Polynomial preserving Virtual Elements with curved edges

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In this paper we tackle the problem of constructing conforming Virtual Element spaces on polygons with curved edges. Unlike previous VEM approaches for curvilinear elements, the present construction ensures that the local VEM spaces contain all the polynomials of a given degree, thus providing the full satisfaction of the patch test. Moreover, unlike standard isoparametric FEM, the present approach allows to deal with curved edges at an intermediate scale, between the small scale (treatable by homogenization) and the bigger one (where a finer mesh would make the curve flatter and flatter) The proposed method is supported by theoretical analysis and numerical tests.

1. Introduction

In this paper we tackle the problem of constructing *conforming* Virtual Element spaces on polygons with curved edges. Apart from the obvious convenience in treating computational domains with curved boundaries with a better accuracy, the interest in using polygons with curved edges could arise in various circumstances, as when a singularity of the solution near the boundary is expected (e.g. in Fig. 1), or in the presence of mixtures of different materials or rough boundaries (as in Fig. 2) when we are still far from the homogenized limit, and so on.

The treatment of curved edges with *nonconforming* approximations (as in nonconforming VEMs and HHO methods, see for instance [7] or [20]) is relatively easier, since the traces of the elements of the Virtual Element spaces do not need to be continuous at the endpoints of edges (and in particular at the endpoints of the *curved* edges). Here instead we want to stick on *conforming* approximations, which (as it is well known) have several advantages in various different circumstances.

In order to simplify the exposition we will limit ourselves, here, to the case of a *single curved edge per element*, but the general philosophy can be easily applied to the case of several curved edges.

Conforming Virtual Element spaces on polygons with curved edges, given in parametric form, have already been introduced in [18], using, on the curved edges, functions that are *polynomials in the parameter t*. Such an approach has several advantages, including the simplicity of the definition and the adaptability to more general situations. The drawback is that the local VEM spaces do not contain all the polynomials of a given degree, and the patch test is not in general satisfied. An extension that guarantees the presence of rigid body motions, thus suitable for solid mechanics problems, was developed in [5]. Instead, the present approach ensures that the local VEM spaces contain all the polynomials of a given degree, thus providing the *full satisfaction of the patch test*. Moreover here the result will not depend (apart from round-off errors) from the choice of the parametrization of the curve.

For another different approach and use of Virtual Elements with curved edges we refer also to [19], where the authors consider a polygonal approximation Ω_h (with straight edges) of the original domain Ω (with curved edges). The discrete solution (say, u_h) is originally defined in Ω_h. But then a smart correction $\tilde{u}_h$ (using a suitable approximation of high order normal derivatives of u_h) is introduced, and used to define a problem in Ω (very much in the spirit of [21], and then [22], [23]).

Another approach to deal with curved edges has been proposed in [2]. This last approach is very similar to the present one (although the two have been developed independently). The (minor!) difference between the two treatments is tricky, and we describe it in detail later on, in Remark 3.4: we anticipate here that the approach of [2] is somehow *simpler*, while ours is *safer* (= less exposed to the possible damages of round-off errors).

We also point out that more traditional ways of dealing with curved edges in Finite Elements (typically, using iso-parametric elements) are based, on the one hand, on the property that *fixed curves tend to be (locally!) straight when the discretization gets finer and finer*, and, on the other hand, on the fact that when the discontinuities of the material (or the wiggles on the boundaries) occur on a very small scale, then one can go for the homogenized limit. Here we want to be able to work somewhat *in between*: when the scale is too big to use the homogenized limit, but too small to be made locally flat. A numerical example in this direction will be given in the last section.

In this presentation it will be convenient to treat different types of curved edges: curved edges internal to the computational domain (typically, to be used in the presence of two different materials separated by a curved interface), curved edges belonging to a part of the boundary where Dirichlet conditions are imposed, and

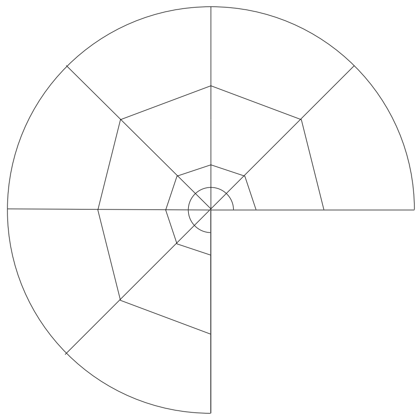

Fig. 1: Taking into account a re-entrant corner

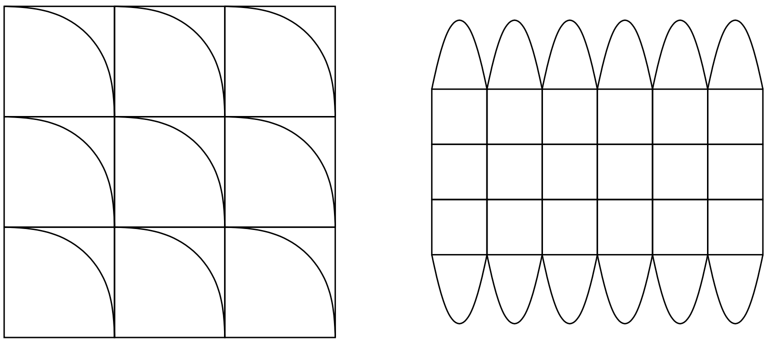

Fig. 2: Other decompositions

curved edges belonging to a part of the boundary where Neumann or Robin boundary conditions are imposed.

We point out that we are ready to accept that an edge which a-priori is declared to be curved might, by chance, be actually straight (or very very near to be straight). On the other hand, there might be parts of the boundaries that are "known to be straight", and these will be treated as straight from the very beginning. Hence, to summarize, we will assume that there are: *i)* edges that are declared as "straight" (and **must** be straight), and *ii)* edges that are declared as "curved" (and **might** be either curved or straight).

The whole treatment here has been concentrated on simple linear elliptic equations in 2 dimensions. Needless to say, there is a lot of interesting work still to be done, in order to extend it to more complex problems, much more interesting for applications. This, apart from the natural need to deal with the three-dimensional case, includes obviously all the subjects where Virtual Elements proved already to be a viable and useful discretization method, as for instance: linear elasticity problems, [26], [6]; incompressible and nearly incompressible materials, [3], [16], [15]; plate bending and more generally fourth order problems, [25], [4] [31]; electromagnetic problems [10] and wave propagation, [17], [28],[17], and so on. Moreover, the use of elements with curved boundary to deal with contact problems (already treated successfully for VEM discretizations with straight edges e.g. in [29], [30]) is a surely challenging topic full of potential troubles and appealing perspectives.

Similarly, in mesh adaptation for dealing with singularity (as in [8]), the use of curved edges could also be very attractive.

On the other hand, more theoretical aspects could also be profitably extended, as the use of $H(\mathrm{div})$ and $H(\mathrm{curl})$ elements (see e.g. [11]), as well as applications to problems with variable coefficients (as e.g.in [13]), or the finer aspects of interpolation errors (see [24], [14]).

Finally, the Serendipity versions (see e.g. [12]) of all these Virtual Element spaces is possibly the most natural extension of the present work.

An outline of the paper is as follows. In Section 2, after recalling some basic notation on polynomial spaces, we will present the model problem: essentially, the simple linear elliptic problem $-\mathrm{div}(\kappa\nabla u) = f$ in a domain Ω, with Dirichlet boundary conditions on one part of the boundary and Robin boundary condition on the remaining part. We will also assume, for simplicity, that the material coefficient κ is piece-wise constant, assuming one constant value in a sub-domain $\Omega_i \subset\subset \Omega$ and another constant value in the remaining part. This trivially extends to the case of several sobdomains Ω_i, each with a different value of the coefficient κ_i. In Section 3 we will introduce the Virtual Element local spaces to be used in the sub-domains of the decomposition. In Sections 4 and 5 we will then introduce the *local* and *global* stiffness matrices, respectively. Finally, in Section 6 we will write the discretized problem and present the corresponding error bounds in terms of the interpolation errors that, in turn, will be discussed in Section 7. Finally, some (academic) numerical experiments will be presented in Section 8.

2. The continuous problem

2.1. *Notation*

Throughout the paper, if d (= *dimension*) is an integer ≥ 1 and k is an integer, $\mathbb{P}_{k,d}$ will denote the space of polynomials of degree $\leq k$ in d dimensions. As usual, $\mathbb{P}_{-1,d} = \{0\}$. In practice, we will consider here only the cases $d = 1$ and $d = 2$. We will denote by π_k^d the dimension of $\mathbb{P}_k$ in d dimensions, so that for $k \geq 0$ we have

$$\pi_k^1 = k + 1 \qquad \text{and} \qquad \pi_k^2 = (k + 1)(k + 2)/2. \tag{2.1}$$

In most cases, when no confusion can occur, we will just use π_k instead of π_k^2.

If $\mathcal{O}$ is a domain in $\mathbb{R}^d$ and k is an integer, we will donote by $\mathbb{P}_k(\mathcal{O})$ the space of the restrictions to $\mathcal{O}$ of the polynomials of $\mathbb{P}_k$. With an abuse of notation, we will often use simply $\mathbb{P}_k$ instead of $\mathbb{P}_k(\mathcal{O})$ when no confusion is likely to occur.

For a 1-d manifold Γ we denote by $|\Gamma|$ its length. For an open set $\omega \subset \mathbb{R}^2$ we denote by $|\omega|$ its area.

Moreover, we will denote by $\Pi_k^{0,\mathcal{O}}$ the orthogonal-projection operator, in $L^2(\mathcal{O})$, onto $\mathbb{P}_k(\mathcal{O})$, defined, as usual, for every $v \in L^2(\mathcal{O})$, by

$$\Pi_k^{0,\mathcal{O}}(v) \in \mathbb{P}_k(\mathcal{O}) \qquad \textit{and} \qquad \int_{\mathcal{O}} (v - \Pi_k^{0,\mathcal{O}}(v))\, q_k \,\mathrm{d}\mathcal{O} = 0 \quad \forall q_k \in \mathbb{P}_k(\mathcal{O}). \tag{2.2}$$

Finally, given a function $\psi \in L^2(\mathcal{O})$ and an integer $s \geq 0$, we recall that the *moments of order* $\leq s$ *of* ψ *on* $\mathcal{O}$ are defined as:

$$\int_{\mathcal{O}} \psi\, q_s \,\mathrm{d}\mathcal{O} \qquad \text{for } q_s \in \mathbb{P}_s(\mathcal{O}). \tag{2.3}$$

Hence *to assign the moments of* ψ *up to the order* s *on* $\mathcal{O}$ will amount to π_s^d conditions. Tipically this will be used when these moments are considered as *degrees of freedom*. Then, we will take in $\mathbb{P}_s$ a basis $\{q_i\}$ such that $\|q_i\|_{L^1} \simeq 1$ for all i.

Throughout the paper we will follow the common notation for scalar products, norms, and seminorms. In particular, $(v,w)_{0,\mathcal{O}}$ (sometimes, just $(v,w)_0$) and $\|v\|_{0,\mathcal{O}}$ (sometimes, just $\|v\|_0$) will denote the L^2 scalar product and norm, $|v|_{1,\mathcal{O}}$ (sometimes, just $|v|_1$) and $\|v\|_{1,\mathcal{O}}$ (sometimes, just $\|v\|_1$) the H^1 semi-norm and norm.

2.2. *The continuous problem*

Let Ω be a bounded open subset of $\mathbb{R}^2$ with Lipschitz continuous boundary and let Ω_i be a subset of Ω, also with a Lipschitz continuous boundary, such that (for simplicity) $\overline{\Omega_i} \subset \Omega$ (implying in particular that $\partial\Omega_i$ cannot touch $\partial\Omega$). See Fig. 3.

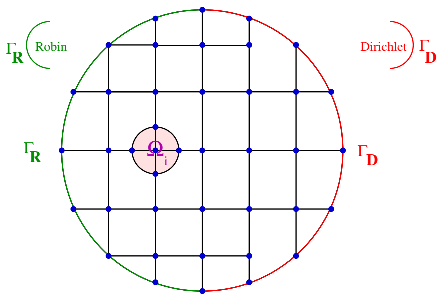

Fig. 3: A decomposition with curved edges

We will assume that the material properties in Ω_i might be different from those in $\Omega \setminus \Omega_i$. With a suitable attention to regularity issues one could also easily treat more general cases.

As already observed in the Introduction, we recall that both on $\partial\Omega_i$ and on $\partial\Omega$ we might have edges that are *declared to be straight*. In such a case it would be natural to treat the corresponding element as a *polygonal element*. On the other hand, it would be cumbersome to check, for every boundary edge, whether it is *straight* (or *very close to straight*) or *curved*. Hence, we will **assume** that each boundary edge belongs to one (and only one) of two distinct sets: the set of edges that we know (from the very beginning) to be straight, and the set of edges that *might be either curved or straight or almost straight*. All the edges (and the corresponding elements) in each set will be treated in the same manner. This implies that our treatment of curved edges should not fail if by chance the edge is straight or almost straight.

We consider the model problem

$$\begin{cases} \text{Find } u \in H^1(\Omega) \text{ such that:} \\ -\operatorname{div}(\kappa \nabla u) = f \ \text{ in } \Omega, \\ u = g_D \text{ on } \Gamma_D \text{ and } (\kappa\nabla u)\cdot \mathbf{n} + \rho u \ = \ g_R \text{ on } \Gamma_R, \end{cases} \tag{2.4}$$

where

- κ assumes two constant values: κ_i in Ω_i and κ_0 in $\Omega \setminus \Omega_i$, both > 0,
- Γ_D and Γ_N are open connected subsets of $\partial\Omega$ with $|\Gamma_D| > 0$, $\overline{\Gamma}_D \cup \overline{\Gamma}_R = \partial\Omega$ and $\Gamma_D \cap \Gamma_R = \emptyset$,
- f is given, say, in $L^2(\Omega)$,
- g_D is given, say, in $H^1(\Gamma_D)$,
- g_R is given, say, in $L^2(\Gamma_D)$,
- $\rho \geq 0$ is given in $L^\infty(\Gamma_R)$. Note that for $\rho = 0$ we are back to Neumann boundary conditions.

It is very well known (and not difficult to see) that defining

$$H^1_{0,\Gamma_D} := \{v \in H^1(\Omega) \text{ such that } v = 0 \text{ on } \Gamma_D\}, \tag{2.5}$$

and

$$H^1_{g_D,\Gamma_D} := \{v \in H^1(\Omega) \text{ such that } v = g_D \text{ on } \Gamma_D\}, \tag{2.6}$$

the solution u of (2.4) coincides with the (unique) solution of the variational problem

$$\begin{cases} \text{find } u \in H^1_{g_D,\Gamma_D} \text{ such that} \\ \displaystyle\int_\Omega \kappa \nabla u \cdot \nabla v \, \mathrm{d}\Omega + \int_{\Gamma_R} \rho\, u\, v \, \mathrm{d}s = \int_\Omega f\, v \, \mathrm{d}\Omega + \int_{\Gamma_R} g_R\, v \, \mathrm{d}s \ \ \forall v \in H^1_{0,\Gamma_D}. \end{cases} \tag{2.7}$$

We also point out that, in a natural way, (2.7) also implies that the co-normal derivative $\kappa\nabla u \cdot \mathbf{n}$ is continuous across the boundary of Ω_i.

