ON SOME QUESTIONS RELATED TO INTEGRABLE GROUPS

RUSSELL D. BLYTH, FRANCESCO FUMAGALLI, AND FRANCESCO MATUCCI

Abstract. A group G is *integrable* if it is isomorphic to the derived subgroup of a group H; that is, if $H' \simeq G$, and in this case H is an *integral* of G. If G is a subgroup of U, we say that G is *integrable within* U if $G = H'$ for some $H \leq U$. In this work we focus on two problems posed in [1]. We classify the almost-simple finite groups G that are integrable, which we show to be equivalent to those integrable within $\mathrm{Aut}(S)$, where S is the socle of G. We then classify all 2-homogeneous subgroups of the finite symmetric group S_n that are integrable within S_n.

This paper is dedicated to the memory of Carlo Casolo

1. Introduction

Recently several articles ([12, 1, 2]) have appeared on the topic of the integrability of a group G, where we say that a group H is an *integral* of a group G if $G \simeq H'$ and we then say that G is *integrable*. Given two groups $G \leq U$, we say that G is *(relatively) integrable within* U if there exists a subgroup $H \leq U$ such that $H' = G$. Burnside [5] was the first to consider integrals of groups, showing, for example, that a nonabelian finite p-group with cyclic center cannot have an integral which is a finite p-group. Since then, several other authors have considered which groups can appear as derived subgroups under certain restrictions. Among known results, we mention that all abelian groups are integrable, while all direct powers of dihedral groups D_{2n} are non-integrable for $n \geq 3$, and that if a finite group has an integral, it also has a finite integral. We leave the reader to explore these results and other prior work (see [1, 2] and their references).

In [1, Section 8.2] the following problem is posed:

Problem 1. *Classify all the almost-simple finite groups G that are integrable within $\mathrm{Aut}(S)$, where S denotes the socle of G.*

More generally, we can ask the following question.

Problem 1′. *Classify all the integrable almost-simple finite groups.*

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In Section 3 we provide the following response.

Theorem A. *A finite almost-simple group G (with socle S) is integrable if and only if it is integrable within* $\mathrm{Aut}(S)$. *Moreover, the subgroups between S and* $\mathrm{Aut}(S)$ *that are integrable within* $\mathrm{Aut}(S)$ *are precisely those contained in* $\mathrm{Aut}(S)'$.

In Section 4 we consider the relative integrability of 2-homogeneous groups. A subgroup G acting on a set Ω is said to be *k-homogeneous* (or *k-set transitive*) if it is transitive on the set $\Omega^{\{k\}}$ of all k element subsets of Ω (for $k \geq 1$). We are interested in the following problem, which is also mentioned in [1, Section 8.2].

Problem 2. *Classify all the 2-homogeneous subgroups G of the finite symmetric group S_n that are integrable within S_n.*

In Section 4 we prove

Theorem B. *Let G be a 2-homogeneous subgroup of the finite symmetric group S_n and let S be its socle. If G is integral within S_n then G lies between $(N_{S_n}(S))''$ and $(N_{S_n}(S))'$; the converse is also true with the exception of a few solvable groups (see Remark 4.5).*

The proof of Theorem B is based on the classification of the 2-homogeneous subgroups of S_n (see Theorem 4.1) and it is completed by considering each possible case appearing in that classification. The proof, of course, relies on the classification of finite simple groups.

We note that the group $\mathrm{PGL}_3(7)$, in its action on the 57 projective points, is an example of a 2-homogeneous subgroup of S_{57} that is not integrable within S_{57}. Nevertheless, the group $\mathrm{PGL}_3(7)$ is integrable as an abstract group (see Example 4.6).

Most of our notation is standard and well-known. We write actions on the right, and for x, y in a group G, we define the conjugation x^y to be $y^{-1}xy$ and the commutator $[x, y]$ to be $x^{-1}y^{-1}xy$. We use the ATLAS [8] notation for some groups and constructions. Hence we use $A.B$ to denote any group that has a normal subgroup isomorphic to A for which the corresponding quotient group is isomorphic to B (an (upward) extension of A by B), and we use $A:B$ to indicate a case of $A.B$ that is a split extension $A \rtimes B$ (a notation we also sometimes use), while $A^{\cdot}B$ denotes a non-split extension. Also, given a positive integer n we denote the cyclic group of order n with both symbols C_n and n.

Finite affine semilinear transformation groups $\mathrm{A\Gamma L}_d(q)$, and their important subgroups (such as $\mathrm{ASL}_d(q)$ and $\mathrm{AGL}_d(q)$) arise in Section 4; as a reference, we suggest [11, Section 2.8].

2. Preliminaries

We start with a simple example that draws a clear distinction between the integrability of a group and the integrability of a particular subgroup within a particular group.

