$, replacing g by a suitable conjugate if necessary, we may suppose that either r is odd and $i = 0$, or r is even and $i \in \{0, 1\}$.

Assume $i = 0$. Let $\{a, b\} \in \mathrm{Fix}(E\Gamma, g)$. As above, replacing a with b if necessary, we may suppose that $a \in \Delta_x$ and $b \in \Delta_{x+1}$, for some $x \in \mathbb{Z}_r$. If $a^g = a$ and $b^g = b$, we have $\Delta_x^g = \Delta_x$ and $\Delta_{x+1}^g = \Delta_{x+1}$. Now, (3.1) yields $-x - s + 1 = x$ and $-(x+1) - s + 1 = x + 1$, that is, $2 = 0$. However, this gives rise to the contradiction $r = 2$. Similarly, if $a^g = b$ and $b^g = a$, we have $\Delta_x^g = \Delta_{x+1}$ and $\Delta_{x+1}^g = \Delta_x$. Now, (3.1) yields $-x - s + 1 = x + 1$ and $-(x+1) - s + 1 = x$, that is, $2x + s = 0$. When r is odd, the equation $2x + s = 0$ has only one solution in $\mathbb{Z}_r$ and, when r is even, the equation $2x + s = 0$ has either zero or two solutions in $\mathbb{Z}_r$ depending on whether s is odd or even. Recalling that the subgraph induced by Γ on $\Delta_x \cup \Delta_{x+1}$ is a disjoint union of cycles of length 4, and noticing that g fixes at most 2 edges of any cycle, we obtain that

$$\mathrm{fpr}(E\Gamma, g) \leq \begin{cases} \frac{|\Delta_x|}{|E\Gamma|} = \frac{1}{2r} & \text{if } r \text{ is odd,} \\ 2 \cdot \frac{|\Delta_x|}{|E\Gamma|} = \frac{1}{r} & \text{if } r \text{ is even.} \end{cases}$$

In both cases, we have $\mathrm{fpr}(E\Gamma, g) \leq 1/4$, which is a contradiction.

Assume $i = 1$. Observe that this implies that r is even. Here the analysis is entirely similar. Let $\{a, b\} \in \mathrm{Fix}(E\Gamma, g)$. As above, replacing a with b if necessary, we may suppose that $a \in \Delta_x$ and $b \in \Delta_{x+1}$, for some $x \in \mathbb{Z}_r$. If $a^g = a$ and $b^g = b$, we have $\Delta_x^g = \Delta_x$ and $\Delta_{x+1}^g = \Delta_{x+1}$. Now, (3.1) yields $-(x+1) - s + 1 = x$ and $-(x+2) - s + 1 = x$, that is, $2 = 0$. However, this gives rise to the usual contradiction $r = 2$. Similarly, if $a^g = b$ and $b^g = a$, we have $\Delta_x^g = \Delta_{x+1}$ and $\Delta_{x+1}^g = \Delta_x$. Now, (3.1) yields $-(x+1) - s + 1 = x + 1$ and $-(x+2) - s + 1 = x$, that is, $2x + s + 1 = 0$. As r is even, the equation $2x + s + 1$ has either zero or two solutions in $\mathbb{Z}_r$ depending on whether s is even or odd. Recalling that the subgraph induced by Γ on $\Delta_x \cup \Delta_{x+1}$ is a disjoint union of cycles of length 4, and noticing that g fixes at most 2 edges of any cycle, we obtain that

$$\mathrm{fpr}(E\Gamma, g) \leq 2 \cdot \frac{|\Delta_x|}{|E\Gamma|} = \frac{1}{r}.$$

Thus, we have $\mathrm{fpr}(E\Gamma, g) \leq 1/4$, which is a contradiction.

Since $g \in K$, if g fixes the edge $\{a, b\} \in E\Gamma$, then g fixes both end-vertices a and b. It remains to show that $3s < 2r - 3$. Notice that τ_i moves precisely those $(s-1)$-arcs of $\mathbf{C}(r, 1)$ that pass through one of the vertices $(i, 0)$ or $(i, 1)$. Therefore, τ_i, as an automorphism of $C(r, s)$, fixes all but $s2^s$ vertices, thus it fixes all but those $(s+1)2^{s+1}$ edges which are incident with such vertices. Since any element in K is obtained as a product of some τ_i, such an element fixes at most as many edges as a single τ_i. Hence

$$\frac{1}{3} < \mathrm{fpr}(E\Gamma, g) \leq \mathrm{fpr}(E\Gamma, \tau_i) = \frac{(r - (s+1))\, 2^{s+1}}{r 2^{s+1}} = \frac{r - s - 1}{r}.$$

□

Lemma 9 *Let $\Gamma = C(r, s)$ be a Praeger–Xu graph, let G be a vertex- and edge-transitive group of automorphism of Γ containing a non-identity element g fixing more than $1/3$ of the edges and with G not 2-arc-transitive. Then G is $\mathrm{Aut}(\Gamma)$-conjugate to a subgroup of H as defined in Sect. 2.3.*

Proof By Lemma 8, $3s < 2r - 3$. If $r \neq 4$, then by Lemma 2 we have $G \leq \mathrm{Aut}(\Gamma) = H$. When $r = 4$, then inequality $3s < 2r - 3$ implies $s = 1$. Now, the veracity of this lemma can be verified with a computation in $\mathrm{Aut}(C(4, 1)) = \mathrm{Aut}(K_{4,4}) = S_4 \wr S_2$. □

Lemma 10 [11, Lemma 1.14] *Let Γ be a finite connected 4-valent graph, let G be a vertex- and edge-transitive group of automorphisms of Γ, and let N be a minimal normal subgroup of G. If N is a 2-group and Γ/N is a cycle of length at least 3, then Γ is isomorphic to a Praeger–Xu graph $C(r, s)$ for some integers r and s.*

4 Proof of Theorem 1

In this section we prove Theorem 1. Our proof is divided into two cases, depending on whether Γ admits a group of automorphisms acting 2-arc-transitively or not.

4.1 Proof of Theorem 1 when Γ is 2-arc-transitive

The following lemma involves four graphs not yet considered in this paper, so it is worth to spend some ink here to describe them.