2.3. *The decomposition*

Let $\mathcal{T}_h$ be a decomposition of Ω into "polygons" $\mathcal{P}$. For simplicity of exposition we assume that each polygon $\mathcal{P}$ has at most one curved edge. We also assume that $\partial\Omega_i$ is all contained in the union of the $\partial\mathcal{P}$ (in other words: the decomposition respects the discontinuities of κ). See Fig. 3.

In order to construct virtual element spaces suitable for the discretization of (2.7) we observe that we should, a-priori, be prepared to distinguish among several different types of elements in $\mathcal{T}_h$:

- $\{\mathbf{0}\}$ Polygons with straight edges. For them we are going to use classical VEMs.
- $\{\mathbf{1}\}$ Elements with a curved edge shared with another element. For instance, referring to Fig.3, the elements having a curved edge that belongs to $\partial\Omega_i$. There are 8 of them in Fig. 4.
- $\{\mathbf{2}\}$ Elements that have a curved edge that belongs entirely to $\overline{\Gamma}_D$. There are 8 of them in our Fig. 4.
- $\{\mathbf{3}\}$ Elements that have a curved edge that belongs entirely to $\overline{\Gamma}_R$. There are 8 of them in our Fig. 4.

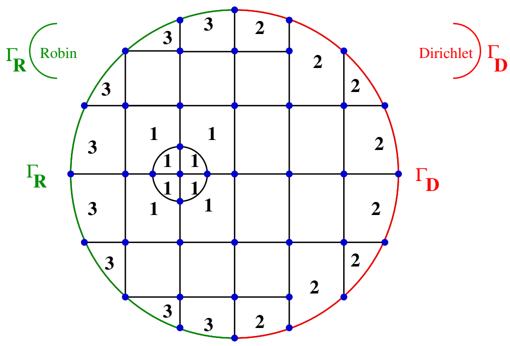

Fig. 4: Types of elements with a curved edge

3. The local spaces

3.1. *Subspaces on polygons with straight edges*

We start by recalling the classical VEMs commonly used in polygonal decompositions. For a given integer $k \geq 1$ we consider the trace space

$$\mathcal{B}_k(\partial\mathcal{P}) := \{v \in C^0(\partial\mathcal{P}) \text{ such that } v_{|e} \in \mathbb{P}_k(e)\ \forall \text{ edge } e \subset \partial\mathcal{P}\}. \tag{3.1}$$

Then, for another integer k_Δ with $-1 \leq k_\Delta \leq k$ we consider the space:

$$\mathcal{V}_{k,k_\Delta}(\mathcal{P}) := \{v \text{ such that } v_{|\partial\mathcal{P}} \in \mathcal{B}_k(\partial\mathcal{P}),\ \text{and } \Delta v \in \mathbb{P}_{k_\Delta}(\mathcal{P})\}. \tag{3.2}$$

The natural set of degrees of freedom for $\mathcal{V}_{k,k_\Delta}(\mathcal{P})$ is given by (see e.g. [1])

- the values at each vertex of $\mathcal{P}$,
- (for $k \geq 2$) the values at the $k-1$ Gauss-Lobatto points of each edge of $\mathcal{P}$,
- (for $k_\Delta \geq 0$) the moments of order $\leq k_\Delta$ inside $\mathcal{P}$.

We point out that polygons having *two or more consecutive edges that belong to the same straight line* are perfectly allowed, so that *vertices and edges*, here, do not coincide with the naive idea that one might have when speaking of *vertices and edges of a polygon*. See the example in Fig. 5.

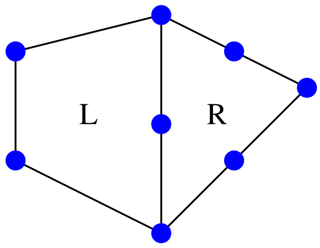

Fig. 5: The Left element has 5 vertices and edges; the Right element has 6 vertices and edges.

Here, for simplicity, we will stick on the simplest case $k_\Delta \equiv k-2$, which is the original choice of [9]. Hence, for $k \geq 1$ we set

$$\mathcal{V}_k(\mathcal{P}) := \mathcal{V}_{k,k-2}(\mathcal{P}) \tag{3.3}$$

and we have therefore the degrees of freedom:

- the values at each vertex of $\mathcal{P}$,
- (for $k \geq 2$) the values at the $k-1$ Gauss-Lobatto points of each edge of $\mathcal{P}$,
- (for $k \geq 2$) the moments of order $\leq k-2$ inside $\mathcal{P}$.

(3.4)

Remark 3.1. In previous works we used the bigger spaces (corresponding to $k_\Delta = k-1$ or $k_\Delta = k$) as starting point for a Serendipity correction (see [12]) that allowed to end up with local VEM spaces smaller than the $\mathcal{V}_k$ defined in (3.3). Here however the Serendipity procedure should be done in different ways for different types of elements (according to our previous classification), and the presentation would become more cumbersome.

3.2. *Subspaces on elements with a curved edge not in* Γ_D

When dealing with elements $\mathcal{P}$ having a curved edge, we would like to follow the same philosophy of the straight polygons. The main feature that we are not willing to give-up here is that the VEM space $\mathcal{V}_k(\mathcal{P})$ contains the polynomial space $\mathbb{P}_k(\mathcal{P})$.

As we already said twice, we are ready to accept that an edge which a-priori could be curved (say, an edge in $\partial\Omega_i$ or in $\partial\Omega$) might, by chance, be straight. However, in the present theoretical treatment we will assume, for simplicity, that all the edges in $\partial\Omega_i$ and in $\partial\Omega$ are declared as "curved", and all the other internal edges are declared as "straight". We recall that, for simplicity of exposition, we also assumed that each element has at most one curved edge.

To start with, we define therefore the space of traces on a curved edge as follows. For a given integer $k \geq 1$, on a given element $\mathcal{P}$ with a curved edge γ, we consider the trace space

$$\begin{aligned}\mathcal{B}_k(\partial\mathcal{P}) := \{v \in C^0(\partial\mathcal{P}) \text{ such that: } v_{|e} \in \mathbb{P}_k(e)\,\forall \text{ straight edge } e \subset \partial\mathcal{P},\\ \text{and } v_{|\gamma} \equiv q_{k\,|\gamma} \text{ for some } q_k \in \mathbb{P}_{k,2}\}.\end{aligned} \tag{3.5}$$

It is clear that:

the trace on $\partial\mathcal{P}$ of every polynomial of $\mathbb{P}_{k,2}$ belongs to $\mathcal{B}_k(\partial\mathcal{P})$.

Then, for every integer $k \geq 1$ we can define (*exactly as before*)

$$\mathcal{V}_k(\mathcal{P}) := \{v \text{ such that } v_{|\partial\mathcal{P}} \in \mathcal{B}_k(\partial\mathcal{P}), \text{ and } \Delta v \in \mathbb{P}_{k-2}(\mathcal{P})\}. \tag{3.6}$$

Here too it is easy to check that for every $k \geq 1$ we have

$$\mathbb{P}_k(\mathcal{P}) \subseteq \mathcal{V}_k(\mathcal{P}). \tag{3.7}$$

The delicate point comes from the choice of the *degrees of freedom*, treated in the next subsection.

3.3. *VEM spaces, degrees of freedom, and "generators"*

In $\mathcal{B}_k(\partial\mathcal{P})$ it seems natural to start taking as degrees of freedom:

- The values of v at the vertices,

and *for every straight edge* e, and $k \geq 2$:

- The values of v at the $k-1$ Gauss-Lobatto points of e.

With that, we took care of the straight edges *and of the values at the endpoints of the curved edge*. But already for $k = 1$ we need some additional information on the edge γ declared as curved. Indeed, once we know the values of a polynomial in $\mathbb{P}_k$ at two distinct points, we still need $\pi_k - 2$ degrees of freedom to determine it uniquely. This will be done by considering the values at $\pi_k - 2$ *fictitious points*, that we call *trace generator points* (or, simply, *tg points*, or even *tgp*) located as in Fig. 6 (essentially: the points that would normally be used to place *the degrees of freedom for $\mathbb{P}_k$ on an ideal triangle having "the segment joining the endpoints of the curved edge" as base*).

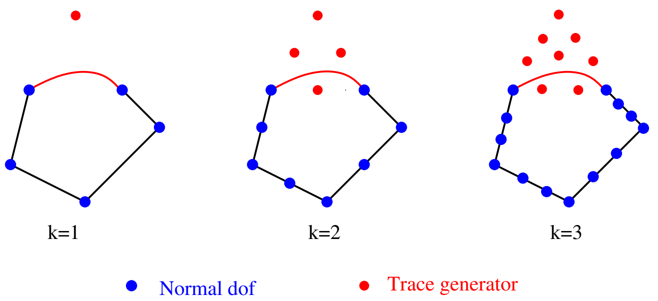

Fig. 6: Normal degrees of freedom and trace generators for $k = 1, 2, 3$

We consider therefore the set of values:

$$\begin{cases} \bullet \text{ the values at each vertex of } \mathcal{P}, \\ \quad \text{for every straight edge } e, \text{ and } k \geq 2 \\ \bullet \text{ the values at the } k-1 \text{ Gauss-Lobatto points of } e, \\ \bullet \text{ the values at the } \pi_k - 2 \text{ trace generator points of the curved edge.} \end{cases} \tag{3.8}$$

We point out that the values indicated in (3.8) can identify uniquely an element of $\mathcal{B}_k(\partial\mathcal{P})$ but cannot be taken as *degrees of freedom in the classical sense*. Indeed, it is very easy to see that on the curved edge γ:

- for every $k \geq 1$, for every set of π_k values (at the trace generators plus the two endpoints), there exists, on γ, a unique function which is "the restriction to γ of a polynomial q_k in $\mathbb{P}_k$ which assumes the given π_k values at the π_k points".
- However, depending on the shape of γ, there might be several different q_k's that have the same restriction to γ (see the example here below).

Indeed, already for $k = 1$, if γ is a straight segment (a case that we **do not** want to forbid!) the restriction of a polynomial of degree ≤ 1 to γ will depend only on the values of the polynomial at the endpoints (= vertices of $\mathcal{P}$) and not on its value at the trace generator. So we will have one tgp plus two endpoints, but the dimension of the space of their restrictions to γ will be only 2.

Summarizing: an element $v \in \mathcal{V}_k(\mathcal{P})$ *could be identified* by

$$\begin{aligned} &\text{- the values at each vertex of } \mathcal{P}, \\ &\qquad \text{for every straight edge } e, \text{ and } k \geq 2 \\ &\text{- the values at the } k-1 \text{ Gauss-Lobatto points of } e, \\ &\text{- the values at the trace generator points of the curved edge } \gamma, \\ &\text{- (for } k \geq 2) \text{ the moments of order } \leq k-2 \text{ inside } \mathcal{P}. \end{aligned} \tag{3.9}$$

However, such an *identification* will not be *injective*, as different sets of the above parameters (3.9) might generate the same element of $\mathcal{V}_k(\mathcal{P})$. Hence it is a natural choice *not to call* them *degrees of freedom*, and *for the quantities* (3.9) *we are going to stick instead to the name* ***generator values****, or simply* ***generators***.

All this might induce a certain amount of confusion: not really in the proofs (as we shall see), and even less in the code (once you know what has to be done). But the *description of the method* might easily become confused. Indeed, our trial and test variables will have *two faces*. We must consider

- The set of generators (3.9): this is inevitably the only thing that will be seen in the code. We will indicate them (both locally and globally) with *small capital* letters: U, V, W, etc.
- To each set of generators we attach a function that leaves in one of our VEM spaces. These functions will be indicated by $\hat{\text{U}}$, $\hat{\text{V}}$, $\hat{\text{W}}$, etc. (or, sometimes, even by u, v, or w), respectively.

Occasionally it will also be convenient to introduce

$$\mathbf{G}\{\mathcal{V}_k(\mathcal{P})\} := \text{ the set of generators of } \mathcal{V}_k(\mathcal{P}) \tag{3.10}$$

that obviously coincides with $\mathbb{R}^{N_k^{\mathcal{P}}}$, where $N_k^{\mathcal{P}}$ is the dimension of the space of generators of $\mathcal{V}_k(\mathcal{P})$.

Remark 3.2. It has to be pointed out that for each set of generators (say, V) we have a unique associated function $\hat{\text{V}}$, but *the converse is not true*, as we have seen. Indeed, there might be sets of generators $\text{V} \neq \mathbf{0}$ whose associated function $\hat{\text{V}}$ is identically zero. In such a case the space of generators $\mathbf{G}\{\mathcal{V}_k(\mathcal{P})\}$ would have a dimension *bigger* than that of the functional space $\mathcal{V}_k(\mathcal{P})$. The unknown solution of the discretized problem, in the computer code, will be a set of generators $\text{U}_h \in \mathbf{G}\{\mathcal{V}_k(\mathcal{P})\}$. This U_h will identify uniquely an element $\hat{\text{U}}_h$ of our VEM space, corresponding to what, in almost all papers on numerical methods for PDE's, would be denoted by u_h.

Remark 3.3. We observe that, whenever a function $v \in \mathcal{V}_k(\mathcal{P})$ is a polynomial, it is always possible to associate with it a set of generators $\text{V} \in \mathbf{G}\{\mathcal{V}_k(\mathcal{P})\}$, defined through (3.9). We denote this by

$$\text{V} = \mathcal{G}v \qquad v \in \mathbb{P}_k(\mathcal{P}). \tag{3.11}$$

We underline the fact that the operator $\mathcal{G}$ is **not** defined on the whole space $\mathcal{V}_k(\mathcal{P})$, but only on $\mathbb{P}_k(\mathcal{P}) \subseteq \mathcal{V}_k(\mathcal{P})$. In particular, having a generic function $v \in \mathcal{V}_k(\mathcal{P})$, in order to reconstruct **a** set of generators V such that $\hat{\text{V}} \equiv v$ we must prescribe the values of V at the trace generator points (obviously, among those that generate the right value of v on γ).

Remark 3.4. At this point we are able to detail in a more precise way the difference between the present approach and that in [2]. Indeed, here (as we have seen) we are going to keep the **idle generators** together with the "working ones", while in [2] the idle generators (identified, roughly speaking, as the ones whose trace on the curved edge is too small) are just eliminated from the set of unknowns. Clearly this is done after a suitable change of basis has been performed in the space of trace generators. We acknowledge the fact that this makes the whole presentation, and partly the code as well, much simpler than our approach. On the other hand we think that the decision on "how lazy a trace generator has to be, in order to justify its elimination" could become delicate, and potentially giving rise to some instabilities in the borderline cases. All considered we believe that having both possibilities at hand could be good for the scientific community.