Example 2.1. Consider the dihedral group $D_8 = \langle r, s \mid r^4 = s^2 = 1, srs^{-1} = r^{-1} \rangle$. Since $D_8' = \{1, r^2\}$, we see immediately that D_8' is the unique subgroup of order 2 of D_8 to be integrable within D_8, while the other four subgroups of order 2 are not integrable within D_8 (although they are all integrable since they are abelian).

The two parts of the next result are fundamental in reducing integrability within certain groups to integrability within suitable quotients.

Lemma 2.2. *Let T be a finite group and G be a subgroup of T.*

1. *If G is not a cyclic p-group and it contains a unique minimal normal subgroup S, then G is integrable within T if and only if G/S is integrable within $N_T(S)/S$.*
2. *If G admits a nontrivial characteristic perfect subgroup K, then G is integrable within T if and only if G/K is integrable within $N_T(K)/K$.*

Proof. (1) Let $H \leq T$ be such that $H' = G$. Of course S, since it is characteristic in H', is normal in H, that is, $H \leq N_T(S)$. Then we have that H/S is a subgroup of $N_T(S)/S$ having derived subgroup equal to G/S. Conversely, assume that $H/S \leq N_T(S)/S$ is such that $(H/S)' = G/S$. Thus $G = H'S \leq HS$, and since $S \leq H$, we have $G \leq H$. Since S is the unique minimal normal subgroup of G and $H' \trianglelefteq G$, we have that either $H' = 1$ or $S \leq H'$. If $H' = 1$ then, since $G \leq H$, G is abelian and therefore G is a cyclic p-group for some prime p, which is a contradiction. Thus $S \leq H'$, and so $G = H'$ is integrable within T.

(2) If $G = H'$ for some $H \leq T$, then since K is characteristic in H', it is normal in H, that is, $H \leq N_T(K)$. But then H/K is a subgroup of $N_T(K)/K$ such that $(H/K)' = G/K$. Conversely, assume now that $G/K = (H/K)'$ for some $H \leq N_T(K)$. Then $G = H'K$. Since $K \leq H$ and K is perfect, we have $K = K' \leq H'$ and so $G = H'$. □

We will need the following result, which does not require the involved groups to be finite.

Lemma 2.3. *Let A be a metacyclic group $A = \langle x, y\rangle$ with $\langle x\rangle \trianglelefteq A$. The subgroups of A that are integrable within A are precisely all the subgroups of A'.*

Proof. Let B be any subgroup of A that is integrable within A. Then there exists some $C \leq A$ such that $B = C'$, and therefore $B \leq A'$. For the converse, write $[x, y] = x^c$ for some $c \in \mathbb{Z}$, so that $A' = \langle x^c\rangle$ by standard commutator calculus. Since $[x, y]$ commutes with x, we have $[x^t, y] = x^{ct}$ for some $t \in \mathbb{Z}$. Therefore if B is an arbitrary subgroup of A', say $B = \langle x^{ct}\rangle$, then if we set $C = \langle x^t, y\rangle$ we have $B = C'$. □

The following example shows that Lemma 2.3 cannot be extended to the classes of supersolvable or nilpotent groups.

Example 2.4. Let G be the standard wreath product $G = D_8 \wr C_2$. Then $G' \simeq D_8 \times C_2$, and therefore not all subgroups of G' are integrable, since D_8 is not integrable (see [1] for a history of this result). In this particular case the subgroups of G' that are not integrable in G are: four copies of D_8 (all maximal in G') and three copies of $C_2 \times C_2$, these last three subgroups are each contained in a unique maximal subgroup of G' of order 8 (one $C_4 \times C_2$ and two $(C_2)^3$).

3. Integrable almost-simple groups

We recall that [2, Theorem 3.2] states that if a group G is integrable, then $\mathrm{Inn}(G) \leq \mathrm{Aut}(G)'$ and that, indeed, $\mathrm{Inn}(G)$ has an integral within $\mathrm{Aut}(G)$. Following the language of [1, Section 5], we recall that a group H is a *reduced integral* of a group G if $H' = G$ and $C_H(G) = 1$. The following result uses ideas from [1, Lemma 5.2] and [2, Theorem 3.2] and does not require the groups to be finite.

Theorem 3.1. *Let S be a nonabelian simple group and let G be a group such that $\mathrm{Inn}(S) \leq G \leq \mathrm{Aut}(S)$. Then G is integrable if and only if G is integrable within $\mathrm{Aut}(S)$.*

Proof. The converse implication is obvious, so we assume now that G is integrable.