- The complete graph K_5 is the only sporadic example arising in Theorem 1, its automorphism group is S_5 and each transposition in S_5 fixes 4 edges out of 10.
- The graph $K_{5,5} - 5K_2$ is obtained deleting a complete matching from the complete bipartite graph $K_{5,5}$, its automorphism group is $S_5 \times C_2$ and every non-identity automorphism fixes at most 6 edges out of 20.
- The hypercube Q_4 is the Cayley graph

$$Q_4 := \mathrm{Cay}(\mathbb{Z}_2^4, \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}).$$

  A non-identity automorphism of Q_4 fixes at most 8 edges out of 32.
- The graph BCH is the bipartite complement of the Heawood graph. The vertices of BCH can be identified with the 7 points and the 7 lines of the Fano plane. The incidence in the graph is given by the anti-flags in the plane, i.e. the point p is adjacent to the line L if, and only if, $p \notin L$. The automorphism group of BCH is isomorphic to $\mathrm{SL}_3(2).2$. A non-identity automorphism of BCH fixes at most 4 edges out of 28.

Lemma 11 *Let Γ be a finite connected 4-valent 2-arc-transitive graph of girth at most 4, i.e. $g(\Gamma) \in \{3, 4\}$. Then one of the following holds:*

1. *$g(\Gamma) = 3$ and Γ is isomorphic to the complete graph K_5;*
2. *$g(\Gamma) = 4$ and Γ is isomorphic to $K_{4,4} \cong C(4, 1)$;*
3. *$g(\Gamma) = 4$ and Γ is isomorphic to $K_{5,5} - 5K_2$, Q_4 or BCH.*

Proof Let v be a vertex, let $\Gamma(v) = \{w_1, w_2, w_3, w_4\}$ be its neighbourhood and let $G := \mathrm{Aut}(\Gamma)$.

First, assume $g(\Gamma) = 3$. Without loss of generality, suppose w_1 and w_2 are adjacent. Since G is 2-arc-transitive, G_v is 2-transitive on $\Gamma(v)$. Hence w_i is adjacent to w_j for any $i \neq j$. Thus $\Gamma \cong K_5$ and part (1) holds.

Now, suppose $g(\Gamma) = 4$. We need to recall the classification arising from [15, Theorem 3.3]. If Δ is a 4-valent edge-transitive graph, then one of the following holds

(i) each vertex in Δ is contained in exactly one 4-cycle,
(ii) there exist two distinct vertices v_1, v_2 with $\Delta(v_1) = \Delta(v_2)$,
(iii) Δ is isomorphic to $K_{5,5} - 5K_2$, Q_4 or BCH.

We consider these three possibilities for Γ in turn. Up to a permutation of the indices, there exists $u \in \Gamma(w_1) \cap \Gamma(w_2)$ such that (v, w_1, u, w_2) is a 4-cycle. Since $G_v^{\Gamma(v)}$ is 2-transitive, there exists $g \in G_v$ with $(w_1, w_2)^g = (w_3, w_4)$. Therefore, $(v, w_1, u, w_2)^g = (v, w_3, u^g, w_4)$ is a 4-cycle different from (v, w_1, u, w_2). Thus part (i) is excluded. If Γ satisfies (ii), then [15, Lemma 4.3] gives that Γ is isomorphic to $C(r, 1)$ for some integer r. From Lemma 2, $C(r, 1)$ is 2-arc-transitive only when $r = 4$; therefore we obtain part (2). If Γ satisfies part (iii), then we obtain the examples in part (3). □

Definition 1 Let Γ be a finite connected 4-valent graph and let g be an automorphism of Γ. We partition $E\Gamma$ with respect to the action of g.

- We let $A(\Gamma, g)$ be the set of edges which are pointwise fixed by g, that is, $\{a, b\} \in A(\Gamma, g)$ if and only if $\{a, b\} \in E\Gamma$, $a^g = a$ and $b^g = b$;
- we let $F(\Gamma, g) := \mathrm{Fix}(E\Gamma, g) \setminus A(\Gamma, g)$, that is, $\{a, b\} \in F(\Gamma, g)$ if and only if $\{a, b\} \in E\Gamma$, $a^g = b$ and $b^g = a$;
- we let $N(\Gamma, g) := E\Gamma \setminus \mathrm{Fix}(E\Gamma, g)$.

We let $\Gamma[g]$ denote the subgraph of Γ induced by Γ on the vertices which are incident with edges in $A(\Gamma, g)$. The edge-set of $\Gamma[g]$ is $A(\Gamma, g)$ and its vertices are 1-, 2- or 4-valent. Given $i \in \{1, 2, 4\}$, we let $V_i(\Gamma, g)$ denote the set of vertices of $\Gamma[g]$ having valency i.

Lemma 12 *Let Γ be a finite connected 4-valent graph of girth $g(\Gamma) \geq 5$ and let g be an automorphism of Γ. Then* $2|F(\Gamma, g)| + 4|V_1(\Gamma, g)| + 3|V_2(\Gamma, g)| + |V_4(\Gamma, g)| \leq |V\Gamma|$.

Proof We let

$$\mathcal{F} := \{v \in V\Gamma \mid \{v, u\} \in F(\Gamma, g) \text{ for some } u \in V\Gamma\},$$
$$\mathcal{N} := \{v \in V\Gamma \setminus (V_1(\Gamma, g) \cup V_2(\Gamma, g)) \mid \{v, u\} \in N(\Gamma, g) \text{ for some } u \in V\Gamma\}.$$

Since $V_1(\Gamma, g), V_2(\Gamma, g), V_4(\Gamma, g), \mathcal{F}, \mathcal{N}$ are pairwise disjoint and since $|\mathcal{F}| = 2|F(\Gamma, g)|$, it suffices to show that $|\mathcal{N}| \geq 3|V_1(\Gamma, g)| + 2|V_2(\Gamma, g)|$.