3.4. *Elements with one edge in* Γ_D

When $\mathcal{P}$ is an element with one curved edge γ in Γ_D we proceed in a different way. For every function $\psi \in H^1(\gamma)$ (including the cases $\psi = 0$, and $\psi = g_D$) we define $\mathcal{V}_{k,\psi}(\mathcal{P})$ as follows.

$$\begin{aligned}\mathcal{V}_{k,\psi}(\mathcal{P}) := \{v \in C^0(\overline{\mathcal{P}}) \text{ such that } v_{|\gamma} = \psi,\ v_{|e} \in \mathbb{P}_k(e) \text{ on edges } e \neq \gamma,\\ \text{and } \Delta v \in \mathbb{P}_{k-2}(\mathcal{P})\}.\end{aligned} \tag{3.12}$$

Clearly, *once* ψ *has been given*, in order to identify an element of $\mathcal{V}_{k,\psi}(\mathcal{P})$ we must prescribe, in addition to the knowledge of ψ,

$$\begin{aligned}&\text{- the values at each vertex of } \mathcal{P} \text{ not on } \gamma,\\ &\qquad \text{for every straight edge } e, \text{ and } k \geq 2\\ &\text{- the values at the } k-1 \text{ Gauss-Lobatto points of } e,\\ &\text{- (for } k \geq 2\text{) the moments of order} \leq k-2 \text{ inside } \mathcal{P}.\end{aligned} \tag{3.13}$$

Note that, in this case, we could actually use the term *degrees of freedom*. Indeed, once ψ has been fixed, the mapping from the above quantities to the elements of $\mathcal{V}_{k,\psi}(\mathcal{P})$ is injective. Note also that for a general ψ the affine manifold $\mathcal{V}_{k,\psi}(\mathcal{P})$ will fail to contain all polynomials of $\mathbb{P}_k$, but whenever ψ is the trace of a polynomial $p_k \in \mathbb{P}_k$ then such a p_k will belong to $\mathcal{V}_{k,\psi}(\mathcal{P})$ (so that the patch-test will not be jeopardized).

For homogeneity of notation, here too we will consider the set of *generators* $\mathbf{G}\{\mathcal{V}_{k,\psi}(\mathcal{P})\}$ as in (3.10), although this, obviously, for $\psi \not\equiv 0$ will not be a linear space, but only an affine manifold.

4. The local VEM stiffness matrices

We define for u and v in $H^1(\Omega)$

$$a^{\mathcal{P}}(u,v) := \int_{\mathcal{P}} \kappa \nabla u \cdot \nabla v \,\mathrm{d}\Omega \ \ \forall \mathcal{P} \in \mathcal{T}_h \quad \text{and} \quad a(u,v) := \sum_{\mathcal{P}} a^{\mathcal{P}}(u,v), \tag{4.1}$$

and observe that, obviously, for all u and v in $H^1(\Omega)$, setting $\kappa^* = \max\{\kappa_i, \kappa_0\}$, and $\kappa_* = \min\{\kappa_i, \kappa_0\}$ we have

$$a^{\mathcal{P}}(u,v) \leq \kappa_{|\mathcal{P}} \|u\|_{1,\mathcal{P}} \, \|v\|_{1,\mathcal{P}}, \quad \text{and} \quad a(u,v) \leq \kappa^* \|u\|_{1,\Omega} \, \|v\|_{1,\Omega}, \tag{4.2}$$

as well as (using Poincaré's inequality)

$$a(v,v) \geq C_* \kappa_* \|v\|^2_{1,\Omega} \qquad \forall v \in H^1_{0,\Gamma_D}. \tag{4.3}$$

In this section we will construct for each element $\mathcal{P}$ a bilinear form $a_h^{\mathcal{P}}(\mathrm{U},\mathrm{V})$, **defined on generators of VEM spaces**, to be used to approximate the *continuous* bilinear form $a^{\mathcal{P}}$. Then, as done for Finite Elements, we will define the *global* virtual element spaces, the *global* bilinear forms, and the *global* right-hand sides summing the contributions of the single elements.

4.1. *The* Π_k^∇ *projection operator*

Our first, fundamental, item will be (as common for Virtual Elements) the construction of the Π_k^∇ projection operator. Given an element $\mathcal{P}$ and a function v in $H^1(\mathcal{P})$, we construct a polynomial $\Pi_k^\nabla v$ in $\mathbb{P}_k(\mathcal{P})$ defined by

$$\begin{cases} \displaystyle\int_{\mathcal{P}} \nabla \Pi_k^\nabla v \cdot \nabla q_k \,\mathrm{d}\mathcal{P} = \int_{\mathcal{P}} \nabla v \cdot \nabla q_k \,\mathrm{d}\mathcal{P} \qquad \forall q_k \in \mathbb{P}_k, \\ \displaystyle\int_{\partial\mathcal{P}} \Pi_k^\nabla v \,\mathrm{d}s = \int_{\partial\mathcal{P}} v \,\mathrm{d}s. \end{cases} \tag{4.4}$$

We point out that for a $v \in \mathcal{V}_k(\mathcal{P})$ (or in $\mathcal{V}_{k,g}(\mathcal{P})$ for some $g \in H^1(\Gamma_D)$) all the terms appearing in (4.4) are actually computable (for all types of elements (3.3), (3.6), and (3.12)) from the knowledge of the *degrees of freedom* (or of the *generators*) (3.4), or (3.9), or (3.13), respectively. Indeed, we first note that both left-hand sides of (4.4) are integrals of polynomials over $\mathcal{P}$ or $\partial\mathcal{P}$, respectively . The right-hand side of the second equation in (4.4) is also computable, since v is known on $\partial\mathcal{P}$ (being either a polynomial, or the trace of a polynomial, or equal to g). Finally, the right-hand side of the first equation is

$$\int_{\mathcal{P}} \nabla v \cdot \nabla q_k \mathrm{d}\mathcal{P} = -\int_{\mathcal{P}} v \Delta q_k \mathrm{d}\mathcal{P} + \int_{\partial\mathcal{P}} v\,(\nabla q_k \cdot \mathbf{n})\mathrm{d}s; \tag{4.5}$$

the first term in the right-hand side of (4.5) is made of moments of v of order $k-2$ on $\mathcal{P}$ (and hence computable from the "degrees of freedom" of v), and the second term is also computable since v is either a polynomial, or a trace of a polynomial, or equal to g on $\partial\mathcal{P}$.

Once we defined the projection Π_k^∇, and checked that it is *computable*, in practice, for every $v \in \mathcal{V}_k(\mathcal{P})$, we can extend it, in the obvious way, to an element V in $\mathbf{G}\{\mathcal{V}_k(\mathcal{P})\}$ by setting

$$\Pi_k^\nabla \mathrm{V} := \Pi_k^\nabla \hat{\mathrm{v}}. \tag{4.6}$$

Then we can follow the usual track of Virtual Elements, setting, for U and V in $\mathbf{G}\{\mathcal{V}_k(\mathcal{P})\}$:

$$a_h^{\mathcal{P}}(\mathrm{U},\mathrm{V}) := \int_{\mathcal{P}} \kappa \nabla \Pi_k^\nabla \mathrm{U} \cdot \nabla \Pi_k^\nabla \mathrm{V} \,\mathrm{d}\mathcal{P} + \mathcal{S}^{\mathcal{P}}((\mathbb{I} - \mathcal{G}\Pi_k^\nabla)\mathrm{U}, (\mathbb{I} - \mathcal{G}\Pi_k^\nabla)\mathrm{V}) \tag{4.7}$$

where $\mathbb{I}$ is the *identity* operator, $\mathcal{G}$ has been defined in (3.11), and, as usual in VEM formulations, $\mathcal{S}^{\mathcal{P}}$ is a symmetric positive semi-definite bilinear form such that there exists a positive constant α_*, independent of h, with

$$\alpha_* a^{\mathcal{P}}(\hat{\mathrm{v}}, \hat{\mathrm{v}}) \leq a_h^{\mathcal{P}}(\mathrm{v}, \mathrm{v}) \quad \forall\, \mathrm{v} \in \mathbf{G}\{\mathcal{V}_k(\mathcal{P})\}. \tag{4.8}$$

We immediately point out that, independently of the choice of $\mathcal{S}^{\mathcal{P}}$, we will always have the consistency property

$$a_h^{\mathcal{P}}(\mathcal{G}p_k, \mathrm{v}) = a^{\mathcal{P}}(p_k, \Pi_k^{\nabla}\hat{\mathrm{v}}) \equiv a^{\mathcal{P}}(p_k, \hat{\mathrm{v}}) \quad \forall p_k \in \mathbb{P}_k,\ \forall \mathrm{v} \in \mathbf{G}\{\mathcal{V}_k(\mathcal{P})\}. \tag{4.9}$$

Coming now to the choice of $\mathcal{S}^{\mathcal{P}}$, we note that there are *many* guidelines available in the VEM literature in order to construct a stabilizing bilinear form $\mathcal{S}^{\mathcal{P}}$. Here, however, we *must* distinguish between bilinear forms that one could apply to generic *elements* v *of the VEM space* $\mathcal{V}_k(\mathcal{P})$, and bilinear forms that can also be applied to the generators $\mathrm{v} \in \mathbf{G}\{\mathcal{V}_k(\mathcal{P})\}$. An example of the first case is the rather common

$$\mathcal{S}^{\mathcal{P}}(u, v) := h_{\mathcal{P}}^{-1} \kappa_{|\mathcal{P}} \int_{\partial\mathcal{P}} u\, v \mathrm{d}\ell \tag{4.10}$$

where $h_{\mathcal{P}}$ is, say, the diameter of $\mathcal{P}$. An example of the second is the (equally classic)

$$\mathcal{S}^{\mathcal{P}}(\mathrm{u}, \mathrm{v}) := \kappa_{|\mathcal{P}} \sum_i \delta_i(\mathrm{u})\delta_i(\mathrm{v}), \tag{4.11}$$

where

- each $\delta_i(\mathrm{v})$ is the *i-th* term of the generator v,
- the sum is extended to all of them.

However, a choice like (4.10) **cannot** be used *here*: indeed, with this choice a v having $\hat{\mathrm{v}} \equiv 0$ would not be stabilized, and (4.8) would not hold. Hence, we will assume here that the stabilizing term $\mathcal{S}^{\mathcal{P}}(\mathrm{u}, \mathrm{v})$ has exactly the form (4.11). In this case, under suitable assumptions on the mesh, that will be discussed in Sect. 7, (4.8) will always hold. At the same time, we also have, with the notation (2.2),

$$a_h^{\mathcal{P}}(\mathrm{v}, \mathrm{v}) + \|\Pi_0^{0,\mathcal{P}}\hat{\mathrm{v}}\|_{0,\mathcal{P}}^2 \geq \sigma_1 \sum_i (\delta_i(\mathrm{v}))^2 \tag{4.12}$$

for some constant $\sigma_1 > 0$ independent of h.

Remark 4.1. We note that (4.8) does not need to hold separately in each element. We just need that **the sum** (over all the elements $\mathcal{P}$ in $\mathcal{T}_h$) satisfies the analogue of (4.8). Actually, it is clear that by applying the stabilization in each element the degrees of freedom common to two or more elements would be "stabilized more than once". This, in general, does not affect the quality of the results, and coding is easier. However, as we shall see in more detail in Sec. 7.1, in some cases (typically when a curved edge separates two elements where κ assumes very different values) stabilizing only **once** is the right choice in order to preserve accuracy.

Remark 4.2. We recall that the classical stability condition for VEM stiffness matrices would be: there exist *two* positive constants α_* and α^* such that

$$\alpha_* a^{\mathcal{P}}(v,v) \leq a_h^{\mathcal{P}}(v,v) \leq \alpha^* a^{\mathcal{P}}(v,v) \quad \forall\, v \in \mathcal{V}_k(\mathcal{P}). \tag{4.13}$$

With the present notation this would be

$$\alpha_* a^{\mathcal{P}}(\hat{\mathrm{v}},\hat{\mathrm{v}}) \leq a_h^{\mathcal{P}}(\mathrm{v},\mathrm{v}) \leq \alpha^* a^{\mathcal{P}}(\hat{\mathrm{v}},\hat{\mathrm{v}}) \quad \forall\, \mathrm{v} \in \mathbf{G}\{\mathcal{V}_k(\mathcal{P})\}. \tag{4.14}$$

However, there are cases where the second inequality in (4.14) would be very difficult, if not definitely impossible, to obtain. This could happen in the presence of **idle generators**. For instance, when γ is straight there will be generators v (corresponding to trace generator points "attached to γ but not belonging to γ") such that $\hat{\mathrm{v}}$ is identically zero on γ (and then in the whole $\mathcal{P}$), producing in (4.14) a left-hand side and a right-hand side both equal to zero. If these dof's were not stabilized, the final stiffness matrix would end up being singular, but stabilizing them (for example as in (4.11)) would make the right inequality in (4.14) impossible.

5. The global spaces and the global a_h

5.1. *Approximations of H^1_{0,Γ_D} and $H^1_{g_D,\Gamma_D}$*

We shall first design the global space and the global *affine manifold* to be used as approximations of H^1_{0,Γ_D} and $H^1_{g_D,\Gamma_D}$ (respectively) in order to discretize (2.7).

For this, we define first, for every function $\psi \in H^1(\Gamma_D)$:

$$\mathcal{V}_{k,\psi}(\Omega) := \{v \in C^0(\overline{\Omega}) \text{ such that } v_{|\mathcal{P}} \in \mathcal{V}_k(\mathcal{P}) \ \forall \mathcal{P} \in \mathcal{T}_h \text{ without edges in } \Gamma_D \\ \text{and } v_{|\mathcal{P}} \in \mathcal{V}_{k,\psi}(\mathcal{P}) \ \forall \mathcal{P} \in \mathcal{T}_h \text{ with an edge in } \Gamma_D\}, \tag{5.1}$$

(where $\mathcal{V}_{k,\psi}(\mathcal{P})$ has been defined in (3.12)) and we observe that, for $\psi = 0$, $\mathcal{V}_{k,0}(\Omega)$ is a linear space. The *generators* for $\mathcal{V}_{k,\psi}(\Omega)$ will be:

- The values at the vertices in $\Omega \cup \Gamma_R$. One unknown per each such vertex. Remember that Γ_R is an *open* subset of $\partial\Omega$, so that the points belonging to the closure of Γ_D are excluded.
- (for $k \geq 2$) The values at the $k-1$ Gauss-Lobatto points of each **straight** edge internal to Ω or in Γ_R. Hence $k-1$ additional unknowns for each such edge.
- The values at the *trace generator points* of each curved edge, on $\partial\Omega_i$ and on Γ_R. These are $\pi_k - 2$ for each such edge.
- (for $k \geq 2$) The moments of order $\leq k-2$ internal to each element (π_{k-2} unknowns per each element).