We recall that we write functions on the right, that is, if g is an arbitrary element of G and s is an arbitrary element of S we write $s.g$ for the image of s under the automorphism g. Also, if A is any group and $a \in A$, denote with $\gamma_a \in \mathrm{Aut}(A)$ the conjugation map $x.\gamma_a = a^{-1}xa$ for all $x \in A$. Since for every $g \in G$ and every $s \in S$ we have that $g^{-1}\gamma_s g = \gamma_{s.g}$, the action of G on S is equivalent to the action by conjugation of G on its normal subgroup $\mathrm{Inn}(S)$. In particular, $C_G(\mathrm{Inn}(S)) = 1$, and $\mathrm{Inn}(S)$ is characteristic in G (since $\mathrm{Inn}(S)$ is the unique minimal normal subgroup of G). Since $Z(G) \leq C_G(\mathrm{Inn}(S)) = 1$, we apply [1, Lemma 5.2] to deduce that G admits a reduced integral H, namely a group H such that $H' = G$ and $C_H(G) = 1$. We extend the faithful action of G on S to a faithful action of H on S by defining, for every $h \in H$ and every $s \in S$, the element $s.h$ to be the unique element of S such that $h^{-1}\gamma_s h = \gamma_{s.h}$. Note that this extension of the action is well defined since $\mathrm{Inn}(S)$ is characteristic in G and H acts faithfully by conjugation on G. Let $K = C_H(S)$ denote the kernel of this action. Then K is normal in H and $K \cap G = C_G(S) = 1$. Therefore $[K, G] = 1$, but then $K \leq C_H(G) = 1$. This shows that the above action of H on S is faithful, that is H is an integral of G within $\mathrm{Aut}(S)$. □

Throughout this section let S be a finite nonabelian simple group and, by identifying S with $\mathrm{Inn}(S)$, let $S \leq G \leq \mathrm{Aut}(S)$. By applying Lemma 2.2 (case (1) with $T := \mathrm{Aut}(S) = N_T(S)$) we have the following crucial fact.

Lemma 3.2. *G is integrable within $\mathrm{Aut}(S)$ if and only if G/S is integrable in the group of outer automorphisms $\mathrm{Out}(S) = \frac{\mathrm{Aut}(S)}{\mathrm{Inn}(S)}$.*

As an immediate consequence of Lemma 3.2 and Lemma 2.3 we have that Theorem A holds for all those S such that $\mathrm{Out}(S)$ is either abelian or metacyclic.

To complete our classification of integrable almost-simple groups we need only focus on those S for which $\mathrm{Out}(S)$ is neither abelian nor metacyclic. Using the classification of finite simple groups (CFSG), such simple groups (and their automorphism groups) are summarized in the following proposition.

Proposition 3.3. *Let S be a finite simple group having outer automorphism group that is neither abelian nor metacyclic. Then $(S, \mathrm{Out}(S))$ must be contained in the following list:*

(1) $\big(A_n(q), d.(f \times 2)\big)$, where $n \geq 2$, $d = (n+1, q-1) > 1$, $q = p^f$ and $f > 1$ is even.

(2) $\big(D_4(q), d.\big(f\times S_3\big)\big)$, *where* $d=(2,q-1)^2$ *and* $q=p^f$.
(3) $\big(D_n(q), d.\big(f\times 2\big)\big)$, *where* $n>4$, $q=p^f$ *with* p *an odd prime, and, respectively,* $d=(2,q-1)^2=2^2$ *when* n *is even, and* $d=(4,q^n-1)$ *when* n *odd.*
(4) $\big(E_6(q), 3.(f\times 2)\big)$, *where* $d=(3,q-1)=3$, $q=p^f$ *and* f *is even.*

Proof. This result is a consequence of the CFSG. The ATLAS [8, Tables 1 and 5] is a good reference. Note that a normal subgroup of order two in any group is always central, which implies that the outer automorphism group of each of the following groups is always abelian or metacyclic: $A_1(q)$, $B_2(q)$ for q odd, $B_n(q)$ and $C_n(q)$ when $n\geq 3$ and $E_7(q)$. Also note that field automorphisms commute with graph automorphisms (see [6]). □

We complete the proof of Theorem A by considering these four remaining cases of Proposition 3.3. For convenience in the following we set $A=\mathrm{Out}(S)$.