We construct an auxiliary graph Δ. The vertex set of Δ is $V_1(\Gamma, g) \cup V_2(\Gamma, g) \cup \mathcal{N}$ and we declare a vertex $v \in V_1(\Gamma, g) \cup V_2(\Gamma, g)$ adjacent to a vertex $u \in \mathcal{N}$ if $\{v, u\} \in E\Gamma$. By construction, Δ is bipartite with parts $V_1(\Gamma, g) \cup V_2(\Gamma, g)$ and $\mathcal{N}$.

Given $v \in V_1(\Gamma, g)$, the automorphism g acts as a 3-cycle on $\Gamma(v)$. Let $v_1, v_2, v_3 \in \Gamma(v)$ forming the 3-cycle of g. Then $\{v, v_1\}, \{v, v_2\}, \{v, v_3\} \in N(\Gamma, g)$ and hence $v_1, v_2, v_3 \in \mathcal{N}$. This shows that each vertex in $V_1(\Gamma, g)$ has three neighbours in $\mathcal{N}$. Similarly, each vertex in $V_2(\Gamma, g)$ has two neighbours in $\mathcal{N}$. As $g(\Gamma) > 4$, we have $g(\Delta) > 4$ and hence $3|V_1(\Gamma, g)| + 2|V_2(\Gamma, g)| \leq |\mathcal{N}|$, because $\Delta(v) \cap \Delta(v') = \emptyset$ for any two distinct vertices $v, v' \in V_1(\Gamma, g) \cup V_2(\Gamma, g)$. □

Let B, L and R be groups, and let $\iota_L : B \to L$ and $\iota_R : B \to R$ be injective homomorphisms of groups. The pair (ι_L, ι_R) is said to be an **amalgam**. When B is a subgroup of both L and R, we can think of ι_L and ι_R as the inclusion mappings. In this case, the amalgam is determined by the triple (L, B, R) and, in this paper, this is the point of view we take.

Let (L, B, R) be an amalgam, we say that its **index** is the couple

$$(|L : B|, |R : B|).$$

Moreover, (L, B, R) is said to be **faithful** if no subgroup of B is normal in L and in R. When the index is precisely $(k, 2)$, for some positive integer k, (L, B, R) is said to be 2-**transitive** if the action of L on the right cosets of B by right multiplication is 2-transitive.

Observe that, if Γ is a finite connected G-arc-transitive graph of valency k, then for any $v \in V\Gamma$ and $w \in \Gamma(v)$, the triplet

$$(G_v, G_{vw}, G_{\{v,w\}})$$

is a faithful amalgam of index $(k, 2)$.

Finite faithful 2-transitive amalgams of index (4, 2) have been studied in detail by Potočnik in [9]. We use this work to deduce some properties on $\mathrm{Fix}(E\Gamma, g)$.

Lemma 13 *Let Γ be a finite connected 4-valent graph, let G be an s-arc-transitive group of automorphisms of Γ with $s \geq 2$ and let $g \in G$ fixing pointwise the s-arc $(v_0, \ldots, v_{s-1})$. If G is not $(s+1)$-arc-transitive and g fixes pointwise $\Gamma(v_0) \cup \Gamma(v_{s-1})$, then $g = 1$.*

Proof If G is s-arc-regular, then $g = 1$ because g fixes an s-arc. Using [9], we see that there are 6 amalgams such that G is not s-arc-regular. For each of these remaining amalgams a case-by-case computation shows that the only automorphism leaving the neighbourhood of each end of a given s-arc fixed is the identity map. □

Lemma 14 *Let Γ be a finite connected 4-valent graph of girth $g(\Gamma) \geq 5$, let G be a 2-arc-transitive group of automorphisms of Γ such that $G_v^{[1]} \cap G_w^{[1]}$ is a 3-group, for any two distinct vertices at distance at most 2, and let $g \in G \setminus \{1\}$. Then $3|V_4(\Gamma, g)| \leq 3|V_1(\Gamma, g)| + |V_2(\Gamma, g)|$.*

Proof Assume that the vertices in $V_4(\Gamma, g)$ are at pairwise distance more than 2. Then any two such vertices share no common neighbour. In particular, $\bigcup_{v \in V_4(\Gamma,g)} \Gamma(v)$ has cardinality $4|V_4(\Gamma, g)|$ and is contained in $V_1(\Gamma, g) \cup V_2(\Gamma, g)$. Therefore,

$4|V_4(\Gamma, g)| \leq |V_1(\Gamma, g)| + |V_2(\Gamma, g)|$ and the lemma immediately follows in this case.

Assume that there exist two distinct vertices v and w of $V_4(\Gamma, g)$ having distance at most 2. In particular, $g \in G_v^{[1]} \cap G_w^{[1]}$ and hence g has order a power of 3, because $G_v^{[1]} \cap G_w^{[1]}$ is a 3-group. Observe that $V_2(\Gamma, g) = \emptyset$ because an element of order 3 in a local group cannot fix exactly two elements. Let $s \geq 2$ such that G is s-arc-transitive, but not $(s+1)$-arc-transitive.

Suppose $\Gamma[g]$ is not a forest. Then $\Gamma[g]$ contains an ℓ-cycle C. As $V_2(\Gamma, g) = \emptyset$, the vertices of C are elements of $V_4(\Gamma, g)$. From Lemma 4, we have $g(\Gamma[g]) \geq g(\Gamma) \geq s+1$ and hence, from C, we can extract an s-arc whose ends lie in $V_4(\Gamma, g)$, contradicting Lemma 13.

Suppose $\Gamma[g]$ is a forest. Let c be the number of connected components of $\Gamma[g]$. From Euler's formula, we have $|V\Gamma[g]| - |E\Gamma[g]| = c$. Clearly, $|V\Gamma[g]| = |V_1(\Gamma, g)| + |V_4(\Gamma, g)|$. Let $\mathcal{S} := \{(v, w) \in V\Gamma[g] \times V\Gamma[g] \mid \{v, w\} \in E\Gamma[g]\}$. Then

$$\begin{aligned} 2|E\Gamma[g]| = |\mathcal{S}| &= \sum_{v \in V\Gamma[g]} |\Gamma[g](v)| \\ &= \sum_{v \in V_1(\Gamma,g)} |\Gamma[g](v)| + \sum_{v \in V_4(\Gamma,g)} |\Gamma[g](v)| \\ &= |V_1(\Gamma, g)| + 4|V_4(\Gamma, g)|. \end{aligned}$$

It follows that $2|V_4(\Gamma, g)| = |V_1(\Gamma, g)| - 2c < |V_1(\Gamma, g)|$. □

Proof (Proof of Theorem 1 when Γ is 2-arc-transitive) Let Γ be a finite connected 4-valent 2-arc-transitive graph admitting a non-identity automorphism g with $\mathrm{fpr}(E\Gamma, g) > 1/3$ and let $G := \mathrm{Aut}(\Gamma)$.