Here too we have that a generator v, together with a function ψ in $H^1(\Gamma_D)$, will identify uniquely an element in $\mathcal{V}_{k,\psi}(\mathcal{P})$, but the converse will not be true.

As common in the Finite Element codes, the affine manifold $\mathcal{V}_{k,g_D}$ could be constructed as

$$\mathcal{V}_{k,g_D}(\Omega) := \overline{g}_D + \mathcal{V}_{k,0}(\Omega) \tag{5.2}$$

where $\overline{g}_D$ is a fixed element of $\mathcal{V}_{k,g_D}$ suitably constructed. Typically, $\overline{g}_D$ is *the* element in $\mathcal{V}_{k,g_D}$ that in each $\mathcal{P}$ has all the degrees of freedom (3.13) equal to zero.

Here, for simplicity, we will treat *every* edge e in Γ_D as if it was *a curved edge*. Indeed, as we already pointed out several times, we want our approach to accept straight boundary edges as particular cases of the curved ones, in particular since we do not want to enter the details of edges that are *only slightly curved*. Needless to say, if a whole part of Γ_D (or even the whole Γ_D) is known to be straight, then the corresponding elements will be treated as *normal polygons*.

5.2. *Approximation of bilinear forms and right-hand sides*

We are now ready to write the *global* quantities. For any function $\psi \in H^1(\Gamma_D)$, and for every U and V in $\mathbf{G}\{\mathcal{V}_{k,\psi}(\Omega)\}$, we set

$$a_h(\mathrm{U},\mathrm{V}) = \sum_{\mathcal{P}\in\mathcal{T}_h} a_h^{\mathcal{P}}(\mathrm{U},\mathrm{V}) \tag{5.3}$$

where $a_h^{\mathcal{P}}$ has been defined in (4.7). The boundary integrals

$$\int_{\Gamma_R} \rho\,\hat{\mathrm{U}}\,\hat{\mathrm{V}}\,\mathrm{d}\ell, \quad \text{and} \quad \int_{\Gamma_R} g_R \hat{\mathrm{V}}\,\mathrm{d}\ell \tag{5.4}$$

are not a difficulty for $\hat{\mathrm{U}}$ and $\hat{\mathrm{V}}$ in $\mathcal{V}_{k,\psi}(\Omega)$. Clearly from (4.8) we immediately have

$$\alpha_* a(\hat{\mathrm{V}},\hat{\mathrm{V}}) \le a_h(\mathrm{V},\mathrm{V}) \quad \forall\,\mathrm{V}\in\mathbf{G}\{\mathcal{V}_{k,0}(\Omega)\}. \tag{5.5}$$

In order to have some upper bound (similar to that in (4.14)) to be used to prove error estimates, we define a new norm

$$\|\mathrm{V}\|_{1,S} := (a_h(\mathrm{V},\mathrm{V}))^{1/2}. \tag{5.6}$$

This allows us to re-write the stability (5.5) as

$$\alpha_* a(\hat{\mathrm{V}},\hat{\mathrm{V}}) \le a_h(\mathrm{V},\mathrm{V}) \equiv \|\mathrm{V}\|_{1,S}^2 \qquad \forall\,\mathrm{V}\in\mathbf{G}\{\mathcal{V}_{k,0}(\Omega)\}. \tag{5.7}$$

We also note that, naturally,

$$a_h(\mathrm{U},\mathrm{V}) \le \|\mathrm{U}\|_{1,S}\,\|\mathrm{V}\|_{1,S} \qquad \forall\,\mathrm{U},\mathrm{V}\in\mathbf{G}\{\mathcal{V}_{k,0}(\Omega)\}. \tag{5.8}$$

Summing (4.12) over the elements, we have

$$a_h(\mathrm{V},\mathrm{V}) + \sum_{\mathcal{P}\in\mathcal{T}_h} \|\Pi_0^{0,\mathcal{P}}\hat{\mathrm{V}}\|_{0,\mathcal{P}}^2 \ge \sigma_1 \sum_i (\delta_i(\mathrm{V}))^2 \qquad \forall\,\mathrm{V}\in\mathbf{G}\{\mathcal{V}_{k,0}(\Omega)\} \tag{5.9}$$

where now the sum is extended to all indices i in $\mathbf{G}\{\mathcal{V}_{k,0}(\Omega)\}$. In turn, using the properties of projection operators and Poincaré inequality one has

$$\sum_{\mathcal{P}\in\mathcal{T}_h} \|\Pi_0^{0,\mathcal{P}}\hat{\mathrm{V}}\|_{0,\mathcal{P}}^2 \le \|\hat{\mathrm{V}}\|_{0,\Omega}^2 \le C\,a(\hat{\mathrm{V}},\hat{\mathrm{V}}) \qquad \forall\,\mathrm{V}\in\mathbf{G}\{\mathcal{V}_{k,0}(\Omega)\} \tag{5.10}$$

for some constant C independent of h. Combining (5.9), (5.10), and (5.7) gives then

$$a_h(\mathrm{V},\mathrm{V}) \ge \sigma_2 \sum_i (\delta_i(\mathrm{V}))^2 \qquad \forall\,\mathrm{V}\in\mathbf{G}\{\mathcal{V}_{k,0}(\Omega)\} \tag{5.11}$$

for some constant σ_2 independent of h.

Next, on every element $\mathcal{P}$, let $\hat{\mathrm{v}}$ be the unique function associated with the set of generators v. We define

$$T_{\mathcal{P}}\hat{\mathrm{v}} := \begin{cases} \Pi_1^{\nabla}\hat{\mathrm{v}} & \text{for } k=1 \\ \Pi_{k-2}^{0,\mathcal{P}}\hat{\mathrm{v}} & \text{for } k\geq 2 \end{cases} \tag{5.12}$$

and

$$T\mathrm{v} \equiv T\hat{\mathrm{v}} := \{T_{\mathcal{P}}\hat{\mathrm{v}} \text{ in every } \mathcal{P}\in\mathcal{T}_h\}. \tag{5.13}$$

Then we set

$$< f_h, \mathrm{v} > := \int_{\Omega} f\, T\hat{\mathrm{v}}\,\mathrm{d}\Omega \equiv (f, T\hat{\mathrm{v}})_0, \tag{5.14}$$

and finally,

$$< g_R, \mathrm{v} > := \int_{\Gamma_R} g_R\,\hat{\mathrm{v}}\,\mathrm{d}\ell. \tag{5.15}$$

6. The discretized problem

We can now write the discretized version of problem (2.7):

$$\begin{cases} \text{find } \mathrm{u}_h \in \mathbf{G}\{\mathcal{V}_{k,g_D}\} \text{ such that} \\ a_h(\mathrm{u}_h,\mathrm{v}) + \displaystyle\int_{\Gamma_R} \rho\,\hat{\mathrm{u}}_h\,\hat{\mathrm{v}}\,\mathrm{d}s = < f_h,\mathrm{v} > + < g_R,\mathrm{v} > \quad \forall \mathrm{v}\in\mathbf{G}\{\mathcal{V}_{k,0}\}. \end{cases} \tag{6.1}$$

We point out that whenever the exact solution u is in $\mathbb{P}_k(\Omega)$, the solution u_h of (6.1) will satisfy $\hat{\mathrm{u}}_h \equiv u$, so that the *patch-test of order* k will hold true.

6.1. *Error estimates*

We have the following error estimates.

Theorem 6.1. *In the above assumptions, problem* (6.1) *has a unique solution* u_h. *Moreover, there exists a constant* C, *depending only on the value of* α_* *in* (5.7) *and on the data* ρ, κ_0 *and* κ_i, *such that: if* u *is the solution of* (2.7), *then for every* u_I *in* $\mathbf{G}\{\mathcal{V}_{k,g_D}\}$ *and for every* u_π *elementwise in* $\mathbb{P}_k(\mathcal{P})$ *we have*

$$\|u-\hat{\mathrm{u}}_h\|_1 \leq C\Big(\|\mathrm{u}_\pi-\mathrm{u}_I\|_{1,S} + \|u-u_\pi\|_{1,h} + \|u-\hat{\mathrm{u}}_I\|_1 + \mathcal{E}_1(f)\Big) \tag{6.2}$$

where $\mathrm{u}_\pi = \mathcal{G}u_\pi$ *(see* (3.11)*),* $\|\cdot\|_{1,h}$ *is the* H^1*-broken norm, and* $\mathcal{E}_1(f)$ *is given by*

$$\mathcal{E}_1(f) := \sup_{\mathrm{v}\in\mathbf{G}\{\mathcal{V}_{k,0}\}} \frac{(f,\hat{\mathrm{v}}) - < f_h,\mathrm{v} >}{\|\hat{\mathrm{v}}\|_1} \equiv \sup_{\hat{\mathrm{v}}\in\mathcal{V}_{k,0}} \frac{(f,\hat{\mathrm{v}}-T\hat{\mathrm{v}})_0}{\|\hat{\mathrm{v}}\|_1}. \tag{6.3}$$

Proof. Uniqueness is an immediate consequence of (5.11). Let then U_h be the solution of (6.1) and let U_I be an element of $\mathbf{G}\{\mathcal{V}_{k,g_D}(\Omega)\}$. We set

$$\Theta := \mathrm{U}_h - \mathrm{U}_I, \tag{6.4}$$

and we recall that

$$\Theta \in \mathbf{G}\{\mathcal{V}_{k,0}\} \text{ while } \quad \hat{\Theta} \in H^1_{0,\Gamma_D}, \tag{6.5}$$

implying in particular, from (4.3) and (5.7),

$$\|\hat{\Theta}\|_1^2 \le C\, a(\hat{\Theta},\hat{\Theta}) \le C\|\Theta\|^2_{1,S}. \tag{6.6}$$

To simplify the notation (and make it similar to more classical ones) we also set

$$u_h := \hat{\mathrm{U}}_h, \qquad u_I := \hat{\mathrm{U}}_I, \qquad \theta := \hat{\Theta} \text{ and } \quad [v,w]_{\Gamma_R} := \int_{\Gamma_R} \rho\, v\, w\, \mathrm{d}s. \tag{6.7}$$

We point out in advance that, since u_π is piecewise in $\mathbb{P}_k$, from (4.9) we have

$$a_h(\mathrm{U}_\pi,\Theta) = \sum_{\mathcal{P}} a^{\mathcal{P}}(u_\pi,\theta). \tag{6.8}$$

Moreover, using classical trace theorems and then (6.6), there exists a constant C, independent of h, such that for all $\mathrm{V} \in \mathbf{G}\{\mathcal{V}_{k,0}\}$

$$\|\hat{\mathrm{V}}\|_{0,\Gamma_R} \le C\|\hat{\mathrm{V}}\|_1 \le C\|\mathrm{V}\|_{1,S}. \tag{6.9}$$

Then we proceed using (successively): (5.6),(6.4); rearranging terms; (6.1) and $\pm\mathrm{U}_\pi$; (6.8); (5.14), $\pm u$ (twice) and (4.1); rearranging; (2.7); (6.3), (5.8), (4.2); finally (6.6) and (6.9). We get:

$$\|\Theta\|^2_{1,S} = a_h(\Theta,\Theta) \le a_h(\Theta,\Theta) + [\theta,\theta]_{\Gamma_R} = a_h(\mathrm{U}_h,\Theta) - a_h(\mathrm{U}_I,\Theta) + [u_h,\theta]_{\Gamma_R} - [u_I,\theta]_{\Gamma_R}$$

$$= a_h(\mathrm{U}_h,\Theta) + [u_h,\theta]_{\Gamma_R} - a_h(\mathrm{U}_I,\Theta) - [u_I,\theta]_{\Gamma_R}$$

$$= < f_h,\Theta > + < g_R,\theta > - a_h(\mathrm{U}_I - \mathrm{U}_\pi,\Theta) - a_h(\mathrm{U}_\pi,\Theta) - [u_I,\theta]_{\Gamma_R}$$

$$= < f_h,\Theta > + < g_R,\theta > - a_h(\mathrm{U}_I - \mathrm{U}_\pi,\Theta) - \sum_{\mathcal{P}} a^{\mathcal{P}}(u_\pi,\theta) - [u_I,\theta]_{\Gamma_R}$$

$$= (f,T\theta)_0 + < g_R,\theta > - a_h(\mathrm{U}_I - \mathrm{U}_\pi,\Theta) - \sum_{\mathcal{P}} a^{\mathcal{P}}(u_\pi - u,\theta)$$
$$\qquad - a(u,\theta) - [u_I - u,\theta]_{\Gamma_R} - [u,\theta]_{\Gamma_R}$$

$$= (f,T\theta)_0 + < g_R,\theta > - a(u,\theta) - [u,\theta]_{\Gamma_R} - a_h(\mathrm{U}_I - \mathrm{U}_\pi,\Theta) - \sum_{\mathcal{P}} a^{\mathcal{P}}(u_\pi - u,\theta)$$
$$\qquad - [u_I - u,\theta]_{\Gamma_R}$$

$$= (f,T\theta-\theta)_0 - a_h(\mathrm{U}_I - \mathrm{U}_\pi,\Theta) - \sum_{\mathcal{P}} a^{\mathcal{P}}(u_\pi - u,\theta) - [u_I - u,\theta]_{\Gamma_R}$$

$$\le C\Big(\mathcal{E}_1(f)\|\theta\|_1 + \|\mathrm{U}_I - \mathrm{U}_\pi\|_{1,S}\|\Theta\|_{1,S} + \|u - u_\pi\|_{1,h}\|\theta\|_1 + \|u_I - u\|_{0,\Gamma_R}\|\theta\|_{0,\Gamma_R}\Big)$$

$$\le C\Big(\mathcal{E}_1(f) + \|\mathrm{U}_I - \mathrm{U}_\pi\|_{1,S} + \|u - u_\pi\|_{1,h} + \|u_I - u\|_1\Big)\,\|\Theta\|_{1,S}.$$

Thus,

$$\|\mathrm{u}_h - \mathrm{u}_I\|_{1,S} \le C\Big(\mathcal{E}_1(f) + \|\mathrm{u}_I - \mathrm{u}_\pi\|_{1,S} + \|u - u_\pi\|_{1,h} + \|u_I - u\|_1\Big).$$

The result follows by the triangle inequality. □

6.2. *Comments on the implementation*

In Sect. 8 we will present various numerical results showing robustness of the present approach. Here we will just give some information on a possible way of treating non-homogeneous Dirichlet boundary conditions. From (5.2) we see that the solution of (6.1) splits as

$$\mathrm{u}_h = \mathrm{u}_h^0 + \overline{g}_D,$$

with $\mathrm{u}_h^0 \in \mathbf{G}\{\mathcal{V}_{k,0}\}$. Consequently, the bilinear form a_h splits into

$$a_h(\mathrm{u}_h^0, \mathrm{v}) + a_h(\overline{g}_D, \mathrm{v}), \quad \mathrm{v} \in \mathbf{G}\{\mathcal{V}_{k,0}\}.$$