Case (1). $S=A_n(q)=\mathrm{PSL}_{n+1}(q)$, with $n\geq 2$, $d=(n+1,q-1)>1$ and $q=p^f$ with $f>1$ even.
In this situation the group A is metabelian isomorphic to $d.(f\times 2)$ and it has the following presentation (see [19, Proposition 2.2.3]):

$$\left\langle \delta,\phi,\iota \,|\, \delta^d=\phi^f=\iota^2=1, \delta^\phi=\delta^p, \delta^\iota=\delta^{-1}, \phi^\iota=\phi\right\rangle.$$

We have that $A'\leq\langle\delta\rangle$, since $A/\langle\delta\rangle$ is abelian. Moreover $A'\geq\left\langle\delta^2,\delta^{p-1}\right\rangle$, as $[\delta,\iota]=\delta^{-2}$ and $[\delta,\phi]=\delta^{p-1}$.
Assume first that p is odd. Then δ^{p-1} is a power of δ^2, forcing $\left\langle\delta^2,\delta^{p-1}\right\rangle=\left\langle\delta^2\right\rangle$. Since $A/\left\langle\delta^2\right\rangle$ is abelian when p is odd, then $A'=\left\langle\delta^2\right\rangle$. Moreover if we set $B=\langle\delta,\iota\rangle$ then B is a metacyclic group whose derived subgroup is $B'=\left\langle\delta^2\right\rangle=A'$. Thus by Lemma 2.3 every subgroup of A' is integrable in B and thus also in A.
Now we assume $p=2$. Since $\left\langle\delta^2,\delta^{p-1}\right\rangle\leq A'$, then $A'=\langle\delta\rangle$. Arguing as in the preceding situation, every subgroup of $\langle\delta\rangle$ is integrable in $\langle\delta,\iota\rangle$ and thus in A too.

We conclude that every subgroup of A' is integrable in A.

Case (2). $S=D_4(q)=P\Omega_8^+(q)$.
By [18, p. 181] we have that

$$\mathrm{Out}(S)\simeq\begin{cases} S_3\times f & \text{if } q \text{ is even,}\\ S_4\times f & \text{if } q \text{ is odd.}\end{cases}$$

So when q is even all subgroups of A' are integrable (these subgroups are just 1 and $A'=A_3$, whose order is three). When q is odd then $A'=A_4$ and again all subgroups H of A' are integrable in A, since if $H=A'$ this is immediate, if $H=V_4$ then $H=A''$, if H has order 3 then $H=(S_3)'$ for some $S_3<S_4<A$, and, finally, if H has order two then $H=(D_8)'$ for some $D_8<S_4<A$.

Case (3). $S=D_n(q)=O_{2n}^+(q)=P\Omega_{2n}^+(q)$, with $n>4$.
When the group A is nonabelian and not metacyclic then, according to [19, Propositions 2.7.3 and 2.8.2], A is isomorphic to $D_8\times f$. Therefore in this case too the subgroups of A that are integrable within A are precisely all the subgroups of A',

namely, 1 and $A' \simeq 2$.

Case (4). $S = E_6(q)$.
When A is nonabelian and not metacyclic then $A \simeq 3.(f \times 2)$ with $f > 1$ even. Then $|A'| = 3$ and, trivially, the subgroups of A that are integrable within A are precisely all the subgroups of A'.

This completes the proof of Theorem A.

4. Integrable 2-homogeneous groups

The enumeration of k-transitive and k-homogeneous subgroups of the finite symmetric group S_n (for $k \geq 2$) has a long history, which is intertwined with the discovery of various finite simple groups, extending back to the work of Mathieu [21, 22] (or even earlier, to notions of transitivity investigated by Cauchy (e.g., [7])). The 2-homogeneous subgroups of S_n have been completely determined as part of the work on the classification of the finite simple groups. The list of 2-homogeneous groups in the following theorem is extracted from Blackburn and Huppert [3, XII, Remark 7.5] and Dixon and Mortimer [11, Section 7.7 and Theorem 9.4B] (with minor errors corrected). We mention several contributions to this classification. Hering's results [13, 14] provide the tools for the classification of 2-transitive groups for which the socle is regular and abelian (Cases (2)-(9) below), building on significant contributions by Huppert [15] (who completed the solvable point stabilizer subgroup case (Case (5) below)), Livingston and Wagner [20], and Kantor [16, 17]. Curtis, Kantor and Seitz [9] classified the 2-transitive groups with socle a nonabelian simple group of Lie type (Cases (10)-(15)). Case (16), when the socle is sporadic, was undertaken by Hering in [14].

We summarize the complete list of the 2-homogeneous subgroups of S_n as follows.