If $g(\Gamma) \leq 4$, then the proof follows from Lemma 11 and from the remarks at the beginning of Sect. 4.1. Therefore, for the rest of the proof we suppose that $g(\Gamma) > 4$. Since $4|V\Gamma| = 2|E\Gamma|$, we have

$$\begin{aligned} \mathrm{fpr}(E\Gamma, g) &= \frac{|F(\Gamma, g)| + |A(\Gamma, g)|}{|E\Gamma|} = \frac{2|F(\Gamma, g)| + 2|A(\Gamma, g)|}{4|V\Gamma|} \\ &\leq \frac{2|F(\Gamma, g)| + |V_1(\Gamma, g)| + 2|V_2(\Gamma, g)| + 4|V_4(\Gamma, g)|}{8|F(\Gamma, g)| + 16|V_1(\Gamma, g)| + 12|V_2(\Gamma, g)| + 4|V_4(\Gamma, g)|}, \end{aligned} \tag{4.1}$$

where in the last inequality we have used Lemma 12.

We claim that, for any two distinct vertices $v, w \in V\Gamma$ at distance at most 2 one of the following holds

(i) $G_v^{[1]} \cap G_w^{[1]}$ is a 3-group;
(ii) the pair (Γ, G) defines the amalgam

$$(S_3 \times S_4, S_3 \times S_3, (S_3 \times S_3) \rtimes C_2),$$

moreover, if $d(v, w) = 1$, then $G_v^{[1]} \cap G_w^{[1]} = 1$ and, if $d(v, w) = 2$, then $G_v^{[1]} \cap G_w^{[1]}$ is isomorphic to C_2.

The claim follows with a case-by-case computation on the finite faithful 2-transitive amalgams of index (4, 2) classified in [9]. We now divide the proof according to (i) and (ii).

Suppose that (i) holds. From Lemma 14, we have $3|V_4(\Gamma, g)| \leq 3|V_1(\Gamma, g)| + |V_2(\Gamma, g)|$. Using this inequality and (4.1), we obtain $\mathrm{fpr}(E\Gamma, g) \leq 1/4 < 1/3$, which is a contradiction.

Suppose that (ii) holds. If there exist two distinct vertices v and w in $V_4(\Gamma, g)$ with $d(v, w) = 1$, then $g \in G_v^{[1]} \cap G_w^{[1]} = 1$, which is a contradiction. Assume there exist two distinct vertices v and w in $V_4(\Gamma, g)$ with $d(v, w) = 2$. Then $g \in G_v^{[1]} \cap G_w^{[1]} \cong C_2$ and hence g has order 2. This implies $V_1(\Gamma, g) = \emptyset$ because an involution in a local group cannot fix only one element. Since the subgraph induced by $\Gamma[g]$ on $V_4(\Gamma, g)$ has no edges and since each vertex in $V_4(\Gamma, g)$ has valency 4, we deduce $4|V_4(\Gamma, g)| \leq |E\Gamma[g]| = |V_2(\Gamma, g)| + 2|V_4(\Gamma, g)|$. Using this inequality and (4.1), we obtain $\mathrm{fpr}(E\Gamma, g) < 1/3$, which is a contradiction.

Finally, assume that the vertices in $V_4(\Gamma, g)$ are at pairwise distance more than 2. Then any two such vertices share no common neighbour. In particular, $\bigcup_{v \in V_4(\Gamma, g)} \Gamma(v)$ has cardinality $4|V_4(\Gamma, g)|$ and is contained in $V_1(\Gamma, g) \cup V_2(\Gamma, g)$. Therefore, $4|V_4(\Gamma, g)| \leq |V_1(\Gamma, g)| + |V_2(\Gamma, g)|$. Using this inequality and (4.1), we obtain $\mathrm{fpr}(E\Gamma, g) \leq 1/4 < 1/3$, which is a contradiction. □

4.2 Proof of Theorem 1 when Γ is not 2-arc-transitive

To conclude the proof of Theorem 1, we argue by induction on $|V\Gamma|$.

Let Γ be a finite connected vertex- and edge-transitive 4-valent graph admitting a non-identity automorphism g fixing more than 1/3 of the edges and with $G := \mathrm{Aut}(\Gamma)$ not 2-arc-transitive. If Γ is isomorphic to a Praeger–Xu graph, then part (2) of Theorem 1 holds. Therefore, for the rest of the argument, we suppose that Γ is not isomorphic to $C(r, s)$, for any choice of r and s with $r \geq 3$ and $1 \leq s \leq r - 1$.

Let $v \in V\Gamma$. Since G is not 2-arc-transitive, $G_v^{\Gamma(v)}$ is not 2-transitive on $\Gamma(v)$. Since G is vertex- and edge-transitive, we obtain that either $G_v^{\Gamma(v)}$ is transitive or $G_v^{\Gamma(v)}$ has two orbits of cardinality 2. In both cases, we deduce that $G_v^{\Gamma(v)}$ is a 2-group. As Γ is connected, it follows that G_v is a 2-group.

If G has no non-identity normal subgroups having cardinality a power of 2, Theorem 3 (applied to the faithful and transitive action of G on $E\Gamma$) contradicts $\mathrm{fpr}(E\Gamma, g) > 1/3$. Thus, G has a minimal normal 2-subgroup N.

As Γ is not isomorphic to a Praeger–Xu graph, Lemma 3 yields that N acts semiregularly on $V\Gamma$. Consider the quotient graph Γ/N and observe that, as G is vertex- and edge-transitive, Γ/N has valency 0, 1, 2 or 4.