The second term is zero on all the elements except on elements $\mathcal{P}$ having an edge on Γ_D. On one such an element we have from definition (4.7)

$$a_h^{\mathcal{P}}(\overline{g}_D, \mathrm{v}) =:= \int_{\mathcal{P}} \kappa \nabla \Pi_k^\nabla \overline{g}_D \cdot \nabla \Pi_k^\nabla \mathrm{v}\, \mathrm{d}\mathcal{P} + \mathcal{S}^{\mathcal{P}}((\mathbb{I} - \mathcal{G}\Pi_k^\nabla)\overline{g}_D, (\mathbb{I} - \mathcal{G}\Pi_k^\nabla)\mathrm{v}).$$

The first term is easily computed, using definition (4.4), that gives

$$\int_{\mathcal{P}} \nabla \Pi_k^\nabla \overline{g}_D \cdot \nabla q_k \,\mathrm{d}\mathcal{P} = -\int_{\mathcal{P}} \overline{g}_D \Delta q_k \,\mathrm{d}\mathcal{P} + \int_{\partial\mathcal{P}} \overline{g}_D (\nabla q_k \cdot \mathbf{n}) \mathrm{d}s = 0 + \int_\gamma g(\nabla q_k \cdot \mathbf{n}) \mathrm{d}s.$$

Once the polynomial $\Pi_k^\nabla \overline{g}_D$ as been computed, the second term is trivial. With this approach, the discrete problem can be rewritten as

$$\begin{cases} \text{find } \mathrm{u}_h^0 \in \mathbf{G}\{\mathcal{V}_{k,0}\} \text{ such that} \\ a_h(\mathrm{u}_h^0, \mathrm{v}) + \displaystyle\int_{\Gamma_R} \rho\, \hat{\mathrm{u}}_h^0\, \hat{\mathrm{v}}\, \mathrm{d}s =< f_h, \mathrm{v} > + < g_R, \mathrm{v} > - a_h(\overline{g}_D, \mathrm{v}) \ \ \forall \mathrm{v} \in \mathbf{G}\{\mathcal{V}_{k,0}\}. \end{cases}$$

7. Interpolation estimates

Looking at Theorem 6.1 we see that in order to get the final error estimate in the usual terms of *powers of* h *and regularity of the solution* we need to choose u_I and u_π and then estimate (for them) the quantities

$$\|u - u_\pi\|_{1,h}, \qquad \|u - u_I\|_1 \qquad \|\mathrm{u}_\pi - \mathrm{u}_I\|_{1,S}, \qquad \text{and } \mathcal{E}_1(f). \tag{7.1}$$

As we shall see, the first two terms and the last one in (7.1) could be treated in a reasonably standard way. The third, however, poses some problems, as the norm $\|\cdot\|_{1,S}$ takes into account the choice of the stabilization, and, as we shall see, care has to be taken in order to choose a u_I and a u_π that make the third term small without affecting the accuracy of the other two.

We make the following mesh assumptions.

A) The boundary of each element $\mathcal{P}$ is piecewise C^1. Moreover, we assume the existence of a positive constant χ such that all elements $\mathcal{P}$ of the mesh are star-shaped with respect to a ball $B_{\mathcal{P}}$ of radius $\xi_{\mathcal{P}} \geq \chi h_{\mathcal{P}}$ (where, as usual, $h_{\mathcal{P}}$ is the diameter of $\mathcal{P}$).

B) There exists a positive constant χ_1 such that the length of each edge of $\mathcal{P}$ is $\geq \chi_1 h_{\mathcal{P}}$.

7.1. *More details on the stabilization*

Before going to the study of the interpolation errors we must spend some more attention to the choice of the stabilization term (that enters in the definition of the $\|\cdot\|_{1,S}$ norm). For simplicity we will only deal with elements with a curved edge, since the case of polygonal elements has already been treated in [14], [24].

Moreover, always for simplicity, we fix our attention on the *dofi-dofi* stabilization (4.11). Under our assumptions on the mesh the validity of (4.8), that is, estimates from below, can be easily proved with the techniques used in [14], [24]. Therefore we only have to deal with estimates from above.

We already pointed out in Remark 4.1 that the degrees of freedom shared by two (or more) elements do not need to be stabilized in each element, but only **once** (in either one of the elements).

We also pointed out, in Remark 4.2, that troubles may arise only when considering the tgp associated with curved edges that separate two elements having a (quite) different material coefficient κ. Indeed, as we shall see in a little while, when a curved edge separates two elements (say, $\mathcal{P}$ and $\mathcal{P}'$) such that the exact solution u is globally regular on $\overline{\mathcal{P}} \cup \overline{\mathcal{P}'}$ the estimate can be performed following classical arguments. However, in the presence of a jump in the coefficient κ, the exact solution will have a jump in the normal derivative at the interface, so that its global regularity on $\mathcal{P} \cup \gamma \cup \mathcal{P}'$ cannot be better than $H^{3/2-\varepsilon}$ with $\varepsilon > 0$. Indeed, as the tgp attached to the curved interface γ, seen from $\mathcal{P}$ and from $\mathcal{P}'$, are *the same*, then the value of u_I (to be chosen) in a tgp cannot be close, at the same time, to the value of u in one element and to a smooth extension of the values that u assumes in the other element, and this is the reason why we **must** stabilize the tgp of the interface γ only **once**.

Hence, for every curved edge γ that separates two elements $\mathcal{P}$ and $\mathcal{P}'$ having a different value of κ we choose, once and for all, one of the two elements (say, $\mathcal{P}$) and we consider the values of the tgp, in the *dofi-dofi* stabilization, only when dealing with the element $\mathcal{P}$. In the description below, we will assume, *to simplify the exposition*, that the element $\mathcal{P}$ chosen for the stabilization is the one that contains "more tg points", and that when considering the values of u at tg points not belonging to $\mathcal{P}$ we will actually consider a smooth extension of $u_{|\mathcal{P}}$. Note that this extension is done just when proving estimates, and (obviously) **not** in the code.

7.2. *Construction of u_I and u_π - Standard case*

For the sake of clarity, we will first discuss the simpler case where either

- the curved edge γ belongs to $\partial\Omega$

or

- γ is internal to Ω but the material coefficient κ varies smoothly from one element to the other.

As we have said already, in such cases we don't have to worry about the effects of the stabilizing terms, and we might assume that (for simplicity) the stabilization is done, independently, in each of the two elements. In this case (that we denoted as *Standard Case*) we proceed as follows. We set:

$$\widetilde{\mathcal{P}} := \mathcal{P} \cup \gamma \cup \mathcal{P}'$$

when the curved edge γ is shared by two elements, and

$$\widetilde{\mathcal{P}} := \mathcal{P}$$

otherwise. We note that, always in the standard case, the exact solution u will be smooth in $\widetilde{\mathcal{P}}$. Then we take a polynomial q_k living on $\widetilde{\mathcal{P}}$ such that $q_k(\nu) = u(\nu)$ at the two endpoint of γ and

$$\|u - q_k\|_{r,\widetilde{\mathcal{P}}} \leq C h_{\widetilde{\mathcal{P}}}^{t-r} |u|_{t,\mathcal{P}} \text{ for all real numbers } 0 \leq r \leq t \leq k+1 \text{ and } t > 1. \quad (7.2)$$

Due to assumption **A** it can be checked that such polynomial exists. Then we take, in both elements $\mathcal{P}$ and $\mathcal{P}'$

$$u_\pi \equiv q_k, \quad (7.3)$$

and this, using (3.11), defines U_π as $\mathcal{G} u_\pi$.

The local interpolant u_I, defined through its generators U_I, will be constructed separately in $\mathcal{P}$ and $\mathcal{P}'$. We will describe its construction in $\mathcal{P}$, and estimate $u - u_I$ in $\mathcal{P}$. The construction and the estimate in $\mathcal{P}'$ are done exactly in the same way.

We take $u_I(\nu) = u(\nu)$ for all points ν that correspond to a vertex of $\mathcal{P}$ or to a Gauss-Lobatto node of a straight edge of $\mathcal{P}$, and

$$u_{I|\gamma} = q_{k|\gamma}. \quad (7.4)$$

Inside $\mathcal{P}$ we require $\Delta u_I = \Pi^0_{k-2}(\Delta u)$. This defines a u_I as an element of $\mathcal{V}_k(\mathcal{P})$. In view of Remark 3.3, in order to define U_I we only need to prescribe its values at the tgp:

$$\mathrm{U}_I(\nu) = q_k(\nu) \quad \text{at all the tgp of the curved edge } \gamma. \quad (7.5)$$

Now we have to prove the interpolation estimate. In the following C will denote as usual a generic constant, uniform in the mesh size and shape, that can possibly change at each occurrence. In what follows all the Sobolev norms on $\partial\mathcal{P}$ are, as usual, defined with respect to the arclength parametrization.

Before delving into the interpolation proofs, we need to introduce some definition and two simple technical lemmas. Let the scaled norms (for all $0 \leq \varepsilon < 1/2$)

$$\begin{aligned} |\!|\!| v |\!|\!|_{1/2+\varepsilon,\partial\mathcal{P}} &= h_{\mathcal{P}}^{-1/2-\varepsilon} \|v\|_{0,\partial\mathcal{P}} + |v|_{1/2+\varepsilon,\partial\mathcal{P}} \\ |\!|\!| v |\!|\!|_{1+\varepsilon,\mathcal{P}} &= h_{\mathcal{P}}^{-1-\varepsilon} \|v\|_{0,\mathcal{P}} + |v|_{1+\varepsilon,\mathcal{P}} \, . \end{aligned} \tag{7.6}$$

Under the hypothesis **A**, following Lemma 3.3 in [18], for every element $\mathcal{P}$ one can build a $W^{1,\infty}$ mapping $\mathbf{F}_{\mathcal{P}}$ from $B_{\mathcal{P}}$ into $\mathcal{P}$, with inverse $\mathbf{F}_{\mathcal{P}}^{-1}$ also in $W^{1,\infty}$, in such a way that the $W^{1,\infty}$ norm of $\mathbf{F}_{\mathcal{P}}$ and $\mathbf{F}_{\mathcal{P}}^{-1}$ are uniformly bounded independently of $\mathcal{P}$.

Lemma 7.1. *Let assumption **A** hold. Then there exists a uniform constant C such that (for all elements $\mathcal{P}$)*

$$\begin{aligned} |v|_{1,\mathcal{P}} &\leq C |v|_{1/2,\partial\mathcal{P}} \qquad \forall v \in H^1(\mathcal{P}) \text{ with } \Delta v = 0 \, , \\ |\!|\!| v |\!|\!|_{1,\mathcal{P}} &\leq C |\!|\!| v |\!|\!|_{1/2,\partial\mathcal{P}} \qquad \forall v \in H^1(\mathcal{P}) \text{ with } \Delta v = 0 \, . \end{aligned} \tag{7.7}$$

Proof. Given any $v \in H^1(\mathcal{P})$ with $\Delta v = 0$, we denote by $\tilde{v}$ the only function in $H^1(B_{\mathcal{P}})$ with $\Delta\tilde{v} = 0$ and such that $\tilde{v} = v \circ \mathbf{F}_{\mathcal{P}}$ on $\partial B_{\mathcal{P}}$. We recall that the map $\mathbf{F}_{\mathcal{P}}$ and its inverse are uniformly in $W^{1,\infty}$. First using that v is harmonic in $\mathcal{P}$ (and thus minimizes the H^1 semi-norm among all H^1 functions with same boundary values), then recalling that $\tilde{v}$ is harmonic in $B_{\mathcal{P}}$, the first bound in (7.7) follows:

$$\begin{aligned} |v|_{1,\mathcal{P}} &\leq |\tilde{v} \circ \mathbf{F}_{\mathcal{P}}^{-1}|_{1,\mathcal{P}} \leq C |\tilde{v}|_{1,B_{\mathcal{P}}} \leq C |\tilde{v}|_{1/2,\partial B_{\mathcal{P}}} \\ &= C |v \circ \mathbf{F}_{\mathcal{P}}|_{1/2,\partial B_{\mathcal{P}}} \leq C |v|_{1/2,\partial\mathcal{P}}. \end{aligned} \tag{7.8}$$

It is well known, see for instance [27], that for all Lipschitz domains ω (and thus in particular for any ball) there exists a constant C such that $\|w\|_{0,\omega} \leq C(|w|_{1,\omega} + \|w\|_{0,\partial\omega})$ for all w in $H^1(\omega)$. By a standard scaling argument with the unitary ball, the previous result immediately yields (for any element $\mathcal{P}$)

$$\xi_{\mathcal{P}}^{-1} \|w\|_{0,B_{\mathcal{P}}} \leq C(|w|_{1,B_{\mathcal{P}}} + \xi_{\mathcal{P}}^{-1/2} \|w\|_{0,\partial B_{\mathcal{P}}}) \quad \forall w \in H^1(B_{\mathcal{P}}) \tag{7.9}$$

with C independent of $\mathcal{P}$. By mapping to the ball, applying result (7.9) and mapping back to $\mathcal{P}$ (we also recall that $\xi_{\mathcal{P}} \sim h_{\mathcal{P}}$) we obtain

$$\begin{aligned} h_{\mathcal{P}}^{-1} \|v\|_{0,\mathcal{P}} &\leq \xi_{\mathcal{P}}^{-1} \|v \circ \mathbf{F}_{\mathcal{P}}^{-1}\|_{0,B_{\mathcal{P}}} \leq C(|v \circ \mathbf{F}_{\mathcal{P}}^{-1}|_{1,B_{\mathcal{P}}} + \xi_{\mathcal{P}}^{-1/2} \|v \circ \mathbf{F}_{\mathcal{P}}^{-1}\|_{0,\partial B_{\mathcal{P}}}) \\ &\leq C(|v|_{1,\mathcal{P}} + h_{\mathcal{P}}^{-1/2} \|v\|_{0,\partial\mathcal{P}}). \end{aligned} \tag{7.10}$$

The second bound in (7.7) follows from (7.6), (7.10) and (7.8). □

Lemma 7.2. *Let assumption **A** hold. Then there exists a uniform constant C such that (for all elements $\mathcal{P}$)*

$$\|v\|_{0,\mathcal{P}} \leq C\Big(h_{\mathcal{P}} |v|_{1,\mathcal{P}} + \Big| \int_{\partial\mathcal{P}} v \Big| \Big) \qquad \forall v \in H^1(\mathcal{P}). \tag{7.11}$$

Proof. We observe first that under our mesh assumptions it holds

$$\|\varphi\|_{0,\mathcal{P}} \leq C\Big|\int_{\partial\mathcal{P}} \varphi\Big| \quad \forall\varphi \text{ constant}. \tag{7.12}$$