Theorem 4.1. *Let G be a 2-homogeneous subgroup of S_n and set $S = \mathrm{soc}(G)$ to be its socle. Then G is one of the groups appearing in the following list:*

- *G is not 2-transitive.*
  This happens precisely when:
  (1) G is a 2-homogeneous subgroup of the affine semilinear group that contains the special affine linear group, $\mathrm{ASL}_1(q) \leq G \leq \mathrm{A\Gamma L}_1(q)$ and $n = q \equiv 3 \pmod 4$.
- *G is 2-transitive and S is regular and abelian.*
  Then $G \leq \mathrm{A\Gamma L}_d(q)$, with degree $n = q^d = |S|$ and point stabilizer G_0 that acts transitively on the set of nonzero vectors in the underlying vector space. This case happens precisely when G is the semidirect product $G = S \rtimes G_0$ and one of the following holds:
  (2) $\mathrm{SL}_d(q) \leq G_0 \leq \Gamma\mathrm{L}_d(q)$.
  (3) $G_0 \leq \Gamma\mathrm{L}_d(q)$ and G_0 contains a copy of $\mathrm{Sp}_d(q)$ as a normal subgroup.
  (4) $G_0 \leq \Gamma\mathrm{L}_6(2^f)$ and G_0 contains as a normal subgroup respectively a copy of $G_2(2^f)$ when $f > 1$, and of $\mathrm{U}_3(3)$ when $f = 1$.
  (5) G_0 is a solvable subgroup of $\Gamma\mathrm{L}_d(q)$ containing a normal extraspecial subgroup E of order 2^{df+1} such that $C_{G_0}(E) = Z(G_0)$ and $G_0/EZ(G_0)$ is faithfully represented on $E/Z(E)$. Moreover, this situation happens

if and only if $G_0 \le \mathrm{GL}_d(q)$ *where:*

$$(d,q) \in \{(2,3),(2,5),(2,7),(2,11),(2,23),(4,3)\}.$$

(6) $(G_0)'' \simeq \mathrm{SL}_2(5)$ *for* $d=2$ *and* $q \in \{9,11,19,29,59\}$.
(7) $G_0 \simeq A_6$, $d=4$, $q=2$.
(8) $G_0 \simeq A_7$, $d=4$, $q=2$.
(9) $G_0 \simeq \mathrm{SL}_2(13)$, $d=6$, $q=3$.

- *G is 2-transitive and S is a nonabelian simple group.*
  This case happens precisely when:
  (10) $S = A_n$ *and* $G \in \{A_n, S_n\}$.
  (11) $S = \mathrm{PSL}_d(q) \le G \le \mathrm{P\Gamma L}_d(q)$ *of degree* $n = (q^d-1)/(q-1)$.
  (12) $S = G = \mathrm{PSp}_{2m}(2)$ *of two possible degrees* $n \in \left\{2^{m-1}(2^m \pm 1)\right\}$.
  (13) $S = \mathrm{U}_3(q) \le G \le \mathrm{P\Gamma U}_3(q)$ *of degree* $n = q^3+1$.
  (14) $S = {}^2B_2(q) \le G \le \mathrm{Aut}(S)$, *with* $q = 2^{2m+1}$ *and degree* $n = q^2+1$.
  (15) $S = {}^2G_2(q) \le G \le \mathrm{Aut}(S)$, *with* $q = 3^{2m+1}$ *and degree* $n = q^3+1$.
  (16) *S is isomorphic to one of:*
    (a) $M_{11}, M_{12}, M_{23}, M_{24}$ *of degree, respectively,* $11, 12, 23$ *and* 24. *Moreover* $G = S$.
    (b) $\mathrm{PSL}_2(11)$ *of degree* 11, *and* $G = S$.
    (c) A_7 *or* $A_8 \simeq \mathrm{PSL}_4(2)$, *both of degree* 15, *and* $G = S$.
    (d) HS *of degree* 176 *and* $G \in \{S, \mathrm{Aut}(S) = S.2\}$.
    (e) Co_3 *of degree* 276 *and* $G = S$.

Remark 4.2. *Note that whenever in the statement of Theorem 4.1 we write $S \le G \le H$, for some suitable $H \le S_n$, then the subgroup H coincides with $N_{S_n}(S)$. This follows from the fact that $S \trianglelefteq H$ for all the groups listed above, and so $H \le N_{S_n}(S)$. Moreover, any group containing a 2-homogeneous subgroup must itself be 2-homogeneous, hence $N_{S_n}(S)$ is a 2-homogeneous subgroup of S_n with socle S and it must be contained in the same H coming from the classification in Theorem 4.1, that is, $N_{S_n}(S) \le H$. A similar situation occurs in Case (2) with G_0 in place of G.*

Note that trivially if G is 2-homogeneous and $G = H'$ (for some $H \le S_n$), then H itself is also 2-homogeneous. Thus our approach to solving Problem 2 is to determine which derived subgroups of 2-homogeneous subgroups of S_n are themselves 2-homogeneous. To obtain such a classification we use Lemma 2.2 (both parts (1) and (2)). The following result guarantees that the hypotheses of Lemma 2.2 are satisfied. Indeed, it is a significant step in the classification of the 2-homogeneous subgroups of S_n (that is, of the proof of Theorem 4.1).