If Γ/N has valency 0, then N is transitive on $V\Gamma$. Thus N is vertex-regular on Γ. As Γ is connected of valency 4, N is generated by at most 4 elements and hence $|V\Gamma| = |N|$ divides 2^4. If Γ/N has valency 1, then N has two orbits on $V\Gamma$. Moreover, [11, Lemma 1.15] implies that $|V\Gamma| = 2|N|$ divides 128. In both cases, the statement can be checked computationally by inspecting the candidate graphs from the census of all 4-valent vertex- and edge-transitive graphs of small order, see [12, 13]. If Γ/N

has valency 2, then we contradict Lemma 10. Therefore, for the rest of the proof, we may suppose that Γ/N has valency 4.

Let K be the kernel of the action of G on $V\Gamma/N$. Since the quotient graph is not degenerate, $K_v = 1$. Thus $K = K_v N = N$. In particular, G/N acts faithfully as a group of automorphisms on Γ/N. Moreover, G/N acts vertex- and edge-transitively on Γ/N, but not 2-arc-transitively. Observe that $g \notin N$, because the elements in N fix no edge of Γ. Thus Ng is not the identity automorphism of Γ/N and, by Lemma 1, we have $\mathrm{fpr}(E\Gamma/N, Ng) > 1/3$. Our inductive hypothesis on $|V\Gamma|$ implies that Γ/N is isomorphic to K_5 or to a Praeger–Xu graph $C(r, s)$ with $3s < 2r - 3$.

Assume $\Gamma/N \cong K_5$. Now, $\mathrm{Aut}(K_5) = S_5$ and S_5 contains a unique conjugacy class of subgroups which are vertex- and edge-transitive, but not 2-transitive (namely, the Frobenius groups of order 20). Therefore, G/N is isomorphic to a Frobenius group of order 20. In particular, as N is an irreducible module for a Frobenius group of order 20, we get $|N| \le 16$. We deduce $|V\Gamma| \le 10 \cdot 16 = 160$ and, as above, the statement can be checked computationally by inspecting the census of all 4-valent vertex- and edge-transitive graphs of small order.

Assume $\Gamma/N \cong C(r, s)$, for some r and s with $3s < 2r - 3$. From Lemma 9, G/N is $\mathrm{Aut}(\Gamma/N)$-conjugate to a subgroup of H as defined in Sect. 2.3. Without loss of generality, we can identify G/N with such a subgroup, so that $G/N \le H$. Now, we first deal with the exceptional case $(r, s) = (4, 1)$. As G/N is a 2-group and N is a minimal normal subgroup of G, we deduce $|N| = 2$ and hence $|V\Gamma| = |V\Gamma/N||N| = 8 \cdot 2 = 16$. Now, the proof follows inspecting the vertex- and edge-transitive graphs of order 16. Therefore, for the rest of the argument, we suppose $(r, s) \ne (4, 1)$. Now, Lemma 8 implies $Ng \in K \le H^+$. Denote by X the group $G/N \cap H^+$. This group is a half-arc-transitive group of automorphisms of Γ/N and, since $|H : H^+| = 2$, we have $|G/N : X| \le 2$. Denote by G^+ the preimage of X with respect to the quotient projection $G \to G/N$, so that $G^+/N \cong X$. Now, G^+ acts half-arc-transitively on Γ and, from $Ng \in X$, we see that $g \in G^+$. In particular, replacing G with G^+ if necessary, in the rest of our argument we may suppose that $G = G^+$, that is, $G/N \le H^+$.

By Lemma 8, all the edges fixed in Γ/N by Ng are fixed as arcs. Therefore, all the edges fixed in Γ by g are fixed as arcs.

Considering the graph induced by Γ on $\mathrm{Fix}(V\Gamma, g)$, we deduce $2|\mathrm{Fix}(E\Gamma, g)| \le 4|\mathrm{Fix}(V\Gamma, g)|$. In particular, if $|\mathrm{Fix}(V\Gamma, g)| \le |V\Gamma|/3$, then

$$\frac{1}{3} < \mathrm{fpr}(E\Gamma, g) = \frac{|\mathrm{Fix}(E\Gamma, g)|}{|E\Gamma|} \le \frac{2|\mathrm{Fix}(V\Gamma, g)|}{|E\Gamma|} \le \frac{2|V\Gamma|}{3|E\Gamma|} = \frac{|E\Gamma|}{3|E\Gamma|} = \frac{1}{3},$$

which is a contradiction. Therefore $\mathrm{fpr}(V\Gamma, g) > 1/3$. Now, the hypothesis of Lemma 2.3 in [11] are satisfied. Therefore, [11, Lemma 2.3] implies that Γ is a Praeger–Xu graph, which is our final contradiction.

5 Proof of Theorem 2

We now turn our attention to finite connected 3-valent vertex-transitive graphs. We divide the proof of Theorem 2 in three cases, which we now describe. Let Γ be a finite connected 3-valent vertex-transitive graph, let $G := \mathrm{Aut}(\Gamma)$ and let $v \in V\Gamma$. The local group $G_v^{\Gamma(v)}$ is a subgroup of the symmetric group of degree 3 and we divide the proof of Theorem 2 depending on the structure of $G_v^{\Gamma(v)}$. When $G_v^{\Gamma(v)} = 1$, the connectivity of Γ implies $G_v = 1$ and hence G acts regularly on $V\Gamma$. In this case an observation of Sabidussi [18] yields that Γ is Cayley graph over G. We deal with this case in Sect. 5.1. When $G_v^{\Gamma(v)}$ is cyclic of order 2, [12] has established a fundamental relation between Γ and a certain finite connected 4-valent graph; in Sect. 5.2, we exploit this relation and Theorem 1 to deal with this case. When $G_v^{\Gamma(v)}$ is transitive, Γ is arc-transitive and we use the amazing result of Tutte concerning the structure of G_v to deal with this case in Sect. 5.3.

5.1 Proof of Theorem 2 when the local group is the identity

Let Γ be a finite connected 3-valent vertex-transitive graph, let $v \in V\Gamma$, let $G := \mathrm{Aut}(\Gamma)$ and let $g \in G \setminus \{1\}$. Assume that $G_v^{\Gamma(v)} = 1$. Lemma 7 yields $\mathrm{fpr}(E\Gamma, g) \le 1/3$ and hence Theorem 2 holds in this case.