Indeed:

$$\|\varphi\|_{0,\mathcal{P}}^2 = |\mathcal{P}|\varphi^2 = \frac{|P|}{|\partial\mathcal{P}|^2}\Big(\int_{\partial\mathcal{P}}\varphi\Big)^2 \leq C\Big(\int_{\partial\mathcal{P}}\varphi\Big)^2,$$

with C independent of $h_{\mathcal{P}}$. For $v \in H^1(\mathcal{P})$, let $\overline{v}$ be its average on $\mathcal{P}$. By standard approximation properties and (7.12) we have

$$\|v\|_{0,\mathcal{P}} \leq \|v - \overline{v}\|_{0,\mathcal{P}} + \|\overline{v}\,\|_{0,\mathcal{P}} \leq C\Big(h_{\mathcal{P}}|v|_{1,\mathcal{P}} + \Big|\int_{\partial\mathcal{P}}\overline{v}\,\Big|\Big). \tag{7.13}$$

Next, by adding and subtracting v and the usual argument mapping "to and from" the ball $B_{\mathcal{P}}$ combined with a scaled trace inequality on the ball, we can easily derive

$$\begin{aligned}\Big|\int_{\partial\mathcal{P}}\overline{v}\,\Big| &\leq \Big|\int_{\partial\mathcal{P}}(\overline{v}-v)\Big| + \Big|\int_{\partial\mathcal{P}} v\,\Big| \leq Ch_{\mathcal{P}}^{1/2}\|\overline{v}-v\|_{0,\partial\mathcal{P}} + \Big|\int_{\partial\mathcal{P}} v\,\Big| \\ &\leq Ch_{\mathcal{P}}^{1/2}\Big(h_{\mathcal{P}}^{-1/2}\|\overline{v}-v\|_{0,\mathcal{P}} + h_{\mathcal{P}}^{1/2}|v|_{1,\mathcal{P}}\Big) + \Big|\int_{\partial\mathcal{P}} v\,\Big| \\ &\leq Ch_{\mathcal{P}}|v|_{1,\mathcal{P}} + \Big|\int_{\partial\mathcal{P}} v\,\Big|.\end{aligned} \tag{7.14}$$

Inserting (7.14) in (7.13) gives the result. □

Corollary 7.3. *An obvious consequence of* (7.11) *is the Poincaré-type inequality*

$$\|v\|_{0,\mathcal{P}} \leq Ch_{\mathcal{P}}|v|_{1,\mathcal{P}} \qquad \forall v \in H^1(\mathcal{P}) \text{ with } \int_{\partial\mathcal{P}} v = 0. \tag{7.15}$$

Proposition 7.4. *Let assumption* ***A*** *hold. Then there exists a positive constant C such that for all $\mathcal{P} \in \mathcal{T}_h$ and any function $u \in H^s(\mathcal{P})$, $s \geq 2$, it holds*

$$|u - u_I|_{1,\mathcal{P}} \leq Ch_{\mathcal{P}}^{s-1}|u|_{s,\mathcal{P}}\,. \tag{7.16}$$

Proof. In the following we assume that s is integer, as the general bound can then be immediately obtained by using classical results of space interpolation theory. As a consequence of the properties of the map $\mathbf{F}_{\mathcal{P}}$, by mapping to $B_{\mathcal{P}}$, applying standard estimates and finally mapping back, one can easily obtain (for all $0 \leq \varepsilon < 1/2$)

$$|\!|\!|v|\!|\!|_{1/2+\varepsilon,\partial\mathcal{P}} \leq C|\!|\!|v|\!|\!|_{1+\varepsilon,\mathcal{P}} \quad \forall v \in H^{1+\varepsilon}(\mathcal{P})\,. \tag{7.17}$$

Let now $\mathbf{n}$ denote the outward unit normal to $\mathcal{P}$ and let γ be the curved edge of $\mathcal{P}$. For any $\boldsymbol{\varphi} \in H(\mathrm{div};\mathcal{P})$ we easily have, also using the second bound in (7.7), and an integration by parts

$$\begin{aligned}|\!|\!|\boldsymbol{\varphi}\cdot\mathbf{n}|\!|\!|_{-1/2,\partial\mathcal{P}} &:= \sup_{w\in H^{1/2}(\partial\mathcal{P})}\frac{<\boldsymbol{\varphi}\cdot\mathbf{n}, w>}{|\!|\!|w|\!|\!|_{1/2,\partial\mathcal{P}}} = \sup_{v\in H^1(\mathcal{P}),\Delta v=0}\frac{<\boldsymbol{\varphi}\cdot\mathbf{n}, v|_{\partial\mathcal{P}}>}{|\!|\!|v|_{\partial\mathcal{P}}|\!|\!|_{1/2,\partial\mathcal{P}}} \\ &\leq C\sup_{v\in H^1(\mathcal{P}),\Delta v=0}\frac{<\boldsymbol{\varphi}\cdot\mathbf{n}, v|_{\partial\mathcal{P}}>}{|\!|\!|v|\!|\!|_{1,\mathcal{P}}} = C\sup_{v\in H^1(\mathcal{P}),\Delta v=0}\frac{\int_{\mathcal{P}}\boldsymbol{\varphi}\cdot\nabla v + \int_{\mathcal{P}} v(\mathrm{div}\boldsymbol{\varphi})}{|\!|\!|v|\!|\!|_{1,\mathcal{P}}},\end{aligned}$$

where (here and in the sequel) the brackets denote a duality on the boundary of $\mathcal{P}$. Therefore, Cauchy-Schwarz inequality and (7.6) ($h_{\mathcal{P}}^{-1}\|v\|_{0,\mathcal{P}} \leq \|\!|v|\!\|_{1,\mathcal{P}}$) yield the uniform bound

$$\|\!|\boldsymbol{\varphi}\cdot\mathbf{n}|\!\|_{-1/2,\partial\mathcal{P}} \leq C\big(\|\boldsymbol{\varphi}\|_{0,\mathcal{P}} + h_{\mathcal{P}}\|\mathrm{div}\boldsymbol{\varphi}\|_{0,\mathcal{P}}\big)\,. \tag{7.18}$$

We can now estimate the interpolation error. Let the constant c be equal to the average of $u-u_I$ on $\partial\mathcal{P}$. First by integrating by parts, then recalling the definition of u_I, we obtain

$$\begin{aligned}
|u-u_I|^2_{1,\mathcal{P}} &= \int_{\mathcal{P}} \nabla(u-u_I)\cdot\nabla(u-u_I-c)\\
&= -\int_{\mathcal{P}}(u-u_I-c)\Delta(u-u_I) + <\nabla(u-u_I)\cdot\mathbf{n}, u-u_I-c>\\
&\leq C\|u-u_I-c\|_{0,\mathcal{P}}\|\Delta u-\Pi^0_{k-2}(\Delta u)\|_{0,\mathcal{P}}\\
&\quad + \|\!|\nabla(u-u_I)\cdot\mathbf{n}|\!\|_{-1/2,\partial\mathcal{P}}\,\|\!|u-u_I-c|\!\|_{1/2,\partial\mathcal{P}} =: I + II.
\end{aligned}$$

Standard approximation estimates on star-shaped domains give

$$I \leq C h_{\mathcal{P}}^{s-1}|u-u_I|_{1,\mathcal{P}}|\Delta u|_{s-2,\mathcal{P}} \leq C h_{\mathcal{P}}^{s-1}|u-u_I|_{1,\mathcal{P}}|u|_{s,\mathcal{P}}, \tag{7.19}$$

while bound (7.18), and using again the definition of u_I, yields

$$\begin{aligned}
II &\leq C\big(|u-u_I|_{1,\mathcal{P}} + h_{\mathcal{P}}\|\Delta(u-u_I)\|_{0,\mathcal{P}}\big)\|\!|u-u_I-c|\!\|_{1/2,\partial\mathcal{P}}\\
&\leq C\big(|u-u_I|_{1,\mathcal{P}} + h_{\mathcal{P}}^{s-1}|u|_{s,\mathcal{P}}\big)\|\!|u-u_I-c|\!\|_{1/2,\partial\mathcal{P}}.
\end{aligned} \tag{7.20}$$

It is now easy to check by some simple algebra that the proof is concluded provided we can show the boundary approximation estimate

$$\|\!|u-u_I-c|\!\|_{1/2,\partial\mathcal{P}} \leq C h_{\mathcal{P}}^{s-1}|u|_{s,\mathcal{P}}. \tag{7.21}$$

First by definition, then by a one dimensional Poincaré inequality (from L^2 into $H^{1/2}$, recalling that c is the average of $u-u_I$ on $\partial\mathcal{P}$) we get

$$\|\!|u-u_I-c|\!\|_{1/2,\partial\mathcal{P}} = h_{\mathcal{P}}^{-1/2}\|u-u_I-c\|_{0,\partial\mathcal{P}} + |u-u_I|_{1/2,\partial\mathcal{P}} \leq C|u-u_I|_{1/2,\partial\mathcal{P}}\ .$$

Since $u-u_I$ vanishes at all vertices of $\mathcal{P}$, a direct argument easily yields

$$\|u-u_I\|_{0,\partial\mathcal{P}} \leq C h_{\mathcal{P}}|u-u_I|_{1,\partial\mathcal{P}}$$

and thus, by space interpolation theory,

$$|u-u_I|_{1/2,\partial\mathcal{P}} \leq C h_{\mathcal{P}}^{\varepsilon}|u-u_I|_{1/2+\varepsilon,\partial\mathcal{P}} \leq C\sum_{e\in\partial\mathcal{P}} h_{\mathcal{P}}^{\varepsilon}|u-u_I|_{1/2+\varepsilon,e} \tag{7.22}$$

with $0<\varepsilon\leq 1/2$, and $e\in\partial\mathcal{P}$ denoting all the edges (curved and straight) of $\mathcal{P}$.

Whenever e in (7.22) is a straight edge, we apply standard interpolation results in one dimension and a trace inequality (for instance from e into the triangle connecting the endpoints of e with the center of the ball $B_{\mathcal{P}}$). We obtain

$$h_{\mathcal{P}}^{\varepsilon}|u-u_I|_{1/2+\varepsilon,e} \leq h_{\mathcal{P}}^{s-1}C|u|_{s-1/2,e} \leq C h_{\mathcal{P}}^{s-1}C|u|_{s,\mathcal{P}}\ . \tag{7.23}$$

Whenever $e = \gamma$ in (7.22) is a curved edge, we first recall the definition of u_I on γ and apply the trace inequality in (7.17), then make use of standard polynomial approximation estimates on star-shaped domains

$$\begin{aligned} h_{\mathcal{P}}^{\varepsilon}|u-u_I|_{1/2+\varepsilon,\gamma} &\le h_{\mathcal{P}}^{\varepsilon}|\!|\!|u-u_I|\!|\!|_{1/2+\varepsilon,\partial\mathcal{P}} \\ &\le h_{\mathcal{P}}^{-1}\|u-q_k\|_{0,\mathcal{P}} + h_{\mathcal{P}}^{\varepsilon}|u-q_k|_{1+\varepsilon,\mathcal{P}} \le h_{\mathcal{P}}^{s-1}|u|_{s,\mathcal{P}}. \end{aligned} \tag{7.24}$$

By inserting (7.23)-(7.24) into (7.22) we obtain (7.21), so that (7.19)-(7.20) give

$$|u-u_I|_{1,\mathcal{P}}^2 \le C h_{\mathcal{P}}^{s-1}|u-u_I|_{1,\mathcal{P}}|u|_{s,\mathcal{P}}, \tag{7.25}$$

and the proof is concluded. □

Corollary 7.5. *Under the same notation and assumptions of Proposition 7.4 and its proof, by choosing $u_\pi = q_k$, for all elements $\mathcal{P}$ it holds*

$$\|\mathrm{u}_I - \mathrm{u}_\pi\|_{1,S(\mathcal{P})}^2 := a_h^{\mathcal{P}}(\mathrm{u}_I - \mathrm{u}_\pi, \mathrm{u}_I - \mathrm{u}_\pi) \le C h_{\mathcal{P}}^{2(s-1)}|u|_{s,\mathcal{P}}^2\,, \tag{7.26}$$

and thus

$$\|\mathrm{u}_I - \mathrm{u}_\pi\|_{1,S} \le C h^{s-1}|u|_{s,\Omega}.$$

Proof. We only show the sketch of the proof. We start by noting that (7.16) combined with a scaled Poincaré-type inequality easily yields

$$\|u-u_I\|_{0,\mathcal{P}} \le C h_{\mathcal{P}}^{s}|u|_{s,\mathcal{P}}\ . \tag{7.27}$$

By definition, the norm $\|\cdot\|_{1,S}$ is split into a consistency term and a stabilization term

$$\begin{aligned} \|\mathrm{u}_I - \mathrm{u}_\pi\|_{1,S(\mathcal{P})}^2 = &\int_{\mathcal{P}} \kappa\nabla\Pi_k^{\nabla}(u_I-u_\pi)\cdot\nabla\Pi_k^{\nabla}(u_I-u_\pi)\,\mathrm{d}\mathcal{P} \\ &+ \mathcal{S}^{\mathcal{P}}((\mathbb{I}-\mathcal{G}\Pi_k^{\nabla})(\mathrm{u}_I-\mathrm{u}_\pi),(\mathbb{I}-\mathcal{G}\Pi_k^{\nabla})(\mathrm{u}_I-\mathrm{u}_\pi)). \end{aligned}$$

The estimate for the first term follows easily from (7.2), (7.16) and the continuity of the Π_k^{∇} operator in the H^1 seminorm (that holds, by definition of Π_k^{∇}, with unitary constant). Adopting definition (4.11), the second term can be split into a part that involves the volume moments and a part that involves a sum of pointwise evaluations, including the *tgp*. Since the novelty here is in the second part (the first one is standard, as for straight polygons) we focus only on the latter term. Let $\mathcal{N}$ denote the set of the boundary nodes of $\mathcal{P}$ plus the tgp nodes. Recalling that u_I and u_π attain the same values on the *tgp*, we thus need to estimate (also using $u_I(\nu) = u(\nu)$ for all nodes $\nu \in \mathcal{N}, \nu \notin tgp$)

$$\begin{aligned} &\sum_{\nu\in\mathcal{N}}((\mathbb{I}-\mathcal{G}\Pi_k^{\nabla})(\mathrm{u}_I-\mathrm{u}_\pi)(\nu))^2 = \sum_{\nu\in\mathcal{N}}((\mathrm{u}_I-\mathrm{u}_\pi)(\nu)-(\Pi_k^{\nabla}u_I-u_\pi)(\nu))^2 \\ &\le 2\Big(\sum_{\nu\in\mathcal{N}}((\mathrm{u}_I-\mathrm{u}_\pi)(\nu))^2 + \sum_{\nu\in\mathcal{N}}(\Pi_k^{\nabla}u_I-u_\pi)(\nu))^2\Big) \\ &= 2\sum_{\nu\in\mathcal{N},\nu\notin tgp}((u-u_\pi)(\nu))^2 + 2\sum_{\nu\in\mathcal{N}}((\Pi_k^{\nabla}u_I-u_\pi)(\nu))^2. \end{aligned}$$

The first term above is bounded by standard polynomial approximation estimates in L^∞. In order to deal with the second term, let $\widetilde{B}$ be the smallest ball that contains $\mathcal{P}$ and all the *tgp*; note that by definition of the *tgp* the radius of such ball is uniformly comparable to that of $B_{\mathcal{P}}$, see assumption **A**. As a consequence, from known properties of polynomial functions and an inverse estimate,

$$\sum_{\nu\in\mathcal{N}}((\Pi_k^\nabla u_I - u_\pi)(\nu))^2 \leq \|\Pi_k^\nabla u_I - u_\pi\|^2_{L^\infty(\widetilde{B})}$$
$$\leq C\|\Pi_k^\nabla u_I - u_\pi\|^2_{L^\infty(\mathcal{P})} \leq Ch_{\mathcal{P}}^{-2}\|\Pi_k^\nabla u_I - u_\pi\|^2_{0,\mathcal{P}}.$$

Now the result follows by adding and subtracting u and $\Pi_k^\nabla u$, using the triangle inequality, bound (7.27), stability properties of Π_k^∇ and standard polynomial approximation estimates. □

7.3. *Construction of u_I for discontinuous $\kappa\mathbf{n}\cdot\nabla u$*

In the case where u is continuous but not C^1 across γ (that is, when κ has two different values across γ), then we have to stabilize $a_h^{\mathcal{P}}$ in only one of the two elements $\mathcal{P}$ and $\mathcal{P}'$. Let's say that we stabilize in $\mathcal{P}$. Then the q_k will be chosen using only the values of u in $\mathcal{P}$ and the estimate (7.2) will be required only in $\mathcal{P}$.