Lemma 4.3. *Let G be a 2-homogeneous subgroup of S_n. Then the socle of G is the unique minimal normal subgroup of G.*

Proof. This follows immediately by [11, Theorem 4.1B] when G is 2-transitive and S is nonabelian, and in the other cases from [11, Theorem 4.3B] and the fact that G is primitive ([11, Exercise 2.1.10]; see also [20, p. 402]). □

As an immediate application of Lemma 2.2 we therefore have the following corollary.

Corollary 4.4. *Let G be a 2-homogeneous subgroup of S_n having socle S. Then G is integrable within S_n if and only if G/S is integrable within $N_{S_n}(S)/S$. Suppose further that G has a nontrivial characteristic perfect subgroup K. Then G is integrable within S_n if and only if G/K is integrable within $N_{S_n}(K)/K$.*

To classify the 2-homogeneous subgroups of degree n that are integrable within S_n (and therefore prove Theorem B) we now take into consideration all the groups G in the list of Theorem 4.1 and apply Corollary 4.4 to them. In the following we will always denote with $q = p^f$ (p a prime) the order of the field on which classical and Lie type groups are defined, except when the groups are unitary in which case we set $q^2 = p^f$.

Case (1). We have that $\mathrm{A\Gamma L}_1(q)/\mathrm{ASL}_1(q)$ is isomorphic to a metacyclic group of order $(q-1) : f$. By Lemma 2.3 we conclude that in this case G is integrable within S_n precisely when G is a subgroup of $(\mathrm{A\Gamma L}_1(q))'$, containing $\mathrm{ASL}_1(q)$.

Case (2). We have that $\mathrm{ASL}_d(q) = S \rtimes \mathrm{SL}_d(q)$ is a characteristic and perfect subgroup of $G = S \rtimes G_0$, and therefore by Corollary 4.4 the integrability of G within S_n is equivalent to the integrability of $G_0/\mathrm{SL}_d(q)$ within $\Gamma\mathrm{L}_d(q)/\mathrm{SL}_d(q)$. Note that this latter group is metacyclic isomorphic to $(q-1) : f$. Therefore Lemma 2.3 implies that G is integrable within S_n if and only if $\mathrm{SL}_d(q) \leq G_0 \leq \Gamma\mathrm{L}_d(q)'$.

Case (3). Let H be a subgroup of $\Gamma\mathrm{L}_d(q)$ isomorphic to $\mathrm{Sp}_d(q)$ and let $H \leq G_0 \leq N_{\Gamma\mathrm{L}_d(q)}(H) \simeq \Gamma\mathrm{Sp}_d(q)$. By [19, Section 2.4] this latter group is isomorphic to

$$\big(\mathrm{Sp}_d(q) : (q-1)\big) : f.$$

In particular $\mathrm{Sp}_d(q)$ is characteristic in $\Gamma\mathrm{Sp}_d(q)$, and thus by Corollary 4.4 the integrable 2-homogeneous subgroups of this case correspond precisely to those G_0 such that G_0/H is isomorphic to an integrable subgroup within $\Gamma\mathrm{Sp}_d(q))/\mathrm{Sp}_d(q)$. Finally, note that $\Gamma\mathrm{Sp}_d(q))/\mathrm{Sp}_d(q)$ is a metacyclic group. Lemma 2.3 completes the proof of this case.

Case (4). In this case $q = 2^f$. It is well-known that the group $\mathrm{Sp}_6(q)$ contains, as a maximal subgroup, a subgroup isomorphic to $G_2(q)$ (see [10] or [23, Theorem 3.7]), which is transitive on the corresponding projective space. We let H be a subgroup of $\mathrm{SL}_6(q)$ isomorphic to $G_2(q)$ and we first determine the structure of $N_{\Gamma\mathrm{L}_6(q)}(H)$. By [4, Table 8.29], we see that a field automorphism ϕ of order f can be taken to normalize H. Thus by Dedekind's modular law we have that

$$N_{\Gamma\mathrm{L}_6(q)}(H) = N_{\mathrm{GL}_6(q)}(H).\, \langle \phi \rangle\, .$$

Also $N_{\mathrm{GL}_6(q)}(H) = ZN_{\mathrm{SL}_6(q)}(H)$, where $Z = Z(\mathrm{GL}_6(q)) \simeq (q-1)$ (here we used again [4, Table 8.29]). We now claim that

$$N_{\mathrm{SL}_6(q)}(H) = Z(\mathrm{SL}_6(q)) \times H \simeq (q-1, 3) \times G_2(q).$$