5.2 Proof of Theorem 2 when the local group is cyclic of order 2

In our proof of this case, we need to refer to two families of 3-valent Cayley graphs. Given $n \in \mathbb{N}$ with $n \ge 3$, the **prism** Pr_n is the Cayley graph

$$\mathrm{Pr}_n = \mathrm{Cay}\left(\mathbb{Z}_n \times \mathbb{Z}_2, \{(0,1), (1,0), (-1,0)\}\right).$$

Similarly, given $n \in \mathbb{N}$ with $n \ge 2$, the **Möbius ladder** Mb_n is the Cayley graph

$$\mathrm{Mb}_n = \mathrm{Cay}\left(\mathbb{Z}_{2n}, \{1, n, -1\}\right).$$

For these two classes of graphs the proof of Theorem 2 follows with a computation. When $n \ne 4$, the automorphism group of Pr_n is isomorphic to $D_n \times C_2$ and, for each $x \in \mathrm{Aut}(\mathrm{Pr}_n)$ with $x \ne 1$, it can be verified that $\mathrm{fpr}(E\mathrm{Pr}_n, x) \le 1/3$, see also Lemma 7. The case $n = 4$ is exceptional, because $\mathrm{Pr}_4 \cong Q_4$ is 2-arc-transitive and hence Pr_4 is of no concern to us here. Similarly, when $n \notin \{2, 3\}$, the automorphism group of Mb_n is isomorphic to D_{2n} and, for each $x \in \mathrm{Aut}(\mathrm{Mb}_n)$ with $x \ne 1$, it can be verified that $\mathrm{fpr}(E\mathrm{Mb}, x) \le 1/3$, again see also Lemma 7. The cases $n \in \{2, 3\}$ are exceptional, because $\mathrm{Mb}_2 \cong K_4$ and Mb_3 are 2-arc-transitive and hence are of no concern to us here.

Now, let Γ be a finite connected 3-valent vertex-transitive graph not isomorphic to Pr_n and not isomorphic to Mb_n, let $v \in V\Gamma$, let $G := \mathrm{Aut}(\Gamma)$ and let $g \in G \setminus \{1\}$ with $\mathrm{fpr}(E\Gamma, g) > 1/3$. Assume that $G_v^{\Gamma(v)}$ is cyclic of order 2.

For a vertex $w \in V\Gamma$, let w' be the neighbour of w such that $\{w'\}$ is the orbit of G_w of length 1. Then clearly $(w')' = w$ and $G_w = G_{w'}$. Hence, the set $\mathcal{M} := \{\{w, w'\} \mid w \in V\Gamma\}$ is a complete matching of Γ, while edges outside $\mathcal{M}$ form a 2-factor $\mathcal{F}$. The group G in its action on $E\Gamma$ fixes setwise both $\mathcal{F}$ and $\mathcal{M}$ and acts transitively on the arcs of each of these two sets. Let $\tilde{\Gamma}$ be the graph with vertex-set $\mathcal{M}$ and two vertices $e_1, e_2 \in \mathcal{M}$ adjacent if and only if they are (as edges of Γ) at distance 1 in Γ. The graph $\tilde{\Gamma}$ is then called the **merge** of Γ. We may also think of Γ as being obtained by contracting all the edges in $\mathcal{M}$. The group G acts as an arc-transitive group of automorphisms on $\tilde{\Gamma}$. Moreover, the connected components of the 2-factor $\mathcal{F}$ gives rise to a decomposition $\mathcal{C}$ of $E\tilde{\Gamma}$ into cycles.

Since we are assuming that Γ is neither a prism nor a Möbius ladder, [12, Lemma 9 and Theorem 10] implies that $\tilde{\Gamma}$ is 4-valent. Moreover, the action of G on $\tilde{\Gamma}$ is faithful, arc-transitive but not 2-arc-transitive.

Noticing that $|E\tilde{\Gamma}| = 2|V\tilde{\Gamma}|$, we can link the fixed-point ratios of Γ and its 4-valent merge $\tilde{\Gamma}$ as follows

$$\begin{aligned}\mathrm{fpr}(E\Gamma, g) &= \frac{|\mathrm{Fix}(V\tilde{\Gamma}, g)| + |\mathrm{Fix}(E\tilde{\Gamma}, g)|}{|V\tilde{\Gamma}| + |E\tilde{\Gamma}|} \\ &= \frac{|\mathrm{Fix}(V\tilde{\Gamma}, g)|}{3|V\tilde{\Gamma}|} + \frac{2|\mathrm{Fix}(E\tilde{\Gamma}, g)|}{3|E\tilde{\Gamma}|} \\ &= \frac{1}{3}\mathrm{fpr}(V\tilde{\Gamma}, g) + \frac{2}{3}\mathrm{fpr}(E\tilde{\Gamma}, g).\end{aligned}$$

Observe that either $\mathrm{fpr}(V\tilde{\Gamma}, g) > 1/3$ or $\mathrm{fpr}(E\tilde{\Gamma}, g) > 1/3$, otherwise

$$\frac{1}{3} < \mathrm{fpr}(E\Gamma, g) \le \frac{1}{3} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{3} = \frac{1}{3}.$$

Using [11, Theorem 1.1] when $\mathrm{fpr}(V\tilde{\Gamma}, g) > 1/3$, and using Theorem 1 when $\mathrm{fpr}(E\tilde{\Gamma}, g) > 1/3$, it follows that either $\tilde{\Gamma} \cong C(r, s)$, with $3s < 2r$, or $|V\tilde{\Gamma}| \le 70$. The latter case yields $|V\Gamma| \le 140$ and the veracity of Theorem 2 follows with an inspection on the connected 3-valent graphs having at most 140 vertices.