In $\mathcal{P}'$ we will choose u_π as the $L^2(\mathcal{P}')$-projection of u onto $\mathbb{P}_k$. In turn, u_I will still be equal to q_k on γ (it must be continuous) and will be defined, on the other edges different from γ, using the values of u itself (and, inside, using $\Delta u_I = \Pi_{k-2}^{0,\mathcal{P}'}(\Delta u)$.

Since the estimate (7.2) still holds on γ, we can deal with the estimate of $u - u_I$ proceeding as before. This will provide, intercalating u, an estimate on

$$\|u_I - u_\pi\|_{1,\mathcal{P}'}.$$

Next, since we **do not** stabilize the tgp degrees of freedom on $\mathcal{P}'$, this will also provide a bound for

$$\|\mathrm{U}_I - \mathrm{U}_\pi\|_{1,S},$$

proceeding, for the other degrees of freedom, as for the usual VEMs and we are done.

7.4. *Estimate of $\mathcal{E}_1(f)$*

For $k = 1$, using the definition (5.12) and noting that for any elements $\mathcal{P}$ and all $v \in H^1(\mathcal{P})$ it holds $\int_{\partial\mathcal{P}}(v - \Pi_1^\nabla v) = 0$ by definition, using bound (7.15) we derive

$$(f, v - Tv)_0 = \sum_{\mathcal{P}}(f, v - \Pi_1^\nabla v)_{0,\mathcal{P}} \leq C\sum_{\mathcal{P}}\|f\|_{0,\mathcal{P}}\, h_{\mathcal{P}}|v - \Pi_1^\nabla v|_{1,\mathcal{P}} \leq Ch\|f\|_0|v|_1,$$

giving

$$\mathcal{E}_1(f) \leq C\, h\|f\|_0 \qquad \text{for } k = 1. \tag{7.28}$$

For $k \geq 2$, from the properties of the L^2-orthogonal projection and standard interpolation estimates we have

$$
\begin{aligned}
(f, v - Tv)_0 &= \sum_{\mathcal{P}} (f, v - \Pi^{0,\mathcal{P}}_{k-2} v)_{0,\mathcal{P}} = \sum_{\mathcal{P}} (f - \Pi^{0,\mathcal{P}}_{k-2} f, v - \Pi^{0,\mathcal{P}}_{k-2} v)_{0,\mathcal{P}} \\
&\leq C \sum_{\mathcal{P}} h_{\mathcal{P}}^{k-1} \|f\|_{k-1,\mathcal{P}} \, h_{\mathcal{P}} |v|_{1,\mathcal{P}} \leq C h^k (\sum_{\mathcal{P}} \|f\|^2_{k-1,\mathcal{P}})^{1/2} \|v\|_1,
\end{aligned}
$$

implying

$$
\mathcal{E}_1(f) \leq C\, h^k (\sum_{\mathcal{P}} \|f\|^2_{k-1,\mathcal{P}})^{1/2} \qquad \text{for } k \geq 2. \tag{7.29}
$$

8. Some numerical experiments

In this section we present some numerical experiments in order to highlight the features of the proposed method. We will show the relative errors in H^1 and L^2. Since the VEM solution u_h is not known inside the polygons we will compute the errors between the exact solution u and both $\Pi^{\nabla}_k u_h$ and $\Pi^0_k u_h$.

8.1. *Problem with discontinuous diffusion*

We begin by dealing with a problem with radial symmetry and discontinuous diffusion. Let Ω be the unit disc, split into the smaller disc Ω_1 of radius $1/2$ and the outer anulus Ω_2 (see Fig. 7), and let κ and f be piecewise constant in Ω:

$$
\kappa_{|\Omega_1} = \kappa_1, \ \kappa_{|\Omega_2} = \kappa_2, \ f_{|\Omega_1} = f_1, \ f_{|\Omega_2} = f_2. \tag{8.1}
$$

Consider the following problem:

$$
\begin{cases} -\mathrm{div}\,(\kappa \nabla u) = f & \text{in } \Omega \\ \qquad\qquad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{8.2}
$$

It can be easily shown that the exact solution of problem (8.2) is

$$
\begin{aligned}
u_1(x,y) &= -\frac{f_1}{4\kappa_1} r^2 + \left[\frac{1}{16} \left(\frac{f_1}{\kappa_1} + \frac{3f_2}{\kappa_2} \right) - \frac{\log(2)}{8\kappa_2} (f_2 - f_1) \right] \quad \text{in } \Omega_1 \\
u_2(x,y) &= -\frac{f_2}{4\kappa_2} r^2 + \left[\frac{1}{8\kappa_2} (f_2 - f_1) \right] \log(r) + \left[\frac{f_2}{4\kappa_2} \right] \quad \text{in } \Omega_2
\end{aligned}
$$

where $r = \sqrt{x^2 + y^2}$. The solution is always a paraboloid in the inner circle Ω_1 while in the outer anulus Ω_2 it is a paraboloid plus a logarithmic term that is present only if $f_1 \neq f_2$. If $k_1 \neq k_2$, the solution always has a discontinuous gradient along the circle of radius $1/2$ (see Figures 8 and 10). In order to satisfy the patch test (in the case $f_1 = f_2$) and to have the optimal rate of convergence (in the general case), the method must treat appropriately the curved boundary and the curved interface.

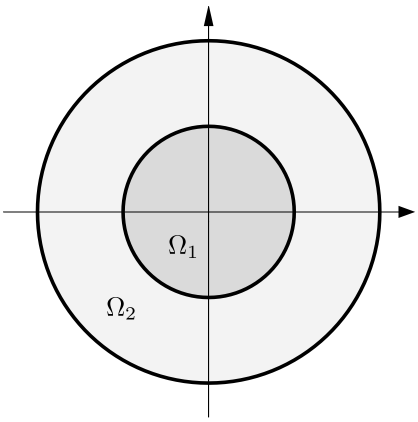

Fig. 7: Computational domain

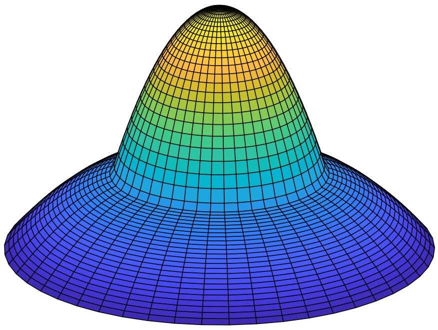

Fig. 8: Exact solution with $\kappa_1 = 1$, $\kappa_2 = 10$, $f_1 = 1$, $f_2 = 2$

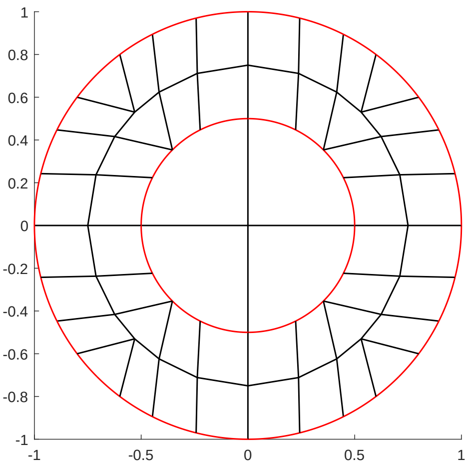

Fig. 9: Mesh

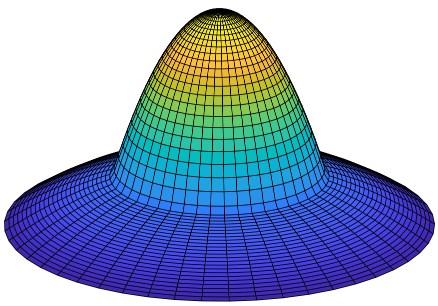

Fig. 10: Exact solution with $\kappa_1 = 1$, $\kappa_2 = 10$, $f_1 = 1$, $f_2 = 1$

8.1.1. *Case* $\kappa_1 = 1$*,* $\kappa_1 = 10$*,* $f_1 = f_2 = 1$*: patch test*

We first deal with the case where the solution in each subregion is a polynomial of degree 2 and κ is discontinous. We check that in this case the VEM solution satisfies the patch test, i.e., we recover the exact solution.

The mesh used is represented in Fig. 9 while in Fig. 10 we report the exact solution. Curved egdes are depicted in red. We point out that in this case the diffusion is discontinuous across the curved edges so that, according to Remark 4.1, we need to stabilize the tgp only once. We present in Fig. 11 the VEM solution obtained in this way for $k = 2$, and in Fig. 12 the (wrong) result obtained stabilizing in the "classical" way (i.e., twice), again for $k = 2$.

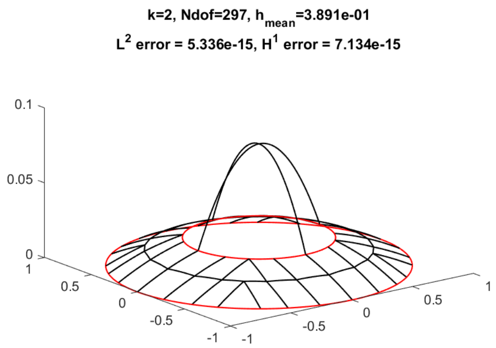

Fig. 11: One-side stabilization

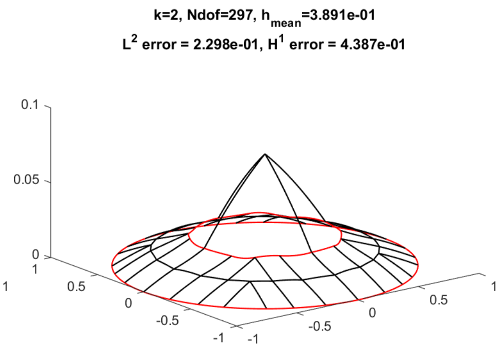

Fig. 12: Two-side stabilization

8.1.2. *Case $\kappa_1 = 1$, $\kappa_1 = 10$, $f_1 = 1$, $f_2 = 2$: convergence*

As stated above, in this case the exact solution (see Fig. 8) is no more a quadratic polynomial in the external anulus Ω_2. We take a sequence of meshes obtained by subdividing recursively the mesh depicted in Fig. 9 and we represent the convergence curves for $k = 3$ in H^1 (Fig. 13) and in L^2 (Fig. 14). The slopes are as expected.

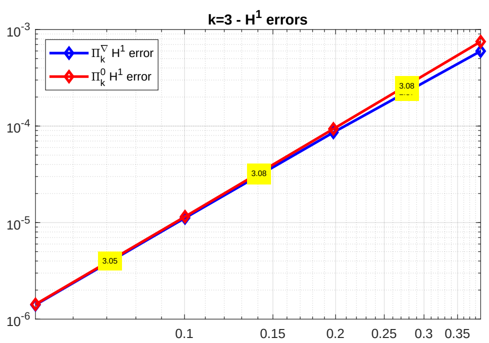

Fig. 13: H^1 error

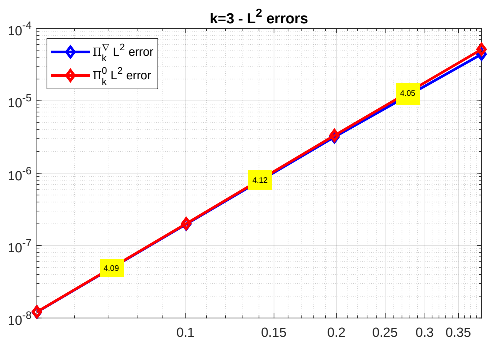

Fig. 14: L^2 error

8.2. *Puzzle mesh*

For this numerical test and for all the subsequent ones we consider the problem

$$-\Delta u = f \quad \text{in } \Omega =]0, 1[\times]0, 1[, \qquad u = g \text{ on } \partial\Omega,$$

with the data f and g chosen case by case. We take a family of meshes whose elements are general polygons with edges given by a parametric cubic curve. The curve has the same shape for all edges, and shrinks uniformly with the edge length. In this case, the method described in [18] would not work. Two typical meshes, with 25 and 100 polygons, are shown in Figures 15 and 16, respectively.

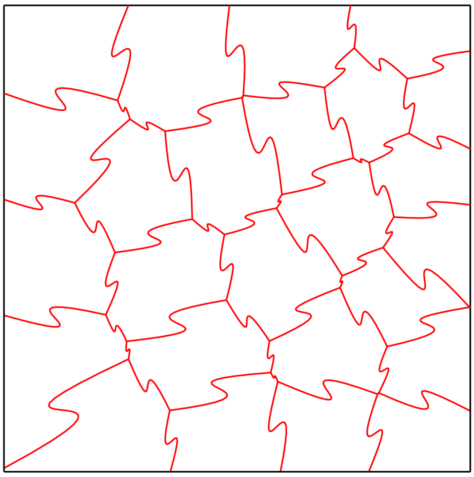

Fig. 15: Mesh with 25 polygons

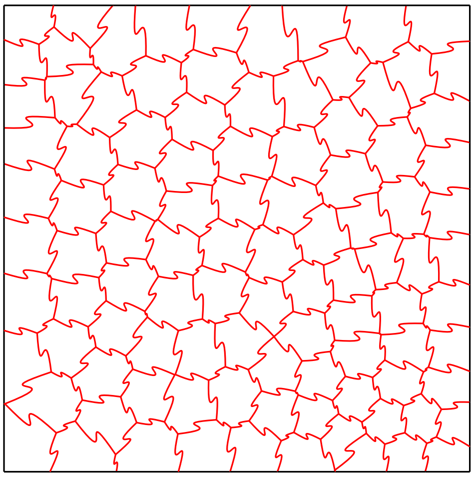

Fig. 16: Mesh with 100 polygons

8.2.1. *Patch test*

We first check that our method passes the patch test. We take as exact solution the third-degree polynomial

$$p_3(x,y) = -9x^3 + \frac{27}{4}x^2y + 9x^2 + \frac{81}{4}xy^2 - \frac{81}{4}xy + x - \frac{207}{8}y^3 + \frac{171}{8}y^2 - \frac{5}{4}y - 1.$$

The data f and g are chosen accordingly. We compute the VEM solution for $k = 3$ on the two meshes given in Figures 15 and 16. The results are shown in Figures 17 and 18 respectively.