This follows from the fact that every maximal subgroup of $\mathrm{SL}_6(q)$ containing a copy of $G_2(q)$ is isomorphic to $(q-1, 3) \times \mathrm{Sp}_6(q)$. To prove this statement we look at the maximal subgroups of $\mathrm{SL}_6(q)$ in [4, Tables 8.24 and 8.25] (eventually also Tables 8.3 and 8.4 to exclude the unique non immediate case of maximal subgroups isomorphic to $\mathrm{SL}_3(q^2).(q+1).2)$. This shows that

$$N_{\Gamma\mathrm{L}_6(q)}(H) \simeq (H \times (q-1)).f$$

and therefore $N_{\Gamma\mathrm{L}_6(q)}(H)/H$ is metacyclic. Lemmas 2.2 and 2.3 complete the proof of this case. We observe that Remark 4.5 summarizes the integrable subgroups for this particular case.

Case (5). The main reference for this case is [15], where the full classification for the 2-transitive groups of this type appear. In particular we have that E is a subgroup of $\mathrm{SL}_d(q)$; moreover, since q is always a prime number, $G_0 \leq \mathrm{GL}_d(q) = \Gamma\mathrm{L}_d(q)$. Now, the integrability of G in S_n, by Corollary 4.4, is equivalent to the integrability of G_0 within $\mathrm{GL}_d(q)$. Thus in particular G_0 must be a subgroup of $\mathrm{SL}_d(q)$ that normalizes E. We examine all possible situations that are classified in [15].
When $d = 2$, according to [15], the subgroups of $\mathrm{SL}_2(q)$ that are transitive on nonzero vectors appear just when $q \in \{3, 5\}$. Moreover, if $q = 3$, since $\mathrm{SL}_2(3) \simeq 2^{\cdot}A_4$, the subgroup E is the unique Sylow 2-subgroup of $\mathrm{SL}_2(3)$. Then either $G_0 = E$ or $G_0 = \mathrm{SL}_2(3)$. Both are integrable in $\mathrm{GL}_2(3)$ since $\mathrm{SL}_2(3) = (\mathrm{GL}_2(3))'$ and $E = (\mathrm{GL}_2(3))''$.
When $q = 5$ then E is a Sylow 2-subgroup of $\mathrm{SL}_2(5)$, since $\mathrm{SL}_2(5) \simeq 2^{\cdot}A_5$. In this case $G_0 = N_{\mathrm{SL}_2(5)}(E) \simeq 2^{\cdot}A_4 \simeq \mathrm{SL}_2(3)$ is transitive on nonzero vectors and also integrable (in $\mathrm{GL}_2(5)$) since it is equal to $(N_{\mathrm{GL}_2(5)}(E))'$.
Finally we consider the case $d = 4$ and $q = 3$. Then E is isomorphic to the central product $D_8 \circ Q_8$. Also $Z(E) = Z(\mathrm{GL}_4(3))$ and up to conjugation there are only three subgroups in $\mathrm{GL}_4(3)$ that are transitive on nonzero vectors. They are all contained in $\mathrm{SL}_4(3)$ and they are respectively isomorphic to $E : 5$, $(E : 5).2$ and $(E : 5).4$. Only the first of these is integrable being the derived subgroup of the other two.

Case (6). Set $L = (G_0)''$ be the second derived subgroup of G_0, so that $L \simeq \mathrm{SL}_2(5) \simeq 2^{\cdot}A_5$ and $L \leq G_0 \leq N_{\Gamma\mathrm{L}_2(q)}(L)$ with q as in the statement. By Corollary 4.4, G is integrable within S_n if and only if G_0/L is integrable within $N_{\Gamma\mathrm{L}_2(q)}(L)/L$, therefore we examine the structure of this last group. By [4, Table 8.2], L is always a maximal subgroup $\mathrm{SL}_2(q)$ and therefore it coincides with its normalizer in $\mathrm{SL}_2(q)$. When $q \neq 9$ we have that q is prime, forcing $\mathrm{GL}_2(q) = \Gamma\mathrm{L}_2(q)$. It follows that $N_{\Gamma\mathrm{L}_2(q)}(L) = LZ$, where $Z = \langle z \rangle$ is the center of $\mathrm{GL}_2(q)$ (where we used again [4, Table 8.2]). The conclusion is that, when $q \neq 9$, only $G = S \rtimes L \simeq S : \mathrm{SL}_2(5)$ is integrable within S_n. Assume now that $q = 9$. Again by [4, 8.2], the subgroup L is maximal in $\mathrm{SL}_2(9)$ and, outside $\mathrm{SL}_2(9)Z$, only field automorphisms of order two normalize it. Therefore $N_{\Gamma\mathrm{L}_2(9)}(L) = LZ.2$ and $N_{\Gamma\mathrm{L}_2(9)}(L)/L \simeq D_8$. We conclude that, for $q = 9$, G_0 is integrable within S_n if it is either L or $L\left\langle z^4 \right\rangle = (N_{\Gamma\mathrm{L}_2(9)}(L))'$.