Therefore, we can suppose $\tilde{\Gamma} \cong C(r, s)$. In view of [12, Theorem 12], the graph Γ can be uniquely reconstructed from $\tilde{\Gamma}$ and the decomposition $\mathcal{C}$ of $E\tilde{\Gamma}$ arising from the 2-factor $\mathcal{F}$ via the splitting operation defined in Sect. 2.4. Hence, $\Gamma \cong S(C(r, s))$. Finally, observe that

$$\frac{1}{3} < \mathrm{fpr}(E\Gamma, g) \le \frac{1}{3}\mathrm{fpr}(V\tilde{\Gamma}, \tau_i) + \frac{2}{3}\mathrm{fpr}(E\tilde{\Gamma}, \tau_i) = \frac{r-s}{3r} + \frac{2(r-s-1)}{3r}.$$

(The τ_i's are defined in Sect. 2.3.) Hence, a direct computation leads to $3s < 2r - 2$.

5.3 Proof of Theorem 2 when the local group is transitive

Let Γ be a finite connected 3-valent vertex-transitive graph, let $v \in V\Gamma$, let $G := \mathrm{Aut}(\Gamma)$ and let $g \in G \setminus \{1\}$ with $\mathrm{fpr}(E\Gamma, g) > 1/3$. Assume that $G_v^{\Gamma(v)}$ is transitive. Let $s \geq 1$ such that G is s-arc-transitive and G is not $(s+1)$-arc-transitive. Tutte's theorem [19] implies that G is s-arc-regular.

Similarly to Definition 1, we partition $E\Gamma$ with respect to the action of g.

- We let $A(\Gamma, g)$ be the set of edges which are pointwise fixed by g, that is, $\{a, b\} \in A(\Gamma, g)$ if and only if $\{a, b\} \in E\Gamma$, $a^g = a$ and $b^g = b$;
- we let $F(\Gamma, g) := \mathrm{Fix}(E\Gamma, g) \setminus A(\Gamma, g)$, that is, $\{a, b\} \in F(\Gamma, g)$ if and only if $\{a, b\} \in E\Gamma$, $a^g = b$ and $b^g = a$;
- we let $N(\Gamma, g) := E\Gamma \setminus \mathrm{Fix}(E\Gamma, g)$.

We let $\Gamma[g]$ denote the subgraph of Γ induced by Γ on the vertices which are incident with edges in $A(\Gamma, g)$.[1] The edge-set of $\Gamma[g]$ is $A(\Gamma, g)$ and its vertices are 1- or 3-valent. Given $i \in \{1, 3\}$, we let $V_i(\Gamma, g)$ denote the set of vertices of $\Gamma[g]$ having valency i.

Suppose $\Gamma[g]$ is not a forest. Then $\Gamma[g]$ contains an ℓ-cycle C. From Lemma 4, we have $g(\Gamma[g]) \geq g(\Gamma) \geq s+1$ and hence, from C, we can extract an s-arc $(v_0, v_1, \ldots, v_{s-1})$. As g fixes this s-arc and as G is s-arc-regular, we deduce $g = 1$, which is a contradiction. Therefore $\Gamma[g]$ is a forest. Before proceeding with the proof of Theorem 2, we prove a preliminary lemma.

Lemma 15 *We have* $2|F(\Gamma, g)| + 3|V_1(\Gamma, g)| + |V_3(\Gamma, g)| \leq |V\Gamma|$.

Proof When $s = 1$, the arc-regularity of G implies $V_1(\Gamma, g) = V_3(\Gamma, g) = \emptyset$ and the proof immediately follows. Hence for the rest of the proof we may suppose $s \geq 2$.

We let

$$\mathcal{F} := \{v \in V\Gamma \mid \{v, u\} \in F(\Gamma, g) \text{ for some } u \in V\Gamma\},$$
$$\mathcal{N} := \{v \in V\Gamma \setminus V_1(\Gamma, g) \mid \{v, u\} \in N(\Gamma, g) \text{ for some } u \in V\Gamma\}.$$

Since $V_1(\Gamma, g), V_3(\Gamma, g), \mathcal{F}, \mathcal{N}$ are pairwise disjoint and since $|\mathcal{F}| = 2|F(\Gamma, g)|$, it suffices to show that $|\mathcal{N}| \geq 2|V_1(\Gamma, g)|$. We divide this proof according to the girth of Γ.

Suppose $g(\Gamma) \geq 5$. Here, we construct an auxiliary graph Δ. The vertex set of Δ is $V_1(\Gamma, g) \cup \mathcal{N}$ and we declare a vertex $v \in V_1(\Gamma, g)$ adjacent to a vertex $u \in \mathcal{N}$ if $\{v, u\} \in E\Gamma$. By construction, Δ is bipartite with parts $V_1(\Gamma, g)$ and $\mathcal{N}$. Given $v \in V_1(\Gamma, g)$, the automorphism g acts as a 2-cycle on $\Gamma(v)$. Let $v_1, v_2 \in \Gamma(v)$ forming the 2-cycle of g. Then $\{v, v_1\}, \{v, v_2\} \in N(\Gamma, g)$ and hence $v_1, v_2 \in \mathcal{N}$. This shows that each vertex in $V_1(\Gamma, g)$ has two neighbours in $\mathcal{N}$. As $g(\Gamma) \geq 5$, we have $g(\Delta) \geq 5$ and hence $2|V_1(\Gamma, g)| \leq |\mathcal{N}|$, because $\Delta(v) \cap \Delta(v') = \emptyset$ for any two distinct vertices $v, v' \in V_1(\Gamma, g)$.

[1] During the revision process of this manuscript, we found out that $\Gamma[g]$ has already been investigated in [6].

Suppose $g(\Gamma) = 3$. Let $\Gamma(v) = \{w_1, w_2, w_3\}$. Without loss of generality, suppose w_1 and w_2 are adjacent. Since G is arc-transitive, w_i is adjacent to w_j for any $i \neq j$. Thus $\Gamma \cong K_4$. The graph K_4 admits no non-identity automorphisms with $\mathrm{fpr}(E\Gamma, g) > 1/3$.