8.2.2. *Convergence*

Now we consider the problem whose exact solution is smooth and is given by

$$u_{\text{ex}}(x,y) = y - x + \log(y^3 + x + 1) - xy - xy^2 + x^2y + x^2 + x^3 + \sin(5x)\sin(7y) - 1.$$

We consider the Virtual Element of degree $k = 4$ and a sequence of meshes with increasing number of polygons, starting from the meshes of Figures 15 and 16. In Figures 19 and 20 we report the approximation error in H^1 and L^2 respectively.

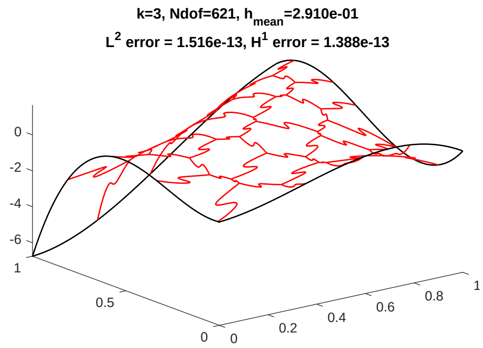

Fig. 17: Mesh with 25 polygons, patch-test of order 3

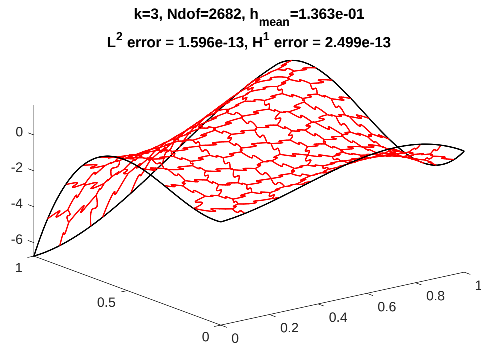

Fig. 18: Mesh with 100 polygons, patch-test of order 3

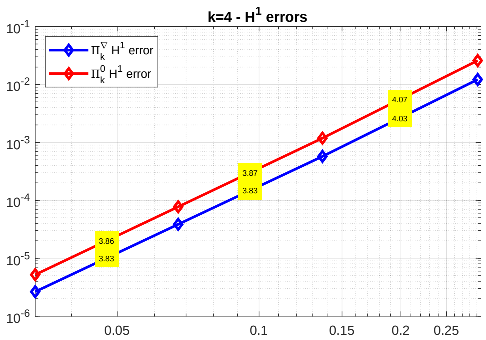

Fig. 19: H^1 error

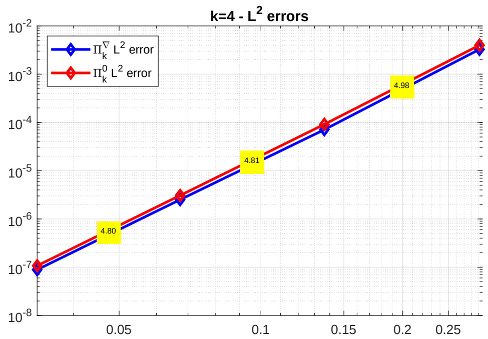

Fig. 20: L^2 error

8.3. *"Sail" mesh*

In this experiment we subdivide the unit square into small squares of side h and in turn we divide each small square into two "triangles" with a curved edge as shown in Fig. 21. The curved edge is a parabola with respect to the diagonal of the small square and the distance of the vertex from the diagonal itself is proportional to h^α, with $\alpha = 1$ or $\alpha = 2$. The two cases are very different:

- Case $\alpha = 1$: in this case all elements are homothetically equivalent, compare Fig. 21 with Fig. 23 where the element has been shrinked by a factor of 10;
- Case $\alpha = 2$: in this case when h gets smaller the elements get flatter, compare Fig. 21 with Fig. 24 where the element has been shrinked by a factor of 10.

We point out that starting from $k = 3$, the space of polynomials of degree less than or equal to k in two variables restricted to the curved edge degenerates, so that the tgp are no more degrees of freedom.

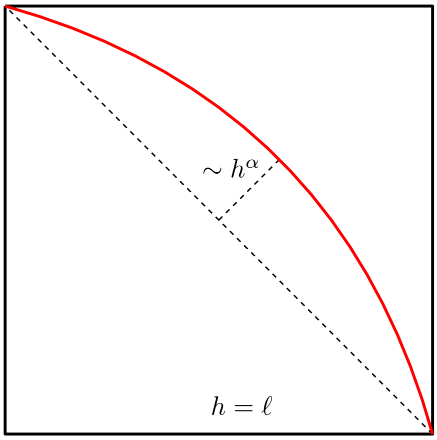

Fig. 21: "Sail" element

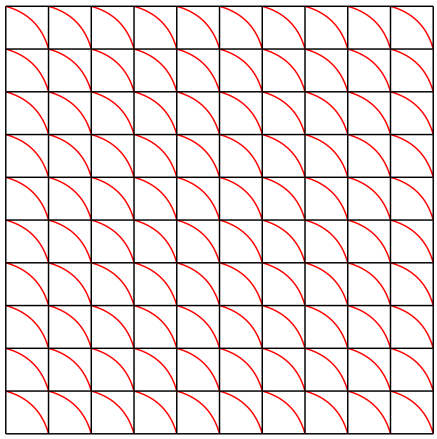

Fig. 22: "Sail" mesh

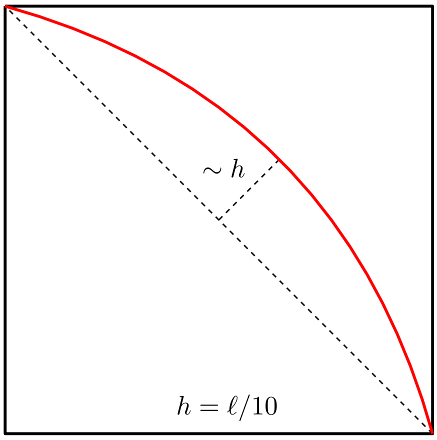

Fig. 23: Case $\alpha = 1$

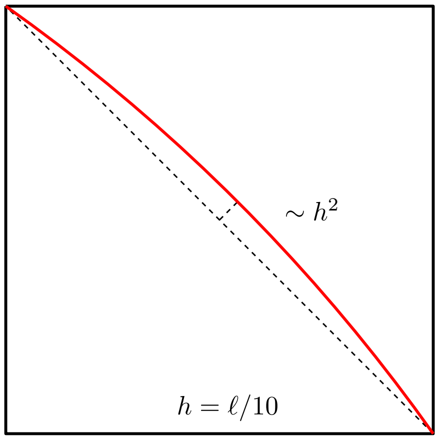

Fig. 24: Case $\alpha = 2$

8.3.1. *Patch test*

We check the patch test in the case $k = 3$ on the mesh obtained dividing the unit square into 10×10 small squares. The exact solution is the same of subsection 8.2.1. We show both the results for $\alpha = 1$ and $\alpha = 2$ in Figures 25 and 26 respectively.

8.3.2. *Convergence*

We take as exact solution the function u_{ex} defined in Subsection 8.2.2 and we consider the sequence of meshes described above obtained by subdividing the unit square into 5×5, 10×10, 20×20 and 40×40 small squares. In Figures 27 and 28 we show the convergence curves for $\alpha = 1$ in the H^1 norm and the L^2 norm

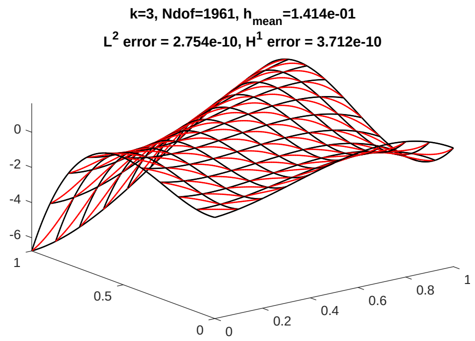

Fig. 25: Case $\alpha = 1$, patch test

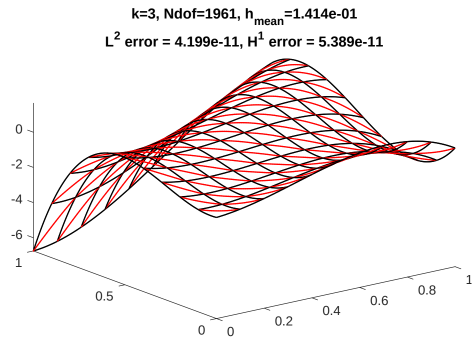

Fig. 26: Case $\alpha = 2$, patch test

respectively, while in Figures 29 and 30 we repeat the same experiments for the meshes with $\alpha = 2$.

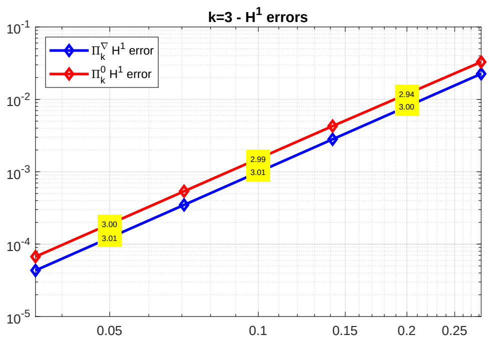

Fig. 27: H^1 error, $\alpha = 1$

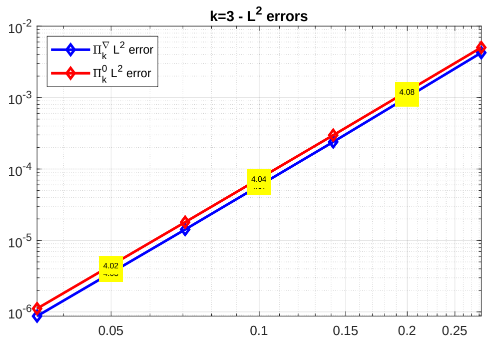

Fig. 28: L^2 error, $\alpha = 1$

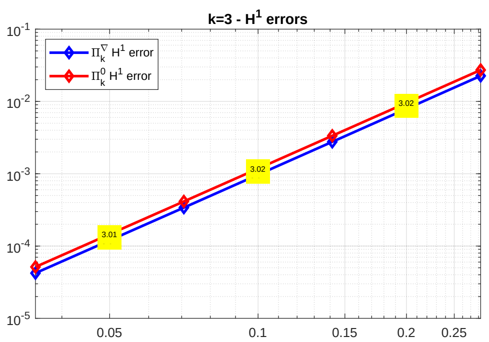

Fig. 29: H^1 error, $\alpha = 2$

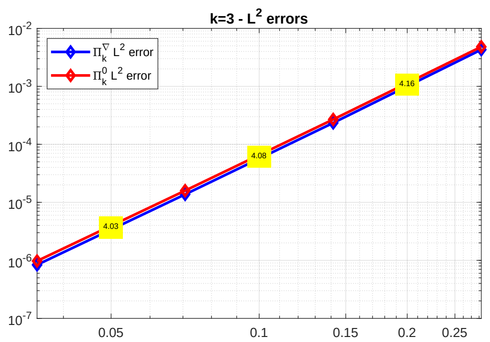

Fig. 30: L^2 error, $\alpha = 2$

8.3.3. *Comparison with the isoparametric Finite Elements*

For this last case we can make a comparison with the isoparametric cubic finite elements for triangles. As expected, Finite Elements lose the optimal order of convergence for $\alpha = 1$ (Figures 31 and 32) while for $\alpha = 2$ they recover convergence with the right rates (Figures 33 and 34). Of course, in this case the patch-test fails.

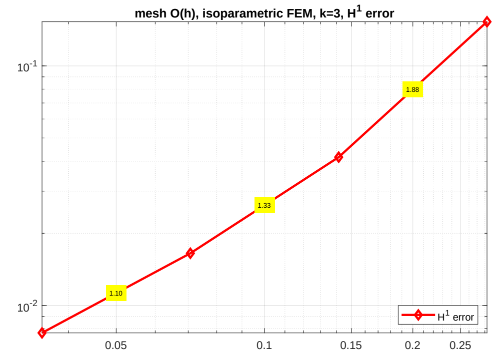

Fig. 31: Isoparametric FEM, H^1 error, $\alpha = 1$

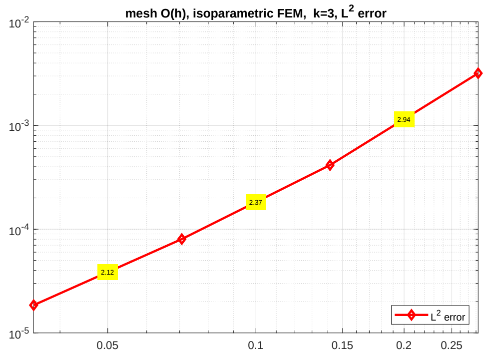

Fig. 32: Isoparametric FEM, L^2 error, $\alpha = 1$

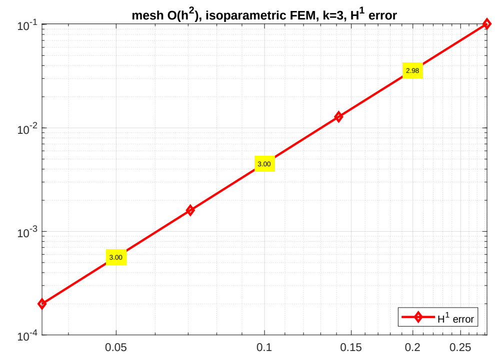

Fig. 33: Isoparametric FEM, H^1 error, $\alpha = 2$

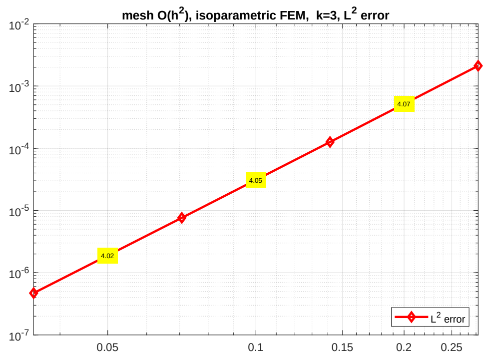

Fig. 34: Isoparametric FEM, L^2 error, $\alpha = 2$

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