Cases (7)-(9). These groups are all integrable, since G_0 (and therefore G) is perfect.

Case (10). Of course $G = A_n$ is integrable within S_n, while $G = S_n$ is not.

Case (11). Since $S = \mathrm{PSL}_d(q)$ and $N_{S_n}(S) = \mathrm{P}\Gamma\mathrm{L}_d(q)$, by [19, Proposition 2.2.3], the group $\mathrm{P}\Gamma\mathrm{L}_d(q)/\mathrm{PSL}_d(q) \leq \mathrm{Out}(S)$ is isomorphic to a finite metacyclic group

$m : f$, where $m = \gcd(d, q-1)$. Lemma 2.3 together with Corollary 4.4 imply that the integrable subgroups in this case are precisely all those G such that $\mathrm{PSL}_d(q) \leq G \leq (\mathrm{P\Gamma L}_d(q))'$.

Case (12). Here we have that $G = S = N_{S_n}(S)$ is a perfect group and therefore trivially integrable.

Case (13). Now we write $q^2 = p^f$, so that $S = \mathrm{U}_3(q)$ and $N_{S_n}(S) = \mathrm{P\Gamma U}_3(q)$. By [19, Proposition 2.3.5], the group $\mathrm{P\Gamma U}_3(q)/\mathrm{U}_3(q)$ is isomorphic to a finite metacyclic group $m : (2f)$, where $m = \gcd(3, q+1)$. By Corollary 4.4, the integrable subgroups in this case are precisely $S = \mathrm{U}_3(q)$ (always) and $\mathrm{PGU}_3(q)$ when $3|(q+1)$ and $\mathrm{P\Gamma U}_3(q)/\mathrm{U}_3(q)$ is not abelian. This latter situation happens if and only if $p \equiv 2 \pmod 3$ and $f/2$ is odd.

Cases (14)-(15). In both cases we have that $\mathrm{Out}(S)$ is a cyclic group of order f. Therefore, by Lemma 2.3, Theorem A and Lemma 3.2, the only integrable groups are $G = S$ in both cases.

Case (16). All the simple groups S listed have trivial or cyclic outer automorphism group. Therefore, by Lemma 2.3, Theorem A and Lemma 3.2, these are the unique integrable subgroups in this case.

It is straightforward now to check that in all cases - except Case (5) - the 2-homogeneous subgroups G that are integrable within S_n are precisely those G such that

$$(N_{S_n}(S))'' \leq G \leq (N_{S_n}(S))',$$

thus proving Theorem B. We leave this verification to the reader. We collect in the following remark the solvable exceptions appearing in Case (5).

Remark 4.5. *Let G be a 2-transitive subgroup of S_n that belongs to Case (5) of Theorem 4.1. Note that this is the unique case in which G is solvable. Then G is integrable within S_n precisely when G is isomorphic to a group in the following table*

isomorphism type of G	degree n
$3^2 : Q_8$	3^2
$\mathrm{ASL}_2(3)$	3^2
$5^2 : \mathrm{SL}_2(3)$	5^2
$3^4 : ((D_8 \circ Q_8) : 5)$	3^4

Note that each of these subgroups is between $(N_{S_n}(S))''$ and $(N_{S_n}(S))'$ and that, apart from when $n = 3^2$, there are also subgroups in this interval that are not integrable (within S_n).

As mentioned in the Introduction, the following example shows that the more general problem of classifying all integrable 2-homogeneous groups (as abstract groups) has a different solution than Problem 2 has.

Example 4.6. The group $G = \mathrm{PGL}_3(7)$ acts 2-transitively on the set of projective points (or, equivalently, on the set of projective lines) whose size is 57 (this group arises in Case (11) of Theorem 4.1). By Theorem B, G is not integrable within S_{57}. Nevertheless, the simple group $\mathrm{PSL}_3(7)$ has outer automorphism group isomorphic to S_3, since it is generated by an outer diagonal automorphism of order three and a graph involution. This implies that $(\mathrm{Aut}(\mathrm{PSL}_3(7)))' = \mathrm{PGL}_3(7)$ which demonstrates the abstract integrability of G.

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Department of Mathematics and Statistics, Saint Louis University, 220 N. Grand Blvd., St. Louis, MO 63103, USA

Email address: russell.blyth@slu.edu

Dipartimento di Matematica e Informatica 'Ulisse Dini', Viale Morgagni 67/A, 50134 Firenze, Italy

Email address: francesco.fumagalli@unifi.it

Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano–Bicocca, Milan 20125, Italy.

Email address: francesco.matucci@unimib.it