Suppose $g(\Gamma) = 4$. Since $s \geq 2$, [8, Theorem 1.1 and Table I] implies that Γ is isomorphic to either $K_{3,3}$ or $K_{4,4} - 4K_2$. In both cases, Γ does not admit a non-identity automorphism g with $\mathrm{fpr}(E\Gamma, g) > 1/3$. □

We now resume our proof of Theorem 2. As $2|E\Gamma| = 3|V\Gamma|$, from Lemma 15, we have

$$\begin{aligned} \mathrm{fpr}(E\Gamma, g) = \frac{|F(\Gamma, g)| + |A(\Gamma, g)|}{|E\Gamma|} &= \frac{2|F(\Gamma, g)| + 2|A(\Gamma, g)|}{3|V\Gamma|} \\ &\leq \frac{2|F(\Gamma, g)| + |V_1(\Gamma, g)| + 3|V_3(\Gamma, g)|}{6|F(\Gamma, g)| + 9|V_1(\Gamma, g)| + 3|V_3(\Gamma, g)|}. \end{aligned} \tag{5.1}$$

Let c be the number of connected components of $\Gamma[g]$. From Euler's formula, we have $|V\Gamma[g]| - |E\Gamma[g]| = c$. Let $\mathcal{S} := \{(v, w) \in V\Gamma[g] \times V\Gamma[g] \mid \{v, w\} \in E\Gamma[g]\}$. Then

$$\begin{aligned} 2|E\Gamma[g]| = |\mathcal{S}| &= \sum_{v \in V\Gamma[g]} |\Gamma[g](v)| \\ &= \sum_{v \in V_1(\Gamma, g)} |\Gamma[g](v)| + \sum_{v \in V_3(\Gamma, g)} |\Gamma[g](v)| \\ &= |V_1(\Gamma, g)| + 3|V_3(\Gamma, g)|. \end{aligned}$$

It follows that $|V_3(\Gamma, g)| = |V_1(\Gamma, g)| - 2c < |V_1(\Gamma, g)|$. Using (5.1) and this inequality, we obtain $\mathrm{fpr}(E\Gamma, g) \leq 1/3$, which is our final contradiction.

Funding Open access funding provided by University degli Studi di Pavia within the CRUI-CARE Agreement.

Data availability The datasets analysed during the current study are available at https://www.fmf.uni-lj.si/\protect\unhbox\voidb@x\penalty\@M\potocnik/work.htm.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

References

1. Biggs, N.: Algebraic Graph Theory, 2nd edn. Cambridge University Press, Cambridge (1993)

2. Gardiner, A., Praeger, C.E.: A characterization of certain families of 4 symmetric graphs. European J. Combin. **15**, 383–397 (1994). https://doi.org/10.1006/eujc.1994.1042
3. Godsil, C., Royle, G.: Algebraic Graph Theory, Graduate Texts in Mathematics, vol. 207. Springer, New York (2001)
4. Jajcay, R., Potočnik, P., Wilson, S.: The Praeger–Xu graphs: cycle structures, maps and semitransitive orientations. Acta Math. Univ. Comenian. **88**, 22–26 (2019)
5. Jajcay, R., Potočnik, P., Wilson, S.: On the cayleyness of Praeger–Xu graphs. J. Combin. Theory Ser. B **152**, 55–79 (2022). https://doi.org/10.1016/j.jctb.2021.09.003
6. Kutnar, K., Marušič, D.: Odd extensions of transitive groups via symmetric graphs – the cubic case. J. Combin. Theory Ser. B **136**, 170–192 (2019). https://doi.org/10.1016/j.jctb.2018.10.003
7. Lehner, F., Potočnik, P., Spiga, P.: On fixity of arc-transitive graphs. Sci. China Math. **64**, 2603–2610 (2021). https://doi.org/10.1007/s11425-020-1825-1
8. Perles, M.A., Martini, H., Kupitz, Y.S.: Locally symmetric graphs of girth 4. J. Graph Theory **73**(1), 44–65 (2013). https://doi.org/10.1002/jgt.21657
9. Potočnik, P.: A list of 4-valent 2-arc-transitive graphs and finite faithful amalgams of index (4,2). European J. Combin. **30**, 1323–1336 (2008). https://doi.org/10.1016/j.ejc.2008.10.001
10. Potočnik, P., Spiga, P.: On the minimal degree of a transitive permutation group with stabilizer a 2-group. J. Group Theory **24**, 619–634 (2021). https://doi.org/10.1515/jgth-2020-0058
11. Potočnik, P., Spiga, P.: On the number of fixed points of automorphisms of vertex-transitive graphs of bounded valency. Combinatorica **41**, 703–747 (2021). https://doi.org/10.1007/s00493-020-4509-y
12. Potočnik, P., Spiga, P., Verret, G.: Cubic vertex-transitive graphs on up to 1280 vertices. J. Symbolic Comput. **50**, 465–477 (2013). https://doi.org/10.1016/j.jsc.2012.09.002
13. Potočnik, P., Spiga, P., Verret, G.: A census of 4-valent half-arc-transitive graphs and arc-transitive digraphs of valency 2. Ars Math. Contemp. **8**, 133–148 (2015). https://doi.org/10.26493/1855-3974.559.c6c
14. Potočnik, P., Toledo, M., Verret, G.: On orders of automorphisms of vertex-transitive graphs. arXiv:2106.06750
15. Potočnik, P., Wilson, S.: Tetravalent edge-transitive graphs of girth at most 4. J. Combin. Theory Ser. B **97**, 217–236 (2007). https://doi.org/10.1016/j.jctb.2006.03.007
16. Praeger, C.E.: Highly arc-transitive digraphs. European J. Combin. **10**, 281–292 (1989). https://doi.org/10.1016/S0195-6698(89)80064-2
17. Praeger, C.E., Xu, M.Y.: A characterization of a class of symmetric graphs of twice prime valency. European J. Combin. **10**, 91–102 (1989). https://doi.org/10.1016/S0195-6698(89)80037-X
18. Sabidussi, G.: On a class of fixed-point-free graphs. Proc. Amer. Math. Soc. **9**, 800–804 (1958). https://doi.org/10.1090/S0002-9939-1958-0097068-7
19. Tutte, W.T.: On the symmetry of cubic graphs. Canad. J. Math. **11**, 621–624 (1959). https://doi.org/10.4153/CJM-1959-057-2

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.